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Dynamic Monetary Risk Measures for Bounded Discrete-Time Processes

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Abstract

We study dynamic monetary risk measures that depend on bounded discrete-time processes describing the evolution of financial values. The time horizon can be finite or infinite. We call a dynamic risk measure time-consistent if it assigns to a process of financial values the same risk irrespective of whether it is calculated directly or in two steps backwards in time. We show that this condition translates into a decomposition property for the corresponding acceptance sets, and we demonstrate how time-consistent dynamic monetary risk measures can be constructed by pasting together one-period risk measures. For conditional coherent and convex monetary risk measures, we provide dual representations of Legendre–Fenchel type based on linear functionals induced by adapted increasing processes of integrable variation. Then we give dual characterizations of time-consistency for dynamic coherent and convex monetary risk measures. To this end, we introduce a concatenation operation for adapted increasing processes of integrable variation, which generalizes the pasting of probability measures. In the coherent case, time-consistency corresponds to stability

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under concatenation in the dual. For dynamic convex monetary risk measures, the dual characterization of time-consistency generalizes to a condition on the family of convex conjugates of the conditional risk measures at different times. The theoretical results are applied by discussing the time-consistency of various specific examples of dynamic monetary risk measures that depend on bounded discrete-time processes.

Key words: Conditional monetary risk measures, Conditional monetary utility functions, Conditional dual representations, Dynamic monetary risk measures, Dynamic monetary utility functions, Time-consistency, Decomposition property of acceptance sets, Concatenation of adapted increasing processes of integrable variation.

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1 Introduction

Motivated by certain shortcomings of traditional risk measures such as Value-at-Risk, Artzner et al. (1997, 1999) gave an axiomatic analysis of capital requirements and introduced the notion of a coherent risk measure. These risk measures were further developed in Delbaen (2000, 2002). In Föllmer and Schied (2002a, 2002b, 2004) and Frittelli and Rosazza Gianin (2002) the more general concepts of convex and monetary risk measures were established. In all these works the setting is static, that is, the risky objects are real-valued random variables describing future financial values and the risk of such financial values is only measured at the beginning of the time-period under consideration. It has been shown that in this framework, monetary risk measures can be characterized by their acceptance sets, and dual representations of Legendre–Fenchel type have been derived for coherent as well as convex monetary risk measures. For relations of coherent and convex monetary risk measures to pricing and hedging in incomplete markets, we refer to Jaschke and Küchler (2001), Carr et al. (2001), Frittelli and Rosazza Gianin (2004), and Staum (2004).

In a multi-period or continuous-time model, the risky objects can be taken to be cash-flow streams or processes that model the evolution of financial values, and risk measurements can be updated as new information is becoming available over time.

The study of dynamic consistency or time-consistency for preferences goes back at least to Koopmans (1960). For further contributions, see for instance, Epstein and Zin (1989), Duffie and Epstein (1992), Wang (2003), Epstein and Schneider (2003).

Examples and characterizations of time-consistent coherent risk measures are given in Artzner et al. (2004), Delbaen (2003), Riedel (2004), Roorda et al. (2005). Weber (2005) studies time-consistent distribution-based convex monetary risk measures that depend on final values. Rosazza Gianin (2003) and Barrieu and El Karoui (2004) give relations between dynamic risk measures and backward stochastic differential equations. Cheridito et al. (2004, 2005) contain representation results for static coherent and convex monetary risk measures that depend on financial values evolving continuously in time. Frittelli and Scandolo (2005) study static risk measures for cash-flow streams in a discrete-time framework.

In this paper we consider dynamic coherent, convex monetary and monetary risk measures for discrete-time processes modelling the evolution of financial values. We simply call these processes value processes. Typical examples are:

- the market value of a firm's equity
- the accounting value of a firm's equity
- the market value of a portfolio of financial securities
- the surplus of an insurance company.

We first introduce coherent, convex monetary and monetary risk measures conditional on the information available at a stopping time and study the relation between such risk measures and their acceptance sets. Then, we pair the space of bounded adapted processes with the space of adapted processes of integrable variation and provide dual representations of Legendre–Fenchel type for conditional coherent and convex monetary risk measures; see

Theorems 3.16 and 3.18. In Definition 3.19, we extend the notion of relevance to our setup. It plays an important role in the consistent updating of risk measures. In Proposition 3.21, Theorem 3.23 and Corollary 3.24, we relate relevance of conditional coherent and convex monetary risk measures to a strict positivity condition in the dual.

A dynamic risk measure is a family of conditional risk measures at different times. We call it time-consistent if it fulfills a dynamic programming type condition; see Definition 4.2. In Theorem 4.6, we show that for dynamic monetary risk measures, the time-consistency condition is equivalent to a simple decomposition property of the corresponding acceptance sets. The ensuing Corollary 4.8 shows that a relevant static monetary risk measure has at most one dynamic extension that is time-consistent. Also, it is shown how arbitrary one-period monetary risk measures can be pasted together to form a time-consistent dynamic risk measures. For dynamic coherent and convex monetary risk measures, we give dual characterizations of time-consistency. To this end, we introduce a concatenation operation for adapted increasing processes of integrable variation. This generalizes the pasting of probability measures as it appears, for instance, in Wang (2003), Epstein and Schneider (2003), Artzner et al. (2004), Delbaen (2003), Riedel (2004), Roorda et al. (2005). In the coherent case, time-consistency corresponds to stability under concatenation in the dual; see Theorems 4.13, 4.15 and their Corollaries 4.14, 4.16. In the convex monetary case, the dual characterization generalizes to a condition on the convex conjugates of the conditional risk measures at different times; see Theorems 4.19, 4.22 and their Corollaries 4.20, 4.23. The paper concludes with a discussion of the time-consistency of various examples of dynamic monetary risk measures for processes.

We also refer to the articles Bion Nadal (2004), Detlefsen and Scandolo (2005), and Ruszczyński and Shapiro (2005), which were written independently of this paper and study conditional dual representations and time-consistency of convex monetary risk measures.

2 The setup and notation

We denote $\mathbb{N} = \{0, 1, 2, \dots\}$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}}, P)$ be a filtered probability space with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. All equalities and inequalities between random variables or stochastic processes are understood in the P-almost sure sense. For instance, if $(X_t)_{t\in\mathbb{N}}$ and $(Y_t)_{t\in\mathbb{N}}$ are two stochastic processes, we mean by $X \geq Y$ that for P-almost all $\omega \in \Omega$, $X_t(\omega) \geq Y_t(\omega)$ for all $t \in \mathbb{N}$. Also, equalities and inclusions between sets in $\mathcal{F}$ are understood in the P-almost sure sense, that is, for $A, B \in \mathcal{F}$, we write $A \subset B$ if $P[A \setminus B] = 0$. By $\mathcal{R}^0$ we denote the space of all adapted stochastic processes $(X_t)_{t\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}}, P)$, where we identify processes that are equal P-almost surely. The two subspaces $\mathcal{R}^\infty$ and $\mathcal{A}^1$ of $\mathcal{R}^0$ are given by

$$\mathcal{R}^\infty := \left\{X \in \mathcal{R}^0 \mid ||X||_{\mathcal{R}^\infty} < \infty\right\} ,$$

where

$$||X||_{\mathcal{R}^\infty} := \inf\left\{m \in \mathbb{R} \mid \sup_{t\in\mathbb{N}} |X_t| \leq m\right\}$$

and

$$\mathcal{A}^1 := \left\{a \in \mathcal{R}^0 \mid ||a||_{\mathcal{A}^1} < \infty\right\} ,$$

where

$$a_{-1} := 0\,, \quad \Delta a_t := a_t - a_{t-1}\,, \text{ for } t \in \mathbb{N}\,, \quad \text{and} \quad ||a||_{\mathcal{A}^1} := \mathrm{E}\left[\sum_{t\in\mathbb{N}} |\Delta a_t|\right].$$

The bilinear form $\langle .,. \rangle$ on $\mathcal{R}^\infty \times \mathcal{A}^1$ is given by

$$\langle X, a\rangle := \mathrm{E}\left[\sum_{t\in\mathbb{N}} X_t\, \Delta a_t\right].$$

$\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$ denotes the coarsest topology on $\mathcal{R}^\infty$ such that for all $a \in \mathcal{A}^1$, $X \mapsto \langle X, a\rangle$ is a continuous linear functional on $\mathcal{R}^\infty$. $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$ denotes the coarsest topology on $\mathcal{A}^1$ such that for all $X \in \mathcal{R}^\infty$, $a \mapsto \langle X, a\rangle$ is a continuous linear functional on $\mathcal{A}^1$.

We call an $(\mathcal{F}_t)$-stopping time τ finite if $\tau < \infty$ and bounded if $\tau \le N$ for some $N \in \mathbb{N}$. For two $(\mathcal{F}_t)$-stopping times τ and θ such that τ is finite and $0 \le \tau \le \theta \le \infty$, we define the projection $\pi_{\tau,\theta} : \mathcal{R}^0 \to \mathcal{R}^0$ by

$$\pi_{\tau,\theta}(X)_t := 1_{\{\tau \le t\}} X_{t\wedge\theta}\,, \quad t \in \mathbb{N}\,.$$

For $X \in \mathcal{R}^\infty$, we set

$$||X||_{\tau,\theta} := \operatorname{ess\,inf}\left\{m \in L^\infty(\mathcal{F}_\tau) \mid \sup_{t\in\mathbb{N}} |\pi_{\tau,\theta}(X)_t| \le m\right\},$$

where ess inf denotes the essential infimum of a family of random variables (see for instance, Proposition VI.1.1 of Neveu, 1975). $||X||_{\tau,\theta}$ is the $\mathcal{R}^\infty$-norm of the projection $\pi_{\tau,\theta}(X)$ conditional on $\mathcal{F}_\tau$. Clearly, $||X||_{\tau,\theta} \le ||X||_{\mathcal{R}^\infty}$. The risky objects considered in this paper are stochastic processes in $\mathcal{R}^\infty$. But since we want to consider risk measurement at different times and discuss time-consistency questions, we also need the subspaces

$$\mathcal{R}^\infty_{\tau,\theta} := \pi_{\tau,\theta}\mathcal{R}^\infty\,.$$

A process $X \in \mathcal{R}^\infty_{\tau,\theta}$ is meant to describe the evolution of a financial value on the discrete time interval $[\tau, \theta] \cap \mathbb{N}$. We assume that there exists a cash account where money can be deposited at a risk-free rate and use it as numeraire, that is, all prices are expressed in multiples of one dollar put into the cash account at time 0. We emphasize that we do not assume that money can be borrowed at the same rate. A conditional monetary risk measure on $\mathcal{R}^\infty_{\tau,\theta}$ is a mapping

$$\rho : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)\,,$$

assigning a value process $X \in \mathcal{R}^\infty_{\tau,\theta}$ a real number that can depend on the information available at the stopping time τ and specifies the minimal amount of money that has to be held in the cash account to make X acceptable at time τ. By our choice of the numeraire, the infusion of an amount of money m at time τ transforms a value process $X \in \mathcal{R}^\infty_{\tau,\theta}$ into $X + m1_{[\tau,\infty)}$ and reduces the risk of X to $\rho(X) - m$.

We find it more convenient to work with negatives of risk measures. If ρ is a conditional monetary risk measure on $\mathcal{R}^\infty_{\tau,\theta}$, we call $\phi = -\rho$ the conditional monetary utility function corresponding to ρ.

3 Conditional monetary utility functions

In this section we extend the concepts of monetary, convex and coherent risk measures to our setup and prove corresponding representation results. In all of Section 3, τ and θ are two fixed $(\mathcal{F}_t)$-stopping times such that $0 \leq \tau < \infty$ and $\tau \leq \theta \leq \infty$.

3.1 Basic definitions and easy properties

In the subsequent definition we extend the axioms of Artzner et al. (1999), Föllmer and Schied (2002a), Frittelli and Rosazza Gianin (2002) for static risk measures to our dynamic framework. Now, the risky objects are value processes instead of random variables, and risk assessment at a finite $(\mathcal{F}_t)$-stopping time τ is based on the information described by $\mathcal{F}_\tau$.

Axiom (M) in Definition 3.1 is the extension of the monotonicity axiom in Artzner et al. (1999) to value processes. (TI), (C) and (PH) are $\mathcal{F}_\tau$-conditional versions of corresponding axioms in Artzner et al. (1999), Föllmer and Schied (2002a), Frittelli and Rosazza Gianin (2002). The normalization axiom (N) is convenient for the purposes of this paper. Differently normalized conditional monetary utility functions on $\mathcal{R}^\infty_{\tau,\theta}$ can be obtained by the addition of an $\mathcal{F}_\tau$-measurable random variable.

Definition 3.1 *We call a mapping $\phi : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ a conditional monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$ if it has the following properties:*

(N) Normalization: $\phi(0) = 0$

(M) Monotonicity: $\phi(X) \leq \phi(Y)$ *for all* $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$ *such that* $X \leq Y$

(TI) $\mathcal{F}_\tau$-Translation Invariance: $\phi(X + m1_{[\tau,\infty)}) = \phi(X) + m$ *for all* $X \in \mathcal{R}^\infty_{\tau,\theta}$ *and* $m \in L^\infty(\mathcal{F}_\tau)$

We call a conditional monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ a conditional concave monetary utility function if it satisfies
(C) $\mathcal{F}_\tau$-Concavity: $\phi(\lambda X + (1-\lambda)Y) \geq \lambda\phi(X) + (1-\lambda)\phi(Y)$ *for all* $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$ *and* $\lambda \in L^\infty(\mathcal{F}_\tau)$ *such that* $0 \leq \lambda \leq 1$

We call a conditional concave monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ a conditional coherent utility function if it satisfies
(PH) $\mathcal{F}_\tau$-Positive Homogeneity: $\phi(\lambda X) = \lambda\phi(X)$ *for all* $X \in \mathcal{R}^\infty_{\tau,\theta}$ *and* $\lambda \in L^\infty_+(\mathcal{F}_\tau) := \{f \in L^\infty(\mathcal{F}_\tau) \mid f \geq 0\}$.

For a conditional monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ and $X \in \mathcal{R}^\infty$, we define $\phi(X) := \phi \circ \pi_{\tau,\theta}(X)$.

A conditional monetary risk measure on $\mathcal{R}^\infty_{\tau,\theta}$ is a mapping $\rho : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ such that $-\rho$ is a conditional monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$. ρ is a conditional convex monetary risk measure if $-\rho$ is a conditional concave monetary utility function and a conditional coherent risk measure if $-\rho$ is a conditional coherent utility function.

Remark 3.2 It is easy to check that a mapping $\phi : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ is a conditional coherent utility function on $\mathcal{R}^\infty_{\tau,\theta}$ if and only if it satisfies (M), (TI) and (PH) of Definition 3.1 together with

(SA) Superadditivity: $\phi(X+Y) \geq \phi(X) + \phi(Y)$ for all $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$.

As in the static case, the axioms (M) and (TI) imply Lipschitz-continuity. But since here, (TI) means $\mathcal{F}_\tau$-translation invariance instead of translation invariance with respect to real numbers, we can derive the stronger $\mathcal{F}_\tau$-Lipschitz continuity (LC) below, which implies the local property (LP). The economic interpretation of (LP) is that a conditional monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ does only depend on future scenarios that have not been ruled out by events that have occurred until time τ.

Proposition 3.3 *Let ϕ be a function from $\mathcal{R}^\infty_{\tau,\theta}$ to $L^\infty(\mathcal{F}_\tau)$ that satisfies* (M) *and* (TI) *of Definition 3.1. Then it also satisfies the following two properties:*

(LC) $\mathcal{F}_\tau$-Lipschitz Continuity: $|\phi(X) - \phi(Y)| \leq ||X - Y||_{\tau,\theta}$, *for all* $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$.

(LP) Local Property: $\phi(1_A X + 1_{A^c} Y) = 1_A \phi(X) + 1_{A^c} \phi(Y)$ *for all* $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$ *and* $A \in \mathcal{F}_\tau$.

Proof. It follows from (M) and (TI) that for all $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$,

$$\phi(X) \leq \phi(Y + ||X - Y||_{\tau,\theta}) = \phi(Y) + ||X - Y||_{\tau,\theta} \,,$$

Hence, $\phi(X) - \phi(Y) \leq ||X - Y||_{\tau,\theta}$, and (LC) follows by exchanging the roles of X and Y. It can be deduced from (LC) that for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $A \in \mathcal{F}_\tau$,

$$|1_A \phi(X) - 1_A \phi(1_A X)| = 1_A \, |\phi(X) - \phi(1_A X)| \leq 1_A ||X - 1_A X||_{\tau,\theta} = 0 \,,$$

which implies (LP). □

Remark 3.4 It can be shown with an approximation argument that every function $\phi : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ satisfying (M), (LP) and the real translation invariance:

(TI') $\phi(X + m 1_{[\tau,\infty)}) = \phi(X) + m$ for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $m \in \mathbb{R}$,

also fulfills the $\mathcal{F}_\tau$-translation invariance (TI). Indeed, it follows from (TI') and (LP) that (TI') also holds for m of the form

$$m = \sum_{k=1}^{K} m_k 1_{A_k} \,, \tag{3.1}$$

where $m_k \in \mathbb{R}$ and $A_k \in \mathcal{F}_\tau$ for $k = 1, \dots, K$. For general $m \in L^\infty(\mathcal{F}_\tau)$, there exists a sequence $(m^n)_{n \geq 1}$ of elements of the form (3.1) such that $||m - m^n||_{L^\infty} \to \infty$, as $n \to \infty$. As in the proof of Proposition 3.3, it can be deduced from (M) and (TI') that

$$|\phi(X) - \phi(Y)| \leq ||X - Y||_{R^\infty} \quad \text{for all } X, Y \in \mathcal{R}^\infty_{\tau,\theta} \,.$$

Hence,

$$\phi(X + m1_{[\tau,\infty)}) = \lim_{n\to\infty} \phi(X + m^n 1_{[\tau,\infty)}) = \lim_{n\to\infty} \phi(X) + m^n = \phi(X) + m\,.$$

Analogously, it can be shown that under (M), (LP) and (TI'), ordinary concavity implies $\mathcal{F}_\tau$-concavity, and positive homogeneity with respect to $\lambda \in \mathbb{R}_+$, implies $\mathcal{F}_\tau$-positive homogeneity.

Next, we introduce the acceptance set $\mathcal{C}_\phi$ of a conditional monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ and show how ϕ can be recovered from $\mathcal{C}_\phi$. In contrast to the static case, everything is done conditionally on $\mathcal{F}_\tau$ and the local property plays an important role.

Definition 3.5 *The acceptance set $\mathcal{C}_\phi$ of a conditional monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ is given by*

$$\mathcal{C}_\phi := \left\{X \in \mathcal{R}^\infty_{\tau,\theta} \mid \phi(X) \geq 0\right\}\,.$$

Proposition 3.6 *The acceptance set $\mathcal{C}_\phi$ of a condtional monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ has the following properties:*

(n) Normalization: $\operatorname{ess\,inf}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid f1_{[\tau,\infty)} \in \mathcal{C}_\phi\right\} = 0$.

(m) Monotonicity: $X \in \mathcal{C}_\phi$, $Y \in \mathcal{R}^\infty_{\tau,\theta}$, $X \leq Y \Rightarrow Y \in \mathcal{C}_\phi$

(cl) $\mathcal{F}_\tau$-Closedness: $(X^n)_{n\in\mathbb{N}} \subset \mathcal{C}_\phi$, $X \in \mathcal{R}^\infty_{\tau,\theta}$, $||X^n - X||_{\tau,\theta} \stackrel{\text{a.s.}}{\to} 0 \Rightarrow X \in \mathcal{C}_\phi$.

(lp) Local Property $1_A X + 1_{A^c} Y \in \mathcal{C}_\phi$ *for all* $X, Y \in \mathcal{C}_\phi$ *and* $A \in \mathcal{F}_\tau$.

If ϕ is a conditional concave monetary utility function, then $\mathcal{C}_\phi$ satisfies

(c) $\mathcal{F}_\tau$-Convexity: $\lambda X + (1-\lambda)Y \in \mathcal{C}_\phi$ *for all* $X, Y \in \mathcal{C}_\phi$ *and* $\lambda \in L^\infty(\mathcal{F}_\tau)$ *such that* $0 \leq \lambda \leq 1$.

If ϕ is a conditional coherent utility function, then $\mathcal{C}_\phi$ satisfies

(ph) $\mathcal{F}_\tau$-Positive Homogeneity: $\lambda X \in \mathcal{C}_\phi$ *for all* $X \in \mathcal{C}_\phi$ *and* $\lambda \in L^\infty_+(\mathcal{F}_\tau)$, *and*

(sa) Stability under addition: $X + Y \in \mathcal{C}_\phi$ *for all* $X, Y \in \mathcal{C}_\phi$.

Proof. (n): It follows from the definition of $\mathcal{C}_\phi$ together with (N) and (TI) of Definition 3.1 that

$$\begin{aligned} & \operatorname{ess\,inf}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid f1_{[\tau,\infty)} \in \mathcal{C}_\phi\right\} = \operatorname{ess\,inf}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid \phi(f1_{[\tau,\infty)}) \geq 0\right\} \\ = \; & \operatorname{ess\,inf}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid \phi(0) + f \geq 0\right\} = \operatorname{ess\,inf}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid f \geq 0\right\} = 0\,. \end{aligned}$$

(m) follows directly from (M) of Definition 3.1.
(cl): Let $(X^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}_\phi$ and $X \in \mathcal{R}^\infty_{\tau,\theta}$ such that $||X^n - X||_{\tau,\theta} \stackrel{\text{a.s.}}{\to} 0$. By (LC) of Proposition 3.3,

$$\phi(X) \geq \phi(X^n) - ||X^n - X||_{\tau,\theta}\,,$$

for all $n \in \mathbb{N}$. Hence, $\phi(X) \geq 0$.
(lp) follows from the fact that ϕ satisfies the local property (LP) of Proposition 3.3.
The remaining statements of the proposition are obvious. □

Definition 3.7 *For an arbitrary subset $\mathcal{C}$ of $\mathcal{R}^\infty_{\tau,\theta}$, we define for all $X \in \mathcal{R}^\infty_{\tau,\theta}$,*

$$\phi_{\mathcal{C}}(X) := \operatorname{ess\,sup}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid X - f\,1_{[\tau,\infty)} \in \mathcal{C}\right\} ,$$

with the convention

$$\operatorname{ess\,sup}\emptyset := -\infty .$$

Remark 3.8 Note that if $\mathcal{C}$ satisfies the local property (lp) of Proposition 3.6 and, for given $X \in \mathcal{R}^\infty_{\tau,\theta}$, the set

$$\left\{f \in L^\infty(\mathcal{F}_\tau) \mid X - f\,1_{[\tau,\infty)} \in \mathcal{C}\right\}$$

is non-empty, then it is directed upwards, and hence, contains an increasing sequence $(f^n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f^n = \phi_{\mathcal{C}}(X)$ almost surely (see Proposition VI.1.1 of Neveu, 1975).

