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Degenerate Variance Control in the One-dimensional Stational Case

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Abstract: We study the problem of stationary control by adaptive choice of the diffusion coefficient in the case that control degeneracy is allowed and the drift admits a unique, asymptotically stable equilibrium point. We characterize the optimal value and obtain it as an Abelian limit of optimal discounted values and as a limiting average of finite horizon optimal values, and we also characterize the optimal stationary strategy. In the case of linear drift, the optimal stationary value is expressed in terms of the solution of an optimal stopping problem. We generalize the above results to allow unbounded cost functions.

Keywords and Phrases: Stochastic control, stationary control, degenerate variance control.

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1 Introduction

In this paper, we study a stationary control problem when the state process is a one dimensional diffusion whose drift admits a unique, asymptotically stable equilibrium point and which is controlled by adaptively choosing the variance of the noise. In particular, the controller may choose a *zero-variance*, or *degenerate*, control. The precise mathematical formulation of the model is specified in section 2 below in (2.1)-(2.4).

The goal is to express the stationary optimal value as an Abelian limit of discounted optimal values and also as an ergodic limit of finite horizon optimal values, and to characterize explicitly the optimal stationary strategy. When the drift is linear, we are also able to express the stationary value and the optimal strategy in terms of an optimal stopping problem for an Ornstein-Uhlenbeck process. The analysis is based in part on our study of the infinite-horizon discounted cost problem in the paper [17].

Work on stationary control of diffusions and its relation to discounted and finite-horizon problems is extensive. We mention only some representative papers. Mandl [16] treats the one dimensional stationary problems in bounded domains with non-degenerate diffusion. Among others, Kushner [13], Borkar and Ghosh [8] treat the multi-dimensional drift control problem with a non-degenerate diffusion coefficient, as do Beneš and Karatzas [4] and Cox and Karatzas [10]. Tarres [18] and Cox [9] study the problem of Abelian limits of discounted cost problems A detailed analysis of the stationary Hamilton-Jacobi-Bellman equation in $I\!R^n$ may be found in Bensoussan and Frehse [5]. Recently, Basak, Borkar, and Ghosh [3] and Borkar [7] consider a stationary problem with drift control and degenerate, but uncontrolled, diffusion coefficient. Further references to variance control for discounted and finite-horizon costs are cited in [17]; we mention also the papers of Kushner [14] and Kurtz and Stockbridge [12] for general results on existence and uniqueness of optimal controls.

The stationary control problem of this paper is in some sense a generalization of a problem posed and solved by Assaf [2] in an investigation of dynamic sampling. Assaf's problem features a state process evolving in $[-1, 1]$, which, when no control is exercised, tends toward the origin, a variance control which can take any value in $[0, \infty)$, a location cost which is largest at the origin, and a control cost. Assaf derives the value function and explicit ϵ-optimal controls. Formally, the optimal control consists of turning on the variance at full strength in an interval about the equilibrium point and otherwise setting the variance to zero. In our model, we restrict to bounded controls, but allow general drifts admitting a unique equilibrium point and consider general location costs. The motivating case again is that of a location cost $c(\cdot)$ in (2.5) which is non-negative and achieves its maximum at the equilibrium point, which we take to be the origin; examples are $c(x) = ke^{-x^2}$ or $c(x) = k/(x^2 + 1)$. A control cost, proportional to the magnitude of the square of the control is added, as in (2.5). With this cost structure, there are two opposing tendencies: if one applies zero variance control, no control cost accumulates, but the state process of (2.1) moves deterministically to the point of highest location cost; on the other hand, if one exercises positive control, there is a control cost, but the state process is spread out stochastically, thus interrupting its steady progression towards the point of highest location cost. The second effect will be of great advantage to the controller in the region around the origin where $c(\cdot)$ is concave. Where $c(\cdot)$ is reasonably flat or convex, the advantage may be outweighed by the control cost. This reasoning provides the intuition behind the form of Assaf's solution: For the above cost functions $c(\cdot)$, our results imply that the same form will be generally valid; we find a *bang-bang* optimal controls described by a feedback policy of the form $\pi^*(x) = \sigma_0 \mathbf{1}_{(p^*,q^*)}(x)$, where $p^* < 0 < q^*$ and $\mathbf{1}_{(p^*,q^*)}(x)$ denotes the indicator function of the interval (p^*, q^*). However, our analysis implies to more general cost functions $c(\cdot)$, see (2.8) and (2.9), in which neither boundedness or even symmetry are assumed. If $c(\cdot)$ is convex, our results yield that the zero control, i.e. $u(t) \equiv 0$ in (2.1), is optimal.

Arisawa and Lions [1] study an ergodic control problem in which both drift and diffusion are controlled and diffusion is allowed to be degenerate. They analyze the related Hamilton-Jacobi-Bellman equation and also establish Abelian limit relationships. In their case, the controlled diffusion takes values in a compact metric space. In contrast, the controlled diffusion process we study here takes values on the real line and the cost function may be unbounded. Also their assumption (9) in [1] fails in our case. In place of the compactness assumption in [1], we assume that the deterministic process obtained by exercising zero variance control admits a unique, asymptotically stable equilibrium point. Because of this, for any admissible control u, the controlled process X_x^u always reaches any small neighborhood of the origin in finite random time. The optimal stationary process for the stationary problem will then converge in distribution to a unique invariant measure. Under some assumptions, this invariant measure will have compact support.

In section 2, we carefully describe the admissible controls, state the basic assumptions, formulate the stationary, discounted and finite-horizon control problems, and present preliminary technical results. In section 3, we study the case of linear drift, $b(x) = -\theta x$, $\theta > 0$. We treat this case separately because it is possible to express the results and their proofs in terms of a simple optimal stopping problem. Section 4 presents a criterion for localization of the control and an application to more general cost functions. Finally, in section 5 we generalize our results to the case of non-linear drift. Throughout, the techniques employed come mostly from elementary differential equations or probabilistic analysis.

2 Problem Statement and Preliminaries

2.1 Problem statement

Throughout the paper, the controlled state process, X_x^u, is a solution of the stochastic differential equation,

$$X_x^u(t) = x + \int_0^t b(X_x^u(s))\, ds + \int_0^t u(s)\, dW(s), \tag{2.1}$$

where:
(i) the control u takes values in $[0, \sigma_0]$, where $\sigma_0 < \infty$ is fixed;
(ii) the drift b is a smooth function that satisfies

$$b(0) = 0 \qquad \text{and} \qquad b'(x) \leq -\delta_0 < 0 \quad \text{for all } x \neq 0. \tag{2.2}$$

Fron (2.2) it follows that $xb(x) < -\delta_0 x^2$ if $x \neq 0$. Assumption (2.2) fixes a unique, exponentially, asymptotically stable equilibrium point at zero for the differential equation, $\dot{X}_x^0(t) = b(X_x^0(t)),\ \ X_x^0(0) = x$, obtained from (2.1) using the null control, $u \equiv 0$.

As is standard, the class of admissible controls should allow choice of the underlying space and filtration on which (2.1) is defined. The quadruple $((\Omega, \mathcal{F}, P), \mathbb{F}, W, u)$ is called an *admissible control system* if: (i) $(\Omega, \mathcal{F}, P)$ is a complete probability space; (ii) $\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$ is a right-continuous, complete filtration; (iii) W is both a scalar, Brownian motion and an $\mathbb{F}$-martingale on $(\Omega, \mathcal{F}, P)$, and; (iv) u is an $\mathbb{F}$-progressively measurable process such that

$$0 \leq u(t) \leq \sigma_0 \qquad \text{for all } t \geq 0. \tag{2.3}$$

Let $\mathcal{U}$ denote the set of all such admissible systems.

A process $u(\cdot)$ from an admissible control system is called an *admissible control.* We shall abuse notation by writing $u \in \mathcal{U}$ to indicate that $u(\cdot)$ is an admissible control, without explicit mention of the admissible control system on which u is defined. Bear in mind that different admissible u may be defined on different probability spaces.

As defined, an admissible control appears to be chosen exogenously. However, it is easy to accommodate controls in feedback form. A function $\pi : \mathbb{R} \to [0, \sigma_0]$ is an *admissible feedback policy (or strategy)* if

$$X_x(t) = x + \int_0^t b(X_x(s))\, ds + \int_0^t \pi(X_x(s))\, dW(s) \tag{2.4}$$

admits at least a weak solution, that is, there is some filtered probability space with Brownian motion $((\Omega, \mathcal{F}, P), \mathbb{F}, W)$ on which (2.4) admits a solution and, on that space, $u(t) = \pi(X_x(t))$ is an admissible control. The optimal feeback controls identified in this paper will be of the form $\pi(x) = \sigma_0 \mathbf{1}_G(x)$, where G is an open set and $\mathbf{1}_G$ its indicator function. In [17], it is shown that such feedback controls are indeed admissible. Moreover, such controls have the following property, which is important in the analysis of the control problems. If $0 \in G$, the corresponding state process solving (2.4) eventually enters the connected component of G containing the equilibrium point $x = 0$ and then remains in that component for all future time; if $0 \notin G$, then $X_x(t) \to 0$ as $t \to \infty$. (See [17], section 2.)

