Electronic Journal of Probability

Vol. 8 (2003) Paper no. 12, pages 1–39.

Journal URL
http://www.math.washington.edu/~ejpecp/

Large Deviation Principle for a Stochastic Heat Equation With Spatially Correlated Noise [1]

David Márquez-Carreras and **Mònica Sarrà**
Facultat de Matemàtiques, Universitat de Barcelona
Gran Via 585, 08007-Barcelona, Spain
E-mails: marquez@mat.ub.es and sarra@mat.ub.es

Abstract. In this paper we prove a large deviation principle (ldp) for a perturbed stochastic heat equation defined on $[0,T]\times[0,1]^d$. This equation is driven by a Gaussian noise, white in time and correlated in space. Firstly, we show the Hölder continuity for the solution of the stochastic heat equation. Secondly, we check that our Gaussian process satisfies a ldp and some requirements on the skeleton of the solution. Finally, we prove the called Freidlin-Wentzell inequality. In order to obtain all these results we need precise estimates of the fundamental solution of this equation.

Keywords and phrases: Stochastic partial differential equation, stochastic heat equation, Gaussian noise, large deviation principle.

Mathematics Subject Classification (2000): 60H07, 60H15, 35R60.

Submitted to EJP on December 4, 2002. Final version accepted on June 20, 2003.

[1] Partially supported by the grant BFM2000-0607

1 Introduction

Consider the following perturbed d-dimensional spatial stochastic heat equation on the compact set $[0,1]^d$

$$\begin{cases} Lu^\varepsilon(t,x) = \varepsilon\alpha(u^\varepsilon(t,x))\dot{F}(t,x) + \beta(u^\varepsilon(t,x)), \ t \geq 0, \ x \in [0,1]^d, \\ u^\varepsilon(t,x) = 0, \qquad x \in \partial([0,1]^d), \\ u^\varepsilon(0,x) = 0, \qquad x \in [0,1]^d, \end{cases} \tag{1.1}$$

with $\varepsilon > 0$, $L = \frac{\partial}{\partial t} - \Delta$ where Δ is the Laplacian on $\mathbb{R}^d$ and $\partial([0,1]^d)$ is the boundary of $[0,1]^d$. We consider null initial conditions and the compact set $[0,1]^d$ instead of $[0,\zeta]^d$, $\zeta > 0$, for the sake of simplicity.
Assume that the coefficients satisfy the following assumptions:

(C) the functions α and β are Lipschitz.

The noise $F = \{F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\}$ is an $L^2(\Omega, \mathcal{F}, P)$-valued Gaussian process with mean zero and covariance functional given by

$$J(\varphi,\psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(s,x) f(x-y) \psi(s,y), \tag{1.2}$$

and $f : \mathbb{R}^d \to \mathbb{R}_+$ is a continuous symmetric function on $\mathbb{R}^d - \{0\}$ such that there exists a non-negative tempered measure λ on $\mathbb{R}^d$ whose Fourier transform is f. The functional J in (1.2) is said to be a covariance functional if all these asumptions are satisfied. Then, in addition,

$$J(\varphi,\psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \lambda(d\xi) \mathcal{F}\varphi(s,\cdot)(\xi) \overline{\mathcal{F}\psi(s,\cdot)(\xi)},$$

where $\mathcal{F}$ is the Fourier transform and $\overline{z}$ is the conjugate complex of z. In (1.2) we could also work with a non-negative and non-negative definite tempered measure, therefore symmetric, instead of the function f but, in this case, all the notation over the sets appearing in the integrals is becoming tedious. In this paper, moreover, we assume the following hypothesis on the measure λ:

$$(\mathrm{H}_\eta) \quad \int_{\mathbb{R}^d} \frac{\lambda(d\xi)}{(1+\|\xi\|^2)^\eta} < \infty,$$

for some $\eta \in (0,1]$. For instance, the function $f(x) = \|x\|^{-\kappa}$, $\kappa \in (0,d)$, satisfies (H_η). As in Dalang [4] (see also Dalang and Frangos [5]) the Gaussian process F can be extended to a worthy martingale measure, in the sense given by Walsh [23],

$$M = \{M_t(A),\, t \in \mathbb{R}_+,\, A \in \mathcal{B}_b(\mathbb{R}^d)\}.$$

Then, following the approach of Walsh [23], one can give a rigorous meaning to (1.1) by means of a weak formulation. Assumptions (C) and (H_1) will ensure the existence and uniqueness of a jointly measurable adapted process $\{u^\varepsilon(t,x),\, (t,x) \in \mathbb{R}_+ \times [0,1]^d\}$ such that

$$\begin{aligned} u^\varepsilon(t,x) = \;& \varepsilon \int_0^t \int_{[0,1]^d} G(t-s,x,y)\alpha(u^\varepsilon(s,y))F(ds,dy) \\ &+ \int_0^t ds \int_{[0,1]^d} dy\, G(t-s,x,y)\beta(u^\varepsilon(s,y)), \end{aligned} \tag{1.3}$$

where $\varepsilon > 0$ and $G(t,x,y)$ denotes the fundamental solution of the heat equation on $[0,1]^d$:

$$\begin{cases} \frac{\partial}{\partial t} G(t,x,y) = \Delta_x G(t,x,y), & t \geq 0,\; x,y \in [0,1]^d, \\ G(t,x,y) = 0, & x \in \partial([0,1]^d), \\ G(0,x,y) = \delta(x-y). \end{cases}$$

The stochastic integral in (1.3) is defined with respect to the $\mathcal{F}_t$-martingale measure M. Denote $D_T^d = [0,T] \times [0,1]^d$. If $d > 1$, the evolution equation can not be driven by the Brownian sheet because G does not belong to $L^2(D_T^d)$ and we need to work with a smoother noise. The study of existence and uniqueness of solution to this sort of equation (1.3) on $\mathbb{R}^d$ has been analyzed by Dalang in [4]. Many other authors have also studied existence and uniqueness of solution to d-dimensional spatial stochastic equations, in particular, wave and heat equations (see, for instance, Dalang and Frangos [5], Karszeswka and Zabszyk [12], Millet and Morien [15], Millet and Sanz-Solé [16], Peszat and Zabszyk [17], [18]).

Assuming the above-mentioned hypothesis, (C) and (H_η) for some $\eta \in [0,1]$, we will also check that the trajectories of the process are (γ_1, γ_2)-Hölder continuous with respect to the parameters t and x, satisfaying $\gamma_1 \in (0, \frac{1-\eta}{2})$ and $\gamma_2 \in (0, 1-\eta)$. The Hölder continuity for the stochastic heat equation on $\mathbb{R}^d$ has been studied by Sanz-Solé and Sarrà [20], [21]. On the other hand, the wave case has been dealt in [15], [16] and [20].

Under (C) and (H_η) for some $\eta \in (0,1)$, we will prove the most important result of this paper: the existence of a large deviation principle (ldp) for the law of the solution u^ε to (1.3) on $\mathcal{C}^{\gamma,\gamma}(D_T^d)$, with $\gamma \in (0, \frac{1-\eta}{4})$. This means that we check the existence of a lower semi-continuous function $I : \mathcal{C}^{\gamma,\gamma}(D_T^d) \to [0,\infty]$, called rate function, such that $\{I \leq a\}$ is compact for any $a \in [0,\infty)$, and

$$\begin{aligned} \varepsilon^2 \log P\{u^\varepsilon \in O\} &\geq -\Lambda(O), \quad \text{for each open set } O, \\ \varepsilon^2 \log P\{u^\varepsilon \in U\} &\leq -\Lambda(U), \quad \text{for each closed set } U, \end{aligned}$$

where, for a given subset $A \in \mathcal{C}^{\gamma,\gamma}(D_T^d)$,

$$\Lambda(A) = \inf_{l \in A} I(l).$$

The proof of this goal is based on a classical result given by Azencott in [1] (see also Priouret [19]), that allow us to go beyond a ldp from εF to u^ε. Azencott's method is the following:

Theorem 1.1 *Let (E_i, d_i), $i = 1, 2$, be two Polish spaces and $X_i : \Omega \to E_i$, $\varepsilon > 0$, $i = 1, 2$, be two families of random variables. Suppose the following requirements:*

1. *$\{X_1^\varepsilon,\ \varepsilon > 0\}$ obeys a ldp with the rate function $I_1 : E_1 \to [0,\infty]$.*

2. *There exists a function $K : \{I_1 < \infty\} \to E_2$ such that, for every $a < \infty$, the function*
$$K : \{I_1 \leq a\} \to E_2$$
is continuous.

3. *For every $R, \rho, a > 0$, there exist $\theta > 0$ and $\varepsilon_0 > 0$ such that, for $h \in E_1$ satisfying $I_1(h) \leq a$ and $\varepsilon \leq \varepsilon_0$, we have*
$$P\Big\{d_2(X_2^\varepsilon, K(h)) \geq \rho,\ d_1(X_1^\varepsilon, h) < \theta\Big\} \leq \exp\left(-\frac{R}{\varepsilon^2}\right). \tag{1.4}$$

Then, the family $\{X_2^\varepsilon,\ \varepsilon > 0\}$ obeys a ldp with the rate function

$$I_2(\phi) = \inf\{I_1(h) :\ K(h) = \phi\}.$$

Consequently, we will need to check that our initial Gaussian process satisfies a ldp, the existence of a function K and, finally, to prove (1.4). We will follow the approach of Freidlin and Wentzell [10] for diffusion process (see also Dembo and Zeitouni [6]). Another remarkable article is Chenal and Millet [3] where they prove the existence of a ldp for a one-dimensional

stochastic heat equation. Sowers has also studied this last equation but using a different method. For more information about the study of one-dimensional stochastic heat equations, we refer to Chenal and Millet [3]. Finally, Chenal [2] has checked a ldp for a stochastic wave equation on $\mathbb{R}^2$.
The paper is organized as follows. In Section 2 we establish a ldp for our Gaussian process (first point of Theorem 1.1). In Section 3 we state some properties on $u^\varepsilon(t,x)$, mainly the Hölder continuity. Section 4 contains the proofs of some requirements on the skeleton of u^ε (second point of Theorem 1.1). Section 5 is devoted to the proof of called Freidlin-Wentzell's inequality (third point of Theorem 1.1). All the arguments of this paper need precise estimates of the fundamental solution G which will be enunciated and proved in an appendix. Moreover, this appendix also contains an exponential inequality used in Section 5.
In this paper we fix $T > 0$ and all constants will be denoted independently of its value. We finish this introduction by giving some basic notations. We write, for functions $\phi, \phi' : D_T^d \to \mathbb{R}$,

$$\|\phi\|_\infty = \sup\{|\phi(t,x)|,\ (t,x) \in D_T^d\},$$

$$\|\phi\|_{\gamma_1,\gamma_2} = \sup\left\{\frac{|\phi(t,x)-\phi(s,y)|}{|t-s|^{\gamma_1} + \|x-y\|^{\gamma_2}},\ (t,x) \neq (s,y) \in D_T^d\right\},$$

$$d_{\gamma_1,\gamma_2}(\phi,\phi') = \|\phi-\phi'\|_\infty + \|\phi-\phi'\|_{\gamma_1,\gamma_2}.$$

Then, we define the topology of (γ_1,γ_2)-Hölder convergence on D_T^d by means of d_{γ_1,γ_2}. Let $\mathcal{C}^{\gamma_1,\gamma_2}(D_T^d)$ be the set of functions $\phi : D_T^d \to \mathbb{R}$ such that $\|\phi\|_\infty + \|\phi\|_{\gamma_1,\gamma_2} < \infty$.
Finally, for $\theta > 0$, a set $A \subset \mathbb{R}^n, n \geq 1$ and $\phi : A \to \mathbb{R}$, we introduce the following notation,

$$\|\phi\|_{\theta,A} = \sup_{w\in A} |\phi(w)| + \sup\left\{\frac{|\phi(w)-\phi(w')|}{|w-w'|^\theta};\ w \neq w',\, w, w' \in A\right\}. \tag{1.5}$$

2 Large deviation principle for the Gaussian process

Let $\mathcal{E}$ be the space of measurable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, |\varphi(y)| f(y-z) |\varphi(z)| < \infty,$$

endowed with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{E}} = \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \varphi(y) f(y-z) \psi(z).$$

Let $\mathcal{H}$ be the completation of $(\mathcal{E}, \langle , \rangle_{\mathcal{E}})$. For $T > 0$, let $\mathcal{H}_T = L^2([0,T]; \mathcal{H})$. This space is a real separable Hilbert space such that, if $\varphi, \psi \in \mathcal{D}([0,T] \times \mathbb{R}^d)$,

$$E\left(F(\varphi)F(\psi)\right) = \int_0^T \langle \varphi(s,\cdot), \psi(s,\cdot) \rangle_{\mathcal{H}} ds = \langle \varphi, \psi \rangle_{\mathcal{H}_T}, \tag{2.1}$$

where F is the noise introduced in Section 1.
For $(t,x), (t',x') \in D_T^d$, set

$$\tilde{\Gamma}((t,x),(t',x')) = \int_0^{t \wedge t'} ds \int_{R(x)} dy \int_{R(x')} dz\, f(y-z),$$

where $R(x)$ is the rectangle $[0,x]$ (here $[0,x]$ is a product of rectangles). We have $\tilde{\Gamma}((t,x),(t',x')) = \langle \varphi_{t,x}, \varphi_{t',x'} \rangle_{\mathcal{H}_T}$ with

$$\varphi_{t,x}(s,y) = 1\!\!1_{([0,t] \times R(x))}(s,y).$$

One can easily check the following three conditions:

(i) $\tilde{\Gamma}$ is symmetric.

