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Convergence of Coalescing Nonsimple Random Walks to the Brownian Web

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Abstract

The Brownian Web (BW) is a family of coalescing Brownian motions starting from every point in space and time $\mathbb{R}\times\mathbb{R}$. It was first introduced by Arratia, and later analyzed in detail by Tóth and Werner. More recently, Fontes, Isopi, Newman and Ravishankar (FINR) gave a characterization of the BW, and general convergence criteria allowing in principle either crossing or noncrossing paths, which they verified for coalescing simple random walks. Later Ferrari, Fontes, and Wu verified these criteria for a two dimensional Poisson Tree. In both cases, the paths are noncrossing. To date, the general convergence criteria of FINR have not been verified for any case with crossing paths, which appears to be significantly more difficult than the noncrossing paths case. Accordingly, in this paper, we formulate new convergence criteria for the crossing paths case, and verify them for non-simple coalescing random walks satisfying a finite fifth moment condition. This is the first time that convergence to the BW has been proved for models with crossing paths. Several corollaries are presented, including an analysis of the scaling limit of voter model interfaces that extends a result of Cox and Durrett.

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1 Introduction and Results

The idea of the Brownian Web dates back to Arratia's thesis [1] in 1979, in which he constructed a process of coalescing Brownian motions starting from every point in space $\mathbb{R}$ at time zero. In a later unpublished manuscript [2], Arratia generalized this construction to a process of coalescing Brownian motions starting from every point in space and time $\mathbb{R} \times \mathbb{R}$, which is essentially what is now often called the Brownian Web (BW). He also defined a dual family of backward coalescing Brownian motions equally distributed (after a time reversal) with the BW which we call the Dual Brownian Web. Unfortunately, Arratia's manuscript was incomplete and never published, and the BW was not studied again until a paper by Tóth and Werner [27], in which they constructed and analyzed versions of the Brownian Web and Dual Brownian Web in great detail and used them to construct a process they call the *True Self Repelling Motion*.

In both Arratia's and Tóth and Werner's constructions of the BW, some semicontinuity condition is imposed to guarantee a unique path starting from every space-time point. More recently, Fontes, Isopi, Newman and Ravishankar [15, 16] gave a different formulation of the BW which provides a more natural setting for weak convergence, and they coined the term Brownian Web. Instead of imposing semicontinuity conditions, multiple paths are allowed to start from the same point. Further, by choosing a suitable topology, the BW can be characterized as a random variable taking values in a complete separable metric space whose elements are compact sets of paths. In [16], they gave general convergence criteria allowing either crossing or noncrossing paths, and they verified the criteria for the noncrossing paths case for coalescing simple random walks. Recently, Ferrari, Fontes, and Wu [14] verified the same criteria for the noncrossing paths case for a two dimensional Poisson tree.

The main goal of this paper is to prove weak convergence to the BW for coalescing *nonsimple* random walks, which are models with crossing paths. Technically, the convergence criteria for the crossing paths case are significantly more difficult to verify than the noncrossing paths case. In the noncrossing paths case, the paths form a totally ordered set and one expects FKG type of positive correlation inequalities to apply; this has been the main tool in the verification of the convergence criteria for coalescing simple random walks [16] and a two dimensional Poisson tree [14]. Also, the noncrossing property of paths gives tightness almost for free. For the crossing paths case, tightness needs to be checked separately. Furthermore, correlation inequalities in general no longer apply, and new ideas are needed to verify a key convergence criterion denoted (B_2'), as formulated in [16]. For coalescing nonsimple random walks, criterion (B_2') turns out to be particularly difficult to verify (and this has not yet been done). Instead, we formulate an alternative criterion (E_1), which we verify along with tightness and the other convergence criteria. Thus the main contributions of this paper are contained in Sections 4 and 6 where tightness and criterion (E_1) are verified. The new convergence criteria and our approach in verifying them should serve as a paradigm for establishing the weak convergence of general models with crossing paths to the Brownian Web. A consequence of the weak convergence of coalescing nonsimple random walks to the Brownian web is that, for the dual one-dimensional voter model with initial condition 1's on the negative axis and 0's on the positive axis, the interface region evolves as a standard Brownian motion after diffusive scaling. This partially recovers and extends a result of Cox and Durrett [9], which required rather difficult calculations.

Coalescing Random Walks: Let Y, a random variable with distribution μ_Y, denote the increment of an irreducible aperiodic random walk on $\mathbb{Z}$. We will always assume $\mathbb{E}[Y] = 0$, $\mathbb{E}[Y^2] = \sigma^2 < +\infty$ throughout this paper, unless a weaker hypotheses is explicitly stated. For our main result, we will also need $\mathbb{E}[|Y|^5] < +\infty$. We are interested in the discrete time process of coalescing random walks with one walker starting from every site on $\mathbb{Z} \times \mathbb{Z}$ (first coordinate space, second coordinate time). All walkers have i.i.d. increments distributed as Y, and two walkers move independently until they first meet, at which time they coalesce. The path of a random walk is defined to be the linear interpolation of the random walk's position at integer times. Note that for non-simple random walks, two random walk paths can cross each other many times before they eventually coalesce. If Y were such that the random walks had period $d \neq 1$, as in the case of simple random walks where $d = 2$, then we would just have d different copies of coalescing random walks on different space-time sublattices, none of which interacts with the other copies.

Let $\mathcal{X}_1$ (with distribution μ_1) denote the random realization of such a collection of coalescing random walk paths on $\mathbb{Z} \times \mathbb{Z}$, and let $\mathcal{X}_\delta$ (with distribution μ_δ), $\delta < 1$, be $\mathcal{X}_1$ rescaled with the usual diffusive scaling of δ/σ in space, δ^2 in time. The main result of our paper (see Theorem 1.5 below) is that if $\mathbb{E}[|Y|^5] < +\infty$, then $\mathcal{X}_\delta$ converges in distribution to the BW as $\delta \to 0$.

We remark that there are natural interacting particle systems constructed out of *simple* random walks, but still with paths crossing each other before eventual coalescence, for which the methods of this paper should be applicable to prove convergence to the BW. For example, there is a coalescing simple random walk model on the space time lattice $\mathbb{Z} \times \mathbb{Z}$, dual to the Stepping Stone Model with no mutation (see, e.g., [21, 28, 13]), which may be defined as follows. One walker starts from every site on the lattice $\mathbb{Z}\times\mathbb{Z}\times\{1, \cdots, M\}$, where the first two coordinates are space and time, and the third coordinate can be regarded as the color. Each walker makes transitions on the space-time lattice as a simple random walk, and at every transition, a color is independently chosen from $\{1, \cdots, M\}$ with equal probability. Two walkers make their space-time transitions and choose their corresponding colors independently until after they meet at the same space-time site and choose the same color, at which time they coalesce. Projecting the random walk paths onto the space-time plane $\mathbb{Z} \times \mathbb{Z}$, and restricting attention to random walks on the even sublattice (i.e., $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ with $x + y$ even), we obtain a system of coalescing simple random walks with crossing paths. A proof of convergence for this and other similar models to the BW would require verification of the hypotheses of Theorem 1.4 below (see also Remark 1.6).

Brownian Web: We recall here Fontes, Isopi, Newman and Ravishankar's [15, 16] choice of the metric space in which the Brownian Web takes its values.

Let $(\bar{\mathbb{R}}^2, \rho)$ be the completion (or compactification) of $\mathbb{R}^2$ under the metric ρ, where

$$\rho((x_1, t_1), (x_2, t_2)) = \left| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \right| \vee |\tanh(t_1) - \tanh(t_2)|. \quad (1.1)$$

$\bar{\mathbb{R}}^2$ can be thought of as the image of $[-\infty, \infty] \times [-\infty, \infty]$ under the mapping

$$(x, t) \rightsquigarrow (\Phi(x, t), \Psi(t)) \equiv \left(\frac{\tanh(x)}{1 + |t|}, \tanh(t) \right). \quad (1.2)$$

For $t_0 \in [-\infty, \infty]$, let $C[t_0]$ denote the set of functions f from $[t_0, \infty]$ to $[-\infty, \infty]$ such that $\Phi(f(t), t)$ is continuous. Then define

$$\Pi = \bigcup_{t_0 \in [-\infty, \infty]} C[t_0] \times \{t_0\}, \quad (1.3)$$

where $(f, t_0) \in \Pi$ represents a path in $\bar{\mathbb{R}}^2$ starting at $(f(t_0), t_0)$. For(f, t_0) in Π, we denote by $\hat{f}$ the function that extends f to all $[-\infty, \infty]$ by setting it equal to $f(t_0)$ for $t < t_0$. Then we take

$$d((f_1,t_1),(f_2,t_2)) = (\sup_t |\Phi(\hat{f}_1(t),t) - \Phi(\hat{f}_2(t),t)|) \vee |\Psi(t_1) - \Psi(t_2)|. \quad (1.4)$$

(Π, d) is a complete separable metric space.

Let now $\mathcal{H}$ denote the set of compact subsets of (Π, d), with $d_{\mathcal{H}}$ the induced Hausdorff metric, i.e.,

$$d_{\mathcal{H}}(K_1,K_2) = \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d(g_1,g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d(g_1,g_2). \quad (1.5)$$

$(\mathcal{H}, d_{\mathcal{H}})$ is also a complete separable metric space. Let $\mathcal{F}_{\mathcal{H}}$ denote the Borel σ-algebra generated by $d_{\mathcal{H}}$.

Lemma 1.1 *Assume* $\mathbb{E}[|Y|] < +\infty$*, then for any* $\delta \in (0,1]$*, the closure of* $\mathcal{X}_\delta$ *in* (Π, d) *is almost surely a compact subset of* (Π, d)*.*

Proof. We prove the lemma only for $\mathcal{X}_1$, since the proof for $\mathcal{X}_\delta$ is identical. Denote the image of $\mathcal{X}_1$ under the mapping (Φ, Ψ) by $\mathcal{X}_1'$. We will show the equicontinuity of paths in $\mathcal{X}_1'$. Note that by the properties of (Φ, Ψ), this reduces to showing the equicontinuity of $\mathcal{X}_1'$ restricted to any time interval $[\Psi(k), \Psi(k+1)]$ with $k \in \mathbb{Z}$, which we denote by $\mathcal{X}_1'|_{\Psi(k)}^{\Psi(k+1)}$. Similarly denote the restriction of $\mathcal{X}_1$ to the time interval $[k, k+1]$ by $\mathcal{X}_1|_k^{k+1}$. Note that $\mathcal{X}_1|_k^{k+1}$ consists of line segments corresponding to random walks jumping from sites in $\mathbb{Z}$ at time k to sites in $\mathbb{Z}$ at time $k+1$. If $\mathcal{X}_1'|_{\Psi(k)}^{\Psi(k+1)}$ is not equicontinuous, then there would exist a sequence of random walk jumps from sites $x_n \to -\infty$ to sites $y_n > L$ for some $L \in \mathbb{R}$ (or from $x_n \to +\infty$ to $y_n < L$). Since $\mathbb{E}[|Y|] < +\infty$, using Borel-Cantelli, it is easily seen that this is an event with probability 0, hence $\mathcal{X}_1'|_{\Psi(k)}^{\Psi(k+1)}$ is almost surely equicontinuous, and this proves the lemma.

Remark 1.1 *Note that the closure of* $\mathcal{X}_\delta$*, which from now on we also denote by* $\mathcal{X}_\delta$*, is obtained from* $\mathcal{X}_\delta$ *by adding all the paths of the form* (f,t) *with* $t \in \delta^2\mathbb{Z} \cup \{+\infty, -\infty\}$ *and* $f \equiv +\infty$ *or* $f \equiv -\infty$*.* $\mathcal{X}_\delta$ *is then a* $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$*-valued random variable.*

In [15, 16], the Brownian Web ($\bar{\mathcal{W}}$ with measure $\mu_{\bar{\mathcal{W}}}$) is constructed as a $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ valued random variable, with the following characterization theorem.

Theorem 1.2 *There is an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\bar{\mathcal{W}}$ whose distribution is uniquely determined by the following three properties.*

(o) from any deterministic point (x,t) in $\mathbb{R}^2$, there is almost surely a unique path $W_{x,t}$ starting from (x,t).

(i) for any deterministic $n, (x_1,t_1), \ldots, (x_n,t_n)$, the joint distribution of $W_{x_1,t_1}, \ldots, W_{x_n,t_n}$ is that of coalescing Brownian motions (with unit diffusion constant), and

(ii) for any deterministic, dense countable subset $\mathcal{D}$ of $\mathbb{R}^2$, almost surely, $\bar{\mathcal{W}}$ is the closure in $(\mathcal{H}, d_{\mathcal{H}})$ of $\{W_{x,t} : (x,t) \in \mathcal{D}\}$.

The $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\bar{\mathcal{W}}$ in Theorem 1.2 is called the standard Brownian Web.

Convergence Criteria and Main Result: In [16], there was also given a set of general convergence criteria for measures supported on compact sets of paths which can cross each other. We now state these criteria as four conditions on our random variables $\mathcal{X}_\delta$.

(I_1) There exist single path valued random variables $\theta_\delta^y \in \mathcal{X}_\delta$, for $y \in \mathbb{R}^2$, satisfying: for $\mathcal{D}$ a deterministic countable dense subset of $\mathbb{R}^2$, for any deterministic $y^1, \ldots, y^m \in \mathcal{D}$, $\theta_\delta^{y^1}, \ldots, \theta_\delta^{y^m}$ converge jointly in distribution as $\delta \to 0^+$ to coalescing Brownian motions (with unit diffusion constant) starting at $y^1, \ldots, y^m$.

Let $\Lambda_{L,T} = [-L, L] \times [-T, T] \subset \mathbb{R}^2$. For $x_0, t_0 \in \mathbb{R}$ and $u, t > 0$, let $R(x_0, t_0; u, t)$ denote the rectangle $[x_0 - u, x_0 + u] \times [t_0, t_0 + t]$ in $\mathbb{R}^2$. Define $A_{t,u}(x_0, t_0)$ to be the event (in $\mathcal{F}_{\mathcal{H}}$) that K (in $\mathcal{H}$) contains a path touching both $R(x_0, t_0; u, t)$ and (at a later time) the left or right boundary of the bigger rectangle $R(x_0, t_0; 17u, 2t)$ (the number 17 is chosen to avoid fractions later). Then the following is a tightness condition for $\{\mathcal{X}_\delta\}$: for every $u, L, T \in (0, +\infty)$,

$$(T_1) \quad \tilde{g}(t, u; L, T) \equiv t^{-1} \limsup_{\delta \to 0^+} \sup_{(x_0,t_0) \in \Lambda_{L,T}} \mu_\delta(A_{t,u}(x_0, t_0)) \to 0 \text{ as } t \to 0^+$$

As shown in [16], if (T_1) is satisfied, one can construct compact sets $G_\epsilon \subset \mathcal{H}$ for each $\epsilon > 0$, such that $\mu_\delta(G_\epsilon^c) < \epsilon$ uniformly in δ. G_ϵ consists of compact subsets of Π whose image under the map (Φ, Ψ) are equicontinuous with a modulus of continuity dependent on ϵ.