Let $c : I\!R \to I\!R$ be a cost function on the location of the state. We shall study the stationary stochastic control problem with optimal value,

$$\lambda_0 \triangleq \inf_{\mathcal{U}} \liminf_{T\to\infty} \frac{1}{T} E\left[\int_0^T [c(X_x^u(t)) + u^2(t)]\, dt\right]. \tag{2.5}$$

Our purpose is to characterize a stationary policy that achieves the value λ_0 and to relate the stationary control problem to the family of discounted control problems,

$$V_\alpha(x) \triangleq \inf_{\mathcal{U}} E\left[\int_0^\infty e^{-\alpha t} \left(c(X_x^u(t)) + u^2(t)\right) dt\right], \qquad \alpha > 0, \tag{2.6}$$

as well as to the family of finite horizon problems,

$$V_0(x,T) \triangleq \inf_{\mathcal{U}} E\left[\int_0^T [c(X_x^u(t)) + u^2(t)]\, dt\right], \qquad T > 0. \tag{2.7}$$

Throughout the paper, we assume the following:

$$c \in C^2(I\!R) \ \text{ and } \ c \ \text{ is non-negative.} \tag{2.8}$$

The main results are derived under the additional assumption that for some finite constant K and non-negative integer m,

$$|c''(x)| \le K\left(1 + |x|^{2m}\right) \qquad \text{for all } x. \tag{2.9}$$

Conditions (2.9) and (2.8), imply there is a constant $\overline{K}$ and a generic, non-negative integer m (possibly different from that in (2.9)), such that

$$c(x) + |c'(x)| + |c''(x)| \le \overline{K}(1 + |x|^{2m}) \qquad \text{for all } x. \tag{2.10}$$

Since adding a constant to the cost c only shifts the value function by a constant in each of the problems (2.5), (2.6), and (2.7), the results of the paper apply immediately if $c \in C^2(I\!R)$ is just bounded below, rather than non-negative; non-negativity is imposed only for convenience of exposition.

The continuity and non-negativity of c, together with the assumption (2.2) on the drift, easily imply finiteness of the value functions in (2.5), (2.6), and (2.7). The C^2 smoothness of c will be utilized in the analysis of the HJB equations characterizing the value functions.

The stationary optimal value λ_0 is independent of the initial state x. Indeed, for any x and any y, there exists an admissible control u such that $X_x^u(t)$ hits y in an almost-surely finite time. For example, the state process $X_x(t) = x + \int_0^t b(X_x(s))\,ds + \int_0^t \sigma_0\,dW(s)$, corresponding to a constant control, is recurrent, because of assumption (2.2), and will reach any y a.s. The cost of using control to get from x to y will disappear in the limit as $T \to \infty$ in (2.5), implying λ_0 is independent of x. Arguments employed later in the paper in the analysis of (2.5)–(2.7) will also imply this non-dependence.

2.2 Preliminary analysis

The discounted variance control problem (2.6) is studied in [17]. In particular, the following variational characterization of V_α is a consequence of the general theories of Krylov [11] and Lions [15].

Theorem 2.1 *Assume (2.8) and assume $b \in C^3$ satisfies (2.2). Then the following holds.*
a) $V_\alpha \in C^1(\mathbb{R})$ and V_α' is absolutely continuous with a locally essentially bounded derivative V_α''. V_α solves the Hamilton-Jacobi-Bellman equation

$$\inf_{u\in[0,\sigma_0]} \frac{u^2}{2}\left(V_\alpha''(x)+2\right) + b(x)V_\alpha'(x) - \alpha V_\alpha(x) + c(x) = 0, \quad \textit{for a.e. } x. \tag{2.11}$$

b) Set $G_\alpha = \{x; b(x)V_\alpha'(x) - \alpha V_\alpha(x) + c(x) > 0\}$. Then $\pi_\alpha(x) \stackrel{\triangle}{=} \sigma_0 \mathbf{1}_{G_\alpha}(x)$ is an optimal feedback policy, that is, $V_\alpha(x) = E\left[\int_0^\infty e^{-\alpha t}\left(c(X_x^*(t)) + (u^*(t))^2\right)dt\right]$, where X_x^* solves (2.4), with π_α in place of π, and where $u^*(t) = \pi_\alpha(X_x^*(t))$.

Actually this theorem was proved in section 3 of [17] under the additional assumption that $c \in C_b^2(\mathbb{R})$, that is, that c and its first and second derivatives are uniformly bounded. However, it is true under the more general assumption (2.8) by a localization argument, which is sketched in the appendix. The smoothness condition on b is a technical assumption imposed so that the solution of (2.1) is C^2 in x.

In the remainder of this section, we state some consequences of assumption (2.2) which will be important for the asymptotic analysis of the discounted and finite horizon problems. The first is a uniform moment bound on controlled processes.

Lemma 2.2 *For $u \in \mathcal{U}$, let X_x^u, $x \in \mathbb{R}$ solve (2.1), where b and δ_0 are as in (2.2).*
(i) For any x and y, $|X_x^u(t) - X_y^u(t)| \le |x-y|e^{-\delta_0 t}$.

(ii) Introduce the sequence $\{D_j(x)\}$ defined recursively by $D_1 = \sigma_0^2/2\delta_0$, and $D_{j+1}(x) = ((j+1)D_j(x) + |x|^{2j}(j+1)^2)\,\sigma_0^2/\delta_0$. Then for every positive integer n

$$E\left[|\; X_x^u(t)\; |^{2n}\right] \leq \left(|x|^{2n} + D_n(x)e^{2\delta_0 nt}\right)e^{-2\delta_0 nt}. \tag{2.12}$$

Proof: To prove part (i), notice that if $x > y$ then $X_x^u(t) \geq X_y^u(t)$ for all $t \geq 0$. The result then follows by applying Itô's rule to $e^{\delta_0 t}\left(X_x^u(t) - X_y^u(t)\right)$. $\diamond$

The proof of (ii) is by induction. For the induction step, one applies Itô's rule to $e^{2n\delta_0 t}|X_x^u(t)|^{2n}$ and uses the fact that $xb(x) \leq -\delta_0 x^2$ for all x.

The lemma implies that for every positive integer n, there exists a finite K_n such that

$$\sup_{u\in\mathcal{U}}\sup_{t\geq 0} E\left[|\; X_x^u(t)\; |^{2n}\right] \leq K_n(1 + |x|^{2n}), \tag{2.13}$$

and this will suffice for our purposes.

The second result states some apriori bounds and convergence results.

Lemma 2.3 *Assume (2.2), (2.8), and (2.9). Then*
(i) There is a finite $\tilde{K}$ independent of α such that

$$|V_\alpha(x) - V_\alpha(y)| \leq \tilde{K}\frac{|x-y|}{2\delta_0}\left(1 + |x|^{2m} + |y|^{2m}\right) \tag{2.14}$$

(ii) $\displaystyle\lim_{\alpha\to 0+}\sup_{|x|\leq M} |\; \alpha V_\alpha(x) - \alpha V_\alpha(0)\; | = 0,$ *for any* $0 < M < \infty$.

(iii) $\displaystyle\Gamma \triangleq \sup_{u\in\mathcal{U},T\geq 0,\alpha>0} E\left[|V_\alpha(X_0^u(T)) - V_\alpha(0)\;|\right] < \infty.$

(iv) $\displaystyle\lim_{T\to\infty}\sup_{|x|\leq M}\frac{|V_0(x,T) - V_0(0,T)|}{T} = 0,$ *for any* $0 < M < \infty$.

Proof: (i) We prove (2.14) when $y = 0$, the general case being similar. From (2.10) it follows that

$$|c(x) - c(y)| \leq \overline{K}|x-y|\left(1 + |x|^{2m} + |y|^{2m}\right). \tag{2.15}$$

Let u be an ϵ-optimal control for $V_\alpha(0)$. Then,

$$V_\alpha(x) - V_\alpha(0) \leq \epsilon + E\left[\int_0^\infty e^{-\alpha t}\left(c(X_x^u(t)) - c(X_0^u(t))\right)dt\right].$$

Using (2.15) together with Lemma 2.2, one then obtains

$$V_\alpha(x) - V_\alpha(0) \leq \epsilon + \frac{|x|}{\alpha+2\delta_0}\overline{K}\left[1 + K_m(2 + |x|^{2m})\right].$$

Taking $\epsilon \downarrow 0$, $V_\alpha(x) - V_\alpha(0) \leq \frac{|x|}{\delta_0}\overline{K}\left[1 + K_m(2 + |x|^{2m})\right]$, for any $\alpha > 0$. A similar argument gives the same estimate for $V_\alpha(0) - V_\alpha(x)$, proving (i).

The proof of (ii) is an immediate Corollary of (i). To prove (iii), use the bound of (i) together with Lemma 2.2, part (ii) and Hölder's inequality. The proof of part (iv) proceeds by deriving a uniform bound similar to that of part (i) for $\mid V_0(x,T) - V_0(0,T) \mid$. The details are omitted. $\diamond$

3 Linear Drift

3.1 Main results

In this section we study the stationary control problem (2.5) in the special case

$$X_x^u(t) = x - \theta \int_0^t X_x^u(s)\, ds + \int_0^t u(s)\, dW(s). \tag{3.1}$$

Here, it is assumed that $\theta > 0$, so that the system with null control, $\dot{x} = -\theta x$, has a globally, asymptotically stable equilibrium point at 0.

Most of the results of this section generalize to controlled state dynamics with nonlinear drift, as discussed in section 5. We single out the linear case first because it can be treated nicely using the following auxiliary family of optimal stopping problems:

$$\begin{aligned} U_\alpha(x) &\triangleq \inf_\tau E\left[\int_0^\tau e^{-(2\theta+\alpha)t}\hat{c}_\alpha(Z_x(t))\, dt\right], \qquad \alpha \geq 0, \qquad \text{where,} \\ \hat{c}_\alpha(x) &\triangleq c''(x) + 2(2\theta + \alpha), \quad \text{and} \qquad Z_x(t) = x - \theta \int_0^t Z_x(s)\, ds + \sigma_0 W(t), \quad t \geq 0. \end{aligned} \tag{3.2}$$

Here Z_x is defined on an admissible control system with Brownian motion W and filtration $\mathbb{F}$, and the infimum in the definition of U_α is taken over all $\mathbb{F}$-stopping times of all admissible control systems. Observe that if $\hat{c}_\alpha$ is non-negative for all x, then U_α is identically zero. This is the case when $c(\cdot)$ is a convex function.