(ii) For any $c_1, \dots, c_n \in \mathbb{R}$, $(t_1,x_1), \dots, (t_n,x_n) \in D_T^d$,

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j \tilde{\Gamma}((t_i,x_i),(t_j,x_j)) \geq 0.$$

(iii) For any (t,x), $(t',x') \in D_T^d$,

$$\tilde{\Gamma}((t,x),(t,x)) + \tilde{\Gamma}((t',x'),(t',x')) - 2\tilde{\Gamma}((t,x),(t',x')) \leq C_T(|t-t'| + \|x-x'\|).$$

Then, Example 1.2 in Watanabe [24] implies that, for any $\gamma' \in [0, \frac{1}{2})$, there exists a Gaussian measure ν on $\mathcal{C}^{\gamma',\gamma'}(D_T^d; \mathbb{R})$ such that

$$\int_{\mathcal{C}^{\gamma',\gamma'}(D_T^d;\mathbb{R})} w(t,x) w(t',x') d\nu(w) = \tilde{\Gamma}((t,x),(t',x')).$$

Moreover, if H denotes the autoreproducing space of ν, $\left(\mathcal{C}^{\gamma',\gamma'}(D_T^d;\mathbb{R}), H, \nu\right)$ is an abstract Wiener space. For any $(t,x) \in D_T^d$, set

$$W_F(t,x) = \int_0^t \int_{R(x)} F(ds,dy).$$

Then, W_F is a Gaussian process with covariance function $\tilde{\Gamma}$. The trajectories of W_F belong to $\mathcal{C}^{\gamma',\gamma'}(D_T^d;\mathbb{R})$, this means that the law of the process W_F is ν. The space H is the set of functions $\tilde{h}$ such that, for $h \in \mathcal{H}_T$ and for any $(t,x) \in D_T^d$,

$$\tilde{h}(t,x) = \langle 1\!\!1_{([0,t]\times R(x))}, h\rangle_{\mathcal{H}_T},$$

with the following scalar product

$$\langle \tilde{h}, \tilde{k}\rangle_H = \langle h, k\rangle_{\mathcal{H}_T}.$$

H is a Hilbert space isomorphic to $\mathcal{H}_T$.
Classical results on Gaussian processes (see, for instance, Theorem 3.4.12 in [7]) show that, for $\gamma' \in [0, \frac{1}{2})$, the family $\{\varepsilon W_F,\ \varepsilon > 0\}$ satisfies a large deviation principle on $\mathcal{C}^{\gamma',\gamma'}(D_T^d;\mathbb{R})$ with rate function

$$\tilde{I}(\tilde{h}) = \begin{cases} \frac{1}{2}\|\tilde{h}\|_H, & \text{if } \tilde{h} \in H, \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark that, if $I(h) = \frac{1}{2}\|h\|_{\mathcal{H}_T}$ for $h \in \mathcal{H}_T$, then $\tilde{I}(\tilde{h}) = I(h)$ for $\tilde{h} \in H$.

3 Properties on the solution

In this section we analyze the existence and uniqueness of solution (1.3) and the Hölder continuity with respect to the parameters.

Proposition 3.1 *Assume* (C) *and* (H$_1$). *Then* (1.3) *has a unique solution. Moreover, for any* $T > 0$, $p \in [1,\infty)$,

$$\sup_{0<\varepsilon\leq 1}\ \sup_{(t,x)\in D_T^d} E(|u^\varepsilon(t,x)|^p) < \infty. \tag{3.1}$$

Proof: Define the following Picard's approximations for $n \geq 1$,

$$u_0^\varepsilon(t,x) = 0,$$

$$u_{n+1}^\varepsilon(t,x) = \varepsilon \int_0^t \int_{[0,1]^d} G(t-s,x,y)\alpha(u_n^\varepsilon(s,y))F(ds,dy)$$
$$+ \int_0^t ds \int_{[0,1]^d} dy\, G(t-s,x,y)\beta(u_n^\varepsilon(s,y)),$$

where $G(t-s,x,y)$ is the fundamental solution to (1.1) described in the Appendix. The proof of this result is almost the same as Proposition 2.4 in [13] but adding the dependence on ε.

$\square$

Theorem 3.2 *Assume* (C) *and* (H_η) *for some* $\eta \in (0,1)$. *Then, for every* $T>0$, $p\in[2,\infty)$, $0<\varepsilon\le 1$, $t,t'\in[0,T]$, $x,x'\in[0,1]^d$, $\gamma_1\in(0,\frac{1-\eta}{2})$ *and* $\gamma_2\in(0,1-\eta)$,

$$E\left(|u^\varepsilon(t,x)-u^\varepsilon(t',x)|^p\right)\le C|t-t'|^{\gamma_1 p}, \tag{3.2}$$

$$E\left(|u^\varepsilon(t,x)-u^\varepsilon(t,x')|^p\right)\le C\|x-x'\|^{\gamma_2 p}. \tag{3.3}$$

Moreover, the trajectories of u^ε *are a.s.* (γ_1,γ_2)*-Hölder continuous in* $(t,x)\in D_T^d$, *for* $\gamma_1\in(0,\frac{1-\eta}{2})$ *and* $\gamma_2\in(0,1-\eta)$.

Proof: We at first prove (3.2). For $0\le t'\le t\le T$, $x\in[0,1]^d$, we have

$$E\left(|u^\varepsilon(t,x)-u^\varepsilon(t',x)|^p\right)\le C\sum_{i=1}^4 A_i,$$

with

$$A_1=E\left(\left|\int_0^{t'}\int_{[0,1]^d}\varepsilon\Big[G(t-s,x,y)-G(t'-s,x,y)\Big]\alpha(u^\varepsilon(s,y))F(ds,dy)\right|^p\right),$$

$$A_2=E\left(\left|\int_{t'}^{t}\int_{[0,1]^d}\varepsilon G(t-s,x,y)\alpha(u^\varepsilon(s,y))F(ds,dy)\right|^p\right),$$

$$A_3=E\left(\left|\int_0^{t'}ds\int_{[0,1]^d}dy\,\Big[G(t-s,x,y)-G(t'-s,x,y)\Big]\beta(u^\varepsilon(s,y))\right|^p\right),$$

$$A_4=E\left(\left|\int_{t'}^{t}ds\int_{[0,1]^d}dy\,G(t-s,x,y)\beta(u^\varepsilon(s,y))\right|^p\right).$$

Burkholder's and Hölder's inequalities, (3.1) and (6.1.7) imply

$$\begin{aligned}
A_1 &\le C\Big[1+\sup_{0<\varepsilon\le 1}\sup_{(t,x)\in D_T^d}E(|u^\varepsilon(t,x)|^p)\Big]\\
&\quad\times\Big[\int_0^{t'}ds\int_{[0,1]^d}dy\int_{[0,1]^d}dz|G(t-s,x,y)-G(t'-s,x,y)|f(x-y)\\
&\quad\times|G(t-s,x,z)-G(t'-s,x,z)|\Big]^{\frac{p}{2}}\\
&\le C|t-t'|^{\gamma_1 p}.
\end{aligned}$$

Using similar arguments together (6.1.8), one can obtain

$$A_2 \leq C|t-t'|^{\gamma_1 p}.$$

In order to finish the proof of (3.2) we notice that A_3 and A_4 can be dealt in a similar but easier way by means of (6.1.9) and (6.1.10), respectively.
We now examine (3.3). For $0 < t < T$, $x, x' \in [0,1]^d$, we write

$$E(|u^\varepsilon(t,x) - u^\varepsilon(t,x')|^p) \leq C(B_1 + B_2),$$

with

$$B_1 = E\left(\left|\int_0^t \int_{[0,1]^d} \varepsilon\Big[G(t-s,x,y) - G(t-s,x',y)\Big]\alpha(u^\varepsilon(s,y))F(ds,dy)\right|^p\right),$$

$$B_2 = E\left(\left|\int_0^t ds \int_{[0,1]^d} dy \,\Big[G(t-s,x,y) - G(t-s,x',y)\Big]\beta(u^\varepsilon(s,y))\right|^p\right).$$

Then, the same steps as before but using (6.1.19) and (6.1.20) conclude the proof of this theorem.

□

4 Continuity of the skeleton

For any $\tilde{h} \in H$, we consider the solution to the deterministic evolution equation

$$\begin{aligned} S^{\tilde{h}}(t,x) = \ & \Big\langle G(t-\cdot,x,*)\alpha(S^{\tilde{h}}(\cdot,*)), h\Big\rangle_{\mathcal{H}_T} \\ & + \int_0^t ds \int_{[0,1]^d} dy \, G(t-s,x,y)\beta(S^{\tilde{h}}(s,y)), \end{aligned} \tag{4.1}$$

for all $(t,x) \in D_T^d$, where $\langle \cdot,\cdot \rangle_{\mathcal{H}_T}$ is defined in (2.1).
Remark. We specify the formulation used in (4.1). Due to the kernel G,

$$\Big\langle G(t-\cdot,x,*)\alpha(S^{\tilde{h}}(\cdot,*)), h\Big\rangle_{\mathcal{H}_T} = \int_0^t ds \int_{[0,1]^d} dy \int_{[0,1]^d} dz \, G(t-s,x,y)\alpha(S^{\tilde{h}}(s,y))f(y-z)h(s,z).$$

Then, in fact, (4.1) can rewrite as follows

$$\begin{aligned} S^{\tilde{h}}(t,x) = & \int_0^t ds \int_{[0,1]^d} dy \int_{[0,1]^d} dz\, G(t-s,x,y)\alpha(S^{\tilde{h}}(s,y))f(y-z)h(s,z) \\ & + \int_0^t ds \int_{[0,1]^d} dy\, G(t-s,x,y)\beta(S^{\tilde{h}}(s,y)). \end{aligned}$$

In this section, we show that the map $\tilde{h} \to S^{\tilde{h}}$ is continuous on $\mathcal{C}^{\gamma_1,\gamma_2}(D_T^d)$ for some (γ_1,γ_2) which will be given later. In order to obtain this goal we will need the next proposition, we omit the proof of this result because it is similar to Proposition 3.1 and Theorem 3.2.

Theorem 4.1 *Suppose* (C).

i) *Assuming* (H_1), *for any* $a \in [0,\infty)$, *we have*

$$\sup_{\tilde{h}:\tilde{I}(\tilde{h})\le a} \sup_{(t,x)\in D_T^d} |S^{\tilde{h}}(t,x)| \le C. \tag{4.2}$$

ii) *Assume* (H_η) *for some* $\eta \in (0,1)$. *For any* $\gamma_1 < \frac{1-\eta}{2}$ *and* $\gamma_2 < 1-\eta$, *there exists a constant* C *such that, for any* $(t,x),(t',x') \in D_T^d$,

$$\sup_{\tilde{h}:\tilde{I}(\tilde{h})\le a} |S^{\tilde{h}}(t,x) - S^{\tilde{h}}(t',x')| \le C(|t-t'|^{\gamma_1} + \|x-x'\|^{\gamma_2}). \tag{4.3}$$

For $s \in [0,T], y \in [0,1]^d$, we will use the following discretization in time and space

$$s_n = \sup\{\tfrac{k-1}{2^n}T;\ \tfrac{k}{2^n}T \le s\} \vee 0,$$

$$y_n^i = \sup\{\tfrac{k}{2^n};\ \tfrac{k}{2^n} \le y^i\},$$

for any $i = 1,\dots,d$.
Set

$$\widehat{t-s_n} = \begin{cases} t-s_n & \text{if} \quad t \ge 2^{-n}T, \\ 2^{-n}T & \text{if} \quad t < 2^{-n}T. \end{cases}$$

We give some obvious properties:

$$s - s_n \le 2^{-(n-1)}, \qquad s \in [0,T], \tag{4.4}$$

$$\widehat{t-s_n} \ge 2^{-n}T, \qquad 0 \le s \le t \le T, \tag{4.5}$$

$$|\widehat{t-u_n} - (\widehat{s-u_n})| \le |t-s|, \qquad 0 \le u \le s \le t \le T, \tag{4.6}$$

$$\|y - y_n\| \le 2^{-n}, \qquad y \in [0,1]^d. \tag{4.7}$$

The kernel is discretized as follows

$$G_n(t-s,x,y) = G(\widehat{t-s_n},x,y_n),$$

and we also use the following notation $S_n^{\tilde{h}}(s,y) = S^{\tilde{h}}(s_n,y_n)$.
The proof of the next theorem is the most important aim of this section.

