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Point shift characterization of Palm measures on Abelian groups

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Abstract

Let **N** denote the space of locally finite simple counting measures on an Abelian topological group G that is assumed to be a locally compact, second countable Hausdorff space. A probability measure on **N** is a canonical model of a random point process on G. Our first aim in this paper is to characterize Palm measures of stationary measures on **N** through point stationarity. This generalizes the main result in (2) from the Euclidean case to the case of an Abelian group. While under a stationary measure a point process looks statistically the same from each site in G, under a point stationary measure it looks statistically the same from each of its points. Even in case $G = \mathbb{R}^d$ our proof will simplify some of the arguments in (2). A new technical result of some independent interest is the existence of a complete countable family of matchings. Using a change of measure we will generalize our results to discrete random measures. In case $G = \mathbb{R}^d$ we will finally treat general random measures by means of a suitable approximation.

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1 Introduction

The general subject of this paper is invariance properties of Palm measures of stationary random measures on an Abelian topological group G. Based on the notion of point-stationarity introduced by Thorisson in (11) we will prove here intrinsic characterizations of Palm measures. The significance of Palm measures both for theory and applications can hardly be overestimated. The modern approach to Palm measures of stationary random measures is mainly due to Matthes (6) and Mecke (8). Extensive expositions can be found, for instance in (7; 10; 1; 4).

As in (8) we assume the Abelian group G to be a locally compact, second countable Hausdorff space and denote the Borel σ-algebra on G by $\mathcal{G}$. A measure μ on $(G, \mathcal{G})$ is called Radon measure, if $\mu(C) < \infty$ for any compact $C \subset G$. On G there exists an invariant measure λ, that is unique up to normalization. We denote by $\mathbf{M}$ the set of all Radon measures on G, and by $\mathcal{M}$ the σ-algebra on $\mathbf{M}$ which is generated by the evaluation functionals $\mu \mapsto \mu(B)$, $B \in \mathcal{G}$. Whenever necessary we will use the notation $(\mathbf{M}(G), \mathcal{M}(G)) := (\mathbf{M}, \mathcal{M})$.

An $\mathbf{M}$-valued random element N is called a *random measure* on G. In this paper we choose a canonical setting, so a random measure on G is defined through a probability measure $\mathbb{P}$ on $(\mathbf{M}, \mathcal{M})$ and the identity map N on $\mathbf{M}$. More generally, we consider σ-finite measures on $(\mathbf{M}, \mathcal{M})$. Such a measure $\mathbb{P}$ is called *stationary*, if it is invariant under the translation operators $\theta_x : \mathbf{M} \to \mathbf{M}$, $x \in G$, which are defined by

$$\theta_x(\mu)(B) := \mu(B + x), \quad B \in \mathcal{G},$$

i.e., $\mathbb{P} = \theta_x(\mathbb{P}) := \mathbb{P} \circ \theta_x^{-1}$. The *Palm measure* $\mathbb{P}^0$ of a σ-finite stationary measure $\mathbb{P}$ is the σ-finite measure on $(\mathbf{M}, \mathcal{M})$ defined by

$$\mathbb{P}^0(A) := \int_{\mathbf{M}} \int_B \mathbf{1}_A(\theta_x \mu) \mu(\mathrm{d}x) \mathbb{P}(\mathrm{d}\mu), \quad A \in \mathcal{M}, \tag{1.1}$$

where $B \in \mathcal{G}$ is a Borel set such that $\lambda(B) = 1$. If $\mathbb{P}$ has a finite and positive *intensity* $\mathbb{P}^0(\mathbf{M})$ and is concentrated on the (measurable) subset $\mathbf{N} \subset \mathbf{M}$ of simple counting measures on G, then N is a simple point process and the normalized version (*Palm distribution*) of the Palm measure $\mathbb{P}^0$ can be interpreted as the distribution that describes the behaviour of N seen from a typical point of the process. The *support* of a measure $\mu \in \mathbf{M}$ is denoted by $\mathrm{supp}(\mu)$. We will identify a simple counting measure $\varphi \in \mathbf{N}$ with its support, and will equally refer to φ as locally finite point set. A random element in the space $\mathbf{M}_{\mathrm{d}} := \{\varphi \in \mathbf{M} : \mathrm{supp}(\varphi) \in \mathbf{N}\} \in \mathcal{M}$ of all Radon measures with discrete support can be interpreted as a point process with weights in $(0, \infty)$. If a σ-finite stationary measure $\mathbb{P}$ is concentrated on $\mathbf{M}_{\mathrm{d}}$ then its Palm measure is concentrated on the set $\mathbf{M}_{d,0} \in \mathcal{M}$ of all measures in $\mathbf{M}_{\mathrm{d}}$ having the neutral element $0 \in G$ in their support.

In his seminal paper (8) Mecke showed that a σ-finite measure $\mathbb{Q}$ on $\mathbf{M}$ is a Palm measure if and only if $\mathbb{Q}$ attributes no positive mass to the zero measure $\mathbf{0}$ on G, i.e. $\mathbb{Q}(\{\mathbf{0}\}) = 0$, and some integral equation is satisfied (cf. Section 4). In particular, no reference is made to the stationary measure. Characterization of Palm measures received new interest, when Thorisson (11) defined the notion of *point-stationarity*. He called a point process point-stationary if it "looks statistically the same from any of its points". We will use this property also for σ-finite measures on $\mathbf{M}_{\mathrm{d}}$ and refer to Section 5 for a formal definition. In (2) it was then shown that, for $G = \mathbb{R}^d$ and $\mathbb{Q}$ concentrated on $\mathbf{N}$, point-stationarity is a sufficient condition for the integral equation in Mecke's theorem to hold. The main purpose of this paper is to extend this result to general Abelian groups G and to measures on $\mathbf{M}_{\mathrm{d}}$:

Theorem 1.1. *Let $\mathbb{Q}$ be a σ-finite measure on $(\mathbf{M}, \mathcal{M})$ satisfying $\mathbb{Q}(\mathbf{M} \setminus \mathbf{M}_{d,0}) = 0$. Then $\mathbb{Q}$ is the Palm measure of some stationary, σ-finite measure $\mathbb{P}$ if and only if $\mathbb{Q}$ is σ-finite and the following measure $\zeta(\mathbb{Q})$ is point-stationary:*

$$\zeta(\mathbb{Q})(A) := \int_{A \cap \mathbf{M}_{d,0}} \varphi(\{0\})^{-1} \mathbb{Q}(\mathrm{d}\varphi), \quad A \in \mathcal{M}.$$

The paper is organized as follows. In Section 2 we introduce *bijective point maps* and *bijective point shifts* on $\mathbf{N}$ in a deterministic setting without any reference to a measure on $\mathbf{N}$. The main result in Section 3, Theorem 3.6, is of some independent interest and provides a *complete* family of *matchings*. Section 4 is a short summary of those facts on Palm measures that are needed in this paper. The proof of Theorem 1.1 is given in Section 5. Even for point processes on $\mathbb{R}^d$ it is a significant simplification of the one in (2). In fact we will show that Palm measures of stationary point processes can already be characterized by matchings.

In the final Section 6 we consider the case $G = \mathbb{R}^d$. Even in this setting, the generalization of the characterization results to Palm measures on $\mathbf{M}$ appears to be a difficult problem. In particular, it is not obvious how to extend the concept of a bijective point map to larger subclasses of the closed subsets of G. To circumvent these difficulties, we will propose here a discretization procedure for Radon measures on $\mathbb{R}^d$ that uses a sequence of finer and finer partitions of $\mathbb{R}^d$ into countably many cubes. A σ-finite measure $\mathbb{Q}$ on $\mathbf{M}$ is then shown to be a Palm measure if and only if all discretized versions of $\mathbb{Q}$ have the properties required by Theorem 1.1.

