Electronic Journal of Probability

Vol. 14 (2009), Paper no. 53, pages 1541–1603.

Journal URL
http://www.math.washington.edu/~ejpecp/

Sharp asymptotics for metastability in the random field Curie-Weiss model*

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Abstract

In this paper we study the metastable behavior of one of the simplest disordered spin system, the random field Curie-Weiss model. We will show how the potential theoretic approach can be used to prove sharp estimates on capacities and metastable exit times also in the case when the distribution of the random field is continuous. Previous work was restricted to the case when the random field takes only finitely many values, which allowed the reduction to a finite dimensional problem using lumping techniques. Here we produce the first genuine sharp estimates in a context where entropy is important.

Key words: Disordered system, Glauber dynamics, metastability, potential theory, capacity.

AMS 2000 Subject Classification: Primary 82C44, 60K35, 60G70.

Submitted to EJP on June 27, 2008, final version accepted May 28, 2009.

*This research was supported through a grant by the German-Israeli Foundation (GIF). The kind hospitality of the Technion, Haifa, and the Weierstrass-Institute for Applied Analysis and Stochastics are gratefully acknowledged

1 Introduction and main results

The simplest example of disordered mean field models is the random field Curie-Weiss model. Here the state space is $\mathscr{S}_N = \{-1, 1\}^N$, where N is the number of particles of the system. Its Hamiltonian is

$$H_N[\omega](\sigma) \equiv -\frac{N}{2}\left(\frac{1}{N}\sum_{i\in\Lambda}\sigma_i\right)^2 - \sum_{i\in\Lambda} h_i[\omega]\sigma_i, \tag{1.1}$$

where $\Lambda \equiv \{1, \ldots, N\}$ and h_i, $i \in \Lambda$, are i.i.d. random variables on some probability space $(\Omega, \mathscr{F}, \mathbb{P}_h)$. For sake of convenience, we will assume throughout this paper that the common distribution of h has bounded support and that it is in a general position in the sense of the assumption on distinct eigenvalues as stated in Lemma 3.2 below.

The dynamics of this model has been studied before: dai Pra and den Hollander studied the short-time dynamics using large deviation results and obtained the analog of the McKeane-Vlasov equations [11]. The dynamics of the metastability for the mean field Curie-Weiss model, without random fields, was studied already by Cassandro et al. in 1984 [13]. Later, Mathieu and Picco [19] and Fontes, Mathieu, and Picco [15], considered convergence to equilibrium in the case where the random field takes only the two values $\pm\varepsilon$. Finally, Bovier et al. [6] analyzed this model in the case when h takes finitely many values, as an example of the use of the potential theoretic approach to metastability. In this article we extend this analysis to the case of random fields with continuous distributions, while at the same time improving the results by giving sharp estimates of transition times between metastable states.

The present paper should be seen, beyond the interest presented by the model as such, as a first case study in the attempt to derive precise asymptotics of metastable characteristics in kinetic Ising models in situations where neither the temperature tends to zero nor an exact reduction to low-dimensional models is possible. As a result one has to control a genuine microscopic evolution in terms of the geometry of macroscopic and/or mesoscopic landscapes. In particular, hitting times of individual microscopic configurations become irrelevant - they live on larger exponential scales. Instead one has to study hitting times of mesoscopic neighborhoods of (mesoscopic) critical points. But this renders useless all the methods which are directly based on the Markovian renewal structure of the microscopic dynamics, and, consequently alternative techniques should be devised. In the sequel we shall refer to the above discussion as to the problem of large entropy of microscopic states. Finding ways to cope with such large entropy is the principal challenge we try to address in this paper. While the RFCW model is certainly one of the simplest examples of this class, we feel that the general methodology developed here will be useful in a much wider class of systems.

1.1 Gibbs measure and order parameter. The static picture

The equilibrium statistical mechanics of the RFCW model was analyzed in detail in [1] and [16]. We give a very brief review of some key features that will be useful later. As usual, we define the Gibbs measure of the model as the random probability measure

$$\mu_{\beta,N}[\omega](\sigma) \equiv \frac{2^{-N}e^{-\beta H_N[\omega](\sigma)}}{Z_{\beta,N}[\omega]}, \tag{1.2}$$

where the partition function is defined as

$$Z_{\beta,N}[\omega] \equiv \mathbb{E}_\sigma e^{-\beta H_N[\omega](\sigma)} \equiv 2^{-N} \sum_{\sigma \in S_N} e^{-\beta H_N[\omega](\sigma)}. \tag{1.3}$$

We define the total magnetization as

$$m_N(\sigma) \equiv \frac{1}{N} \sum_{i\in\Lambda} \sigma_i. \tag{1.4}$$

The magnetization will be the *order parameter* of the model, and we define its distribution under the Gibbs measures as the *induced measure*,

$$\mathcal{Q}_{\beta,N} \equiv \mu_{\beta,N} \circ m_N^{-1}, \tag{1.5}$$

on the set of possible values $\Gamma_N \equiv \{-1, -1+2/N, \dots, 1\}$.

Let us begin by writing

$$Z_{\beta,N}[\omega]\mathcal{Q}_{\beta,N}[\omega](m) = \exp\left(\frac{N\beta}{2} m^2\right) Z^1_{\beta,N}[\omega](m) \tag{1.6}$$

where

$$Z^1_{\beta,N}[\omega](m) \equiv \mathbb{E}_\sigma \exp\left(\beta \sum_{i\in\Lambda} h_i \sigma_i\right) \mathbb{1}_{\{N^{-1}\sum_{i\in\Lambda}\sigma_i = m\}} \equiv \mathbb{E}^h_\sigma \mathbb{1}_{\{N^{-1}\sum_{i\in\Lambda}\sigma_i = m\}}. \tag{1.7}$$

For simplicity we will in the sequel identify functions defined on the discrete set Γ_N with functions defined on $[-1,1]$ by setting $f(m) \equiv f([2Nm]/2N)$. Then, for $m \in (-1,1)$, $Z^1_N(m)$ can be expressed, using sharp large deviation estimates (see e.g. (1.2.27) in [12]), as

$$Z^1_{\beta,N}[\omega](m) = \frac{\exp\left(-NI_N[\omega](m)\right)}{\sqrt{\frac{N\pi}{2}/I''_N[\omega](m)}} (1 + \mathrm{o}(1)), \tag{1.8}$$

where $\mathrm{o}(1)$ goes to zero as $N \uparrow \infty$. This means that we can express the right-hand side in (1.6) as

$$Z_{\beta,N}[\omega]\mathcal{Q}_{\beta,N}[\omega](m) = \sqrt{\frac{2I''_N[\omega](m)}{N\pi}} \exp\left(-N\beta F_{\beta,N}[\omega](m)\right)(1+\mathrm{o}(1)), \tag{1.9}$$

where

$$F_{\beta,N}[\omega](m) \equiv -\frac{1}{2}m^2 + \frac{1}{\beta} I_N[\omega](m). \tag{1.10}$$

Here $I_N[\omega](y)$ is the Legendre-Fenchel transform of the log-moment generating function

$$\begin{aligned} U_N[\omega](t) &\equiv \frac{1}{N} \ln \mathbb{E}^h_\sigma \exp\left(t \sum_{i\in\Lambda} \sigma_i\right) \\ &= \frac{1}{N} \sum_{i\in\Lambda} \ln\cosh\left(t + \beta h_i\right). \end{aligned} \tag{1.11}$$

Above we have indicated the random nature of all functions that appear by making their dependence on the random parameter ω explicit. To simplify notation, in the sequel this dependence will mostly be dropped.

We are interested in the behavior of this function near critical points of $F_{\beta,N}$. An important consequence of Equations (1.6) through (1.11) is that if m^* is a critical point of $F_{\beta,N}$, then for $|v| \leq N^{-1/2+\delta}$,

$$\frac{\mathcal{Q}_{\beta,N}(m^*+v)}{\mathcal{Q}_{\beta,N}(m^*)} = \exp\left(-\frac{\beta N}{2}a(m^*)v^2\right)(1+o(1)), \tag{1.12}$$

with

$$a(m^*) \equiv F''_{\beta,N}(m^*) = -1 + \beta^{-1} I''_N(m^*). \tag{1.13}$$

Now, if m^* is a critical point of $F_{\beta,N}$, then

$$m^* = \beta^{-1} I'_N(m^*) \equiv \beta^{-1} t^*, \tag{1.14}$$

or

$$\beta m^* = I'_N(m^*) = t^*. \tag{1.15}$$

Since I_N is the Legendre-Fenchel transform of U_N, $I'_N(x) = U'^{-1}_N(x)$, so that

$$m^* = U'_N(\beta m^*) \equiv \frac{1}{N}\sum_{i\in\Lambda} \tanh(\beta(m^*+h_i))). \tag{1.16}$$

Finally, using that at a critical point, $I''_{N,\ell}(m^*) = \frac{1}{U''_{N,\ell}(t^*)}$, we get the alternative expression

$$a(m^*) = -1 + \frac{1}{\beta U''_N(\beta m^*)} = -1 + \frac{1}{\frac{\beta}{N}\sum_{i\in\Lambda}\left(1-\tanh^2(\beta(m^*+h_i))\right)}. \tag{1.17}$$

We see that, by the law of large numbers, the set of critical points converges, $\mathbb{P}_h$-almost surely, to the set of solutions of the equation

$$m^* = \mathbb{E}_h \tanh\left(\beta\left(m^*+h\right)\right), \tag{1.18}$$

and the second derivative of $F_{\beta,N}(m^*)$ converges to

$$\lim_{N\to\infty} F''_{\beta,N}(m^*) = -1 + \frac{1}{\beta\mathbb{E}_h\left(1-\tanh^2(\beta(m^*+h))\right)}. \tag{1.19}$$

Thus, m^* is a local minimum if

$$\beta\mathbb{E}_h\left(1-\tanh^2(\beta(m^*+h))\right) < 1, \tag{1.20}$$

and a local maximum if

$$\beta\mathbb{E}_h\left(1-\tanh^2(\beta(m^*+h))\right) > 1. \tag{1.21}$$

(The cases where $\beta\mathbb{E}_h\left(1-\tanh^2(\beta(m^*+h))\right) = 1$ correspond to second order phase transitions and will not be considered here).