For $K \in \mathcal{H}$ a compact set of paths in Π, define the counting variable $\mathcal{N}_{t_0,t}([a,b])$ for $a, b, t_0, t \in \mathbb{R}, a < b, t > 0$ by

$$\begin{aligned}\mathcal{N}_{t_0,t}([a,b]) \;=\; &\{y \in \mathbb{R} \,|\, \exists\, x \in [a,b] \text{ and a path in } K \text{ which touches} \\ &\qquad \text{both } (x,t_0) \text{ and } (y, t_0+t)\}. \end{aligned} \tag{1.6}$$

Let l_{t_0} (resp. r_{t_0}) denote the leftmost (resp. rightmost) value in $[a,b]$ with some path in K touching (l_{t_0}, t_0) (resp. (r_{t_0}, t_0)). Also define $\mathcal{N}^+_{t_0,t}([a,b])$ (resp. $\mathcal{N}^-_{t_0,t}([a,b])$) to be the subset of $\mathcal{N}_{t_0,t}([a,b])$ due to paths in K that touch (l_{t_0}, t_0) (resp. (r_{t_0}, t_0)). The last two conditions for the convergence of $\{\mathcal{X}_\delta\}$ to the Brownian Web are

$$(B_1') \;\forall \beta > 0, \limsup_{\delta \to 0^+} \sup_{t > \beta} \sup_{t_0, a \in \mathbb{R}} \mu_\delta(|\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon])| > 1) \to 0 \text{ as } \epsilon \to 0^+$$

$$\begin{aligned}(B_2') \;\forall \beta > 0, \frac{1}{\epsilon} \limsup_{\delta \to 0^+} \sup_{t > \beta} \sup_{t_0, a \in \mathbb{R}} \mu_\delta(&\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon]) \neq \mathcal{N}^+_{t_0,t}([a-\epsilon, a+\epsilon]) \\ &\cup \mathcal{N}^-_{t_0,t}([a-\epsilon, a+\epsilon])) \to 0 \text{ as } \epsilon \to 0^+.\end{aligned}$$

The general convergence theorem of [16] is the following, where we have replaced our family $\{\mathcal{X}_\delta\}$ with $\delta \to 0$ by a general sequence $\{X_n\}$ with $n \to +\infty$:

Theorem 1.3 *Let $\{X_n\}$ be a family of $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ valued random variables satisfying conditions $(I_1), (T_1), (B_1')$ and (B_2'), then X_n converges in distribution to the standard Brownian Web $\bar{\mathcal{W}}$.*

Condition (B_1') guarantees that for any subsequential limit X of $\{X_n\}$, and for any deterministic point $y \in \mathbb{R}^2$, there is μ_X almost surely at most one path starting from y. Together with condition (I_1), this implies that for a deterministic countable dense set $\mathcal{D} \subset \mathbb{R}^2$, the distribution of paths in X starting from finite subsets of $\mathcal{D}$ is that of coalescing Brownian motions. Conditions (B_1') and (B_2') together imply that for the family of counting random variables $\eta(t_0, t; a, b) = |\mathcal{N}_{t_0,t}([a,b])|$, we have $\mathbb{E}[\eta_X(t_0,t;a,b)] \leq \mathbb{E}[\eta_{\bar{\mathcal{W}}}(t_0,t;a,b)] = 1 + \frac{b-a}{\sqrt{\pi t}}$ for all $t_0, t, a, b \in \mathbb{R}$ with $t > 0, a < b$. By Theorem 4.6 in [16] and the remark following it, X is then equidistributed with $\bar{\mathcal{W}}$. For the process of coalescing random walks $\mathcal{X}_\delta$, we have not yet been able to verify condition (B_2'), but an examination of the proof of Theorem 4.6 in [16] shows that we can also use the dual family of counting random variables

$$\hat{\eta}_X(t_0, t; a, b) = |\{x \in (a, b) \;\;|\;\; \exists \text{ a path in } X \text{ touching both } \mathbb{R} \times \{t_0\} \text{ and } (x, t_0 + t)\}|. \tag{1.7}$$

By a duality argument [27] (see also [1, 2, 16]), $\hat{\eta}$ and $\eta - 1$ are equally distributed for the Brownian Web $\bar{\mathcal{W}}$. We can then replace (B_2') by

(E_1) If $\mathcal{X}$ is any subsequential limit of $\{\mathcal{X}_\delta\}$, then $\forall t_0, t, a, b \in \mathbb{R}$ with $t > 0$ and $a < b$, $\mathbb{E}[\hat{\eta}_{\mathcal{X}}(t_0, t; a, b)] \leq \mathbb{E}[\hat{\eta}_{\bar{\mathcal{W}}}(t_0, t; a, b)] = \frac{b-a}{\sqrt{\pi t}}$.

With this change, we immediately obtain our *modified general convergence theorem*,

Theorem 1.4 *Let* $\{X_n\}$ *be a family of* $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ *valued random variables satisfying conditions* $(I_1), (T_1), (B_1')$ *and* (E_1)*, then* X_n *converges in distribution to the standard Brownian Web* $\bar{\mathcal{W}}$.

The main result of this paper is

Theorem 1.5 *If the random walk increment* Y *satisfies* $\mathbb{E}[|Y|^5] < +\infty$*, then* $\{\mathcal{X}_\delta\}$ *satisfy the conditions of Theorem* 1.4*, and hence converges in distribution to* $\bar{\mathcal{W}}$.

Remark 1.6 *The main difficulty lies in the verification of the tightness condition* (T_1) *and condition* (E_1)*. Condition* (E_1) *in our general convergence result, Theorem 1.4, may seem strong since it requires an upper bound that is actually exact, but as we will show in our proof of* (E_1) *for* $\{\mathcal{X}_\delta\}$ *in Section 6, all we need are the Markov property of the random walks and an upper bound of the type* $\limsup_{\delta \downarrow 0} \mathbb{E}[\hat{\eta}_{\mathcal{X}_\delta}(t_0, t; a, b)] \leq C$ *for some finite* C *depending on* t, a, b.

Remark 1.7 *Recently, Belhaouari et al. [8] have succeeded in verifying a version of the tightness criterion* (T_1) *in the context of voter model interfaces under a finite* $3 + \epsilon$ *moment assumption on* Y*. This establishes the convergence of* $\mathcal{X}_\delta$ *to the Brownian web also under a finite* $3+\epsilon$ *moment assumption since the other convergence criteria require either tightness or at most finite second moment of* Y*.*

In Section 2, we list some basic facts about random walks, then in sections 3 to 6, we proceed to verify condition (B_1'), (T_1), (I_1) and (E_1). In section 7, we present some corollaries for one dimensional coalescing random walks and the dual non-nearest-neighbor voter models. In particular, we prove that under the assumption $\mathbb{E}[|Y|^5] < +\infty$, the point process at rescaled time 1 of coalescing nonsimple random walks starting from $(\delta/\sigma)\mathbb{Z}$ at time 0 converges weakly to the point process at time 1 of coalescing Brownian motions starting from $\mathbb{R}$ at time 0. This extends a result of Arratia [1], and it follows that the density of coalescing nonsimple random walks starting from $\mathbb{Z}$ at time 0 decays as $1/(\sigma\sqrt{\pi t})$, extending a result of Bramson and Griffeath [5]. Another corollary that follows from the convergence of the coalescing random walks $\mathcal{X}_\delta$ to the Brownian Web is that, for the dual voter model $\phi_t^{\mathbb{Z}^-}(x)$ with initial configuration $\phi_0^{\mathbb{Z}^-}(x) \equiv 1$ for $x \in \mathbb{Z}^-$ and $\phi_0^{\mathbb{Z}^-}(x) \equiv 0$ for $x \in \mathbb{Z}^+ \cup \{0\}$, under diffusive scaling, the time evolution of the interface between the all 0 region and the all 1 region converges in distribution to a Brownian motion starting from the origin at time 0. This partially recovers and also extends a result of Cox and Durrett [9], in which they proved that under the assumption $\mathbb{E}[|Y|^3] < +\infty$, the interface region is of size $O(1)$ as $t \to +\infty$, and under diffusive scaling, the position at rescaled time 1 of the interface converges in distribution to a standard Gaussian. In Section 7, we also discuss how our proof can be modified to establish the weak convergence of continuous time coalescing nonsimple random walks to the Brownian Web.

2 Random Walk Estimates

In this section, we introduce some notation that will be used throughout the paper, and we list some basic facts about random walks that will be used in later sections. The results are all standard, but for self-containedness we include them here.

Given macroscopic space and time coordinates $(x,t) \in \mathbb{R}^2$, define their microscopic counterparts before diffusive scaling by $\tilde{t} = t\delta^{-2}$ and $\tilde{x} = x\sigma\delta^{-1}$. Quantities such as $\tilde{u}, \tilde{t}_0$ are defined from u, t_0 similarly. Since μ_δ and μ_1 are related by diffusive scaling, we will do most of our analysis using μ_1, with x, t, u, t_0 for μ_δ replaced by $\tilde{x}, \tilde{t}, \tilde{u}, \tilde{t}_0$ for μ_1.

Let ξ_s^A denote the state at time s of a system of coalescing random walks starting from a subset $A \subset \mathbb{Z} \times \mathbb{Z}$. In the special case when $A = B \times \{t_0\}$ for some $B \subset \mathbb{Z}$, we will denote it by ξ_s^{B,t_0}, and when $t_0 = 0$, we may simply denote it by ξ_s^B, as in the case of $B = \mathbb{Z}$. We will use $\mu_1(\cdot)$ to denote the probability of events for systems of coalescing random walks on $\mathbb{Z} \times \mathbb{Z}$ since they are marginals of $\mathcal{X}_1$.

Denote the linearly interpolated path of a random walk starting at some point $(x, t_0) \in \mathbb{Z} \times \mathbb{Z}$ by $\pi^{x,t_0}(s)$. Denote the event that a random walk starting at (x, t_0) stays inside the interval $[a, b]$ containing x up to time t by $B^{x,t_0}_{[a,b],t}$.

Given $r \in \mathbb{Z}$, define stopping times

$$\tau_r^{x,t_0} = \inf\{n \geq t_0, n \in \mathbb{Z} \mid \pi^{x,t_0}(n) = r\}, \tag{2.1}$$

$$\tau_{r^+}^{x,t_0} = \inf\{n \geq t_0, n \in \mathbb{Z} \mid \pi^{x,t_0}(n) \geq r\}. \tag{2.2}$$

When the time coordinate in the superscripts of $\xi, \pi, B, \tau, \tau_+$ is 0, we will suppress it. We will use $\mathbb{P}_x$ and $\mathbb{E}_x$ to denote probability and expectation for a random walk process starting from x at time 0. $\mathbb{P}_{x,y}$ and $\mathbb{E}_{x,y}$ will correspond to two independent random walks starting at x and y at time 0.

Recall that we always assume the random walk increment Y is distributed such that the random walk is irreducible and aperiodic with $\mathbb{E}(Y) = 0$ and $\mathbb{E}[Y^2] < +\infty$, unless a different moment condition is explicitly stated.

Lemma 2.1 *Let π^x, π^y be two independent random walks with increment Y starting at $x, y \in \mathbb{Z}$ at time 0. Let $\tau_{x,y}$ be the integer stopping time when the two walkers first meet, and let $l(x, y) = \sup_{n\in[0,\tau_{x,y}]} |\pi^x(n) - \pi^y(n)|$. Then $\tau_{x,y}$ and $l(x, y)$ are almost surely finite.*

Proof. Let $\bar{\pi}^{y-x}(n) = \pi^y(n) - \pi^x(n)$. Then $\bar{\pi}^{y-x}$ is an irreducible aperiodic symmetric random walk starting at $y - x$ with increment distributed as $\mu_Y * \mu_{-Y}$. The lemma is simply a consequence of the recurrence of $\bar{\pi}^{y-x}$, which requires less than a finite second moment of Y.

Lemma 2.2 *Let π^x, π^y, $\tau_{x,y}$ be as in Lemma 2.1. Then $\mathbb{P}_{x,y}(\tau_{x,y} > t) \leq \frac{C}{\sqrt{t}}|x - y|$ for some constant C independent of t, x and y.*

Proof. Let $\bar{\pi}^{y-x}(n) = \pi^y(n) - \pi^x(n)$ as in the proof of Lemma 2.1. Let $\bar{\mathbb{P}}_{y-x}$ denote probability for this random walk, and let $\bar{\tau}_0^{y-x}$ denote the first integer time when $\bar{\pi}^{y-x} = 0$. Then $\mathbb{P}_{x,y}(\tau_{x,y} > t) = \bar{\mathbb{P}}_{y-x}(\bar{\tau}_0^{y-x} > t)$. When $|x - y| = 1$, it is a standard fact (see e.g. Proposition 32.4 in [24]) that this probability is bounded by $\frac{C}{\sqrt{t}}$. When $|x - y| > 1$, without loss of generality assume $x < y$, and regard π^x and π^y as a subset of the system of coalescing random walks $\xi^{\{x,x+1,\ldots,y\}}$. Then

$$\begin{aligned} & \mathbb{P}_{x,y}(\tau_{x,y} > t) \leq \mu_1(|\xi_t^{\{x,\ldots,y\}}| > 1) \\ = \;& \mu_1(\bigcup_{i=x}^{y-1}\{\tau_{i,i+1} > t\}) \leq (y-x)\mathbb{P}_{0,1}(\tau_{0,1} > t) \leq \frac{C(y-x)}{\sqrt{t}}. \end{aligned}$$

This establishes the lemma.

Lemma 2.3 *Let $u > 0$ and $t > 0$ be fixed, and let $\pi(s) = \pi^{0,0}(s)$ be a random walk starting from the origin at time 0. Let $\tilde{u}, \tilde{t}$ and the event $B^0_{[-\tilde{u},\tilde{u}],\tilde{t}}$ be defined as at the beginning of this section (note that they depend on δ), and let $(B^0_{[-\tilde{u},\tilde{u}],\tilde{t}})^c$ be the complement of $B^0_{[-\tilde{u},\tilde{u}],\tilde{t}}$. If $\mathcal{B}_s$ is a standard Brownian motion starting from 0, then*

$$0 < \lim_{\delta\to 0^+} \mathbb{P}_0((B^0_{[-\tilde{u},\tilde{u}],\tilde{t}})^c) = \mathbb{P}(\sup_{s\in[0,t]} |\mathcal{B}_s| > u) < 4e^{-\frac{u^2}{2t}}.$$

Proof. The limit follows from Donsker's invariance principle [11] for random walks. The first inequality is trivial, and the second inequality follows from a well-known computation for Brownian motion using the reflection principle.

Lemma 2.4 *Let $u, t, \tilde{u}, \tilde{t}$ be as before. Let π^x, π^y and $\tau_{x,y}$ be as in Lemmas $2.1-2.2$ with $x < y$. Let $\tau_{x,y,\tilde{u}^+}$ be the first integer time when $\pi^x(n) - \pi^y(n) \geq \tilde{u}$. If $\mathbb{E}[|Y|^3] < +\infty$, and δ is sufficiently small, we then have*

$$\mathbb{P}_{x,y}(\tau_{x,y,\tilde{u}^+} < (\tau_{x,y} \wedge \tilde{t})) < C(t,u)\delta$$

for some constant $C(t,u)$ depending only on t and u.

Proof. Let $z = x - y < 0$. Note that x, y, z are fixed while $\tilde{u}, \tilde{t} \to +\infty$ as $\delta \to 0$. For the difference of the two walks $\bar{\pi}^z(n) = \pi^x(n) - \pi^y(n)$, we still denote the first integer time when $\bar{\pi}^z = 0$ by $\bar{\tau}^z_0$, and the first integer time when $\bar{\pi}^z \geq \tilde{u}$ by $\bar{\tau}^z_{\tilde{u}^+}$. Note we are using the bar $\bar{\cdot}$ to emphasize the fact that we are studying the symmetrized random walks. The inequality then becomes

$$\bar{\mathbb{P}}_z(\bar{\tau}^z_{\tilde{u}^+} < (\bar{\tau}^z_0 \wedge \tilde{t})) < C(t,u)\delta. \tag{2.3}$$

First we will prove that, for δ sufficiently small,

$$\bar{\mathbb{P}}_w(\bar{\tau}^w_{\tilde{u}^+} < (\bar{\tau}^w_0 \wedge \tilde{t})) < C'(t,u)|w|\delta. \tag{2.4}$$

By the strong Markov property,

$$\begin{aligned}
&\bar{\mathbb{P}}_w(\bar{\tau}^w_0 > \tilde{t}) \\
\geq\ & \sum_{k=\lceil \tilde{u}\rceil}^{+\infty} \bar{\mathbb{P}}_w(\bar{\tau}^w_{\tilde{u}^+} < (\bar{\tau}^w_0 \wedge \tilde{t}), \bar{\pi}^w(\bar{\tau}^w_{\tilde{u}^+}) = k, B^{k,\bar{\tau}^w_{\tilde{u}^+}}_{[k-\frac{\tilde{u}}{2},k+\frac{\tilde{u}}{2}],\tilde{t}}) \\
=\ & \sum_{k=\lceil \tilde{u}\rceil}^{+\infty} \sum_{n=0}^{\lfloor \tilde{t}\rfloor} \bar{\mathbb{P}}_w(\bar{\tau}^w_{\tilde{u}^+} = n, n < \bar{\tau}^w_0, \bar{\pi}^w(n) = k)\bar{\mathbb{P}}_k(B^k_{[k-\frac{\tilde{u}}{2},k+\frac{\tilde{u}}{2}],\tilde{t}-n}) \\
\geq\ & \sum_{k=\lceil \tilde{u}\rceil}^{+\infty} \sum_{n=0}^{\lfloor \tilde{t}\rfloor} \bar{\mathbb{P}}_w(\bar{\tau}^w_{\tilde{u}^+} = n, n < \bar{\tau}^w_0, \bar{\pi}^w(n) = k)\bar{\mathbb{P}}_k(B^k_{[k-\frac{\tilde{u}}{2},k+\frac{\tilde{u}}{2}],\tilde{t}}) \\
\geq\ & C''(t,u)\bar{\mathbb{P}}_w(\bar{\tau}^w_{\tilde{u}^+} < (\bar{\tau}^w_0 \wedge \tilde{t})).
\end{aligned}$$

If δ is sufficiently small, the last inequality is valid by Lemma 2.3. Also by Lemma 2.2,

$$\bar{\mathbb{P}}_w(\bar{\tau}^w_0 > \tilde{t}) < \frac{C}{\sqrt{\tilde{t}}}|w| = \frac{C|w|}{\sqrt{t}}\delta,$$

together they give (2.4).