2 Point maps and point shifts

A point map can be thought of in two (equivalent) ways, either as a measurable function that maps locally finite sets with a point in 0 (the system of those sets is denoted by $\mathbf{N}_0$) to one of their points, or as an equivariant mapping $\mathbf{N} \times G \to G$ that, given an arbitrary $\varphi \in \mathbf{N}$, defines a function $G \to G$ that is stable on φ. When referring to the latter mapping we will use the term extended point map. More formally, we define:

Definition 2.1. A measurable mapping $\sigma : \mathbf{N} \to G$ such that $\sigma(\varphi) \in \varphi$ for all $\varphi \in \mathbf{N}_0$ and $\sigma(\varphi) = 0$ for all $\varphi \in \mathbf{N} \setminus \mathbf{N}_0$ is called a *point map*. The associated *extended point map* $\tilde{\sigma} : \mathbf{N} \times G \to G$ is defined by $\tilde{\sigma}(\varphi, x) := \sigma(\theta_x \varphi) + x$ for all $\varphi \in \mathbf{N}$ and $x \in G$.

With a point map σ we associate the mapping $\theta_\sigma : \mathbf{N} \to \mathbf{N}$ that shifts the simple point measure φ in such a way that $\sigma(\varphi)$ is translated to the origin.

Definition 2.2. Let $\sigma : \mathbf{N} \to G$ be a point map. The composed mapping $\theta_\sigma : \mathbf{N} \to \mathbf{N}$ defined by

$$\theta_\sigma(\varphi) := \theta_{\sigma(\varphi)} \varphi$$

is called the *point shift* associated with σ.

It follows straight from the definition of a point map σ that $\theta_\sigma(\varphi) \in \mathbf{N}_0$ whenever $\varphi \in \mathbf{N}_0$. We use the associated point shift to define the composition of two point maps σ and τ by

$$\sigma \circ \tau(\varphi) := \sigma(\theta_\tau(\varphi)) + \tau(\varphi).$$

It can easily be shown that the composition of point maps is associative.

In the sequel, the subclass of point maps that define a bijection on the points of any $\varphi \in \mathbf{N}$ is of particular interest.

Definition 2.3. We call a point map σ *bijective* if, for any $\varphi \in \mathbf{N}$, the induced mapping $\tilde{\sigma}(\varphi, \cdot) : G \to G$ is a bijection.

The composed point map $\sigma \circ \tau$ of two bijective point maps σ and τ is also bijective. A trivial example of a bijective point map is the mapping ν that maps any $\varphi \in \mathbf{N}$ to 0. This point map is the neutral element with respect to the composition of two point maps. Indeed, it is easy to verify that $\nu \circ \sigma = \sigma \circ \nu = \sigma$ for all point maps σ.

Definition 2.4. A point map τ is called the *inverse point map* σ^{-1} of the point map σ if $\sigma \circ \tau = \tau \circ \sigma = \nu$. A self-inverse point map is called a *matching*.

Given a bijective point map σ, we can define an inverse element with respect to the composition by letting $\tau(\varphi, x)$ be the unique $y \in G$ that satisfies $\sigma(\varphi, y) = 0$. Conversely if σ is a point map and τ the inverse point map then both σ and τ are bijective. Moreover, the following computational rules hold.

Lemma 2.5. *Let $\sigma, \tau : \mathbf{N} \to G$ be point maps and $\pi : \mathbf{N} \to G$ a matching. Then*

(a) $\theta_{\tilde{\sigma}(\varphi,x)}(\varphi) = \theta_\sigma(\theta_x \varphi), \quad \varphi \in \mathbf{N}, x \in G,$

(b) $\theta_{\sigma \circ \tau} = \theta_\sigma \circ \theta_\tau,$

(c) $\pi \circ \theta_\pi(\varphi) = -\pi(\varphi), \quad \varphi \in \mathbf{N}.$

The proof of the lemma is straightforward and can be omitted here. We have established that the set Π of bijective point maps equipped with the composition $\circ$ is a group. Moreover, from $\theta_\nu = \mathrm{id}_{\mathbf{N}}$ and Lemma 2.5 (b), we deduce that the *bijective point shifts* $\{\theta_\sigma : \sigma \in \Pi\}$ equipped with the composition also form a group, which acts (in the sense of a group operation) on the simple counting measures $\mathbf{N}$. Invariant measures under (slightly generalized) bijective point shifts will be the subject of Sections 5 and 6.

We will conclude the section with two examples of bijective point maps. For $G = \mathbb{R}$, a simple, yet important, example of a bijective point map is defined by associating the first strictly positive point in φ to any $\varphi \in \mathbf{N}_0$ that is neither upper bounded nor lower bounded and 0 to all other elements of $\mathbf{N}$.

In higher dimensions, the definition of non-trivial bijective point maps requires more care, and the first example of such a point map, known as *mutual nearest neighbour matching*, was given by Olle Häggström. The point map $\pi : \mathbf{N} \to G$ is defined by $\pi(\varphi) := x$ if $0, x \in \varphi$ are mutual unique nearest neighbours in φ, and by $\pi(\varphi) := 0$ otherwise. The symmetry in the condition ensures that π is a matching and, hence, a bijective point map.

3 A maximal family of matchings

We will now provide tools for the definition of more sophisticated bijective point maps and begin with index functions, that were first introduced on point processes by Holroyd and Peres in (3). They defined (isometry-invariant) index functions in order to generalize a tree construction for random graphs. A more detailed account on the subject was given by Timar in (12). Here we will only postulate invariance under shift operators and adapt the concept to our deterministic setting.

Definition 3.1. An injective, measurable function $f : \mathbf{N} \to [0, \infty)$ is called *index function.* The associated *extended index function* $\tilde{f} : \mathbf{N} \times G \to [0, \infty)$ is defined as the composed mapping

$$\tilde{f}(\varphi, x) := f(\theta_x \varphi), \quad \varphi \in \mathbf{N}, x \in G.$$

Assume that f is an index function and let $\varphi \in \mathbf{N}$. The mapping

$$x \mapsto \tilde{f}(\varphi, x) \tag{3.1}$$

is an injective function from G to $[0, \infty)$ if and only if, for all $x, y \in G,\ x \neq y$, we have $\theta_x \varphi \neq \theta_y \varphi$. In this case we say that φ is *aperiodic.* Locally finite sets φ such that $\theta_x \varphi = \theta_y \varphi$ for some distinct $x, y \in G$ are called *periodic.* In (3) and (12), almost sure injectivity of the function given in (3.1) is a defining property for the index function of a random point set.

The following example of an index function is constructed along the lines that Holroyd and Peres propose in (3). Let $\mathcal{B} = (B_n)$ denote a countable base of the topology of G, where $B_n, n \in \mathbb{N}$, are open sets with compact closure. We then define the *standard index function* $I : \mathbf{N} \to [0, \infty)$ by

$$I(\varphi) := \sum_{n \in \mathbb{N}} 2^{-n}\, \mathbf{1}\{\varphi(B_n) \neq 0\}. \tag{3.2}$$

Lemma 3.2. *The function I is an index function. In particular, for $\varphi, \eta \in \mathbf{N}$, we have $\varphi = \eta$ if and only if $I(\varphi) = I(\eta)$.*

Proof: As a pointwise limit function of measurable functions, I is also measurable. For $\varphi, \eta \in \mathbf{N}$ such that $\varphi \not\subset \eta$, there exists $x \in G$ such that $x \in \varphi$ and $x \notin \eta$. Since $G \setminus \eta$ is an open subset of G, there exists $n \in \mathbb{N}$ such that $x \in B_n \subset G \setminus \eta$. Hence, $\varphi \cap B_n \neq \emptyset$ and $\eta \cap B_n = \emptyset$, and we conclude that $I(\varphi) \neq I(\eta)$. □

Index functions can be applied to choose a subset ψ from a locally finite set $\varphi \in \mathbf{N}$ in a measurable and equivariant manner.