Because of the linearity of the drift in the state dynamics (3.1), there is a simple relation between U_α and the value functions V_α for $\alpha > 0$, namely, $U_\alpha = V_\alpha'' + 2$ on the component of G_α containing the equilibrium point 0, where G_α is as in Theorem 2.1. This relation, proved under more restrictive conditions on c in [17], is generalized in Lemma 3.6 below and is key to the analysis of the stationary control problem. In particular, it enables us to relate the value of λ_0 in (2.5) to the value $U_0(0)$.

The statement of the main theorem follows.

Theorem 3.1 *Assume (2.8)-(2.9) and let λ_0 be the value defined by (2.5). Then*

(i) $\lambda_0 = (\sigma_0^2/2)U_0(0) + c(0)$.

(ii) *For every finite, positive* M,

$$\lim_{\alpha \to 0+} \sup_{|x| \leq M} \left| \alpha V_\alpha(x) - \lambda_0 \right| = 0. \tag{3.3}$$

(ii) *If* $U_0(0) = 0$, *then* $u^* \equiv 0$ *is optimal for the stationary control problem (2.5). If* $U_0(0) < 0$, *let* (p_0, q_0) *be the connected component of the open set* $\{x\,;\, U_0(x) < 0\}$ *with* $p_0 < 0 < q_0$. *Then* $\pi^*(x) = \sigma_0 \mathbf{1}_{(p_0,q_0)}(x)$ *is an optimal stationary control for the problem (2.5).*

(iv) $\displaystyle\lim_{T \to \infty} \sup_{|x| \leq M} \left| \frac{V_0(x,T)}{T} - \lambda_0 \right| = 0$, *for every* $0 < M < \infty$.

In the course of the proof we also derive:

Theorem 3.2 *If (2.8)–(2.9) hold,* $\displaystyle\inf_{u \in \mathcal{U}} \limsup_{T \to \infty} \frac{1}{T} E\left[\int_0^T [c(X_x^u(t)) + u^2(t)]\, dt\right] = \lambda_0.$

The proofs are completed in section 3.4. Sections 3.2 and 3.3 establish the necessary preliminary analysis.

3.2 The value function U_α

In this section it is often only necessary to assume in addition to (2.8), that

$$c''(x) \geq -K\left(1 + |x|^{2m}\right) \qquad \text{for all } x, \tag{3.4}$$

for some constant K and non-negative integer m, rather than the stronger condition (2.9).

The process Z_x defined in (3.2) is the solution to equation (3.1) with the constant control $u \equiv \sigma_0$. Therefore, Lemma 2.2 applies, and, from (2.13),

$$\sup_t E\left[|Z_x(t)|^{2m}\right] \leq K_m(1+|x|^{2m}) \tag{3.5}$$

Therefore, a simple argument using assumption (3.4) gives

$$E\left[\int_0^\tau e^{-(2\theta+\alpha)t} \hat{c}_\alpha(Z_x(t))\, dt\right] \geq -K\left[1 + K_m(1+|x|^{2m})\right](2\theta+\alpha)^{-1} \tag{3.6}$$

for any stopping time τ and $\alpha \geq 0$, where $\hat{c}_\alpha$ is given in (3.2). (It is possible that the expectation is $+\infty$.) Therefore, it is clear that the value $U_\alpha(x)$ is defined and finite for all $\alpha \geq 0$. Indeed, by (3.6), $0 \geq U_\alpha(x) \geq -K\left[1 + K_m(1+|x|^{2m})\right](2\theta+\alpha)^{-1}$ for all x.

We shall state the main properties of U_α. The first is its characterization as the solution of a variational equation. For each $\alpha \geq 0$, define the differential operator

$$L_\alpha \triangleq \frac{\sigma_0^2}{2}\frac{d^2}{dx^2} - \theta x\frac{d}{dx} - (2\theta+\alpha).$$

Proposition 3.3 *Assume (2.8) and (3.4). Then U_α is the unique function such that $U_\alpha \in C^1(I\!R)$, $|U_\alpha(x)|$ grows at most at a polynomial rate, U'_α is absolutely continuous, U''_α is locally essentially bounded, and*

$$\min\{L_\alpha U_\alpha(x) + \hat{c}_\alpha(x), -U_\alpha(x)\} = 0 \qquad a.e. \tag{3.7}$$

The set $\{x; U_\alpha(x) = 0\}$ is an optimal stopping region.

See the appendix for the statement of the general theorem from which Proposition 3.3 derives and for references. The fact that $U_\alpha \in C^1(I\!R)$ is a rigorous statement of the smooth-fit principle for this problem.

The optimal continuation region $\{x;\, U_\alpha(x) < 0\}$ is an open set in $I\!R$, and, as such, is a countable union of disjoint, open intervals. The optimal stopping strategy in one of these intervals is to stop upon exit from the interval. To express this, let τ_{ab} be the first time at which Z_x exits (a,b), where $-\infty \leq a < b \leq \infty$, and define

$$U_{\alpha,ab}(x) \triangleq E\left[\int_0^{\tau_{ab}} e^{-(2\theta+\alpha)t}\hat{c}_\alpha(Z_x(t))\,dt\right], \qquad a \leq x \leq b, \quad |x| < \infty, \tag{3.8}$$

For convenience of notation, we abbreviate $U_{\alpha,ab}$ by U_{ab}, when the value of α is clear from context. By (3.6), U_{ab} is always well-defined, if the value $+\infty$ is allowed. Because U_α is the optimal value, $U_\alpha(x) \leq U_{ab}(x)$ for any $-\infty \leq a < b \leq \infty$ and any finite x in $[a,b]$.

The next result gives a useful characterization of the open connected components of the optimal continuation region.

Lemma 3.4 *Let (p,q) be a connected component of $\{x\,;\, U_\alpha(x) < 0\}$. Then for every finite x in $[p,q]$:*

$$U_\alpha(x) = U_{pq}(x). \tag{3.9}$$

Any connected component (p,q) has the following characterization. Let (ℓ,m) be any connected component of $\{x\,;\, \hat{c}_\alpha(x) < 0\}$ contained in (p,q); at least one such (ℓ,m) exists. Then

$$p = \inf\{a\,;\, \exists b \text{ such that } (\ell,m) \subset (a,b) \text{ and } U_{ab} < 0 \text{ on } (a,b)\}\,; \tag{3.10}$$

$$q = \sup\{b\,;\, \exists a \text{ such that } (\ell,m) \subset (a,b) \text{ and } U_{ab} < 0 \text{ on } (a,b)\}\,. \tag{3.11}$$

Proof: Proposition 3.3 says that $\tau^* = \inf\{t\,;\, U_\alpha(Z_x(t)) = 0\}$ is an optimal stopping time. If $x \in [p,q]$, then $\tau^* = \tau_{pq}$, which proves (3.9).

If $\hat{c}_\alpha \geq 0$ on an interval (a,b), then it is clear from (3.8) that $U_{ab}(x) \geq 0$ for x in (a,b). Thus, if (p,q) is a connected component of $\{x; U_\alpha(x) < 0\}$, (p,q) must contain points x for which $\hat{c}_\alpha(x) < 0$, because $U_\alpha(x) = U_{pq}(x) < 0$ on (p,q). At the same time, if (ℓ, m) is a connected component of $\{x; \hat{c}_\alpha(x) < 0\}$, definition (3.8) implies $U_\alpha(x) \leq U_{\ell m}(x) < 0$ on (ℓ, m), so (ℓ, m) is contained in $\{x; U_\alpha(x) < 0\}$.

To prove (3.10) and (3.11), Finally, notice that if $(\ell, m) \subset (p,q)$, where (p,q) is a connected component of $\{x; U_\alpha(x) < 0\}$, and if $(\ell, m) \subset (a,b)$ where $U_{ab} < 0$ on (a,b), then $(a,b) \subset \{x; U_\alpha(x) < 0\}$ also. Thus $(a,b) \subset (p,q)$, and (3.10) and (3.11) hold. ◇

In the appendix, it is shown how one can take (3.10)-(3.11) as the starting point for an elementary proof of Proposition 3.3.

The value function U_0 for $\alpha = 0$ can be used to construct a solution of the formal HJB equation for the stationary problem (2.5) on an interval about the origin. If $U_0(0) < 0$, let (p_0, q_0) denote the connected component of $\{x; U_0(x) < 0\}$ containing 0, and define

$$Q(x) \triangleq \int_0^x \int_0^y (U_0(z) - 2)\ dz\, dy + \theta^{-1}\left[\frac{\sigma_0^2}{2}U_0'(0) + c'(0)\right] x$$

The possibility that (p_0, q_0) is unbounded has not been excluded. We shall use the notation $\partial(p_0, q_0)$ to denote the boundary of (p_0, q_0), that is, the set of finite endpoints of (p_0, q_0).

Lemma 3.5 *Let $U_0(0) < 0$. Then Q satisfies*

$$\begin{aligned}
\frac{\sigma_0^2}{2}\left(Q''(x) + 2\right) - \theta x Q'(x) + c(x) \quad &= \quad \frac{\sigma_0^2}{2}U_0(0) + c(0) \qquad x \in (p_0, q_0) && (3.12)\\
(Q'' + 2)\mid_{\partial(p_0,q_0)} = 0 \qquad\qquad Q'' + 2 \quad &< \quad 0 \quad \text{on } (p_0, q_0). && (3.13)
\end{aligned}$$

Proof: Since $Q'' + 2 = U_0$, (3.13) is immediate from (3.9).

Let Φ denote the left-hand side of (3.12). By direct calculation, $\Phi''(x) = L_0 U_0(x) + \hat{c}_0(x) = 0$ on (p_0, q_0), where L_0 is defined as in Proposition 3.3, and $\Phi'(0) = 0$. Hence $\Phi(x) = \Phi(0) = (\sigma_0^2/2)U_0(0) + c(0)$ on (p_0, q_0), which proves (3.12).