To show (2.3), we condition on the first integer time when $\bar{\pi}^z(n) \geq 0$, which we denote by $\bar{\tau}^z_{0+}$. Then by the strong Markov property and (2.4),

$$\begin{aligned}
&\bar{\mathbb{P}}_z(\bar{\tau}^z_{\tilde{u}^+} < (\bar{\tau}^z_0 \wedge \tilde{t})) \\
=\ & \sum_{w=1}^{+\infty} \sum_{n=0}^{\lfloor \tilde{t}\rfloor} \bar{\mathbb{P}}_z[\bar{\tau}^z_{0+} = n, \bar{\pi}^z(n) = w]\bar{\mathbb{P}}_w[\bar{\tau}^w_{\tilde{u}^+} < (\bar{\tau}^w_0 \wedge (\tilde{t}-n))] \\
<\ & \sum_{w=1}^{+\infty} \sum_{n=0}^{\lfloor \tilde{t}\rfloor} \bar{\mathbb{P}}_z[\bar{\tau}^z_{0+} = n, \bar{\pi}^z(n) = w]\ C'(t,u)|w|\ \delta \\
<\ & C'(t,u)\ \delta\ \bar{\mathbb{E}}_z[\bar{\pi}^z(\bar{\tau}^z_{0+})] < C(t,u)\delta.
\end{aligned}$$

The last inequality follows from our assumption $\mathbb{E}[|Y|^3] < +\infty$ and the following two lemmas.

Lemma 2.5 *Let π^x be a random walk with increment Y starting from $x < 0$ at time 0. If $\mathbb{E}[Y^2] < +\infty$, then the overshoot $\pi^x(\tau^x_{0+})$ has a limiting distribution as $x \to -\infty$. In terms of the ladder variable $Z = \pi^0(\tau^0_{1+})$,*

$$\lim_{x\to-\infty} \mathbb{P}[\pi^x(\tau^x_{0+}) = k] = \frac{\mathbb{P}[Z \geq k+1]}{\mathbb{E}[Z]}.$$

Proof. This is a standard fact from renewal theory, see e.g. Proposition 24.7 in [24].

Lemma 2.6 *Let π^x, Y and Z be as in the previous lemma. If $\mathbb{E}[|Y|^{r+2}] < +\infty$ for some $r > 0$, then $[\pi^x(\tau^x_{0+})]^r$ is uniformly integrable in $x \in \mathbb{Z}_-$, and*

$$\lim_{x\to-\infty} \mathbb{E}\big[[\pi^x(\tau^x_{0+})]^r\big] = \frac{1}{\mathbb{E}[Z]} \sum_{k=1}^{+\infty} k^r \mathbb{P}[Z \geq k+1] < +\infty.$$

Proof. Note that if we let γ^x denote the random walk starting from $x < 0$ at time 0 with increment distributed as Z, then γ^x simply records the successive maxima of the random walk π^x, and so the overshoots $\pi^x(\tau^x_{0+})$ and $\gamma^x(\tau^x_{0+})$ are equally distributed. By a last passage decomposition for γ^x,

$$\mathbb{P}[\gamma^x(\tau^x_{0+}) = k] = \sum_{i=x}^{-1} G_\gamma(x,i)\mathbb{P}[Z = k-i] \leq \mathbb{P}[Z \geq k+1],$$

where $G_\gamma(x,i)$ is the probability γ^x will ever visit i. Since $\mathbb{E}[|Y|^{r+2}] < +\infty$ implies $\mathbb{E}[Z^{r+1}] < +\infty$ (see e.g. problem 6 in Chapter IV of [24]), we have $\mathbb{E}\big[[\pi^x(\tau^x_{0+})]^r\big] \leq \sum_{k=1}^{+\infty} k^r \mathbb{P}[Z \geq k+1] < +\infty$, giving uniform integrability. The rest then follows from Lemma 2.5 and dominated convergence.

Lemma 2.7 *Let $\xi^{\mathbb{Z}}_n$ be a system of coalescing random walks starting from every site on $\mathbb{Z}$ at time 0, whose random walk increments are distributed as Y with $\mathbb{E}[Y^2] < +\infty$. Then $p_n \equiv \mu_1(0 \in \xi^{\mathbb{Z}}_n) \leq \frac{C}{\sqrt{n}}$ for some constant C independent of the time n.*

Remark 2.1 *The proof we present here is an adaptation of the argument used by Bramson and Griffeath [5] to establish similar upper bounds for continuous time coalescing simple random walks in $\mathbb{Z}^d, d \geq 2$. In Corollary 7.1 below, we will prove that in fact $p_n \sim 1/(\sigma\sqrt{\pi n})$ as $n \to +\infty$ under a stronger moment assumption.*

Proof. Let $B_M = [0, M-1] \cap \mathbb{Z}$, and let $e_n(B_M) = \mathbb{E}[|\xi_n^{\mathbb{Z}} \cap B_M|]$. By translation invariance, $e_n(B_M) = p_n M$, and

$$e_n(B_M) \leq \sum_{k\in\mathbb{Z}} \mathbb{E}[|\xi_n^{B_M+kM} \cap B_M|] = \sum_{k\in\mathbb{Z}} \mathbb{E}[|\xi_n^{B_M} \cap (B_M + kM)|] = \mathbb{E}[|\xi_n^{B_M}|].$$

Since $M - |\xi_n^{B_M}|$ is at least as large as the number of nearest neighbor pairs in B_M that have coalesced by time n, we may take expectation and apply Lemma 2.2 to obtain

$$\mathbb{E}[|\xi_n^{B_M}|] \leq M-(M-1)\mu_1(|\xi_n^{\{0,1\}}| = 1) \leq M-(M-1)(1-\frac{C}{\sqrt{n}}) < 1+M\frac{C}{\sqrt{n}}.$$

Therefore $p_n < 1/M + C/\sqrt{n}$. Since M can be arbitrarily large for any fixed n, we obtain $p_n \leq C/\sqrt{n}$.

Lemma 2.8 *For any $A \subset \mathbb{Z}$, let ξ_n^A be a system of discrete time coalescing random walks on $\mathbb{Z}$ starting at time 0 with one walker at every site in A, where all the random walks have increments distributed as some arbitrary $\mathbb{Z}$-valued random variable Y. Then for any pair of disjoint sets $B, C \subset \mathbb{Z}$, and for any time $n \geq 0$,*

$$\mathbb{P}(\xi_n^A \cap B \neq \emptyset, \xi_n^A \cap C \neq \emptyset) \leq \mathbb{P}(\xi_n^A \cap B \neq \emptyset)\mathbb{P}(\xi_n^A \cap C \neq \emptyset). \tag{2.5}$$

In particular, if x, y are any two distinct sites in $\mathbb{Z}$, we have

$$\mu_1(x \in \xi_n^{\mathbb{Z}}, y \in \xi_n^{\mathbb{Z}}) \leq \mu_1(x \in \xi_n^{\mathbb{Z}})\mu_1(y \in \xi_n^{\mathbb{Z}}). \tag{2.6}$$

Proof. The continuous time version of this lemma is due to Arratia (see Lemma 1 in [3]). Our proof for the discrete time case is an adaptation of Arratia's proof for the continuous time case. Arratia's proof uses a theorem of Harris [18], which breaks down for discrete time because there are transitions between states that are not comparable to each other. However, this can be easily remedied by using an induction argument.

We can assume A, B, C are all finite sets, since otherwise we can approximate by finite sets, and the relevant probabilities will all converge. The main tool in the proof is the duality between coalescing random walks and voter models. For any pair of finite disjoint sets $B, C \subset \mathbb{Z}$, let $\phi_n^{B,C}$, with distribution $\nu_n^{B,C}$, be a discrete time multitype voter model on $\mathbb{Z}$ defined as follows. The state space is $X = \{-1, 0, 1\}^{\mathbb{Z}}$ with product topology. At time

0, $\phi_0^{B,C}(x) = 0$ if $x \in (B \cup C)^c$; $\phi_0^{B,C}(x) = 1$ if $x \in B$; $\phi_0^{B,C}(x) = -1$ if $x \in C$. At time $n \geq 1$, we update $\phi_n^{B,C}$ by setting $\phi_n^{B,C}(x) = \phi_{n-1}^{B,C}(x + Y_{x,n})$, where $\{Y_{x,n}\}_{x\in\mathbb{Z},n\in\mathbb{N}}$ are i.i.d. integer-valued random variables distributed as Y.

Let $E_A^+ \subset X$ (resp., $E_A^- \subset X$) be the event that some site in A is assigned the value $+1$ (resp., -1). Then by the duality between voter models and coalescing random walks (see, e.g., [22]), $\mathbb{P}(\xi_n^A \cap B \neq \emptyset) = \nu_n^{B,C}(E_A^+)$, $\mathbb{P}(\xi_n^A \cap C \neq \emptyset) = \nu_n^{B,C}(E_A^-)$, and $\mathbb{P}(\xi_n^A \cap B \neq \emptyset, \xi_n^A \cap C \neq \emptyset) = \nu_n^{B,C}(E_A^+ \cap E_A^-)$. The correlation inequality (2.5) then becomes

$$\nu_n^{B,C}(E_A^+ \cap E_A^-) \leq \nu_n^{B,C}(E_A^+)\nu_n^{B,C}(E_A^-). \tag{2.7}$$

We can define a partial order on the state space X by setting $\eta \leq \zeta \in X$ whenever $\eta(x) \leq \zeta(x)$ for all $x \in \mathbb{Z}^d$.A function $f : X \to \mathbb{R}$ is called increasing (resp., decreasing) if for any $\eta \leq \zeta$, $f(\eta) \leq f(\zeta)$ (resp., $f(\eta) \geq f(\zeta)$). An event E is called increasing (resp., decreasing) if 1_E is an increasing (resp., decreasing) function. Clearly, for finite A, $1_{E_A^+}$ is an increasing function and $1_{E_A^-}$ is a decreasing function. Inequality (2.7) will follow if we show that $\nu_n^{B,C}$ has the FKG property (see, e.g, [17, 22]), i.e., for any two increasing functions f and g, $\int fg\, d\nu_n^{B,C} \geq \int f\, d\nu_n^{B,C} \int g\, d\nu_n^{B,C}$.

We prove this by induction. For any pair of finite disjoint sets $B, C \subset \mathbb{Z}$, $\nu_0^{B,C}$ has the FKG property because the measure is concentrated at a single configuration. Observe that $\nu_1^{B,C}$ is a product measure and therefore also has the FKG property (this is the consequence of a simple special case of the main result of [17]). We proceed to the induction step, which is a fairly standard argument [19]. Assume that for all disjoint finite sets B and C, and for all $0 \leq k \leq n-1$, $\nu_k^{B,C}$ has the FKG property. Let us denote the collection of sites in $\mathbb{Z}$ where $\phi_{n-1}^{B,C}(x) = 1$ by B_{n-1}, and where $\phi_{n-1}^{B,C}(x) = -1$ by C_{n-1}. Then for any two increasing functions f and g, conditioning on $\phi_{n-1}^{B,C}$, we have by the Markov property,

$$\begin{aligned}
\int fg\, d\nu_n^{B,C} &= \int\int fg\, d\nu_1^{B_{n-1},C_{n-1}}\, d\nu_{n-1}^{B,C} \\
&\geq \int\int f\, d\nu_1^{B_{n-1},C_{n-1}} \int g\, d\nu_1^{B_{n-1},C_{n-1}}\, d\nu_{n-1}^{B,C} \\
&\geq \int\int f\, d\nu_1^{B_{n-1},C_{n-1}} d\nu_{n-1}^{B,C} \int\int g\, d\nu_1^{B_{n-1},C_{n-1}}\, d\nu_{n-1}^{B,C} \\
&= \int f\, d\nu_n^{B,C} \int g\, d\nu_n^{B,C},
\end{aligned}$$

where we have used the FKG property for both $\nu_{n-1}^{B,C}$ and for $\nu_1^{B_{n-1},C_{n-1}}$, and the observation that the conditional expectations $\int f d\nu_1^{B_{n-1},C_{n-1}}, \int g d\nu_1^{B_{n-1},C_{n-1}}$ conditioned on $\phi_{n-1}^{B,C}$ are still increasing functions. Therefore $\nu_n^{B,C}$ also has the FKG property. This concludes the induction proof, and establishes the lemma.

Remark 2.2 *Lemma 2.8 is also valid for random walks in $\mathbb{Z}^d$ by the same argument.*

3 Verification of condition (B_1')

Let's fix $t_0, a \in \mathbb{R}$, $\beta > 0$, $t > \beta$, $\epsilon > 0$. Also fix a δ and let $\tilde{t}_0, \tilde{t}, \tilde{a}$ and $\tilde{\epsilon}$ be defined from t_0, t, a and ϵ by diffusive scaling. Then we have

$$\mu_\delta(|\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon])| > 1) = \mu_1(|\mathcal{N}_{\tilde{t}_0,\tilde{t}}([\tilde{a}-\tilde{\epsilon}, \tilde{a}+\tilde{\epsilon}])| > 1).$$

If $\tilde{t}_0 = n_0 \in \mathbb{Z}$, then the contribution to $\mathcal{N}$ is all due to walkers starting from $[\tilde{a}-\tilde{\epsilon}, \tilde{a}+\tilde{\epsilon}] \cap \mathbb{Z}$ at time n_0. Thus we have

$$\begin{aligned}
&\mu_1(|\mathcal{N}_{n_0,\tilde{t}}([\tilde{a}-\tilde{\epsilon}, \tilde{a}+\tilde{\epsilon}])| > 1) = \mu_1(|\xi_{n_0+\tilde{t}}^{[\tilde{a}-\tilde{\epsilon},\tilde{a}+\tilde{\epsilon}]\cap\mathbb{Z},n_0}| > 1) \\
&\le \sum_{i=\lceil \tilde{a}-\tilde{\epsilon}\rceil}^{\lfloor \tilde{a}+\tilde{\epsilon}\rfloor -1} \mu_1(|\xi_{n_0+\tilde{t}}^{\{i,i+1\},n_0}| > 1) \\
&\le 2\tilde{\epsilon}\mu_1(|\xi_{\tilde{t}}^{\{0,1\},0}| > 1) \le 2\tilde{\epsilon}\frac{C}{\sqrt{\tilde{t}}} = \frac{2C\sigma\epsilon}{\sqrt{t}} < \frac{2C\sigma\epsilon}{\sqrt{\beta}}
\end{aligned} \tag{3.1}$$

The first inequality follows from the observation that if the collection of walkers starting from $[\tilde{a}-\tilde{\epsilon}, \tilde{a}+\tilde{\epsilon}] \cap \mathbb{Z}$ at n_0 has not coalesced into a single walker by $n_0+\tilde{t}$, then there is at least one adjacent pair of such walkers which has not coalesced by $n_0+\tilde{t}$. The next inequality follows from Lemma 2.2.