Definition 3.3. A *selection function* is a measurable mapping $S : \mathbf{N} \to \mathbf{N}$ such that $\theta_x \circ S(\varphi) = S \circ \theta_x(\varphi)$ and $S(\varphi) \subset \varphi$ for all $\varphi \in \mathbf{N}$.

It is easy to prove that the composition $S \circ T$ of two selection functions S, T is again a selection function. More importantly, the composition of a (bijective) point map with a selection function is again a (bijective) point map.

Proposition 3.4. *Let σ be a point map resp. bijective point map and T a selection function. Then $\sigma \circ T$ is a point map resp. bijective point map.*

Proof: The mapping $\sigma \circ T$ is measurable as a composition of measurable mappings. Assume that $\varphi \in \mathbf{N}_0$. If $T(\varphi) \notin \mathbf{N}_0$ then we have $\sigma(T(\varphi)) = 0 \in \varphi$. Otherwise, we have $T(\varphi) \in \mathbf{N}_0$ and then

$$\sigma \circ T(\varphi) = \sigma(T(\varphi)) \in T(\varphi) \subset \varphi.$$

Also, $\varphi \notin \mathbf{N}_0$ yields $T(\varphi) \notin \mathbf{N}_0$ and hence $\sigma \circ T(\varphi) = 0$. Hence, $\sigma \circ T$ is a point map.

If σ is bijective then there exists an inverse point map σ^{-1}, and we claim that $\sigma^{-1} \circ T$ is the inverse point map of $\sigma \circ T$. Indeed, we have for all $\varphi \in \mathbf{N}$ that

$$\begin{aligned}(\sigma \circ T) \circ (\sigma^{-1} \circ T)(\varphi) &= \sigma \circ T(\theta_{\sigma^{-1} \circ T}(\varphi)) + \sigma^{-1} \circ T(\varphi) \\ &= \sigma \circ T \circ \theta_{\sigma^{-1} \circ T(\varphi)}(\varphi) + \sigma^{-1} \circ T(\varphi) \\ &= \sigma \circ \theta_{\sigma^{-1} \circ T(\varphi)} \circ T(\varphi) + \sigma^{-1} \circ T(\varphi) \\ &= \sigma \circ \theta_{\sigma^{-1}}(T(\varphi)) + \sigma^{-1}(T(\varphi)) \\ &= \sigma \circ \sigma^{-1}(T(\varphi)) = 0.\end{aligned}$$

We have proved that the point map $\sigma^{-1} \circ T$ is the left inverse point map of $\sigma \circ T$, and, exchanging the roles of σ and σ^{-1}, we obtain that it is also the right inverse point map of $\sigma \circ T$. Hence, $(\sigma \circ T)^{-1} = \sigma^{-1} \circ T$ and, in particular, $\sigma \circ T$ is a bijective point map. □

For a specific choice of $\varphi \in \mathbf{N}$, extended point maps may act trivially on the points of φ. Consider, for example, $\varphi \in \mathbf{N}$ consisting of a collection of infinite descending chains, i.e., a point set without mutual nearest neighbours. Even though the extended point map associated with mutual nearest neighbour matching (cf. Section 2) satisfies $\pi(\varphi, x) = x$ for all $x \in G$, the extended point map is non-trivial on subsets of φ, which do contain mutual nearest neighbours. The following proposition shows that selection functions are an efficient tool to throw away points, which may impede a point map σ from mapping φ to a point other than the origin.

Proposition 3.5. *There exists a countable family of selection functions $\{S_n : n \in \mathbb{N}\}$ such that, for any $B \in \mathcal{G}$ with compact closure, $\varphi \in \mathbf{N}$ and $x, y \in \varphi$, there exists $n \in \mathbb{N}$ with*

$$\tilde{I}(\varphi, z) \in \{\tilde{I}(\varphi, x), \tilde{I}(\varphi, y)\} \tag{3.3}$$

for all $z \in S_n(\varphi) \cap B$, where $\tilde{I}$ denotes the extended version of the standard index function defined in (3.2).

Proof: Let $\{(q_n, r_n, s_n) : n \in \mathbb{N}\}$ be dense in $[0,1]^3$. Define the selection functions $S_n : \mathbf{N} \to \mathbf{N}$, $n \in \mathbb{N}$, by

$$S_n(\varphi) := \{z \in \varphi : \min\{|\tilde{I}(\varphi, z) - q_n|, |\tilde{I}(\varphi, z) - r_n|\} \le s_n\}.$$

A point $z \in \varphi$ is in $S_n(\varphi)$ if and only if the extended index function $\tilde{I}$ applied to (φ, z) is close enough to q_n or r_n. The measurability of the selection function S_n follows from the measurability of $\tilde{I}$.

Fix $B \in \mathcal{G}$ with compact closure, $\varphi \in \mathbf{N}$ and $x, y \in \varphi$. From the local finiteness of φ we deduce that the set

$$\{\tilde{I}(\varphi, z) : z \in \varphi \cap B\}$$

is a finite subset of $[0, 1]$. Hence, there exists $\varepsilon > 0$ such that

$$\{|\tilde{I}(\varphi, z) - \tilde{I}(\varphi, x)|, |\tilde{I}(\varphi, z) - \tilde{I}(\varphi, y)| : z \in \varphi \cap B\} \cap (0, \varepsilon) = \emptyset.$$

Moreover, there exists $n \in \mathbb{N}$ such that $0 < s_n < \varepsilon/2$ and both $|\tilde{I}(\varphi, x) - q_n| < s_n$ and $|\tilde{I}(\varphi, y) - r_n| < s_n$ hold, finishing the proof of the proposition. □

A moderate variation of mutual nearest neighbour matching, was defined in (2), Section 4, the matchings by *symmetric area search*. In this example, the symmetric nearest neighbour condition is replaced by a unique neighbour condition in a symmetric area as follows.

Fix a Borel set $B \in \mathcal{G}$. Then call $y \in \varphi$ a *B-neighbour* of $x \in \varphi$ if $y \in x + B$. Clearly, y is a B-neighbour of x if and only if x is a B^*-neighbour of y, where the reflected set B^* is defined by $B^* := \{-x : x \in B\}$. We say that y is the *unique B-neighbour* of x if $(x + B) \cap \varphi = \{y\}$. A point y can be the unique B-neighbour of x, and x a non-unique B^*-neighbour of y. We then define a point map $\pi_B : \mathbf{N} \to G$ by

$$\pi_B(\varphi) := \begin{cases} x & \text{if } 0 \text{ and } x \text{ are mutual unique } B \cup B^*\text{-neighbours,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

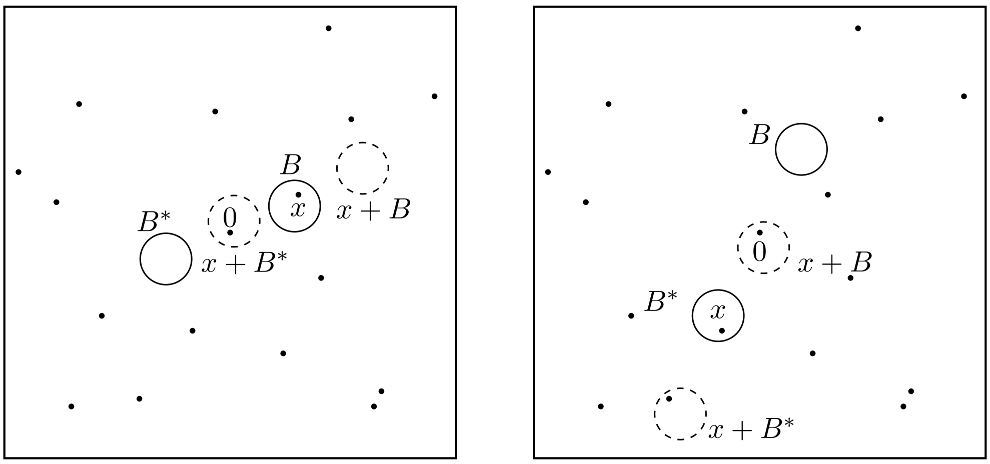

Figure 1: Matching by symmetric area search

In Figure 1, we illustrate on the left hand side the case where x is a unique $B \cup B^*$-neighbour of 0 and 0 a unique $B \cup B^*$-neighbour of x. Not so on the right hand side, where we have $\text{card}(x + (B \cup B^*)) = 2$. We will now combine the selection functions from Proposition 3.5 and the concept of symmetric area search to prove our first main result.