Remarks:
(i) Assume (2.9) and suppose that (p_0, q_0) is not bounded. By (3.6), $|U_0|$ grows at most at a polynomial rate. Because $Q'' = U_0 - 2$, Q also admits at most polynomial growth.
(ii) Whenever $\{a_n\}$ and $\{b_n\}$ are sequences of finite numbers with $a_n \downarrow a$ and $b_n \uparrow b$, where $-\infty \leq a < b \leq \infty$, $U_{ab}(x) = \lim_{n\to\infty} U_{a_n b_n}(x)$. This is a consequence of the fact that $\tau_{a_n b_n} \uparrow \tau_{ab}$, almost-surely. Decompose $\hat{c}_\alpha$ into the difference of its positive and negative parts, $\hat{c}_\alpha = \hat{c}_\alpha^+ - \hat{c}_\alpha^-$, express U_{ab} as the difference of two terms, one involving $\hat{c}_\alpha^+(Z_x(t))$, the other $\hat{c}_\alpha^-(Z_x(t))$, and apply the monotone convergence theorem in each term to obtain the result.

3.3 Relation of U_α to V_α

Recall from Theorem 2.1 that V_α is continuously differentiable and that the optimal feedback policy is given by $\pi_\alpha(x) \triangleq \sigma_0 \mathbf{1}_{G_\alpha}(x)$, where G_α is the open set $G_\alpha = \{x; b(x)V_\alpha'(x) - \alpha V_\alpha(x) + c(x) > 0\}$. If $\alpha V_\alpha(0) < c(0)$, one sees immediately that $0 \in G_\alpha$.

We shall use the following notation: (r_α, s_α), denotes the connected component of G_α containing 0, if it exists, otherwise (r_α, s_α) is empty; on the other hand, (p_α, q_α) denotes the connected component of the open set $\{x; U_\alpha(x) < 0\}$ containing 0, if it exists, otherwise, (p_α, q_α) is be empty. In section 3 of [17] it is shown that (r_α, s_α) is necessarily bounded for each $\alpha > 0$.

Lemma 3.6 *Let $\alpha > 0$. Assume (2.8), (2.9), and (3.4). The following are equivalent:*

(i) $\alpha V_\alpha(0) < c(0)$.

(ii) (r_α, s_α) *is non-empty.*

(iii) (p_α, q_α) *is non-empty, i.e.* $U_\alpha(0) < 0$.

When these equivalent conditions hold,

$$(r_\alpha, s_\alpha) = (p_\alpha, q_\alpha) \quad \text{and} \quad V_\alpha'' + 2 = U_\alpha \quad \text{on } (r_\alpha, s_\alpha). \tag{3.14}$$

In all cases,

$$\alpha V_\alpha(0) = c(0) + \frac{\sigma_0}{2} U_\alpha(0). \tag{3.15}$$

Proof: Fix $\alpha > 0$. To simplify notation and to reserve subscripts for other uses, write V for V_α and U for U_α.

We have explained in defining (r_α, s_α) above that (i) and (ii) are equivalent.

To proceed further, it is necessary to develop some facts about V from [17]. For $a < 0 < b$, where a and b are finite, let V_{ab} denote the discounted cost associated to the feedback control $\sigma_0 \mathbf{1}_{(a,b)}(x)$. That is

$$V_{ab}(x) \triangleq E\left[\int_0^\infty e^{-\alpha t} \left(c(Y_x(t)) + \sigma_0^2 \mathbf{1}_{(a,b)}(Y_x(t))\right) dt\right],$$

where $dY_x(t) = -\theta Y_x(t)\,dt + \sigma_0 \mathbf{1}_{(a,b)}(Y_x(t))\,dW(t)$, and $Y_x(0) = x$. Observe that $Y_x(t)$ remains in $[a, b]$ for all time $t \geq 0$ if $x \in [a, b]$; see [17], section 2.

Fact 1:

$$V_{ab}'' + 2 = U_{ab} \qquad \text{on } [a, b]. \tag{3.16}$$

Indeed, it is shown in Theorem 2.2 of [17] that

$$\begin{aligned}\frac{\sigma_0^2}{2}\left(V''_{ab}(x)+2\right)-\theta x V'_{ab}(x)-\alpha V_{ab}(x)+c(x) &= 0 \qquad x\in(a,b) \\ V''_{ab}(a+)+2=0 \qquad V''_{ab}(b-)+2=0 &\end{aligned} \tag{3.17}$$

Let $v \triangleq V''_{ab}+2$. Differentiate (3.17) twice and use the above boundary conditions. Then

$$\frac{\sigma_0}{2}v''(x)-\theta x v'(x)-(2\theta+\alpha)v(x)+\hat{c}_\alpha(x)=0, \qquad a<x<b \tag{3.18}$$

$$v(a)=0 \qquad v(b)=0. \tag{3.19}$$

Therefore, $v=U_{ab}$ on $[a,b]$, where U_{ab} is as in (3.8), by the Feynman-Kac formula. This is the step that uses crucially the linearity of the drift.
Fact 2: By definition of V as the least discounted cost under all controls,

$$V(x)\le V_{ab}(x) \quad \text{on } I\!R \text{ for any } a<0<b. \tag{3.20}$$

Fact 3: (See Theorems 3.1 and 3.2 in [17].) If (r_α, s_α) is non-empty, it is bounded, and

$$V(x)=V_{r_\alpha s_\alpha}(x) \quad \text{on } [r_\alpha,s_\alpha] \text{ and } \quad V''_{r_\alpha s_\alpha}(x)+2<0 \quad \text{on } (r_\alpha,s_\alpha). \tag{3.21}$$

From (3.21), (3.17) and (3.16),

$$\alpha V(x)=c(x)-\theta x V'(x)+\frac{\sigma_0^2}{2}U_{r_\alpha s_\alpha}(x) \qquad \text{on } [r_\alpha,s_\alpha]. \tag{3.22}$$

Assume that (ii) holds. Then, (3.16) and (3.21) imply that $U_{r_\alpha s_\alpha}(x)<0$ on (r_α,s_α). Thus $U_\alpha(0)\le U_{r_\alpha s_\alpha}(0)<0$, and statement (iii) holds.

Now assume that (iii) holds. When (p_α,q_α) is bounded, we can use (3.20), (3.16), and (3.17) to conclude:

$$\alpha V(0)\le \alpha V_{p_\alpha q_\alpha}(0)=c(0)+\frac{\sigma_0^2}{2}U_{p_\alpha q_\alpha}(0)<c(0),$$

whence follows (i).

However, we do not know a priori that (p_α,q_α) is bounded. Even if it is not, $U_{p_\alpha q_\alpha}(0)<0$ by assumption, and by (3.4)-(3.6) and the argument below (3.6), $U_{p_\alpha q_\alpha}(x)\ge U_\alpha(x)>-\infty$ for all x. We saw in the previous section that

$$\lim_n U_{a_n b_n}(x)=U_{p_\alpha q_\alpha}(x) \qquad \text{for all } x\in(p_\alpha,q_\alpha), \tag{3.23}$$

whenever $\{a_n\}$ and $\{b_n\}$ are sequences of finite numbers with $a_n < b_n$ for all n, such that $a_n \downarrow p_\alpha$ and $b_n \uparrow q_\alpha$ as $n \to \infty$. Choose n large enough that $U_{a_nb_n}(0) < 0$. Then, arguing as above, $\alpha V(0) \leq \alpha V_{a_nb_n}(0) = c(0) + (\sigma_0^2/2)U_{a_nb_n}(0) < c(0)$, which proves statement (i).

Finally, assume that the equivalent conditions (i)-(iii) hold. By (3.16), (3.17), (3.21), and the characterizations (3.10) and (3.11) of p_α and q_α, it is clear that $(r_\alpha, s_\alpha) \subset (p_\alpha, q_\alpha)$. We shall argue the reverse inclusion by contradiction. First, we claim that if (r_α, s_α) is a proper subset of (p_α, q_α), then $U_{p_\alpha q_\alpha}(x) < U_{r_\alpha s_\alpha}(x)$ on (r_α, s_α). Indeed, on (r_α, s_α), both $U_{p_\alpha q_\alpha}$ and $U_{r_\alpha q_\alpha}$ are solutions of $L_\alpha w(x) + \hat{c}_\alpha(x) = 0$, and $U_{p_\alpha q_\alpha}(r_\alpha) \leq 0 = U_{r_\alpha s_\alpha}(r_\alpha)$, $U_{p_\alpha q_\alpha}(s_\alpha) \leq 0 = U_{r_\alpha s_\alpha}(s_\alpha)$. At least one of these inequalities is strict if (r_α, s_α) is properly contained in (p_α, q_α). The maximum principle and uniqueness of solutions then implies $U_{p_\alpha q_\alpha}(x) < U_{r_\alpha s_\alpha}(x)$ on (r_α, s_α). Thus, by (3.23), there exist finite $a < 0 < b$ such that $U_{ab}(0) < U_{r_\alpha s_\alpha}(0)$.

From (3.16) and (3.21), $U_{ab}(0) = V''_{ab} + 2$, and $U_{r_\alpha s_\alpha}(0) = V''(0) + 2$. Hence

$$\alpha V(0) \leq \alpha V_{ab}(0) = c(0) + \frac{\sigma_0^2}{2}U_{ab}(0) < c(0) + \frac{\sigma_0^2}{2}(V''(0) + 2) = \alpha V(0).$$

This is a contradiction, and so we conclude that (r_α, s_α) and (p_α, q_α) are identical. This completes the proof of (3.14); (3.15) is a direct consequence of (3.14) and (3.17).

3.4 Proofs of Theorems 3.1 and 3.2

Because of Lemma 3.6, the asymptotic behavior of $\alpha V_\alpha(0)$ as $\alpha \downarrow 0$ is determined by the asymptotic behavior of $U_\alpha(0)$, which is easily computed.

Lemma 3.7 *Assume (2.8)–(2.9). Then* $\lim_{\alpha \to 0+} \sup_{|x| \leq M} | U_\alpha(x) - U_0(x) | = 0$, *for every finite, positive* M.