Proposition 3.9 *Let ϕ be a conditional monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$. Then $\phi_{\mathcal{C}_\phi} = \phi$.*

Proof. For all $X \in \mathcal{R}^\infty_{\tau,\theta}$,

$$\begin{aligned}
\phi_{\mathcal{C}_\phi}(X) &= \operatorname{ess\,sup}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid X - f1_{[\tau,\infty)} \in \mathcal{C}_\phi\right\}\\
&= \operatorname{ess\,sup}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid \phi(X - f1_{[\tau,\infty)}) \geq 0\right\}\\
&= \operatorname{ess\,sup}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid \phi(X) \geq f\right\} = \phi(X) .
\end{aligned}$$

□

Proposition 3.10 *If $\mathcal{C}$ is a subset of $\mathcal{R}^\infty_{\tau,\theta}$ that satisfies* (n) *and* (m) *of Proposition 3.6, then $\phi_{\mathcal{C}}$ is a conditional monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$ and $\mathcal{C}_{\phi_{\mathcal{C}}}$ is the smallest subset of $\mathcal{R}^\infty_{\tau,\theta}$ that contains $\mathcal{C}$ and satisfies the conditions* (cl) *and* (lp) *of Proposition 3.6.*
If $\mathcal{C}$ satisfies (n), (m) *and* (c) *of Proposition 3.6, then $\phi_{\mathcal{C}}$ is a conditional concave monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$.*
If $\mathcal{C}$ satisfies (n), (m), (c) *and* (ph) *or* (n), (m), (a) *and* (ph) *of Proposition 3.6, then $\phi_{\mathcal{C}}$ is a conditional coherent utility function on $\mathcal{R}^\infty_{\tau,\theta}$.*

Proof. (N) of Definition 3.1 follows from (n) of Proposition 3.6, and (M) of Definition 3.1 from (m) of Proposition 3.6. (TI) of Definition 3.1 follows directly from the definition of $\phi_{\mathcal{C}}$. By Proposition 3.6, $\mathcal{C}_{\phi_{\mathcal{C}}}$ satisfies the conditions (cl) and (lp), and it obviously contains $\mathcal{C}$. To show that $\mathcal{C}_{\phi_{\mathcal{C}}}$ is the smallest subset of $\mathcal{R}^\infty_{\tau,\theta}$ that contains $\mathcal{C}$ and satisfies the properties (cl) and (lp) of Proposition 3.6, we introduce the set

$$\tilde{\mathcal{C}} := \left\{\sum_{k=1}^K 1_{A_k} X^k \mid K \geq 1\,,\ (A_k)_{k=1}^K \text{ an } \mathcal{F}_\tau\text{-partition of } \Omega\,,\ X^k \in \mathcal{C} \text{ for all } k\right\} .$$

It is the smallest subset of $\mathcal{R}^\infty_{\tau,\theta}$ containing $\mathcal{C}$ and satisfying (lp). Obviously, $\tilde{\mathcal{C}}$ inherits from $\mathcal{C}$ the monotonicity property (m) of Proposition 3.6, and by Remark 3.8, there exists for every $X \in \mathcal{C}_{\phi_{\mathcal{C}}}$, an increasing sequence $(f^n)_{n\in\mathbb{N}}$ in $L^\infty(\mathcal{F}_\tau)$ such that $X - f^n 1_{[\tau,\infty)} \in \tilde{\mathcal{C}}$

and $f^n \overset{\text{a.s.}}{\to} \phi_{\mathcal{C}}(X) \geq 0$. Set $g^n := f^n \wedge 0$. Then, $X - g^n 1_{[\tau,\infty)} \in \tilde{\mathcal{C}}$, and $g^n \to 0$ almost surely, which shows that $\mathcal{C}_{\phi_{\mathcal{C}}}$ is the smallest subset of $\mathcal{R}^\infty_{\tau,\theta}$ that satisfies the condition (cl) of Proposition 3.6 and contains $\tilde{\mathcal{C}}$. It follows that $\mathcal{C}_{\phi_{\mathcal{C}}}$ is the smallest subset of $\mathcal{R}^\infty_{\tau,\theta}$ containing $\mathcal{C}$ and satisfying the conditions (cl) and (lp) of Proposition 3.6. □

3.2 Dual representations of conditional concave monetary and coherent utility functions on $\mathcal{R}^\infty_{\tau,\theta}$

In this subsection we generalize duality results of Artzner et al. (1999), Delbaen (2002), Föllmer and Schied (2002a), and Frittelli and Rosazza Gianin (2002). Similar results for coherent risk measures have been obtained by Riedel (2004) and Roorda et al. (2005).

We work with conditional positive linear functionals on $\mathcal{R}^\infty_{\tau,\theta}$ that are induced by elements in $\mathcal{A}^1$. More precisely, we define

$$\langle X, a \rangle_{\tau,\theta} := \mathrm{E}\left[\sum_{t \in [\tau,\theta] \cap \mathbb{N}} X_t \Delta a_t \mid \mathcal{F}_\tau\right], \quad X \in \mathcal{R}^\infty, \, a \in \mathcal{A}^1,$$

and introduce the following subsets of $\mathcal{A}^1$:

$$\begin{aligned} \mathcal{A}^1_+ &:= \left\{a \in \mathcal{A}^1 \mid \Delta a_t \geq 0 \text{ for all } t \in \mathbb{N}\right\} \\ (\mathcal{A}^1_{\tau,\theta})_+ &:= \pi_{\tau,\theta}\mathcal{A}^1_+ \\ \text{and} \quad \mathcal{D}_{\tau,\theta} &:= \left\{a \in (\mathcal{A}^1_{\tau,\theta})_+ \mid \langle 1, a \rangle_{\tau,\theta} = 1\right\}. \end{aligned}$$

Processes in $\mathcal{D}_{\tau,\theta}$ can be viewed as conditional probability densities on the product space $\Omega \times \mathbb{N}$ and will play the role played by ordinary probability densities in the static case.

By $\bar{L}(\mathcal{F})$ we denote the space of all measurable functions from $(\Omega, \mathcal{F})$ to $[-\infty, \infty]$, where we identify two functions when they are equal P-almost surely, and we set

$$\bar{L}_-(\mathcal{F}) := \left\{f \in \bar{L}(\mathcal{F}) \mid f \leq 0\right\}.$$

Definition 3.11 *A penalty function γ on $\mathcal{D}_{\tau,\theta}$ is a mapping from $\mathcal{D}_{\tau,\theta}$ to $\bar{L}_-(\mathcal{F}_\tau)$ with the following property:*

$$\operatorname*{ess\,sup}_{a \in \mathcal{D}_{\tau,\theta}} \gamma(a) = 0. \tag{3.2}$$

We say that a penalty function γ on $\mathcal{D}_{\tau,\theta}$ has the local property if

$$\gamma(1_A a + 1_{A^c} b) = 1_A \gamma(a) + 1_{A^c} \gamma(b),$$

for all $a, b \in \mathcal{D}_{\tau,\theta}$ and $A \in \mathcal{F}_\tau$.

It is easy to see that for any penalty function γ on $\mathcal{D}_{\tau,\theta}$, the conditional Legendre–Fenchel type transform

$$\phi(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{\langle X, a \rangle_{\tau,\theta} - \gamma(a)\right\}, \quad X \in \mathcal{R}^\infty_{\tau,\theta}, \tag{3.3}$$

is a conditional concave monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$. Condition (3.2) corresponds to the normalization $\phi(0) = 0$. In the following we are going to show that every conditional concave monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ satisfying the upper semicontinuity condition of Definition 3.15 below, has a representation of the form (3.3) for $\gamma = \phi^\#$, where $\phi^\#$ is defined as follows:

Definition 3.12 *For a conditional concave monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ and $a \in \mathcal{A}^1$, we define*

$$\phi^\#(a) := \operatorname*{ess\,inf}_{X \in \mathcal{C}_\phi} \langle X, a \rangle_{\tau,\theta}$$

and

$$\phi^*(a) := \operatorname*{ess\,inf}_{X \in \mathcal{R}^\infty_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi(X) \right\} .$$

Remarks 3.13
Let ϕ be a conditional concave monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$.
1. Obviously, for all $a \in \mathcal{A}^1$, $\phi^\#(a)$ and $\phi^*(a)$ belong to $\bar{L}_-(\mathcal{F}_\tau)$, and

$$\phi^\#(a) \geq \phi^*(a) \quad \text{for all } a \in \mathcal{A}^1 .$$

Moreover,

$$\phi^\#(a) = \phi^*(a) \quad \text{for all } a \in \mathcal{D}_{\tau,\theta} \tag{3.4}$$

because for $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $a \in \mathcal{D}_{\tau,\theta}$,

$$\langle X, a \rangle_{\tau,\theta} - \phi(X) = \left\langle X - \phi(X) 1_{[\tau,\infty)}, a \right\rangle_{\tau,\theta} , \text{ and } \quad X - \phi(X) 1_{[\tau,\infty)} \in \mathcal{C}_\phi .$$

2. It can easily be checked that

$$\phi^\#(\lambda a + (1-\lambda) b) \geq \lambda \phi^\#(a) + (1-\lambda) \phi^\#(b) ,$$

for all $a, b \in \mathcal{A}^1$ and $\lambda \in L^\infty(\mathcal{F}_\tau)$ such that $0 \leq \lambda \leq 1$, and

$$\phi^\#(\lambda a) = \lambda \phi^\#(a) \quad \text{for all } a \in \mathcal{A}^1 \text{ and } \lambda \in L^\infty_+(\mathcal{F}_\tau) . \tag{3.5}$$

It follows from (3.5) that $\phi^\#$ satisfies the local property:

$$\phi^\#(1_A a + 1_{A^c} b) = 1_A \phi^\#(a) + 1_{A^c} \phi^\#(b)$$

for all $a, b \in \mathcal{A}^1$ and $A \in \mathcal{F}_\tau$.

In addition to the properties of Remarks 3.13, $\phi^\#$ fulfills the following two conditional versions of $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-upper semicontinuity:

Proposition 3.14 *Let ϕ be a conditional concave monetary utility function ϕ on $\mathcal{R}^{\infty}_{\tau,\theta}$. Then*
1. *For all $A \in \mathcal{F}_\tau$ and $m \in \mathbb{R}$,*

$$\left\{a \in \mathcal{D}_{\tau,\theta} \mid \mathrm{E}\left[1_A \phi^{\#}(a)\right] \geq m\right\}$$

is a $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed subset of $\mathcal{A}^1$.
2. *For every $f \in \bar{L}_-(\mathcal{F}_\tau)$,*

$$\left\{a \in \mathcal{D}_{\tau,\theta} \mid \phi^{\#}(a) \geq f\right\}$$

is a $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed subset of $\mathcal{A}^1$.

Proof.
1. Let $(a^\mu)_{\mu \in M}$ be a net in $\left\{a \in \mathcal{D}_{\tau,\theta} \mid \mathrm{E}\left[1_A \phi^{\#}(a)\right] \geq m\right\}$ and $a^0 \in \mathcal{A}^1$ such that $a^\mu \to a^0$ in $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$. Then, $a^0 \in \mathcal{D}_{\tau,\theta}$, and for all $X \in \mathcal{C}_\phi$ and $\mu \in M$,

$$\langle 1_A X, a^\mu \rangle = \mathrm{E}\left[1_A \langle X, a^\mu \rangle_{\tau,\theta}\right] \geq \mathrm{E}\left[1_A \phi^{\#}(a^\mu)\right] \geq m\,.$$

Hence, $\left\langle 1_A X, a^0 \right\rangle \geq m$ for all $X \in \mathcal{C}_\phi$. Since $\mathcal{C}_\phi$ has the local property (lp), the set $\left\{\left\langle X, a^0 \right\rangle_{\tau,\theta} \mid X \in \mathcal{C}_\phi\right\}$ is directed downwards, and therefore it follows from Beppo Levi's monotone convergence theorem that

$$\begin{aligned} \mathrm{E}\left[1_A \phi^{\#}(a^0)\right] &= \mathrm{E}\left[1_A \operatorname*{ess\,inf}_{X \in \mathcal{C}_\phi} \left\langle X, a^0 \right\rangle_{\tau,\theta}\right] \\ = \quad \inf_{X \in \mathcal{C}_\phi} \mathrm{E}\left[1_A \left\langle X, a^0 \right\rangle_{\tau,\theta}\right] &= \inf_{X \in \mathcal{C}_\phi} \left\langle 1_A X, a^0 \right\rangle \geq m\,. \end{aligned}$$

2. Let $(a^\mu)_{\mu \in M}$ be a net in $\left\{a \in \mathcal{D}_{\tau,\theta} \mid \phi^{\#}(a) \geq f\right\}$ and $a^0 \in \mathcal{A}^1$ such that $a^\mu \to a^0$ in $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$. Then, $a^0 \in \mathcal{D}_{\tau,\theta}$, and for all $X \in \mathcal{C}_\phi$, $\mu \in M$ and $A \in \mathcal{F}_\tau$,

$$\langle 1_A X, a^\mu \rangle = \mathrm{E}\left[1_A \langle X, a^\mu \rangle_{\tau,\theta}\right] \geq \mathrm{E}\left[1_A \phi^{\#}(a^\mu)\right] \geq \mathrm{E}\left[1_A f\right]\,.$$

Hence,

$$\mathrm{E}\left[1_A \left\langle X, a^0 \right\rangle_{\tau,\theta}\right] = \left\langle 1_A X, a^0 \right\rangle \geq \mathrm{E}\left[1_A f\right]\,,$$

which shows that

$$\left\langle X, a^0 \right\rangle_{\tau,\theta} \geq f\,, \quad \text{for all } X \in \mathcal{C}_\phi$$

and therefore, $\phi^{\#}(a^0) \geq f$. □

In the representation results, Theorem 3.16 and Theorem 3.18 below, the following upper semicontinuity property for conditional utility functions plays an important role.

Definition 3.15 *We call a function* $\phi : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ *continuous for bounded decreasing sequences if*

$$\lim_{n\to\infty} \phi(X^n) = \phi(X) \quad \textit{almost surely}$$

for every decreasing sequence $(X^n)_{n\in\mathbb{N}}$ *in* $\mathcal{R}^\infty_{\tau,\theta}$ *and* $X \in \mathcal{R}^\infty_{\tau,\theta}$ *such that*

$$X^n_t \overset{\text{a.s.}}{\to} X_t \quad \textit{for all } t \in \mathbb{N}\,.$$

Theorem 3.16 *The following are equivalent:*

(1) ϕ *is a mapping defined on* $\mathcal{R}^\infty_{\tau,\theta}$ *that can be represented as*

$$\phi(X) = \operatorname*{ess\,inf}_{a\in\mathcal{D}_{\tau,\theta}} \left\{ \langle X, a\rangle_{\tau,\theta} - \gamma(a) \right\} , \quad X \in \mathcal{R}^\infty_{\tau,\theta}\,, \tag{3.6}$$

for a penalty function γ *on* $\mathcal{D}_{\tau,\theta}$*.*

(2) ϕ *is a conditional concave monetary utility function on* $\mathcal{R}^\infty_{\tau,\theta}$ *whose acceptance set* $\mathcal{C}_\phi$ *is a* $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$*-closed subset of* $\mathcal{R}^\infty$*.*

(3) ϕ *is a conditional concave monetary utility function on* $\mathcal{R}^\infty_{\tau,\theta}$ *that is continuous for bounded decreasing sequences.*

Moreover, if (1)–(3) *are satisfied, then* $\phi^\#$ *is a penalty function on* $\mathcal{D}_{\tau,\theta}$*, the representation* (3.6) *also holds with* $\phi^\#$ *instead of* γ*,* $\gamma(a) \le \phi^\#(a)$ *for all* $a \in \mathcal{D}_{\tau,\theta}$*, and* $\gamma = \phi^\#$ *provided that* γ *is concave, has the local property of Definition* 3.11 *and* $\{a \in \mathcal{D}_{\tau,\theta} \mid \mathrm{E}\,[1_A \gamma(a)] \ge m\}$ *is a* $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$*-closed subset of* $\mathcal{A}^1$ *for all* $A \in \mathcal{F}_\tau$ *and* $m \in \mathbb{R}$*.*

Proof.
(1) $\Rightarrow$ (3): If ϕ has a representation of the form (3.6), then it obviously is a conditional concave monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$. To show that it is continuous for bounded decreasing sequences, let $(X^n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{R}^\infty_{\tau,\theta}$ and $X \in \mathcal{R}^\infty_{\tau,\theta}$ such that

$$\lim_{n\to\infty} X^n_t = X_t \quad \text{almost surely, for all } t \in \mathbb{N}\,. \tag{3.7}$$

It follows from Beppo Levi's monotone convergence theorem that for every fixed $a \in \mathcal{D}_{\tau,\theta}$,

$$\lim_{n\to\infty} \langle X^n, a\rangle_{\tau,\theta} = \langle X, a\rangle_{\tau,\theta}\ ,$$

and therefore,

$$\begin{aligned}
&\lim_{n\to\infty} \phi(X^n) = \inf_{n\in\mathbb{N}} \phi(X^n) = \inf_{n\in\mathbb{N}} \operatorname*{ess\,inf}_{a\in\mathcal{D}_{\tau,\theta}} \left\{ \langle X^n, a\rangle_{\tau,\theta} - \gamma(a) \right\} \\
= \;& \operatorname*{ess\,inf}_{a\in\mathcal{D}_{\tau,\theta}} \inf_{n\in\mathbb{N}} \left\{ \langle X^n, a\rangle_{\tau,\theta} - \gamma(a) \right\} = \operatorname*{ess\,inf}_{a\in\mathcal{D}_{\tau,\theta}} \left\{ \langle X, a\rangle_{\tau,\theta} - \gamma(a) \right\} = \phi(X)\,.
\end{aligned}$$

(3) ⇒ (2): follows from Lemma 3.17 below.
(2) ⇒ (1): By (3.4) and the definition of ϕ^*,

$$\phi^{\#}(a) = \phi^*(a) \le \langle X, a \rangle_{\tau,\theta} - \phi(X)$$

for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $a \in \mathcal{D}_{\tau,\theta}$. Hence,

$$\phi(X) \le \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \right\} \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta} . \tag{3.8}$$

To show the converse inequality, fix $X \in \mathcal{R}^\infty_{\tau,\theta}$, let $m \in L^\infty(\mathcal{F}_\tau)$ with

$$m \le \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \right\} , \tag{3.9}$$

and assume that $Y = X - m 1_{[\tau,\infty)} \notin \mathcal{C}_\phi$. Since $\mathcal{C}_\phi$ is a convex, $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed subset of $\mathcal{R}^\infty$, it follows from the separating hyperplane theorem that there exists an $a \in \mathcal{A}^1$ such that

$$\langle Y, a \rangle < \inf_{Z \in \mathcal{C}_\phi} \langle Z, a \rangle . \tag{3.10}$$

Since Y and all the processes in $\mathcal{C}_\phi$ are in $\mathcal{R}^\infty_{\tau,\theta}$, the process a can be chosen in $\mathcal{A}^1_{\tau,\theta}$. As $\mathcal{C}_\phi$ has the monotone property (m), it follows from (3.10) that a has to be in $(\mathcal{A}^1_{\tau,\theta})_+$. Now, we can write the two sides of (3.10) as

$$\langle Y, a \rangle = \mathrm{E}\left[\langle Y, a \rangle_{\tau,\theta} \right] \quad \text{and} \quad \inf_{Z \in \mathcal{C}_\phi} \langle Z, a \rangle = \mathrm{E}\left[\operatorname*{ess\,inf}_{Z \in \mathcal{C}_\phi} \langle Z, a \rangle_{\tau,\theta} \right] ,$$

where the second equality follows from Beppo Levi's monotone convergence theorem because $\mathcal{C}_\phi$ has the local property (lp), and therefore, the set $\left\{ \langle Z, a \rangle_{\tau,\theta} \mid Z \in \mathcal{C}_\phi \right\}$ is directed downwards. Hence, it follows from (3.10) that $P[B] > 0$, where

$$B := \left\{ \langle Y, a \rangle_{\tau,\theta} < \operatorname*{ess\,inf}_{Z \in \mathcal{C}_\phi} \langle Z, a \rangle_{\tau,\theta} \right\} .$$

Note that for $A = \left\{ \langle 1, a \rangle_{\tau,\theta} = 0 \right\}$,

$$1_A \left| \langle Z, a \rangle_{\tau,\theta} \right| \le 1_A \langle |Z| , a \rangle_{\tau,\theta} \le 1_A ||Z||_{\tau,\theta} \langle 1, a \rangle_{\tau,\theta} = 0 \quad \text{for all} \quad Z \in \mathcal{R}^\infty_{\tau,\theta} .$$

Hence, $B \subset \left\{ \langle 1, a \rangle_{\tau,\theta} > 0 \right\}$. Define the process $b \in \mathcal{D}_{\tau,\theta}$ as follows:

$$b := 1_B \frac{a}{\langle 1, a \rangle_{\tau,\theta}} + 1_{B^c} 1_{[\tau,\infty)} .$$

By definition of the set B,

$$\langle X, b \rangle_{\tau,\theta} - m = \langle Y, b \rangle_{\tau,\theta} < \operatorname*{ess\,inf}_{Z \in \mathcal{C}_\phi} \langle Z, b \rangle_{\tau,\theta} = \phi^{\#}(b) \quad \text{on} \quad B .$$

But this contradicts (3.9). Hence, $X - m1_{[\tau,\infty)} \in \mathcal{C}_\phi$, and therefore, $\phi(X) \geq m$ for all $m \in L^\infty(\mathcal{F}_\tau)$ satisfying (3.9). It follows that

$$\phi(X) \geq \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \right\} .$$

Together with (3.8), this proves that

$$\phi(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \right\} \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta} .$$

In particular,

$$0 = \phi(0) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ -\phi^{\#}(a) \right\} .$$

This shows that (2) implies (1) and that $\phi^{\#}$ is a penalty function on $\mathcal{D}_{\tau,\theta}$.