Now suppose $\tilde{t}_0 \in (n_0, n_0+1)$ for some $n_0 \in \mathbb{Z}$. Note that a walker's path can only cross $[\tilde{a}-\tilde{\epsilon}, \tilde{a}+\tilde{\epsilon}] \times \{\tilde{t}_0\}$ due to the increment at time n_0. After the increment, at time n_0+1, it will either land in $[\tilde{a}-2\tilde{\epsilon}, \tilde{a}+2\tilde{\epsilon}]$, or else outside that interval. In the first case, the contribution of the walker's path to $\mathcal{N}$ is included in $\xi_{\tilde{t}_0+\tilde{t}}^{[\tilde{a}-2\tilde{\epsilon},\tilde{a}+2\tilde{\epsilon}]\cap\mathbb{Z},n_0+1}$, the probability of which by our previous argument is bounded by $\frac{4C\sigma\epsilon}{\sqrt{t}}$ times a prefactor which approaches 1 as $\delta \to 0$. In the second case, either a walker in $(-\infty, \tilde{a}+\tilde{\epsilon}]$ jumps to the right of $\tilde{a} + 2\tilde{\epsilon}$, or a walker in $[\tilde{a} - \tilde{\epsilon}, +\infty)$ jumps to the left of $\tilde{a} - 2\tilde{\epsilon}$, the probability of which is bounded by

$$\begin{aligned}
&\sum_{x=\lceil \tilde{a}-\tilde{\epsilon}\rceil}^{+\infty} \mathbb{P}(Y \leq \tilde{a} - 2\tilde{\epsilon} - x) + \sum_{x=-\infty}^{\lfloor \tilde{a}+\tilde{\epsilon}\rfloor} \mathbb{P}(Y \geq \tilde{a} + 2\tilde{\epsilon} - x) \\
\leq\ & \sum_{k=0}^{+\infty} \mathbb{P}(|Y| \geq k + \tilde{\epsilon}) \ \leq\ \sum_{k=0}^{+\infty} \frac{\mathbb{E}[Y^2, |Y| \geq k + \tilde{\epsilon}]}{(k+\tilde{\epsilon})^2} \\
\leq\ & \sum_{k=0}^{+\infty} \frac{\mathbb{E}[Y^2, |Y| \geq \tilde{\epsilon}]}{(k+\tilde{\epsilon})^2} \ \leq\ \frac{2\mathbb{E}[Y^2, |Y| \geq \tilde{\epsilon}]}{\tilde{\epsilon}} \leq \frac{4\sigma^2}{\tilde{\epsilon}} = \frac{4\sigma\delta}{\epsilon}. \qquad (3.2)
\end{aligned}$$

The next to last inequality in (3.2) is valid if we take δ to be sufficiently small. The bounds in (3.1) and (3.2) are independent of $t_0, t > \beta$ and a. Taking the supremum over $t > \beta, t_0$ and a, and letting $\delta \to 0^+$, we obtain (B_1').

Corollary 3.1 *Assume $\mathcal{X}$ (with distribution μ) is a subsequential limit of $\mathcal{X}_\delta$, then for any deterministic point $y \in \mathbb{R}^2$, $\mathcal{X}$ has almost surely at most one path starting from y.*

Proof. It was shown in the proof of Theorem 5.3 in [16] that (B_1') implies

$$(B_1'')\ \forall \beta > 0, \sup_{t>\beta} \sup_{t_0, a \in \mathbb{R}} \mu(|\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon])| > 1) \to 0 \text{ as } \epsilon \to 0^+$$

and the corollary then follows.

Remark 3.1 *Note that if $Z_\delta^{A_\delta}$ is the process of coalescing random walks starting from a subset A_δ of the rescaled lattice, and $Z_\delta^{A_\delta}$ converges in distribution to a limit Z, then by the same argument as above, for any deterministic point $y \in \mathbb{R}^2$, Z has almost surely at most one path starting from y.*

4 Verification of tightness condition (T_1)

Define $A^+_{t,u}(x_0, t_0)$ to be the event that K contains a path touching both $R(x_0, t_0; u, t)$ and (at a later time) the right boundary of the bigger rectangle $R(x_0, t_0; 17u, 2t)$, and similarly define the event $A^-_{t,u}(x_0, t_0)$ corresponding to the left boundary of the bigger rectangle. Then $A = (A^+ \cup A^-)$, and writing (T_1) in terms of μ_1, we argue that it is sufficient to prove

$$(T_1^+) \qquad \tilde{g}(t,u;L,T) \equiv t^{-1} \limsup_{\delta \to 0^+} \ \mu_1(A^+_{\tilde{t},\tilde{u}}(0,0)) \to 0 \text{ as } t \to 0^+.$$

The sup over x_0, t_0 has been safely omitted because μ_1 is invariant under translation by integer units of space and time. When $\tilde{x}_0, \tilde{t}_0 \notin \mathbb{Z}$, we can bound the probability from above by using larger rectangles with vertices in $\mathbb{Z} \times \mathbb{Z}$ and base centered at $(0,0)$. Since the argument establishing the analogous tightness condition (T_1^-) for the event A^- is identical to that for (T_1^+), (T_1) follows from (T_1^+). In the ensuing discussions, we will simply write $A^+_{t,u}(0,0)$ as $A^+_{t,u}$ or just A^+, and $R(0,0;u,t)$ as $R(u,t)$.

Before we prove (T_1^+), and hence (T_1), we introduce some simplifying notation. Denote random walks starting at time 0 from $x_1 = \lceil 3\tilde{u} \rceil$, $x_2 = \lceil 7\tilde{u} \rceil$, $x_3 = \lceil 11\tilde{u} \rceil, x_4 = \lceil 15\tilde{u} \rceil$ by $\pi_1, \pi_2, \pi_3, \pi_4$. Denote the event that π_i, $(i = 1,2,3,4)$ stays within a distance $\tilde{u}$ of x_i up to time $2\tilde{t}$ by B_i (see Figure 1). For a random walk starting from $(x,m) \in R(\tilde{u},\tilde{t})$, denote the integer stopping times that the walker first exceeds $5\tilde{u}, 9\tilde{u}, 13\tilde{u}$ and $17\tilde{u}$ by $\tau_1^{x,m}, \tau_2^{x,m}, \tau_3^{x,m}$ and $\tau_4^{x,m}$. We also define $\tau_0^{x,m} = m$, and $\tau_5^{x,m} = 2\tilde{t}$. Denote the event that $\pi^{x,m}$ does not coalesce with π_i before time $2\tilde{t}$ by $C_i(x,m)$. As we shall see, the reason for choosing four paths π_i is because each path contributes a factor of δ to our estimate of the μ_1 probability in (T_1^+), and an overall factor of δ^4 is needed to outweigh the $O(\delta^{-3})$ number of lattice points in the rectangle $R(\tilde{u},\tilde{t})$ from where a random walk can start. We are now ready to prove (T_1^+).

Proof of (T_1^+). First we can assume $\tilde{t} \in \mathbb{Z}$, since we can always replace $\tilde{t}$ by $\lceil \tilde{t} \rceil$ which only enlarges the event A^+. The contribution to the event A^+ is either due to random walk paths that originate from within $R(\tilde{u},\tilde{t})$, or paths that cross $R(\tilde{u},\tilde{t})$ without landing inside it after the crossing. Denote the latter event by $D(\tilde{u},\tilde{t})$. Then

$$\begin{aligned}\mu_1(A^+_{\tilde{t},\tilde{u}}) \le \mu_1(\bigcup_{i=1}^4 B_i^c) + \mu_1(D(\tilde{u},\tilde{t}))\\ + \mu_1(\bigcap_{i=1}^4 B_i;\ \exists (x,m) \in R(\tilde{u},\tilde{t}) \ s.t. \ \bigcap_{i=1}^4 C_i(x,m), \tau_4^{x,m} < 2\tilde{t}). \quad (4.1)\end{aligned}$$

By Lemma 2.3, the first term on the right hand side of (4.1) is of order $o(t)$ after taking the limit $\delta \to 0$. For the second event in (4.1) to occur,

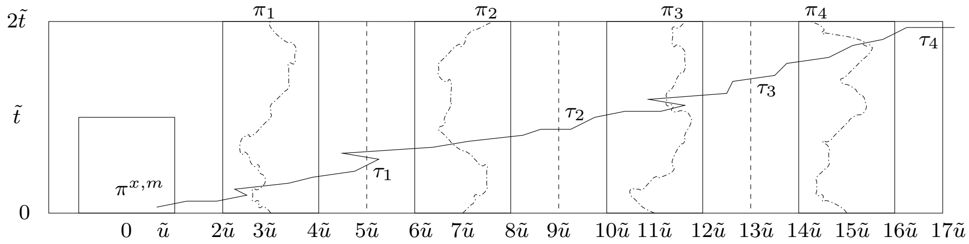

Figure 1: The random walks π_1, π_2, π_3 and π_4 start from $3\tilde{u}, 7\tilde{u}, 11\tilde{u}$ and $15\tilde{u}$ at time 0 and each stays within a distance of $\tilde{u}$. The random walk $\pi^{x,m}$ starts from (x, m) inside the rectangle $R(0, 0; \tilde{u}, \tilde{t})$ and exits the right boundary of the rectangle $R(0, 0; 17\tilde{u}, 2\tilde{t})$ at time τ_4 without coalescing with π_1, π_2, π_3 and π_4 on the way.

either a walker at a site in $(-\infty, -\tilde{u}] \times \{n\}$ jumps to a site in $[\tilde{u}, +\infty)$ in one step, or a walker in $[\tilde{u}, +\infty) \times \{n\}$ jumps to a site in $(-\infty, -\tilde{u}]$ in one step for some $n \in [0, \tilde{t} - 1] \cap \mathbb{Z}$. Denote the event just described by $D'(\tilde{u}, n)$, then $\mu_1(D(\tilde{u}, \tilde{t})) \leq \sum_{n=0}^{\tilde{t}-1} \mu_1(D'(\tilde{u}, n))$. From this we see that repeating the calculations in (3.2) for (B_1') under the assumption $\mathbb{E}[|Y|^3] < +\infty$ will guarantee that $\mu_1(D(\tilde{u}, \tilde{t})) \to 0$ as $\delta \to 0^+$.

To estimate the third term in (4.1) (see Figure 1 for an illustration of the event), we first treat the case of a fixed $(x, m) \in R(\tilde{u}, \tilde{t})$. Suppressing (x, m) from $\pi^{x,m}$, $C_i(x, m)$ and $\tau_i^{x,m}$, we have

$$\begin{aligned}
&\mu_1\{\text{for } (x,m) \text{ fixed}, \bigcap_{i=1}^{4} B_i, \bigcap_{i=1}^{4} C_i, \ \tau_4 < 2\tilde{t}\} \\
\leq\ &\mu_1\{\pi(\tau_1) > 5\frac{1}{2}\tilde{u} \ \ or\ \pi(\tau_2) > 9\frac{1}{2}\tilde{u} \ \ or\ \pi(\tau_3) > 13\frac{1}{2}\tilde{u}\} \\
+\ &\mu_1\{\pi(\tau_1) \leq 5\frac{1}{2}\tilde{u}, \pi(\tau_2) \leq 9\frac{1}{2}\tilde{u}, \pi(\tau_3) \leq 13\frac{1}{2}\tilde{u}, \tau_4 < 2\tilde{t}, \bigcap B_i, \bigcap C_i\}. \quad (4.2)
\end{aligned}$$

The first part is bounded by

$$\begin{aligned}
3 \sup_{x \in \mathbb{Z}^-} \mathbb{P}_x[\pi^x(\tau_{0+}) > \frac{\tilde{u}}{2}] &\leq 3\,(\frac{2}{\tilde{u}})^3 \sup_{x \in \mathbb{Z}^-} \mathbb{E}_x[(\pi^x(\tau_{0+}))^3, \pi^x(\tau_{0+}) > \frac{\tilde{u}}{2}] \\
&\leq \frac{24}{u^3\sigma^3}\,\delta^3\omega(\delta), \qquad (4.3)
\end{aligned}$$

where $\omega(\delta) \to 0$ as $\delta \to 0$. The last inequality is due to the uniform integrability of the third moment of the overshoot distribution, which follows from our assumption $\mathbb{E}[Y^5] < +\infty$ and Lemma 2.6.

For the second μ_1 probability in (4.2), denote the event that none of the conditions listed are violated by time t by G_t. If $\tau_1 > t$, we interpret an inequality like $\pi(\tau_1) \leq 5\frac{1}{2}\tilde{u}$ as not having been violated by time t. G_t is then a nested family of events, and the second probability in (4.2) becomes

$$\mu_1(G_{2\tilde{t}}) = \mu_1(G_{\tau_5}) = \mu_1(G_m) \prod_{k=1}^{5} \mu_1(G_{\tau_k}|G_{\tau_{k-1}}) < \prod_{k=1}^{4} \mu_1(G_{\tau_k}|G_{\tau_{k-1}}),$$

since $G_{\tau_k} \subset G_{\tau_{k-1}}$. Denote the history of the random walks $\pi^{x,m}, \pi^1, \pi^2, \pi^3$ and π^4 up to time t by Π_t, and denote expectation with respect to the conditional distribution of Π_t conditioned on the event G_t by $\mathbb{E}_t$. Then for $k = 1, 2, 3, 4$,

$$\mu_1(G_{\tau_k}|G_{\tau_{k-1}}) = \mathbb{E}_{\tau_{k-1}}\big[\mu_1(G_{\tau_k}|\Pi_{\tau_{k-1}} \in G_{\tau_{k-1}})\big],$$

where the μ_1 probability on the right hand side is conditioned on a given realization of $\Pi_{\tau_{k-1}} \in G_{\tau_{k-1}}$, which is a positive probability event. For any $\Pi_{\tau_{k-1}} \in G_{\tau_{k-1}}$, we have by the strong Markov property that

$$\begin{aligned} \mu_1(G_{\tau_k}|\Pi_{\tau_{k-1}} \in G_{\tau_{k-1}}) &= \mu_1[G_{\tau_k}|\pi^{x,m}(\tau_{k-1}), \pi^i(\tau_{k-1}), i = 1,2,3,4] \\ &\leq C(t,u)\delta, \end{aligned} \tag{4.4}$$

where the inequality follows from Lemma 2.4 for δ sufficiently small. Thus $\mu_1(G_{\tau_k}|G_{\tau_{k-1}}) \leq C(t,u)\delta$, and $\mu_1(G_{2\tilde{t}}) \leq C^4(t,u)\delta^4$. We then have

$$\begin{aligned} & \mu_1\Big(\bigcap_{i=1}^{4} B_i;\ \exists (x,m) \in R(\tilde{u},\tilde{t}),\ s.t.\ \bigcap_{i=1}^{4} C_i(x,m), \tau_4^{x,m} < 2\tilde{t}\Big) \\ \leq\ & \sum_{x\in[-\tilde{u},\tilde{u}]\cap\mathbb{Z}} \ \sum_{m\in[0,\tilde{t}]\cap\mathbb{Z}} \mu_1\Big[\text{for } (x,m) \text{ fixed}, \bigcap_{i=1}^{4} B_i,\ \bigcap_{i=1}^{4} C_i,\ \tau_4^{x,m} < 2\tilde{t}\Big] \\ \leq\ & \Big[\frac{24}{u^3\sigma^3}\,\delta^3\omega(\delta) + C^4(t,u)\delta^4\Big]\ 2\tilde{u}\ \tilde{t} = \omega'(\delta), \end{aligned} \tag{4.5}$$

where $2\tilde{u}\tilde{t} = O(\delta^{-3})$ and hence $\omega'(\delta) \to 0$ as $\delta \to 0$. Thus the last two terms in (4.1) go to 0 as $\delta \to 0$, and the first term is of order $o(t)$ after taking the limit $\delta \to 0$. Together they give (T_1^+).

Remark 4.1 *The only place in this paper where we need the assumption* $\mathbb{E}[|Y|^5] < +\infty$ *is in* (4.3)*. We need* $\mathbb{E}[|Y|^3] < +\infty$ *to estimate* $\mu_1(D(\tilde{u}, \tilde{t}))$ *in* (4.1)*, and to apply Lemma* 2.4 *in* (4.4)*. A finite third moment is the minimal moment condition for the convergence of* $\mathcal{X}_\delta$*. For any* $\epsilon > 0$*, there are choices of* Y *satisfying* $\mathbb{E}[|Y|^{3-\epsilon}] < +\infty$*, but with* $\mu_1(D(\tilde{u}, \tilde{t})) \to 1$ *as* $\delta \to 0$ *for all* $t > 0$*, which implies* $\{\mathcal{X}_\delta\}$ *is not tight.*

5 Verification of condition (I_1)

Our verification of (I_1) is essentially a careful application of Donsker's invariance principle and the Continuous Mapping Theorem for weak convergence. We define three sets of random walks: $\{\pi^i_\delta\}_{1\le i\le m}$, a family of m independent random walks on the rescaled lattice $(\delta/\sigma)\mathbb{Z} \times \delta^2\mathbb{Z}$; $\{\pi^i_{\delta,f}\}_{1\le i\le m}$, the family of m coalescing random walks constructed from $\{\pi^i_\delta\}$ by applying a mapping f to $\{\pi^i_\delta\}$; and $\{\pi^i_{\delta,g}\}_{1\le i\le m}$, an auxiliary family of m coalescing walks constructed by applying a mapping g to $\{\pi^i_\delta\}$ such that two walks coalesce as soon as their paths intersect (note that random walks in $\{\pi^i_{\delta,g}\}$ coalesce earlier than they do in $\{\pi^i_{\delta,f}\}$). By Donsker's invariance principle, $\{\pi^i_\delta\}$ will converge weakly to a family of independent Brownian motions $\{\mathcal{B}^i\}_{1\le i\le m}$. As we will see, the mapping g is almost surely continuous with respect to $\{\mathcal{B}^i\}$, and $\{\mathcal{B}^i_g\}_{1\le i\le m}$ is distributed as coalescing Brownian motions. Therefore by the Continuous Mapping Theorem for weak convergence, $\{\pi^i_{\delta,g}\}$ converges weakly to the coalescing Brownian motions $\{\mathcal{B}^i_g\}$. Finally to show that $\{\pi^i_{\delta,f}\}$ also converges weakly to $\{\mathcal{B}^i_g\}$, we will prove that the distance between the two versions of coalescing walks $\{\pi^i_{\delta,f}\}$ and $\{\pi^i_{\delta,g}\}$ converges to 0 in probability.