Collecting all these observations, we get the following:

Proposition 1.1. *Let m^* be a critical point of $\mathcal{Q}_{\beta,N}$. Then, $\mathbb{P}_h$-almost surely,*

$$Z_{\beta,N}\mathcal{Q}_{\beta,N}(m^*) = \frac{\exp\left(-\beta N F_{\beta,N}(m^*)\right)(1+o(1))}{\sqrt{\frac{N\pi}{2}\left|\mathbb{E}\left(1-\tanh^2(\beta(m^*+h))\right)\right|}} \tag{1.22}$$

with

$$F_{\beta,N}(m^*) = \frac{(m^*)^2}{2} - \frac{1}{\beta N}\sum_{i\in\Lambda}\ln\cosh\left(\beta(m^*+h_i)\right). \tag{1.23}$$

From this discussion we get a very precise picture of the distribution of the order parameter.

1.2 Glauber dynamics

We will consider for definiteness discrete time Glauber dynamics with Metropolis transition probabilities

$$p_N[\omega](\sigma,\sigma') \equiv \frac{1}{N}\exp\left(-\beta[H_N[\omega](\sigma') - H_N[\omega](\sigma)]_+\right), \tag{1.24}$$

if σ and σ' differ on a single coordinate,

$$p_N[\omega](\sigma,\sigma) \equiv 1 - \sum_{\sigma'\sim\sigma}\frac{1}{N}\exp\left(-\beta[H_N[\omega](\sigma') - H_N[\omega](\sigma)]_+\right), \tag{1.25}$$

and $p_N(\sigma,\sigma') = 0$ in all other cases. We will denote the Markov chain corresponding to these transition probabilities $\sigma(t)$ and write $\mathbb{P}_\nu[\omega] \equiv \mathbb{P}_\nu$, for the law of this chain with initial distribution ν, and we will set $\mathbb{P}_\sigma \equiv \mathbb{P}_{\delta_\sigma}$. As is well known, this chain is ergodic and reversible with respect to the Gibbs measure $\mu_{\beta,N}[\omega]$, for each ω. Note that we might also study chains with different transition probabilities that are reversible with respect to the same measures. Details of our results will depend on this choice. The transition matrix associated with these transition probabilities will be called P_N, and we will denote by $L_N \equiv P_N - \mathbb{1}$ the (discrete) generator of the chain.

Our main result will be sharp estimates for mean hitting times between minima of the function $F_{\beta,N}(m)$ defined in (1.10).

More precisely, for any subset $A \subset S_N$, we define the stopping time

$$\tau_A \equiv \inf\{t > 0 | \sigma(t) \in A\}. \tag{1.26}$$

We also need to define, for any two subsets $A, B \subset S_N$, the probability measure on A given by

$$\nu_{A,B}(\sigma) = \frac{\mu_{\beta,N}(\sigma)\mathbb{P}_\sigma[\tau_B < \tau_A]}{\sum_{\sigma\in A}\mu_{\beta,N}(\sigma)\mathbb{P}_\sigma[\tau_B < \tau_A]}. \tag{1.27}$$

We will be mainly concerned with sets of configurations with given magnetization. For any $I \in \Gamma_N$, we thus introduce the notation $S[I] \equiv \{\sigma \in S_N : m_N(\sigma) \in I\}$ and state the following:

Theorem 1.2. *Assume that β and the distribution of the magnetic field are such that there exist more than one local minimum of $F_{\beta,N}$. Let m^* be a local minimum of $F_{\beta,N}$, $M \equiv M(m^*)$ be the set of minima* *of $F_{\beta,N}$ such that $F_{\beta,N}(m) < F_{\beta,N}(m^*)$, and z^* be the minimax between m and M, i.e. the lower of the highest maxima separating m from M to the left respectively right. Then, $\mathbb{P}_h$-almost surely,*

$$\begin{aligned}\mathbb{E}_{\nu_{S[m^*],S[M]}} \tau_{S[M]} &= \exp\left(\beta N\left[F_{\beta,N}(z^*) - F_{\beta,N}(m^*)\right]\right) \\ &\times \frac{2\pi N}{\beta|\bar{\gamma}_1|}\sqrt{\frac{\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(z^*+h)\right)\right)-1}{1-\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(m^*+h)\right)\right)}}(1+o(1)),\end{aligned} \tag{1.28}$$

where $\bar{\gamma}_1$ is the unique negative solution of the equation

$$\mathbb{E}_h\left[\frac{\left(1-\tanh(\beta(z^*+h))\right)\exp\left(-2\beta\left[z^*+h\right]_+\right)}{\frac{\exp\left(-2\beta[z^*+h]_+\right)}{\beta\left(1+\tanh(\beta(z^*+h))\right)}-2\gamma}\right] = 1. \tag{1.29}$$

Note that we have the explicit representation for the random quantity

$$\begin{aligned}F_{\beta,N}(z^*) - F_{\beta,N}(m^*) &= \frac{(z^*)^2 - (m^*)^2}{2} \\ &- \frac{1}{\beta N}\sum_{i\in\Lambda}\left[\ln\cosh\left(\beta(z^*+h_i)\right) - \ln\cosh\left(\beta(m^*+h_i)\right)\right].\end{aligned} \tag{1.30}$$

Remark. Note that the $\beta \to \infty$ limit in (1.18) leads to the following limiting relation for critical points,

$$m^* = \mathbb{P}_h(h > -m^*) - \mathbb{P}_h(h < -m^*) \tag{1.31}$$

It is easy to construct distributions of h for which the above equation has a prescribed (odd) number of solutions. If h has a compact support (which does not contain ± 1 as its inf and sup) then the number of solutions to (1.18) obviously equals to the number of solutions to (1.31) for all β large enough. In other words, the assumptions of our Theorem are naturally satisfied by a large family of random fields.

Remark. Two obvious questions are: Does the conclusion of Theorem 1.2 still holds if we start the dynamics not from $\nu_{S[m^*],S[M]}$, but rather from an arbitrary microscopic point $\sigma \in S[m^*]$? Also is there an exponential scaling law for the escape times? These issues are addressed in the forthcoming [3]. The answer to the first question is "yes", but so far for a specific choice of the dynamics. The proof involves a coupling construction which is inspired by the recent paper [17]. Although we have little doubt that the answer to the second question should be "yes" as well, for the moment there is still a technical issue to settle.

The proof of Theorem 1.2 on mean transition times relies on the following result on *capacities* (for a definition see Eq. (2.5) in Section 2 below).

Theorem 1.3. *With the same notation as in Theorem 1.2 we have that*

$$Z_{\beta,N}\operatorname{cap}\left(S[m^*], S[M]\right) = \frac{\beta|\bar{\gamma}_1|}{2\pi N}\frac{\exp\left(-\beta N F_{\beta,N}(z^*)\right)(1+o(1))}{\sqrt{\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(z^*+h)\right)\right)-1}}. \tag{1.32}$$

The proof of Theorem 1.3 is the core of the present paper. As usual, the proof of an upper bound of the form (1.32) will be relatively easy. The main difficulty is to prove a corresponding lower bound. The main contribution of this paper is to provide a method to prove such a lower bound in a situation where the entropy of paths cannot be neglected.

Before discussing the methods of proof of these results, it will be interesting to compare this theorem with the prediction of the simplest uncontrolled approximation.

The naive approximation. A widespread heuristic picture for metastable behavior of systems like the RFCW model is based on replacing the full Markov chain on S_N by an effective Markov chain on the order parameter, i.e. by a nearest neighbor random walk on Γ_N with transition probabilities that are reversible with respect to the induced measure, $\mathcal{Q}_{\beta,N}$. The ensuing model can be solved exactly. In the absence of a random magnetic field, this replacement is justified since the image of $\sigma(t)$, $m(t) \equiv m_N(\sigma(t))$, is a Markov chain reversible w.r.t. $\mathcal{Q}_{\beta,N}$; unfortunately, this fact relies on the perfect permutation symmetry of the Hamiltonian of the Curie-Weiss model and fails to hold in the presence of random field.

A natural choice for the transition rates of the heuristic dynamics is

$$r_N[\omega](m,m') \equiv \frac{1}{\mathcal{Q}_{\beta,N}[\omega](m)} \sum_{\sigma:m_N(\sigma)=m} \mu_{\beta,N}[\omega](\sigma) \sum_{\sigma':m_N(\sigma')=m'} p_N[\omega](\sigma,\sigma'), \tag{1.33}$$

which are different from zero only if $m' = m \pm 2/N$ or if $m = m'$. The ensuing Markov process is a one-dimensional nearest neighbor random walk for which most quantities of interest can be computed quite explicitly by elementary means (see e.g. [20; 4]). In particular, it is easy to show that for this dynamics,

$$\begin{aligned} \mathbb{E}_{\nu_{S[m^*],S[M]}} \tau_{S[M]} &= \exp\left(\beta N \left[F_{\beta,N}(z^*) - F_{\beta,N}(m^*)\right]\right) \\ &\quad \times \frac{2\pi N}{\beta|a(z^*)|} \sqrt{\frac{\beta \mathbb{E}_h\left(1-\tanh^2\left(\beta(z^*+h)\right)\right)-1}{1-\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(m^*+h)\right)\right)}}\,(1+o(1)), \end{aligned}$$

where $a(z^*)$ is defined in (1.19).