Theorem 4.2 *Assume* (C) *and* (H$_\eta$) *for some* $\eta \in (0,1)$. *Fix* $T>0$ *and* $a>0$. *The map* $\tilde{h} \to S^{\tilde{h}}$ *is continuous from* $\{\tilde{I} \le a\}$ *to* $\mathcal{C}^{\gamma_1,\gamma_2}(D_T^d)$, $(\gamma_1,\gamma_2) \in (0,\frac{1-\eta}{4}) \times (0,\frac{1-\eta}{2})$, *where* $\{\tilde{I} \le a\}$ *is endowed with the topology of uniform convergence.*

Proof: In order to prove the continuity of $S^{\tilde{h}}$ we decompose $d_{\gamma_1,\gamma_2}(S^{\tilde{h}},S^{\tilde{g}})$ into two parts:

$$\sup_{x\in[0,1]^d} \|S^{\tilde{h}}(\cdot,x) - S^{\tilde{g}}(\cdot,x)\|_{\gamma_1,[0,T]}, \tag{4.8}$$

and

$$\sup_{t\in[0,T]} \|S^{\tilde{h}}(t,\cdot) - S^{\tilde{g}}(t,\cdot)\|_{\gamma_2,[0,1]^d}. \tag{4.9}$$

We first examine (4.8). According to (1.5), if we want to study (4.8), we need to deal with the following two functions, for $t \in [0,T]$,

$$\begin{aligned}
\psi(t) &= \sup_{(s,x)\in[0,t]\times[0,1]^d} |S^{\tilde{h}}(s,x) - S^{\tilde{g}}(s,x)|, \\
\psi_{\gamma_1}(t) &= \sup_{x\in[0,1]^d} \sup_{\substack{s,r\in[0,t]\\ s\neq r}} \frac{|S^{\tilde{h}}(s,x) - S^{\tilde{g}}(s,x) - S^{\tilde{h}}(r,x) + S^{\tilde{g}}(r,x)|}{|s-r|^{\gamma_1}}.
\end{aligned}$$

We start the proof studying $\psi(T)$. For $(s,x) \in D_T^d$, we have

$$\begin{aligned}
S^{\tilde{h}}(s,x) - S^{\tilde{g}}(s,x) = &\int_0^s dr \int_{[0,1]^d} dy\, G(s-r,x,y)\Big[\beta(S^{\tilde{h}}(r,y)) - \beta(S^{\tilde{g}}(r,y))\Big] \\
&+\Big\langle G(s-\cdot,x,*)\Big[\alpha(S^{\tilde{h}}(\cdot,*)) - \alpha(S^{\tilde{g}}(\cdot,*))\Big],g\Big\rangle_{\mathcal{H}_T} \\
&+\sum_{i=1}^{3} A_i(s,x),
\end{aligned}$$

with

$$
\begin{aligned}
A_1(s,x) &= \Big\langle (G-G_n)(s-\cdot,x,*)\alpha(S_n^{\tilde{h}}(\cdot,*)), h-g \Big\rangle_{\mathcal{H}_T},\\
A_2(s,x) &= \Big\langle G(s-\cdot,x,*)\Big[\alpha(S^{\tilde{h}}(\cdot,*)) - \alpha(S_n^{\tilde{h}}(\cdot,*))\Big], h-g \Big\rangle_{\mathcal{H}_T},\\
A_3(s,x) &= \Big\langle G_n(s-\cdot,x,*)\alpha(S_n^{\tilde{h}}(\cdot,*)), h-g \Big\rangle_{\mathcal{H}_T}.
\end{aligned}
$$

The usual argument based on Hölder's inequality together with (C) and the estimates (6.1.5) and (6.1.6) imply that, for $t \in [0,T]$ and $\tilde{h}, \tilde{g} \in \{\tilde{I} \le a\}$,

$$
\psi(t)^2 \le C(1+\|g\|^2_{\mathcal{H}_T}) \int_0^t \psi(s)^2 ds + C \sum_{i=1}^{3} \sup_{s\le t} \sup_{x\in[0,1]^d} A_i^2(s,x).
$$

Then, Gronwall's lemma yields

$$
\psi(T) = \sup_{(t,x)\in D_T^d} |S^{\tilde{h}}(t,x) - S^{\tilde{g}}(t,x)| \le C \sum_{i=1}^{3} \sup_{(t,x)\in D_T^d} A_i(t,x). \tag{4.10}
$$

The rest of the study of $\psi(T)$ consists in dealing with A_i, for $i = 1,2,3$. For $\xi_1 \in (0, \frac{1-\eta}{2})$, from hypothesis (C) and the estimates (4.2), (6.1.28) and (6.1.29) it follows that

$$
\begin{aligned}
\sup_{(t,x)\in D_T^d} |A_1(t,x)| &\le C\|h-g\|_{\mathcal{H}_T} \sup_{(t,x)\in D_T^d} \|(G-G_n)(t-\cdot,x,*)\|_{\mathcal{H}_T}\\
&\le C 2^{-\xi_1(n-1)}.
\end{aligned}
\tag{4.11}
$$

For $\xi_2 \in (0, \frac{1-\eta}{2})$ and $\xi_3 \in (0, 1-\eta)$, using (C), (4.3), (4.4), (4.7) and (6.1.5), we have

$$
\begin{aligned}
\sup_{(t,x)\in D_T^d} |A_2(t,x)| &\le C\|h-g\|_{\mathcal{H}_T} \sup_{(t,x)\in D_T^d} |S^{\tilde{h}}(t,x) - S_n^{\tilde{h}}(t,x)|\\
&\le C(2^{-\xi_2(n-1)} + 2^{-\xi_3 n}).
\end{aligned}
\tag{4.12}
$$

For $i_1, \dots, i_d \in \{0, \dots, 2^n - 1\}$, denote

$$
R_{i_1,\dots,i_d} = \Big\{ y = (y_1,\dots,y_d) \in [0,1]^d,\ \frac{i_j}{2^k} \le y_j \le \frac{i_j+1}{2^n},\ j = 1,\dots,d \Big\}.
$$

Then,

$$
\begin{aligned}
A_3(t,x) &= \sum_{i_1,\dots,i_d=0}^{2^n-1} G(2^{-n}T,x,(i_1,\dots,i_d))\alpha(S^{\tilde h}(0,(i_1,\dots,i_d)))\\
&\quad \times\left\langle 1\!\!1_{[0,t]\times R_{i_1,\dots,i_d}}, h-g\right\rangle_{\mathcal{H}_T} 1\!\!1_{\{t<2^{-n}T\}}\\
&+ \Bigg[\sum_{i_1,\dots,i_d=0}^{2^n-1} G(t,x,(i_1,\dots,i_d))\alpha(S^{\tilde h}(0,(i_1,\dots,i_d)))\\
&\quad \times\left\langle 1\!\!1_{[0,\frac{T}{2^n}]\times R_{i_1,\dots,i_d}}, h-g\right\rangle_{\mathcal{H}_T}\\
&+ \sum_{i=0}^{2^n-2}\sum_{i_1,\dots,i_d=0}^{2^n-1} G\left(t-\frac{iT}{2^n},x,(i_1,\dots,i_d)\right)\alpha\left(S^{\tilde h}\left(\frac{iT}{2^n},(i_1,\dots,i_d)\right)\right)\\
&\quad \times\left\langle 1\!\!1_{[\frac{(i+1)T}{2^n}\wedge t,\frac{(i+2)T}{2^n}\wedge t]\times R_{i_1,\dots,i_d}}, h-g\right\rangle_{\mathcal{H}_T}\Bigg] 1\!\!1_{\{t\geq 2^{-n}T\}}.
\end{aligned}
\tag{4.13}
$$

By (C), (4.2) and (6.1.27),

$$
\sup_{(t,x)\in D_T^d} |A_3(t,x)| \leq C2^{\frac{nd}{2}}2^{(d+1)n}\|\tilde h-\tilde g\|_\infty. \tag{4.14}
$$

Then, (4.10) together with (4.11)-(4.14) ensure

$$
\psi(T) \leq C\left(2^{-n\xi}+2^{n(\frac{3d}{2}+1)}\|\tilde h-\tilde g\|_\infty\right), \tag{4.15}
$$

for some $\xi>0$.
We now analyze $\psi_{\gamma_1}(T)$. Fix $x\in[0,1]^d$, assume $t,t'\in[0,T]$, $t'<t$. Then, from the Lipschitz property of β, it follows

$$
\left|(S^{\tilde h}-S^{\tilde g})(t,x)-(S^{\tilde h}-S^{\tilde g})(t',x)\right| \leq C\sum_{i=1}^{7} B_i, \tag{4.16}
$$

with

$$
\begin{aligned}
B_1 &= \int_{t'}^{t} ds\int_{[0,1]^d} dy\, G(t-s,x,y)|S^{\tilde h}(s,y)-S^{\tilde g}(s,y)|\\
&\quad +\int_0^{t'} ds\int_{[0,1]^d} dy\,|G(t-s,x,y)-G(t'-s,x,y)||S^{\tilde h}(s,y)-S^{\tilde g}(s,y)|,\\
B_2 &= \left|\left\langle\Big[G(t-\cdot,x,*)-G(t'-\cdot,x,*)\Big]\Big[\alpha(S^{\tilde h}(\cdot,*))-\alpha(S_n^{\tilde h}(\cdot,*))\Big],h\right\rangle_{\mathcal{H}_T}\right|,
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= \Big| \Big\langle \Big[G(t-\cdot,x,*) - G_n(t-\cdot,x,*) - G(t'-\cdot,x,*) + G_n(t'-\cdot,x,*) \Big] \\
&\quad \times \alpha(S^{\tilde h}(\cdot,*)), h \Big\rangle_{\mathcal{H}_T} \Big|, \\
B_4 &= \Big| \Big\langle \Big[G_n(t-\cdot,x,*) - G_n(t'-\cdot,x,*) \Big] \alpha(S_n^{\tilde h}(\cdot,*)), h-g \Big\rangle_{\mathcal{H}_T} \Big|, \\
B_5 &= \Big| \Big\langle \Big[G_n(t-\cdot,x,*) - G_n(t'-\cdot,x,*) \Big] \Big[\alpha(S_n^{\tilde h}(\cdot,*)) - \alpha(S_n^{\tilde g}(\cdot,*)) \Big], g \Big\rangle_{\mathcal{H}_T} \Big|,
\end{aligned}
$$

B_6 and B_7 are the same terms as B_2 and B_3, changing h and $\tilde h$ for g and $\tilde g$, respectively.
Applying (4.15), (6.1.9) and (6.1.10), we have, for all $\tilde\gamma_1 \in (0, \frac{1}{2})$,

$$
B_1 \le C \Big[2^{-n\xi} + 2^{n(\frac{3d}{2}+1)} \|\tilde h - \tilde g\|_\infty \Big] |t-t'|^{2\tilde\gamma_1}, \tag{4.17}
$$

for some $\xi > 0$.
In the sequel, $\xi, \gamma_1' \in (0, \frac{1-\eta}{2})$. Schwarz's inequality, the Lipschitz property of α, and the estimates (6.1.7), (6.1.8), (4.3), (4.4) and (4.7), imply

$$
B_2 + B_6 \le C 2^{-n\xi} |t-t'|^{\gamma_1'}, \qquad \xi > 0. \tag{4.18}
$$

From (4.2), (6.1.37) and (6.1.39),

$$
B_3 + B_7 \le C 2^{-\xi(n-1)} |t-t'|^{\frac{\gamma_1'}{2}}, \qquad \xi > 0. \tag{4.19}
$$

Using (4.15), (6.1.38) and (6.1.40), we obtain

$$
B_5 \le C \Big[2^{-n\xi} + 2^{n(\frac{3d}{2}+1)} \|\tilde h - \tilde g\|_\infty \Big] |t-t'|^{\gamma_1'}, \qquad \xi > 0. \tag{4.20}
$$

Finally, we bound B_4 as follows

$$
B_4 \le C \left(B_{4,1} + B_{4,2} \right),
$$

with

$$
\begin{aligned}
B_{4,1} = & \Big| \int_{t'}^{t} ds \int_{[0,1]^d} dy \int_{[0,1]^d} dz \, G(\widehat{t-s_n}, x, y_n) \alpha(S_n^{\tilde h}(s,y)) f(y-z) \\
& \qquad \times [h(s,z) - g(s,z)] \Big|, \\
B_{4,2} = & \Big| \int_0^{t'} ds \int_{[0,1]^d} dy \int_{[0,1]^d} dz \, \Big[G(\widehat{t-s_n}, x, y_n) - G(\widehat{t'-s_n}, x, y_n) \Big] \\
& \qquad \times \alpha(S_n^{\tilde h}(s,y)) f(y-z) [h(s,z) - g(s,z)] \Big|.
\end{aligned}
$$