Proof: From (2.9) and (3.5), it follows that $\sup_{t \geq 0} E[|\hat{c}_0(Z_x(t))|] \leq \tilde{K}(1 + |x|^{2m})$ for some finite $\tilde{K}$. Let τ^* be the optimal stopping time that achieves the value $U_0(x)$. Observe that $\hat{c}_\alpha = \hat{c}_0 + 2\alpha$. Then

$$U_\alpha(x) - U_0(x) \leq E\left[\int_0^{\tau^*} (e^{-2\theta t}[e^{-\alpha t} - 1]\hat{c}_0(Z_x(t)) + e^{-(2\theta+\alpha)t}2\alpha)\, dt\right] \leq \frac{2\alpha + \tilde{K}(1+|x|^{2m})\alpha(2\theta)^{-1}}{2\theta+\alpha}.$$

This last expression is bounded by $\dfrac{\alpha}{2\theta}(4 + \dfrac{\tilde{K}(1 + |x|^{2m})}{2\theta})$. A similar inequality holds for $U_0(x) - U_\alpha(x)$, which implies the lemma. $\diamond$

Let $\Lambda_0 \stackrel{\triangle}{=} c(0) + (\sigma_0^2/2)U_0(0)$. By (3.15) of Lemma 3.6 and Lemma 3.7, $\lim_{\alpha\to 0+} \alpha V_\alpha(0) = c(0) + (\sigma_0^2/2)U_0(0)$. It follows from Lemma 2.3 part (ii) that

$$\lim_{\alpha\to 0^+} \sup_{|x|\le M} \mid \alpha V_\alpha(x) - \Lambda_0 \mid = 0.$$

We shall show that $\lambda_0 = \Lambda_0$, thereby completing the proof of Theorem 3.1, parts (i) and (ii). In the course of this argument, we establish part (iii) also.

Observe

$$\lambda_0 = \inf_{\mathcal{U}} \liminf_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(c(X_x^u(t)) + u^2(t)\right) dt\right] \ge \liminf_{T\to\infty} \frac{1}{T} V_0(x,T).$$

We shall first establish that for any x,

$$\liminf_{T\to\infty} \frac{1}{T} V_0(x,T) \ge \Lambda_0, \tag{3.24}$$

which implies $\lambda_0 \ge \Lambda_0$. Because of Lemma 2.3 (iv), it is enough to consider the case $x = 0$. We want to apply Ito's rule to $V_\alpha(X_0^u(t))$, for arbitrary $u \in \mathcal{U}$ and arbitrary $\alpha > 0$. We do this using the approximation procedure described in Theorem 3.1 of [17] and the HJB equation (2.11) for V_α. We obtain

$$E\left[\int_0^T c(X_0^u(t)) + u^2(t)\, dt\right] \ge E\left[\int_0^T \alpha V_\alpha(X_0^u(t))\, dt\right] - \Gamma, \tag{3.25}$$

where Γ is as in Lemma 2.3, part (iii). But using (2.13) and (2.14),

$$E\left[\mid V_\alpha(X_0^u(t)) - V_\alpha(0) \mid\right] \le K_1 E\left[1 + |X_0^u(t)|^{2m} + |X_0^u(t)|^{2m+2}\right] \le K_2,$$

where K_2 is a finite constant, independent of α but depending on the constants in (2.13) and (2.14). Hence, $E\left[\int_0^T \mid \alpha V_\alpha(X_0^u(t)) - \alpha V_\alpha(0) \mid\, dt\right] \le \alpha K_2 T$. Therefore, it follows from (3.25) that

$$\frac{V_0(0,T)}{T} \ge \Lambda_0 - \frac{|\alpha V_\alpha(0) - \Lambda_0|}{T} - \alpha K_2 - \frac{\Gamma}{T}.$$

Letting $T \to \infty$ and then $\alpha \downarrow 0$ proves the claim (3.24).

Next, we show that the controls defined in part (iii) of Theorem 3.1 achieve the value Λ_0 in the stationary control problem. This shows that $\Lambda_0 \ge \lambda_0$, and hence, because of (3.24), that $\Lambda_0 = \lambda_0$. It also proves optimality of the controls achieving Λ_0.

If $U_0(0) = 0$, then $\Lambda_0 = c(0)$. In this case, set $u^* \equiv 0$. Then $X_x^{u^*}(t) = xe^{-\theta t} \to 0$ as $t \to \infty$, and so

$$\lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(c(X_x^{u^*}(t)) + (u^*)^2(t)\right) dt\right] = c(0) = \Lambda_0. \tag{3.26}$$

If $U_0(0) < 0$, there is a connected component (p_0, q_0) of $\{x\,;\,U_0(x) < 0\}$ with $p_0 < 0 < q_0$. We shall use the feedback policy $\sigma_0 \mathbf{1}_{(p_0,q_0)}(x)$. The state process corresponding to this control is $Y_x^*(t)$, where

$$dY_x^*(t) = -\theta Y_x^*(t)\,dt + \sigma_0 \mathbf{1}_{(p_0,q_0)}(Y_x^*(t))\,dW(t), \qquad \text{and} \qquad Y_x^*(0) = x.$$

Let $u^*(t) = \sigma_0 \mathbf{1}_{(p_0,q_0)}(Y_x^*(t))$ denote the corresponding control process. If $x \in [p_0, q_0]$, the process $Y_x^*(t)$ remains in $[p_0, q_0]$ for all time. Applying Itô's rule to $Q(Y_x^*(t))$, where Q is given as in Lemma 3.5, and using the equation (3.12) yields,

$$E\left[\int_0^T \Big(c(Y_x^*(t)) + (u^*)^2(t)\Big)\,dt\right] = \Lambda_0 T + Q(x) - Q(Y_x^*(T)). \tag{3.27}$$

From Remark (i) of section 3.2, Q has at most polyomial growth, and by the moment bound (2.13), $E\left[Q(Y_x^*(T))\right]$ is bounded uniformly in T. Thus,

$$\lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \Big(c(Y_x^*(t)) + (u^*)^2(t)\Big)\,dt\right] = \Lambda_0, \tag{3.28}$$

which is what we wanted to prove.

If $x \notin [p_0, q_0]$, then since no variance control is exercised unless the process enters $[p_0, q_0]$, $Y_x^*(t) = xe^{-\theta t}$ until it hits $[p_0, q_0]$, which it does in a finite time, after which it proceeds as if started at one of the endpoints. The cost accumulated up to the hitting time is lost in the averaged limit as $T \to \infty$ and, so (3.28) is still valid. This completes the proof of Theorem 3.1 (i)–(iii).

Notice that limits and not limit infima appear in (3.26) and (3.28). Thus, we have also proved,

$$\inf_{u\in\mathcal{U}} \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T \Big(c(X_x^u(t)) + u^2(t)\Big)\,dt\right] = \Lambda_0 = \lambda_0.$$

Besides verifying Theorem 3.2, this remark completes the proof of Theorem 3.1, part (iv), when $x = 0$, because it implies

$$\limsup_{T\to\infty} \frac{V_0(0,T)}{T} \le \inf_{u\in\mathcal{U}} \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T \Big(c(X_x^u(t)) + u^2(t)\Big)\,dt\right] = \lambda_0,$$

which, in conjunction with (3.24), implies $\lim_{T\to\infty} T^{-1} V_0(0,T) = \lambda_0$. Theorem 3.1, part (iv), now follows using Lemma 2.3, part (iv).

4 Localization and general unbounded cost

The aim of this section is to give a simple condition under which the interval of non-trivial variance control about the origin for the discounted problem is bounded uniformly with respect to α. When this is the case, only the behavior of c in a bounded interval is relevant to the analysis of the asymptotic behavior of the value functions. This effectively localizes the analysis of the limit and allows one to obtain an extension of the main results without a global growth condition on c.

We shall prove uniform localization under the condition

$$\text{there exists } \; R > 0 \text{ such that } \; \hat{c}_0(x) = c''(x) + 4\theta \geq 0 \; \text{ for all } |x| \geq R. \tag{4.1}$$

In what follows, it is assumed also that

$$\inf_{\boldsymbol{R}} \hat{c}_0(x) = \inf_{\boldsymbol{R}} c''(x) + 4\theta < 0 \tag{4.2}$$

Otherwise, the optimal controls are always $u^* \equiv 0$ and the proof is trivial. Observe for later reference that (4.1) easily implies $\inf_{\alpha \geq 0} \inf_x U_\alpha(x) > -\infty$; see (3.6).

Theorem 4.1 *Assume that c is a non-negative, twice-differentiable cost function satisfying (4.1). Then Theorems 3.1 and 3.2 hold. Moreover, in the optimal feedback strategy $\sigma_0 \mathbf{1}_{(p_0,q_0)}(x)$, given in part (iii) of Theorem 3.1, the interval (p_0, q_0) is bounded.*

The idea of the proof is to show that the various value functions are equal locally to value functions in which the cost function is in $C_b^2(\mathbb{R})$, as assumed in section 3. Then the theorems of section 3 are applied locally to deduce Theorem 4.1.

The first step is to localize the action of the optimal feedback strategies. Recall that there is an optimal feedback strategy for the discounted control problem (2.6) of the form $\sigma_0 \mathbf{1}_{G_\alpha}(x)$, where G_α is the open set defined in Theorem 2.1.

Lemma 4.2 *Let c be non-negative, twice-continuously differentiable and satisfy (4.1) and (4.2). There exists a constant K_0 depending only on $\inf_x c''(x)$, θ, $\sup_{|x|\leq R} |c(x)| + |c'(x)|$, and R, such that $U_\alpha(x) = 0$ on $[-K_0, K_0]^c$ and $G_\alpha \subset [-K_0, K_0]$ for all $\alpha \geq 0$.*

Proof. From assumption (4.1) and the definition of U_α it is easy to see that $\ell_1 \triangleq \sup_x |U_\alpha(x)| \leq |\inf_x c''(x)| < \infty$. One can also show by an apriori bound that $\ell_2 \triangleq \sup_{\alpha \geq 0} \sup_{|x|\leq R} |U'_\alpha(x)| < \infty$ and that the bound depends only on θ, R, and $\sup_{|x|\leq R} |c''(x)|$; we defer the proof momentarily.