We start with some notation. Let $\mathcal{D}$ be any deterministic countable dense subset of $\mathbb{R}^2$. Let $y^1 = (x^1, t^1), \ldots, y^m = (x^m, t^m) \in \mathcal{D}$ be fixed, and let $\mathcal{B}^1, ..., \mathcal{B}^m$ be independent Brownian motions starting from $y^1, ..., y^m$. For a fixed δ, denote $\lceil \tilde{y}^i \rceil = (\lceil \tilde{x}^i \rceil, \lceil \tilde{t}^i \rceil)$, where $\tilde{x}^i = \sigma\delta^{-1}x^i$ and $\tilde{t} = \delta^{-2}t^i$ as defined in Section 2, and let y^i_δ denote $\lceil \tilde{y}^i \rceil$'s space-time position after diffusive scaling on the rescaled lattice $(\delta/\sigma)\mathbb{Z} \times \delta^2\mathbb{Z}$. Let $\tilde{\pi}^i$ $(i = 1, \cdots, m)$ be independent random walks in the $\mathbb{Z} \times \mathbb{Z}$ lattice starting from $\lceil \tilde{y}^i \rceil$. We regard $(\mathcal{B}^1, ..., \mathcal{B}^m)$, and $(\tilde{\pi}^1, ..., \tilde{\pi}^m)$ as random variables in the product metric space (Π^m, d^{*m}), where

$$d^{*m}[(\xi^1, \ldots, \xi^m), (\zeta^1, \ldots, \zeta^m)] = \max_{1\le i\le m} d(\xi^i, \zeta^i) \tag{5.1}$$

and d is defined in (1.4); thus d^{*m} gives the product topology on Π^m. We will also need the metric

$$\bar{d}((f_1,t_1),(f_2,t_2)) = \sup_t |\hat{f}_1(t) - \hat{f}_2(t)| \vee |t_1 - t_2| \tag{5.2}$$

and $\bar{d}^{*m}$ is defined in a similar way as d^{*m}.

We now define a mapping g from (Π^m, d^{*m}) to (Π^m, d^{*m}) that constructs coalescing paths from independent paths. The construction is such that when two paths first intersect, the path with the higher index will be replaced by the path with the lower index after the time of intersection. This procedure is then iterated until no more intersections take place. To be explicit, we give the following algorithmic construction.

Let $(\xi^1, \ldots, \xi^m) \in \Pi^m$, and start with equivalence relations on the set $\{1, \ldots, m\}$ by letting $i \nsim j \ \forall\ i \neq j$. Define a one step iteration Γ on $(\xi^1, \ldots, \xi^m)$ and the equivalence relations by

$$\tau_g = \inf\{t \in \mathbb{R} \mid \exists\, i,j \in \{1,\ldots,m\},\ i \nsim j,\ \xi^i(t) = \xi^j(t)\}, \tag{5.3}$$

$$i^* = \min\{j \mid 1 \leq j \leq m,\ \xi^i(\tau_g) = \xi^j(\tau_g)\}, \tag{5.4}$$

$$\Gamma\xi^i(t) = \begin{cases} \xi^i(t) & \text{if } t \leq \tau_g, \\ \xi^{i^*}(t) & \text{if } t > \tau_g, \end{cases} \tag{5.5}$$

and update equivalence relations by assigning $i \sim i^*$. Iterate the mapping Γ, and label the successive intersection times τ_g by τ_g^k. Then the iteration stops when $\tau_g^k = +\infty$ for some $k \in \{1, 2, \ldots, m\}$, i.e., either there is no more intersection among the different equivalence classes of paths, or all the paths have coalesced and formed a single equivalence class. Denote the final collection of paths by $g(\xi^1, \ldots, \xi^m) = (\xi_g^1, \ldots, \xi_g^m)$. Then it's clear by the strong Markov property, that $(\mathcal{B}_g^1, \ldots, \mathcal{B}_g^m)$ has the distribution of coalescing Brownian motions. However $(\tilde{\pi}_g^1, \ldots, \tilde{\pi}_g^m)$ is not distributed as coalescing random walks, because for nonsimple random walks, paths can cross before the random walks actually coalesce (by being at the same space-time lattice site).

To construct coalescing random walk paths from independent random walks in $\mathbb{Z} \times \mathbb{Z}$, we define another mapping f from (Π^m, d^{*m}) to (Π^m, d^{*m}) in a similar way as we defined g, except in (5.3)–(5.5), the time of first intersection τ_g is replaced by the time of first coincidence on the unscaled lattice,

$$\tau_f = \min\{\ n \in \mathbb{Z} \mid \exists\, i,j \in \{1,\ldots,m\},\ i \nsim j,\ \xi^i(n) = \xi^j(n)\}. \tag{5.6}$$

We will label the successive coincidence times by τ_f^k. Also denote $f(\xi^1, \cdots, \xi^m)$ by $(\xi_f^1, \cdots, \xi_f^m)$. It is then clear that $(\tilde{\pi}_f^1, \ldots, \tilde{\pi}_f^m)$ is distributed as coalescing random walks starting from $(\lceil \tilde{y}^1 \rceil, \ldots, \lceil \tilde{y}^m \rceil)$ in the unscaled lattice $\mathbb{Z} \times \mathbb{Z}$. We shall denote the diffusively rescaled versions of $(\tilde{\pi}^1, \ldots, \tilde{\pi}^m)$, $(\tilde{\pi}_g^1, \ldots, \tilde{\pi}_g^m)$ and $(\tilde{\pi}_f^1, \ldots, \tilde{\pi}_f^m)$ by $(\pi_\delta^1, \ldots, \pi_\delta^m)$, $(\pi_{\delta,g}^1, \ldots, \pi_{\delta,g}^m)$ and $(\pi_{\delta,f}^1, \ldots, \pi_{\delta,f}^m)$. We need the following two lemmas to prove (I_1).

Lemma 5.1 *Let $(\mathcal{B}^1, ..., \mathcal{B}^m)$ be m independent Brownian motions starting from $(y^1, ..., y^m)$, and let $(\pi_\delta^1, ..., \pi_\delta^m)$ be independent random walks starting from $(y_\delta^1, ..., y_\delta^m)$ in the rescaled lattice as defined before. Then $(\pi_{\delta,g}^1, ..., \pi_{\delta,g}^m)$ converges in distribution to $(\mathcal{B}_g^1, ..., \mathcal{B}_g^m)$ as $\delta \to 0^+$.*

Proof. Clearly $(y_\delta^1, \ldots, y_\delta^m)$ converge to $(y^1, \ldots, y^m)$. By Donsker's invariance principle [11], $(\pi_\delta^1, ..., \pi_\delta^m)$ converges weakly to $(\mathcal{B}^1, ..., \mathcal{B}^m)$ as $\delta \to 0^+$. From standard properties of Brownian motion, it is easy to see that the mapping g is almost surely continuous with respect to $(\mathcal{B}^1, ..., \mathcal{B}^m)$. Also note that $(\pi_{\delta,g}^1, ..., \pi_{\delta,g}^m)$ is the same as applying g to $(\pi_\delta^1, ..., \pi_\delta^m)$, therefore by the Continuous Mapping Theorem for weak convergence (see, e.g., Section 8.1 of [12]), $(\pi_{\delta,g}^1, ..., \pi_{\delta,g}^m)$ converges in distribution to $(\mathcal{B}_g^1, ..., \mathcal{B}_g^m)$ as $\delta \to 0^+$.

Lemma 5.2 *$\forall\ \epsilon > 0$, $\mathbb{P}\{d^{*m}[(\pi_{\delta,f}^1, \ldots, \pi_{\delta,f}^m), (\pi_{\delta,g}^1, \ldots, \pi_{\delta,g}^m)] \geq \epsilon\} \to 0$ as $\delta \to 0^+$.*

Proof. From the definition of d and $\bar{d}$ in (1.4) and (5.2), it is clear that $d((f_1, t_1), (f_2, t_2)) \leq \bar{d}((f_1, t_1), (f_2, t_2))$ for any $(f_1, t_1), (f_2, t_2) \in \Pi$. Therefore it is sufficient to prove the lemma with d^{*m} replaced by $\bar{d}^{*m}$. In terms of random walks in $\mathbb{Z} \times \mathbb{Z}$, the lemma can be stated as

$$\forall\ \epsilon > 0, \mathbb{P}\{\bar{d}^{*m}[(\tilde{\pi}_f^1, \ldots, \tilde{\pi}_f^m), (\tilde{\pi}_g^1, \ldots, \tilde{\pi}_g^m)] \geq \tilde{\epsilon}\} \to 0 \text{ as } \delta \to 0^+. \tag{5.7}$$

We first prove (5.7) for $m = 2$. Note that for $m = 2$, $\tilde{\pi}_f^1 = \tilde{\pi}_g^1 = \tilde{\pi}^1$, hence $\bar{d}^{*2}[(\tilde{\pi}_f^1, \tilde{\pi}_f^2), (\tilde{\pi}_g^1, \tilde{\pi}_g^2)] = \bar{d}\,(\tilde{\pi}_f^2, \tilde{\pi}_g^2)$. Let $\tilde{T}_g^{1,2}$ denote the first integer time when $\tilde{\pi}^1$ and $\tilde{\pi}^2$ coincide or interchange relative ordering, and let $\tilde{T}_f^{1,2}$ denote the first integer time when the two walks coincide. Also let $l(0, n)$ denote the maximum distance over all time between two coalescing random walks starting at 0 and n at time 0. Then by the strong Markov property, and conditioning at time $\tilde{T}_g^{1,2}$,

$$\mathbb{P}[\ \bar{d}(\tilde{\pi}_f^2, \tilde{\pi}_g^2)\ \geq \tilde{\epsilon}\ ] \leq \sum_{n=1}^{+\infty} \mathbb{P}[|\tilde{\pi}^1(\tilde{T}_g^{1,2}) - \tilde{\pi}^2(\tilde{T}_g^{1,2})| = n]\ \mathbb{P}[l(0, n) \geq \tilde{\epsilon}]. \tag{5.8}$$

The first probability in the summand converges to a limiting probability distribution as $\delta \to 0$ by applying Lemma 2.5 to $(\tilde{\pi}^1 - \tilde{\pi}^2)$. The second probability converges to 0 for every fixed n by Lemma 2.1. This proves (5.7) for $m = 2$.

For $m > 2$, let $\tilde{T}_f^{i,j}$ and $\tilde{T}_g^{i,j}$ denote respectively the first integer time when the two independent walks $\tilde{\pi}^i$ and $\tilde{\pi}^j$ coincide or interchange relative ordering. As usual, let $T_{\delta,f}^{i,j} = \delta^2 \tilde{T}_f^{i,j}$ and $T_{\delta,g}^{i,j} = \delta^2 \tilde{T}_g^{i,j}$. Then by the weak convergence of $(\pi_\delta^1, \cdots, \pi_\delta^m)$ to $(\mathcal{B}^1, \cdots, \mathcal{B}^m)$ and the Continuous Mapping Theorem, $\{T_{\delta,g}^{i,j}\}_{1\leq i<j\leq m}$ converge jointly in distribution to $\{\tau^{i,j}\}_{1\leq i<j\leq m}$, the associated pairwise first intersection times for $\{\mathcal{B}^1, \cdots, \mathcal{B}^m\}$. By the standard properties of Brownian motion, $\{\tau^{i,j}\}_{1\leq i<j\leq m}$ are almost surely all distinct. By an argument similar to (5.8), we also have $\sup_{1\leq i<j\leq m} |T_{\delta,f}^{i,j} - T_{\delta,g}^{i,j}| \to 0$ in probability. Note that in our definition of the mapping g that constructs $(\tilde{\pi}_g^1, \cdots, \tilde{\pi}_g^m)$ from $(\tilde{\pi}^1, \cdots, \tilde{\pi}^m)$, the successive times of intersection $\{\tau_g^k\}_{1\leq k\leq m-1}$, are all times of first intersection between independent paths, i.e., $\{\lceil \tau_g^k \rceil\}_{1\leq k\leq m-1} \subset \{\tilde{T}_g^{i,j}\}_{1\leq i<j\leq m}$. The event in (5.7) can only occur due to: either (1) for some τ_g^k in the definition of g, with $\lceil \tau_g^k \rceil = \tilde{T}_g^{i,j}$ for some i and j, $\tau_g^{k+1} \leq \tilde{T}_f^{i,j}$; or else, (2) whenever a coalescing takes place between two paths $\tilde{\pi}^i, \tilde{\pi}^j$ in the mapping g, the same two paths will coalesce in the mapping f before another coalescing takes place in the mapping g, and the event in (5.7) occurs because for some τ_g^k with $\lceil \tau_g^k \rceil = \tilde{T}_g^{i,j}$, the distance between the two paths $\tilde{\pi}^i$ and $\tilde{\pi}^j$ during the time interval $[\tilde{T}_g^{i,j}, \tilde{T}_f^{i,j}]$ exceeds $\tilde{\epsilon}$. The probability of the event (1) tends to 0 as $\delta \to 0$ by our observations that $\{T_{\delta,g}^{i,j}\}_{1\leq i<j\leq m}$ converges jointly in distribution to $\{\tau^{i,j}\}_{1\leq i<j\leq m}$, which are almost surely all distinct, and the fact that $\sup_{1\leq i<j\leq m} |T_{\delta,f}^{i,j} - T_{\delta,g}^{i,j}| \to 0$ in probability. The probability of the event (2) tends to 0 by our proof of (5.7) for $m = 2$. This proves (5.7) and Lemma 5.2.

Proof of ($\mathbf{I_1}$). Lemmas 5.1 and 5.2 imply that $(\pi_{\delta,f}^1, \ldots, \pi_{\delta,f}^m)$ converge in distribution to $(\mathcal{B}_g^1, ..., \mathcal{B}_g^m)$ as $\delta \to 0^+$ by converging together lemma (see, e.g., Section 8.1 of [12]). Certainly $\{\pi_{\delta,f}^1, \ldots, \pi_{\delta,f}^m\} \subset \mathcal{X}_\delta$, therefore (I_1) is proved.

6 Verification of condition (E_1)

As usual, we start with some notation. For an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable X, define X^{s^-} to be the subset of paths in X which start before or at time s, and for $s \leq t$ define X^{s^-, t_T} to be the set of paths in X^{s^-} *truncated* before time t, i.e., replacing each path in X^{s^-} by its restriction to time greater than or equal to t. When $s = t$, we denote X^{s^-, s_T} simply by X^{s_T}. Also let $X(t) \subset \mathbb{R}$ denote the set of values at time t of all paths in X. Note that $\hat{\eta}_X(t_0, t; a, b) = |X^{t_0^-}(t_0 + t) \cap (a, b)|$. We may sometimes abuse the notation and use $X(t)$ also to denote the set of points $X(t) \times \{t\} \subset \mathbb{R}^2$.

We recall here the definition of stochastic domination as given in [16]. For two measures μ_1 and μ_2 on $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$, μ_2 is stochastically dominated by μ_1 ($\mu_2 << \mu_1$) if for any bounded increasing function f on $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$, (i.e. $f(K) \leq f(K')$ if $K \subset K'$), $\mathbb{E}_{\mu_2}[f] \leq \mathbb{E}_{\mu_1}[f]$. When μ_1, μ_2 are the distributions of two $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables X_1 and X_2, we will also denote the stochastic domination by $X_2 << X_1$. The first step of our proof is to reduce (E_1) to the following condition (E_1'), which singles out the set of paths that are of interest to us (i.e., the set of paths starting before time t_0).