Theorem 3.6. *There exists a countable family of matchings (π_n) that satisfies*

$$\{\pi_n(\varphi) : n \in \mathbb{N}\} = \{x \in \varphi : \tilde{I}(\varphi, x) \neq \tilde{I}(\varphi, 0)\} \cup \{0\} \tag{3.5}$$

for all $\varphi \in \mathbf{N}_0$, i.e., for any $\varphi \in \mathbf{N}_0$ and $x \in \varphi$ with $\varphi \neq \varphi - x$, there exists $n \in \mathbb{N}$ such that $\pi_n(\varphi) = x$.

Lemma 3.7. *There exists a countable family of matchings (σ_n) such that*

$$\{\sigma_n(\varphi) : n \in \mathbb{N}\} \supset \{x \in \varphi : \tilde{I}(\varphi, x) \neq \tilde{I}(\varphi, 0) \text{ and } \tilde{I}(\varphi, x) \neq \tilde{I}(\varphi, -x)\} \cup \{0\} \tag{3.6}$$

for all $\varphi \in \mathbf{N}_0$.

Proof: Let $\{B_m\}$ be a countable base of the topology of G. We consider the matchings $\tau_m := \pi_{B_m}, m \in \mathbb{N}$, defined in (3.4) and the family of selection functions $\{S_k : k \in \mathbb{N}\}$ from Proposition 3.5.

Let $\varphi \in \mathbf{N}_0$ and $x \in \varphi$ such that $\tilde{I}(\varphi, x) \neq \tilde{I}(\varphi, 0)$ and $\tilde{I}(\varphi, x) \neq \tilde{I}(\varphi, -x)$. There exists an open set U with compact closure in G such that $0, x, -x, 2x \in U$. By Proposition 3.5 there exists $k \in \mathbb{N}$ such that, for all $z \in S_k(\varphi) \cap U$, we have $\tilde{I}(\varphi, z) \in \{\tilde{I}(\varphi, 0), \tilde{I}(\varphi, x)\}$. From $\tilde{I}(\varphi, -x) \notin \{\tilde{I}(\varphi, 0), \tilde{I}(\varphi, x)\}$ we obtain $\varphi \neq \varphi + x$ and $\varphi - x \neq \varphi + x$, and we deduce $\tilde{I}(\varphi, 2x) \notin \{\tilde{I}(\varphi, 0), \tilde{I}(\varphi, x)\}$, so $-x, 2x \notin S_k(\varphi)$. Since $\{B_m\}$ is a base of the topology of G there exists $m \in \mathbb{N}$ such that $B_m \cap \varphi = \{x\}$, $x + B_m^* \cap \varphi = \{0\}$, $B_m^* \cap \varphi \subset \{-x\}$ and $x + B_m \cap \varphi \subset \{2x\}$. We conclude that $\tau_m \circ S_k(\varphi) = x$. Reenumeration of the countable family of matchings $\{\tau_m \circ S_k : m, k \in \mathbb{N}\}$ yields a family of matchings $\{\sigma_n : n \in \mathbb{N}\}$ that satisfies (3.6). □

Lemma 3.8. *There exists a countable family of matchings* (σ_n) *such that*

$$\{\sigma_n(\varphi) : n \in \mathbb{N}\} \supset \{x \in \varphi : \tilde{I}(\varphi, x) \neq \tilde{I}(\varphi, 0) \text{ and } \tilde{I}(\varphi, x) = \tilde{I}(\varphi, -x)\} \tag{3.7}$$

for all $\varphi \in \mathbf{N}_0$.

Proof: For $n \in \mathbb{N}$, define a mapping $\sigma_n : \mathbf{N} \to G$ by

$$\sigma_n(\varphi) := \begin{cases} x & \text{if } \varphi \cap B_n = \{x\}, \varphi \cap B_n^* = \{-x\}, \varphi \cap (x + B_n^*) = \{0\}, \\ & \varphi \cap (x + B_n) = \{2x\} \text{ and } \tilde{I}(\varphi, -x) = \tilde{I}(\varphi, x) > \tilde{I}(\varphi, 0), \\ x & \text{if } \varphi \cap B_n^* = \{x\}, \varphi \cap B_n = \{-x\}, \varphi \cap (x + B_n) = \{0\}, \\ & \varphi \cap (x + B_n^*) = \{2x\} \text{ and } \tilde{I}(\varphi, -x) = \tilde{I}(\varphi, x) < \tilde{I}(\varphi, 0), \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

If $\varphi \in \mathbf{N}_0$ and $x \in \varphi$ such that $\tilde{I}(\varphi, x) = \tilde{I}(\varphi, -x)$, then $-x, 2x \in \varphi$. Moreover, there exists $n \in \mathbb{N}$ such that x is the unique B_n-neighbour of 0 in φ, $-x$ the unique B_n^*-neighbour of 0, 0 the unique B_n^*-neighbour of x in φ and $2x$ the unique B_n-neighbour of x, hence, $\sigma_n(\varphi) = x$. As $\tilde{I}(\varphi, x) = \tilde{I}(\varphi, -x) > \tilde{I}(\varphi, 0)$ implies $\tilde{I}(\theta_x\varphi, -x) = \tilde{I}(\theta_x\varphi, x) < \tilde{I}(\theta_x\varphi, 0)$, it easily follows that σ_n is a matching.

Analogously, if $\tilde{I}(\varphi, x) = \tilde{I}(\varphi, -x) < \tilde{I}(\varphi, 0)$, there exists $n \in \mathbb{N}$ such that $x \in B_n^*$ and $\sigma_n(\varphi) = x$. □

Lemma 3.9. *Let* $\varphi \in \mathbf{N}_0$ *and* $x \in \varphi \backslash \{0\}$ *and assume that* $\varphi = \varphi - x$. *Then there is no matching* π *that satisfies* $\pi(\varphi) = x$.

Proof: Let π be a point map such that $\pi(\varphi) = x$. Then

$$\pi^2(\varphi) = \pi(\theta_\pi\varphi) + \pi(\varphi) = \pi(\theta_x\varphi) + x = 2x,$$

hence, π is not a matching. □

Proof of Theorem 3.6: Let $\{\pi_n : n \in \mathbb{N}\}$ be the family of matchings containing the point maps defined in Lemma 3.7 and Lemma 3.8. Then the inclusion "$\supset$" in (3.5) is a consequence of these two Lemmas and the reverse inclusion follows from Lemma 3.9. □

4 Palm measures

We will recall a few basic results on Palm measures referring to (9; 10) for more details. We denote by λ a (non-trivial) invariant measure on G. For measurable $f : \mathbf{M} \times G \to [0, \infty]$, we have

$$\int_{\mathbf{M}} \int_G f(\theta_x \mu, x) \mu(\mathrm{d}x) \mathbb{P}(\mathrm{d}\mu) = \int_{\mathbf{M}} \int_G f(\mu, x) \lambda(\mathrm{d}x) \mathbb{P}^0(\mathrm{d}\mu). \tag{4.1}$$