The prediction of the naive approximation is slightly different from the exact answer, albeit only by a wrong prefactor. One may of course consider this as a striking confirmation of the quality of the naive approximation; from a different angle, this shows that a true understanding of the details of the dynamics is only reached when the prefactors of the exponential rates are known (see [18] for a discussion of this point).

The picture above is in some sense generic for a much wider class of metastable systems: on a heuristic level, one wants to think of the dynamics on metastable time scales to be well described by a diffusion in a double (or multi) well potential. While this cannot be made rigorous, it should be possible to find a family of mesoscopic variables with corresponding (discrete) diffusion dynamics that asymptotically reproduce the metastable behavior of the true dynamics. The main message of this paper is that such a picture can be made rigorous within the potential theoretic approach.

Acknowledgements. The authors thank Alexandre Gaudillière, Frank den Hollander, and Cristian Spitoni for useful discussions on metastability.

2 Some basic concepts from potential theory

Our approach to the analysis of the dynamics introduced above will be based on the ideas developed in [6; 7; 8] to analyze metastability through a systematic use of classical potential theory. Let us recall the basic notions we will need. In the sequel we shall frequently use the short-hand notation $\mu \equiv \mu_{\beta,N}$.

For two disjoint sets $A, B \subset S_N$, the equilibrium potential, $h_{A,B}$, is the harmonic function, i.e. the solution of the equation

$$(L_N h_{A,B})(\sigma) = 0, \quad \sigma \notin A \cup B, \tag{2.1}$$

with boundary conditions

$$h_{A,B}(\sigma) = \begin{cases} 1, & \text{if } \sigma \in A \\ 0, & \text{if } \sigma \in B \end{cases}. \tag{2.2}$$

The *equilibrium measure* is the function

$$e_{A,B}(\sigma) \equiv -(L_N h_{A,B})(\sigma) = (L_N h_{B,A})(\sigma), \tag{2.3}$$

which clearly is non-vanishing only on A and B. An important formula is the discrete analog of the first Green's identity: Let $D \subset S_N$ and $D^c \equiv S_N \setminus D$. Then, for any function f, we have

$$\begin{aligned} &\frac{1}{2} \sum_{\sigma,\sigma' \in S_N} \mu(\sigma) p_N(\sigma,\sigma')[f(\sigma) - f(\sigma')]^2 \\ &= -\sum_{\sigma \in D} \mu(\sigma) f(\sigma)(L_N f)(\sigma) - \sum_{\sigma \in D^c} \mu(\sigma) f(\sigma)(L_N f)(\sigma). \end{aligned} \tag{2.4}$$

In particular, for $f = h_{A,B}$, we get that

$$\begin{aligned} &\frac{1}{2} \sum_{\sigma,\sigma' \in S_N} \mu(\sigma) p_N(\sigma,\sigma')[h_{A,B}(\sigma) - h_{A,B}(\sigma')]^2 \\ &= \sum_{\sigma \in A} \mu(\sigma) e_{A,B}(\sigma) \equiv \mathrm{cap}(A,B), \end{aligned} \tag{2.5}$$

where the right-hand side is called the *capacity* of the capacitor A, B. The functional appearing on the left-hand sides of these relations is called the *Dirichlet form* or *energy*, and denoted

$$\Phi_N(f) \equiv \frac{1}{2} \sum_{\sigma,\sigma' \in S_N} \mu(\sigma) p_N(\sigma,\sigma')[f(\sigma) - f(\sigma')]^2. \tag{2.6}$$

As a consequence of the *maximum principle*, the function $h_{A,B}$ is the unique minimizer of Φ_N with boundary conditions (2.2), which implies the *Dirichlet principle*:

$$\mathrm{cap}(A,B) = \inf_{h \in \mathcal{H}_{A,B}} \Phi_N(h), \tag{2.7}$$

where $\mathcal{H}_{A,B}$ denotes the space of functions satisfying (2.2).

Equilibrium potential and equilibrium measure have an immediate

$$\mathbb{P}_\sigma[\tau_A < \tau_B] = \begin{cases} h_{A,B}(\sigma), & \text{if } \sigma \notin A \cup B \\ e_{B,A}(\sigma), & \text{if } \sigma \in B. \end{cases} \tag{2.8}$$

An important observation is that equilibrium potentials and equilibrium measures also determine the Green's function. In fact (see e.g. [7; 5]),

$$h_{A,B}(\sigma) = \sum_{\sigma' \in A} G_{S_N \setminus B}(\sigma, \sigma') e_{A,B}(\sigma') \tag{2.9}$$

In the case then A is a single point, this relation can be solved for the Green's function to give

$$G_{S_N \setminus B}(\sigma, \sigma') = \frac{\mu(\sigma') h_{\sigma,B}(\sigma')}{\mu(\sigma) e_{\sigma,B}(\sigma)}. \tag{2.10}$$

This equation is perfect if the cardinality of the state space does not grow too fast. In our case, however, it is of limited use since it is not possible to give precise enough estimates on the functions $h_{\sigma,B}(\sigma')$ and $e_{\sigma,B}(\sigma)$.

But (2.9) remains useful. In particular, it gives the following representation for mean hitting times

$$\sum_{\sigma \in A} \mu(\sigma) e_{A,B}(\sigma) \mathbb{E}_\sigma \tau_B = \sum_{\sigma' \in S_N} \mu(\sigma') h_{A,B}(\sigma'), \tag{2.11}$$

or, using definition (1.27),

$$\mathbb{E}_{\nu_{A,B}} \tau_B = \frac{1}{\mathrm{cap}(A,B)} \sum_{\sigma' \in S_N} \mu(\sigma') h_{A,B}(\sigma'). \tag{2.12}$$

From these equations we see that our main task will be to obtain precise estimates on capacities and some reasonably accurate estimates on equilibrium potentials. In previous applications [6; 7; 8; 10; 9], three main ideas were used to obtain such estimates:

(i) Upper bounds on capacities can be obtained using the Dirichlet variational principle with judiciously chosen test functions.

(ii) Lower bounds were usually obtained using the monotonicity of capacities in the transition probabilities (Raighley's principle). In most applications, reduction of the network to a set of parallel 1-dimensional chains was sufficient to get good bounds.

(iii) The simple renewal estimate $h_{A,B}(x) \leq \frac{\mathrm{cap}(x,A)}{\mathrm{cap}(x,B)}$ was used to bound the equilibrium potential through capacities again.

These methods were sufficient in previous applications essentially because entropy were not an issue there. In the models at hand, entropy is important, and due to the absence of any symmetry, we cannot use the trick to deal with entropy by a mapping of the model to a low-dimensional one, as can be done in the standard Curie-Weiss model and in the RFCW model when the magnetic field takes only finitely many values [19; 6].

Thus we will need to improve on these ideas. In particular, we will need a new approach to lower bounds for capacities. This will be done by exploiting a dual variational representation of capacities in terms of flows, due to Berman and Konsowa [2]. Indeed, one of the main messages of this paper is to illustrate the power of this variational principle.

Random path representation and lower bounds on capacities. It will be convenient to think of the quantities $\mu(\sigma)p_N(\sigma,\sigma')$ as *conductances*, $c(\sigma,\sigma')$, associated to the edges $e=(\sigma,\sigma')$ of the graph of allowed transitions of our dynamics. This interpretation is justified since, due to reversibility, $c(\sigma,\sigma')=c(\sigma',\sigma)$ is symmetric.

For purposes of the exposition, it will be useful to abstract from the specific model and to consider a general finite connected graph, $(S,\mathscr{E})$ such that whenever $e=(a,b)\in\mathscr{E}$, then also $-e\equiv(b,a)\in\mathscr{E}$. Let this graph be endowed with a symmetric function, $c:\mathscr{E}\to\mathbb{R}_+$, called conductance.

Given two disjoint subsets $A,B\subset S$ define the capacity,

$$\mathrm{cap}(A,B)=\frac{1}{2}\min_{h|_A=0,\ h|_B=1}\sum_{e=(a,b)\in\mathscr{E}}c(a,b)(h(b)-h(a))^2\,. \tag{2.13}$$

Definition 2.1. Given two disjoint sets, $A,B\subset S$, a non-negative, cycle free *unit flow*, f, from A to B is a function $f:\mathscr{E}\to\mathbb{R}_+\cup\{0\}$, such that the following conditions are verified:

(i) if $f(e)>0$, then $f(-e)=0$;

(ii) f satisfies *Kirchoff's law*, i.e. for any vertex $a\in S\setminus(A\cup B)$,

$$\sum_b f(b,a)=\sum_d f(a,d); \tag{2.14}$$

(iii)

$$\sum_{a\in A}\sum_b f(a,b)=1=\sum_a\sum_{b\in B}f(a,b); \tag{2.15}$$

(iv) any path, γ, from A to B such that $f(e)>0$ for all $e\in\gamma$, is self-avoiding.

We will denote the space of non-negative, cycle free unit flows from A to B by $\mathbb{U}_{A,B}$.

An important example of a unit flow can be constructed from the equilibrium potential, h^*, i.e. the unique minimizer of (2.13). Since h^* satisfies, for any $a\in S\setminus(A\cup B)$,

$$\sum_b c(a,b)(h^*(b)-h^*(a))=0, \tag{2.16}$$

one verifies easily that the function, f^*, defined by

$$f^*(a,b)\equiv\frac{1}{\mathrm{cap}(A,B)}c(a,b)\left(h^*(a)-h^*(b)\right)_+, \tag{2.17}$$

is a non-negative unit flow from A to B. We will call f^* the *harmonic flow*.