To prove the last two statements of the theorem, we assume that ϕ is a conditional concave monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$ with a representation of the form (3.6). Then,

$$\gamma(a) \leq \langle X, a \rangle_{\tau,\theta} - \phi(X) \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta} \text{ and } a \in \mathcal{D}_{\tau,\theta} ,$$

and it immediately follows that $\gamma(a) \leq \phi^*(a) = \phi^{\#}(a)$ for all $a \in \mathcal{D}_{\tau,\theta}$. On the other hand, suppose that γ is concave, has the local property, $\{a \in \mathcal{D}_{\tau,\theta} \mid \mathrm{E}\left[1_A \gamma(a)\right] \geq m\}$ is for all $A \in \mathcal{F}_\tau$ and $m \in \mathbb{R}$ a $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed subset of $\mathcal{A}^1$, and there exists an $a^0 \in \mathcal{D}_{\tau,\theta}$ such that

$$P[\gamma(a^0) < \phi^{\#}(a^0)] > 0 . \tag{3.11}$$

Note that

$$\left\{ \gamma(a^0) < \phi^{\#}(a^0) \right\} = \bigcup_{k \geq 1} B_k , \quad \text{where } B_k := \left\{ (-k) \vee \gamma(a^0) < \phi^{\#}(a^0) \right\} .$$

Hence, there exists a $K \geq 1$, such that $P[B_K] > 0$, and therefore,

$$-\infty \leq \tilde{\gamma}(a^0) < \tilde{\phi}^{\#}(a^0) \leq 0 , \tag{3.12}$$

where

$$\tilde{\gamma}(a) := \mathrm{E}\left[1_{B_K} \gamma(a)\right] \quad \text{and} \quad \tilde{\phi}^{\#}(a) := \mathrm{E}\left[1_{B_K} \phi^{\#}(a)\right] , \, a \in \mathcal{D}_{\tau,\theta} .$$

Let $X \in \mathcal{R}^\infty_{\tau,\theta}$ and note that since γ and $\phi^{\#}$ have the local property, the sets

$$\left\{ \langle X, a \rangle_{\tau,\theta} - \gamma(a) \mid a \in \mathcal{D}_{\tau,\theta} \right\} \quad \text{and} \quad \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \mid a \in \mathcal{D}_{\tau,\theta} \right\}$$

are directed downwards. Hence,

$$\begin{aligned} \inf_{a \in D_{\tau,\theta}} \left\{ \langle 1_{B_K} X, a \rangle - \tilde{\gamma}(a) \right\} &= \mathrm{E}\left[1_{B_K} \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \gamma(a) \right\} \right] \\ &= \mathrm{E}\left[1_{B_K} \phi(X)\right] \\ &= \mathrm{E}\left[1_{B_K} \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \right\} \right] \\ &= \inf_{a \in D_{\tau,\theta}} \left\{ \langle 1_{B_K} X, a \rangle - \tilde{\phi}^{\#}(a) \right\} . \end{aligned} \tag{3.13}$$

In particular, $\operatorname{ess\,sup}_{a\in\mathcal{D}_{\tau,\theta}} \tilde{\gamma}(a) = 0$. By assumption, the function $\tilde{\gamma} : \mathcal{D}_{\tau,\theta} \to [-\infty, 0]$ is concave and $\{a \in \mathcal{D}_{\tau,\theta} \mid \tilde{\gamma}(a) \geq m\}$ is for all $m \in \mathbb{R}$, a $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed subset of $\mathcal{A}^1$. Therefore, the set

$$C := \{(a, z) \in \mathcal{D}_{\tau,\theta} \times \mathbb{R} \mid \tilde{\gamma}(a) \geq z\}$$

is a non-empty, convex, $\sigma(\mathcal{A}^1 \times \mathbb{R}, \mathcal{R}^\infty \times \mathbb{R})$-closed subset of $\mathcal{A}^1 \times \mathbb{R}$ which, by (3.12), does not contain $(a^0, \tilde{\phi}^\#(a^0))$. Since γ has the local property, $(a, z) \in \mathcal{D}_{\tau,\theta} \times \mathbb{R}$ is in C if and only if $(1_{B_K} a + 1_{B_K^c} 1_{[\tau,\infty)}, z)$ is in C. Therefore, it follows from the separating hyperplane theorem that there exists $(Y, y) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathbb{R}$ such that

$$\left\langle 1_{B_K} Y, a^0 \right\rangle + y\tilde{\phi}^\#(a^0) < \inf_{(a,z)\in C} \{\langle 1_{B_K} Y, a\rangle + yz\} \leq \inf_{a\in D_{\tau,\theta}\,,\,\tilde{\gamma}(a)>-\infty} \{\langle 1_{B_K} Y, a\rangle + y\tilde{\gamma}(a)\} \,. \tag{3.14}$$

The first inequality in (3.14) and the form of the set C imply that $y \leq 0$. If $y < 0$, then it follows from (3.14) that

$$\left\langle -1_{B_K} \frac{1}{y} Y, a^0 \right\rangle - \tilde{\phi}^\#(a^0) < \inf_{a\in\mathcal{D}_{\tau,\theta}} \left\{ \left\langle -1_{B_K} \frac{1}{y} Y, a \right\rangle - \tilde{\gamma}(a) \right\} .$$

But this contradicts (3.13). If $y = 0$, there exists a $\lambda > 0$ such that

$$\left\langle 1_{B_K} \lambda Y, a^0 \right\rangle - \tilde{\phi}^\#(a^0) < \inf_{a\in D_{\tau,\theta}\,,\,\tilde{\gamma}(a)>-\infty} \langle 1_{B_K} \lambda Y, a\rangle \leq \inf_{a\in D_{\tau,\theta}\,,\,\tilde{\gamma}(a)>-\infty} \{\langle 1_{B_K} \lambda Y, a\rangle - \tilde{\gamma}(a)\} \,,$$

which again contradicts (3.13). Hence, $\gamma = \phi^\#$ if γ is concave, has the local property of Definition (3.11) and $\{a \in \mathcal{D}_{\tau,\theta} \mid \mathrm{E}[1_A \gamma(a)] \geq m\}$ is a $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed subset of $\mathcal{A}^1$ for all $A \in \mathcal{F}_\tau$ and $m \in \mathbb{R}$. □

Lemma 3.17 *Let ϕ be an increasing concave function from $\mathcal{R}^\infty_{\tau,\theta}$ to $L^\infty(\mathcal{F}_\tau)$ that is continuous for bounded decreasing sequences. Then $\mathcal{C}_\phi := \left\{X \in \mathcal{R}^\infty_{\tau,\theta} \mid \phi(X) \geq 0\right\}$ is a $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed subset of $\mathcal{R}^\infty$.*

Proof. Let $(X^\mu)_{\mu\in M}$ be a net in $\mathcal{C}_\phi$ and $X \in \mathcal{R}^\infty$ such that $X^\mu \to X$ in $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$. It follows that $X \in \mathcal{R}^\infty_{\tau,\theta}$. Assume

$$\phi(X) < 0 \quad \text{on } A \quad \text{for some } A \in \mathcal{F}_\tau \text{ with } P[A] > 0\,. \tag{3.15}$$

The map $\tilde{\phi} : \mathcal{R}^\infty \to \mathbb{R}$ given by

$$\tilde{\phi}(X) = \mathrm{E}\left[1_A\, \phi \circ \pi_{\tau,\theta}(X)\right] , \quad X \in \mathcal{R}^\infty ,$$

is increasing, concave and continuous for bounded decreasing sequences. Denote by $\mathcal{G}$ the sigma-algebra on $\Omega \times \mathbb{N}$ generated by all the sets $B \times \{t\}$, $t \in \mathbb{N}$, $B \in \mathcal{F}_t$, and by ν the measure on $(\Omega \times \mathbb{N}, \mathcal{G})$ given by

$$\nu(B \times \{t\}) = 2^{-(t+1)} P[B]\,,\, t \in \mathbb{N}\,,\, B \in \mathcal{F}_t\,.$$

Then $\mathcal{R}^\infty = L^\infty(\Omega \times \mathbb{N}, \mathcal{G}, \nu)$ and $\mathcal{A}^1$ can be identified with $L^1(\Omega \times \mathbb{N}, \mathcal{G}, \nu)$. Hence, it can be deduced from the Krein–Šmulian theorem that $\mathcal{C}_{\tilde{\phi}} := \left\{ X \in \mathcal{R}^\infty \mid \tilde{\phi}(X) \geq 0 \right\}$ is a $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed subset of $\mathcal{R}^\infty$ (see the proof of Theorem 3.2 in Delbaen (2002) or Remark 4.3 in Cheridito et al. (2004)). Since $(X^\mu)_{\mu \in M} \subset \mathcal{C}_{\tilde{\phi}}$, it follows that $\mathrm{E}\left[1_A\, \phi(X)\right] \geq 0$, which contradicts (3.15). Hence, $\phi(X) \geq 0$. $\square$

Theorem 3.18 *The following are equivalent:*

(1) *ϕ is a mapping defined on $\mathcal{R}^\infty_{\tau,\theta}$ that can be represented as*

$$\phi(X) = \operatorname*{ess\,inf}_{a \in \mathcal{Q}} \langle X, a \rangle_{\tau,\theta}\,, \quad X \in \mathcal{R}^\infty_{\tau,\theta}\,, \tag{3.16}$$

for a non-empty subset $\mathcal{Q}$ of $\mathcal{D}_{\tau,\theta}$.

(2) *ϕ is a conditional coherent utility function on $\mathcal{R}^\infty_{\tau,\theta}$ whose acceptance set $\mathcal{C}_\phi$ is a $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed subset of $\mathcal{R}^\infty$.*

(3) *ϕ is a conditional coherent utility function on $\mathcal{R}^\infty_{\tau,\theta}$ that is continuous for bounded decreasing sequences.*

Moreover, if (1)–(3) *are satisfied, then the set*

$$\mathcal{Q}^0_\phi := \left\{ a \in \mathcal{D}_{\tau,\theta} \mid \phi^\#(a) = 0 \right\}$$

is equal to the smallest $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed, $\mathcal{F}_\tau$-convex subset of $\mathcal{D}_{\tau,\theta}$ that contains $\mathcal{Q}$, and the representation (3.16) *also holds with $\mathcal{Q}^0_\phi$ instead of $\mathcal{Q}$.*

Proof. If (1) holds, then it follows from Theorem 3.16 that ϕ is a conditional concave monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$ that is continuous for bounded decreasing sequences, and it is clear that ϕ is coherent. This shows that (1) implies (3). The implication (3) $\Rightarrow$ (2) follows directly from Theorem 3.16. If (2) holds, then Theorem 3.16 implies that $\phi^\#$ is a penalty function on $\mathcal{D}_{\tau,\theta}$, and

$$\phi(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^\#(a) \right\} \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta}\,. \tag{3.17}$$

Since $\phi^\#$ has the local property, the set $\left\{ \phi^\#(a) \mid a \in \mathcal{D}_{\tau,\theta} \right\}$ is directed upwards and there exists a sequence $(a^k)_{k \in \mathbb{N}}$ in $\mathcal{D}_{\tau,\theta}$ such that almost surely,

$$\phi^\#(a^k) \nearrow \operatorname*{ess\,sup}_{a \in \mathcal{D}_{\tau,\theta}} \phi^\#(a) = 0\,, \quad \text{as } k \to \infty\,.$$

It can easily be deduced from the coherency of ϕ that for all $a \in \mathcal{D}_{\tau,\theta}$,

$$\left\{ \phi^\#(a) = 0 \right\} \cup \left\{ \phi^\#(a) = -\infty \right\} = \Omega\,.$$

Hence, the sets $A_k := \left\{\phi^{\#}(a^k) = 0\right\}$ are increasing in k, and $\bigcup_{k\in\mathbb{N}} A_k = \Omega$. Therefore,

$$a^* := 1_{A_0} a^0 + \sum_{k\geq 1} 1_{A_k\setminus A_{k-1}} a^k \in \mathcal{D}_{\tau,\theta}\,,$$

and $\phi^{\#}(a^*) = 0$ by the local property of $\phi^{\#}$. Note that for all $a \in \mathcal{D}_{\tau,\theta}$,

$$1_{\left\{\phi^{\#}(a)=0\right\}} a + 1_{\left\{\phi^{\#}(a)=-\infty\right\}} a^* \in \mathcal{Q}^0_\phi\,.$$

Hence, it follows from (3.17) that

$$\phi(X) = \operatorname*{ess\,inf}_{a\in\mathcal{Q}^0_\phi} \langle X, a\rangle_{\tau,\theta}\,, \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta}\,. \tag{3.18}$$

It remains to show that $\mathcal{Q}^0_\phi$ is equal to the $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed $\mathcal{F}_\tau$-convex hull $\langle\mathcal{Q}\rangle_\tau$ of $\mathcal{Q}$. It follows from Theorem 3.16 that $\phi^{\#}$ is the largest among all penalty functions on $\mathcal{D}_{\tau,\theta}$ that induce ϕ. This implies $\mathcal{Q} \subset \mathcal{Q}^0_\phi$. By Remark 3.13.2 and Proposition 3.14.2, $\mathcal{Q}^0_\phi$ is $\mathcal{F}_\tau$-convex and $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed. Hence, $\langle\mathcal{Q}\rangle_\tau \subset \mathcal{Q}^0_\phi$. Now, assume that there exists a $b \in \mathcal{Q}^0_\phi \setminus \langle\mathcal{Q}\rangle_\tau$. Then, it follows from the separating hyperplane theorem that there exists an $X \in \mathcal{R}^\infty_{\tau,\theta}$ such that

$$\langle X, b\rangle < \inf_{a\in\langle\mathcal{Q}\rangle_\tau} \langle X, a\rangle = \mathrm{E}\left[\operatorname*{ess\,inf}_{a\in\langle\mathcal{Q}\rangle_\tau} \langle X, a\rangle_{\tau,\theta}\right] = \mathrm{E}\left[\operatorname*{ess\,inf}_{a\in\mathcal{Q}} \langle X, a\rangle_{\tau,\theta}\right] = \mathrm{E}\left[\phi(X)\right] \tag{3.19}$$

(the first equality holds because $\langle\mathcal{Q}\rangle_\tau$ is $\mathcal{F}_\tau$-convex, and therefore, the set $\{\langle X, a\rangle_{\tau,\theta} \mid a \in \langle\mathcal{Q}\rangle_\tau\}$ is directed downwards). But, by (3.18),

$$\langle X, b\rangle - \mathrm{E}\left[\phi(X)\right] = \mathrm{E}\left[\langle X, b\rangle_{\tau,\theta} - \phi(X)\right] \geq 0$$

for all $b \in \mathcal{Q}^0_\phi$, which contradicts (3.19). Hence, $\mathcal{Q}^0_\phi \setminus \langle\mathcal{Q}\rangle_\tau$ is empty, that is, $\mathcal{Q}^0_\phi \subset \langle\mathcal{Q}\rangle_\tau$.□

3.3 Relevance

In this subsection we generalize the relevance axiom of Artzner et al. (1999) to our framework and show representation results for relevant conditional concave monetary and coherent utility functions. In Artzner et al. (1999) a monetary risk measure is called relevant if it is positive for future financial positions that are non-positive and negative with positive probability. In the following definition we give an $\mathcal{F}_\tau$-conditional version of this concept. It has consequences for the dual representation of conditional concave monetary and coherent utility functions and plays an important role for the uniqueness of time-consistent dynamic extensions of static monetary risk measures; see Proposition 4.8 below.

Definition 3.19 *Let ϕ be a conditional monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$. We call ϕ θ-relevant if*

$$A \subset \left\{\phi(-\varepsilon 1_A 1_{[t\wedge\theta,\infty)}) < 0\right\}$$

for all $\varepsilon > 0$, $t \in \mathbb{N}$ and $A \in \mathcal{F}_{t\wedge\theta}$, and we define

$$\mathcal{D}^{\mathrm{rel}}_{\tau,\theta} := \left\{ a \in \mathcal{D}_{\tau,\theta} \mid P\left[\sum_{j \geq t\wedge\theta} \Delta a_j > 0\right] = 1 \text{ for all } t \in \mathbb{N} \right\} .$$

Remarks 3.20
1. If ϕ is a θ-relevant conditional monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$ and ξ is an $(\mathcal{F}_t)$-stopping time such that $\tau \leq \xi \leq \theta$, then, obviously, the restriction of ϕ to $\mathcal{R}^\infty_{\tau,\xi}$ is ξ-relevant.
2. Assume that θ is finite. Then it can easily be checked that a conditional monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ is θ-relevant if and only if

$$A \subset \left\{\phi(-\varepsilon 1_A 1_{[\theta,\infty)}) < 0\right\}$$

for all $\varepsilon > 0$ and $A \in \mathcal{F}_\theta$. Also, in this case,

$$\mathcal{D}^{\mathrm{rel}}_{\tau,\theta} = \{a \in \mathcal{D}_{\tau,\theta} \mid P\left[\Delta a_\theta > 0\right] = 1\} .$$

Proposition 3.21 *Let $\mathcal{Q}^{\mathrm{rel}}$ be a non-empty subset of $\mathcal{D}^{\mathrm{rel}}_{\tau,\theta}$. Then*

$$\phi(X) = \operatorname*{ess\,inf}_{a \in \mathcal{Q}^{\mathrm{rel}}} \langle X, a\rangle_{\tau,\theta} , \quad X \in \mathcal{R}^\infty_{\tau,\theta}$$

is a θ-relevant conditional coherent utility function on $\mathcal{R}^\infty_{\tau,\theta}$.

Proof. That ϕ is a conditional coherent utility function on $\mathcal{R}^\infty_{\tau,\theta}$ follows from Theorem 3.18. To show that it is θ-relevant, let $\varepsilon > 0$, $t \in \mathbb{N}$, $A \in \mathcal{F}_{t\wedge\theta}$ and choose $a \in \mathcal{Q}^{\mathrm{rel}}$. Then

$$\phi(-\varepsilon 1_A 1_{[t\wedge\theta,\infty)}) \leq -\varepsilon \left\langle 1_A 1_{[t\wedge\theta,\infty)}, a\right\rangle_{\tau,\theta} = -\varepsilon\, \mathrm{E}\left[1_A \sum_{j \geq t\wedge\theta} \Delta a_j \mid \mathcal{F}_\tau\right] ,$$

and it remains to show that

$$\mathrm{E}\left[1_A \sum_{j \geq t\wedge\theta} \Delta a_j \mid \mathcal{F}_\tau\right] > 0 \quad \text{on } A . \tag{3.20}$$

Denote $B = \left\{\mathrm{E}\left[1_A \sum_{j \geq t\wedge\theta} \Delta a_j \mid \mathcal{F}_\tau\right] = 0\right\}$ and note that

$$0 = \mathrm{E}\left[1_B \mathrm{E}\left[1_A \sum_{j \geq t\wedge\theta} \Delta a_j \mid \mathcal{F}_\tau\right]\right] = \mathrm{E}\left[1_B 1_A \sum_{j \geq t\wedge\theta} \Delta a_j\right] .$$

This implies $B \cap A = \emptyset$, and therefore, (3.20). $\square$

To prove the converse of Proposition 3.21 we introduce for a conditional concave monetary utility function ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ and a constant $K \geq 0$, the set

$$\mathcal{Q}^K_\phi := \left\{ a \in \mathcal{D}_{\tau,\theta} \mid \phi^\#(a) \geq -K \right\} .$$

Note that it follows from Remark 3.13.2 and Proposition 3.14.2 that $\mathcal{Q}^K_\phi$ is $\mathcal{F}_\tau$-convex and $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed.