Equation (4.1) is one cornerstone of Palm theory, known as the *refined Campbell formula*. When applied to the function $f(\mu, x) := \mathbf{1}_A(\mu)\, \mathbf{1}_B(x)$ for some $B \in \mathcal{B}$ with $\lambda(B) = 1$ and some $A \in \mathcal{M}$, it reduces to the defining equation (1.1) of Palm measures. Also, for measurable functions $h : \mathbf{M} \times G \to [0, \infty]$ such that $\int h(\mu, x)\mu(\mathrm{d}x) = 1$ for all $\mu \in \mathbf{M}$ and $f : \mathbf{M} \to [0, \infty]$ such that $f(\mathbf{0}) = 0$, where $\mathbf{0}$ denotes the zero measure on G, we obtain the *inversion formula*

$$\int_{\mathbf{M}} f(\mu) \mathbb{P}(\mathrm{d}\mu) = \int_{\mathbf{M}} \int_G h(\theta_{-x}\mu, x) f(\theta_{-x}\mu) \lambda(\mathrm{d}x) \mathbb{P}^0(\mathrm{d}\mu), \tag{4.2}$$

that allows to express the restriction of the stationary measure $\mathbb{P}$ to $\mathbf{M} \setminus \{\mathbf{0}\}$ by means of the Palm measure $\mathbb{P}^0$. In the special case, where $\mathbb{P}$ is the distribution of a simple point process with finite intensity, a function h can be defined using stationary partitions (e.g. the Voronoi mosaic) providing a probabilistic interpretation of the inversion formula (cf. (5; 11)). An explicit definition of a function h as in (4.2) can be found in Section 2 of (8).

A converse of the refined Campbell formula is given in Satz 2.3 in (8). Indeed, if $\mathbb{P}$ and $\mathbb{Q}$ are σ-finite measures on $(\mathbf{M}, \mathcal{M})$ and (4.1) holds for all measurable $f : \mathbf{M} \times G \to [0, \infty]$ then $\mathbb{P}$ is stationary and $\mathbb{Q} = \mathbb{P}^0$ is the associated Palm measure. In particular, the stationarity of $\mathbb{P}$ follows in a straightforward manner from (4.1). This joint characterization of stationary and Palm measure and the explicit definition of a function h as in (4.2) can be used to prove that a σ-finite measure $\mathbb{Q}$ on $(\mathbf{M}, \mathcal{M})$ is the Palm measure of some σ-finite stationary measure $\mathbb{P}$ on $(\mathbf{M}, \mathcal{M})$ if and only if $\mathbb{Q}(\{\mathbf{0}\}) = 0$ and

$$\int_{\mathbf{M}} \int_G f(\theta_x \mu, -x) \mu(\mathrm{d}x) \mathbb{Q}(\mathrm{d}\mu) = \int_{\mathbf{M}} \int_G f(\mu, x) \mu(\mathrm{d}x) \mathbb{Q}(\mathrm{d}\mu) \tag{4.3}$$

holds for all measurable $f : \mathbf{M} \times G \to [0, \infty]$. This intrinsic characterization of Palm measures is given in Satz 2.5 in (8).

Often the following slight generalisation of the above concepts is useful. Let $M : \mathbf{M} \to \mathbf{M}$ be a measurable mapping that is equivariant, i.e.

$$M(\theta_x \mu) = \theta_x M(\mu), \quad \mu \in \mathbf{M}, x \in G. \tag{4.4}$$

Consider a σ-finite and stationary measure $\mathbb{P}$ on $(\mathbf{M}, \mathcal{M})$. Then we can define a σ-finite measure $\mathbb{P}^0_M$ (the Palm measure of M w.r.t. $\mathbb{P}$) on $(\mathbf{M}, \mathcal{M})$ satisfying

$$\int_{\mathbf{M}} \int_G f(\theta_x \mu, x) M(\mu)(\mathrm{d}x) \mathbb{P}(\mathrm{d}\mu) = \int_{\mathbf{M}} \int_G f(\mu, x) \lambda(\mathrm{d}x) \mathbb{P}^0_M(\mathrm{d}\mu). \tag{4.5}$$

With M fixed, the appropriate version of (4.3) is still characteristic for Palm measures. Applications require to go even a step further and to replace $(\mathbf{M}, \mathcal{M})$ by an abstract measurable space equipped with a flow indexed by the group (see e.g. (10; 2)). With obvious modifications all results of this paper remain valid in this framework.

5 Characterization of Palm measures

In this section we give the precise definition of point-stationarity and prove Theorem 1.1. The spaces $\mathbf{M}_\mathrm{d}$, $\mathbf{M}_{d,0}$, and $\mathbf{N}$ (defined in the introduction) are measurable subsets of $\mathbf{M}$ and the mapping $\mathrm{supp} : \mathbf{M}_\mathrm{d} \to \mathbf{N}, \varphi \mapsto \mathrm{supp}(\varphi)$, that maps a measure in $\mathbf{M}_\mathrm{d}$ to its support, is measurable with respect to the restrictions of the cylindrical σ-field on $\mathbf{M}$ to $\mathbf{M}_\mathrm{d}$ resp. $\mathbf{N}$. This allows for the following generalisation of point maps and point shifts.

Definition 5.1. Let σ be a point map and θ_σ the associated point shift. The domain of σ is extended to $\mathbf{M}$ by

$$\sigma(\varphi) := \begin{cases} \sigma(\mathrm{supp}(\varphi)) & \text{if } \mathrm{supp}(\varphi) \in \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Accordingly, the domain and range of the associated point shift are extended to $\mathbf{M}$ by $\theta_\sigma(\varphi) := \theta_{\sigma(\varphi)}(\varphi)$.

Definition 5.2. A σ-finite measure $\mathbb{Q}$ on $(\mathbf{M}, \mathcal{M})$ is *point-stationary* if $\mathbb{Q}(\mathbf{M} \setminus \mathbf{M}_{d,0}) = 0$ and $\mathbb{Q}$ is invariant under bijective point shifts, i.e., $\theta_\sigma(\mathbb{Q}) = \mathbb{Q}$ for all bijective point maps σ.

In the special case $G = \mathbb{R}^d$, it has been shown that point-stationarity is a characterizing property of Palm measures concentrated on $\mathbf{N}$ (cf. Theorem 4.2 in (2)). The Palm measure of a stationary measure on $\mathbf{M}_d$ is in general not point-stationary. A simple σ-finite counterexample is provided by "randomizing" a measure in $\mathbf{N}$ that puts different masses on two distinct points. Therefore we will use the following transformation of σ-finite measures on $(\mathbf{M}, \mathcal{M})$.

Definition 5.3. The *zero compensated version* $\zeta(\mathbb{Q})$ of a σ-finite measure $\mathbb{Q}$ on $(\mathbf{M}, \mathcal{M})$ is defined by

$$\zeta(\mathbb{Q})(A) := \mathbb{Q}(A \setminus \mathbf{M}_{d,0}) + \int_{A \cap \mathbf{M}_{d,0}} \varphi(\{0\})^{-1} \mathbb{Q}(\mathrm{d}\varphi), \quad A \in \mathcal{M}.$$

The zero compensated version $\zeta(\mathbb{Q})$ is σ-finite resp. concentrated on $\mathbf{M}_{d,0}$ whenever $\mathbb{Q}$ is σ-finite resp. concentrated on $\mathbf{M}_{d,0}$. To see that this is a natural transformation we introduce a measurable mapping $M : \mathbf{M} \to \mathbf{M}$ by $M(\mu)(B) := \mathrm{card}(\mathrm{supp}(\mu) \cap B)$, $B \in \mathcal{G}$, if $\mu \in \mathbf{M}_\mathrm{d}$ and $M(\mu) := \mu$ otherwise. This mapping satisfies (4.4). If $\mathbb{P}$ be a σ-finite and stationary measure on $(\mathbf{M}, \mathcal{M})$, then (4.5) implies that the Palm measure $\mathbb{P}^0_M$ of M is just $\zeta(\mathbb{P}^0)$. If $\mathbb{P}$ is concentrated on $\mathbf{M}_\mathrm{d}$, then Satz 4.3 in (9) (see also Theorem 9.4.1 in (11) and Theorem 3.1 in (2)) implies that $\mathbb{P}^0_M = \zeta(\mathbb{P}^0)$ is point-stationary.