The key observation is that *any* $f\in\mathbb{U}_{A,B}$ gives rise to a *lower bound* on the capacity $\mathrm{cap}(A,B)$, and that this bound becomes sharp for the harmonic flow. To see this we construct from f a stopped Markov chain $\mathbb{X}=\left(\mathbb{X}_0,\ldots,\mathbb{X}_\tau\right)$ as follows: For each $a\in S\setminus B$ define $F(a)=\sum_b f(a,b)$.

We define the initial distribution of our chain as $\mathbb{P}^f(a)=F(a)$, for $a\in A$, and zero otherwise. The transition probabilities are given by

$$q^f(a,b)=\frac{f(a,b)}{F(a)}, \tag{2.18}$$

for $a \notin B$, and the chain is stopped on arrival in B. Notice that by our choice of the initial distribution and in view of (2.18) $\mathbb{X}$ will never visit sites $a \in S \setminus B$ with $F(a) = 0$.

Thus, given a trajectory $\mathscr{X} = (a_0, a_1, \dots, a_r)$ with $a_0 \in A$, $a_r \in B$ and $a_\ell \in S \setminus (A \cup B)$ for $\ell = 0, \dots, r-1$,

$$\mathbb{P}^f(\mathbb{X} = \mathscr{X}) = \frac{\prod_{\ell=0}^{r-1} f(e_\ell)}{\prod_{\ell=0}^{r-1} F(a_\ell)}, \tag{2.19}$$

where $e_\ell = (a_\ell, a_{\ell+1})$ and we use the convention $0/0 = 0$. Note that, with the above definitions, the probability that $\mathbb{X}$ passes through an edge e is

$$\mathbb{P}^f(e \in \mathbb{X}) = \sum_{\mathscr{X}} \mathbb{P}^f(\mathscr{X}) \mathbb{1}_{\{e \in \mathscr{X}\}} = f(e). \tag{2.20}$$

Consequently, we have a partition of unity,

$$\mathbb{1}_{\{f(e)>0\}} = \sum_{\mathscr{X}} \frac{\mathbb{P}^f(\mathscr{X}) \mathbb{1}_{\{e \in \mathscr{X}\}}}{f(e)}. \tag{2.21}$$

We are ready now to derive our f-induced lower bound: For every function h with $h|_A = 0$ and $h|_B = 1$,

$$\begin{aligned} \frac{1}{2} \sum_e c(e) \left(\nabla_e h\right)^2 \quad &\geq \quad \sum_{e: f(e)>0} c(e) \left(\nabla_e h\right)^2 \\ &= \quad \sum_{\mathscr{X}} \sum_{e \in \mathscr{X}} \mathbb{P}^f(\mathscr{X}) \frac{c(e)}{f(e)} \left(\nabla_e h\right)^2. \end{aligned}$$

As a result, interchanging the minimum and the sum,

$$\begin{aligned} \operatorname{cap}(A,B) \quad &\geq \quad \sum_r \sum_{\mathscr{X}=(a_0,\dots,a_r)} \mathbb{P}^f(\mathscr{X}) \min_{h(a_0)=0,\ h(a_r)=1} \sum_0^{r-1} \frac{c(a_\ell, a_{\ell+1})}{f(a_\ell, a_{\ell+1})} \left(h(a_{\ell+1}) - h(a_\ell)\right)^2 \\ &= \quad \sum_{\mathscr{X}} \mathbb{P}^f(\mathscr{X}) \left[\sum_{e \in \mathscr{X}} \frac{f(e)}{c(e)}\right]^{-1}. \end{aligned} \tag{2.22}$$

Since for the equilibrium flow, f^*,

$$\sum_{e \in \mathscr{X}} \frac{f^*(e)}{c(e)} = \frac{1}{\operatorname{cap}(A,B)}, \tag{2.23}$$

with $\mathbb{P}^{f^*}$-probability one, the bound (2.22) is sharp.

Thus we have proven the following result from [2]:

Proposition 2.2. *Let $A, B \subset S$. Then, with the notation introduced above,*

$$\operatorname{cap}(A,B) = \sup_{f \in \mathbb{U}_{A,B}} \mathbb{E}^f \left[\sum_{e \in \mathscr{X}} \frac{f(e)}{c(e)}\right]^{-1} \tag{2.24}$$

3 Coarse graining and the mesoscopic approximation

The problem of entropy forces us to investigate the model on a coarse grained scale. When the random fields take only finitely many values, this can be done by an exact mapping to a low-dimensional chain. Here this is not the case, but we can will construct a sequence of approximate mappings that in the limit allow to extract the exact result.

3.1 Coarse graining

Let I denote the support of the distribution of the random fields. Let I_ℓ, with $\ell \in \{1,\dots,n\}$, be a partition of I such that, for some $C<\infty$ and for all ℓ, $|I_\ell| \leq C/n \equiv \varepsilon$.

Each realization of the random field $\{h_i[\omega]\}_{i\in\mathbb{N}}$ induces a random partition of the set $\Lambda \equiv \{1,\dots,N\}$ into subsets

$$\Lambda_k[\omega] \equiv \{i \in \Lambda : h_i[\omega] \in I_k\}. \tag{3.1}$$

We may introduce n order parameters

$$\boldsymbol{m}_k[\omega](\sigma) \equiv \frac{1}{N} \sum_{i\in\Lambda_k[\omega]} \sigma_i. \tag{3.2}$$

We denote by $\boldsymbol{m}[\omega]$ the n-dimensional vector $(\boldsymbol{m}_1[\omega],\dots,\boldsymbol{m}_n[\omega])$. In the sequel we will use the convention that bold symbols denote n-dimensional vectors and their components, while the sum of the components is denoted by the corresponding plain symbol, e.g. $m \equiv \sum_{\ell=1}^n \boldsymbol{m}_\ell$. $\boldsymbol{m}$ takes values in the set

$$\Gamma_N^n[\omega] \equiv \times_{k=1}^n \left\{-\rho_{N,k}[\omega], -\rho_{N,k}[\omega]+\tfrac{2}{N},\dots,\rho_{N,k}[\omega]-\tfrac{2}{N},\rho_{N,k}[\omega]\right\}, \tag{3.3}$$

where

$$\rho_k \equiv \rho_{N,k}[\omega] \equiv \frac{|\Lambda_k[\omega]|}{N}. \tag{3.4}$$

We will denote by $\boldsymbol{e}_\ell$, $\ell = 1,\dots,n$, the lattice vectors of the set Γ_N^n, i.e. the vectors of length $2/N$ parallel to unit vectors.

Note that the random variables $\rho_{N,k}$ concentrate exponentially (in N) around their mean values $\mathbb{E}_h \rho_{N,k} = \mathbb{P}_h[h_i \in I_k] \equiv p_k$.

Notational warning: To simplify statements in the remainder of the paper, we will henceforth assume that all statements involving random variables on $(\Omega,\mathscr{F},\mathbb{P}_h)$ hold true with $\mathbb{P}_h$-probability one, for all but finitely many values of N.

We may write the Hamiltonian in the form

$$H_N[\omega](\sigma) = -NE(\boldsymbol{m}[\omega](\sigma)) + \sum_{\ell=1}^n \sum_{i\in\Lambda_\ell} \sigma_i \tilde{h}_i[\omega], \tag{3.5}$$

where $E : \mathbb{R}^n \to \mathbb{R}$ is the function

$$E(\boldsymbol{x}) \equiv \frac{1}{2}\left(\sum_{k=1}^n \boldsymbol{x}_k\right)^2 + \sum_{k=1}^n \bar{h}_k \boldsymbol{x}_k, \tag{3.6}$$

with

$$\bar{h}_\ell \equiv \frac{1}{|\Lambda_\ell|}\sum_{i\in\Lambda_\ell} h_i, \quad \text{and} \quad \tilde{h}_i \equiv h_i - \bar{h}_\ell. \tag{3.7}$$

Note that if $h_i = \bar{h}_\ell$ for all $i \in \Lambda_\ell$, which is the case when h takes only finitely many values and the partition I_ℓ is chosen suitably, then the Glauber dynamics under the family of functions $\boldsymbol{m}_\ell$ is again Markovian. This fact was exploited in [19; 6]. Here we will consider the case where this is not the case. However, the idea behind our approach is to exploit that by choosing n large we can get to a situation that is rather close to that one.

Let us define the equilibrium distribution of the variables $\boldsymbol{m}[\sigma]$

$$\begin{aligned}\mathcal{Q}_{\beta,N}[\omega](\boldsymbol{x}) &\equiv \mu_{\beta,N}[\omega](\boldsymbol{m}[\omega](\sigma) = \boldsymbol{x}) \\ &= \frac{1}{Z_N[\omega]} e^{\beta N E(\boldsymbol{x})} \mathbb{E}_\sigma \mathbb{1}_{\{\boldsymbol{m}[\omega](\sigma)=\boldsymbol{x}\}} e^{\sum_{\ell=1}^n \sum_{i\in\Lambda_\ell} \sigma_i (h_i - \bar{h}_\ell)}\end{aligned} \tag{3.8}$$

where $Z_N[\omega]$ is the normalizing partition function. Note that with some abuse of notation, we will use the same symbols $\mathcal{Q}_{\beta,N}$, $F_{\beta,N}$ as in Section 1 for functions defined on the n-dimensional variables $\boldsymbol{x}$. Since we distinguish the vectors from the scalars by use of bold type, there should be no confusion possible. Similarly, for a mesoscopic subset $\boldsymbol{A} \subseteq \Gamma_N^n[\omega]$, we define its microscopic counterpart,

$$A = \mathcal{S}_N[\boldsymbol{A}] = \{\sigma \in \mathcal{S}_N \ : \ \boldsymbol{m}(\sigma) \in \boldsymbol{A}\}. \tag{3.9}$$

3.2 The landscape near critical points.

We now turn to the precise computation of the behavior of the measures $\mathcal{Q}_{\beta,N}[\omega](\boldsymbol{x})$ in the neighborhood of the critical points of $F_{\beta,N}[\omega](\boldsymbol{x})$. We will see that this goes very much along the lines of the analysis in the one-dimensional case in Section 1.