Computing as in (4.14) we can get

$$B_{4,1} \leq C 2^{n(\frac{3d}{2}+1)} \|\tilde{h} - \tilde{g}\|_\infty .$$

Then, Schwarz's inequality and (6.1.27) imply

$$\begin{aligned} B_{4,1} &= B_{4,1}^{\frac{1}{2}} B_{4,1}^{\frac{1}{2}} \\ &\leq C 2^{\frac{n}{2}(\frac{3d}{2}+1)} \|\tilde{h} - \tilde{g}\|_\infty^{\frac{1}{2}} B_{4,1}^{\frac{1}{2}} \\ &\leq C 3^{\frac{n}{2}(2d+1)} \|\tilde{h} - \tilde{g}\|_\infty^{\frac{1}{2}} |t - t'|^{\frac{1}{4}} . \end{aligned} \tag{4.21}$$

By (6.1.40), $B_{4,2}$ can be easily bounded as follows

$$B_{4,2} \leq 2^{-n\xi} |t - t'|^{\gamma_1'}, \qquad \xi > 0. \tag{4.22}$$

Then, for $\gamma_1 \in (0, \frac{1-\eta}{4})$, (4.15)-(4.22) yield

$$\sup_{x \in [0,1]^d} \|S^{\tilde{h}}(\cdot, x) - S^{\tilde{g}}(\cdot, x)\|_{\gamma_1 \times [0,T]} \leq C \Big[2^{-n\xi} + 2^{k_1(d)n} \|\tilde{h} - \tilde{g}\|_\infty^{\frac{1}{2}} \Big], \tag{4.23}$$

for some $\xi > 0$ and a positive constant $k_1(d)$ depending on the spatial dimension.
We have just studied (4.8) and we now have to examine (4.9). From (6.1.19), (6.1.20), (6.1.53) and (6.1.54), we can study (4.9) by means of similar arguments as before and to prove that, for $\gamma_2 \in (0, \frac{1-\eta}{2})$,

$$\sup_{t \in [0,T]} \|S^{\tilde{h}}(t, \cdot) - S^{\tilde{g}}(t, \cdot)\|_{\gamma_2, [0,1]^d} \leq C \Big[2^{-n\xi'} + 2^{k_2(d)n} \|h - g\|_\infty \Big], \tag{4.24}$$

for some $\xi' \in (0, \frac{1-\eta}{2})$ and $k_2(d) > 0$.
Finally, fixing $\rho > 0$ and choosing n_0 such that, for any $n \geq n_0$, $C(2^{-n\xi} + 2^{-n\xi'}) < \frac{\rho}{2}$, and $\delta \leq \frac{\rho}{4}(2^{-nk_1(d)} \wedge 2^{-nk_2(d)})$, inequalities (4.23) and (4.24) imply

$$\sup_{\|\tilde{h} - \tilde{g}\|_\infty \leq \delta} d_{\gamma_1, \gamma_2}(S^{\tilde{h}}, S^{\tilde{g}}) \leq \rho.$$

This conclude the proof of this theorem.

$\square$

5 The Freidlin-Wentzell inequality

In this section we will prove the inequality (1.4) of Theorem 1.1.

Proposition 5.1 *Assume* (C) *and* (H_η) *for some* $\eta \in (0,1)$. *For all* $R, \rho, a > 0$, $\gamma' \in (0, \frac{1}{2})$ *and* $\gamma \in (0, \frac{1-\eta}{4} \wedge \gamma')$, *there exists* $\delta > 0$ *and* $\varepsilon_0 > 0$ *such that, for any* $\tilde{h} \in H$ *satisfying* $\tilde{I}(\tilde{h}) \leq a$ *and* $\varepsilon < \varepsilon_0$, *we have*

$$P\Big\{d_{\gamma,\gamma}(u^\varepsilon, S^{\tilde{h}}) \geq \rho,\ d_{\gamma',\gamma'}(\varepsilon W_F, \tilde{h}) < \delta\Big\} \leq \exp\left(-\frac{R}{\varepsilon^2}\right).$$

Section 3, Theorem 4.2 and Proposition 5.1, by means of Theorem 1.1, allow us to deduce the next theorem.

Theorem 5.2 *Assume* (C) *and* (H_η) *for some* $\eta \in (0,1)$. *Then, the law of the solution* u^ε *satisfies on* $\mathcal{C}^{\gamma,\gamma}(D_T^d)$, $\gamma \in (0, \frac{1-\eta}{4})$, *a large deviation principle with the rate function*

$$S(g) = \begin{cases} \inf\{\tilde{I}(\tilde{h});\, S^{\tilde{h}} = g\}, & \textit{if } g \in Im(S^\cdot), \\ +\infty, & \textit{otherwise}. \end{cases}$$

Proof of Proposition 5.1: We need to prove that,

$$\begin{aligned} &\forall R, \rho, a > 0, \gamma' \in (0, \tfrac{1}{2}) \text{ and } \gamma \in (0, \tfrac{1-\eta}{4} \wedge \gamma'), \\ &\exists \delta, \varepsilon_0 > 0, \text{ such that } \forall \tilde{h} \in H \text{ satisfying } \tilde{I}(\tilde{h}) \leq a \text{ and } \forall \varepsilon < \varepsilon_0, \end{aligned} \tag{5.1}$$

we have

$$P\Big\{\|u^\varepsilon - S^{\tilde{h}}\|_{\gamma,\gamma} \geq \rho,\ \|\varepsilon W_F - \tilde{h}\|_{\gamma',\gamma'} < \delta\Big\} \leq \exp\left(-\frac{R}{\varepsilon^2}\right). \tag{5.2}$$

In all the following steps of this proof we will assume the above-mentioned conditions (5.1). We follow the classical proof of this kind of inequalities, which follows several different steps.

Step 1. Using the stopping time

$$\tau^\varepsilon = \inf\left\{t,\ \sup_{s \leq t} \sup_{x \in [0,1]^d} \left|u^\varepsilon(s,x) - S^{\tilde{h}}(s,x)\right| \geq \rho\right\} \wedge T,$$

and by applying a usual localization procedure (see, for instance, Proposition 3.9 in [14]) we can assume that the coefficients α and β are bounded.

Step 2. Given $\tilde{h} \in H$ such that $\tilde{I}(\tilde{h}) \leq a$ and $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned} v^{\varepsilon,\tilde{h}}(t,x) = \; & \varepsilon \int_0^t \int_{[0,1]^d} G(t-s,x,y)\alpha(v^{\varepsilon,\tilde{h}}(s,y))F(ds,dy) \\ & + \Big\langle G(t-\cdot,x,*)\alpha(v^{\varepsilon,\tilde{h}}(\cdot,*)), h \Big\rangle_{\mathcal{H}_T} \\ & + \int_0^t ds \int_{[0,1]^d} dy \, G(t-s,x,y)\beta(v^{\varepsilon,\tilde{h}}(s,y)). \end{aligned}$$

An extension of Girsanov's Theorem (see, for instance, Section IV.5 in [2]) allows us to reduce the proof of (5.2) to establishing the following

$$P\Big\{\|v^{\varepsilon,\tilde{h}} - S^{\tilde{h}}\|_{\gamma,\gamma} \geq \rho, \; \|\varepsilon W_F\|_{\gamma',\gamma'} < \delta\Big\} \leq \exp\left(-\frac{R}{\varepsilon^2}\right). \tag{5.3}$$

Step 3. We will now observe that (5.3) is equivalent to

$$P\Big\{\|\varepsilon \mathcal{K}^{\varepsilon,\tilde{h}}\|_{\gamma,\gamma} \geq \rho, \; \|\varepsilon W_F\|_{\gamma',\gamma'} < \delta\Big\} \leq \exp\left(-\frac{R}{\varepsilon^2}\right), \tag{5.4}$$

where

$$\mathcal{K}^{\varepsilon,\tilde{h}}(t,x) = \int_0^t \int_{[0,1]^d} G(t-s,x,y)\alpha(v^{\varepsilon,\tilde{h}}(s,y))F(ds,dy),$$

given $(t,x) \in D_T^d$. In order to check this equivalence we proceed as in Theorem 4.2. We will give the basic ideas of the proof. Schwarz's inequality, the Lipschitz property on α and β, (6.1.5), (6.1.6) and Gronwall's Lemma yield

$$\|v^{\varepsilon,\tilde{h}} - S^{\tilde{h}}\|_\infty \leq C\|\varepsilon \mathcal{K}^{\varepsilon,\tilde{h}}\|_\infty. \tag{5.5}$$

It only remains to study the Hölder property, that means to deal with

$$\left| v^{\varepsilon,\tilde{h}}(t,x) - S^{\tilde{h}}(t,x) - v^{\varepsilon,\tilde{h}}(t',x') + S^{\tilde{h}}(t',x') \right|,$$

for $(t,x) \neq (t',x') \in D_T^d$. Then, Schwarz's inequality, the Lipschitz property, (5.5), (6.1.7)-(6.1.10), (6.1.19) and (6.1.20) imply the equivalence between (5.3) and (5.4).

Step 4. Here the stochastic integral $\mathcal{K}^{\varepsilon,\tilde{h}}$ will be discretized as follows

$$K_n^{\varepsilon,\tilde{h}}(t,x) = \int_0^t \int_{[0,1]^d} G(\widehat{t-s_n},x,y_n)\alpha(v^{\varepsilon,\tilde{h}}(s_n,y_n))F(ds,dy).$$

We will also consider the sets

$$A_n^\varepsilon = \left\{ \|\varepsilon(\mathcal{K}^{\varepsilon,\tilde{h}} - \mathcal{K}_n^{\varepsilon,\tilde{h}})\|_{\gamma,\gamma} > \tfrac{\rho}{2} \right\},$$

$$B_n^\varepsilon = \left\{ \|\varepsilon K_n^{\varepsilon,\tilde{h}}\|_{\gamma,\gamma} > \tfrac{\rho}{2},\ \|\varepsilon W_F\|_{\gamma',\gamma'} < \delta \right\}.$$

It is immediate to check that

$$\left\{ \|\varepsilon \mathcal{K}^{\varepsilon,\tilde{h}}\|_{\gamma,\gamma} \geq \rho,\ \|\varepsilon W_F\|_{\gamma',\gamma'} < \delta \right\} \subset A_n^\varepsilon \cup B_n^\varepsilon.$$

In order to conclude the proof of this proposition we need to prove the next two facts:

1) There exists n_0 such that, for $n \geq n_0$,

$$P\left(A_n^\varepsilon\right) \leq \exp\left(-\frac{R}{\varepsilon^2}\right). \tag{5.6}$$

2) For any $n > 0$, we can choose $\delta > 0$ such that, considering

$$B_n^\varepsilon(\delta) = \left\{ \|\varepsilon \mathcal{K}_n^{\varepsilon,\tilde{h}}\|_{\gamma,\gamma} > \frac{\rho}{2},\ \|\varepsilon W_F\|_{\gamma',\gamma'} < \delta \right\}, \tag{5.7}$$

then $B_n^\varepsilon(\delta) = \emptyset$.