Proof of Theorem 1.1: If $\mathbb{Q}$ is the Palm measure of some stationary and σ-finite measure $\mathbb{P}$ then $\mathbb{Q}$ is also σ-finite and, as argued above, $\zeta(\mathbb{Q})$ is point-stationary. A short and explicit proof can be given by suitably modifying the proof of Theorem 3.1 in (2).

Let us now assume that $\mathbb{Q}$ is σ-finite and $\zeta(\mathbb{Q})$ point-stationary. Then

$$\mathbb{Q}(\mathbf{M} \setminus \mathbf{M}_{d,0}) = \zeta(\mathbb{Q})(\mathbf{M} \setminus \mathbf{M}_{d,0}) = 0,$$

and $\zeta(\mathbb{Q})$ is invariant under bijective point shifts. By Mecke's characterization theorem we have to show that, for any measurable $f : \mathbf{M} \times G \to [0, \infty]$, the characterizing equation (4.3) holds.

We fix such an f and a matching π. The invariance of $\zeta(\mathbb{Q})$ under the associated bijective point shift θ_π yields

$$\begin{aligned}
&\int_{\mathbf{M}} \mathbf{1}\{\theta_\pi(\varphi) \neq \varphi\} f(\varphi, \pi(\varphi))\varphi(\{\pi(\varphi)\})\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}} \mathbf{1}\{\theta_\pi(\varphi) \neq \varphi\} f(\varphi, \pi(\varphi))\varphi(\{\pi(\varphi)\})\varphi(\{0\})\zeta(\mathbb{Q})(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}} \mathbf{1}\{\varphi \neq \theta_\pi(\varphi)\} f(\theta_\pi(\varphi), -\pi(\varphi))\theta_\pi(\varphi)(\{-\pi(\varphi)\})\theta_\pi(\varphi)(\{0\})\zeta(\mathbb{Q})(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}} \mathbf{1}\{\varphi \neq \theta_\pi(\varphi)\} f(\theta_\pi(\varphi), -\pi(\varphi))\varphi(\{0\})\varphi(\{\pi(\varphi)\})\zeta(\mathbb{Q})(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}} \mathbf{1}\{\varphi \neq \theta_\pi(\varphi)\} f(\theta_\pi(\varphi), -\pi(\varphi))\varphi(\{\pi(\varphi)\})\mathbb{Q}(\mathrm{d}\varphi),
\end{aligned}$$

where we have used parts (a) and (c) of Lemma 2.5. Using the family $\{\pi_n : n \in \mathbb{N}\}$ of matchings defined in Theorem 3.6 and exploiting the completeness property (3.5), we obtain

$$\begin{aligned}
&\int_{\mathbf{M}} \int_G \mathbf{1}\{\theta_x\varphi \neq \varphi\} f(\varphi, x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}} \sum_{n\in\mathbb{N}} \mathbf{1}\{\pi_n(\varphi) \neq 0\}\, \mathbf{1}\{\pi_n(\varphi) \neq \pi_m(\varphi) : 1 \leq m < n\} f(\varphi, \pi_n(\varphi))\varphi(\pi_n(\varphi))\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}} \sum_{n\in\mathbb{N}} \mathbf{1}\{\pi_n(\varphi) \neq 0\}\, \mathbf{1}\{\pi_n(\varphi) \neq \pi_m(\varphi) : 1 \leq m < n\} f(\theta_{\pi_n}\varphi, -\pi_n(\varphi))\theta_{\pi_n}\varphi(0)\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}} \int_G \mathbf{1}\{\theta_x\varphi \neq \varphi\} f(\theta_x\varphi, -x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi), \qquad (5.1)
\end{aligned}$$

where, in the second step, we have used that

$$-\pi_n(\varphi) \neq \pi_m(\theta_{\pi_n}\varphi) \iff \pi_m(\theta_{\pi_n}\varphi) + \pi_n(\varphi) \neq 0 \iff \pi_m \circ \pi_n(\varphi) \neq 0 \iff \pi_m(\varphi) \neq \pi_n(\varphi).$$

Since, for $\varphi \in \mathbf{M}_{d,0}$ and $x \in G$ such that $\theta_x\varphi = \varphi$, we have $\theta_{-x}\varphi = \varphi$ and, in particular, $\varphi(\{-x\} = \varphi(\{x\})$, we obtain

$$\begin{aligned}
\int_{\mathbf{M}} \int_G \mathbf{1}\{\theta_x\varphi = \varphi\} f(\varphi, x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi) &= \int_{\mathbf{M}} \int_G \mathbf{1}\{\theta_x\varphi = \varphi\} f(\theta_{-x}\varphi, x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}} \int_G \mathbf{1}\{\theta_x\varphi = \varphi\} f(\theta_x\varphi, -x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi).
\end{aligned}$$

Combined with (5.1) this implies (4.3), concluding the proof of the theorem. □

6 Discretisation of general random measures

The generalization of the point shift characterization to general Palm measures on $\mathbf{M}$ appears to be a difficult problem. In particular, only little is known about bijective point maps that are defined on the support of general random measures, i.e., closed subsets of G. So neither formulation nor proof of Theorem 1.1 do easily transfer to the general case.

In this section, we will restrict our attention to the case $G = \mathbb{R}^d$ and circumvent the difficulties described above by a discretization procedure. We denote by $\mathbb{D}_n$ the subgroup of $(\mathbb{R}^d, +)$ formed by the dyadic numbers of order $n \in \mathbb{N}$, i.e.,

$$\mathbb{D}_n = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i = k_i/2^n, (k_1, \ldots, k_d) \in \mathbb{Z}^d\}.$$

Equipped with the discrete topology $(\mathbb{D}_n, +)$ is a lcscH group. Let $C_n := [-2^{-n-1}, 2^{-n-1})^d$ be the product of d half-open intervals $[-2^{-n-1}, 2^{-n-1})$ and $d_n : \mathbb{R}^d \to \mathbb{D}_n$ the mapping such that, for $x \in \mathbb{R}^d$, $d_n(x)$ is the (unique) point in $\mathbb{D}_n$ such that $x \in C_n + d_n(x)$. Conversely, we denote by $i_n : \mathbb{D}_n \to \mathbb{R}^d$ the embedding of $\mathbb{D}_n$ into $\mathbb{R}^d$.

The counting measure is the (up to a constant) unique translation invariant measure on $\mathbb{D}_n$ and we define the normalization λ_n of the counting measure, in such a way that $\lambda_n(\mathbb{D}_n \cap [0,1)^d) = 1$. For $n \in \mathbb{N}$, a discretisation operator D_n is defined by

$$D_n(\mu) := d_n(\mu) = \sum_{x \in \mathbb{D}_n} \mu(C_n + x)\delta_x,$$

and called *dyadic lattice discretisation operator* of order n. Conversely, an embedding operator $I_n : \mathbf{M}(\mathbb{D}_n) \to \mathbf{M}(\mathbb{R}^d)$ is defined by

$$I_n(\mu) := i_n(\mu) = \sum_{x \in \mathbb{D}_n} \mu(\{x\})\delta_{i_n(x)}.$$

We have $D_n(\lambda^d) = \lambda_n$ and vague convergence $\lambda_n \xrightarrow{v} \lambda^d$ for $n \to \infty$, where λ^d is the Lebesgue measure on $\mathbb{R}^d$ and where we refer to the Appendix A2 in (4) for basic facts on vague convergence. More generally, the operators D_n and I_n have the following properties.

Lemma 6.1. *The composition $D_n \circ I_n$ is the identity mapping on $\mathbf{M}(\mathbb{D}_n)$. Conversely, for $\varphi \in \mathbf{M}(\mathbb{R}^d)$, the sequence of measures $(I_n \circ D_n(\varphi))$ converges vaguely towards φ, i.e., $I_n \circ D_n(\varphi) \xrightarrow{v} \varphi$ for $n \to \infty$.*

Proof: The first claim follows immediately from the definitions of I_n and D_n. The second is a consequence of the fact that continuous functions with compact support on $\mathbb{R}^d$ are Riemann integrable. □

The dominated convergence theorem yields the following corollary on weak convergence of finite measures on $\mathbf{M}(\mathbb{R}^d)$ (see (7) or (1) for an extensive discussion of this weak convergence).