Suppose there exists a z in $\{x\,;\, U_\alpha(x) < 0\}$ such that $z > R$, and let (p, q), with $p < z < q$, be the connected component of $\{x\,;\, U_\alpha(x) < 0\}$ containing z. Since (p, q) must intersect

$\{x\,;\,\hat{c}(x) < 0\}$ in at least one point, and since $\{x\,;\,\hat{c}(x) < 0\} \subset (-R, R)$, it must be true that $p < R$. Now integrate (3.7) twice from R to q. The result, after some rearrangement of terms, is

$$\frac{\sigma_0^2}{2}\left[U_\alpha(q) - U_\alpha(R)\right] - c(R) + \left[(\theta R - \frac{\sigma_0^2}{2})U'_\alpha(R) - c'(R)\right](q - R) + (2\theta + \alpha)(q - R)^2 = (\theta + \alpha)\int_R^q \int_R^y U_\alpha(x)\,dx\,dy - c(q)$$

The right-hand side is negative. On the other hand, there is a $\bar{K}_0$, depending only on R, ℓ_1, ℓ_2, $c(R)$, and $c'(R)$, and not on α, such that the left-hand side is positive if $q > \bar{K}_0$. Hence we obtain a contradiction unless $q \leq \bar{K}_0$. By applying the same reasoning to $(-\infty, -R)$, one obtains a K_0 such that $U_\alpha(x) = 0$ for all $|x| \geq K_0$ and all $\alpha \geq 0$.

Next fix an arbitrary $\alpha > 0$ and consider any connected component (r, s) of G_α, the optimal feedback set for the discounted problem. The component (r, s) is necessarily bounded. As a consequence of Theorem 2.1, $u \stackrel{\triangle}{=} V''_\alpha + 2$ is a solution of

$$(\sigma_0^2/2)u'' - \theta x u' - (2\theta + \alpha)u + \hat{c}_\alpha = 0 \quad x \in (r, s) \qquad u(p) = u(q) = 0,$$

and, so $u(x) = U_{rs}(x)$. In addition, $u < 0$ on (r, s). Hence $(r, s) \subset \{x\,;\,U_\alpha(x) < 0\}$, which was just shown to be contained in $[-K_0, K_0]$. Thus $G_\alpha \subset [-K_0, K_0]$.

It remains to prove the boundedness of ℓ_2. There is a finite α_0 such that, for $\alpha \geq \alpha_0$, $\hat{c}_\alpha \geq 0$ and hence $U_\alpha \equiv 0$. Therefore, one can restrict attention to $0 \leq \alpha \leq \alpha_0$. In any interval $(b, b+1)$, the Mean Value Theorem provides a point $x_0 \in (b, b+1)$, for which $|U'_\alpha(x_0)| \leq \ell_1$. By solving (3.7) for U'_α, it follows that $|U'_\alpha(x)|$ is bounded uniformly for $0 \leq \alpha \leq \alpha_0$ and $b \leq x \leq b+1$ by a constant depending only on b, θ, ℓ_1, and $\sup_{b<x<b+1}|c''(x)|$. It follows that for any M, $\sup_{0\leq\alpha}\sup_{|x|\leq M}|U'_\alpha(x)|$ is bounded by a constant depending only on ℓ_1, θ, and M. $\diamond$

Proof of Theorem 4.1: Let R be as in (4.1). For each positive integer n, it is easy to see that there is a non-negative, twice-continuously differentiable function c_n which is equal to c on $[-n, n]$, which satisfies $c_n(x) \leq c(x)$ for all x, and which also satisfies both the polynomial growth bound (2.9) and the condition (4.1) for the given R.

Let $V_\alpha^{(n)}$ and $U_\alpha^{(n)}$ be the value functions for the discounted control problem and the associated optimal stopping problem, respectively. Let G_α be as in Theorem 2.1. Let G_α^n be the set defining the optimal feedback policy $\pi^n(x) = \sigma_0 \mathbf{1}_{G_\alpha^n}(x)$ for $V_\alpha^{(n)}$. Finally, let λ_0^n denote the stationary value associated to cost c_n. Because $c_n \leq c$,

$$\lambda_0^n \leq \lambda_0 \qquad \text{for all } n. \tag{4.3}$$

By Lemma 4.2, there is a K_0 such that the interval $[-K_0, K_0]$ contains all of the following sets $\{x\,;U_\alpha(x) < 0\}$, G_α, $\{x\,;U_\alpha^{(n)}(x) < 0\}$ for all n, and G_α^n for all n. This has the following consequences. First, $U_\alpha^{(n)} = U_\alpha$ for all n, $n > K_0$, and all $\alpha \geq 0$, because for such n, $c_n = c$ on $[-K_0, K_0]$, which contains the optimal continuation region for all n. Second, if $n > K_0$, then $V_\alpha^{(n)}(x) = V_\alpha(x)$ on $[-n, n]$ for all $\alpha > 0$. To see this, recall that if G is an open set contained in $[-K_0, K_0]$ and $n > K_0$, the solution to

$$X_x(t) = x - \theta \int_0^t X_x(s)\, ds + \sigma_0 \int_0^t \mathbf{1}_G(X_x(s))\, dW(s)$$

remains in $[-n, n]$ if the initial point x is in $[-n, n]$. Thus for any $n > K_0$ the optimal processes for both $V_\alpha^{(n)}$ and V_α remain in $[-n, n]$ if they start there. Since c and c_n are equal on $[-n, n]$, it follows that the two value functions are equal on $[-n, n]$.

Theorems 3.1 and 3.2 are valid for each c_n for, except that λ_0 must be replaced by λ_0^n, where $n > K_0$ but is otherwise arbitrary. We show that $\lambda_0 = \lambda_0^n$ for $n > K_0$. Because of (4.3) we need only show $\lambda_0 \leq \lambda_0^n$. However, for any $n > K_0$, the optimal feedback policy $\sigma_0 \mathbf{1}_{(p_0,q_0)}(x)$ achieves the value λ_0^n when the cost function is c_n. Moreover, when $|x| \leq n$, the optimal state process remains in $[-n, n]$ where c_n equals c, so the value λ_0^n can also be achieved when the cost function is c. Since λ_0 is the optimal stationary value, $\lambda_0 \leq \lambda_0^n$. Theorem 3.1, part (i), now follows by taking $n > K_0$, as does part (iii). For Theorem 3.1 part (ii) and part (iv), take $n > \max\{K_0, M\}$ for each $M > 0$. Theorem 3.1, part (iv) and Theorem 3.2 is proved as before.

◇

5 Nonlinear drift

In this section, we sketch results for the problems (2.5)–(2.7) assuming only the condition (2.2) on the drift b and the polynomial growth condition (2.9) on the second derivative of the location cost c. Observe that, using the zero control, one can obtain $\lambda_0 \leq c(0)$, where λ_0 is defined as in (2.5). The main result is:

Theorem 5.1 *Assume (2.2), (2.8) and (2.9). Let λ_0 be given by (2.5). Then for every finite, positive M,*

$$\lim_{\alpha\to 0^+} \sup_{|x|\leq M} \mid \alpha V_\alpha(x) - \lambda_0 \mid = 0 \tag{5.1}$$

If $\lambda_0 = c(0)$ then $u^* \equiv 0$ is an optimal control for the stationary problem. If $\lambda_0 < c(0)$. there is an optimal feedback policy for the stationary problem of the form $\pi^*(x) = \sigma_0 \mathbf{1}_{(p^*,q^*)}$, where $p^* < 0 < q^*$.

Finally,

$$\lim_{T\to\infty} \sup_{|x|\leq M} \frac{1}{T} \mid V_0(x, T) - \lambda_0 \mid = 0, \qquad \textit{for every finite } M > 0. \tag{5.2}$$

The proof uses Lemma 2.3 heavily. First note that as a consequence of Lemma 2.3, part (i), there exists a finite constant K_1, independent of α, such that

$$\mid V_\alpha(x) - V_\alpha(y) \mid \leq |x-y| K_1 \left(1 + |x|^{2m} + |y|^{2m}\right) \qquad \text{for all } \alpha > 0. \tag{5.3}$$

As a consequence,

$$\mid V_\alpha'(x) \mid \leq 2K_1 \left(1 + |x|^{2m}\right) \qquad \text{for all } \alpha > 0,\ x \in I\!R. \tag{5.4}$$

To fix terminology, let us say that a control u achieves the stationary value γ if

$$\liminf_{T\to\infty} \frac{1}{T} E\left[\int_0^T [c(X_x^u(t)) + u^2(t)]\, dt\right] = \gamma$$

Let $\{\alpha_n\}$ be a sequence of positive discount factors with $\alpha_n \downarrow 0$ as $n \to \infty$, such that $\Lambda_0 \stackrel{\triangle}{=} \lim_{\alpha_n\to\infty} \alpha_n V_{\alpha_n}(0)$ exists. Because $0 \leq \alpha V_\alpha(0) \leq c(0)$ such sequences exist, and for any such sequence, $0 \leq \Lambda_0 \leq c(0)$. The argument which proves (3.24) applies without change to the sequence α_n, because it depends only on Lemma 2.3. We therefore obtain

$$\lambda_0 \geq \Lambda_0 \tag{5.5}$$

If one then exhibits a control u^* which achieves Λ_0, it follows at once from the definition of λ_0 as the optimal stationary value that $\lambda_0 = \Lambda_0$. By applying Lemma 2.3 (i), one then obtains

$$\lim_{n\to\infty} \sup_{|x|\leq M} \mid \alpha_n V_{\alpha_n}(x) - \lambda_0 \mid = 0 \tag{5.6}$$

In particular, we know already that $u^* \equiv 0$ achieves the stationary value $c(0)$. Hence if $\Lambda_0 = c(0)$, it follows that $\lambda_0 = c(0)$ and that (5.6) holds.