We first show (5.6). Before we need to see that, for any $\mu > 0$, there exists $\tilde{n}_0$ such that, for $n \geq \tilde{n}_0$,

$$P\left\{ \sup_{(t,x)\in D_T^d} \left| v^{\varepsilon,\tilde{h}}(t,x) - v^{\varepsilon,\tilde{h}}(t_n,x_n) \right| \geq \mu \right\} \leq \exp\left(-\frac{R}{\varepsilon^2}\right). \tag{5.8}$$

Set

$$\left| v^{\varepsilon,\tilde{h}}(t,x) - v^{\varepsilon,\tilde{h}}(t_n,x_n) \right| \leq \sum_{i=1}^{3} L_{i,n}^{\varepsilon,\tilde{h}}(t,x),$$

with

$$L_{1,n}^{\varepsilon,\tilde{h}}(t,x) = \int_0^t ds \int_{[0,1]^d} dy \left| G(t-s,x,y) - G(t_n - s, x_n, y) \right| \beta(v^{\varepsilon,\tilde{h}}(s,y)),$$

$$L_{2,n}^{\varepsilon,\tilde{h}}(t,x) = \left| \left\langle \left[G(t-\cdot,x,*) - G(t_n - \cdot, x_n, *) \right] \alpha(v^{\varepsilon,\tilde{h}}(\cdot,*)), h \right\rangle_{\mathcal{H}_T} \right|,$$

$$L_{3,n}^{\varepsilon,\tilde{h}}(t,x) = \left| \varepsilon(\mathcal{K}^{\varepsilon,\tilde{h}}(t,x) - \mathcal{K}^{\varepsilon,\tilde{h}}(t_n,x_n)) \right|.$$

Since β is bounded, from (6.1.9), (6.1.10), (6.1.20), (4.4) and (4.7), we obtain

$$\sup_{(t,x)\in D_T^d} L_{1,n}^{\varepsilon,\tilde{h}}(t,x) \leq C2^{-\theta_1 n}\|\beta\|_\infty,$$

for $\theta_1 \in (0,1)$.
On the other hand, Schwarz's inequality together with (6.1.7), (6.1.8), (6.1.19), (4.4) and (4.7) imply

$$\begin{aligned} L_{2,n}^{\varepsilon,\tilde{h}}(t,x) &\leq C\|\alpha\|_\infty \Big\|G(t-\cdot,x,*) - G(t_n-\cdot,x_n,*)\Big\|_{\mathcal{H}_T} \\ &\leq C^{-\theta_2 n}\|\alpha\|_\infty, \end{aligned}$$

for $\theta_2 \in (0,\frac{1-\eta}{2})$.
Then, assuming $C\left(2^{-\theta_1 n}\|\beta\|_\infty + 2^{-\theta_2 n}\|\alpha\|_\infty\right) \leq \frac{\mu}{2}$ and using these two last bounds, we have

$$\begin{aligned} \left\{\sup_{(t,x)\in D_T^d}\left|v^{\varepsilon,\tilde{h}}(t,x) - v^{\varepsilon,\tilde{h}}(t_n,x_n)\right| \geq \mu\right\} &\subset \left\{\sup_{(t,x)\in D_T^d} L_{3,n}^{\varepsilon,\tilde{h}}(t,x) \geq \frac{\mu}{2}\right\} \\ &\subset \left\{\|\varepsilon\mathcal{K}^{\varepsilon,\tilde{h}}\|_{\theta_3,\theta_3} \geq \frac{\mu}{3\cdot 2^{1-n\theta_3}}\right\}, \end{aligned}$$

with $\theta_3 \in (0,\frac{1-\eta}{2})$.
So, (5.8) will follows from the bound of this last set $\left\{\|\varepsilon\mathcal{K}^{\varepsilon,\tilde{h}}\|_{\theta_3,\theta_3} \geq \frac{\mu}{3\cdot 2^{1-n\theta_3}}\right\}$.
For any $(t,x),(t',x') \in D_T^d$, (6.1.7), (6.1.8) and (6.1.19) yield

$$\begin{aligned} &\left\|\Big[G(t-\cdot,x,*) - G(t'-\cdot,x',*)\Big]\alpha(v^{\varepsilon,\tilde{h}}(\cdot,*))\right\|_{\mathcal{H}_T}^2 \\ &\qquad \leq C\|\alpha\|_\infty^2\Big[|t-t'|^{2\theta_3} + \|x-x'\|^{2\theta_3}\Big], \end{aligned}$$

then, if $\frac{\mu}{3\cdot 2^{1-n\theta_3}} \geq 2C^{\frac{1}{2}}\|\alpha\|_\infty C(\theta_3)\tilde{C}(\theta_3)$ and $0<\varepsilon<1$, Lemma 6.2.1 for $\tau = T$ ensures

$$\begin{aligned} &P\left\{\sup_{(t,x)\in D_T^d}\left|v^{\varepsilon,\tilde{h}}(t,x) - v^{\varepsilon,\tilde{h}}(t_n,x_n)\right| \geq \mu\right\} \\ &\qquad \leq K\exp\left(-\frac{\mu^2}{3^2\cdot 2^4\cdot C\|\alpha\|_\infty^2 C^2(\theta_3)2^{-2n\theta_3}\varepsilon^2}\right). \end{aligned}$$

One can choose $\tilde{n}_0$ such that for any $n \geq \tilde{n}_0$, (5.8) is satisfied.

Now, we will use (5.8) to prove (5.6). Let

$$\mathcal{K}^{\varepsilon,\tilde{h}}(t,x) - \mathcal{K}_n^{\varepsilon,\tilde{h}}(t,x) = \mathcal{K}_{1,n}^{\varepsilon,\tilde{h}}(t,x) + \mathcal{K}_{2,n}^{\varepsilon,\tilde{h}}(t,x),$$

with

$$\begin{aligned}\mathcal{K}_{1,n}^{\varepsilon,\tilde{h}}(t,x) = & \int_0^t ds \int_{[0,1]^d} G(t-s,x,y)\Big[\alpha(v^{\varepsilon,\tilde{h}}(s,y)) \\ & - \alpha(v^{\varepsilon,\tilde{h}}(s_n,y_n))\Big] F(ds,dy),\\ \mathcal{K}_{2,n}^{\varepsilon,\tilde{h}}(t,x) = & \int_0^t ds \int_{[0,1]^d} \Big[G(t-s,x,y) - G(\widehat{t-s_n},x,y_n)\Big] \\ & \times \alpha(v^{\varepsilon,\tilde{h}}(s_n,y_n)) F(ds,dy).\end{aligned}$$

Define

$$\begin{aligned} A_{n,1}^{\varepsilon} &= \Big\{\|\varepsilon\mathcal{K}_{1,n}^{\varepsilon,\tilde{h}}\|_{\gamma,\gamma} \geq \tfrac{\rho}{4}\Big\},\\ A_{n,2}^{\varepsilon} &= \Big\{\|\varepsilon\mathcal{K}_{2,n}^{\varepsilon,\tilde{h}}\|_{\gamma,\gamma} \geq \tfrac{\rho}{4}\Big\},\\ D_n^{\varepsilon} &= \left\{\sup_{(t,x)\in D_T^d} \left|v^{\varepsilon,\tilde{h}}(t,x) - v^{\varepsilon,\tilde{h}}(t_n,x_n)\right| \leq \mu\right\}.\end{aligned}$$

It is obvious that

$$A_n^{\varepsilon} \subset A_{n,1}^{\varepsilon} \cup A_{n,2}^{\varepsilon} \subset \left(A_{n,1}^{\varepsilon} \cap D_n^{\varepsilon}\right) \cup (D_n^{\varepsilon})^c \cup A_{n,2}^{\varepsilon}.$$

The set $(D_n^{\varepsilon})^c$ has alredy bounded in (5.8). Proving that there exists $\hat{n}_0$ such that, for any $n \geq \hat{n}_0$,

$$P\left(A_{n,1}^{\varepsilon} \cap D_n^{\varepsilon}\right) \leq \exp\left(-\frac{R}{\varepsilon^2}\right), \tag{5.9}$$

$$P\left(A_{n,2}^{\varepsilon}\right) \leq \exp\left(-\frac{R}{\varepsilon^2}\right), \tag{5.10}$$

we can conclude (5.6) for any $n \geq n_0$ being $n_0 = \tilde{n}_0 \vee \hat{n}_0$.
Set

$$\tau_n^{\varepsilon} = \inf\Big\{t \geq 0,\, \exists (s,x) \in D_T^d, s \leq t, |v^{\varepsilon,\tilde{h}}(s,x) - v^{\varepsilon,\tilde{h}}(s_n,x_n)| \geq \mu\Big\} \wedge T.$$

For any $(t,x),(t',x') \in D^d_{\tau^\varepsilon_n}$, by (6.1.7), (6.1.8) and (6.1.19) we have

$$\left\| \Big[G(t-\cdot,x,*) - G(t'-\cdot,x',*)\Big] \Big[\alpha(v^{\varepsilon,\tilde{h}}(\cdot,*)) - \alpha(v_n^{\varepsilon,\tilde{h}}(\cdot,*))\Big] \right\|^2_{\mathcal{H}_T} \\ \leq C\mu^2 \Big[|t-t'|^{2\gamma} + \|x-x'\|^{2\gamma}\Big],$$

for $\gamma \in (0, \frac{1-\eta}{2})$ and with $v_n^{\varepsilon,\tilde{h}}(s,y) = v^{\varepsilon,\tilde{h}}(s_n,y_n)$. Then, if $\frac{\rho}{4} \geq 2C^{\frac{1}{2}}\mu C(\gamma)\tilde{C}(\gamma)$ and $0<\varepsilon<1$, Lemma 6.2.1 gives

$$P\left(A^\varepsilon_{n,1} \cap D^\varepsilon_n\right) \leq K \exp\left(-\frac{\rho^2}{2^8 C\mu^2 C^2(\gamma)\varepsilon^2}\right).$$

For any $R>0$ we can choose $\mu>0$ such that (5.9) is proved.
For any $(t,x),(t',x') \in D^d_T$, (6.1.39) and (6.1.53) yield

$$\left\| \Big[G(t-\cdot,x,*) - G_n(t-\cdot,x,*) - G(t'-\cdot,x',*) + G_n(t'-\cdot,x',*)\Big] \alpha(v_n^{\varepsilon,\tilde{h}}(\cdot,*)) \right\|^2_{\mathcal{H}_T} \\ \leq C\|\alpha\|^2_\infty \Big[|t-t'|^{\gamma_0} + \|x-x'\|^{\gamma_0}\Big] 2^{-\gamma_0(n-1)},$$

for $\gamma_0 \in (0, \frac{1-\eta}{2})$. Then, if $\frac{\rho}{4} \geq 2C^{\frac{1}{2}}\|\alpha\|_\infty C(\gamma,\gamma_0)\tilde{C}(\gamma,\gamma_0)2^{-\frac{\gamma_0}{2}(n-1)}$ and $0<\varepsilon<1$, Lemma 6.2.1 implies

$$P\left(A^\varepsilon_{n,2}\right) \leq K \exp\left(-\frac{\rho^2}{2^8 C\|\alpha\|^2_\infty C(\gamma,\gamma_0)2^{-\gamma_0(n-1)}\varepsilon^2}\right),$$

for $\gamma \in (0, \frac{\gamma_0}{2})$ what means $\gamma \in (0, \frac{1-\eta}{4})$. Choosing n'_0 large enough, for any $n \geq n'_0$, we have (5.10).
Consequently, (5.6) is completely proved. In order to conclude the proof of Proposition 5.1 we have to check (5.7), that means that, for any $n>0$, we can find $\delta>0$ such that

$$B^\varepsilon_n(\delta) = \left\{ \|\varepsilon\mathcal{K}_n^{\varepsilon,\tilde{h}}\|_{\gamma,\gamma} > \frac{\delta}{2},\ \|\varepsilon W_F\|_{\gamma',\gamma'} < \delta \right\} = \emptyset. \tag{5.11}$$

The proof of (5.11) is similar in some aspects to the argument used in Theorem 4.2. We give a brief sketch of this proof. We have to work with the difference

$$\varepsilon\left(\mathcal{K}_n^{\varepsilon,\tilde{h}}(t,x) - \mathcal{K}_n^{\varepsilon,\tilde{h}}(t',x')\right),$$

for $x,x' \in [0,1]^d, t,t' \in [0,T]$. We assume $x=x'$ and $t'<t$ for the sake of simplicity. Then,

$$\varepsilon\left(\mathcal{K}_n^{\varepsilon,\tilde{h}}(t,x) - \mathcal{K}_n^{\varepsilon,\tilde{h}}(t',x)\right) = M^{\varepsilon,\tilde{h}}_{1,n} + M^{\varepsilon,\tilde{h}}_{2,n},$$

with

$$
\begin{aligned}
M_{1,n}^{\varepsilon,\tilde{h}} &= \varepsilon \int_{t'}^{t} \int_{[0,1]^d} G(\widehat{t-s_n}, x, y_n)\alpha(v^{\varepsilon,\tilde{h}}(s_n, y_n))F(ds, dy), \\
M_{2,n}^{\varepsilon,\tilde{h}} &= \varepsilon \int_{0}^{t'} \int_{[0,1]^d} \left[G(\widehat{t-s_n}, x, y_n) - G(\widehat{t'-s_n}, x, y_n)\right] \\
&\quad \times \alpha(v^{\varepsilon,\tilde{h}}(s_n, y_n))F(ds, dy).
\end{aligned}
$$

Decompositions similar to (4.13) for stochastic integrals allow us to study these two terms and to obtain

$$
\left|\varepsilon\left(\mathcal{K}_n^{\varepsilon,\tilde{h}}(t,x) - \mathcal{K}_n^{\varepsilon,\tilde{h}}(t',x')\right)\right| \le C 2^{k(d)n}\left[|t-t'|^{\gamma'} + \|x-x'\|^{\gamma'}\right]\|\varepsilon W_F\|_{\gamma',\gamma'},
$$

for a positive constant $k(d)$ depending on the spatial dimension. In this last inequality we can not get an estimation in terms of $\|\varepsilon W_F\|_\infty$, that is the unique difference between the proof of (4.13) and the study of (5.11).
This argument finishes the proof of Proposition 5.1.