Corollary 6.2. *Let $\mathbb{P}$ be a finite measure on $(\mathbf{M}(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d))$. Then the sequence of measures $(I_n \circ D_n(\mathbb{P}))$ converges weakly towards $\mathbb{P}$.*

For $x \in \mathbb{D}_n$ the relation $\theta_x \circ D_n = D_n \circ \theta_x$ yields that, for a stationary measure $\mathbb{P}$ on $(\mathbf{M}(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d))$, $D_n(\mathbb{P})$ is a stationary measure on $(\mathbf{M}(\mathbb{D}_n), \mathcal{M}(\mathbb{D}_n))$. It is natural to ask for the connection between the Palm measure $(D_n(\mathbb{P}))^0$ of $D_n(\mathbb{P})$ and the Palm measure $\mathbb{P}^0$ of $\mathbb{P}$. However, σ-finiteness of the stationary measure is a necessary condition for the definition of the associated Palm measure. We will see in the following example that σ-finiteness is a property that is, in general, not preserved by the discretisation operators.

Example 6.3. For $c \in [0,1]$ we define a density function $f_c : \mathbb{R} \to [0,\infty)$ by $f_c(x) = c + (1-c)2(2x - \lfloor 2x \rfloor)$, where $\lfloor \cdot \rfloor$ denotes the Gaussian brackets, that map a real number x to the biggest integer that is smaller than x. Then we define locally finite measures μ_c on $\mathbb{R}$ by $\mu_c := f_c \cdot \lambda^1$, where λ^1 is the Lebesgue measure on $\mathbb{R}$. A σ-finite measure $\mathbb{P}$ on $(\mathbf{M}(\mathbb{R}), \mathcal{M}(\mathbb{R}))$ is then defined by

$$\mathbb{P}(A) := \int_{[0,1]} \mathbf{1}\{\mu_c \in A\} c^{-1} \lambda^1(\mathrm{d}c), \quad A \in \mathcal{M}(\mathbb{R}).$$

However, since $D_1(\mu_c) = \lambda_1$ for all $c \in [0,1]$, the measure $D_1(\mathbb{P})$ is not σ-finite, but the degenerate measure $D_1(\mathbb{P})(\cdot) = \infty \cdot \delta_{\lambda_1}(\cdot)$.

Proposition 6.4. *Let $\mathbb{P}$ be a σ-finite, stationary measure on $(\mathbf{M}(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d))$ and $\mathbb{P}^0$ the corresponding Palm measure. Define a measure $\mathbb{Q}_n$ by*

$$\mathbb{Q}_n(A) := \frac{1}{\lambda^d(C_n)} \int_{C_n} D_n \circ \theta_{-z}(\mathbb{P}^0)(A) \lambda^d(\mathrm{d}z), \quad A \in \mathcal{M}(\mathbb{D}_n).$$

Then we have for all measurable $f : \mathbf{M}(\mathbb{D}_n) \times \mathbb{D}_n \to [0,\infty]$ that

$$\int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} f(\theta_x \mu, x) \mu(\mathrm{d}x) D_n(\mathbb{P})(\mathrm{d}\mu) = \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} f(\mu, y) \lambda_n(\mathrm{d}y) \mathbb{Q}_n(\mathrm{d}\mu). \tag{6.1}$$

If $D_n(\mathbb{P})$ is σ-finite, then $\mathbb{Q}_n$ is the Palm measure of $D_n(\mathbb{P})$.

Proof: For f as in (6.1) we have

$$\begin{aligned}
\int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} f(\theta_x \mu, x) \mu(\mathrm{d}x) D_n(\mathbb{P})(\mathrm{d}\mu) &= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{D}_n} f(\theta_x \circ D_n(\mu), x) D_n(\mu)(\mathrm{d}x) \mathbb{P}(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} f(D_n \circ \theta_{i_n \circ d_n(x)}(\mu), d_n(x)) \mu(\mathrm{d}x) \mathbb{P}(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} f(D_n \circ \theta_{i_n \circ d_n(x) - x}(\mu), d_n(x)) \lambda^d(\mathrm{d}x) \mathbb{P}^0(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \sum_{y \in \mathbb{D}_n} \left(\int_{C_n} f(D_n \circ \theta_{-z}(\mu), y) \lambda^d(\mathrm{d}z) \right) \mathbb{P}^0(\mathrm{d}\mu) \\
&= \int_{C_n} \int_{\mathbf{M}(\mathbb{R}^d)} \sum_{y \in \mathbb{D}_n} f(D_n \circ \theta_{-z}(\mu), y) \mathbb{P}^0(\mathrm{d}\mu) \lambda^d(\mathrm{d}z) \\
&= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} f(\mu, y) \lambda_n(\mathrm{d}y) D_n \circ \theta_{-z}(\mathbb{P}^0)(\mathrm{d}\mu) \lambda^d(\mathrm{d}z) \\
&= \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} f(\mu, y) \lambda_n(\mathrm{d}y) \mathbb{Q}_n(\mathrm{d}\mu),
\end{aligned}$$

where we have used the refined Campbell formula (4.1) for the fourth equation. If $D_n(\mathbb{P})$ is σ-finite, then Satz 2.3 in (8) (see also Section 5) yields that $D_n(\mathbb{P})$ is a stationary measure with Palm measure $\mathbb{Q}_n$. This concludes the proof of the theorem. □

We have seen in (4.2) how the restriction of the stationary measure $\mathbb{P}$ to $\mathbf{M} \setminus \{\mathbf{0}\}$ can be retrieved from the Palm measure $\mathbb{P}^0$. Following the proof of Satz 2.1 in (8), we will now make the choice of the function h on $\mathbb{R}^d \times \mathbf{M}(\mathbb{R}^d)$ more explicit. Let (u_n) be an enumeration of the elements of $\mathbb{Z}^d$ and define $G_n := u_n + C_1$. Clearly, (G_n) is a partition of $\mathbb{R}^d$ into relatively compact sets. Then define a function $\bar{h} : \mathbf{M}(\mathbb{R}^d) \times \mathbb{R}^d \to [0, \infty]$ by

$$\bar{h}(\mu, x) := \sum_{n=1}^{\infty} 2^{-n} (\mu(G_n))^{-1} \mathbf{1}_{G_n}(x), \tag{6.2}$$

where $(\mu(G_n))^{-1} := \infty$ if $\mu(G_n) = 0$. The function $h : \mathbf{M}(\mathbb{R}^d) \times \mathbb{R}^d \to [0, \infty]$ defined by

$$h(\mu, x) := \left(\int_{R^d} \bar{h}(\mu, y) \mu(\mathrm{d}y) \right)^{-1} \bar{h}(\mu, x) \tag{6.3}$$

is measurable, and has the properties $\int h(\mu, x)\mu(\mathrm{d}x) = 1$ for all $\mu \in \mathbf{M}(\mathbb{R}^d) \backslash \{\mathbf{0}\}$ and $h(\mathbf{0}, x) = \infty$ for all $x \in \mathbb{R}^d$. Moreover, h satisfies the following invariance relation with respect to discretization.