(E_1') If Z_{t_0} is any subsequential limit of $\{\mathcal{X}_\delta^{t_0^-}\}$ for any $t_0 \in \mathbb{R}$, then $\forall t, a, b \in \mathbb{R}$ with $t > 0$ and $a < b$, $\mathbb{E}[\hat{\eta}_{Z_{t_0}}(t_0, t; a, b)] \leq \mathbb{E}[\hat{\eta}_{\bar{\mathcal{W}}}(t_0, t; a, b)] = \frac{b-a}{\sqrt{\pi t}}$.

Lemma 6.1 (E_1') *implies* (E_1).

Proof. Let $t_0 \in \mathbb{R}, t > 0$ be fixed, and let $\mathcal{X}$ be the weak limit of $\mathcal{X}_{\delta_n}$ for a sequence $\delta_n \downarrow 0$. To prove the Lemma, it is sufficient to show that for any $0 < \epsilon < t$, there is a subsequence $\{\delta_n'\}$ such that $\mathcal{X}_{\delta_n'}^{(t_0+\epsilon)^-}$ converges weakly to a limit $Z_{t_0+\epsilon}$, and $\mathcal{X}^{t_0^-} << Z_{t_0+\epsilon}$. Because then we have $\mathbb{E}[\hat{\eta}_{\mathcal{X}}(t_0, t; a, b)] \leq \mathbb{E}[\hat{\eta}_{Z_{t_0+\epsilon}}(t_0 + \epsilon, t - \epsilon; a, b)] \leq \frac{b-a}{\sqrt{\pi(t-\epsilon)}}$ by (E_1'), and letting $\epsilon \to 0$ establishes (E_1).

To prove the existence of $\{\delta_n'\}$, $Z_{t_0+\epsilon}$, and the stochastic domination, we use a coupling argument. Define $(\mathcal{H} \times \mathcal{H}, d_{\mathcal{H}}^*)$-valued random variables $W_{\delta_n} = (\mathcal{X}_{\delta_n}, \mathcal{X}_{\delta_n}^{(t_0+\epsilon)^-})$, where $d_{\mathcal{H}}^*$ is given by

$$d_{\mathcal{H}}^*[(K_1, K_2), (K_1', K_2')] = \max\{d_{\mathcal{H}}(K_1, K_1'), d_{\mathcal{H}}(K_2, K_2')\}$$

for $K_1, K_2, K_1', K_2' \in \mathcal{H}$. $(\mathcal{H} \times \mathcal{H}, d_{\mathcal{H}}^*)$ is a complete separable metric space. Since $\{\mathcal{X}_{\delta_n}\}$ is tight, and $\{\mathcal{X}_{\delta_n}^{(t_0+\epsilon)^-}\}$ is almost surely a compact subset of $\mathcal{X}_{\delta_n}$

for all δ_n, $\{\mathcal{X}_{\delta_n}^{(t_0+\epsilon)^-}\}$ and $\{W_{\delta_n}\}$ are also tight. Therefore we can choose a subsequence δ'_n such that $W_{\delta'_n}$ converges weakly to a limit W. By Skorohod's representation theorem (see, e.g., [4, 12]), we can define random variables $W'_{\delta'_n} = (\mathcal{X}'_{\delta'_n}, \mathcal{X}'^{(t_0+\epsilon)^-}_{\delta'_n})$ and $W' = (\mathcal{X}', Z_{t_0+\epsilon})$ on the proability space $[0,1]$ with Lebesgue measure, such that they are equally distributed with $W_{\delta'_n}$ and W and the convergence of $W'_{\delta'_n}$ to W' is almost sure. Clearly $\mathcal{X}'$ is equally distributed with $\mathcal{X}$ and $\mathcal{X}_{\delta'_n}^{(t_0+\epsilon)^-}$ converges weakly to $Z_{t_0+\epsilon}$. Almost surely, any path $(f,t) \in \mathcal{X}'$ with $t \leq t_0$ is the limit of a sequence of paths $(f_n, t_n) \in \mathcal{X}'_{\delta'_n}$ with $t_n \to t$. Since (f_n, t_n) is eventually in $\mathcal{X}'^{(t_0+\epsilon)^-}_{\delta'_n}$, we also have $(f,t) \in Z_{t_0+\epsilon}$. Therefore $\mathcal{X}'^{t_0^-} \subset Z_{t_0+\epsilon}$ almost surely, and $\mathcal{X}^{t_0^-} << Z_{t_0+\epsilon}$. This finishes the proof of Lemma 6.1.

Condition (E'_1) would follow if we knew that asymptotically the density of coalescing random walks is $p_n \sim 1/(\sigma\sqrt{\pi n})$ (see Lemma 2.7). But in the literature this result seems only to have been established for continuous time coalescing *simple* random walks. Instead of trying to directly establish the exact asymptotic density, we make the following observation. The coarse bound provided by Lemma 2.7 implies that $Z_{t_0}(t_0+\epsilon)$ is locally finite for any positive time ϵ. Then we expect $Z_{t_0}^{(t_0+\epsilon)_T}$, the set of paths in Z_{t_0} (which all start at time $t \leq t_0$) truncated before time $t_0+\epsilon$, to be distributed as coalescing Brownian motions starting from the random set $Z_{t_0}(t_0+\epsilon) \subset \mathbb{R}^2$. If we can verify that, which should be relatively easy because of the local finiteness of $Z_{t_0}(t_0+\epsilon)$, then condition (E'_1) will follow since the system of coalescing Brownian motions starting from a random locally finite set on $\mathbb{R}$ at time $t_0+\epsilon$ is certainly stochastically dominated by the system of coalescing Brownian motions starting from every point on $\mathbb{R}$ at time $t_0+\epsilon$, and for the latter we know how to compute $\mathbb{E}[\hat{\eta}]$. Sending ϵ to 0 will then establish (E'_1). In Corollary 7.1 below, we will show that the convergence of $\mathcal{X}_\delta$ to the Brownian web also implies that the density of coalescing random walks is asymptotically $1/(\sigma\sqrt{\pi n})$, so the coarse bound of Lemma 2.7 in fact leads to the exact asymptotics.

We now make everything precise. Let Z_{t_0} be as in (E'_1). (E'_1) follows from the next two lemmas, which are also what one needs to check in order to verify condition (E'_1) for general models other than coalescing random walks.

Lemma 6.2 *Let $Z_{t_0}(t_0+\epsilon) \subset \mathbb{R} \times \{t_0+\epsilon\}$ be the intersections of paths in*

Z_{t_0} *with the line* $t = t_0 + \epsilon$. *Then for any* $\epsilon > 0$, $Z_{t_0}(t_0 + \epsilon)$ *is almost surely locally finite.*

Lemma 6.3 *For any* $\epsilon > 0$, $Z_{t_0}^{(t_0+\epsilon)_T}$, *the set of paths in* Z_{t_0} *(which all start at time* $t \leq t_0$*) truncated before time* $t_0 + \epsilon$, *is distributed as* $\mathcal{B}^{Z_{t_0}(t_0+\epsilon)}$, *i.e., coalescing Brownian motions starting from the random set* $Z_{t_0}(t_0 + \epsilon) \subset \mathbb{R}^2$.

Proof of (E_1')**.** Assume Lemmas 6.2 and 6.3 for the moment. Since $\mathcal{B}^{Z_{t_0}(t_0+\epsilon)} << \bar{\mathcal{W}}$, we have for $0 < \epsilon < t$

$$\begin{aligned}
& \mathbb{E}[\hat{\eta}_{Z_{t_0}}(t_0, t; a, b)] \\
= \; & \mathbb{E}[\hat{\eta}_{Z_{t_0}^{(t_0+\epsilon)_T}}(t_0 + \epsilon, t - \epsilon; a, b)] = \mathbb{E}[\hat{\eta}_{\mathcal{B}^{Z_{t_0}(t_0+\epsilon)}}(t_0 + \epsilon, t - \epsilon; a, b)] \\
\leq \; & \mathbb{E}[\hat{\eta}_{\bar{\mathcal{W}}}(t_0 + \epsilon, t - \epsilon; a, b)] = \frac{b - a}{\sqrt{\pi(t - \epsilon)}} \, .
\end{aligned}$$

Since $0 < \epsilon < t$ is arbitrary, letting $\epsilon \to 0$ establishes (E_1').

Lemma 6.2 will be a consequence of the following:

Lemma 6.4 $\forall\ t_0, t, a, b \in \mathbb{R}$ *with* $t > 0$ *and* $a < b$, *we have*

$$\limsup_{\delta \to 0^+} \mathbb{E}[\hat{\eta}_{\mathcal{X}_\delta}(t_0, t; a, b)] \leq \frac{C(b - a)}{\sqrt{t}}$$

for some $0 < C < +\infty$ *independent of* t_0, t, a *and* b.

Proof. By space-time lattice translation invariance and elementary arguments, we can assume $t_0 = 0$ and $(a, b) = (-r, r)$. Note that $\mathbb{E}[\hat{\eta}_{\mathcal{X}_\delta}(0, t; -r, r)]$ $= \mathbb{E}[\hat{\eta}_{\mathcal{X}_1}(0, \tilde{t}; -\tilde{r}, \tilde{r})]$. If $\tilde{t} \in \mathbb{Z}$, then

$$\mathbb{E}[\hat{\eta}_{\mathcal{X}_1}(0, \tilde{t}; -\tilde{r}, \tilde{r})] \leq (2\tilde{r} + 1)\mu_1(0 \in \xi_{\tilde{t}}^{\mathbb{Z}}) \tag{6.1}$$

by translation invariance. If $\tilde{t} \notin \mathbb{Z}$,

$$\begin{aligned}
& \mathbb{E}[\hat{\eta}_{\mathcal{X}_1}(0, \tilde{t}; -\tilde{r}, \tilde{r})] \\
\leq \; & \mathbb{E}[\hat{\eta}_{\mathcal{X}_1}(0, \lfloor \tilde{t} \rfloor; -2\tilde{r}, 2\tilde{r}) + \sum_{x=-\infty}^{-2\tilde{r}} \mathbb{P}(Y > -\tilde{r} - x) + \sum_{x=2\tilde{r}}^{+\infty} \mathbb{P}(Y < \tilde{r} - x) \\
\leq \; & (4\tilde{r} + 1)\mu_1(0 \in \xi_{\lfloor \tilde{t} \rfloor}^{\mathbb{Z}}) + \sum_{k=0}^{+\infty} \mathbb{P}(|Y| > \tilde{r} + k), \qquad (6.2)
\end{aligned}$$

where by Lemma 2.7 the first quantity is bounded by $(4\tilde{r}+1)(C/\sqrt{\lfloor\tilde{t}\rfloor})$, which tends to $2C\sigma(b-a)/\sqrt{t}$ as $\delta \to 0$, and the second probability goes to 0 as $\delta \to 0$ by the same calculation as in (3.2). The lemma then follows.

Proof of Lemma 6.2. Let Z_{t_0} be the weak limit of a sequence $\{\mathcal{X}_{\delta_n}^{t_0^-}\}$. Define another counting random variable $\hat{\eta}'_X(t_0,t;a,b) = \lim_{s\downarrow 0}\hat{\eta}_X(t_0-s,t+s;a,b)$. Note that $\{K \in \mathcal{H}|\hat{\eta}'_K(t_0,t;a,b) \geq k\}$ is an open set for all $k \in \mathbb{N}$. For $0<\alpha<\epsilon$ and $a<b$, we then have by weak convergence and Lemma 6.4,

$$
\begin{aligned}
& \mathbb{E}[\hat{\eta}_{Z_{t_0}}(t_0,\epsilon;a,b)] \\
\leq\ & \mathbb{E}[\hat{\eta}'_{Z_{t_0}}(t_0+\alpha,\epsilon-\alpha;a,b)] = \sum_{k=1}^{+\infty}\mathbb{P}[\hat{\eta}'_{Z_{t_0}}(t_0+\alpha,\epsilon-\alpha;a,b)\geq k] \\
\leq\ & \sum_{k=1}^{+\infty}\liminf_{\delta_n\downarrow 0}\mathbb{P}[\hat{\eta}'_{\mathcal{X}_{\delta_n}^{t_0^-}}(t_0+\alpha,\epsilon-\alpha;a,b)\geq k] \\
\leq\ & \liminf_{\delta_n\downarrow 0}\mathbb{E}[\hat{\eta}'_{\mathcal{X}_{\delta_n}^{t_0^-}}(t_0+\alpha,\epsilon-\alpha;a,b)] \\
\leq\ & \liminf_{\delta_n\downarrow 0}\mathbb{E}[\hat{\eta}_{\mathcal{X}_{\delta_n}}(t_0+\alpha,\epsilon-\alpha;a,b)] \leq \frac{C(b-a)}{\sqrt{\epsilon-\alpha}} < +\infty.
\end{aligned}
$$

Since $a<b$ is arbitrary, the lemma then follows.

It only remains to prove Lemma 6.3. We need one more lemma. Denote the space of compact subsets of $(\bar{\mathbb{R}}^2,\rho)$ by $(\mathcal{P},\rho_{\mathcal{P}})$, with $\rho_{\mathcal{P}}$ the induced Hausdorff metric, i.e., for $A_1,A_2\in\mathcal{P}$,

$$
\rho_{\mathcal{P}}(A_1,A_2) = \sup_{z_1\in A_1}\inf_{z_2\in A_2}\rho(z_1,z_2) \vee \sup_{z_2\in A_2}\inf_{z_1\in A_1}\rho(z_1,z_2). \tag{6.3}
$$

Note that $(\mathcal{P},\rho_{\mathcal{P}})$ is a complete separable metric space.

Lemma 6.5 *Let A_δ and A be $(\mathcal{P},\rho_{\mathcal{P}})$-valued random variables, where A is almost surely a locally finite set, A_δ is almost surely a subset of $(\delta\mathbb{Z}/\sigma)\times(\delta^2\mathbb{Z})$, and A_δ converges in distribution to A as $\delta\to 0$. Conditioned on A_δ, let $\mathcal{X}_\delta^{A_\delta}$ be the process of coalescing random walks on $(\delta\mathbb{Z}/\sigma)\times(\delta^2\mathbb{Z})$ starting from the point set A_δ. Then as $\delta\to 0$, $\mathcal{X}_\delta^{A_\delta}$ converges in distribution to $\mathcal{B}^A$, the process of coalescing Brownian motions starting from a random point set distributed as A.*

Proof. We first treat the case where A and A_δ are deterministic and $\rho_{\mathcal{P}}(A_\delta,A)\to 0$ as $\delta\to 0$. Note that $\{\mathcal{X}_\delta^{A_\delta}\}$ is tight since $\mathcal{X}_\delta^{A_\delta}$ is almost surely a subset of $\mathcal{X}_\delta$ and $\{\mathcal{X}_\delta\}$ is tight. If Z is a subsequential limit of $\mathcal{X}_\delta^{A_\delta}$, then by (I_1) and the remark following Corollary 3.1, there is μ_Z almost surely exactly one path starting from every $y \in A$, and the finite dimensional distributions of Z are those of coalescing Brownian motions. Therefore Z is equidistributed with $\mathcal{B}^A$, which proves the deterministic case.

For the nondeterministic case, it suffices to show $\mathbb{E}[f(\mathcal{X}_\delta^{A_\delta})] \to \mathbb{E}[f(\mathcal{B}^A)]$ as $\delta \to 0$ for any bounded continuous function f on $(\mathcal{H}, d_{\mathcal{H}})$. If we denote $f_\delta(A_\delta) = \mathbb{E}[f(\mathcal{X}_\delta^{A_\delta})|A_\delta]$, and $f_{\mathcal{B}}(A) = \mathbb{E}[f(\mathcal{B}^A)|A]$, then $\mathbb{E}[f(\mathcal{X}_\delta^{A_\delta})] = \mathbb{E}[f_\delta(A_\delta)]$ and $\mathbb{E}[f(\mathcal{B}^A)] = \mathbb{E}[f_{\mathcal{B}}(A)]$. Since A_δ converges in distribution to A, by Skorohod's representation theorem [4, 12], we can construct random variables A'_δ and A' which are equidistributed with A_δ and A, such that $A'_\delta(\omega) \to A'(\omega)$ in $\rho_{\mathcal{P}}$ almost surely. Then for almost every ω in the probability space where A'_δ and A' are defined, by the part of the proof already done (for deterministic A_δ and A), $\mathcal{X}_\delta^{A'_\delta(\omega)}$ converges in distribution to $\mathcal{B}^{A'(\omega)}$. Thus $f_\delta(A'_\delta(\omega)) = \mathbb{E}[f(\mathcal{X}_\delta^{A'_\delta(\omega)})] \to f_{\mathcal{B}}(A'(\omega)) = \mathbb{E}[f(\mathcal{B}^{A'(\omega)})]$ for almost every ω. By the bounded convergence theorem, $\mathbb{E}[f_\delta(A'_\delta)] \to \mathbb{E}[f_{\mathcal{B}}(A')]$ as $\delta \to 0$. Since A'_δ and A' are equidistributed with A_δ and A, the lemma follows.