Let us begin by writing

$$Z_{\beta,N}[\omega]\mathcal{Q}_{\beta,N}[\omega](\boldsymbol{x}) = \exp\left(N\beta\left(\frac{1}{2}\left(\sum_{\ell=1}^n \boldsymbol{x}_\ell\right)^2 + \sum_{\ell=1}^n \boldsymbol{x}_\ell \bar{h}_\ell\right)\right)\prod_{\ell=1}^n Z^\ell_{\beta,N}[\omega](\boldsymbol{x}_\ell/\rho_\ell), \tag{3.10}$$

where

$$Z^\ell_{\beta,N}[\omega](y) \equiv \mathbb{E}_{\sigma_{\Lambda_\ell}} \exp\left(\beta\sum_{i\in\Lambda_\ell}\tilde{h}_i\sigma_i\right)\mathbb{1}_{\left\{|\Lambda_\ell|^{-1}\sum_{i\in\Lambda_\ell}\sigma_i = y\right\}} \equiv \mathbb{E}^{\tilde{h}}_{\sigma_{\Lambda_\ell}}\mathbb{1}_{\left\{|\Lambda_\ell|^{-1}\sum_{i\in\Lambda_\ell}\sigma_i = y\right\}}. \tag{3.11}$$

For $y \in (-1,1)$, these Z^ℓ_N can be expressed, using sharp large deviation estimates [12], as

$$Z^\ell_{\beta,N}[\omega](y) = \frac{\exp\left(-|\Lambda_\ell| I_{N,\ell}[\omega](y)\right)}{\sqrt{\frac{\pi}{2}|\Lambda_\ell|/I''_{N,\ell}[\omega](y)}}(1+o(1)), \tag{3.12}$$

where $o(1)$ goes to zero as $|\Lambda_\ell| \uparrow \infty$. Note that as in the one-dimensional case, we identify functions on Γ_N^n with their natural extensions to $\mathbb{R}^n$. This means that we can express the right-hand side in

(3.10) as

$$Z_{\beta,N}[\omega]\mathcal{Q}_{\beta,N}[\omega](\boldsymbol{x}) = \prod_{\ell=1}^{n} \sqrt{\frac{\left(I''_{N,\ell}[\omega](\boldsymbol{x}_\ell/\rho_\ell)/\rho_\ell\right)}{N\pi/2}} \exp\left(-N\beta F_{\beta,N}[\omega](\boldsymbol{x})\right)(1+o(1)), \tag{3.13}$$

where

$$F_{\beta,N}[\omega](\boldsymbol{x}) \equiv -\frac{1}{2}\left(\sum_{\ell=1}^{n}\boldsymbol{x}_\ell\right)^2 - \sum_{\ell=1}^{n}\boldsymbol{x}_\ell\bar{h}_\ell + \frac{1}{\beta}\sum_{\ell=1}^{n}\rho_\ell I_{N,\ell}[\omega](\boldsymbol{x}_\ell/\rho_\ell). \tag{3.14}$$

Here $I_{N,\ell}[\omega](y)$ is the Legendre-Fenchel transform of the log-moment generating function,

$$\begin{aligned} U_{N,\ell}[\omega](t) &\equiv \frac{1}{|\Lambda_\ell|}\ln\mathbb{E}^{\tilde{h}}_{\sigma_{\Lambda_\ell}}\exp\left(t\sum_{i\in\Lambda_\ell}\sigma_i\right) \\ &= \frac{1}{|\Lambda_\ell|}\sum_{i\in\Lambda_\ell}\ln\cosh\left(t+\beta\tilde{h}_i\right). \end{aligned} \tag{3.15}$$

We again analyze our functions near critical points, $\boldsymbol{z}^*$, of $F_{\beta,N}$. Equations (3.10)-(3.15) imply: if $\boldsymbol{z}^*$ is a critical point, then, for $\|\boldsymbol{v}\| \le N^{-1/2+\delta}$,

$$\frac{\mathcal{Q}_{\beta,N}(\boldsymbol{z}^*+\boldsymbol{v})}{\mathcal{Q}_{\beta,N}(\boldsymbol{z}^*)} = \exp\left(-\frac{\beta N}{2}(\boldsymbol{v},\mathbb{A}(\boldsymbol{z}^*)\boldsymbol{v})\right)(1+o(1)), \tag{3.16}$$

with

$$(\mathbb{A}(\boldsymbol{z}^*))_{k\ell} = \frac{\partial^2 F_{\beta,N}(\boldsymbol{z}^*)}{\partial\boldsymbol{z}_k\partial\boldsymbol{z}_\ell} = -1 + \delta_{k,\ell}\beta^{-1}\rho_\ell^{-1}I''_{N,\ell}(\boldsymbol{z}^*_\ell/\rho_\ell) \equiv -1 + \delta_{\ell,k}\hat{\lambda}_\ell. \tag{3.17}$$

Now, if $\boldsymbol{z}^*$ is a critical point of $F_{\beta,N}$,

$$\sum_{j=1}^{n}\boldsymbol{z}^*_j + \bar{h}_\ell = \beta^{-1}I'_{N,\ell}(\boldsymbol{z}^*_\ell/\rho_\ell) \equiv \beta^{-1}t^*_\ell, \tag{3.18}$$

or, with $z^* = \sum_{j=1}^{n}\boldsymbol{z}^*_\ell$,

$$\beta\left(z^*+\bar{h}_\ell\right) = I'_{N,\ell}(\boldsymbol{z}^*_\ell/\rho_\ell) = t^*_\ell. \tag{3.19}$$

By standard properties of Legendre-Fenchel transforms, we have that $I'_{N,\ell}(x) = U'^{-1}_{N,\ell}(x)$, so that

$$\boldsymbol{z}^*_\ell/\rho_\ell = U'_{N,\ell}(\beta(z^*+h_\ell)) \equiv \frac{1}{|\Lambda_\ell|}\sum_{i\in\Lambda_\ell}\tanh(\beta(z^*+h_i))). \tag{3.20}$$

Summing over ℓ, we see that z^* must satisfy the equation

$$z^* = \frac{1}{N}\sum_{i\in\Lambda}\tanh(\beta(z^*+h_i)), \tag{3.21}$$

which nicely does not depend on our choice of the coarse graining (and hence on n).

Finally, using that at a critical point $I''_{N,\ell}(\boldsymbol{z}^*_\ell/\rho_\ell) = \frac{1}{U''_{N,\ell}(t^*_\ell)}$, we get the explicit expression for the random numbers $\hat{\lambda}_\ell$ on the right hand side of (3.17)

$$\hat{\lambda}_\ell = \frac{1}{\beta\rho_\ell U''_{N,\ell}(\beta(z^*+\bar{h}_\ell))} = \frac{1}{\frac{\beta}{N}\sum_{i\in\Lambda_\ell}\left(1-\tanh^2(\beta(z^*+h_i))\right)}. \tag{3.22}$$

The determinant of the matrix $\mathbb{A}(\boldsymbol{z}^*)$ has a simple expression of the form

$$\begin{aligned} \det\left(\mathbb{A}(\boldsymbol{z}^*)\right) &= \left(1-\sum_{\ell=1}^n \frac{1}{\hat{\lambda}_\ell}\right)\prod_{\ell=1}^n \hat{\lambda}_\ell \\ &= \left(1-\frac{\beta}{N}\sum_{i\in\Lambda}\left(1-\tanh^2(\beta(z^*+h_i))\right)\right)\prod_{\ell=1}^n \hat{\lambda}_\ell \\ &= \left(1-\beta\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right)\right)\prod_{\ell=1}^n \hat{\lambda}_\ell\,(1+o(1)), \end{aligned} \tag{3.23}$$

where o(1) $\downarrow 0$, a.s., as $N\uparrow\infty$. Combining these observations, we arrive at the following proposition.

Proposition 3.1. *Let $\boldsymbol{z}^*$ be a critical point of $\mathcal{Q}_{\beta,N}$. Then $\boldsymbol{z}^*$ is given by (3.20) where z^* is a solution of (3.21). Moreover,*

$$\begin{aligned} Z_{\beta,N}\mathcal{Q}_{\beta,N}(\boldsymbol{z}^*) &= \frac{\sqrt{|\det(\mathbb{A}(\boldsymbol{z}^*))|}}{\sqrt{\left(\frac{N\pi}{2\beta}\right)^n\left|\beta\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right)-1\right|}} \\ &\times \exp\left(\beta N\left(-\frac{(z^*)^2}{2}+\frac{1}{\beta N}\sum_{i\in\Lambda}\ln\cosh\left(\beta(z^*+h_i)\right)\right)\right)(1+o(1)). \end{aligned} \tag{3.24}$$

Proof. We only need to examine (3.13) at a critical point $\boldsymbol{z}^*$. The equation for the prefactor follows by combining (3.12) with (3.23). As for the exponential term, $F_{\beta,N}$, notice that by convex duality

$$I_{N,\ell}(\boldsymbol{z}^*_\ell/\rho_\ell) = t^*_\ell\boldsymbol{z}^*_\ell/\rho_\ell - U_{N,\ell}(t^*_\ell) = \beta(z^*+\bar{h}_\ell)\boldsymbol{z}^*_\ell/\rho_\ell - U_{N,\ell}\left(\beta(z^*+\bar{h}_\ell)\right). \tag{3.25}$$

Hence (3.14) equals

$$\begin{aligned} &-\frac{1}{2}(z^*)^2 - \sum_{\ell=1}^n \boldsymbol{z}^*_\ell\bar{h}_\ell + \frac{1}{\beta}\sum_{\ell=1}^n\left[\rho_\ell\beta(z^*+\bar{h}_\ell)\boldsymbol{z}^*_\ell/\rho_\ell - \rho_\ell U_{N,\ell}\left(\beta(z^*+\bar{h}_\ell)\right)\right] \\ &= -\frac{1}{2}(z^*)^2 - \sum_{\ell=1}^n\left[\boldsymbol{z}^*_\ell\bar{h}_\ell - z^*\boldsymbol{z}^*_\ell - \bar{h}\boldsymbol{z}^*_\ell + \frac{1}{\beta N}\sum_{i\in\Lambda_\ell}\ln\cosh\left(\beta(z^*+h_i)\right)\right] \\ &= \frac{1}{2}(z^*)^2 - \frac{1}{\beta N}\sum_{i\in\Lambda}\ln\cosh\left(\beta(z^*+h_i)\right). \end{aligned} \tag{3.26}$$

□

Remark. The form given in Proposition 3.1 is highly suitable for our purposes as the dependence on n appears only in the denominator of the prefactor. We will see that this is just what we need to get a formula for capacities that is independent of the choice of the partition of I and has a limit as $n\uparrow\infty$.