Lemma 3.22 *Let ϕ be a conditional concave monetary utility function on $\mathcal{R}^\infty_{\tau,\theta}$ that is continuous for bounded decreasing sequences and θ-relevant. Then*

$$\mathcal{Q}^K_\phi \cap \mathcal{D}^{\text{rel}}_{\tau,\theta} \quad \textit{is non-empty for all} \quad K > 0 .$$

Proof. Fix $K > 0$ and $t \in \mathbb{N}$. For $a \in \mathcal{D}_{\tau,\theta}$, we denote

$$e_t(a) := \sum_{j \geq t \wedge \theta} \Delta a_j ,$$

and we define

$$\alpha_t := \sup_{a \in \mathcal{Q}^K_\phi} P\left[e_t(a) > 0\right] . \tag{3.21}$$

Let $(a^{t,n})_{n \in \mathbb{N}}$ be a sequence in $\mathcal{Q}^K_\phi$ with

$$\lim_{n \to \infty} P\left[e_t(a^{t,n}) > 0\right] = \alpha_t .$$

Since $\mathcal{Q}^K_\phi$ is convex and $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed,

$$a^t := \sum_{n \geq 1} 2^{-n} a^{t,n} \quad \text{is still in} \quad \mathcal{Q}^K_\phi ,$$

and, obviously,

$$P\left[e_t(a^t) > 0\right] = \alpha_t .$$

In the next step we show that $\alpha_t = 1$. Assume to the contrary that $\alpha_t < 1$ and denote $A_t := \left\{e_t(a^t) = 0\right\}$. Since ϕ is θ-relevant,

$$A_t \subset \left\{\phi(-K 1_{A_t} 1_{[t \wedge \theta, \infty)}) < 0\right\} ,$$

and therefore also,

$$\hat{A}_t := \bigcap_{B \in \mathcal{F}_\tau,\, A_t \subset B} B \ \subset \ \left\{\phi(-K 1_{A_t} 1_{[t \wedge \theta, \infty)}) < 0\right\} .$$

By Theorem 3.16,

$$\phi(-K 1_{A_t} 1_{[t \wedge \theta, \infty)}) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \left\langle -K 1_{A_t} 1_{[t \wedge \theta, \infty)}, a \right\rangle_{\tau,\theta} - \phi^\#(a) \right\} .$$

Hence, there must exist an $a \in \mathcal{D}_{\tau,\theta}$ with $P\left[A_t \cap \{e_t(a) > 0\}\right] > 0$ and $\phi^{\#}(a) \geq -K$ on $\hat{A}_t$. We then have that

$$b^t := 1_{\hat{A}_t}\, a + 1_{\hat{A}^c}\, a^t \in \mathcal{Q}_\phi^K \,, \quad c^t := \frac{1}{2} b^t + \frac{1}{2} a^t \in \mathcal{Q}_\phi^K \,,$$

and $P\left[e_t(c^t) > 0\right] > P\left[e_t(a^t) > 0\right] = \alpha_t$. This contradicts (3.21). Therefore, we must have $\alpha_t = 1$ for all $t \in \mathbb{N}$. Finally, set

$$a^* = \sum_{t \geq 1} 2^{-t} a^t \,,$$

and note that $a^* \in \mathcal{Q}_\phi^K \cap \mathcal{D}_{\tau,\theta}^{\mathrm{rel}}$. □

Theorem 3.23 *Let ϕ be a conditional concave monetary utility function on $\mathcal{R}_{\tau,\theta}^\infty$ that is continuous for bounded decreasing sequences and θ-relevant. Then*

$$\phi(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}^{\mathrm{rel}}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \right\} \,, \quad \textit{for all } X \in \mathcal{R}_{\tau,\theta}^\infty \,.$$

Proof. By Theorem 3.16,

$$\phi(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \right\} \,, \quad \text{for all } X \in \mathcal{R}_{\tau,\theta}^\infty \,,$$

which immediately shows that

$$\phi(X) \leq \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}^{\mathrm{rel}}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \right\} \,, \quad \text{for all } X \in \mathcal{R}_{\tau,\theta}^\infty \,.$$

To show the converse inequality, we choose $b \in \mathcal{D}_{\tau,\theta}$. It follows from Lemma 3.22 that there exists a process $c \in \mathcal{Q}_\phi^1 \cap \mathcal{D}_{\tau,\theta}^{\mathrm{rel}}$. Then, for all $n \geq 1$,

$$b^n := (1 - \frac{1}{n}) b + \frac{1}{n} c \in \mathcal{D}_{\tau,\theta}^{\mathrm{rel}} \,,$$

$$\lim_{n \to \infty} \langle X, b^n \rangle_{\tau,\theta} = \lim_{n \to \infty} \left\{ (1 - \frac{1}{n}) \langle X, b \rangle_{\tau,\theta} + \frac{1}{n} \langle X, c \rangle_{\tau,\theta} \right\} = \langle X, b \rangle_{\tau,\theta} \quad \text{almost surely,}$$

and

$$\begin{aligned} \phi^{\#}(b^n) &= \operatorname*{ess\,inf}_{X \in \mathcal{C}_\phi} \langle X, b^n \rangle_{\tau,\theta} \geq (1 - \frac{1}{n}) \operatorname*{ess\,inf}_{X \in \mathcal{C}_\phi} \langle X, b \rangle_{\tau,\theta} + \frac{1}{n} \operatorname*{ess\,inf}_{X \in \mathcal{C}_\phi} \langle X, c \rangle_{\tau,\theta} \\ &= (1 - \frac{1}{n}) \phi^{\#}(b) + \frac{1}{n} \phi^{\#}(c) \quad \to \quad \phi^{\#}(b) \quad \text{almost surely} \,. \end{aligned}$$

This shows that

$$\langle X, b \rangle_{\tau,\theta} - \phi^{\#}(b) \geq \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}^{\mathrm{rel}}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \right\} \,,$$

and therefore,

$$\phi(X) \geq \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}^{\mathrm{rel}}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^{\#}(a) \right\} \,,$$

which completes the proof. □

Corollary 3.24 *Let ϕ be a conditional coherent utility function on $\mathcal{R}^{\infty}_{\tau,\theta}$ that is continuous for bounded decreasing sequences and θ-relevant. Then*

$$\phi(X) = \operatorname*{ess\,inf}_{a \in \mathcal{Q}^{0,\mathrm{rel}}_{\phi}} \langle X, a \rangle_{\tau,\theta} \,, \quad X \in \mathcal{R}^{\infty}_{\tau,\theta} \,,$$

where $\mathcal{Q}^{0,\mathrm{rel}}_{\phi} := \left\{ a \in \mathcal{D}^{\mathrm{rel}}_{\tau,\theta} \mid \phi^{\#}(a) = 0 \right\}$.

Proof. This corollary can either be deduced from Theorem 3.18 and Lemma 3.22 like Theorem 3.23 from Theorem 3.16 and Lemma 3.22, or from Theorem 3.23 with the arguments used in the proof of the implication (2) $\Rightarrow$ (1) of Theorem 3.18. □

4 Dynamic monetary utility functions

In this section we introduce a time-consistency condition for dynamic monetary utility functions. We show that it is equivalent to a decomposition property of the corresponding acceptance sets. For dynamic coherent and concave monetary utility functions we give dual characterizations of time-consistency.

In the whole section we fix $S \in \mathbb{N}$ and $T \in \mathbb{N} \cup \{\infty\}$ such that $S \leq T$.

Definition 4.1 *Assume that for all $t \in [S,T] \cap \mathbb{N}$, $\phi_{t,T}$ is a conditional monetary utility function on $\mathcal{R}^{\infty}_{t,T}$ with acceptance set $\mathcal{C}_{t,T}$. Then we call $(\phi_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ a dynamic monetary utility function and $(\mathcal{C}_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ the corresponding family of acceptance sets.*

For τ and θ two $(\mathcal{F}_t)$-stopping times such that τ is finite and $S \leq \tau \leq \theta \leq T$, we define the mapping $\phi_{\tau,\theta} : \mathcal{R}^{\infty}_{\tau,\theta} \to L^{\infty}(\mathcal{F}_{\tau})$ by

$$\phi_{\tau,\theta}(X) := \sum_{t \in [S,T] \cap \mathbb{N}} \phi_{t,T}(1_{\{\tau = t\}} X) \,, \tag{4.1}$$

and the set $\mathcal{C}_{\tau,\theta} \subset \mathcal{R}^{\infty}_{\tau,\theta}$ by

$$\mathcal{C}_{\tau,\theta} := \left\{ X \in \mathcal{R}^{\infty}_{\tau,\theta} \mid 1_{\{\tau = t\}} X \in \mathcal{C}_{t,T} \text{ for all } t \in [S,T] \cap \mathbb{N} \right\} . \tag{4.2}$$

It can easily be checked that $\phi_{\tau,\theta}$ defined by (4.1) is a conditional monetary utility function on $\mathcal{R}^{\infty}_{\tau,\theta}$ and that the set $\mathcal{C}_{\tau,\theta}$ given in (4.2) is the acceptance set of $\phi_{\tau,\theta}$. Moreover, if all $\phi_{t,T}$ are concave monetary, then so is $\phi_{\tau,\theta}$; if all $\phi_{t,T}$ are coherent, then $\phi_{\tau,\theta}$ is coherent too; and if all $\phi_{t,T}$ are continuous for bounded decreasing sequences, then so is $\phi_{\tau,\theta}$.

4.1 Time-consistency

Condition (4.3) in the following definition is the time-consistency condition we work with in this paper. In Proposition 4.4 we show that it is equivalent to the more intuitive condition (4.4)–(4.5). Proposition 4.5 shows that condition (4.3) can be slightly weakened if the time horizon T is finite or the conditional monetary utility functions are continuous for bounded decreasing sequences.

Dynamic consistency conditions equivalent or similar to (4.3) or (4.4)–(4.5) have been studied in different contexts; see, for instance, Koopmans (1960), Kreps and Porteus (1978), Epstein and Zin (1989), Duffie and Epstein (1992), Wang (2003), Epstein and Schneider (2003), Delbaen (2003), Artzner et al. (2004), Riedel (2004), Peng (2004), Roorda et al. (2005), Weber (2005).

Definition 4.2 *We call a dynamic monetary utility function $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ time-consistent if*

$$\phi_{t,T}(X) = \phi_{t,T}(X1_{[t,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)}) \tag{4.3}$$

for each $t \in [S,T] \cap \mathbb{N}$, every finite $(\mathcal{F}_t)$-stopping time θ such that $t \leq \theta \leq T$ and all processes $X \in \mathcal{R}^\infty_{t,T}$.

Remark 4.3 Let $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ be a time-consistent dynamic monetary utility function. Then it can easily be seen from Definition 4.1 that

$$\phi_{\tau,T}(X) = \phi_{\tau,T}(X1_{[\tau,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)})$$

for every pair of finite $(\mathcal{F}_t)$-stopping times τ and θ such that $S \leq \tau \leq \theta \leq T$ and all processes $X \in \mathcal{R}^\infty_{\tau,T}$.

Proposition 4.4 *Let τ and θ be finite $(\mathcal{F}_t)$-stopping times such that $0 \leq \tau \leq \theta \leq T$. Let $\phi_{\tau,T}$ be a conditional monetary utility function on $\mathcal{R}^\infty_{\tau,T}$ and $\phi_{\theta,T}$ a conditional monetary utility function on $\mathcal{R}^\infty_{\theta,T}$. Then the following two conditions are equivalent:*

(1) $\phi_{\tau,T}(X) = \phi_{\tau,T}(X1_{[\tau,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)})$

(2) *If X and Y are two processes in $\mathcal{R}^\infty_{\tau,T}$ such that*

$$X1_{[\tau,\theta)} = Y1_{[\tau,\theta)} \quad \textit{and} \quad \phi_{\theta,T}(X) \leq \phi_{\theta,T}(Y)\,, \tag{4.4}$$

then

$$\phi_{\tau,T}(X) \leq \phi_{\tau,T}(Y)\,. \tag{4.5}$$

Proof. (1) $\Rightarrow$ (2):
If X and Y are two processes in $\mathcal{R}^\infty_{\tau,T}$ that satisfy (4.4), then

$$\phi_{\tau,T}(X) = \phi_{\tau,T}(X1_{[\tau,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)}) \leq \phi_{\tau,T}(Y1_{[\tau,\theta)} + \phi_{\theta,T}(Y)1_{[\theta,\infty)}) = \phi_{\tau,T}(Y)\,.$$

(2) $\Rightarrow$ (1):
Choose $X \in \mathcal{R}^\infty_{\tau,\theta}$ and define

$$Y := X1_{[\tau,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)}\,.$$

Then,

$$X1_{[\tau,\theta)} = Y1_{[\tau,\theta)} \quad \text{and} \quad \phi_{\theta,T}(X) = \phi_{\theta,T}(Y)\,.$$

Hence, it follows from (2) that

$$\phi_{\tau,T}(X) = \phi_{\tau,T}(Y) = \phi_{\tau,T}(X1_{[\tau,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)}) \,.$$

$\square$

Condition (2) of Proposition 4.4 requires that if a process X coincides with another process Y between times τ and $\theta-1$ and at θ, the capital requirement for X is bigger than for Y in every possible state of the world, then also at time τ, the capital requirement for X should be bigger than for Y. A violation of this condition clearly leads to capital requirements that are inconsistent over time.

The following proposition gives two conditions under which one-time-step time-consistency implies time-consistency.

Proposition 4.5 *Let $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ be a dynamic monetary utility function that satisfies*

$$\phi_{t,T}(X) = \phi_{t,T}(X1_{\{t\}} + \phi_{t+1,T}(X)1_{[t+1,\infty)}) \quad \text{for all } S \leq t < T \text{ and } X \in \mathcal{R}^{\infty}_{t,T} \tag{4.6}$$

and at least one of the following two conditions:
(i) $T \in \mathbb{N}$
(ii) *all $\phi_{t,T}$ are continuous for bounded decreasing sequences.*
Then $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ is time-consistent.

Proof. Let us first assume that (4.6) and (i) are satisfied. Then, for $t \in [S,T]\cap\mathbb{N}$, an $(\mathcal{F}_t)$-stopping time θ such that $t \leq \theta \leq T$ and a process $X \in \mathcal{R}^{\infty}_{t,T}$, we denote $Y = X1_{[t,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)}$ and show

$$\phi_{t,T}(X) = \phi_{t,T}(Y) \tag{4.7}$$

by induction. For $t = T$, (4.7) is obvious. If $t \leq T-1$, we assume that

$$\phi_{t+1,T}(Z) = \phi_{t+1,T}(Z1_{[t+1,\xi)} + \phi_{\xi,T}(Z)1_{[\xi,\infty)}) \,,$$

for all $Z \in \mathcal{R}^{\infty}_{t+1,T}$ and every $(\mathcal{F}_t)$-stopping time ξ such that $t+1 \leq \xi \leq T$. Then it follows from the normalization (N) and local property (LP) of ϕ that

$$1_{\{\theta\geq t+1\}}\phi_{t+1,T}(X) = \phi_{t+1,T}(1_{\{\theta\geq t+1\}}X) = \phi_{t+1,T}(1_{\{\theta\geq t+1\}}Y) = 1_{\{\theta\geq t+1\}}\phi_{t+1,T}(Y) \,.$$

This and assumption (4.6), together with (LP) and (N) imply

$$\begin{aligned}
\phi_{t,T}(Y) &= \phi_{t,T}\left(1_{\{\theta=t\}}\phi_{t,T}(X)1_{[t,\infty)} + 1_{\{\theta\geq t+1\}}Y\right)\\
&= 1_{\{\theta=t\}}\phi_{t,T}(X) + 1_{\{\theta\geq t+1\}}\phi_{t,T}(Y)\\
&= 1_{\{\theta=t\}}\phi_{t,T}(X) + 1_{\{\theta\geq t+1\}}\phi_{t,T}(Y1_{\{t\}} + \phi_{t+1,T}(Y)1_{[t+1,\infty)})\\
&= 1_{\{\theta=t\}}\phi_{t,T}(X) + 1_{\{\theta\geq t+1\}}\phi_{t,T}(X1_{\{t\}} + \phi_{t+1,T}(X)1_{[t+1,\infty)})\\
&= 1_{\{\theta=t\}}\phi_{t,T}(X) + 1_{\{\theta\geq t+1\}}\phi_{t,T}(X)\\
&= \phi_{t,T}(X) \,.
\end{aligned}$$

If (4.6) and (ii) hold but $T = \infty$, we choose $t \in [S,\infty) \cap \mathbb{N}$, a process $X \in \mathcal{R}^\infty_{t,\infty}$ and a finite $(\mathcal{F}_t)$-stopping time $\theta \geq t$. For all $N \in [t,\infty) \cap \mathbb{N}$ we introduce the process

$$X^N := X 1_{[t,N)} + ||X||_{N,\infty} 1_{[N,\infty)} .$$

By the first part of the proof,

$$\begin{aligned} &\phi_{t,\infty}(X^N) = \phi_{t,N}(X^N) = \phi_{t,N}(X^N 1_{[t,\theta\wedge N)} + \phi_{\theta\wedge N,N}(X^N) 1_{[\theta\wedge N,\infty)}) \\ &= \phi_{t,\infty}(X^N 1_{[t,\theta)} + \phi_{\theta,\infty}(X^N) 1_{[\theta,\infty)}) . \end{aligned} \tag{4.8}$$

Clearly, the sequence (X^N) is decreasing and $X^N_t \to X_t$ almost surely for all $t \in \mathbb{N}$. Therefore,

$$\phi_{t,\infty}(X^N) \to \phi_{t,\infty}(X) \quad \text{almost surely.} \tag{4.9}$$

As mentioned after Definition 4.1, $\phi_{\theta,\infty}$ is also continuous for bounded decreasing sequences. It follows that $\phi_{\theta,\infty}(X^N) \to \phi_{\theta,\infty}(X)$ almost surely, and hence,

$$\phi_{t,\infty}(X^N 1_{[t,\theta)} + \phi_{\theta,\infty}(X^N) 1_{[\theta,\infty)}) \to \phi_{t,\infty}(X 1_{[t,\theta)} + \phi_{\theta,\infty}(X) 1_{[\theta,\infty)}) ,$$

which, together with (4.8) and (4.9), shows that

$$\phi_{t,\infty}(X) = \phi_{t,\infty}(X 1_{[t,\theta)} + \phi_{\theta,\infty}(X) 1_{[\theta,\infty)}) .$$

□

The next result characterizes time-consistency in terms of acceptance sets. Depending on the point of view, condition (2) of Theorem 4.6 can be seen as an additivity or decomposition property of the family of acceptance sets corresponding to a dynamic utility function. In Section 7 of Delbaen (2003), this property is studied for dynamic coherent utility functions that depend on random variables.

Theorem 4.6 *Let τ and θ be finite $(\mathcal{F}_t)$-stopping times such that $0 \leq \tau \leq \theta \leq T$. Let $\phi_{\tau,T}$ be a conditional monetary utility function on $\mathcal{R}^\infty_{\tau,T}$ with acceptance set $\mathcal{C}_{\tau,T}$ and $\phi_{\theta,T}$ a conditional monetary utility function on $\mathcal{R}^\infty_{\theta,T}$ with acceptance set $\mathcal{C}_{\theta,T}$. Denote $\mathcal{C}_{\tau,\theta} := \mathcal{C}_{\tau,T} \cap \mathcal{R}^\infty_{\tau,\theta}$. Then the following two conditions are equivalent:*

(1) $\phi_{\tau,T}(X) = \phi_{\tau,T}(X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)})$ *for all* $X \in \mathcal{R}^\infty_{\tau,T}$.

(2) $\mathcal{C}_{\tau,T} = \mathcal{C}_{\tau,\theta} + \mathcal{C}_{\theta,T}$.

Proof.

(1) ⇒ (2):

Assume $Y \in \mathcal{C}_{\tau,\theta}$ and $Z \in \mathcal{C}_{\theta,T}$. Then $X = Y + Z \in \mathcal{R}^\infty_{\tau,T}$, $X 1_{[\tau,\theta)} = Y 1_{[\tau,\theta)}$ and $\phi_{\theta,T}(X) = Y_\theta + \phi_{\theta,T}(Z) \geq Y_\theta$. Therefore,

$$\phi_{\tau,T}(X) = \phi_{\tau,T}(X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}) \geq \phi_{\tau,T}(Y) \geq 0 .$$

This shows that $\mathcal{C}_{\tau,\theta} + \mathcal{C}_{\theta,T} \subset \mathcal{C}_{\tau,T}$.

To show $\mathcal{C}_{\tau,T} \subset \mathcal{C}_{\tau,\theta} + \mathcal{C}_{\theta,T}$, let $X \in \mathcal{C}_{\tau,T}$ and set $Z := (X - \phi_{\theta,T}(X))1_{[\theta,\infty)}$ and $Y := X - Z = X1_{[\tau,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)}$. It follows directly from the translation invariance of $\phi_{\theta,T}$ that $Z \in \mathcal{C}_{\theta,T}$. Moreover, $\phi_{\tau,T}(Y) = \phi_{\tau,T}(X) \geq 0$, which shows that $Y \in \mathcal{C}_{\tau,\theta}$.

(2) $\Rightarrow$ (1):

Let $X \in \mathcal{R}^\infty_{\tau,T}$ and $f \in L^\infty(\mathcal{F}_\tau)$ such that $X - f1_{[\tau,\infty)} \in \mathcal{C}_{\tau,T}$. Since

$$\phi_{\theta,T}(X) = \operatorname{ess\,sup}\left\{g \in L^\infty(\mathcal{F}_\theta) \mid (X - g)1_{[\theta,\infty)} \in \mathcal{C}_{\theta,T}\right\}$$

and

$$\mathcal{C}_{\tau,T} \subset \mathcal{C}_{\tau,\theta} + \mathcal{C}_{\theta,T}\,,$$

the process

$$X1_{[\tau,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)} - f1_{[\tau,\infty)} = X - f1_{[\tau,\infty)} - (X - \phi_{\theta,T}(X))1_{[\theta,\infty)}$$

has to be in $\mathcal{C}_{\tau,\theta}$. This shows that

$$\phi_{\tau,T}(X1_{[\tau,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)}) \geq \phi_{\tau,T}(X)\,.$$

On the other hand, if $X \in \mathcal{R}^\infty_{\tau,T}$ and $f \in L^\infty(\mathcal{F}_\tau)$ such that

$$X1_{[\tau,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)} - f1_{[\tau,\infty)} \in \mathcal{C}_{\tau,\theta}\,,$$

then also $X - f1_{[\tau,\infty)} \in \mathcal{C}_{\tau,T}$ because $(X - \phi_{\theta,T}(X))1_{[\theta,\infty)} \in \mathcal{C}_{\theta,T}$ and $\mathcal{C}_{\tau,\theta} + \mathcal{C}_{\theta,T} \subset \mathcal{C}_{\tau,T}$. It follows that

$$\phi_{\tau,T}(X) \geq \phi_{\tau,T}(X1_{[\tau,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)})\,.$$

□

Proposition 4.7 *Let $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ be a time-consistent dynamic monetary utility function with corresponding family of acceptance sets $(\mathcal{C}_{t,T})_{t\in[S,T]\cap\mathbb{N}}$, and let τ and θ be two finite $(\mathcal{F}_t)$-stopping times such that $S \leq \tau \leq \theta \leq T$. Then*

1. *$1_A X \in \mathcal{C}_{\tau,T}$ for all $X \in \mathcal{C}_{\theta,T}$ and $A \in \mathcal{F}_\theta$.*

2. *If $\phi_{\tau,\theta}$ is θ-relevant, and X is a process in $\mathcal{R}^\infty_{\theta,T}$ such that $1_A X \in \mathcal{C}_{\tau,T}$ for all $A \in \mathcal{F}_\theta$, then $X \in \mathcal{C}_{\theta,T}$.*

3. *If ξ is an $(\mathcal{F}_t)$-stopping time such that $\theta \leq \xi \leq T$ and $\phi_{\tau,\xi}$ is ξ-relevant, then $\phi_{\theta,\xi}$ is ξ-relevant too. In particular, if $\phi_{S,T}$ is T-relevant, then $\phi_{\tau,T}$ is T-relevant for every finite $(\mathcal{F}_t)$-stopping time such that $S \leq \tau \leq T$.*

Proof.