□

6 Appendix

6.1 Study of the kernel

Here we will enunciate and prove all the results on the kernel G used in this paper. Recall that G denotes the fundamental solution of the heat equation

$$
\begin{cases}
\frac{\partial}{\partial t}G(t,x,y) = \Delta_x G(t,x,y), \qquad t \ge 0,\ x, y \in [0,1]^d, \\
G(t,x,y) = 0, \qquad x \in \partial([0,1]^d), \\
G(0,x,y) = \delta(x-y).
\end{cases}
\tag{6.1.1}
$$

The details about the construction of the fundamental solution G can be found in Chapter 1 of [11], more specifically in Section 4. This fundamental solution G is non-negative and can be decomposed into different terms as follows:

$$
G(t-s,x,y) = H(t-s,x,y) + R(t-s,x,y) + \sum_{i \in I_d} H_i(t-s, x-y), \tag{6.1.2}
$$

where H is the heat kernel on $\mathbb{R}^d$

$$
H(t,x) = \left(\frac{1}{\sqrt{4\pi t}}\right)^d \exp\left(-\frac{\|x\|^2}{4t}\right),
$$

R is a Lipschitz function, $I_d = \Big\{i = (i_1, \ldots, i_d) \in \{-1, 0, 1\}^d - \{(0, \ldots, 0)\}\Big\}$ and

$$H_i(t-s, x-y) = (-1)^k H(t-s, x-y^i),$$

with $k = \sum_{j=1}^{d} |i_j|$ and

$$\begin{cases} y_j^i = y_j, & \text{if} \quad i_j = 0, \\ y_j^i = -y_j, & \text{if} \quad i_j = 1, \\ y_j^i = 2 - y_j, & \text{if} \quad i_j = -1. \end{cases}$$

This decomposition is given in [9].
Finally, it is well-known that

$$\mathcal{F}H(t, \cdot)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} H(t, x) dx = \exp\left(-4\pi^2 t \|\xi\|^2\right), \tag{6.1.3}$$

where $\mathcal{F}$ is the Fourier transform.
The proof of following bounds of $G, D_t G$ and $D_x G$ can be found in Theorem 1.1 of [8].

Lemma 6.1.1 *There exists a positive constant C such that*

$$\begin{aligned} G(t-s, x, y) &\leq CH(t-s, x-y), \qquad (6.1.4) \\ |D_t G(t-s, x, y)| &\leq C(t-s)^{-\frac{d+2}{2}} \exp\left(-\frac{\|x-y\|^2}{4(t-s)}\right), \\ \Big\|D_x G(t-s, x, y)\Big\| &\leq C(t-s)^{-\frac{d+1}{2}} \exp\left(-\frac{\|x-y\|^2}{4(t-s)}\right). \end{aligned}$$

In the sequel we give new results on G.

Lemma 6.1.2 *Assume* (H_1). *There exist positive constants C_1 and C_2 such that, for any $0 \leq s \leq t \leq T$, $x \in [0,1]^d$,*

$$\Big\|G(t - \cdot, x, *)\Big\|^2_{\mathcal{H}_T} \leq C_1, \tag{6.1.5}$$

$$\int_{[0,1]^d} dy\, G(t-s, x, y) \leq C_2. \tag{6.1.6}$$

Proof: The first bound (6.1.5) is a consequence of (6.1.4) and Example 8 in [4]. The estimate (6.1.6) becomes from (6.1.4).

$\square$

Lemma 6.1.3 *Assume* (H_η) *for some* $\eta \in (0,1)$. *There exists a positive constant* C *such that, for any* $0 \le t' \le t \le T$, $x \in [0,1]^d$, $\gamma_1 \in (0, \frac{1-\eta}{2})$ *and* $\gamma_2 \in (0, \frac{1}{2})$,

$$\int_0^{t'} ds \left\| G(t-s,x,*) - G(t'-s,x,*) \right\|_{\mathcal{H}}^2 \le C|t-t'|^{2\gamma_1}, \tag{6.1.7}$$

$$\int_{t'}^{t} ds \left\| G(t-s,x,*) \right\|_{\mathcal{H}}^2 \le C|t-t'|^{2\gamma_1}, \tag{6.1.8}$$

$$\int_0^{t'} ds \int_{[0,1]^d} dy \left| G(t-s,x,y) - G(t'-s,x,y) \right| \le C|t-t'|^{2\gamma_2}, \tag{6.1.9}$$

$$\int_{t'}^{t} ds \int_{[0,1]^d} dy \, G(t-s,x,y) \le C|t-t'|. \tag{6.1.10}$$

Remark. Recall that the fundamental solution G of (6.1.1) satisfies

$$\begin{aligned} \left\| G(t-\cdot,x,*) - G(t'-\cdot,x,*) \right\|_{\mathcal{H}_T}^2 &= \int_{t'}^{t} ds \, \|G(t-s,x,*)\|_{\mathcal{H}}^2 \\ &+ \int_0^{t'} ds \left\| G(t-s,x,*) - G(t'-s,x,*) \right\|_{\mathcal{H}}^2. \end{aligned}$$

Proof of Lemma 6.1.3: Observe that (6.1.2) implies

$$\begin{aligned} &\left\| G(t-\cdot,x,*) - G(t'-s,x,*) \right\|_{\mathcal{H}}^2 \\ &\quad \le (3^d+1) \Bigg[\left\| H(t-s,x-*) - H(t'-s,x-*) \right\|_{\mathcal{H}}^2 \\ &\qquad + \left\| R(t-s,x-*) - R(t'-s,x-*) \right\|_{\mathcal{H}_T}^2 \\ &\qquad + \sum_{i \in I_d} \left\| H_i(t-s,x-*) - H_i(t'-s,x-*) \right\|_{\mathcal{H}}^2 \Bigg]. \end{aligned} \tag{6.1.11}$$

In order to check (6.1.7) we only need to bound the right-hand side of (6.1.11). The terms H_i can be studied using the same arguments as H.

Let L be the Lipschitz constant of the function R. Clearly

$$\int_0^{t'} ds \Big\| R(t-s, x-*) - R(t'-s, x-*) \Big\|_{\mathcal{H}}^2 \leq L^2 \ \Theta \ T|t-t'|^2, \qquad (6.1.12)$$

where

$$\Theta = \int_{[0,1]^d} dy \int_{[0,1]^d} dz f(y-z),$$

and f is defined in Section 1.
Now we analyze the term with H. First of all, from $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is a scalar product, we have

$$\int_0^{t'} ds \Big\| H(t-s, x-*) - H(t'-s, x-*) \Big\|_{\mathcal{H}}^2 \leq 2(A_1 + A_2), \qquad (6.1.13)$$

with

$$\begin{aligned}
A_1 &= \int_0^{t'} ds \int_{[0,1]^d} dy \int_{[0,1]^d} dz f(y-z) \left(\frac{1}{4\pi(t-s)} \right)^d \\
&\quad \times \left[\exp\left(-\frac{\|x-y\|^2}{4(t-s)} \right) - \exp\left(-\frac{\|x-y\|^2}{4(t'-s)} \right) \right] \\
&\quad \times \left[\exp\left(-\frac{\|x-z\|^2}{4(t-s)} \right) - \exp\left(-\frac{\|x-z\|^2}{4(t'-s)} \right) \right], \\
A_2 &= \int_0^{t'} ds \int_{[0,1]^d} dy \int_{[0,1]^d} dz f(y-z) \left[\left(\frac{1}{4\pi(t-s)} \right)^{\frac{d}{2}} - \left(\frac{1}{4\pi(t'-s)} \right)^{\frac{d}{2}} \right]^2 \\
&\quad \times \exp\left(-\frac{\|x-y\|^2}{4(t'-s)} \right) \exp\left(-\frac{\|x-z\|^2}{4(t'-s)} \right).
\end{aligned}$$

Since $t' < t$, using basic tools of mathematical calculus and (6.1.3), we obtain

$$\begin{aligned}
A_1 &\leq 2 \int_0^{t'} ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz H(t-s, x-y) f(y-z) H(t-s, x-z) \\
&\qquad -2 \int_0^{t'} ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left(\frac{t'-s}{t-s} \right)^d H(t'-s, x-y) \\
&\qquad \times f(y-z) H(t'-s, x-z) \\
&\leq 2 \int_0^{t'} ds \int_{\mathbb{R}^d} \lambda(d\xi) \exp\left(-4\pi^2(t'-s)\|\xi\|^2\right) \left(1 - \left(\frac{t'-s}{t-s} \right)^d \right)
\end{aligned}$$

$$= \quad 2\int_0^{t'} ds \int_{\mathbb{R}^d} \lambda(d\xi) \exp\left(-4\pi^2(t'-s)\|\xi\|^2\right)\left(1-\frac{t'-s}{t-s}\right)$$
$$\times\left(1+\frac{t'-s}{t-s}+\left(\frac{t'-s}{t-s}\right)^2+\ldots+\left(\frac{t'-s}{t-s}\right)^{d-1}\right).$$

For $\gamma_1 \in (0, \frac{1-\eta}{2})$ and $s < t'$, we have

$$\left(\frac{t-t'}{t-s}\right)^{2\gamma_1} < \frac{t-t'}{t-s} < 1,$$

and then,

$$\begin{aligned} A_1 \quad & \leq 2\, d\, |t-t'|^{2\gamma_1} \int_0^{t'} ds \int_{\mathbb{R}^d} \lambda(d\xi) \exp\left(-4\pi^2(t'-s)\|\xi\|^2\right)(t'-s)^{-2\gamma_1} \\ & \leq C|t-t'|^{2\gamma_1}\Gamma(1-2\gamma_1)\int_{\mathbb{R}^d} \lambda(d\xi)\frac{1}{\|\xi\|^{2(1-2\gamma_1)}} \\ & \leq C|t-t'|^{2\gamma_1}, \end{aligned} \tag{6.1.14}$$

where Γ is the Gamma function.
On the other hand, (6.1.3) again yields

$$\begin{aligned} A_2 \quad & \leq \int_0^{t'} ds \int_{\mathbb{R}^d} \lambda(d\xi)(t'-s)^d\left(\left(\frac{1}{t-s}\right)^{\frac{d}{2}}-\left(\frac{1}{t'-s}\right)^{\frac{d}{2}}\right)^2 \\ & \quad \times \exp\left(-4\pi^2(t'-s)\|\xi\|^2\right) \\ & = \int_0^{t'} ds \int_{\mathbb{R}^d} \lambda(d\xi) \exp\left(-4\pi^2(t'-s)\|\xi\|^2\right)\left(\frac{\sqrt{t'-s}}{\sqrt{t-s}}-1\right)^2 \\ & \quad \times\left(1+\frac{\sqrt{t'-s}}{\sqrt{t-s}}+\left(\frac{\sqrt{t'-s}}{\sqrt{t-s}}\right)^2+\cdots+\left(\frac{\sqrt{t'-s}}{\sqrt{t-s}}\right)^{d-1}\right)^2. \end{aligned}$$

The next inequality

$$\sqrt{t-s}-\sqrt{t'-s} \leq \sqrt{t-t'},$$

and similar computations to the study of A_1 show that

$$A_2 \leq C|t-t'|^{2\gamma_1}, \tag{6.1.15}$$

for all $\gamma_1 \in (0, \frac{1-\eta}{2})$.

Then, (6.1.7) follows from (6.1.12)-(6.1.15).
Now we prove (6.1.8). Set

$$\int_{t'}^{t} ds \Big\| G(t-s,x,*) \Big\|_{\mathcal{H}}^2 \leq (3^d+1)[B_1+B_2+B_3], \tag{6.1.16}$$

with

$$\begin{aligned}
B_1 &= \int_{t'}^{t} ds \Big\| R(t-s,x-*) \Big\|_{\mathcal{H}}^2, \\
B_2 &= \int_{t'}^{t} ds \Big\| H(t-s,x-*) \Big\|_{\mathcal{H}}^2, \\
B_3 &= \sum_{i \in I_d} \int_{t'}^{t} ds \Big\| H_i(t-s,x-*) \Big\|_{\mathcal{H}}^2.
\end{aligned}$$

As before the study of B_1 is obvious and we can deal with B_3 by means of B_2. Then, we will only need to work with B_2.
From (6.1.3), by integrating we obtain

$$\begin{aligned}
B_2 &\leq \int_{t'}^{t} ds \int_{\mathbb{R}^d} \lambda(d\xi) \exp\left(-4\pi^2 (t-s)\|\xi\|^2\right) \\
&= \int_{\mathbb{R}^d} \lambda(d\xi) \left(1 - \exp\left(-4\pi^2(t-t')\|\xi\|^2\right)\right) \frac{1}{4\pi^2 \|\xi\|^2}.
\end{aligned}$$

Since

$$1 - e^{-x} \leq x, \qquad \forall x > 0, \tag{6.1.17}$$

the hypothesis (H_η), implies, for $\gamma_1 \in (0, \frac{1-\eta}{2})$,

$$\begin{aligned}
B_2 &\leq \frac{1}{4\pi^2} \int_{\mathbb{R}^d} \lambda(d\xi) \left(1 - \exp\left(-4\pi^2(t-t')\|\xi\|^2\right)\right)^{2\gamma_1} \frac{1}{\|\xi\|^2} \\
&\leq \frac{1}{4\pi^2} \int_{\mathbb{R}^d} \lambda(d\xi) \frac{|t-t'|^{2\gamma_1} \|\xi\|^{4\gamma_1} (4\pi^2)^{2\gamma_1}}{\|\xi\|^2} \\
&\leq C|t-t'|^{2\gamma_1}.
\end{aligned} \tag{6.1.18}$$

Then, (6.1.8) follows from (6.1.16) and (6.1.18).
Finally, the proof of (6.1.9) is similar to (6.1.7) and the estimate (6.1.10) is an immediate consequence of (6.1.4).