Eigenvalues of the Hessian. We now describe the eigenvalues of the Hessian matrix $\mathbb{A}(\boldsymbol{z}^*)$.

Lemma 3.2. *Let z^* be a solution of the equation (3.21). Assume in addition that the distribution of magnetic fields is in a general position in the sense that all numbers $\hat{\lambda}_k$ (see (3.22)) are $\mathbb{P}_h$-a.s. distinct . Then γ is an eigenvalue of $\mathbb{A}(\boldsymbol{z}^*)$ if and only if it is a solution of the equation*

$$\sum_{\ell=1}^{n} \frac{1}{\frac{1}{\frac{\beta}{N}\sum_{i\in\Lambda_\ell}\left(1-\tanh^2\left(\beta\left(z^*+h_i\right)\right)\right)}-\gamma} = 1. \tag{3.27}$$

Moreover, (3.27) has at most one negative solution, and it has such a negative solution if and only if

$$\frac{\beta}{N}\sum_{i=1}^{N}\left(1-\tanh^2\left(\beta\left(z^*+h_i\right)\right)\right) > 1. \tag{3.28}$$

Remark. To analyze the case when some $\hat{\lambda}_k$ coincide is also not difficult. See Lemma 7.2 of [6].

Proof. Due to the particular form of the matrix $\mathbb{A}$, we get that $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ is an eigenvector of $\mathbb{A}$ with eigenvalue γ if

$$-\sum_{\ell=1}^{n} u_\ell + (\hat{\lambda}_k - \gamma)u_k = 0, \forall\ k = 1, \dots, n. \tag{3.29}$$

From the assumption that all $\hat{\lambda}_k$ take distinct values, it is easy to check that the set of equations (3.29) has no non-trivial solution for $\gamma = \hat{\lambda}_k$.

Since none of the $\hat{\lambda}_k = \gamma$, to find the eigenvalues of $\mathbb{A}$ we just replace $\hat{\lambda}_k$ by $\hat{\lambda}_k - \gamma$ in the first line of (3.23). This gives

$$\det\left(\mathbb{A}(\boldsymbol{z}^*) - \gamma\right) = \left(1 - \sum_{\ell=1}^{n}\frac{1}{\hat{\lambda}_\ell - \gamma}\right)\prod_{\ell=1}^{n}(\hat{\lambda}_\ell - \gamma). \tag{3.30}$$

Equation (3.27) is then just the demand that the first factor on the right of (3.30) vanishes. It is easy to see that, under the hypothesis of the lemma, this equation has n solutions, and that exactly one of them is negative under the hypothesis (3.28). ☐

Topology of the landscape. From the analysis of the critical points of $F_{\beta,N}$ it follows that the landscape of this function is in correspondence with the one-dimensional landscape described in Section 1 (see also Figure 1.). We collect the following features:

(i) Let $m_1^* < z_1^* < m_2^* < z_2^* < \cdots < z_k^* < m_{k+1}^*$ be the sequence of minima resp. maxima of the one-dimensional function $F_{\beta,N}$ defined in (1.10). Then to each minimum, m_i^*, corresponds a minimum, $\boldsymbol{m}_i^*$ of $F_{\beta,N}$, such that $\sum_{\ell=1}^n \boldsymbol{m}_{i,\ell}^* = m_i^*$, and to each maximum, z_i^*, corresponds a saddle point, $\boldsymbol{z}_i^*$ of $F_{\beta,N}$, such that $\sum_{\ell=1}^n \boldsymbol{z}_{i,\ell}^* = z_i^*$.

(ii) For any value m of the total magnetization, the function $F_{\beta,N}(\boldsymbol{x})$ takes its relative minimum on the set $\{\boldsymbol{y} : \sum \boldsymbol{y}_\ell = m\}$ at the point $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ determined (coordinate-wise) by the equation

$$\hat{\boldsymbol{x}}_\ell(m) = \frac{1}{N}\sum_{i\in\Lambda_\ell}\tanh\left(\beta\left(m + a + h_i\right)\right), \tag{3.31}$$

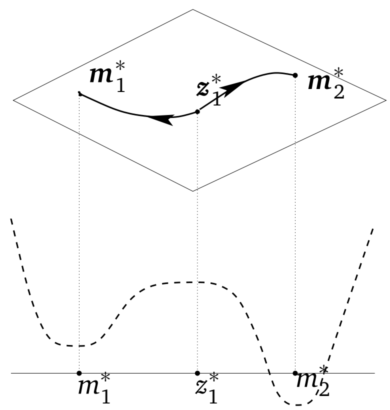

Figure 1: Correspondence between one and n-dimensional landscapes

where $a = a(m)$ is recovered from

$$m = \frac{1}{N} \sum_{i \in \Lambda} \tanh\left(\beta\left(m + a + h_i\right)\right). \tag{3.32}$$

Moreover, taking into account that the cardinality of $\{\boldsymbol{y} : \sum \boldsymbol{y}_\ell = m\}$ is $O(N^n)$, we infer,

$$F_{\beta,N}(m) \leq F_{\beta,N}(\hat{\boldsymbol{x}}) \leq F_{\beta,N}(m) + O(n \ln N / N). \tag{3.33}$$

Remark. Note that the minimal energy curves $\hat{\boldsymbol{x}}(\cdot)$ defined by (3.31) pass through the minima and saddle points, but are in general not the integral curves of the gradient flow connecting them. Note also that since we assume that random fields $\{h_i(\omega)\}$ have bounded support, for every $\delta > 0$ there exist two universal constants $0 < c_1 \leq c_2 < \infty$, such that

$$c_1 \rho_\ell \leq \frac{\mathrm{d}\hat{\boldsymbol{x}}_\ell(m)}{\mathrm{d}m} \leq c_2 \rho_\ell, \tag{3.34}$$

uniformly in N, $m \in [-1+\delta, 1-\delta]$ and in $\ell = 1, \ldots, n$.

4 Upper bounds on capacities

This and the next section are devoted to proving Theorem 1.3. In this section we derive upper bounds on capacities between two local minima. The procedure to obtain these bounds has two steps. First, we show that using test functions that only depend on the block variables $\boldsymbol{m}(\sigma)$, we can always get upper bounds in terms of a finite dimensional Dirichlet form. Second, we produce a good test function for this Dirichlet form.

4.1 First blocking.

Let us consider two sets, $A, B \subset \mathcal{S}_N$, that are defined in terms of block variables $\boldsymbol{m}$. This means that for some $\boldsymbol{A}, \boldsymbol{B} \subseteq \Gamma_N^n$, $A = \mathcal{S}_N[\boldsymbol{A}]$ and $B = \mathcal{S}_N[\boldsymbol{B}]$. Later we will be interested in pre-images of two minima of the function $F_{\beta,N}$. We get the obvious upper bound

$$\begin{aligned}
\mathrm{cap}(A,B) &= \inf_{h\in\mathscr{H}_{A,B}} \frac{1}{2}\sum_{\sigma,\sigma'\in\mathscr{S}_N} \mu_{\beta,N}[\omega](\sigma)p(\sigma,\sigma')\left[h(\sigma)-h(\sigma')\right]^2 \\
&\leq \inf_{u\in\mathscr{G}_{\boldsymbol{A},\boldsymbol{B}}} \frac{1}{2}\sum_{\sigma,\sigma'\in\mathscr{S}_N} \mu_{\beta,N}[\omega](\sigma)p(\sigma,\sigma')\left[u(\boldsymbol{m}(\sigma))-u(\boldsymbol{m}(\sigma'))\right]^2 \\
&= \inf_{u\in\mathscr{G}_{\boldsymbol{A},\boldsymbol{B}}} \sum_{\boldsymbol{x},\boldsymbol{x}'\in\Gamma_N^n} \left[u(\boldsymbol{x})-u(\boldsymbol{x}')\right]^2 \sum_{\sigma\in\mathscr{S}_N[\boldsymbol{x}]} \mu_{\beta,N}[\omega](\sigma) \sum_{\sigma'\in\mathscr{S}_N[\boldsymbol{x}']} p(\sigma,\sigma') \\
&\equiv \inf_{u\in\mathscr{G}_{\boldsymbol{A},\boldsymbol{B}}} \sum_{\boldsymbol{x},\boldsymbol{x}'\in\Gamma_N^n} \mathscr{Q}_{\beta,N}[\omega](\boldsymbol{x})r_N(\boldsymbol{x},\boldsymbol{x}')\left[u(\boldsymbol{x})-u(\boldsymbol{x}')\right]^2 \\
&\equiv \mathbb{C}\mathrm{ap}_N^n(\boldsymbol{A},\boldsymbol{B}). \qquad (4.1)
\end{aligned}$$

with

$$r_N(\boldsymbol{x},\boldsymbol{x}') \equiv \frac{1}{\mathscr{Q}_{\beta,N}[\omega](\boldsymbol{x})} \sum_{\sigma\in\mathscr{S}_N[\boldsymbol{x}]} \mu_{\beta,N}[\omega](\sigma) \sum_{\sigma'\in\mathscr{S}_N[\boldsymbol{x}']} p(\sigma,\sigma'). \qquad (4.2)$$