1. If $X \in \mathcal{C}_{\theta,T}$ and $A \in \mathcal{F}_\theta$, then also $1_A X \in \mathcal{C}_{\theta,T}$. Since $0 \in \mathcal{C}_{\tau,\theta}$, it follows from Theorem 4.6 that $1_A X = 0 + 1_A X \in \mathcal{C}_{\tau,T}$.

2. Assume $1_A X \in \mathcal{C}_{\tau,T}$ for all $A \in \mathcal{F}_\theta$ but $X \notin \mathcal{C}_{\theta,T}$. Then there exists an $\varepsilon > 0$ such that $P[A] > 0$, where $A = \{\phi_{\theta,T}(X) \leq -\varepsilon\}$. We get from (LP) and (N) that $\phi_{\theta,T}(1_A X) = 1_A\phi_{\theta,T}(X) \leq -\varepsilon 1_A$. By Theorem 4.6, there exist $Y \in \mathcal{C}_{\tau,\theta}$ and $Z \in \mathcal{C}_{\theta,T}$ such that

$1_A X = Y + Z$. Since $1_A X$ is in $\mathcal{R}^\infty_{\theta,T}$, the process Y must be of the form $Y = \eta 1_{[\theta,\infty)}$, for some $\eta \in L^\infty(\mathcal{F}_\theta)$. Hence,

$$0 \le \phi_{\theta,T}(Z) = \phi_{\theta,T}(1_A X - \eta 1_{[\theta,\infty)}) = \phi_{\theta,T}(1_A X) - \eta \le -\varepsilon 1_A - \eta\,,$$

and therefore, $\eta \le -\varepsilon 1_A$. But then, since $\phi_{\tau,\theta}$ is θ-relevant, Y cannot be in $\mathcal{C}_{\tau,\theta}$, which is a contradiction.
3. Let $\varepsilon > 0$, $t \in \mathbb{N}$ and $A \in \mathcal{F}_{t\wedge\xi}$. Set

$$B := A \cap \left\{\phi_{\theta,\xi}(-\varepsilon 1_A 1_{[t\wedge\xi,\infty)}) = 0\right\}$$

and note that

$$0 \ge 1_B\, \phi_{\theta,\xi}(-\varepsilon 1_B 1_{[t\wedge\xi,\infty)}) \ge 1_B\, \phi_{\theta,\xi}(-\varepsilon 1_A 1_{[t\wedge\xi,\infty)}) = 0\,.$$

It follows that

$$\phi_{\theta,\xi}(-\varepsilon 1_B 1_{[t\wedge\xi,\infty)}) = 0 \quad \text{on } \hat{B} := \bigcap_{C\in\mathcal{F}_\theta\,;\,B\subset C} C\,.$$

On the other hand, by (N) and (LP),

$$1_{\hat{B}^c}\, \phi_{\theta,\xi}(-\varepsilon 1_B 1_{[t\wedge\xi,\infty)}) = \phi_{\theta,\xi}(-\varepsilon 1_{\hat{B}^c} 1_B 1_{[t\wedge\xi,\infty)}) = 0\,,$$

and therefore, $\phi_{\theta,\xi}(-\varepsilon 1_B 1_{[t\wedge\xi,\infty)}) = 0$. Hence, $-\varepsilon 1_B 1_{[t\wedge\xi,\infty)} \in \mathcal{C}_{\theta,\xi}$, which by part 1 implies $-\varepsilon 1_B 1_{[t\wedge\xi,\infty)} \in \mathcal{C}_{\tau,\xi}$. Since $\phi_{\tau,\xi}$ is ξ-relevant, we must have $P[B] = 0$. This shows that $\phi_{\theta,\xi}$ is ξ-relevant. □

Corollary 4.8 *Let ϕ be a T-relevant conditional monetary utility function on $\mathcal{R}^\infty_{S,T}$. Then there exists at most one time-consistent dynamic monetary utility function $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ with $\phi_{S,T} = \phi$.*

Proof. Let $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ be a time-consistent dynamic monetary utility function with $\phi_{S,T} = \phi$ and $(\mathcal{C}_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ the corresponding family of acceptance sets. By Proposition 4.7.3, $\phi_{t,T}$ is T-relevant for all $t \in [S,T]\cap\mathbb{N}$. Therefore, it follows from 1. and 2. of Proposition 4.7 that for all $t \in [S,T]\cap\mathbb{N}$, a process $X \in \mathcal{R}^\infty_{t,T}$ is in $\mathcal{C}_{t,T}$ if and only if $1_A X \in \mathcal{C}_{S,T}$ for all $A \in \mathcal{F}_t$. This shows that $\mathcal{C}_{t,T}$ is uniquely determined by the acceptance set $\mathcal{C}_{S,T}$ of ϕ. Hence, $\phi_{t,T}$ is uniquely determined by ϕ. □

4.2 Consistent extension of the time horizon

Proposition 4.9 *Let $(\phi^1_{t,S})_{t\in[0,S]\cap\mathbb{N}}$ and $(\phi^2_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ be two time-consistent dynamic monetary utility functions with corresponding family of acceptance sets $(\mathcal{C}^1_{t,S})_{t\in[0,S]\cap\mathbb{N}}$ and $(\mathcal{C}^2_{t,T})_{t\in[S,T]\cap\mathbb{N}}$. Define the conditional utility functions $\phi_{t,T}$, $t \in [0,T]\cap\mathbb{N}$, by*

$$\phi_{t,T}(X) := \begin{cases} \phi^1_{t,S}\left(X 1_{[t,S)} + \phi^2_{S,T}(X) 1_{[S,\infty)}\right) & \text{if } t < S \\ \phi^2_{t,T}(X) & \text{if } t \ge S \end{cases}\,, \quad X \in \mathcal{R}^\infty_{t,T}\,, \tag{4.10}$$

and the sets $\mathcal{C}_{t,T}$, $t \in [0,T] \cap \mathbb{N}$ by

$$\mathcal{C}_{t,T} := \begin{cases} \mathcal{C}^1_{t,S} + \mathcal{C}^2_{S,T} & \text{if } t < S \\ \mathcal{C}^2_{t,T} & \text{if } t \geq S \end{cases} \tag{4.11}$$

Then $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is a time-consistent dynamic monetary utility functions with corresponding family of acceptance sets $(\mathcal{C}_{t,T})_{t \in [0,T] \cap \mathbb{N}}$. If $(\phi^1_{t,S})_{t \in [0,S] \cap \mathbb{N}}$ and $(\phi^2_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ are dynamic concave monetary utility functions, then so is $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$. If $(\phi^1_{t,S})_{t \in [0,S] \cap \mathbb{N}}$ and $(\phi^2_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ are dynamic coherent utility functions, then so is $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$. If $(\phi^1_{t,S})_{t \in [0,S] \cap \mathbb{N}}$ and $(\phi^2_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ consist of conditional monetary utility functions that are continuous for bounded decreasing sequences, then $\phi_{t,T}$ is continuous for bounded decreasing sequences for all $t \in [0,T] \cap \mathbb{N}$.

Proof. It can easily be checked that for all $t \in [0,S] \cap \mathbb{N}$, the mapping $\phi_{t,T}$ defined in (4.10) is a conditional monetary utility function on $\mathcal{R}^\infty_{t,T}$ with acceptance set $\mathcal{C}_{t,T}$ given by (4.11). Also, it is obvious that $\phi_{t,T}$ inherits concavity, positive homogeneity, and continuity for bounded decreasing sequences. To prove that $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is time-consistent, we fix $t \in [0,S)$, a process $X \in \mathcal{R}^\infty_{t,T}$ and a finite $(\mathcal{F}_t)$-stopping time θ such that $t \leq \theta \leq T$. Then, it follows from definition (4.10) and the time-consistency of $(\phi^1_{t,S})_{t \in [0,S] \cap \mathbb{N}}$ and $(\phi^2_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ that

$$\begin{aligned}
& \phi_{t,T}(X 1_{[t,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}) \\
= \; & \phi^1_{t,S}(X 1_{[t,\theta \wedge S)} + 1_{\{\theta \leq S\}} \phi_{\theta,T}(X) 1_{[\theta,\infty)} + 1_{\{\theta > S\}} \phi^2_{S,T}(X 1_{[S,\theta)} + \phi^2_{\theta,T}(X) 1_{[\theta,\infty)}) 1_{[S,\infty)}) \\
= \; & \phi^1_{t,S}(X 1_{[t,\theta \wedge S)} + 1_{\{\theta \leq S\}} \phi_{\theta,T}(X) 1_{[\theta,\infty)} + 1_{\{\theta > S\}} \phi^2_{S,T}(X) 1_{[S,\infty)}) \\
= \; & \phi^1_{t,S}(X 1_{[t,\theta \wedge S)} + \phi_{\theta \wedge S,T}(X) 1_{[\theta \wedge S,\infty)}) \\
= \; & \phi^1_{t,S}(X 1_{[t,\theta \wedge S)} + \phi^1_{\theta \wedge S,S}(X 1_{[\theta \wedge S,S)} + \phi^2_{S,T}(X) 1_{[S,\infty)}) 1_{[\theta \wedge S,\infty)}) \\
= \; & \phi^1_{t,S}(X 1_{[t,S)} + \phi^2_{S,T}(X) 1_{[S,\infty)}) \\
= \; & \phi_{t,T}(X) \, .
\end{aligned}$$

□

For $T \in \mathbb{N}$, time-consistent dynamic monetary utility functions can be defined by backwards induction as follows: For all $t = 0, \ldots, T$, there exists only one conditional monetary utility function $\phi_{t,t}$ on $\mathcal{R}^\infty_{t,t}$. It is given by

$$\phi_{t,t}(m 1_{[t,\infty)}) = m \, , \quad \text{for } m \in L^\infty(\mathcal{F}_t) \, ,$$

and its acceptance set is

$$C_{t,t} = \left\{ m 1_{[t,\infty)} \mid m \in L^\infty_+(\Omega, \mathcal{F}_t, P) \right\} \, .$$

Now, for every $t = 0, \ldots T-1$, let $\phi_{t,t+1}$ be an arbitrary conditional monetary utility function on $\mathcal{R}^\infty_{t,t+1}$ with acceptance set $\mathcal{C}_{t,t+1}$. It can easily be checked that for all $t =$

$0, \dots T-1$, the dynamic monetary utility function $(\phi_{s,t+1})_{s=t}^{t+1}$ is time-consistent. Therefore, it follows from Proposition 4.9 that a time-consistent dynamic monetary utility function $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ can be defined recursively by

$$\phi_{t,T}(X) := \phi_{t,t+1}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}) \quad \text{for } t \leq T-2 \text{ and } X \in \mathcal{R}^\infty_{t,T} .$$

The corresponding acceptance sets are given by

$$\mathcal{C}_{t,T} := \mathcal{C}_{t,t+1} + \mathcal{C}_{t+1,t+2} + \cdots + \mathcal{C}_{T-1,T} , \quad t = 0, 1, \dots, T-1 .$$

4.3 Concatenation of elements in $\mathcal{A}^1_+$

For the dual characterization of time-consistency of dynamic coherent and concave monetary utility functions, the following concatenation operation in $\mathcal{A}^1_+$ will play an important role:

Definition 4.10 *Let $a, b \in \mathcal{A}^1_+$, θ a finite $(\mathcal{F}_t)$-stopping time and $A \in \mathcal{F}_\theta$. Then we define the concatenation $a \oplus^\theta_A b$ as follows:*

$$(a \oplus^\theta_A b)_t := \begin{cases} a_t & \text{on } \{t < \theta\} \cup A^c \cup \left\{ \langle 1, b \rangle_{\theta,\infty} = 0 \right\} \\ a_{\theta-1} + \frac{\langle 1, a \rangle_{\theta,\infty}}{\langle 1, b \rangle_{\theta,\infty}} (b_t - b_{\theta-1}) & \text{on } \{t \geq \theta\} \cap A \cap \left\{ \langle 1, b \rangle_{\theta,\infty} > 0 \right\} \end{cases} ,$$

where we set $a_{-1} = b_{-1} = 0$.

We call a subset $\mathcal{Q}$ of $\mathcal{A}^1_+$ c1*-stable if*

$$a \oplus^s_A b \in \mathcal{Q} \quad \text{for all } a, b \in \mathcal{Q} ,\ s \in \mathbb{N} \text{ and all } A \in \mathcal{F}_s .$$

We call a subset $\mathcal{Q}$ of $\mathcal{A}^1_+$ c2*-stable if*

$$a \oplus^\theta_A b \in \mathcal{Q} \quad \text{for all } a, b \in \mathcal{Q} ,\ \text{every finite } (\mathcal{F}_t)\text{-stopping time } \theta \text{ and all } A \in \mathcal{F}_\theta .$$

Remark 4.11
Let $\mathcal{Q}$ be a c1-stable subset of $\mathcal{A}^1_+$. Then

$$a \oplus^\theta_A b \in \mathcal{Q} \quad \text{for all } a, b \in \mathcal{Q}, \text{ each bounded } (\mathcal{F}_t)\text{-stopping time } \theta, \text{ and } A \in \mathcal{F}_\theta ,$$

and

$$a \oplus^\theta_A b \quad \text{is in the } ||.||_{\mathcal{A}^1}\text{-closure of } \mathcal{Q}$$

for all $a, b \in \mathcal{Q}$, each finite $(\mathcal{F}_t)$-stopping time θ and $A \in \mathcal{F}_\theta$.

Indeed, if $\mathcal{Q}$ is c1-stable, set for each $(\mathcal{F}_t)$-stopping time θ and $A \in \mathcal{F}_\theta$, $A_n := A \cap \{\theta = n\}$, $n \in \mathbb{N}$. Then all of the following processes are in $\mathcal{Q}$:

$$a^0 := a \oplus^0_{A_0} b , \quad a^n := a^{n-1} \oplus^n_{A_n} b , \quad n \geq 1 .$$

If θ is bounded, then $a^n = a \oplus^\theta_A b$ for all n such that $n \geq \theta$. If θ is finite, then $a^n \to a \oplus^\theta_A b$ in $||.||^1_{\mathcal{A}}$, as $n \to \infty$.

4.4 Time-consistent dynamic coherent utility functions

In this subsection we show how time-consistency of dynamic coherent utility functions is related to stability under concatenation in $\mathcal{A}^1_+$. This generalizes results of Artzner et al. (2004), Riedel (2004) and Roorda et al. (2005). Examples will be discussed in Subsections 5.1, 5.2, 5.4 and 5.5 below.

Remark 4.12 Let $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ be a time-consistent dynamic coherent utility function such that for all $t \in [0,T] \cap \mathbb{N}$ and $X \in \mathcal{R}^\infty_{t,T}$,

$$\phi_{t,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{Q}_{t,T}} \langle X, a \rangle_{t,T} \ ,$$

for a non-empty subset $\mathcal{Q}_{t,T}$ of $\mathcal{D}_{t,T}$. Then, it can easily be checked that for all finite stopping times $\tau \leq T$,

$$\phi_{\tau,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{Q}_{\tau,T}} \langle X, a \rangle_{\tau,T} \ , \quad X \in \mathcal{R}^\infty_{\tau,T} \ ,$$

where $\mathcal{Q}_{\tau,T}$ is given by

$$\mathcal{Q}_{\tau,T} := \left\{ a \in \mathcal{D}_{\tau,T} \mid a = \sum_{t \in [0,T]\cap\mathbb{N}} 1_{\{\tau = t\}} a^t \, , \, a^t \in \mathcal{Q}_{t,T} \text{ for all } t \right\} .$$

In the following theorem and corollary, we provide necessary dual conditions for time-consistency of dynamic coherent utility functions $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ such that all $\phi_{t,T}$ are continuous for bounded decreasing sequences.

Theorem 4.13 *Let $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ be a time-consistent dynamic coherent utility function such that for all $t \in [0,T] \cap \mathbb{N}$ and $X \in \mathcal{R}^\infty_{t,T}$,*

$$\phi_{t,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{Q}^0_{t,T}} \langle X, a \rangle_{t,T} \ ,$$

where for all finite $(\mathcal{F}_t)$-stopping times $\tau \leq T$, the set $\mathcal{Q}^0_{\tau,T}$ is given by

$$\mathcal{Q}^0_{\tau,T} := \left\{ a \in \mathcal{D}_{\tau,T} \mid \phi^\#_{\tau,T}(a) = 0 \right\} .$$

Then, for every pair of finite stopping times τ and θ with $0 \leq \tau \leq \theta \leq T$, the following hold:

1. *For every $a \in \mathcal{Q}^0_{\tau,T}$ there exists $b \in \mathcal{Q}^0_{\theta,T}$ such that*

$$\frac{a}{\langle 1, a \rangle_{\theta,T}} 1_{[\theta,\infty)} = b \quad \text{on the set } \left\{ \langle 1, a \rangle_{\theta,T} > 0 \right\} .$$

2.

$$a \oplus^\theta_A b \in \mathcal{Q}^0_{\tau,T} \quad \text{for all } a \in \mathcal{Q}^0_{\tau,T} \, , \, b \in \mathcal{Q}^0_{\theta,T} \text{ and } A \in \mathcal{F}_\theta \, .$$

3.

$$a \oplus^\theta_A b \in \mathcal{Q}^0_{\tau,T} \quad \text{for all } a, b \in \mathcal{Q}^0_{\tau,T} \text{ and } A \in \mathcal{F}_\theta \, .$$

Corollary 4.14 *Let $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ be a time-consistent dynamic coherent utility function such that $\phi_{0,T}$ is T-relevant and continuous for bounded decreasing sequences. Then the sets*

$$\mathcal{Q}^0_{0,T} := \left\{a \in \mathcal{D}_{0,T} \mid \phi^{\#}_{0,T}(a) = 0\right\} \quad \textit{and} \quad \mathcal{Q}^{0,\mathrm{rel}}_{0,T} := \mathcal{Q}^0_{0,T} \cap \mathcal{D}^{\mathrm{rel}}_{0,T}\,,$$

are c2*-stable, and for every finite $(\mathcal{F}_t)$-stopping time $\tau \le T$ and $X \in \mathcal{R}^\infty_{\tau,T}$,*

$$\phi_{\tau,T}(X) = \operatorname*{ess\,inf}_{a\in\mathcal{Q}^0_{0,T}} \frac{\langle X, a\rangle_{\tau,T}}{\langle 1, a\rangle_{\tau,T}} = \operatorname*{ess\,inf}_{a\in\mathcal{Q}^{0,\mathrm{rel}}_{0,T}} \frac{\langle X, a\rangle_{\tau,T}}{\langle 1, a\rangle_{\tau,T}}\,,$$

where

$$\frac{\langle X, a\rangle_{\tau,T}}{\langle 1, a\rangle_{\tau,T}} \quad \textit{is understood to be } \infty \textit{ on } \left\{\langle 1, a\rangle_{\tau,T} = 0\right\}.$$

Proof of Theorem 4.13.
1. Let $a \in \mathcal{Q}^0_{\tau,T}$ and denote

$$\mathcal{C}_{\tau,T} := \left\{X \in \mathcal{R}^\infty_{\tau,T} \mid \phi_{\tau,T}(X) \ge 0\right\} \quad \text{and} \quad \mathcal{C}_{\theta,T} := \left\{X \in \mathcal{R}^\infty_{\theta,T} \mid \phi_{\theta,T}(X) \ge 0\right\}.$$

Choose $X \in \mathcal{C}_{\theta,T}$ and $A \in \mathcal{F}_\theta$. It follows from Proposition 4.7.1 that $1_A X \in \mathcal{C}_{\tau,T}$. Hence, by Remark 4.12,

$$\operatorname*{ess\,inf}_{c\in\mathcal{Q}^0_{\tau,T}} \langle 1_A X, c\rangle_{\tau,T} = \phi_{\tau,T}(1_A X) \ge 0\,,$$

and therefore,

$$\mathrm{E}\left[1_A \langle X, a\rangle_{\theta,T}\right] = \mathrm{E}\left[\langle 1_A X, a\rangle_{\tau,T}\right] \ge \mathrm{E}\left[\phi_{\tau,T}(1_A X)\right] \ge 0\,.$$

This shows that $\langle X, a\rangle_{\theta,T} \ge 0$ for all $X \in \mathcal{C}_{\theta,T}$. Take $d \in \mathcal{Q}^0_{\theta,T}$ and set

$$b := 1_{\left\{\langle 1,a\rangle_{\theta,T} > 0\right\}} \frac{a}{\langle 1, a\rangle_{\theta,T}} + 1_{\left\{\langle 1,a\rangle_{\theta,T} = 0\right\}} d\,.$$

Then $b \in \mathcal{Q}^0_{\theta,T}$ and

$$\frac{a}{\langle 1, a\rangle_{\theta,T}} = b \quad \text{on the set } \left\{\langle 1, a\rangle_{\theta,T} > 0\right\}.$$

2. Let $a \in \mathcal{Q}^0_{\tau,T}$, $b \in \mathcal{Q}^0_{\theta,T}$, $A \in \mathcal{F}_\theta$ and $X \in \mathcal{C}_{\tau,T}$. By 1.,

$$\frac{a}{\langle 1, a\rangle_{\theta,T}} = c \quad \text{for some } c \in \mathcal{Q}^0_{\theta,T} \text{ on the set } \left\{\langle 1, a\rangle_{\theta,T} > 0\right\}.$$