Recall that the optimal control for the discounted problem with discount $\alpha > 0$ has the form $\pi_\alpha(x) = \sigma_0 \mathbf{1}_{G_\alpha}(x)$ for some open set G_α. As in section 3, let (r_α, s_α) denote the connected component of G_α containing the origin, if such a component exists; otherwise consider (r_α, s_α) to be empty. For each α, (r_α, s_α) is a bounded interval since $V_\alpha'' \leq -2$ on (r_α, s_α) and V_α is a non-negative function.

There are several possibilites.

First, there may exist a subsequence $\{\beta_n\}$ of $\{\alpha_n\}$ such $(r_{\beta_n}, s_{\beta_n})$ is always empty. By the definition of G_α and equation (2.11), $\beta_n V_{\beta_n}(0) = c(0)$ along this subsequence. It follows that $\Lambda_0 = c(0)$ and so, as we argued above, $\lambda_0 = c(0)$ and (5.6) holds.

Second, there may exist a subsequence $\{\beta_n\}$ such that either $r_{\beta_n} \to 0$ or $s_{\beta_n} \to 0$. Without loss of generality, suppose the former. From (2.11), we have

$$b(r_{\beta_n}) V_{\beta_n}'(r_{\beta_n}) - \beta_n V_{\beta_n}(r_{\beta_n}) + c(r_{\beta_n}) = 0.$$

Since (5.4) implies that $V'_{\beta_n}(r_{\beta_n})$ is uniformly bounded, and since $b(0)=0$, it follows that $\lim_{n\to\infty}\beta_n V_{\beta_n}(r_{\beta_n}) = c(0)$. So, again $\lambda_0 = c(0)$ and (5.6) holds.

If neither of the first two cases holds, there is an $\epsilon_0 > 0$ such that $r_{\alpha_n} \leq -\epsilon_0 < \epsilon_0 \leq s_{\alpha_n}$ for all n. We shall construct a function Q that is precisely analogous to the Q found in section 3. To do this, use the uniform boundedness of $V'_{\alpha_n}(0)$ proved in (5.4) to choose a subsequence β_n such that

$$p^* := \lim_{n\to\infty} r_{\beta_n}, \qquad q^* := \lim_{n\to\infty} q_{\beta_n}, \qquad \mu_0 := \lim_{n\to\infty} V'_{\beta_n}(0) \tag{5.7}$$

all exist and μ_0 is finite. (We do not exclude the possibility of infinite limits.) From (2.11), $V''_\alpha(x)+2<0$ on (r_α, s_α) and V_α solves

$$\begin{aligned} \frac{\sigma_0^2}{2}\left(V''_\alpha(x)+2\right) + b(x)V'_\alpha(x) - \alpha V_\alpha(x) + c(x) &= 0 \qquad \text{for } x \text{ in } (r_\alpha, s_\alpha), && (5.8)\\ V''_\alpha(r_{\beta_n}) + 2 = 0 \qquad V''_\alpha(s_\alpha)+2 = 0, & && (5.9) \end{aligned}$$

By the variation of constants formula

$$V'_\alpha(x) = e^{-A(x)}\left[V'_\alpha(0) + \int_0^x e^{A(y)}(2/\sigma_0^2)\left(\alpha V_\alpha(y) - c(y) - \sigma_0^2\right)\, dy\right],$$

where $A(x) = (2/\sigma_0^2)\int_0^x b(r)\,dr$. Define Q so that $Q(0)=0$ and

$$Q'(x) = e^{-A(x)}\left[\mu_0 + \int_0^x e^{A(y)}(2/\sigma_0^2)\left(\Lambda_0 - c(y) - \sigma_0^2\right)\, dy\right],$$

Clearly V'_{β_n} converges to Q' uniformly on compact subsets. By direct differentiation, and limit analysis, Q solves

$$\begin{aligned} \frac{\sigma_0^2}{2}\left(Q''(x)+2\right) + b(x)Q'(x) + c(x) &= \Lambda_0 \qquad \text{on } (p^*, q^*), && (5.10)\\ Q''(x)+2 \leq 0 \ \text{ on } (p^*,q^*), \quad Q''+2\,|_{\partial(p^*,q^*)} &= 0 && (5.11) \end{aligned}$$

Furthermore Q' inherits the polynomial growth bound (5.4). Hence Q grows at most at a polynomial rate also. Therefore, we can follow the argument of section 3.4 to show that the feedback control $\pi^*(x) = \sigma_0 \mathbf{1}_{(p^*,q^*)}$ achieves Λ_0 as a stationary value. We have again proved (5.6).

We have shown that (5.6) is true for any sequence $\{\alpha_n\}$ converging to 0 along which $\alpha_n V_{\alpha_n}(0)$ is converges to a finite limit. We now argue that if $\alpha_n \to 0^+$, then $\alpha_n V_{\alpha_n}(0)$ must converge to a finite limit. Suppose not. Since $0 \leq \alpha V_\alpha(0) \leq c(0)$ for all $\alpha > 0$, the sequence $\{\alpha_n V_{\alpha_n}\}$ is bounded. If it does not converge it must contain two subsequences converging to different limits. But this contradicts what we have proved above. Therefore (5.6) is true for every sequence α_n converging to 0, which proves (5.1). We have also identified the optimal stationary controls claimed by Theorem 5.1 in the convergence arguments above. The statement about the convergence of $T^{-1}V_0(x,T)$ is proved exactly as in section 3. $\diamond$

We conclude with some sufficient conditions for optimality of the null control in the stationary problem and for the existence of an optimal stationary feedback policy $\pi(x) = \sigma_0 \mathbf{1}_{(p^*,q^*)}(x)$ where (p^*, q^*) is bounded. We only sketch the proofs. For the statement of these results we state the extension of $\hat{c}_\alpha$, defined in (3.2) for the analysis of the linear case, to the nonlinear case. Using the same notation, this extension is

$$\hat{c}_\alpha(x) \triangleq c''(x) + 2(\alpha - 2b'(x)).$$

We define also,

$$g_\alpha(x) := \hat{c}_\alpha(x) + b''(x)\frac{c'(x) - 2b(x)}{\alpha - b'(x)}.$$

Theorem 5.2 *Assume (2.2) and (2.9). If for some $\alpha_0 > 0$, $g_\alpha(x) \geq 0$ for all x and all $0 < \alpha \leq \alpha_0$, then $u^* \equiv 0$ is the optimal stationary control.*

Proof: It is shown in section 5 of [17] that $g_\alpha(x) \geq 0$ implies that the null control is optimal for the discounted control problem (2.6) with parameter α. Thus $\alpha V_\alpha(0) = c(0)$ for all $0 < \alpha \leq \alpha_0$. By the proof of Theorem 5.1, $\lambda_0 = c(0)$ and the null control is optimal for the stationary problem. $\diamond$

Theorem 5.3 *Assume (2.2) and (2.9). Suppose*

$$\begin{aligned} b'(0)\left(c''(0) - 4b'(0)\right) \quad &> \quad b''(0)c'(0), & (5.12)\\ \limsup_{x\to-\infty}\left(c'(x) - 2b(x)\right) < \quad 0 \quad &< \liminf_{x\to\infty}\left(c'(x) - 2b(x)\right). & (5.13)\end{aligned}$$

Then the optimal feedback control has the form $\pi^*(x) = \sigma_0 \mathbf{1}_{(p^*,q^*)}(x)$ where $p^* < 0 < q^*$ and (p^*, q^*) is bounded.

Proof: (Sketch) Condition (5.12) implies that for some $\alpha_0 > 0$ and $\epsilon_0 > 0$, we have that $\sup_{|x|\leq\epsilon_0}\sup_{0<\alpha<\alpha_0} g_\alpha(x) < 0$. Then it can be shown that

$$V''_{-\epsilon_0\epsilon_0}(x) + 2 < 0 \qquad \text{for all } x \text{ in } (-\epsilon_0, \epsilon_0) \text{ uniformly in } 0 < \alpha < \alpha_0. \tag{5.14}$$

Here $V_{-\epsilon_0\epsilon_0}$ is defined as in section 3.3, except that the drift b is used in place of $-\theta x$. One proves (5.14) by considering the differential equation satisfied by $W \triangleq V''_{-\epsilon_0\epsilon_0} + 2$ (see (5.5) in [17]) and applying the maximum principle . On the other hand $V_\alpha = V_{r_\alpha s_\alpha}$ on (r_α, s_α), where (r_α, s_α) is the connected component of G_α containing 0. One then compares $W^* \triangleq V''_{r_\alpha s_\alpha} + 2$ to W to show (r_α, s_α) contains $(-\epsilon_0, \epsilon_0)$.

Next we show that there is a constant K_2 such that $(r_\alpha, s_\alpha) \subset (-K_2, K_2)$ for all $0 < \alpha < \alpha_0$. Let $M \stackrel{\triangle}{=} \sup_{0<\alpha<\alpha_0} |V'_\alpha(0)|$, which by (5.4) is finite. On the one hand,

$$V'_\alpha(s_\alpha) < M - 2s_\alpha,$$

because $V''_\alpha < -2$ on (r_α, s_α). On the other hand, the the C^1 smoothness of V_α and equation (2.11) imply,

$$(\alpha - b'(s_\alpha))\, V'_\alpha(s_\alpha) = c'(s_\alpha) - 2b(s_\alpha).$$

By assumption (5.13), there is a K_3 such that $c'(x) - 2b(x) > 0$ for $x > K_3$. Since $\alpha - b'(s_\alpha) > 0$, $s_\alpha > K_3$ can occur only if $V'_\alpha(s_\alpha) > 0$, which requires $s_\alpha < M/2$. Therefore $s_\alpha \leq \max(M/2, K_3)$. By a similar argument, $|r_\alpha| \leq \max(M/2, K_4)$, where $c'(x) - 2b(x) < 0$ for $x < K_4$. These bounds are independent of α. Hence $(r_\alpha, s_\alpha) \subset (-K_2, K_2)$ for all $0 < \alpha < \alpha_0$, where $K_2 = \max(M/2, K_3, K_4)$.