□

Lemma 6.1.4 *Assume* (H_η) *for some* $\eta \in (0,1)$. *There exists a positive constant* C *such that, for any* $0 \le t \le T$, $x, x' \in [0,1]^d$, $\gamma_1 \in (0, 1-\eta)$ *and* $\gamma_2 \in (0, \frac{1}{2})$,

$$\int_0^t ds \Big\| G(t-s, x, *) - G(t-s, x', *) \Big\|_{\mathcal{H}}^2 \le C \|x - x'\|^{2\gamma_1}, \tag{6.1.19}$$

$$\int_0^t ds \int_{[0,1]^d} dy \Big| G(t-s, x, *) - G(t-s, x', *) \Big|_{\mathcal{H}} \le C \|x - x'\|^{2\gamma_2}. \tag{6.1.20}$$

Proof: As in (6.1.11) we only need to check (6.1.19) replacing G by H. Let

$$D = \int_0^t ds \Big\| H(t-s, x-*) - H(t-s, x'-*) \Big\|_{\mathcal{H}}^2.$$

If $\gamma_1 \in (0, 1-\eta)$, then

$$D \le C(D_1 + D_2), \tag{6.1.21}$$

with

$$\begin{aligned} D_1 &= \int_0^t ds \bigg\| \Big| H(t-s, x-*) - H(t-s, x'-*) \Big|^{\gamma_1} \Big| H(t-s, x-*) \Big|^{1-\gamma_1} \bigg\|_{\mathcal{H}}^2, \\ D_2 &= \int_0^t ds \bigg\| \Big| H(t-s, x-*) - H(t-s, x'-*) \Big|^{\gamma_1} \Big| H(t-s, x'-*) \Big|^{1-\gamma_1} \bigg\|_{\mathcal{H}}^2. \end{aligned} \tag{6.1.22}$$

We first analyze D_1. We have

$$\begin{aligned} D_1 \ &= \int_0^t ds \int_{[0,1]^d} dy \int_{[0,1]^d} dz f(y-z) \left(\frac{1}{4\pi(t-s)} \right)^d \\ &\times \left| \exp\left(-\frac{\|x-y\|^2}{4(t-s)} \right) - \exp\left(-\frac{\|x'-y\|^2}{4(t-s)} \right) \right|^{\gamma_1} \exp\left(-\frac{(1-\gamma_1)\|x-y\|^2}{4(t-s)} \right) \\ &\times \left| \exp\left(-\frac{\|x-z\|^2}{4(t-s)} \right) - \exp\left(-\frac{\|x'-z\|^2}{4(t-s)} \right) \right|^{\gamma_1} \exp\left(-\frac{(1-\gamma_1)\|x-z\|^2}{4(t-s)} \right). \end{aligned}$$

It is well-known that, for any $x > 0$,

$$\begin{aligned} x \ &\le \ e^x, &(6.1.23) \\ e^{-x} \ &\le \ 1. &(6.1.24) \end{aligned}$$

By the mean-value theorem, (6.1.23), (6.1.24) and (H_η), we get

$$\begin{aligned}
D_1 &\leq C\|x-x'\|^{2\gamma 1}\int_0^t ds\int_{[0,1]^d} dy\int_{[0,1]^d} dz(t-s)^{-\gamma_1}f(y-z)\\
&\times\left(\frac{1}{4\pi(t-s)}\right)^d \exp\left(-\frac{(1-\gamma_1)\|x-y\|^2}{4(t-s)}\right)\exp\left(-\frac{(1-\gamma_1)\|x-z\|^2}{4(t-s)}\right)\\
&\leq C\|x-x'\|^{2\gamma_1}\int_0^t ds\int_{\mathbb{R}^d}\lambda(d\xi)(t-s)^{-\gamma_1}\exp\left(-4\pi^2(t-s)\|\xi\|^2\right)\\
&\leq C\|x-x'\|^{2\gamma_1}\Gamma(1-\gamma_1)\int_{\mathbb{R}^d}\frac{\lambda(d\xi)}{\|\xi\|^{2(1-\gamma_1)}}\\
&\leq C\|x-x'\|^{2\gamma_1}.
\end{aligned}\tag{6.1.25}$$

Analogously,

$$D_2 \leq C\|x-x'\|^{2\gamma_1}. \tag{6.1.26}$$

Then, (6.1.19) follows from (6.1.21)-(6.1.26).
The proof of (6.1.20) is very similar to (6.1.19).

□

Consider the same discretization in time and space as in the Section 4. By (6.1.4) and (4.5) we have

$$|G_n(t-s,x,y)| = |G(\widehat{t-s_n},x,y_n)| \leq C|\widehat{t-s_n}|^{-\frac{d}{2}} \leq C2^{n\frac{d}{2}}, \tag{6.1.27}$$

for any $0\leq s<t\leq T$, $x,y\in[0,1]^d$.

Lemma 6.1.5 *Assume* (H_η) *for some* $\eta\in(0,1)$. *There exists a positive constant* C *such that, for any* $(t,x)\in D_T^d$ *and* $\gamma\in(0,\frac{1-\eta}{2})$,

$$\int_0^t ds\left\|G(\widehat{t-s_n},x,*)-G(t-s,x,*)\right\|_{\mathcal{H}}^2 \leq C2^{-2\gamma(n-1)}, \tag{6.1.28}$$

$$\int_0^t ds\left\|G_n(t-s,x,*)-G(\widehat{t-s_n},x,*)\right\|_{\mathcal{H}}^2 \leq C2^{-2\gamma n}. \tag{6.1.29}$$

Proof: As in (6.1.7) we can prove (6.1.28). In order to check (6.1.29) we will only need the next inequality, for all $\gamma\in(0,\frac{1-\eta}{2})$,

$$E=\int_0^t ds\left\|H_n(t-s,x-*)-H(\widehat{t-s_n},x-*)\right\|_{\mathcal{H}}^2 \leq C2^{-2\gamma n}, \tag{6.1.30}$$

where $H_n(t-s, x-y) = H(\widehat{t-s_n}, x-y_n)$.
If $\gamma \in (0,1)$, then

$$\begin{aligned} E &\leq \int_0^t ds \Big\| \Big| H_n(t-s, x-*) - H(\widehat{t-s_n}, x-*) \Big|^{\gamma} \\ &\qquad \times \Big| H_n(t-s, x-*) - H(\widehat{t-s_n}, x-*) \Big|^{1-\gamma} \Big\|_{\mathcal{H}}^2 \\ &\leq C(E_1+E_2), \end{aligned} \tag{6.1.31}$$

where

$$E_1 = \int_0^t ds \Big\| \Big| H_n(t-s, x-*) - H(\widehat{t-s_n}, x-*) \Big|^{\gamma} \Big| H(\widehat{t-s_n}, x-*) \Big|^{1-\gamma} \Big\|_{\mathcal{H}}^2,$$

$$E_2 = \int_0^t ds \Big\| \Big| H_n(t-s, x-*) - H(\widehat{t-s_n}, x-*) \Big|^{\gamma} \Big| H_n(t-s, x-*) \Big|^{1-\gamma} \Big\|_{\mathcal{H}}^2.$$

Similar arguments to the computation of the estimate (6.1.25) together with (4.7) imply

$$E_1 \leq C 2^{-2\gamma n}, \tag{6.1.32}$$

for all $\gamma \in (0, \frac{1-\eta}{2})$.
Now we examine E_2. First of all, we can ensure the existence of a positive constant C such that

$$\|y_n - x\|^2 \geq C\|y-x\|^2, \qquad \forall y \in B(x, 3\cdot 2^{-n})^c. \tag{6.1.33}$$

Then, the mean-value theorem and (6.1.23) ensure

$$E_2 \leq C(E_{21}+E_{22}), \tag{6.1.34}$$

where

$$\begin{aligned} E_{21} &\leq C 2^{-2\gamma n} \int_0^t ds \int_{B(x,3\cdot 2^{-n})^c} dy \int_{B(x,3\cdot 2^{-n})^c} dz \; (\widehat{t-s_n})^{-\gamma} f(y-z) \\ &\times \left(\frac{1}{4\pi(\widehat{t-s_n})} \right)^d \exp\left(-\frac{(1-\gamma)\|x-y_n\|^2}{4(\widehat{t-s_n})} \right) \exp\left(-\frac{(1-\gamma)\|x-z_n\|^2}{4(\widehat{t-s_n})} \right), \\ E_{22} &\leq C 2^{-2\gamma n} \int_0^t ds \int_{B(x,3\cdot 2^{-n})} dy \int_{B(x,3\cdot 2^{-n})} dz \; (\widehat{t-s_n})^{-\gamma} f(y-z) \\ &\times \left(\frac{1}{4\pi(\widehat{t-s_n})} \right)^d \exp\left(-\frac{(1-\gamma)\|x-y_n\|^2}{4(\widehat{t-s_n})} \right) \exp\left(-\frac{(1-\gamma)\|x-z_n\|^2}{4(\widehat{t-s_n})} \right). \end{aligned}$$

By (6.1.33) we can easily study E_{21}. Indeed, (6.1.33) yields

$$\begin{aligned} E_{21} &\leq C2^{-2\gamma n}\int_0^t ds \int_{B(x,3\cdot 2^{-n})^c} dy \int_{B(x,3\cdot 2^{-n})^c} dz\ (\widehat{t-s_n})^{-\gamma} f(y-z) \\ &\times \left(\frac{1}{4\pi(\widehat{t-s_n})}\right)^d \exp\left(-\frac{(1-\gamma)\|x-y\|^2}{4(\widehat{t-s_n})}\right)\exp\left(-\frac{(1-\gamma)\|x-z\|^2}{4(\widehat{t-s_n})}\right), \end{aligned}$$

Now, same computations as in (6.1.25) give, for all $\gamma \in (0, \frac{1-\eta}{2})$,

$$E_{21} \leq C2^{-2\gamma n}. \tag{6.1.35}$$

Let $\gamma \in (0, \frac{1-\eta}{2})$. From (6.1.24) and (4.5), we obtain by integrating with respect to s and changing to polar coordinates,

$$\begin{aligned} E_{22} &\leq C2^{-2\gamma n}\int_0^t ds \int_{B(x,3\cdot 2^{-n})} dy \int_{B(x,3\cdot 2^{-n})} dz f(y-z)(\widehat{t-s_n})^{-d-\gamma} \\ &\leq C2^{-2\gamma n+dn}\int_0^t ds (t-s)^{-\gamma}\int_{B(x,3\cdot 2^{-n})} dy \int_{B(x,3\cdot 2^{-n})} dz f(y-z) \\ &\leq C2^{-2\gamma n+dn}\int_{B(x,3\cdot 2^{-n})} dy \int_{B(x,3\cdot 2^{-n})} dz f(y-z) \\ &= C2^{-2\gamma n+dn-dn}\int_0^1 r^{-(d-1)} f(r) dr \\ &\leq C2^{-2\gamma n}. \end{aligned} \tag{6.1.36}$$

Hence, the estimates (6.1.31)-(6.1.36) prove (6.1.30).