Proof of Lemma 6.3. Let Z_{t_0} be the weak limit of $\{\mathcal{X}_{\delta_n}^{t_0^-}\}$ for a sequence of $\delta_n \downarrow 0$. By Skorohod's representation theorem, we can assume the convergence is almost sure. Then almost surely, $\rho_{\mathcal{P}}(\mathcal{X}_{\delta_n}^{t_0^-}(t_0+\epsilon), Z_{t_0}(t_0+\epsilon)) \to 0$, and $d_{\mathcal{H}}(\mathcal{X}_{\delta_n}^{t_0^-,(t_0+\epsilon)_T}, Z_{t_0}^{(t_0+\epsilon)_T}) \to 0$. Let $m_\delta = \delta^2\lceil \tilde{t}_0 + \tilde{\epsilon}\rceil$, the first time on the rescaled lattice greater than or equal to $t_0+\epsilon$. Using the fact that the image of $Z_{t_0}^{(t_0+\epsilon)_T}$ under (Φ, Ψ) is almost surely equicontinuous, it is not difficult to see that $\rho_{\mathcal{P}}(\mathcal{X}_{\delta_n}^{t_0^-}(m_{\delta_n}), Z_{t_0}(t_0+\epsilon)) \to 0$ and $d_{\mathcal{H}}(\mathcal{X}_{\delta_n}^{t_0^-,(m_{\delta_n})_T}, Z_{t_0}^{(t_0+\epsilon)_T}) \to 0$ almost surely. On the other hand, $Z_{t_0}(t_0+\epsilon)$ is almost surely locally finite by Lemma 6.2, and $\mathcal{X}_{\delta_n}^{t_0^-,(m_{\delta_n})_T}$ is distributed as coalescing random walks on the rescaled lattice starting from $\mathcal{X}_{\delta_n}^{t_0^-}(m_{\delta_n}) \subset (\delta\mathbb{Z}/\sigma) \times (\delta^2\mathbb{Z})$. Therefore by Lemma 6.5, $\mathcal{X}_{\delta_n}^{t_0^-,(m_{\delta_n})_T}$ converges weakly to $\mathcal{B}^{Z_{t_0}(t_0+\epsilon)}$, and $Z_{t_0}^{(t_0+\epsilon)_T}$ is equally distributed with $\mathcal{B}^{Z_{t_0}(t_0+\epsilon)}$. This concludes the proof of (E'_1) and Theorem 1.5.

Remark 6.1 *The key to the proof of Lemma 6.3 is to approximate* $\mathcal{X}_{\delta_n}^{t_0^-,(t_0+\epsilon)_T}$ *by a Markov process with random initial conditions such that Lemma 6.5 can be applied. For discrete time coalescing random walks, the natural choice is* $\mathcal{X}_{\delta_n}^{t_0^-,(m_{\delta_n})_T}$*, while for continuous time, as we will discuss at the end of the* *next section, the natural choice is to take the piecewise constant version of* $\mathcal{X}_{\delta_n}^{t_0^-,(t_0+\epsilon)_T}$.

7 Further Results

Theorem 7.1 *If the random walk increment satisfies* $\mathbb{E}[|Y|^5] < +\infty$*, then* $\mathcal{X}_\delta^{0_T}$*, the set of coalescing random walk paths on the rescaled lattice starting from* $\delta\mathbb{Z}/\sigma$ *at time 0, converges in distribution to* $\bar{\mathcal{W}}^0$*, the subset of paths in* $\bar{\mathcal{W}}$ *starting at time 0.*

Proof. Note that for any countable dense subset $\mathcal{D}^0 \subset \mathbb{R} \times \{0\}$, $\bar{\mathcal{W}}(\mathcal{D}^0)$, the closure in (Π, d) of coalescing Brownian motion paths starting from $\mathcal{D}^0$ is equidistributed with $\bar{\mathcal{W}}^0$ by properties of the Brownian Web [16]. As in the case of the convergence of $\mathcal{X}_\delta$ to $\bar{\mathcal{W}}$, we need to show (T_1),(I_1), (B_1') and (E_1), where in (I_1), the countable dense set $\mathcal{D} \subset \mathbb{R}^2$ is now replaced by $\mathcal{D}^0$, and in (B_1') and (E_1), t_0 is set to 0. All these conditions have been verified in the preceding sections.

Remark 7.2 *The assumption* $\mathbb{E}[|Y|^5] < +\infty$ *can be weakened to* $\mathbb{E}[|Y|^2] < +\infty$*. All we need to check is tightness. A version of the tightness criterion* (T_1) *for* $\{\mathcal{X}_\delta^{0_T}\}$ *can be established using a very recent result of Belhaouari and Mountford [7], which improves a result of Cox and Durrett [9] on voter model interfaces from a finite 3rd moment assumption to a finite 2nd moment assumption.*

A direct consequence of Theorem 7.1 is the following:

Convergence of $\mathcal{X}_\delta^{0_T}(1)$: In [1], Arratia proved that, for coalescing simple random walks, $\mathcal{X}_\delta^{0_T}(1)$ as a point process on $\mathbb{R}$ converges in distribution to $\bar{\mathcal{W}}^0(1)$, which is a stationary simple point process with intensity $1/\sqrt{\pi}$. In [2], Arratia stated the analogous result for nonsimple walks with zero mean and finite variance for its increment, but a proof was not given. In this section, we give a proof for the case of random walks with mean zero and finite fifth moment for its increment (by the last remark, also valid under a finite third moment assumption), which follows as a corollary of Theorem 7.1.

Let $(\hat{\mathcal{N}}, \mathcal{B}_{\hat{\mathcal{N}}})$ be the space of locally finite counting measures on $\mathbb{R}$, where $\mathcal{B}_{\hat{\mathcal{N}}}$ is the Borel σ-algebra generated by the vague topology on $\hat{\mathcal{N}}$, (for more background on random measures and the vague topology, see [10, 20]). The vague topology on $\hat{\mathcal{N}}$ can be metrized so that $(\hat{\mathcal{N}}, \mathcal{B}_{\hat{\mathcal{N}}})$ is a complete separable metric space.

Theorem 7.3 *If $\mathbb{E}[|Y|^5] < +\infty$, then $\mathcal{X}_\delta^{0_T}(1)$ as an $(\hat{\mathcal{N}}, \mathcal{B}_{\hat{\mathcal{N}}})$-valued random variable converges weakly to $\bar{\mathcal{W}}^0(1)$ as $\delta \to 0$.*

Proof. Since $\bar{\mathcal{W}}^0(1)$ is a simple point process, to prove weak convergence of $\mathcal{X}_\delta^{0_T}(1)$ to $\bar{\mathcal{W}}^0(1)$, it is sufficient to show: (i) tightness; (ii) any subsequential limit of $\mathcal{X}_\delta^{0_T}(1)$ is a simple point process; (iii) convergence of the avoidance (zero) functions, i.e., for any disjoint union of a finite number of finite intervals $A = \bigcup_{i=1}^n [a_i, b_i]$, $\mathbb{P}[\mathcal{X}_\delta^{0_T}(1) \cap A = \emptyset] \to \mathbb{P}[\bar{\mathcal{W}}^0(1) \cap A = \emptyset]$ as $\delta \to 0$. (See, e.g., Sections 7.3 and 9.1 in [10]).

To prove that $\{\mathcal{X}_\delta^{0_T}(1)\}$ is tight, it is sufficient to show that for any finite closed interval $[a,b]$, $\sup_{0<\delta<1} \mathbb{E}_{\mathcal{X}_\delta^{0_T}(1)}\big[\zeta[a,b]\big] < C$, where $\zeta[a,b]$ is the measure of $[a,b]$ with respect to an element $\zeta \in \hat{\mathcal{N}}$, and $C < +\infty$ is a constant depending on $[a,b]$. Since $\mathbb{E}_{\mathcal{X}_\delta^{0_T}(1)}[\zeta[a,b]] < \mathbb{E}[\hat{\eta}_{\mathcal{X}_\delta}(0,1;a-1,b+1)]$, tightness follows from Lemma 6.4.

Let Z be a weak limit of $\{\mathcal{X}_{\delta_n}^{0_T}(1)\}$ in $(\hat{\mathcal{N}}, \mathcal{B}_{\hat{\mathcal{N}}})$ with distribution μ_Z. For any $\zeta \in \hat{\mathcal{N}}$, let $\zeta_{[m,n]}$ denote ζ restricted to $[m,n]$. Also let $A^m_{N,i} = [m+(i-1)2^{-N}, m+i2^{-N}]$. Then for all $N \in \mathbb{N}$, $m < n \in \mathbb{Z}$,

$$
\begin{aligned}
\mu_Z\{\zeta \mid \zeta_{[m,n]} \text{ not simple }\} &\leq \mu_Z\Big(\bigcup_{i=1}^{(m-n)2^N} \{\zeta(A^m_{N,i}) \geq 2\}\Big) \\
&\leq (m-n)2^N \sup_{a\in[m,n]} \mu_Z(\zeta[a, a+2^{-N}] \geq 2). \quad (7.1)
\end{aligned}
$$

To prove Z is a simple point process, it is then sufficient to show that

$$
\limsup_{\epsilon \to 0^+} \ \frac{1}{\epsilon} \sup_{a\in\mathbb{R}} \mu_Z\big[\zeta[a, a+\epsilon] \geq 2\big] = 0, \qquad (7.2)
$$

since this implies that $\zeta_{[m,n]}$ is μ_Z almost surely a simple counting measure by taking $N \to +\infty$ in (7.1). Letting $m \to +\infty$ and $n \to -\infty$ then implies that Z is almost surely a simple counting measure.

Note that $\mu_Z[\zeta[a, a+\epsilon] \geq 2] \ \leq \ \mu_Z[\zeta(a-\epsilon, a+2\epsilon) \geq 2]$, and $\{\zeta | \zeta(a-\epsilon, a+2\epsilon) \geq 2\}$ is an open set in $(\hat{\mathcal{N}}, \mathcal{B}_{\hat{\mathcal{N}}})$. By the weak convergence of $\mathcal{X}_{\delta_n}^{0_T}(1)$ to Z, and recalling that ξ_t^Z denotes the positions at time t of coalescing random walks starting from $\mathbb{Z}$ at time 0, we have that

$$\begin{aligned}
& \mu_Z[\zeta(a-\epsilon, a+2\epsilon) \geq 2] \\
\leq\ & \liminf_{\delta_n \downarrow 0} \mu_{\mathcal{X}^{0_T}_{\delta_n}(1)}[\zeta(a-\epsilon, a+2\epsilon) \geq 2] \\
=\ & \liminf_{\delta_n \downarrow 0} \mu_1(|\xi^{\mathbb{Z}}_{\delta_n^{-2}} \cap (\tilde{a}-\tilde{\epsilon}, \tilde{a}+2\tilde{\epsilon})| \geq 2) \\
\leq\ & \liminf_{\delta_n \downarrow 0} \sum_{i,j=\lceil \tilde{a}-\tilde{\epsilon}\rceil, i\neq j}^{\lfloor \tilde{a}+2\tilde{\epsilon}\rfloor} \mu_1(i,j \in \xi^{\mathbb{Z}}_{\delta_n^{-2}}) \\
\leq\ & \liminf_{\delta_n \downarrow 0} \sum_{i,j=\lceil \tilde{a}-\tilde{\epsilon}\rceil, i\neq j}^{\lfloor \tilde{a}+2\tilde{\epsilon}\rfloor} \mu_1(i \in \xi^{\mathbb{Z}}_{\delta_n^{-2}})\mu_1(j \in \xi^{\mathbb{Z}}_{\delta_n^{-2}}) \qquad (7.3) \\
\leq\ & \liminf_{\delta_n \downarrow 0} (3\tilde{\epsilon}+1)^2 \mu_1[0 \in \xi^{\mathbb{Z}}_{\delta_n^{-2}}]^2 \ \leq\ 9\, C^2\sigma^2\epsilon^2, \qquad (7.4)
\end{aligned}$$

where in (7.3), we applied Lemma 2.8, and in (7.4), we applied Lemma 2.7. To apply Lemma 2.8, we have implicitly assumed $\delta_n^{-2} \in \mathbb{N}$. If $\delta_n^{-2} \notin \mathbb{N}$, then we need to approximate and use an argument similar to the one leading to the computation in (3.2). This establishes (7.2), thus proving any subsequential limit of $\mathcal{X}^{0_T}_{\delta_n}(1)$ must be a simple point process.

We now show the convergence of the avoidance functions. Let $A = \bigcup_{i=1}^n [a_i, b_i]$ be the disjoint union of a finite number of finite intervals. By Theorem 7.1, $\mathcal{X}^{0_T}_{\delta}$ converges weakly to $\bar{\mathcal{W}}^0$ as $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables, so by Skorohod's representation theorem, we can define random variables $\mathcal{X}^{'0_T}_{\delta}$ and $\bar{\mathcal{W}}'^0$ equally distributed with $\mathcal{X}^{0_T}_{\delta}$ and $\bar{\mathcal{W}}^0$ such that $\mathcal{X}^{'0_T}_{\delta}$ converges almost surely to $\bar{\mathcal{W}}'^0$ in $(\mathcal{H}, d_{\mathcal{H}})$. In particular, $\mathcal{X}^{'0_T}_{\delta}(1)$ converges almost surely to $\bar{\mathcal{W}}'^0(1)$ in $\rho_{\mathcal{P}}$ as defined in (6.3). Since $\bar{\mathcal{W}}'^0(1)$ is a stationary simple point process with intensity $1/\sqrt{\pi}$, $\mathbb{P}[\partial A \cap \bar{\mathcal{W}}'^0(1) \neq \emptyset] = 0$. It is then easy to see that $\mathbf{1}_{\mathcal{X}^{'0_T}_{\delta}(1)\cap A=\emptyset} \to \mathbf{1}_{\bar{\mathcal{W}}'^0(1)\cap A=\emptyset}$ almost surely. By the bounded convergence theorem, $\lim_{\delta\downarrow 0} \mathbb{P}(\mathcal{X}^{'0_T}_{\delta}(1) \cap A = \emptyset) = \mathbb{P}(\bar{\mathcal{W}}'^0(1) \cap A = \emptyset)$. Since $X^{'0_T}_{\delta}(1)$ and $\bar{\mathcal{W}}'^0(1)$ are equidistributed with $\mathcal{X}^{0_T}_{\delta}(1)$ and $\bar{\mathcal{W}}^0(1)$, this proves the convergence of avoidance functions and the theorem.

Corollary 7.1 *Let $\xi^{\mathbb{Z}}_n$ be the process of coalescing random walks starting from $\mathbb{Z}$ at time 0 where all random walk increments are distributed as Y with $\mathbb{E}[|Y|^5] < +\infty$. Then $\mathbb{P}(0 \in \xi^{\mathbb{Z}}_n) \sim 1/(\sigma\sqrt{\pi n})$ as $n \to +\infty$.*

Proof. Let $\delta_n = 1/\sqrt{n}$, and denote $p_n = \mathbb{P}(0 \in \xi^{\mathbb{Z}}_n)$. Let $f_\epsilon(x)$, for $\epsilon > 0$, be a continuous function with support on $[-\epsilon, 1+\epsilon]$, with $0 \leq f_\epsilon \leq 1$, and

$f_\epsilon \equiv 1$ on $[0,1]$. By Theorem 7.3, $\lim_{n\to+\infty} \mathbb{E}_{\mathcal{X}^{0T}_{\delta_n}(1)}[\int f_\epsilon d\zeta] = \mathbb{E}_{\bar{\mathcal{W}}^0(1)}[\int f_\epsilon d\zeta]$. Since

$$\frac{1}{\sqrt{\pi}} < \mathbb{E}_{\bar{\mathcal{W}}^0(1)}[\int f_\epsilon d\zeta] < \frac{(1+2\epsilon)}{\sqrt{\pi}},$$

and

$$\sigma\sqrt{n}p_n \leq \mathbb{E}_{\mathcal{X}^{0T}_{\delta_n}(1)}[\int f_\epsilon d\zeta] \leq (1+2\epsilon)\sigma\sqrt{n}p_n,$$

it follows that

$$\frac{1}{(1+2\epsilon)\sqrt{\pi}} \leq \liminf_{n\to+\infty} \sigma\sqrt{n}p_n \leq \limsup_{n\to+\infty} \sigma\sqrt{n}p_n \leq \frac{(1+2\epsilon)}{\sqrt{\pi}}.$$

Since $\epsilon > 0$ is arbitrary, letting $\epsilon \to 0$ establishes the corollary.