Hence,

$$\begin{aligned}
& 1_{A^c} \langle X, a\rangle_{\theta,T} + 1_A \langle 1, a\rangle_{\theta,T} \langle X, b\rangle_{\theta,T} \\
= \; & 1_{A^c} \langle 1, a\rangle_{\theta,T} \langle X, c\rangle_{\theta,T} + 1_A \langle 1, a\rangle_{\theta,T} \langle X, b\rangle_{\theta,T} \\
\ge \; & \langle 1, a\rangle_{\theta,T}\, \phi_{\theta,T}(X)\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left\langle X, a \oplus_A^\theta b \right\rangle_{\tau,T} &= \mathrm{E}\left[\sum_{t=\tau}^{\theta-1} X_t \Delta a_t + 1_{A^c} \sum_{t\in[\theta,T]\cap\mathbb{N}} X_t \Delta a_t + 1_A \langle 1, a\rangle_{\theta,T} \sum_{t\in[\theta,T]\cap\mathbb{N}} X_t \Delta b_t \mid \mathcal{F}_\tau\right]\\
&= \mathrm{E}\left[\sum_{t=\tau}^{\theta-1} X_t \Delta a_t + 1_{A^c} \langle X, a\rangle_{\theta,T} + 1_A \langle 1, a\rangle_{\theta,T} \langle X, b\rangle_{\theta,T} \mid \mathcal{F}_\tau\right]\\
&\geq \mathrm{E}\left[\sum_{t=\tau}^{\theta-1} X_t \Delta a_t + \langle 1, a\rangle_{\theta,T}\, \phi_{\theta,T}(X) \mid \mathcal{F}_\tau\right]\\
&= \mathrm{E}\left[\sum_{t=\tau}^{\theta-1} X_t \Delta a_t + \phi_{\theta,T}(X) \sum_{t\in[\theta,T]\cap\mathbb{N}} \Delta a_t \mid \mathcal{F}_\tau\right]\\
&\geq \phi_{\tau,T}(X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}) = \phi_{\tau,T}(X) \geq 0\,.
\end{aligned}$$

This shows that $\phi^{\#}_{\tau,T}(a \oplus_A^\theta b) = 0$, and therefore, $a \oplus_A^\theta b \in \mathcal{Q}^0_{\tau,T}$.
3. follows directly from 1. and 2. □

Proof of Corollary 4.14.
It follows from Theorem 3.18 and Corollary 3.24 that

$$\phi_{0,T}(X) = \operatorname*{ess\,inf}_{a\in\mathcal{Q}^0_{0,T}} \langle X, a\rangle_{0,T} = \operatorname*{ess\,inf}_{a\in\mathcal{Q}^{0,\mathrm{rel}}_{0,T}} \langle X, a\rangle_{0,T}\;.$$

By Theorem 4.13.3, $\mathcal{Q}^0_{0,T}$ is c2-stable, which immediately implies that also $\mathcal{Q}^{0,\mathrm{rel}}_{0,T}$ is c2-stable. To prove the rest of the corollary, let $(\mathcal{C}_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ be the family of acceptance sets corresponding to $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ and fix a finite $(\mathcal{F}_t)$-stopping time $\tau \leq T$. Parts 1 and 2 of Proposition 4.7 imply that for every $X \in \mathcal{R}^\infty_{\tau,T}$,

$$X \in \mathcal{C}_{\tau,T} \;\Leftrightarrow\; 1_A X \in \mathcal{C}_{0,T} \quad \text{for all } A \in \mathcal{F}_\tau\,,$$

and therefore,

$$\begin{aligned}
X \in \mathcal{C}_{\tau,T} \quad &\Leftrightarrow \quad \langle 1_A X, a\rangle_{0,T} \geq 0 \quad \text{for all } A \in \mathcal{F}_\tau \text{ and } a \in \mathcal{Q}^{0,\mathrm{rel}}_{0,T}\\
&\Leftrightarrow \quad \langle X, a\rangle_{\tau,T} \geq 0 \quad \text{for all } a \in \mathcal{Q}^{0,\mathrm{rel}}_{0,T}\;.
\end{aligned}$$

This shows that $\phi_{\tau,T}$ and the conditional coherent utility function

$$\operatorname*{ess\,inf}_{a\in\mathcal{Q}^{0,\mathrm{rel}}_{\tau,T}} \frac{\langle X, a\rangle_{\tau,T}}{\langle 1, a\rangle_{\tau,T}}\,, \quad X \in \mathcal{R}^\infty_{\tau,T}$$

have the same acceptance set. Hence, they must be equal. It is clear that

$$\operatorname*{ess\,inf}_{a\in\mathcal{Q}^0_{0,T}} \frac{\langle X, a\rangle_{\tau,T}}{\langle 1, a\rangle_{\tau,T}} \leq \operatorname*{ess\,inf}_{a\in\mathcal{Q}^{0,\mathrm{rel}}_{0,T}} \frac{\langle X, a\rangle_{\tau,T}}{\langle 1, a\rangle_{\tau,T}}\,, \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,T}\,.$$

On the other hand, since $X - \phi_{\tau,T}(X)1_{[\tau,\infty)} \in \mathcal{C}_{\tau,T}$, it follows that

$$\left\langle 1_A \left(X - \phi_{\tau,T}(X)1_{[\tau,\infty)}\right), a\right\rangle_{0,T} \geq 0\,, \quad \text{for all } A \in \mathcal{F}_\tau \text{ and } a \in \mathcal{Q}^0_{\tau,T}\,,$$

and therefore,

$$\frac{\left\langle \left(X - \phi_{\tau,T}(X)1_{[\tau,\infty)}\right), a\right\rangle_{\tau,T}}{\langle 1, a\rangle_{\tau,T}} \geq 0\,, \quad \text{for all } a \in \mathcal{Q}^0_{\tau,T}\,,$$

which shows that

$$\operatorname*{ess\,inf}_{a \in \mathcal{Q}^0_{\tau,T}} \frac{\langle X, a\rangle_{\tau,T}}{\langle 1, a\rangle_{\tau,T}} \geq \phi_{\tau,T}(X)\,, \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,T}\,.$$

□

The subsequent theorem and its corollary give sufficient dual conditions for time-consistency of dynamic coherent utility functions.

Theorem 4.15 *Let $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ be a dynamic coherent utility function such that for all $t \in [0,T] \cap \mathbb{N}$ and $X \in \mathcal{R}^\infty_{t,T}$,*

$$\phi_{t,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{Q}_{t,T}} \langle X, a\rangle_{t,T}\,,$$

for a non-empty subset $\mathcal{Q}_{t,T}$ of $\mathcal{D}_{t,T}$. Set

$$\mathcal{Q}^0_{t,T} := \left\{a \in \mathcal{D}_{t,T} \mid \phi^\#_{t,T}(a) = 0\right\} \quad \textit{for all } t \in [0,T] \cap \mathbb{N}\,,$$

and assume that for each $t \in [0,T) \cap \mathbb{N}$, the following two conditions are satisfied:

(i) *For every $a \in \mathcal{Q}_{t,T}$ there exists $b \in \mathcal{Q}^0_{t+1,T}$ such that*

$$\frac{a}{\langle 1, a\rangle_{t+1,T}} 1_{[t+1,\infty)} = b \quad \textit{on the set } \left\{\langle 1, a\rangle_{t+1,T} > 0\right\}$$

(ii)

$$a \oplus^{t+1}_A b \in \mathcal{Q}^0_{t,T} \quad \textit{for all } a \in \mathcal{Q}_{t,T}\,,\ b \in \mathcal{Q}_{t+1,T} \textit{ and } A \in \mathcal{F}_{t+1}\,.$$

Then $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is time-consistent.

Corollary 4.16 *Let $\mathcal{Q}^{\mathrm{rel}}$ be a non-empty subset of $\mathcal{D}^{\mathrm{rel}}_{0,T}$ such that for all $a, b \in \mathcal{Q}^{\mathrm{rel}}$, every $s \in (0,T] \cap \mathbb{N}$ and $A \in \mathcal{F}_s$,*

$$a \oplus^s_A b \in \left\langle \mathcal{Q}^{\mathrm{rel}}\right\rangle\,,$$

where $\left\langle \mathcal{Q}^{\mathrm{rel}}\right\rangle$ is the $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed, convex hull of $\mathcal{Q}^{\mathrm{rel}}$. Then

$$\phi_{t,T}(X) := \operatorname*{ess\,inf}_{a \in \mathcal{Q}^{\mathrm{rel}}} \frac{\langle X, a\rangle_{t,T}}{\langle 1, a\rangle_{t,T}}\,, \quad t \in [0,T] \cap \mathbb{N}\,,\ X \in \mathcal{R}^\infty_{t,T}\,,$$

defines a time-consistent dynamic coherent utility function such that

$$\phi_{\tau,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{Q}^{\mathrm{rel}}} \frac{\langle X, a\rangle_{\tau,T}}{\langle 1, a\rangle_{\tau,T}} \tag{4.12}$$

for every finite $(\mathcal{F}_t)$-stopping time $\tau \leq T$ and $X \in \mathcal{R}^\infty_{\tau,T}$. In particular, $\phi_{\tau,T}(X)$ is T-relevant for every finite $(\mathcal{F}_t)$-stopping time $\tau \leq T$.

Proof of Theorem 4.15.
By Theorem 3.18, all $\phi_{t,T}$ are continuous for bounded decreasing sequences. Hence, by Proposition 4.5, it is enough to show that

$$\phi_{t,T}(X) = \phi_{t,T}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)})$$

for each $t \in [0,T) \cap \mathbb{N}$ and $X \in \mathcal{R}^\infty_{t,T}$. Let $a \in \mathcal{Q}_{t,T}$. By condition (i), there exists $b \in \mathcal{Q}^0_{t+1,T}$ such that

$$\frac{a}{\langle 1, a\rangle_{t+1,T}} 1_{[t+1,\infty)} = b \quad \text{on the set } \left\{\langle 1, a\rangle_{t+1,T} > 0\right\}.$$

Hence, $\langle X, a\rangle_{t+1,T} = \langle 1, a\rangle_{t+1,T} \langle X, b\rangle_{t+1,T} \geq \langle 1, a\rangle_{t+1,T} \, \phi_{t+1,T}(X)$, and therefore,

$$\begin{aligned}
\langle X, a\rangle_{t,T} &= \mathrm{E}\left[X_t \Delta a_t + \sum_{j \in [t+1,T] \cap \mathbb{N}} X_j \Delta a_j \mid \mathcal{F}_t\right] = \mathrm{E}\left[X_t \Delta a_t + \langle X, a\rangle_{t+1,T} \mid \mathcal{F}_t\right] \\
&\geq \mathrm{E}\left[X_t \Delta a_t + \langle 1, a\rangle_{t+1,T} \, \phi_{t+1,T}(X) \mid \mathcal{F}_t\right] = \left\langle X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}, a\right\rangle_{t,T},
\end{aligned}$$

which shows that

$$\phi_{t,T}(X) \geq \phi_{t,T}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}).$$

To show the converse inequality, we introduce the set

$$\tilde{\mathcal{Q}}_{t+1,T} := \left\{\sum_{k=1}^K 1_{B_k} b^k \mid K \geq 1, b^k \in \mathcal{Q}_{t+1,T}, (B_k)_{k=1}^K \text{ an } \mathcal{F}_{t+1}\text{-measurable partition of } \Omega\right\}$$

and note that it induces $\phi_{t+1,T}$. Also, it follows from condition (ii) that

$$a \oplus^{t+1}_\Omega b \in \mathcal{Q}^0_{t,T} \quad \text{for all } a \in \mathcal{Q}_{t,T} \text{ and } b \in \tilde{\mathcal{Q}}_{t+1,T}.$$

Choose $X \in \mathcal{R}^\infty_{t+1,T}$. The set $\left\{\langle X, b\rangle_{t+1,T} \mid b \in \tilde{\mathcal{Q}}_{t+1,T}\right\}$ is directed downwards. Therefore, there exists a sequence $(b^n)_{n\geq 1}$ in $\tilde{\mathcal{Q}}_{t+1,T}$ such that

$$\langle X, b^n\rangle_{t+1,T} \searrow \phi_{t+1,T}(X) \quad \text{almost surely,}$$

and hence, for every $a \in \mathcal{Q}_{t,T}$,

$$\phi_{t,T}(X) \leq \left\langle X, a \oplus^{t+1}_\Omega b^n\right\rangle_{t,T} \searrow \left\langle X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}, a\right\rangle_{t,T} \quad \text{almost surely,}$$

which shows that

$$\phi_{t,T}(X) \leq \phi_{t,T}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)})\,.$$

□

Proof of Corollary 4.16.
Time-consistency follows from Theorem 4.15, the representation (4.12) from Remark 4.12, and T–relevance from Proposition 3.21. □

As a consequence of Corollaries 4.14 and 4.16 we get the following stability result for subsets of $\mathcal{D}^{\mathrm{rel}}_{0,T}$.

Corollary 4.17 *Let $\mathcal{Q}^{\mathrm{rel}}$ be a non-empty subset of $\mathcal{D}^{\mathrm{rel}}_{0,T}$ and denote by $\langle \mathcal{Q}^{\mathrm{rel}} \rangle$ the $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed, convex hull of $\mathcal{Q}^{\mathrm{rel}}$. If*

$$a \oplus^s_A b \in \left\langle \mathcal{Q}^{\mathrm{rel}} \right\rangle \quad \textit{for all } a, b \in \mathcal{Q}^{\mathrm{rel}},\ s \in (0,T] \cap \mathbb{N} \textit{ and } A \in \mathcal{F}_s\,,$$

then the sets $\langle \mathcal{Q}^{\mathrm{rel}} \rangle$ and $\langle \mathcal{Q}^{\mathrm{rel}} \rangle \cap \mathcal{D}^{\mathrm{rel}}_{0,T}$ are c2*-stable.*
In particular, if $\mathcal{Q}^{\mathrm{rel}}$ is c1*-stable, then $\langle \mathcal{Q}^{\mathrm{rel}} \rangle$ and $\langle \mathcal{Q}^{\mathrm{rel}} \rangle \cap \mathcal{D}^{\mathrm{rel}}_{0,T}$ are* c2*-stable.*

Proof. Define the dynamic coherent utility function $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ by

$$\phi_{t,T}(X) := \inf_{a \in \mathcal{Q}^{\mathrm{rel}}} \frac{\langle X, a \rangle_{t,T}}{\langle 1, a \rangle_{t,T}}\,, \quad t \in [0,T] \cap \mathbb{N}\,,\ X \in \mathcal{R}^\infty_{t,T}\,.$$

By Corollary 4.16, $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is time-consistent. Hence, it follows from Corollary 4.14 that $\langle \mathcal{Q}^{\mathrm{rel}} \rangle$ and $\langle \mathcal{Q}^{\mathrm{rel}} \rangle \cap \mathcal{D}^{\mathrm{rel}}_{0,T}$ are c2-stable. □

4.5 Time-consistent dynamic concave monetary utility functions

In this subsection we give necessary and sufficient conditions for time-consistency of dynamic concave monetary utility functions in terms of penalty functions.

We use the following standard conventions for the definition of conditional expectations of random variables that are not necessarily integrable:
Let $f \in \bar{L}(\mathcal{F})$ and τ a finite $(\mathcal{F}_t)$-stopping time . If there exists a $g \in L^1(\mathcal{F})$ such that $f \geq g$, we define

$$\mathrm{E}\left[f \mid \mathcal{F}_\tau\right] := \lim_{n \to \infty} \mathrm{E}\left[f \wedge n \mid \mathcal{F}_\tau\right]\,.$$

If there exists a $g \in L^1(\mathcal{F})$ such that $f \leq g$, we define

$$\mathrm{E}\left[f \mid \mathcal{F}_\tau\right] := \lim_{n \to -\infty} \mathrm{E}\left[f \vee n \mid \mathcal{F}_\tau\right]\,.$$

If X is an adapted process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{N}}, P)$ taking values in the interval $[m, \infty]$ for some $m \in \mathbb{R}$, we define for all $a \in \mathcal{A}^1_+$,

$$\langle X, a \rangle_{\tau,\theta} := \lim_{n \to \infty} \langle X \wedge n, a \rangle_{\tau,\theta}\,.$$

If X takes values in $[-\infty, m]$ for some $m \in \mathbb{R}$, we define for all $a \in \mathcal{A}^1_+$,

$$\langle X, a \rangle_{\tau,\theta} := \lim_{n \to -\infty} \langle X \vee n, a \rangle_{\tau,\theta} \,.$$

Remark 4.18 Let $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ be a dynamic concave monetary utility function such that for each $t \in [0,T] \cap \mathbb{N}$, $\phi_{t,T}$ is given by

$$\phi_{t,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{t,T}} \left\{ \langle X, a \rangle_{t,T} - \gamma_{t,T}(a) \right\} , \quad X \in \mathcal{R}^\infty_{t,T} \,,$$

for a penalty function $\gamma_{t,T}$ on $\mathcal{D}_{t,T}$ that satisfies the local property. Then it can easily be checked that for all finite $(\mathcal{F}_t)$-stopping times $\tau \leq T$,

$$\phi_{\tau,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,T}} \left\{ \langle X, a \rangle_{\tau,T} - \gamma_{\tau,T}(a) \right\} , \quad X \in \mathcal{R}^\infty_{\tau,T} \,,$$

where $\gamma_{\tau,T}$ is the penalty function on $\mathcal{D}_{\tau,T}$ given by

$$\gamma_{\tau,T}(a) := \sum_{t \in [0,T] \cap \mathbb{N}} 1_{\{\tau = t\}} \gamma_t (1_{\{\tau = t\}} a + 1_{\{\tau \neq t\}} 1_{[t,\infty)}) \,, \quad a \in \mathcal{D}_{\tau,T} \,. \tag{4.13}$$

The subsequent theorem and corollary give necessary dual conditions for time-consistency of dynamic concave monetary utility functions $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ such that all $\phi_{t,T}$ are continuous for bounded decreasing sequences.

Theorem 4.19 *Let $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ be a time-consistent dynamic concave monetary utility function such that for all $t \in [0,T] \cap \mathbb{N}$,*

$$\phi_{t,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{t,T}} \left\{ \langle X, a \rangle_{t,T} - \phi^\#_{t,T}(a) \right\} , \quad X \in \mathcal{R}^\infty_{t,T} \,.$$

Then

$$\phi^\#_{\tau,T}(a) = \operatorname*{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^\#_{\tau,T} \left(a \oplus^\theta_\Omega b \right) + \mathrm{E} \left[\phi^\#_{\theta,T}(a) \mid \mathcal{F}_\tau \right] , \tag{4.14}$$

for every pair of finite $(\mathcal{F}_t)$-stopping times τ, θ such that $0 \leq \tau \leq \theta \leq T$ and all $a \in \mathcal{D}_{\tau,T}$.

Corollary 4.20 *Let $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ be a time-consistent dynamic concave monetary utility function such that $\phi_{0,T}$ is T-relevant and continuous for bounded decreasing sequences. Then*

$$\phi_{\tau,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,T}} \left\{ \langle X, a \rangle_{\tau,T} - \phi^\#_{\tau,T}(a) \right\} = \operatorname*{ess\,inf}_{a \in \mathcal{D}^{\mathrm{rel}}_{\tau,T}} \left\{ \langle X, a \rangle_{\tau,T} - \phi^\#_{\tau,T}(a) \right\} ,$$

for every finite $(\mathcal{F}_t)$-stopping time $\tau \leq T$, and

$$\begin{aligned}
\phi^\#_{\tau,T}(a) &= \operatorname*{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^\#_{\tau,T} \left(a \oplus^\theta_\Omega b \right) + \mathrm{E} \left[\phi^\#_{\theta,T}(a) \mid \mathcal{F}_\tau \right] \\
&= \operatorname*{ess\,sup}_{b \in \mathcal{D}^{\mathrm{rel}}_{\theta,T}} \phi^\#_{\tau,T} \left(a \oplus^\theta_\Omega b \right) + \mathrm{E} \left[\phi^\#_{\theta,T}(a) \mid \mathcal{F}_\tau \right] ,
\end{aligned}$$

for every pair of finite $(\mathcal{F}_t)$-stopping times τ, θ such that $0 \leq \tau \leq \theta \leq T$ and all $a \in \mathcal{D}_{\tau,\theta}$.