Here

$$\mathscr{H}_{A,B} \equiv \{h : S_N \to [0,1] : \forall\sigma\in A, h(\sigma)=1, \forall\sigma\in B, h(\sigma)=0\} \qquad (4.3)$$

and

$$\mathscr{G}_{\boldsymbol{A},\boldsymbol{B}} \equiv \{u : \Gamma_N^n \to [0,1] : \forall\boldsymbol{x}\in\boldsymbol{A},\ u(\boldsymbol{x})=1, \forall\boldsymbol{x}\in\boldsymbol{B},\ u(\boldsymbol{x})=0\}. \qquad (4.4)$$

4.2 Sharp upper bounds for saddle point crossings

Let now $\boldsymbol{z}^*$ be a saddle point, i.e. a critical point of $\mathscr{Q}_{\beta,N}$ such that the matrix $\mathbb{A}(\boldsymbol{z}^*)$ has exactly one negative eigenvalue and that all its other eigenvalues are strictly positive. Let $\boldsymbol{A},\boldsymbol{B}$ be two disjoint neighborhoods of minima of $F_{\beta,N}$ that are connected through $\boldsymbol{z}^*$, i.e. $\boldsymbol{A}$ and $\boldsymbol{B}$ are strictly contained in two different connected components of the level set $\{\boldsymbol{x} : F_{\beta,N}(\boldsymbol{x}) < F_{\beta,N}(\boldsymbol{z}^*)\}$, and there exists a path γ from $\boldsymbol{A}$ to $\boldsymbol{B}$ such that $\max_{\boldsymbol{x}\in\gamma} F_{\beta,N}(\boldsymbol{x}) = F_{\beta,N}(\boldsymbol{z}^*)$.

To estimate such capacities it suffices to compute the capacity of some small set near the saddle point (see e.g. [4] or [8] for an explanation). For a given (small) constant $\rho=\rho(N)\ll 1$, we define

$$D_N(\rho) \equiv \{\boldsymbol{x}\in\Gamma_N^n : |\boldsymbol{z}_\ell^* - \boldsymbol{x}_\ell| \leq \rho, \forall 1\leq\ell\leq n\}, \qquad (4.5)$$

In this section we will later choose $\rho = C\sqrt{\ln N/N}$, with $C<\infty$. $D_N(\rho)$ is the hypercube in Γ_N^n centered in $\boldsymbol{z}^*$ with sidelenght 2ρ. For a fixed vector, $\boldsymbol{v}\in\Gamma_N^n$, consider three disjoint subsets,

$$\begin{aligned}
W_0 &= \{\boldsymbol{x}\in\Gamma_N^n : |(\boldsymbol{v},(\boldsymbol{x}-\boldsymbol{z}^*))| < \rho\} \\
W_1 &= \{\boldsymbol{x}\in\Gamma_N^n : (\boldsymbol{v},(\boldsymbol{x}-\boldsymbol{z}^*)) \leq -\rho\} \\
W_2 &= \{\boldsymbol{x}\in\Gamma_N^n : (\boldsymbol{v},(\boldsymbol{x}-\boldsymbol{z}^*)) \geq \rho\}. \qquad (4.6)
\end{aligned}$$

We will compute the capacity of the Dirichlet form restricted to the set $D_N(\rho)$ with boundary conditions zero and one, respectively, on the sets $W_1\cap D_N(\rho)$ and $W_2\cap D_N(\rho)$. This will be done by exhibiting an approximately harmonic function with these boundary conditions. Before doing this, it will however be useful to slightly simplify the Dirichlet form we have to work with.

Cleaning of the Dirichlet form. One problem we are faced with in our setting is that the transition rates $r_N(\boldsymbol{x},\boldsymbol{x}')$ are given in a somewhat unpleasant form. At the same time it would be nicer to be able to replace the measure $\mathcal{Q}_{\beta,N}$ by the approximation given in (3.16). That we are allowed to do this follows from the simple assertion below, that is an immediate consequence of the positivity of the terms in the Dirichlet form, and of the Dirichlet principle.

Lemma 4.1. *Let $\Phi_N, \widetilde{\Phi}_N$ be two Dirichlet forms defined on the same space, Γ, corresponding to the measure $\mathcal{Q}$ and transition rates r, respectively $\widetilde{\mathcal{Q}}$ and $\tilde{r}$. Assume that, for all $\boldsymbol{x},\boldsymbol{x}' \in \Gamma$,*

$$\left|\frac{\mathcal{Q}(\boldsymbol{x})}{\widetilde{\mathcal{Q}}(\boldsymbol{x})}-1\right| \le \delta, \quad , \left|\frac{r(\boldsymbol{x},\boldsymbol{x}')}{\widetilde{r}(\boldsymbol{x},\boldsymbol{x}')}-1\right| \le \delta. \tag{4.7}$$

Then for any sets $\boldsymbol{A},\boldsymbol{B}$

$$(1-\delta)^2 \le \frac{\mathbb{C}\mathrm{ap}_N^n(\boldsymbol{A},\boldsymbol{B})}{\widetilde{\mathbb{C}\mathrm{ap}}_N^n(\boldsymbol{A},\boldsymbol{B})} \le (1-\delta)^{-2}. \tag{4.8}$$

Proof. Note that $\mathbb{C}\mathrm{ap}_N^n(\boldsymbol{A},\boldsymbol{B}) \equiv \inf_{u\in\mathcal{G}_{A,B}} \Phi_N(u) = \Phi_N(u^*)$, and
$\widetilde{\mathbb{C}\mathrm{ap}}_N^n(\boldsymbol{A},\boldsymbol{B}) \equiv \inf_{u\in\mathcal{G}_{A,B}} \widetilde{\Phi}_N(u) = \widetilde{\Phi}_N(\tilde{u}^*)$. But clearly

$$\begin{aligned}
\Phi_N(u^*) &= \frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{x}'\in\Gamma} \widetilde{\mathcal{Q}}(\boldsymbol{x})\frac{\mathcal{Q}(\boldsymbol{x})}{\widetilde{\mathcal{Q}}(\boldsymbol{x})}\tilde{r}(\boldsymbol{x},\boldsymbol{x}')\frac{r(\boldsymbol{x},\boldsymbol{x}'))}{\widetilde{r}(\boldsymbol{x},\boldsymbol{x}')}\left(u^*(\boldsymbol{x})-u^*(\boldsymbol{x}')\right) \\
&\ge \frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{x}'\in\Gamma} \widetilde{\mathcal{Q}}(\boldsymbol{x})(1-\delta)\tilde{r}(\boldsymbol{x},\boldsymbol{x}')(1-\delta)\left(u^*(\boldsymbol{x})-u^*(\boldsymbol{x}')\right) \\
&\ge (1-\delta)^2 \inf_{u\in\mathcal{G}_{A,B}} \frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{x}'\in\Gamma} \widetilde{\mathcal{Q}}(\boldsymbol{x})\tilde{r}(\boldsymbol{x},\boldsymbol{x}')\left(u(\boldsymbol{x})-u(\boldsymbol{x}')\right) \\
&= (1-\delta)^2\widetilde{\mathbb{C}\mathrm{ap}}_N^n(\boldsymbol{A},\boldsymbol{B}).
\end{aligned} \tag{4.9}$$

By the same token,

$$\widetilde{\Phi}_N(u^*) \quad \ge \quad (1-\delta)^2\mathbb{C}\mathrm{ap}_N^n(\boldsymbol{A},\boldsymbol{B}). \tag{4.10}$$

The claimed relation follows. □

To make use of this observation, we need to define suitable modified measure and rates, in order to control the rates $r_N(\boldsymbol{x},\boldsymbol{x}')$ and the measure $\mathcal{Q}_{\beta,N}(\boldsymbol{x})$. Let us define the modified measure

$$\widetilde{\mathcal{Q}}_{\beta,N}(\boldsymbol{x}) \equiv \mathcal{Q}_{\beta,N}(\boldsymbol{z}^*)\exp\left(-\frac{\beta N}{2}((\boldsymbol{x}-\boldsymbol{z}^*),\mathbb{A}(\boldsymbol{z}^*)(\boldsymbol{x}-\boldsymbol{z}^*))\right). \tag{4.11}$$

Making a second-order Taylor expansion of the exponent of $\mathcal{Q}_{\beta,N}(\boldsymbol{x})$, $F_{\beta,N}(\boldsymbol{x})$, around $\boldsymbol{z}^*$, and from the definition (4.11) of $\widetilde{\mathcal{Q}}_{\beta,N}(\boldsymbol{x})$, we get that for all $\boldsymbol{x}\in D_N(\rho)$ and for some $K<\infty$ it holds

$$\left|\frac{\mathcal{Q}_{\beta,N}(\boldsymbol{x})}{\widetilde{\mathcal{Q}}_{\beta,N}(\boldsymbol{x})}-1\right| \le KNn^3\rho^3, \tag{4.12}$$

which follows easily from a rough estimate of the error term in the Taylor expansion of $F_{\beta,N}(\boldsymbol{x})$.