Lemma 6.5. *For $\mu \in \mathbf{M}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we have*

$$h(\mu, x) = h(\mu, i_n \circ d_n(x)) = h(I_n \circ D_n(\mu), i_n \circ d_n(x)). \tag{6.4}$$

Proof: For $\bar{h}$ defined in (6.2) and $\mu \in \mathbf{M}(\mathbb{R}^d)$, we have

$$\begin{aligned}
\bar{h}(\mu, x) &= \sum_{m \in \mathbb{N}} 2^{-m} (\mu(G_m))^{-1} \mathbf{1}\{x \in G_m\} = \sum_{m \in \mathbb{N}} 2^{-m} (\mu(G_m))^{-1} \mathbf{1}\{i_n \circ d_n(x) \in G_m\} \\
&= \bar{h}(\mu, i_n \circ d_n(x)) \\
&= \sum_{m \in \mathbb{N}} 2^{-m} (I_n \circ D_n(\mu)(G_m))^{-1} \mathbf{1}\{i_n \circ d_n(x) \in G_m\} \\
&= \bar{h}(I_n \circ D_n(\mu), i_n \circ d_n(x)).
\end{aligned}$$

We deduce that

$$\int \bar{h}(\mu, x)\mu(\mathrm{d}x) = \int \bar{h}(I_n \circ D_n \mu, x)\mu(\mathrm{d}x),$$

so the lemma is proved. □

We are now prepared to state and prove a converse result of Proposition 6.4, which is the main theorem of this section.

Theorem 6.6. *Let $\mathbb{Q}$ be a measure on $(\mathbf{M}(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d))$ satisfying*

$$\int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} h(\theta_{-x}\mu, x) \lambda^d(\mathrm{d}x) \mathbb{Q}(\mathrm{d}\mu) < \infty. \tag{6.5}$$

Define for each $n \in \mathbb{N}$ a measure $\mathbb{Q}_n$ by

$$\mathbb{Q}_n(A) := \frac{1}{\lambda^d(C_n)} \int_{C_n} D_n \circ \theta_{-z}(\mathbb{Q})(A) \lambda^d(\mathrm{d}z), \quad A \in \mathcal{M}(\mathbb{D}_n).$$

Then $\mathbb{Q}$ is a Palm measure if and only if, for all $n \in \mathbb{N}$, the zero-compensated version of $\mathbb{Q}_n$ is point-stationary.

PROOF: In view of Proposition 6.4 we have to show that the condition of the theorem is sufficient. We fix $n \in \mathbb{N}$. Using the characterization of Palm measures on lcscH groups given in Theorem 1.1 we will show that $\mathbb{Q}_n$ is a Palm measure on $(\mathbf{M}(\mathbb{D}_n), \mathcal{M}(\mathbb{D}_n))$. We have to show that $\mathbb{Q}_n$ is σ-finite. Indeed, the function $u_n : \mathbf{M}(\mathbb{D}_n) \to [0, \infty)$ defined by

$$u_n(\mu) := \int_{\mathbb{D}_n} h_n(\theta_{-x}\mu, x)\lambda_n(\mathrm{d}x)$$

is strictly positive on $\mathbf{M}(\mathbb{D}_n) \setminus \{\mathbf{0}\}$. We have

$$\begin{aligned}
\int_{\mathbf{M}(\mathbb{D}_n)} u_n(\mu)\mathbb{Q}_n(\mathrm{d}\mu) &= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} h_n(\theta_{-x}\mu, x)\lambda_n(\mathrm{d}x) D_n \circ \theta_{-z}(\mathbb{Q})(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} h(\theta_{-i_n(x)}(I_n(\mu)), i_n(x))\lambda_n(\mathrm{d}x) D_n \circ \theta_{-z}(\mathbb{Q})(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} h(\theta_{-x} \circ I_n \circ D_n \circ \theta_{-z}(\mu), x) I_n(\lambda_n)(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \frac{1}{\lambda^d(C_n)} \int_{\mathbf{M}(\mathbb{R}^d)} \int_{C_n} \int_{\mathbb{R}^d} h(I_n \circ D_n \circ \theta_{-x-z}(\mu), x) I_n(\lambda_n)(\mathrm{d}x)\lambda^d(\mathrm{d}z)\mathbb{Q}(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{C_n} \sum_{x \in \mathbb{D}_n} h(I_n \circ D_n \circ \theta_{-i_n(x)-z}(\mu), i_n(x) + z)\lambda^d(\mathrm{d}z)\mathbb{Q}(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} h(\theta_{-y}(\mu), y)\lambda^d(\mathrm{d}y)\mathbb{Q}(\mathrm{d}\mu) < \infty,
\end{aligned}$$

where we have used (6.4) and (6.5) for the last equalities. Hence, there exists a σ-finite, stationary measure $\mathbb{P}_n$ on $(\mathbf{M}(\mathbb{D}_n), \mathcal{M}(\mathbb{D}_n))$ such that $\mathbb{Q}_n$ is the Palm measure of $\mathbb{P}_n$. In particular, the inversion formula (cf. (4.2)) yields

$$\int_{\mathbf{M}(\mathbb{D}_n)} f(\mu)\mathbb{P}_n(\mathrm{d}\mu) = \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} h_n(\theta_{-x}\mu, x) f(\theta_{-x}\mu)\lambda_n(\mathrm{d}x)\mathbb{Q}_n(\mathrm{d}\mu), \tag{6.6}$$

and from the special case $f \equiv 1$ we deduce that all $\mathbb{P}_n, n \in \mathbb{N}$, are finite measures. Let us now show that $(I_n(\mathbb{P}_n))$ is a weakly convergent sequence of finite measures, and that the limit measure is given by

$$\mathbb{P}(A) := \int \int h(\theta_{-x}\mu, x)\, \mathbf{1}\{\theta_{-x}\mu \in A\}\lambda^d(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\mu), \quad A \in \mathcal{M}(\mathbb{R}^d). \tag{6.7}$$

For an arbitrary measurable and bounded function $f : \mathbf{M}(\mathbb{R}^d) \to \mathbb{R}$, we have

$$\int_{\mathbf{M}(\mathbb{R}^d)} f(\mu) I_n(\mathbb{P}_n)(\mathrm{d}\mu) = \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} h(\theta_{-y}(\mu), y) f(I_n \circ D_n \circ \theta_{-y}(\mu))\lambda^d(\mathrm{d}y)\mathbb{Q}(\mathrm{d}\mu), \tag{6.8}$$

where we leave the straightforward details of the calculation to the reader. Assume now that f is a continuous, bounded function on $\mathbf{M}(\mathbb{R}^d)$. Then, by Proposition 6.2, the sequence $f(I_n \circ D_n \circ \theta_{-y}(\mu))$ tends to $f(\theta_{-y}(\mu))$ for $n \to \infty$. By (6.5) we can use dominated convergence to conclude that the left-hand side of (6.8) converges to $\int_{\mathbf{M}(\mathbb{R}^d)} f(\mu)\mathbb{P}(\mathrm{d}\mu)$, where $\mathbb{P}$ is defined in (6.7). Finally, for $n \in \mathbb{N}$, the measures $I_m(\mathbb{P}_m), m \geq n$, are invariant under θ_x for all $x \in i_n(\mathbb{D}_n) \subset \mathbb{R}^d$, and we infer that the limit measure $\mathbb{P}$ is also invariant under θ_x for all $x \in i_n(\mathbb{D}_n)$. Clearly, $\bigcup_{n\in\mathbb{N}} i_n(\mathbb{D}_n)$ is dense in $\mathbb{R}^d$. Hence, for an arbitrary $y \in \mathbb{R}^d$, there exists a sequence $y_n \in i_n(\mathbb{D}_n), n \in \mathbb{N}$, with limit y, and $\theta_{y_n}(\mathbb{P}) = \mathbb{P}$. On the other hand we clearly have for all $\varphi \in \mathbf{M}(\mathbb{R}^d)$ that $\theta_{y_n}\varphi \xrightarrow{v} \varphi$, so that $\theta_{y_n}(\mathbb{P})$ converges weakly towards $\theta_y(\mathbb{P})$. Hence $\theta_y(\mathbb{P}) = \mathbb{P}$ and we conclude that $\mathbb{P}$ is stationary and $\mathbb{Q}$ is the Palm measure of $\mathbb{P}$. □

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