Proof of Theorem 4.19.
Let τ and θ be two finite $(\mathcal{F}_t)$-stopping times such that $0 \le \tau \le \theta \le T$, and $(\mathcal{C}_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ the acceptance sets corresponding to $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$. It follows from Remark 4.3 and Theorem 4.6 that for all $a \in \mathcal{D}_{\tau,T}$,

$$\begin{aligned}
\phi^{\#}_{\tau,T}(a) &= \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\tau,T}} \langle X, a \rangle_{\tau,T} \\
&= \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\theta,T}} \langle X, a \rangle_{\tau,T} \\
&= \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \mathrm{E}\left[\operatorname*{ess\,inf}_{X \in \mathcal{C}_{\theta,T}} \langle X, a \rangle_{\theta,T} \mid \mathcal{F}_\tau\right] \\
&= \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \mathrm{E}\left[\phi^{\#}_{\theta,T}(a) \mid \mathcal{F}_\tau\right] , \qquad (4.15)
\end{aligned}$$

and for all $a \in \mathcal{D}_{\tau,T}$ and $b \in \mathcal{D}_{\theta,T}$,

$$\begin{aligned}
\phi^{\#}_{\tau,T}(a \oplus^{\theta}_{\Omega} b) &= \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\tau,T}} \left\langle X, a \oplus^{\theta}_{\Omega} b \right\rangle_{\tau,T} \\
&= \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \left\langle X, a \oplus^{\theta}_{\Omega} b \right\rangle_{\tau,T} + \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\theta,T}} \left\langle X, a \oplus^{\theta}_{\Omega} b \right\rangle_{\tau,T} \\
&= \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \mathrm{E}\left[\operatorname*{ess\,inf}_{X \in \mathcal{C}_{\theta,T}} \langle X, b \rangle_{\theta,T} \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_\tau\right] \\
&= \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \mathrm{E}\left[\phi^{\#}_{\theta,T}(b) \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_\tau\right] .
\end{aligned}$$

By Remark 4.18,

$$\phi_{\theta,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\theta,T}} \left\{ \langle X, a \rangle_{\theta,T} - \phi^{\#}_{\theta,T}(a) \right\} , \quad X \in \mathcal{R}^{\infty}_{\theta,T} ,$$

which implies,

$$\operatorname*{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^{\#}_{\theta,T}(b) = 0 .$$

Since $\phi^{\#}_{\theta,T}$ has the local property, the set $\left\{\phi^{\#}_{\theta,T}(b) \mid b \in \mathcal{D}_{\theta,T}\right\}$ is directed upwards, and therefore,

$$\operatorname*{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \mathrm{E}\left[\phi^{\#}_{\theta,T}(b) \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_\tau\right] = \mathrm{E}\left[\operatorname*{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^{\#}_{\theta,T}(b) \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_\tau\right] = 0 .$$

Hence,

$$\operatorname*{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^{\#}_{\tau,T}\left(a \oplus^{\theta}_{\Omega} b\right) = \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} ,$$

which together with (4.15), proves (4.14). $\square$

Proof of Corollary 4.20.
By Theorem 3.16, $\mathcal{C}_{0,T}$ is a $\sigma(\mathcal{R}^{\infty}, \mathcal{A}^1)$-closed subset of $\mathcal{R}^{\infty}$. Let $\tau \le T$ be a finite $(\mathcal{F}_t)$-stopping time and $(X^{\mu})_{\mu \in M}$ a net in $\mathcal{C}_{\tau,T}$ such that $X^{\mu} \to X$ in $\sigma(\mathcal{R}^{\infty}, \mathcal{A}^1)$ for some

$X \in \mathcal{R}^\infty$. Then, $X \in \mathcal{R}^\infty_{\tau,T}$, and for each $A \in \mathcal{F}_\tau$, $1_A X^\mu \to 1_A X$ in $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$. By Proposition 4.7.1, $(1_A X^\mu)_{\mu \in M}$ is a net in $\mathcal{C}_{0,T}$. Hence, $1_A X \in \mathcal{C}_{0,T}$, which by Proposition 4.7.2, implies that $X \in \mathcal{C}_{\tau,T}$. This shows that $\mathcal{C}_{\tau,T}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed. Hence, it follows from Theorem 3.16 that

$$\phi_{\tau,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}_{\tau,T}} \left\{ \langle X, a \rangle_{\tau,T} - \phi^\#_{\tau,T}(a) \right\} . \tag{4.16}$$

By Proposition 4.7.3, $\phi_{\tau,T}$ is T-relevant, which by Theorem 3.23, implies that

$$\phi_{\tau,T}(X) = \operatorname*{ess\,inf}_{a \in \mathcal{D}^{\mathrm{rel}}_{\tau,T}} \left\{ \langle X, a \rangle_{\tau,T} - \phi^\#_{\tau,T}(a) \right\} . \tag{4.17}$$

By Theorem 4.19, it follows from (4.16) that

$$\phi^\#_{\tau,T}(a) = \operatorname*{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^\#_{\tau,T}\left(a \oplus^\theta_\Omega b\right) + \mathrm{E}\left[\phi^\#_{\theta,T}(a) \mid \mathcal{F}_\tau\right] ,$$

for every pair of finite $(\mathcal{F}_t)$-stopping times τ, θ such that $0 \le \tau \le \theta \le T$ and all $a \in \mathcal{D}_{\tau,\theta}$. In the proof of Theorem 4.19 we showed that for all $a \in \mathcal{D}_{\tau,T}$ and $b \in \mathcal{D}_{\theta,T}$,

$$\phi^\#_{\tau,T}(a \oplus^\theta_\Omega b) = \operatorname*{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \mathrm{E}\left[\phi^\#_{\theta,T}(b) \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_\tau\right] ,$$

and it follows from (4.16) and (4.17) that

$$\operatorname*{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^\#_{\theta,T}(b) = \operatorname*{ess\,sup}_{b \in \mathcal{D}^{\mathrm{rel}}_{\theta,T}} \phi^\#_{\theta,T}(b) = 0 .$$

Hence,

$$\operatorname*{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \mathrm{E}\left[\phi^\#_{\theta,T}(b) \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_\tau\right] = \operatorname*{ess\,sup}_{b \in \mathcal{D}^{\mathrm{rel}}_{\theta,T}} \mathrm{E}\left[\phi^\#_{\theta,T}(b) \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_\tau\right] = 0 ,$$

and therefore,

$$\operatorname*{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^\#_{\tau,T}\left(a \oplus^\theta_\Omega b\right) = \operatorname*{ess\,sup}_{b \in \mathcal{D}^{\mathrm{rel}}_{\theta,T}} \phi^\#_{\tau,T}\left(a \oplus^\theta_\Omega b\right) .$$

□

In the next theorem and corollary we give sufficient dual conditions for time-consistency of dynamic concave monetary utility functions. For their formulation we need the following notation:

Definition 4.21 *Let θ be a finite $(\mathcal{F}_t)$-stopping time such that $\theta \le T$.*

For every $a \in \mathcal{D}_{0,T}$, we define the process $\overset{\rightarrow\theta}{a} \in \mathcal{D}_{\theta,T}$ as follows:

$$\overset{\rightarrow\theta}{a} := \begin{cases} \frac{a}{\langle 1, a \rangle_{\theta,T}} 1_{[\theta,\infty)} & \text{on } \left\{ \langle 1, a \rangle_{\theta,T} > 0 \right\} \\ 1_{[\theta,\infty)} & \text{on } \left\{ \langle 1, a \rangle_{\theta,T} = 0 \right\} \end{cases}$$

If $\gamma_{\theta,T}$ is a penalty function on $\mathcal{D}_{\theta,T}$, we extend it to $\mathcal{D}_{0,T}$ by setting

$$\gamma^{\mathrm{ext}}_{\theta,T}(a) := \begin{cases} \langle 1, a\rangle_{\theta,T}\, \gamma_{\theta,T}\left(\overset{\rightarrow\theta}{a}\right) & on \ \left\{\langle 1, a\rangle_{\theta,T} > 0\right\} \\ 0 & on \ \left\{\langle 1, a\rangle_{\theta,T} = 0\right\} \end{cases}, \ a \in \mathcal{D}_{0,T}\,.$$

Theorem 4.22 *Let $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ be a dynamic concave monetary utility function such that for every $t \in [0,T]\cap\mathbb{N}$,*

$$\phi_{t,T}(X) = \operatorname*{ess\,inf}_{a\in\mathcal{D}_{t,T}} \left\{\langle X, a\rangle_{t,T} - \gamma_{t,T}(a)\right\}, \quad X \in \mathcal{R}^\infty_{t,T}$$

for a penalty function $\gamma_{t,T}$ on $\mathcal{D}_{t,T}$ with the local property. If for each $t \in [0,T)\cap\mathbb{N}$ and $a \in \mathcal{D}_{t,T}$,

$$\gamma_{t,T}(a) = \operatorname*{ess\,sup}_{b\in\mathcal{D}_{t+1,T}} \gamma_{t,T}(a \oplus^{t+1}_{\Omega} b) + \mathrm{E}\left[\gamma^{\mathrm{ext}}_{t+1,T}(a) \mid \mathcal{F}_t\right],$$

then $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is time-consistent.

Corollary 4.23 *Let $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ be a dynamic concave monetary utility function such that for all $t \in [0,T]\cap\mathbb{N}$,*

$$\phi_{t,T}(X) = \operatorname*{ess\,inf}_{a\in\mathcal{D}^{\mathrm{rel}}_{t,T}} \left\{\langle X, a\rangle_{t,T} - \gamma_{t,T}(a)\right\}, \quad X \in \mathcal{R}^\infty_{t,T}$$

for a penalty function $\gamma_{t,T}$ on $\mathcal{D}^{\mathrm{rel}}_{t,T}$ with the local property. If for all $t \in [0,T)\cap\mathbb{N}$ and $a \in \mathcal{D}^{\mathrm{rel}}_{t,T}$,

$$\gamma_{t,T}(a) = \operatorname*{ess\,sup}_{b\in\mathcal{D}^{\mathrm{rel}}_{t+1,T}} \gamma_{t,T}(a \oplus^{t+1}_{\Omega} b) + \mathrm{E}\left[\gamma^{\mathrm{ext}}_{t+1,T}(a) \mid \mathcal{F}_t\right],$$

then $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is time-consistent.

Proof of Theorem 4.22.
Fix $t \in [0,T)\cap\mathbb{N}$ and $X \in \mathcal{R}^\infty_{t,T}$. Note that

$$\gamma_{t,T}(a) \geq \operatorname*{ess\,sup}_{b\in\mathcal{D}_{t+1,T}} \gamma_{t,T}(a \oplus^{t+1}_{\Omega} b) + \mathrm{E}\left[\gamma^{\mathrm{ext}}_{t+1,T}(a) \mid \mathcal{F}_t\right], \quad \text{for all } a \in \mathcal{D}_{t,T}\,,$$

implies that

$$\gamma_{t,T}(a \oplus^{t+1}_{\Omega} b) \geq \gamma_{t,T}(a) + \mathrm{E}\left[\gamma^{\mathrm{ext}}_{t+1,T}(a \oplus^{t+1}_{\Omega} b) \mid \mathcal{F}_t\right]. \tag{4.18}$$

for all $a \in \mathcal{D}_{t,T}$ and $b \in \mathcal{D}_{t+1,T}$.

Since $\gamma_{t+1,T}$ has the local property, the set

$$\left\{\langle X, b\rangle_{t+1,T} - \gamma_{t+1,T}(b) \mid b \in \mathcal{D}_{t+1,T}\right\}$$

is directed downwards. Hence, there exists a sequence $(b^n)_{n\geq 1}$ in $\mathcal{D}_{t+1,T}$ such that

$$\langle X, b^n\rangle_{t+1,T} - \gamma_{t+1,T}(b^n) \searrow \phi_{t+1,T}(X) \quad \text{almost surely.}$$

It can easily be checked that for all $a \in \mathcal{D}_{t,T}$ and $b \in \mathcal{D}_{t+1,T}$,

$$\left\langle X 1_{\{t\}} + \langle X, b \rangle_{t+1,T} 1_{[t+1,\infty)}, a \right\rangle_{t,T} = \left\langle X, a \oplus_{\Omega}^{t+1} b \right\rangle_{t,T} \tag{4.19}$$

and

$$\left\langle \gamma_{t+1,T}(b) 1_{[t+1,\infty)}, a \right\rangle_{t,T} = \mathrm{E}\left[\gamma^{\mathrm{ext}}_{t+1,T}(a \oplus_{\Omega}^{t+1} b) \mid \mathcal{F}_t\right] . \tag{4.20}$$

Therefore, it follows from (4.18) that for all $a \in \mathcal{D}_{t,T}$ and $n \geq 1$,

$$\begin{aligned}
& \left\langle X 1_{\{t\}} + \left[\langle X, b^n \rangle_{t+1,T} - \gamma_{t+1,T}(b^n)\right] 1_{[t+1,\infty)}, a \right\rangle_{t,T} - \gamma_{t,T}(a) \\
= \;& \left\langle X, a \oplus_{\Omega}^{t+1} b^n \right\rangle_{t,T} - \mathrm{E}\left[\gamma^{\mathrm{ext}}_{t+1,T}(a \oplus_{\Omega}^{t+1} b^n) \mid \mathcal{F}_t\right] - \gamma_{t,T}(a) \\
\geq \;& \left\langle X, a \oplus_{\Omega}^{t+1} b^n \right\rangle_{t,T} - \gamma_{t,T}(a \oplus_{\Omega}^{t+1} b^n) \\
\geq \;& \phi_{t,T}(X) .
\end{aligned}$$

This shows that for all $a \in \mathcal{D}_{t,T}$,

$$\left\langle X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}, a \right\rangle_{t,T} - \gamma_{t,T}(a) \geq \phi_{t,T}(X) ,$$

and therefore,

$$\phi_{t,T}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty]}) \geq \phi_{t,T}(X) . \tag{4.21}$$

It follows from (4.19) and (4.20) that for all $a \in \mathcal{D}_{t,T}$ and $b \in \mathcal{D}_{t+1,T}$,

$$\left\langle X 1_{\{t\}} + \left\langle X, \overset{\to t+1}{a} \right\rangle_{t+1,T} 1_{[t+1,\infty)}, a \oplus_{\Omega}^{t+1} b \right\rangle_{t,T} = \langle X, a \rangle_{t,T}$$

and

$$\left\langle \gamma_{t+1,T}(\overset{\to t+1}{a}) 1_{[t+1,\infty)}, a \oplus_{\Omega}^{t+1} b \right\rangle_{t,T} = \mathrm{E}\left[\gamma^{\mathrm{ext}}_{t+1,T}(a) \mid \mathcal{F}_t\right] .$$

Hence, for fixed $a \in \mathcal{D}_{t,T}$, the inequality

$$\gamma_{t,T}(a) \leq \operatorname*{ess\,sup}_{b \in \mathcal{D}_{t+1,T}} \gamma_{t,T}(a \oplus_{\Omega}^{t+1} b) + \mathrm{E}\left[\gamma^{\mathrm{ext}}_{t+1,T}(a) \mid \mathcal{F}_t\right]$$

implies that

$$\begin{aligned}
& \langle X, a \rangle_{t,T} - \gamma_{t,T}(a) \\
\geq \;& \langle X, a \rangle_{t,T} - \mathrm{E}\left[\gamma^{\mathrm{ext}}_{t+1,T}(a) \mid \mathcal{F}_t\right] - \operatorname*{ess\,sup}_{b \in \mathcal{D}_{t+1,T}} \gamma_{t,T}(a \oplus_{\Omega}^{t+1} b) \\
= \;& \operatorname*{ess\,inf}_{b \in \mathcal{D}_{t+1,T}} \Bigg\{ \left\langle X 1_{\{t\}} + \left[\left\langle X, \overset{\to t+1}{a} \right\rangle_{t+1,T} - \gamma_{t+1,T}\left(\overset{\to t+1}{a}\right)\right] 1_{[t+1,\infty)}, a \oplus_{\Omega}^{t+1} b \right\rangle_{t,T} \\
& \qquad - \gamma_{t,T}\left(a \oplus_{\Omega}^{t+1} b\right) \Bigg\} \\
\geq \;& \operatorname*{ess\,inf}_{b \in \mathcal{D}_{t+1,T}} \left\{ \left\langle X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}, a \oplus_{\Omega}^{t+1} b \right\rangle_{t,T} - \gamma_{t,T}(a \oplus_{\Omega}^{t+1} b) \right\} \\
\geq \;& \phi_{t,T}\left(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}\right) .
\end{aligned}$$

This shows that

$$\phi_{t,T}(X) \geq \phi_{t,T}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}) \,. \tag{4.22}$$

By Theorem 3.16, all $\phi_{t,T}$ are continuous for bounded decreasing sequences. Hence, it follows from (4.21), (4.22) and Proposition 4.5 that $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is time-consistent. $\square$

Proof of Corollary 4.23.
The proof of Corollary 4.23 is exactly the same as the proof of Theorem 4.22. $\square$

5 Special cases and examples

In much of this section the pasting of probability measures plays an important role. It can be viewed as a special case of the concatenation operation introduced in Subsection 4.3 and has appeared under different names in various contexts; see for instance, Wang (2003), Epstein and Schneider (2003), Artzner et al. (2004), Delbaen (2003), Riedel (2004), Roorda et al. (2005).

We describe probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to P by their Radon–Nikodym derivatives $dQ/dP \in \left\{h \in L^1(\mathcal{F}) \mid h \geq 0 \,, \mathrm{E}\,[h] = 1\right\}$. We recall that a probability measure Q absolutely continuous with respect to P is equivalent to P if and only if $f = dQ/dP > 0$, in which case, for finite $(\mathcal{F}_t)$-stopping time θ, the conditional expectation $\mathrm{E}_Q[Y \mid \mathcal{F}_\theta]$ of a random variable $Y \in L^\infty(\mathcal{F})$ is given by

$$\frac{\mathrm{E}\,[f\,Y \mid \mathcal{F}_\theta]}{\mathrm{E}\,[f \mid \mathcal{F}_\theta]} \,.$$

For all $T \in \mathbb{N} \cup \{\infty\}$, we denote

$$\tilde{\mathcal{D}}_T := \left\{h \in L^1(\mathcal{F}_T) \mid h \geq 0 \,, \mathrm{E}\,[h] = 1\right\}$$

and

$$\tilde{\mathcal{D}}_T^{\mathrm{rel}} := \left\{h \in L^1(\mathcal{F}_T) \mid h > 0 \,, \mathrm{E}\,[h] = 1\right\} \,,$$

where $\mathcal{F}_\infty$ is the sigma-algebra generated by $\bigcup_{t\in\mathbb{N}} \mathcal{F}_t$.

Definition 5.1 *For $T \in \mathbb{N} \cup \{\infty\}$, $f, g \in \tilde{\mathcal{D}}_T$, a finite $(\mathcal{F}_t)$-stopping time $\theta \leq T$ and $A \in \mathcal{F}_\theta$, we define the pasting $f \otimes_A^\theta g$ by*

$$f \otimes_A^\theta g := \begin{cases} f & \text{on } A^c \cup \{\mathrm{E}\,[g \mid \mathcal{F}_\theta] = 0\} \\ \mathrm{E}\,[f \mid \mathcal{F}_\theta]\, \frac{g}{\mathrm{E}[g \mid \mathcal{F}_\theta]} & \text{on } A \cap \{\mathrm{E}\,[g \mid \mathcal{F}_\theta] > 0\} \end{cases} \,. \tag{5.1}$$

We call a subset $\mathcal{P}$ of $\tilde{\mathcal{D}}_T$ m1*-stable if it contains $f \otimes_A^s g$ for all $f, g \in \mathcal{P}$, every $s \in [0,T] \cap \mathbb{N}$ and $A \in \mathcal{F}_s$, and* m2*-stable if it contains $f \otimes_A^\theta g$ for all $f, g \in \mathcal{P}$, every finite $(\mathcal{F}_t)$-stopping time $\theta \leq T$ and $A \in \mathcal{F}_\theta$.*

Remark 5.2 It can be shown as in Remark 4.11 that for $T \in \mathbb{N}$, m1-stability is equivalent to m2-stability.

5.1 Dynamic coherent utility functions that depend on final values

Let $T \in \mathbb{N}$ and $\mathcal{P}$ a non-empty subset of $\tilde{D}_T$. Then

$$\mathcal{Q}(\mathcal{P}) := \left\{ f \, 1_{[T,\infty)} \mid f \in \mathcal{P} \right\}$$

is a non-empty subset of $\mathcal{D}_{0,T}$, and the concatenation of two elements

$$a = f 1_{[T,\infty)} \quad \text{and} \quad b = g 1_{[T,\infty)}$$

in $\mathcal{Q}(\mathcal{P})$ at an $(\mathcal{F}_t)$-stopping time $\theta \le T$ for a set $A \in \mathcal{F}_\theta$, is equal to

$$\left(f \otimes_A^\theta g \right) 1_{[T,\infty)} \, .$$

This shows that $\mathcal{Q}(\mathcal{P})$ is c1-stable if and only if $\mathcal{P}$ is m1-stable. (For $T \in \mathbb{N}$, c1-stability is equivalent to c2-stability and m1-stability equivalent to m2-stability).

If $\mathcal{P}^{\mathrm{rel}}$ is a non-empty subset of $\tilde{\mathcal{D}}_T^{\mathrm{rel}}$, then $\mathcal{Q}(\mathcal{P}^{\mathrm{rel}})$ is a non-empty subset of $\mathcal{D}_{0,T}^{\mathrm{rel}}$, and

$$\phi_{t,T}(X) := \operatorname*{ess\,inf}_{a \in \mathcal{Q}(\mathcal{P}^{\mathrm{rel}})} \frac{\langle X, a \rangle_{t,T}}{\langle 1, a \rangle_{t,T}} = \operatorname*{ess\,inf}_{f \in \mathcal{P}^{\mathrm{rel}}} \frac{\mathrm{E}\left[f X_T \mid \mathcal{F}_t \right]}{\mathrm{E}\left[f \mid \mathcal{F}_t \right]} \, , \; t = 0, \dots, T \, , \; X \in \mathcal{R}_{t,T}^\infty \, ,$$

defines a dynamic coherent utility function such that $\phi_{t,T}$ is T-relevant for every $t = 0, \dots, T$.

If $\mathcal{P}^{\mathrm{rel}}$ is m1-stable, it follows from Corollary 4.16 that $(\phi_{t,T})_{t=0}^T$ is time-consistent. On the other hand, if $(\phi_{t,T})_{t=0}^T$ is time consistent, then by Corollary 4.14, the $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed convex hull of $\mathcal{Q}(\mathcal{P}^{\mathrm{rel}})$ is c1-stable, which implies that the $\sigma(L^1, L^\infty)$-closed, convex hull of $\mathcal{P}^{\mathrm{rel}}$ is m1-stable.

This class of time-consistent dynamic coherent utility functions appears in Artzner et al. (2004), Riedel (2004), Roorda et al. (2005) and in a continuous-time setup, in Delbaen (2003).

5.2 Dynamic coherent utility functions defined by worst stopping

Let $T \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{P}^{\mathrm{rel}}$ a non-empty m1-stable subset of $\tilde{\mathcal{D}}_T^{\mathrm{rel}}$. For all $t \in [0, T] \cap \mathbb{N}$, define

$$\psi_t(Y) := \operatorname*{ess\,inf}_{f \in \mathcal{P}^{\mathrm{rel}}} \frac{\mathrm{E}\left[f \, Y \mid \mathcal{F}_t \right]}{\mathrm{E}\left[f \mid \mathcal{F}_t \right]} \, , \quad Y \in L^\infty(\mathcal{F}_T) \, ,$$

and for all $X \in \mathcal{R}_{t,T}^\infty$,

$$\phi_{t,T}(X) := \operatorname*{ess\,inf} \left\{ \psi_t(X_\xi) \mid \xi \text{ a finite } (\mathcal{F}_t)\text{-stopping time such that } t \le \xi \le T \right\} . \quad (5.2)$$

Then $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is a time-consistent dynamic coherent utility function such that every $\phi_{t,T}$ is T-relevant.