We have shown that

$$(-\epsilon_0, \epsilon_0) \subset (p^*_\alpha, q^*_\alpha) \subset (-K_2, K_2) \qquad \text{for all } 0 < \alpha < \alpha_0.$$

In the proof of Theorem 5.1, we obtained (p^*, q^*) as a limit of a convergent subsequence of $(r_{\alpha_n}, s_{\alpha_n})$ where $\alpha_n \downarrow 0$. Thus $(-\epsilon_0, \epsilon_0) \subset (p^*, q^*) \subset (-K_2, K_2)$. $\diamond$

Remarks: The condition (5.12) implies that $\sup_{0<\alpha\leq\beta} g_\alpha(0) < 0$ for some $\beta > 0$.

The condition (5.13), which implies the boundedness of (p^*, q^*), is weaker than condition (4.1), which was used to get localization of the region of positive variance control in the linear drift case. In this section, we took advantage of a prior bound on the derivative of V_α which followed from the assumption of polynomial growth of c''. With a more careful construction of approximations to c in section 4, one can prove Theorem 4.1 in the linear case when only (5.13) holds.

6 Appendix

Proof of Theorem 2.1 (Sketch) Theorem 2.1 is proved for cost functions $c \in C^2_b(I\!R)$ in Theorems 3.1 and 3.2 in [17]. We extend it to the generality of assumption (2.8) by a localization argument.

Lemma 6.1 *Let $c \in C^2_b$ and let G_α be the optimal stopping set for the corresponding discounted problem (2.6). Let $\eta(x) \stackrel{\triangle}{=} (2\alpha)^{-1} \sup_{|y|\leq x+1} |c(y)|$, for $x > 0$. Then for any $x_0 > 0$.*

$$[x_0 - 1, x_0 + \eta(x_0) + 2] \cap G^c_\alpha \quad \textit{is non-empty.} \tag{6.1}$$

Proof: By using zero control ($u \equiv 0$) and non-negativity of c,

$$0 \leq V_\alpha(x) \leq \frac{1}{\alpha} \sup_{|y|\leq x} |c(y)|.$$

Pick a point $x_0 > 0$. By the mean value theorem and the previous estimate, there is a point x_1 in (x_0-1, x_0+1) such that $|V'_\alpha(x_1)| \leq \eta(x_0)$. Now suppose that the interval $[x_0-1, x_0+1]$ is contained in the component (p,q) of G_α. Since $V''_\alpha < -2$ on G_α, it follows that for any x in (p,q),

$$V_\alpha(x) \leq \eta(x_0)\,[1 + |x - x_1|] - (x - x_1)^2$$

An easy calculation then shows that $|x - x_1| < \eta(x_0) + 1$ for any x in (p,q). Now if $q \leq x_0 + \eta(x_0) + 2$, the result is trivial. If $q > x_0 + \eta(x_0) + 2$, we obtain a contradiction by choosing $x \in (p,q)$ so that $|x - x_1| > \eta(x_0) + 1$. $\diamond$

Now take a non-negative function c in C^2. We continue to define $\eta(x)$ as above. Pick a strictly increasing sequence $\{a_N\}$ such that $a_N > N + \eta(N) + 3$ for each positive integer N. Then construct an associated sequence of costs $\{c_N\}$ such that

(i) $c_N \leq c_{N+1} \leq c$ for every positive integer N;

(ii) $c_N(x) = c(x)$ on $[-a_N, a_N]$ for every positive integer N; and,

(iii) $c_N \in C_b^2(I\!R)$. for every positive integer N.

For each N, let $G_{\alpha,N}$ be the set defining the optimal discounted control associated to the cost c_N. By applying (6.1) to c_N and using (ii) and $a_N > N + 1$, we see that $[N, N + \eta(N+1) + 3] \cap G^c_{\alpha,N}$ for every N. Similarly $[-(N+\eta(N+1)+3), -N] \cap G^c_{\alpha,N}$ is non-empty for every N. For each N, let k_N be a point in $[N, N + \eta(N+1) + 2] \cap G^c_{\alpha,N}$, and let ℓ_N be a point in $[-(N+\eta(N)+3), -N] \cap G^c_{\alpha,N}$. Note that $-a_N < \ell_N < -N < N < k_N < a_N$.

Let V_α denote the value function defined at (2.6) using the cost function c, and, for each N, let $V_{\alpha,N}(x)$ be the value function defined using c_N. Theorems 3.1 and 3.2 of [17] state that Theorem 2.1 holds for each $V_{\alpha,N}$, because $c_N \in C_b^2(I\!R)$. Let X_x^N be the process obtained by using the feedback policy $\pi_{\alpha,N}(x) = \sigma_0 \mathbf{1}_{G_{\alpha,N}}(x)$; X_x^N is the optimal process achieving the value $V_{\alpha_N}(x)$. Because ℓ_n and k_N are not in $G_{\alpha,N}$ and the drift always points toward the origin, $X_x^N(t)$ will remain in $[\ell_N, k_N]$ for all time if the initial condition x is in $[\ell_N, k_N]$. By property (ii) of the sequence $\{c_N\}$, $c_N = c$ on $[\ell_N, k_N]$, and hence $V_{\alpha,N}(x) \geq V_\alpha(x)$ on $[\ell_N, k_N]$. However, by property (i), $V_{\alpha,N}(x) \leq V_\alpha(x)$ for all x. Thus $V_{\alpha,N}(x) = V_\alpha(x)$ on $[\ell_N, k_N]$ and hence V_α satisfies (2.11) on $[\ell_N, k_N]$. Since, by construction, $\lim_N \ell_N = -\infty$ and $\lim_N k_N = \infty$, Theorem 2.1 follows.

Let f be a continuously differentiable function on $I\!R$. Let Z_x solve the stochastic differential equation

$$Z_x(t) = x + \int_0^t f(Z_x(s))\,ds + \sigma_0 W(t),$$

where W is a Brownian motion from an admissible control system. Assume that the lifetime of Z_x is a.s. infinite for all x.

Theorem 6.2 *Let ℓ be a continuous function such that*

$$\inf_y \ell(y) > -\infty. \tag{6.2}$$

Let $\lambda > 0$ and define $U(x) = \inf_\tau E\left[\int_0^\tau e^{-\lambda t}\ell(Z_x(t))\,dt\right]$, where the infimum is over all $I\!F$ stopping times. Then U is the unique solution of

$$U \in C^1(I\!R) \cap L^\infty(I\!R), \quad U' \text{ is absolutely continuous and } U'' \in L^\infty_{loc}(I\!R), \tag{6.3}$$

$$\min\left\{\frac{\sigma_0^2}{2}U''(x) + f(x)U'(x) - \lambda U(x) + \ell(x), -U(x)\right\} = 0, \quad a.e. \text{ in } I\!R. \tag{6.4}$$

The set $\{x; U(x) = 0\}$ is the optimal stopping region.

Suppose, instead of (6.2), it is assumed that for some positive constants M, $K > 0$, and $p > 0$,

$$\ell^-(x) \le K\left(1 + |x|^p\right), \qquad xf(x) \le M, \qquad x \in I\!R, \tag{6.5}$$

where ℓ^- is the negative part of ℓ. Then U is the unique solution of (6.3′) and (6.4), where (6.3′) is the same as (6.3), except that the condition that $U \in L^\infty$ is replaced by the condition that $|U(x)|$ grows at most polynomially in x.

This result is entirely standard, with the exception that the problem is considered over an unbounded domain and the drift coefficient is unbounded. The requirement that $U \in C^1(I\!R)$ is the famous *smooth-fit principle*, which is here a rigorous statement.

Bensoussan and Lions [6], p. 398, provide a proof of a multi-dimensional extension of Thereom 6.2 under the additional assumption of polynomial growth on ℓ and its first derivative. However, the one-dimensional case is particularly simple and it is easy to extend the result to cover the assumptions of Theorem 6.2.

Here is an elementary approach to a proof. Rather than derive Lemma 3.4 from Proposition 3.3 one can take (3.10)-(3.11) as the starting point for an elementary proof of Proposition 3.3. For each connected component of $\{x; \hat{c}_\alpha(x) < 0\}$ define p and q as in (3.10)-(3.11). Let $\{(p_i, q_i); i \le I\}$, where I is countable, be an enumeration of the countable set of *distinct* intervals so constructed. By elementary differential equations arguments, it may be shown that (p_i, q_i) and (p_j, q_j) are disjoint if $i \ne j$, and, for each i, that

$$\frac{\sigma_0^2}{2}U''_{p_iq_i}(x) - \theta x U'_{p_iq_i}(x) - (2\theta+\alpha)U_{p_iq_i}(x) + \hat{c}_\alpha(x) = 0 \quad p_i < x < q_i \tag{6.6}$$

$$U_{p_iq_i}\big|_{\partial(p_i,q_i)} = 0 \qquad U'_{p_iq_i}\big|_{\partial(p_i,q_i)} = 0. \tag{6.7}$$

In (6.7), $\partial(p,q)$ denotes the boundary of the interval, that is, the set of finite endpoints. Set $U(x) = \sum_i \mathbf{1}_{(p_i,q_i)}(x) U_{p_i q_i}(x)$. Then, because of the boundary conditions in (6.7), $U \in C^1$. It may be checked directly that U is a solution of (3.7); a verification argument using Itô's rule then shows that $U \equiv U_\alpha$ and that $\{x; U_\alpha(x) = 0\}$ is an optimal stopping region.

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