Let ϕ^0_n be a discrete time one-dimensional voter model with state space $\{0,1\}^{\mathbb{Z}}$ and initial configuration $\phi^0_0(x) = 0$ for $x \in \mathbb{Z}\backslash\{0\}$, and $\phi^0_0(0) = 1$. The dynamics of the model is described in the proof of Lemma 2.8. Then by the duality between voter models and coalescing random walks (see, e.g.,[22]), $\mathbb{P}(\phi^0_n \not\equiv 0) = \mathbb{P}(0 \in \xi^{\mathbb{Z}}_n)$. Corollary 7.1 is then equivalent to

Corollary 7.2 *Let ϕ^0_n be the voter model defined above. If $\mathbb{E}[|Y|^5] < +\infty$, then $\mathbb{P}(\phi^0_n \not\equiv 0) \sim 1/(\sigma\sqrt{\pi n})$ as $n \to +\infty$.*

Remark 7.4 *Corollaries 7.1 and 7.2 partially extend a result of Bramson and Griffeath [5]. They proved that, for continuous time coalescing* **simple** *random walks in $\mathbb{Z}^d$, $\xi^{\mathbb{Z}^d}_t$, and the dual voter model $\phi^{0,d}_t$ with initial configuration all $0'$s except for a 1 at the origin, $p_t = \mathbb{P}(0 \in \xi^{\mathbb{Z}^d}_t) = \mathbb{P}(\phi^{0,d}_t \not\equiv 0)$ decays asymptotically as $1/(\sqrt{\pi t})$ for $d = 1$, $\log t/(\pi t)$ for $d = 2$, and $1/(\gamma_d t)$ for $d \geq 3$. For $d \geq 2$, their proof also works for discrete time random walks, and as pointed out in the remark before Lemma 2 in [6], can be easily extended to much more general random walks (see [6] for more details).*

Remark 7.5 *The following correlation inequality is valid for the point process $\bar{\mathcal{W}}^0(1)$. Let A, B be two disjoint open sets in $\mathbb{R}$, and let $O_A = \{\zeta \in \hat{\mathcal{N}} \,|\, \zeta(A) \geq 1\}$ and $O_B = \{\zeta \in \hat{\mathcal{N}} \,|\, \zeta(B) \geq 1\}$. Then*

$$\mu_{\bar{\mathcal{W}}^0(1)}(O_A \cap O_B) \leq \mu_{\bar{\mathcal{W}}^0(1)}(O_A)\ \mu_{\bar{\mathcal{W}}^0(1)}(O_B).$$

This negative correlation inequality for $\bar{\mathcal{W}}^0(1)$ is implicit in the work of Arratia [3]; it is also a direct consequence of Lemma 2.8 and Theorem 7.3. By similar arguments, the same correlation inequality holds for point processes generated at time 1 by coalescing Brownian motion paths starting from any closed space-time region strictly below time 1.

Voter Model Interface: Let $\phi_n^{\mathbb{Z}^-}$ be a voter model defined like near the end of Section 1, with state space $\{0,1\}^{\mathbb{Z}}$ and initial configuration $\phi_0^{\mathbb{Z}^-}(x) \equiv 1$ for $x \in \mathbb{Z}^-$ and $\phi_0^{\mathbb{Z}^-}(x) \equiv 0$ for $x \in \mathbb{Z}^+ \cup \{0\}$. At time $n \in \mathbb{N}$, $\phi_n^{\mathbb{Z}^-}$ will contain a leftmost 0 and a rightmost 1, whose positions we denote by l_n and r_n. Then $\phi_n^{\mathbb{Z}^-}(x) = 1$ for $x < l_n$, $\phi_n^{\mathbb{Z}^-}(x) = 0$ for $x > r_n$, and the configuration of $\phi_n^{\mathbb{Z}^-}$ between l_n and r_n defines what is called the interface process, $\alpha_n = \phi_n^{\mathbb{Z}^-}(x + l_n), 1 \leq x$, which is a random variable taking values in $\left\{\xi : \mathbb{Z}^+ \to \{0,1\}, \sum_{x\in\mathbb{Z}^+} \xi(x) < +\infty\right\}$. In [9], Cox and Durrett proved that, for the continuous time analogue $\tilde{\phi}_t^{\mathbb{Z}^-}$ under a finite third moment assumption on Y, the interface process α_t is an irreducible positive recurrent Markov chain. Hence the size of the interface $r_t - l_t$ is of $O(1)$ as $t \to +\infty$. They also proved that $l_t/(\sigma\sqrt{t})$ and $r_t/(\sigma\sqrt{t})$ converge in distribution to standard Gaussian variables as $t \to +\infty$. Their result should also be valid for the discrete time model $\phi_n^{Z^-}$.

The weak convergence of $\mathcal{X}_\delta$ to the Brownian Web $\bar{\mathcal{W}}$ recovers Cox and Durrett's result under the stronger assumption that $\mathbb{E}[|Y|^5] < +\infty$, but it also establishes that the time evolutions of l_n and r_n converge weakly to the same Brownian motion. In the following discussion, we will let l_t, r_t, for $t \geq 0$, denote the continuous paths constructed from $l_n, r_n, n \in \mathbb{Z}^+ \cup \{0\}$ by linear interpolation.

Theorem 7.6 *Let $\phi_n^{\mathbb{Z}^-}$, l_t and r_t be as defined above, and let $l_{t,\delta} = \delta\sigma^{-1} l_{t\delta^{-2}}$ and $r_{t,\delta} = \delta\sigma^{-1} r_{t\delta^{-2}}$. If $\mathbb{E}[|Y|^5] < +\infty$, then $\{(l_{t,\delta}, 0), (r_{t,\delta}, 0)\}$ as $(\mathcal{H}, d_{\mathcal{H}})$ valued random variables converge in distribution to $(\mathcal{B}_t^0, 0)$, a standard Brownian motion starting at the origin at time 0.*

Proof. Let $\hat{\mathcal{X}}_\delta = \{(f(-t), t \leq -t_0) \mid (f(t), t \geq t_0) \in \mathcal{X}_\delta\}$, i.e. the coalescing random walks running backward in time on the rescaled lattice. By the duality between voter models and coalescing random walks (see, e.g., [22]), there is a natural coupling between $\hat{\mathcal{X}}_1$ and $\phi_n^{\mathbb{Z}^-}$ such that $\forall\, x \in \mathbb{Z}, n \geq 0$, $\phi_n^{\mathbb{Z}^-}(x) = \phi_0^{\mathbb{Z}^-}(\hat{\pi}_0^{x,n})$, where $\hat{\pi}_0^{x,n}$ is the location at time 0 of the backward random walk path in $\hat{\mathcal{X}}_1$ starting at x at time n.

By Theorem 1.5, $\hat{\mathcal{X}}_\delta$ converges weakly to $\hat{\bar{\mathcal{W}}}$ (the backward Brownian Web) as $\delta \to 0$; and by Skorohod's representation theorem, we may assume this convergence is almost sure. $\hat{\bar{\mathcal{W}}}$ uniquely determines a dual (forward) Brownian web $\bar{\mathcal{W}}$, which is equally distributed with the standard Brownian web. The pair $(\bar{\mathcal{W}}, \hat{\bar{\mathcal{W}}})$ forms what is called the *double Brownian web* $\bar{\mathcal{W}}^D$ with the property that, almost surely, paths in $\bar{\mathcal{W}}$ and $\hat{\bar{\mathcal{W}}}$ *reflect* off each other and never cross [16, 27, 25, 2].

Let $\phi^{\mathbb{Z}^-}_{t,\delta}$ be the diffusively rescaled voter model dual to $\hat{\mathcal{X}}_\delta$ with the natural coupling between the two models. We will overload the notation and let $l_{t,\delta}$ and $r_{t,\delta}$ also denote the corresponding interface boundary lines. To prove the theorem, it is then sufficient to show that almost surely, $l_{t,\delta}$ and $r_{t,\delta}$ converge in (Π, d) to $\pi^{0,0}_t \in \bar{\mathcal{W}}$, the unique path in the forward Brownian web starting at 0 at time 0, which is distributed as a standard Brownian motion.

For a fixed point ω in the probability space of $\bar{\mathcal{W}}^D$, if $l_{t,\delta}$ does not converge to $\pi^{0,0}_t$ in (Π, d), then there exists $\epsilon_\omega > 0$ and a sequence $\delta_n \downarrow 0$, such that $d(l_{t,\delta_n}, \pi^{0,0}_t) > \epsilon_\omega$. In particular, for δ_n sufficiently small, there exists $(x_n, t_n) \in (\delta_n/\sigma)\mathbb{Z} \times \delta_n^2\mathbb{Z}$ such that $\sup_n t_n < +\infty$, $l_{t_n,\delta_n} = x_n$ and $|x_n - \pi^{0,0}_{t_n}| > \epsilon_\omega/2$. Since x_n is the position of the leftmost zero at time t_n for the rescaled voter model $\phi^{\mathbb{Z}^-}_{t,\delta_n}$, by duality, the backward random walk paths in $\hat{\mathcal{X}}_{\delta_n}$ starting at x_n and $x_n - \delta_n/\sigma$ at time t_n satisfy $\hat{\pi}^{x_n,t_n}_{0,\delta_n} \geq 0$ and $\hat{\pi}^{x_n-\delta_n/\sigma,t_n}_{0,\delta_n} < 0$. But by the almost sure convergence of $\hat{\mathcal{X}}_{\delta_n}$ to $\hat{\bar{\mathcal{W}}}$, there exists a subsequence $\delta_{n'}$, such that $\hat{\pi}^{x_{n'},t_{n'}}_{t,\delta_{n'}}$ and $\hat{\pi}^{x_{n'}-\delta_{n'}/\sigma,t_{n'}}_{t,\delta_{n'}}$ converge respectively to $\hat{\pi}^{x_0,t_0}_t$ and $\hat{\pi}^{x_0^-,t_0}_t \in \hat{\bar{\mathcal{W}}}$ for some (x_0, t_0), two paths in $\hat{\bar{\mathcal{W}}}$ both starting at (x_0, t_0) with $0 \leq t_0 < +\infty$, $|x_0 - \pi^{0,0}_{t_0}| > \epsilon_\omega/2$, $\hat{\pi}^{x_0,t_0}_0 \geq 0$ and $\hat{\pi}^{x_0^-,t_0}_0 \leq 0$. (Note that if $x_0 = \pm\infty$, then the only path that could start from (x_0, t_0) is (f, t_0) with $f \equiv +\infty$ or $f \equiv -\infty$.) By the non-crossing property of the double Brownian web, $\hat{\pi}^{x_0,t_0}_t$ and $\hat{\pi}^{x_0^-,t_0}_t$ cannot cross $\pi^{0,0}_t$, hence either $\hat{\pi}^{x_0,t_0}_0 = 0$ or $\hat{\pi}^{x_0^-,t_0}_0 = 0$. Therefore

$$\{l_{t,\delta} \not\to \pi^{0,0}_t\} \subset \{\,\exists\, \hat{\pi}^{x_0,t_0}_t \in \hat{\bar{\mathcal{W}}} \text{ with } 0 \leq t_0 < +\infty \text{ and } \hat{\pi}^{x_0,t_0}_0 = 0\}.$$

The second event in this inclusion has probability zero for the double Brownian web; therefore almost surely, $l_{t,\delta} \to \pi^{0,0}_t$ as $\delta \to 0$. The same is true for $r_{t,\delta}$, and this proves the theorem.

Remark 7.7 *In some sense, the convergence of $\mathcal{X}_\delta$ to the Brownian web is equivalent to the convergence of the voter model interface boundary to a* *standard Brownian motion. The proof of Theorem 7.6 shows the implication in the forward direction. Conversely, if the voter model interface boundary converges to a Brownian motion, then together with a finite third moment assumption on Y, one can show that $\{\mathcal{X}_\delta\}$ is a tight family, and hence converges to the Brownian web. Finite third moment is seen to be the least moment condition necessary for the convergence in both cases. For any $\epsilon > 0$, one can find choices of Y with $\mathbb{E}[|Y|^{3-\epsilon}] < \infty$ and $\mathbb{E}[|Y|^3] = \infty$, such that in the diffusive scaling limit the voter model interface boundary reaches ∞ instantly. Just as in the case of $\{\mathcal{X}_\delta\}$, the failure of convergence is due to the presence of many large jumps which destroys tightness.*

Remark 7.8 *Recently, Belhaouari et al. [8] proved Theorem 7.6 under a finite $3+\epsilon$ moment assumption on Y for any $\epsilon > 0$. Their tightness condition for the voter model interface boundary is in fact slightly stronger than the tightness criterion (T_1) for $\{\mathcal{X}_\delta\}$. Thus, their result improves the convergence of $\{\mathcal{X}_\delta\}$ to the Brownian web also to a finite $3+\epsilon$ moment assumption. Finite third moment for Y (with possible lower order corrections) is thus both necessary and sufficient for the convergence of $\mathcal{X}_\delta$ to the Brownian web and the voter model interface boundary to a Brownian motion.*

Continuous Time: We define the continuous time analog of $\mathcal{X}_\delta$, $\tilde{\mathcal{X}}_\delta$ as follows. A walker starts from every point on $\mathbb{Z} \times \mathbb{R}$ and undergoes rate 1 jumps with increments distributed as Y. The jump times (clock rings) are given by independent rate 1 Poisson point processes on $\{i\} \times \mathbb{R}$ for each integer i. Two walkers coalesce when they first meet, and clearly all walkers starting at the same site between two consecutive Poisson clock rings will have coalesced by the time of the second clock ring. If we call the time and location at which a Poisson event occurs a *jump point*, then we define the path of a given walker to be the polygonal path consisting of first a constant-position line segment connecting the point of birth to the first jump point, and then linear segments connecting consecutive jump points. For random walks born at a jump point, we will take two paths, one starting with a constant-position line segment, the other without. $\tilde{\mathcal{X}}_1$ is then defined to be random variable consisting of all the coalescing random walk paths, and $\tilde{\mathcal{X}}_\delta$ is $\tilde{\mathcal{X}}_1$ diffusively rescaled. $\hat{\mathcal{X}}_\delta$ also converges weakly to the Brownian web $\bar{\mathcal{W}}$ under the finite fifth moment assumption, and all the corollaries we have stated so far are also valid for continuous time. We will only briefly outline here the technical issues that need to be addressed. For more details, see [26].

To establish the weak convergence of $\tilde{\mathcal{X}}_\delta$ to $\bar{\mathcal{W}}$, we would need to show the almost sure precompactness of $\tilde{\mathcal{X}}_\delta$ in (Π, d) (Lemma 1.1), and verify conditions (B_1'), (T_1), (I_1) and (E_1). Two technical details stand out. First, by our definition of random walk paths as polygonal curves, the times at which two random walk paths intersect are not stopping times, and the distribution of the paths is no longer Markovian at all time. To circumvent this problem, we may take the piecewise constant version of the coalescing random walk paths, which is Markovian. The time when two paths first intersect or interchange ordering will be the stopping times we use, and then the random walk estimates in Section 2, the proof of (B_1'), and the stopping time arguments used in the proof of (T_1) all carry over to the continuous time case. The proof of (I_1) and (E_1) will require approximating the interpolated paths by the piecewise constant paths. The second technical detail is that, for continuous time random walks, the lattice is no longer equally spaced in time. Conditioned on the realization of the Poisson point processes on $\{i\} \times \mathbb{R}$ for all $i \in \mathbb{Z}$, we can regard the coalescing random walks as walks on a random space-time lattice, where the lattice sites are the *jump points*. The proofs of Lemma 1.1 and (T_1), which use the regularity of the lattice, can then be adapted for the continuous time case by suitably conditioning on the realization of the Poisson point processes.

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