To see this, note that $\phi_{0,T}$ is a T-relevant coherent utility function on $\mathcal{R}_{0,T}^\infty$ that can be represented as

$$\phi_{0,T}(X) = \inf_{a \in \mathcal{Q}(\mathcal{P}^{\mathrm{rel}})} \langle X, a \rangle_{0,T} \, , \quad X \in \mathcal{R}_{0,T}^\infty \, ,$$

where $\mathcal{Q}(\mathcal{P}^{\mathrm{rel}})$ is the non-empty subset of $\mathcal{D}_{0,T}$ given by

$$\mathcal{Q}(\mathcal{P}^{\mathrm{rel}}) := \left\{ \mathrm{E}\left[f \mid \mathcal{F}_\xi\right] 1_{[\xi,\infty)} \mid f \in \mathcal{P}^{\mathrm{rel}}\,,\, \xi \le T \text{ a finite } (\mathcal{F}_t)\text{-stopping time} \right\}.$$

Note that, unless $T \in \mathbb{N}$ and $\xi = T$, an element of $\mathcal{Q}(\mathcal{P}^{\mathrm{rel}})$ of the form $\mathrm{E}\left[f \mid \mathcal{F}_\xi\right] 1_{[\xi,\infty)}$ does not belong to $\mathcal{D}^{\mathrm{rel}}_{0,T}$. But it follows from Theorem 3.18 and Corollary 3.24 that $\phi_{0,T}$ can also be represented as

$$\phi_{0,T}(X) = \inf_{a \in \mathcal{Q}^0_{0,T}} \langle X, a \rangle_{0,T} = \inf_{a \in \mathcal{Q}^{0,\mathrm{rel}}_{0,T}} \langle X, a \rangle_{0,T}\,, \quad X \in \mathcal{R}^\infty_{0,T}\,,$$

where

$$\mathcal{Q}^0_{0,T} := \left\{ a \in \mathcal{D}_{0,T} \mid \phi^{\#}_{0,T}(a) = 0 \right\} \quad \text{and} \quad \mathcal{Q}^{0,\mathrm{rel}}_{0,T} := \mathcal{Q}^0_{0,T} \cap \mathcal{D}^{\mathrm{rel}}_{0,T}\,.$$

Let $\theta \le T$ be a finite $(\mathcal{F}_t)$-stopping time, $A \in \mathcal{F}_\theta$ and a, b two processes in $\mathcal{Q}(\mathcal{P}^{\mathrm{rel}})$ of the form

$$a = f_a 1_{[\xi_a,\infty)} \quad \text{and} \quad b = f_b 1_{[\xi_b,\infty)}\,,$$

where $\xi_a \le T$ and $\xi_b \le T$ are finite $(\mathcal{F}_t)$-stopping times, $f_a = \mathrm{E}\left[\hat{f}_a \mid \mathcal{F}_{\xi_a}\right]$ and $f_b = \mathrm{E}\left[\hat{f}_b \mid \mathcal{F}_{\xi_b}\right]$ for $\hat{f}_a, \hat{f}_b \in \mathcal{P}^{\mathrm{rel}}$. Then

$$\begin{aligned}
(a \oplus^\theta_A b)_t &= 1_{B^c}\, f_a\, 1_{\{t \ge \xi_a\}} + 1_B \frac{\mathrm{E}\left[f_a \mid \mathcal{F}_\theta\right]}{\mathrm{E}\left[f_b \mid \mathcal{F}_\theta\right]} f_b\, 1_{\{t \ge \xi_b\}} \\
&= 1_{B^c}\, f_a\, 1_{\{t \ge \xi_a\}} + 1_B \frac{\mathrm{E}\left[\hat{f}_a \mid \mathcal{F}_\theta\right]}{\mathrm{E}\left[\hat{f}_b \mid \mathcal{F}_\theta\right]} f_b\, 1_{\{t \ge \xi_b\}} \\
&= \mathrm{E}\left[\hat{f} \mid \mathcal{F}_\xi\right] 1_{\{t \ge \xi\}}\,,
\end{aligned}$$

where

$$B = A \cap \{t \ge \theta\} \cap \{\xi_b \ge \theta\} \cap \{\xi_a \ge \theta\} \in \mathcal{F}_{\theta \wedge \xi_a \wedge \xi_b}\,,$$
$$\hat{f} = \hat{f}_a \otimes^\theta_B \hat{f}_b \quad \text{and} \quad \xi = 1_{B^c}\xi_a + 1_B \xi_b\,.$$

It follows from the m1-stability of $\mathcal{P}^{\mathrm{rel}}$ that $\mathcal{Q}(\mathcal{P}^{\mathrm{rel}})$ is c1-stable. By Theorem 3.18, $\mathcal{Q}^0_{0,T}$ is the $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed, convex hull of $\mathcal{Q}(\mathcal{P}^{\mathrm{rel}})$. Hence, it follows from Corollary 4.17 that $\mathcal{Q}^0_{0,T}$ and $\mathcal{Q}^{0,\mathrm{rel}}_{0,T}$ are c2-stable. Therefore, by Corollary 4.16,

$$\tilde{\phi}_{t,T}(X) := \operatorname*{ess\,inf}_{a \in \mathcal{Q}^{0,\mathrm{rel}}_{0,T}} \frac{\langle X, a \rangle_{t,T}}{\langle 1, a \rangle_{t,T}}\,, \quad t \in [0,T] \cap \mathbb{N}\,,\, X \in \mathcal{R}^\infty_{t,T}\,,$$

defines a time-consistent dynamic coherent utility function such that every $\tilde{\phi}_{t,T}$ is T-relevant. It can easily be checked that $\tilde{\phi}_{t,T} = \phi_{t,T}$ for all $t \in [0,T] \cap \mathbb{N}$.

For finite time horizon T, this class of time-consistent dynamic coherent utility functions is also discussed in Artzner et al. (2004) and in a continuous-time setup in Delbaen (2003). Up to signs they are of the same form as the super-hedging prices of American contingent claims discussed in Karatzas and Kou (1998), see also Sections 6.5, 7.3, 9.3 and 9.4 in Föllmer and Schied (2004).

5.3 Dynamic monetary utility functions defined by worst stopping

The time-consistent dynamic utility functions of Subsection 5.2 can be generalized as follows:

Let $T \in \mathbb{N} \cup \{\infty\}$. For all $t \in [0,T] \cap \mathbb{N}$, let ψ_t be a mapping from $L^\infty(\mathcal{F}_T)$ to $L^\infty(\mathcal{F}_t)$ that satisfies the following conditions:

(N) $\psi_t(0) = 0$

(M) $\psi_t(Y) \le \psi_t(Z)$ for all $Y, Z \in L^\infty(\mathcal{F}_T)$ such that $Y \le Z$

(TI) $\psi_t(Y + m) = \psi_t(Y) + m$ for all $Y \in L^\infty(\mathcal{F}_T)$ and $m \in L^\infty(\mathcal{F}_t)$

(TC) $\psi_t(\psi_{t+1}(Y)) = \psi_t(Y)$, for all $Y \in L^\infty(\mathcal{F}_T)$ and $t \in [0,T) \cap \mathbb{N}$.

By Proposition 3.3, ψ_t also satisfies

(LP) $\psi_t(1_A Y + 1_{A^c} Z) = 1_A \psi_t(X) + 1_{A^c} \psi_t(Y)$ for all $Y, Z \in L^\infty(\mathcal{F}_T)$ and $A \in \mathcal{F}_t$.

Denote by $\Theta_{t,T}$ the set of all $(\mathcal{F}_t)$-stopping times ξ such that $t \le \xi \le T$, and define a dynamic monetary utility function by

$$\phi_{t,T}(X) := \operatorname*{ess\,inf}_{\xi \in \Theta_{t,T}} \psi_t(X_\xi)\,, \quad t \in [0,T] \cap \mathbb{N}\,,\; X \in \mathcal{R}^\infty_{t,T}\,.$$

Note that except linearity and σ-additivity, the operators ψ_t have all the properties of conditional expectations.

For $T < \infty$, we can proceed as in Section VI.1 of Neveu (1975) and define for all $X \in \mathcal{R}^\infty_{0,T}$ the process $(S_t(X))_{t=0}^T$ recursively by

$$\begin{aligned} S_T(X) &:= X_T \\ S_t(X) &:= X_t \wedge \psi_t(S_{t+1}(X))\,, \quad \text{for } t \le T-1\,. \end{aligned} \tag{5.3}$$

For all $t = 0, \dots, T$, let the stopping time ξ^t be given by

$$\xi^t := \inf\{j = t, \dots, T \mid S_j(X) = X_j\}\,.$$

It can easily be checked by backwards induction that

$$S_t(X) = \psi_t(X_{\xi^t}) = \phi_{t,T}(X) \quad \text{for all } t = 0, \dots, T\,.$$

Hence, it follows from (5.3) that for all $t = 0, \dots, T-1$ and $X \in \mathcal{R}^\infty_{t,T}$,

$$\phi_{t,T}(X) = X_t \wedge \psi_t(\phi_{t+1,T}(X)) = \phi_{t,T}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)})\,,$$

which by Proposition 4.5 shows that $(\phi_{t,T})_{t=0}^T$ is time-consistent.

If $T = \infty$ and all ψ_t are continuous for bounded decreasing sequences, it can be shown as in Proposition VI.1.2 of Neveu (1975) that for all $X \in \mathcal{R}^\infty_{t,\infty}$,

$$\phi_{t,\infty}(X) = X_t \wedge \psi_t(\phi_{t+1,\infty}(X)) \quad \text{for all } t \in \mathbb{N}\,.$$

This implies

$$\phi_{t,T}(X) = \phi_{t,T}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)})\,,$$

for all $t \in \mathbb{N}$ and $X \in \mathcal{R}^\infty_{t,\infty}$. Since $\phi_{t,\infty}$ inherits the continuity for bounded decreasing sequences from ψ_t, it again follows from Proposition 4.5 that $(\phi_{t,\infty})_{t \in \mathbb{N}}$ is time-consistent.

5.4 Dynamic coherent utility functions that depend on the infimum over time

Let $T \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{P}^{\mathrm{rel}}$ a non-empty subset of $\tilde{\mathcal{D}}_T^{\mathrm{rel}}$. For all $t \in [0,T] \cap \mathbb{N}$, define

$$\psi_t(Y) := \operatorname*{ess\,inf}_{f \in \mathcal{P}^{\mathrm{rel}}} \frac{\mathrm{E}\left[f\, Y \mid \mathcal{F}_t\right]}{\mathrm{E}\left[f \mid \mathcal{F}_t\right]}\,, \quad Y \in L^\infty(\mathcal{F}_T)\,,$$

and

$$\phi_{t,T}(X) := \psi_t\left(\inf_{s \in [t,T] \cap \mathbb{N}} X_s\right)\,, \quad X \in \mathcal{R}^\infty_{t,T}\,.$$

Then $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is a dynamic coherent utility function such that every $\phi_{t,T}$ is T-relevant. But even if $\mathcal{P}^{\mathrm{rel}}$ is m2-stable, $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is in general not time-consistent.

For an easy counter-example, consider a probability space of the form $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with $P[\omega_j] = \frac{1}{4}$ for all $j = 1, \dots, 4$. Let $T = 2$ and assume that the filtration $(\mathcal{F}_t)_{t=0}^2$ is given as follows: $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1$ is generated by the set $\{\omega_1, \omega_2\}$ and $\mathcal{F}_2$ is generated by the sets $\{\omega_j\}$, $j = 1, \dots, 4$. If $\mathcal{P}^{\mathrm{rel}} = \{1\}$. Then, for $t \in \{0,1,2\}$ and $X \in \mathcal{R}^\infty_{t,2}$,

$$\phi_{t,2}(X) = \mathrm{E}\left[\inf_{t \le s \le 2} X_s \mid \mathcal{F}_t\right]\,.$$

If $X_0 = 2$, $X_1(\omega_1) = X_1(\omega_2) = 4$, $X_1(\omega_3) = X_1(\omega_4) = 1$, $X_2(\omega_1) = 5$, $X_2(\omega_2) = 1$, $X_2(\omega_3) = 2$ and $X_2(\omega_4) = -1$, then $\phi_{0,2}(X) = \frac{3}{4}$. On the other hand, $\phi_{1,2}(X) = \frac{5}{2}$ on $\{\omega_1, \omega_2\}$ and $\phi_{1,2}(X) = 0$ on $\{\omega_3, \omega_4\}$. Hence, $\phi_{0,2}(X 1_{\{0\}} + \phi_{1,2}(X) 1_{[1,2]}) = 1$.

5.5 Dynamic coherent utility functions that depend on an average over time

Let $T \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{P}^{\mathrm{rel}}$ a non-empty subset of $\tilde{\mathcal{D}}_T^{\mathrm{rel}}$. For all $t \in [0,T] \cap \mathbb{N}$, define

$$\psi_t(Y) := \operatorname*{ess\,inf}_{f \in \mathcal{P}^{\mathrm{rel}}} \frac{\mathrm{E}\left[f\, Y \mid \mathcal{F}_t\right]}{\mathrm{E}\left[f \mid \mathcal{F}_t\right]}\,, \quad Y \in L^\infty(\mathcal{F}_T)\,,$$

and

$$\phi_{t,T}(X) := \psi_t\left(\frac{\sum_{s \in [t,T] \cap \mathbb{N}} \mu_s X_s}{\sum_{s \in [t,T] \cap \mathbb{N}} \mu_s}\right)\,, \quad X \in \mathcal{R}^\infty_{t,T}\,,$$

where $(\mu_s)_{s \in [0,T] \cap \mathbb{N}}$ is a sequence of non-negative numbers such that

$$\sum_{s \in [0,T] \cap \mathbb{N}} \mu_s = 1 \quad \text{and} \quad \sum_{s \in [t,T] \cap \mathbb{N}} \mu_s > 0 \quad \text{for all } t \in [0,T] \cap \mathbb{N}\,.$$

Then $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is a dynamic coherent utility function such that every $\phi_{t,T}$ is T-relevant. If $\mathcal{P}^{\mathrm{rel}}$ is m1-stable, then $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is time-consistent.

To show the latter, we denote for $f \in \mathcal{P}^{\mathrm{rel}}$ by $J(f)$ the process $a \in \mathcal{D}^{\mathrm{rel}}_{0,T}$ given by

$$\Delta a_t := \mu_t \mathrm{E}\left[f \mid \mathcal{F}_t\right] \quad \text{for all } t \in [0,T] \cap \mathbb{N}\,.$$

Clearly,

$$\phi_{t,T}(X) = \operatorname*{ess\,inf}_{a \in J(\mathcal{P}^{\mathrm{rel}})} \frac{\langle X, a\rangle_{t,T}}{\langle 1, a\rangle_{t,T}} \quad \text{for all } t \in [0,T] \cap \mathbb{N} \quad \text{and} \quad X \in \mathcal{R}^\infty_{t,T}\,,$$

and it can easily be checked that for all $f, g \in \mathcal{P}^{\mathrm{rel}}$, every $s \in (0,T] \cap \mathbb{N}$ and $A \in \mathcal{F}_s$,

$$J(f) \oplus^s_A J(g) = J(f \otimes^s_A g)\,.$$

Hence, $J(\mathcal{P}^{\mathrm{rel}})$ is c1-stable, and it follows from Corollary 4.16 that $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is time-consistent.

5.6 Dynamic robust entropic utility functions

Let $T \in \mathbb{N}$, $\mathcal{P}^{\mathrm{rel}}$ a non-empty subset of $\tilde{\mathcal{D}}^{\mathrm{rel}}_T$ and $\alpha > 0$. For $t = 0, \dots, T$ and $X \in \mathcal{R}^\infty_{t,T}$, define

$$\phi_{t,T}(X) := \operatorname*{essinf}_{f \in \mathcal{P}^{\mathrm{rel}}} \left\{ -\frac{1}{\alpha} \log \frac{\mathrm{E}\left[f \exp(-\alpha X_T) \mid \mathcal{F}_t\right]}{\mathrm{E}\left[f \mid \mathcal{F}_t\right]} \right\}\,, \quad X \in \mathcal{R}^\infty_{0,T}\,.$$

Then, for all $t = 0, \dots, T$, $\phi_{t,T}$ is a T-relevant conditional concave monetary utility function on $\mathcal{R}^\infty_{t,T}$ that is continuous for bounded decreasing sequences, and $(\phi_{t,T})_{t=0}^T$ is time-consistent if $\mathcal{P}^{\mathrm{rel}}$ is m1-stable.

Indeed, it is clear that for all $t = 0, \dots, T$, $\phi_{t,T}$ is a T-relevant conditional monetary utility function on $\mathcal{R}^\infty_{t,T}$. To show the other assertions, we set for $f \in \tilde{\mathcal{D}}^{\mathrm{rel}}_T$ and $t = 0, \dots, T$,

$$f_t := \frac{f}{\mathrm{E}\left[f \mid \mathcal{F}_t\right]}\,,$$

and introduce the mappings

$$\psi^f_t(Y) := -\frac{1}{\alpha} \log \mathrm{E}\left[f_t \exp(-\alpha Y) \mid \mathcal{F}_t\right]\,, \quad Y \in L^\infty(\mathcal{F}_T)$$

and

$$\psi_t(Y) := \operatorname*{ess\,inf}_{f \in \mathcal{P}^{\mathrm{rel}}} \psi^f_t(Y)\,, \quad Y \in L^\infty(\mathcal{F}_T)\,.$$

For $g, f \in \tilde{\mathcal{D}}^{\mathrm{rel}}_T$, we denote by $H_t(g \mid f)$ the conditional relative entropy $\mathrm{E}\left[g_t \log \frac{g_t}{f_t} \mid \mathcal{F}_t\right]$. It follows from Jensen's inequality that

$$H_t(g \mid f) = \mathrm{E}\left[f_t \frac{g_t}{f_t} \log \frac{g_t}{f_t} \mid \mathcal{F}_t\right] \geq 0$$

and $H_t(g \mid f) = 0$ if and only if $g_t = f_t$. For $Y \in L^\infty(\mathcal{F}_T)$, we define

$$f^Y_t := \frac{e^{-\alpha Y}}{\mathrm{E}\left[f_t\, e^{-\alpha Y} \mid \mathcal{F}_t\right]}\, f_t\,.$$

Then, for $g \in \tilde{\mathcal{D}}_T^{\mathrm{rel}}$ and $f \in \mathcal{P}^{\mathrm{rel}}$,

$$\begin{aligned} \frac{1}{\alpha} H_t(g \mid f) &= \mathrm{E}\left[g_t \log \frac{g_t}{f_t^Y} \frac{f_t^Y}{f_t} \mid \mathcal{F}_t\right] = \frac{1}{\alpha} H_t(g \mid f_t^Y) + \frac{1}{\alpha}\mathrm{E}\left[g_t \log \frac{f_t^Y}{f_t} \mid \mathcal{F}_t\right] \\ \geq \quad & \frac{1}{\alpha}\mathrm{E}\left[g_t \log \frac{f_t^Y}{f_t} \mid \mathcal{F}_t\right] = -\mathrm{E}\left[g_t Y \mid \mathcal{F}_t\right] - \frac{1}{\alpha} \log \mathrm{E}\left[f_t e^{-\alpha Y} \mid \mathcal{F}_t\right] \end{aligned}$$

with equality if $g_t = f_t^Y$. This shows that for all $Y \in L^\infty(\mathcal{F}_T)$,

$$\operatorname*{ess\,inf}_{g \in \tilde{\mathcal{D}}_T^{\mathrm{rel}}} \left\{ \mathrm{E}\left[g_t Y \mid \mathcal{F}_t\right] + \frac{1}{\alpha} H_t(g \mid f) \right\} = -\frac{1}{\alpha} \log \mathrm{E}\left[f_t\, e^{-\alpha Y} \mid \mathcal{F}_t\right] ,$$

(compare to Example 4.33 in Föllmer and Schied, 2004). Hence,

$$\psi_t(Y) = \operatorname*{ess\,inf}_{f \in \mathcal{P}^{\mathrm{rel}},\, g \in \tilde{\mathcal{D}}_T^{\mathrm{rel}}} \left\{ \mathrm{E}\left[g_t Y \mid \mathcal{F}_t\right] + \frac{1}{\alpha} H_t(g \mid f) \right\} ,$$

and it follows from Theorem 3.16 that ψ_t is $\mathcal{F}_t$-concave and continuous for bounded decreasing sequences.

Now, assume that $\mathcal{P}^{\mathrm{rel}}$ is m1-stable. Then, for all $t = 0, \ldots, T$ and $Y \in L^\infty(\mathcal{F}_T)$, the set

$$\left\{ \psi_t^f(Y) \mid f \in \mathcal{P}^{\mathrm{rel}} \right\}$$

is directed downwards because for all $f, g \in \mathcal{P}^{\mathrm{rel}}$,

$$\psi_t^f(Y) \wedge \psi_t^g(Y) = \psi_t^h(Y) ,$$

where

$$h = f \otimes_A^t g \quad \text{for} \quad A = \left\{ \psi_t^g(Y) < \psi_t^f(Y) \right\} .$$

Hence, there exists a sequence $(f^k)_{k \in \mathbb{N}}$ in $\mathcal{P}^{\mathrm{rel}}$ such that almost surely,

$$\psi_t^{f^k}(Y) \searrow \psi_t(T) , \quad \text{as } k \to \infty .$$

Next, note that for all $t = 0, \ldots, T-1$, $f, g \in \mathcal{P}^{\mathrm{rel}}$ and $Y \in L^\infty(\mathcal{F}_T)$,

$$\psi_t^f(\psi_{t+1}^g(Y)) = \psi_t^h(Y) ,$$

where

$$h = f \otimes_\Omega^{t+1} g .$$

It follows that

$$\psi_t(\psi_{t+1}(Y)) = \psi_t(Y) ,$$

for all $t = 0, \ldots, T-1$ and $Y \in L^\infty(\mathcal{F}_T)$, and therefore,

$$\phi_{t,T}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}) = \phi_{t,T}(X) ,$$

for all $t = 0, \ldots, T-1$ and $X \in \mathcal{R}^{\infty}_{t,T}$, which by Proposition 4.5, implies that $(\phi_{t,T})_{t=0}^{T}$ is time-consistent.

The functions ψ_t are conditional robust versions of the mapping

$$C : L^{\infty}(\mathcal{F}_T) \to \mathbb{R}\,, \quad Y \mapsto -\frac{1}{\alpha} \log \mathrm{E}\left[\exp(-\alpha Y)\right]\,,$$

which assigns a random variable $Y \in L^{\infty}(\mathcal{F}_T)$ its certainty equivalent under expected exponential utility. For the relation of entropic utility functions to pricing in incomplete markets we refer to Frittelli (2000), Rouge and El Karoui (2000), and Delbaen et al. (2002). Entropic risk measures can be found in Föllmer and Schied (2002a, 2004) and Weber (2005). Conditional entropic risk measures and their dynamic properties are also studied in Frittelli and Rosazza Gianin (2004), Barrieu and El Karoui (2004), Mania and Schweizer (2005), and Detlefsen and Scanolo (2005).

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