□

Lemma 6.1.6 *Assume* (H$_\eta$) *for some* $\eta \in (0,1)$. *There exists a positive constant* C *such that, for any* $0 \leq t' \leq t \leq T$, $x \in [0,1]^d$ *and* $\gamma \in (0, \frac{1-\eta}{2})$,

$$\int_{t'}^t ds \Big\| G(t-s,x,*) - G_n(t-s,x,*)\Big\|_{\mathcal{H}}^2 \leq C2^{-2\gamma(n-1)}|t-t'|^\gamma, \tag{6.1.37}$$

$$\int_{t'}^t ds \Big\| G_n(t-s,x,*)\Big\|_{\mathcal{H}}^2 \leq C|t-t'|^{2\gamma}, \tag{6.1.38}$$

$$\begin{aligned} \int_0^{t'} ds \Big\| &G(t-s,x,*) - G_n(t-s,x,*) - G(t'-s,x,*) \\ &+ G_n(t'-s,x,*)\Big\|_{\mathcal{H}}^2 \leq C2^{-\gamma(n-1)}|t-t'|^\gamma, \end{aligned} \tag{6.1.39}$$

$$\int_0^{t'} ds \Big\| G_n(t-s,x,*) - G_n(t'-s,x,*) \Big\|_{\mathcal{H}}^2 \le C|t-t'|^{2\gamma}. \tag{6.1.40}$$

Proof: As usual, we will only check the inequalities (6.1.37)-(6.1.40) changing G by H. We start proving (6.1.37). Clearly

$$\int_{t'}^{t} ds \Big\| H(t-s,x-*) - H_n(t-s,x-*) \Big\|_{\mathcal{H}}^2 \le (F_1+F_2), \tag{6.1.41}$$

where

$$\begin{aligned} F_1 &= \int_{t'}^{t} ds \Big\| H(t-s,x-*) - H(\widehat{t-s_n},x-*) \Big\|_{\mathcal{H}}^2, \\ F_2 &= \int_{t'}^{t} ds \Big\| H(\widehat{t-s_n},x-*) - H_n(t-s,x-*) \Big\|_{\mathcal{H}}^2. \end{aligned} \tag{6.1.42}$$

Basic calculus as in Lemma 6.1.3 give

$$F_1 \le C(F_{11}+F_{12}), \tag{6.1.43}$$

where

$$\begin{aligned} F_{11} = \;& \int_{t'}^{t} ds \int_{[0,1]^d} dy \int_{[0,1]^d} dz \left(\frac{1}{4\pi(\widehat{t-s_n})} \right)^d f(y-z) \\ & \times \left[\exp\left(-\frac{\|x-y\|^2}{4(\widehat{t-s_n})} \right) - \exp\left(-\frac{\|x-y\|^2}{4(t-s)} \right) \right] \\ & \times \left[\exp\left(-\frac{\|x-z\|^2}{4(\widehat{t-s_n})} \right) - \exp\left(-\frac{\|x-z\|^2}{4(t-s)} \right) \right], \\ F_{12} = \;& \int_{t'}^{t} ds \int_{[0,1]^d} dy \int_{[0,1]^d} dz \left[\left(\frac{1}{4\pi(\widehat{t-s_n})} \right)^{\frac{d}{2}} - \left(\frac{1}{4\pi(\widehat{t-s_n})} \right)^{\frac{d}{2}} \right]^2 \\ & \times f(y-z) \exp\left(-\frac{\|x-y\|^2}{4(t-s)} \right) \exp\left(-\frac{\|x-z\|^2}{4(t-s)} \right). \end{aligned}$$

As in (6.1.7) we can obtain

$$F_{11} \le 2 \int_{t'}^{t} ds \int_{\mathbb{R}^d} \lambda(d\xi) \exp\left(-4\pi^2 (t-s)\|\xi\|^2 \right) \frac{|(\widehat{t-s_n}) - (t-s)|}{\widehat{t-s_n}}.$$

Notice that, for all $x>0$

$$e^{-x} \le x^{-p}, \qquad p>0. \tag{6.1.44}$$

Since $\gamma \in (0, \frac{1-\eta}{2})$, (6.1.44), by integrating with respect to the time and the assumption (H_η) ensure

$$
\begin{aligned}
F_{11} &\le C 2^{d-\gamma(n-1)} \int_{t'}^{t} ds \int_{\mathbb{R}^d} \lambda(d\xi) \exp\left(-4\pi^2 (t-s)\|\xi\|^2\right)(t-s)^{-\gamma} \\
&\le C 2^{d-\gamma(n-1)} \int_{t'}^{t} ds \int_{\mathbb{R}^d} \lambda(d\xi) \frac{(t-s)^{-\gamma}}{\left(4\pi^2(t-s)\|\xi\|^2\right)^{1-2\gamma}} \\
&\le C 2^{d-\gamma(n-1)} \int_{\mathbb{R}^d} \frac{\lambda(d\xi)}{\|\xi\|^{2(1-2\gamma)}} \int_{t'}^{t} ds (t-s)^{-1+\gamma} \\
&\le C 2^{d-\gamma(n-1)} |t-t'|^{\gamma} \int_{\mathbb{R}^d} \frac{\lambda(d\xi)}{\|\xi\|^{2(1-2\gamma)}} \\
&\le C 2^{-\gamma(n-1)} |t-t'|^{\gamma}.
\end{aligned}
\tag{6.1.45}
$$

Analogously, we can prove

$$F_{12} \le C 2^{-\gamma(n-1)} |t-t'|^{\gamma} \tag{6.1.46}$$

and, moreover,

$$F_{2} \le C 2^{-\gamma(n-1)} |t-t'|^{\gamma}. \tag{6.1.47}$$

Then, the estimate (6.1.37) follows from (6.1.41)-(6.1.47).
We now prove (6.1.38). Using the set $B(x, 3\cdot 2^{-n})$, we have

$$\int_{t'}^{t} ds \left\| G_n(t-s, x, *) \right\|_{\mathcal{H}}^2 \le J_1 + J_2, \tag{6.1.48}$$

where

$$
\begin{aligned}
J_1 &= \int_{t'}^{t} ds \int_{B(x,3\cdot 2^{-n})} dy \int_{B(x,3\cdot 2^{-n})} dz f(y-z) G(\widehat{t-s_n}, x, y_n) \\
&\quad \times G(\widehat{t-s_n}, x, z_n), \\
J_2 &= \int_{t'}^{t} ds \int_{B(x,3\cdot 2^{-n})^c} dy \int_{B(x,3\cdot 2^{-n})^c} dz f(y-z) G(\widehat{t-s_n}, x, y_n) \\
&\quad \times G(\widehat{t-s_n}, x, z_n).
\end{aligned}
$$

Now, (6.1.4), (6.1.24) and (4.5) yield

$$\begin{aligned} J_1 &\le C\int_{t'}^{t} ds\int_{B(x,3\cdot 2^{-n})} dy\int_{B(x,3\cdot 2^{-n})} dz(\widehat{t-s_n})^{-d} f(y-z)\\ &\le C2^{dn}|t-t'|\int_{B(x,3\cdot 2^{-n})} dy\int_{B(x,3\cdot 2^{-n})} dz f(y-z)\\ &\le C|t-t'|. \end{aligned} \tag{6.1.49}$$

On the other hand, since $s_n < s$, (6.1.4) again, (6.1.3), (6.1.44) and (H$_\eta$) ensure, for all $\gamma\in(0,\frac{1-\eta}{2})$,

$$\begin{aligned} J_2 &\le \int_{t'}^{t} ds\int_{B(x,3\cdot 2^{-n})^c} dy\int_{B(x,3\cdot 2^{-n})^c} dz f(y-z)H(\widehat{t-s_n},x-y)f(y-z)\\ &\quad\times H(\widehat{t-s_n},x-z)\\ &\le C\int_{t'}^{t} ds\int_{\mathbb{R}^d}\lambda(d\xi)\exp\left(-4\pi\|\xi\|^2|t-s_n|\right)\\ &\le C\int_{t'}^{t} ds\int_{\mathbb{R}^d}\lambda(d\xi)\exp\left(-4\pi\|\xi\|^2|t-s|\right)\\ &\le C\int_{t'}^{t} ds\int_{\mathbb{R}^d}\frac{\lambda(d\xi)}{\|\xi\|^{2(1-2\gamma)}(t-s)^{1-2\gamma}}\\ &\le C|t-t'|^{2\gamma}. \end{aligned} \tag{6.1.50}$$

Then, the estimates (6.1.48)-(6.1.50) imply (6.1.38).
Finally, we check (6.1.39). Let

$$K=\int_0^{t'} ds\Big\|G(t-s,x,*)-G_n(t-s,x,*)-G(t'-s,x,*)+G_n(t'-s,x,*)\Big\|^2_{\mathcal{H}}.$$

On the one hand, we have

$$K=K_1+\tilde K_1$$

where

$$\begin{aligned} K_1 &= \int_0^{t'} ds\Big\|G(t-s,x,*)-G(t'-s,x,*)\Big\|^2_{\mathcal{H}},\\ \tilde K_1 &= \int_0^{t'} ds\Big\|G_n(t-s,x,*)-G_n(t'-s,x,*)\Big\|^2_{\mathcal{H}}. \end{aligned}$$

On the other hand, we also have

$$K = K_2 + \tilde{K}_2,$$

where

$$K_2 = \int_0^{t'} ds \Big\| G(t-s,x,*) - G_n(t-s,x,*) \Big\|_{\mathcal{H}}^2,$$

$$\tilde{K}_2 = \int_0^{t'} ds \Big\| G(t'-s,x,*) - G_n(t'-s,x,*) \Big\|_{\mathcal{H}}^2.$$

As in (6.1.7) we can obtain, for all $\gamma \in (0, \frac{1-\eta}{2})$,

$$K = K_1 + \tilde{K}_1 \leq C|t-t'|^{2\gamma}, \tag{6.1.51}$$

$$K = K_2 + \tilde{K}_2 \leq C 2^{-2\gamma(n-1)}. \tag{6.1.52}$$

Hence, Schwarz's inequality together with (6.1.51) and (6.1.52) imply (6.1.39). The analysis of $\tilde{K}_1$ also gives (6.1.40).

□

Finally, we enunciate a lemma that can be proved using similar arguments as before.

Lemma 6.1.7 *Assume* (H$_\eta$) *for some* $\eta \in (0,1)$. *There exists a positive constant* C *such that, for any* $t \in [0,T]$, $x, x' \in [0,1]^d$ *and* $\gamma \in (0, \frac{1-\eta}{2})$,

$$\int_0^t ds \Big\| G(t-s,x,*) - G_n(t-s,x,*) - G(t-s,x',*) + G_n(t-s,x',*) \Big\|_{\mathcal{H}}^2$$
$$\leq C 2^{-\gamma(n-1)} \|x-x'\|^{\gamma}, \tag{6.1.53}$$

$$\int_0^t ds \Big\| G_n(t-s,x,*) - G_n(t-s,x',*) \Big\|_{\mathcal{H}}^2 \leq C\|x-x'\|^{2\gamma}. \tag{6.1.54}$$

□

6.2 Extension of Garsia's Lemma

In this subsection we give a new version of Garsia's Lemma that is a very slight modification of a result given by Chenal in [2].

Lemma 6.2.1 *Let $F = \{F(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\}$ be an $L^2(\Omega, \mathcal{F}, P)$- valued Gaussian process with mean zero and covariance functional given by*

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(s,x) f(x-y) \psi(s,y),$$

where f has the same condition as in (1.2).
Denote by $\mathcal{F}_t$ the σ-field generated by $\{F([0,s] \times A);\ s \le t,\ A \in \mathcal{B}(\mathbb{R}^d)\}$. Consider

$$Z : \Omega \times (\mathbb{R}_+ \times \mathbb{R}^d)^2 \to \mathbb{R}$$

an a.s. continous $\mathcal{F}_t$-adapted process such that a.s

$$Z(w,t,x,s,y) \equiv 0 \qquad \text{when } s > t \text{ or } x \in ((0,1)^d)^c \text{ or } y \in ((0,1)^d)^c.$$

Let $\tau \le T$ be an $\mathcal{F}'_t$-stopping time, $C > 0$ and $\gamma_0 > 0$ satisfying, for any $(t,x), (t',x') \in D_T^d$,

$$\begin{aligned} &\left\| 1\!\!1_{[0,\tau]}(\cdot) \Big[Z(w,t,x,\cdot,*) - Z(w,t',x',\cdot,*) \Big] \right\|_{\mathcal{H}_T}^2 \\ &\qquad \le C \Big[|t-t'|^{2\gamma_0} + \|x-x'\|^{2\gamma_0} \Big], \end{aligned} \tag{6.2.1}$$

with $\mathcal{H}_T$ defined in Section 3. For $(t,x) \in D_T^d$, let

$$I(t,x) = \int_0^\tau \int_{[0,1]^d} Z(\cdot,t,x,s,y) F(ds,dy).$$

Then, for any $\gamma \in (0,\gamma_0)$, there exist strictly positive constants $C(\gamma,\gamma_0)$ and $\tilde{C}(\gamma,\gamma_0)$ such that for $M > 2C^{\frac{1}{2}} C(\gamma,\gamma_0) \tilde{C}(\gamma,\gamma_0)$,

$$P\Big\{ \|I\|_{\gamma,\gamma,\tau} \ge M \Big\} \le K \exp\left(-\frac{M^2}{4CC^2(\gamma,\gamma_0)} \right), \tag{6.2.2}$$

where

$$\|I\|_{\gamma,\gamma,\tau} = \sup \left\{ \frac{|I(t,x) - I(t',x')|}{|t-t'|^\gamma + \|x-y\|^\gamma},\ (t,x) \neq (t',x') \in D_\tau^d \right\}.$$

Proof: It is a particular case of Corollary B.5 of [2].

□

Remark. In some cases, we will see that (6.2.1) will be satisfied for $\gamma \in (0, \frac{1-\eta}{2})$ and then, we will be able to ensure (6.2.2) for any $\gamma \in (0, \frac{1-\eta}{2})$. In these cases, to lighten notation, we will denote by $C(\gamma)$ and $\tilde{C}(\gamma)$ the constants $C(\gamma,\gamma_0)$ and $\tilde{C}(\gamma,\gamma_0)$, respectively.

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