For that concerns the rates, we first define, for $\sigma \in \mathscr{S}_N$,

$$\Lambda_k^{\pm}(\sigma) \equiv \{i \in \Lambda_k \;:\; \sigma(i) = \pm 1\}. \tag{4.13}$$

For all $\boldsymbol{x} \in \Gamma_N^n$, we then have

$$\begin{aligned} r_N(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{e}_\ell) &= \mathscr{Q}_{\beta,N}(\boldsymbol{x})^{-1} \sum_{\sigma \in \mathscr{S}_N[\boldsymbol{x}]} \mu_{\beta,N}[\omega](\sigma) \sum_{i \in \Lambda_\ell^-(\sigma)} p(\sigma, \sigma^i) \\ &= \mathscr{Q}_{\beta,N}(\boldsymbol{x})^{-1} \sum_{\sigma \in \mathscr{S}_N[\boldsymbol{x}]} \mu_{\beta,N}[\omega](\sigma) \sum_{i \in \Lambda_\ell^-(\sigma)} \tfrac{1}{N} e^{-2\beta\left[m(\sigma) - \frac{1}{N} + h_i\right]_+}. \end{aligned} \tag{4.14}$$

Notice that for all $\sigma \in \mathscr{S}_N(\boldsymbol{x})$, $|\Lambda_\ell^-(\sigma)|$ is a constant just depending on $\boldsymbol{x}$. Using that $h_i = \bar{h}_\ell + \widetilde{h}_i$, with $\widetilde{h}_i \in [-\varepsilon, \varepsilon]$, we get the bounds

$$r_N(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{e}_\ell) = \frac{|\Lambda_\ell^-(\boldsymbol{x})|}{N} e^{-2\beta\left[m(\sigma)+\bar{h}_\ell\right]_+} (1 + O(\varepsilon)). \tag{4.15}$$

It follows easily that, for all $\boldsymbol{x} \in D_N(\rho)$,

$$\left| \frac{r_N(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{e}_\ell)}{r_N(\boldsymbol{z}^*, \boldsymbol{z}^*+\boldsymbol{e}_\ell)} - 1 \right| \leq c\beta(\varepsilon + n\rho), \tag{4.16}$$

for some finite constant $c > 0$.

With this in mind, we let $\widetilde{r}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{e}_\ell) \equiv r_N(\boldsymbol{z}^*, \boldsymbol{z}^*+\boldsymbol{e}_\ell) \equiv r_\ell$ and $\widetilde{r}(\boldsymbol{x}+\boldsymbol{e}_\ell, \boldsymbol{x}) \equiv r_\ell \frac{\widetilde{\mathscr{Q}}_{\beta,N}(\boldsymbol{x})}{\widetilde{\mathscr{Q}}_{\beta,N}(\boldsymbol{x}+\boldsymbol{e}_\ell)}$ be the modified rates of a dynamics on $D_N(\rho)$ reversible w.r.t. the measure $\widetilde{\mathscr{Q}}_{\beta,N}(\boldsymbol{x})$, and let $\widetilde{L}_N$ denote the correspondent generator.

For $u \in \mathscr{G}_{\boldsymbol{A},\boldsymbol{B}}$, we write the corresponding Dirichlet form as

$$\widetilde{\Phi}_{D_N}(u) \equiv \mathscr{Q}_{\beta,N}(\boldsymbol{z}^*) \sum_{\boldsymbol{x} \in D_N(\rho)} \sum_{\ell=1}^{n} r_\ell e^{-\beta N((\boldsymbol{x}-\boldsymbol{z}^*), \mathbb{A}(\boldsymbol{z}^*)(\boldsymbol{x}-\boldsymbol{z}^*))} \left(u(\boldsymbol{x}) - u(\boldsymbol{x}+\boldsymbol{e}_\ell)\right)^2. \tag{4.17}$$

4.3 Approximated harmonic functions for $\widetilde{\Phi}_{D_N}$

We will now describe a function that we will show to be almost harmonic with respect to the Dirichlet form $\widetilde{\Phi}_{D_N}$. Define the matrix $\mathbb{B}(\boldsymbol{z}^*) \equiv \mathbb{B}$ with elements

$$\mathbb{B}_{\ell,k} \equiv \sqrt{r_\ell}\, \mathbb{A}(\boldsymbol{z}^*)_{\ell,k} \sqrt{r_k}. \tag{4.18}$$

Let $\hat{\boldsymbol{v}}^{(i)}$, $i = 1, \ldots, n$ be the normalized eigenvectors of $\mathbb{B}$, and $\hat{\gamma}_i$ be the corresponding eigenvalues. We denote by $\hat{\gamma}_1$ the unique negative eigenvalue of $\mathbb{B}$, and characterize it in the following lemma.

Lemma 4.2. *Let z^* be a solution of the equation (3.21) and assume in addition that*

$$\frac{\beta}{N} \sum_{i=1}^{N} \left(1 - \tanh^2\left(\beta\left(z^* + h_i\right)\right)\right) > 1. \tag{4.19}$$

Then, $\boldsymbol{z}^*$ defined through (3.20) is a saddle point and the unique negative eigenvalue of $\mathbb{B}(\boldsymbol{z}^*)$ is the unique negative solution, $\hat{\gamma}_1 \equiv \hat{\gamma}_1(N,n)$, of the equation

$$\sum_{\ell=1}^{n} \rho_\ell \frac{\frac{1}{|\Lambda_\ell|}\sum_{i\in\Lambda_\ell}\left(1-\tanh(\beta(z^*+h_i))\right)\exp\left(-2\beta\left[z^*+\bar{h}_\ell\right]_+\right)}{\frac{\frac{1}{|\Lambda_\ell|}\sum_{i\in\Lambda_\ell}\left(1-\tanh(\beta(z^*+h_i))\right)\exp\left(-2\beta\left[z^*+\bar{h}_\ell\right]_+\right)}{\frac{\beta}{|\Lambda_\ell|}\sum_{i\in\Lambda_\ell}\left(1-\tanh^2(\beta(z^*+h_i))\right)}-2\gamma} = 1. \tag{4.20}$$

Moreover, we have that

$$\lim_{n\uparrow\infty}\lim_{N\uparrow\infty}\hat{\gamma}_1(N,n) \equiv \bar{\gamma}_1, \tag{4.21}$$

where $\bar{\gamma}_1$ is the unique negative solution of the equation

$$\mathbb{E}_h\left[\frac{\left(1-\tanh(\beta(z^*+h))\right)\exp\left(-2\beta\left[z^*+h\right]_+\right)}{\frac{\exp\left(-2\beta[z^*+h]_+\right)}{\beta\left(1+\tanh(\beta(z^*+h))\right)}-2\gamma}\right] = 1. \tag{4.22}$$

Proof. The particular form of the matrix $\mathbb{B}$ allows to obtain a simple characterization of all eigenvalues and eigenvectors. Explicitly, any eigenvector $u = (u_1, \dots, u_n)$ of $\mathbb{B}$ with eigenvalue γ, should satisfy the set of equations

$$-\sum_{\ell=1}^{n}\sqrt{r_\ell r_k}u_\ell + (r_k\hat{\lambda}_k - \gamma)u_k = 0, \forall\ k = 1,\dots,n. \tag{4.23}$$

Assume for simplicity that all $r_k\hat{\lambda}_k$ take distinct values (see [6], Lemma 7.2 for the general case). Then the above set of equations has no non-trivial solution for $\gamma = r_k\hat{\lambda}_k$, and we can assume that $\sum_{\ell=1}^{n}\sqrt{r_\ell}u_\ell \neq 0$. Thus,

$$u_k = \frac{\sqrt{r_k}\sum_{\ell=1}^{n}\sqrt{r_\ell}u_\ell}{r_k\hat{\lambda}_k - \gamma}. \tag{4.24}$$

Multiplying by $\sqrt{r_k}$ and summing over k, u_k is a solution if and only if γ satisfies the equation

$$\sum_{k=1}^{n}\frac{r_k}{r_k\hat{\lambda}_k - \gamma} = 1. \tag{4.25}$$

Using (4.15) and noticing that $\frac{|\Lambda_k^-|}{N} = \frac{1}{2}(\rho_k - \boldsymbol{z}_k^*)$, we get

$$r_k = \tfrac{1}{2}(\rho_k - \boldsymbol{z}_k^*)\exp\left(-2\beta\left[m(\sigma)+\bar{h}_k\right]_+\right)(1+O(\varepsilon)). \tag{4.26}$$

Inserting the expressions for $\boldsymbol{z}_k^*/\rho_k$ and $\hat{\lambda}_k$ given by (3.20) and (3.22) into (4.26) and substituting the result into (4.25), we recover (4.20).

Since the left-hand side of (4.25) is monotone decreasing in γ as long as $\gamma \geq 0$, it follows that there can be at most one negative solution of this equation, and such a solution exists if and only if left-hand side is larger than 1 for $\gamma = 0$. The claimed convergence property (4.21) follows easily. $\square$

We continue our construction defining the vectors $\boldsymbol{v}^{(i)}$ by

$$\boldsymbol{v}_\ell^{(i)} \equiv \hat{\boldsymbol{v}}_\ell^{(i)} / \sqrt{r_\ell}, \tag{4.27}$$

and the vectors $\check{\boldsymbol{v}}^{(i)}$ by

$$\check{\boldsymbol{v}}_\ell^{(i)} \equiv \hat{\boldsymbol{v}}_\ell^{(i)} \sqrt{r_\ell} = r_\ell \boldsymbol{v}_\ell^{(i)}. \tag{4.28}$$

We will single out the vectors $\boldsymbol{v} \equiv \boldsymbol{v}^{(1)}$ and $\check{\boldsymbol{v}} \equiv \check{\boldsymbol{v}}^{(1)}$. The important facts about these vectors is that

$$\mathbb{A}\check{\boldsymbol{v}}^{(i)} = \hat{\gamma}_i \boldsymbol{v}^{(i)}, \tag{4.29}$$

and that

$$(\check{\boldsymbol{v}}^{(i)}, \boldsymbol{v}^{(j)}) = \delta_{ij}. \tag{4.30}$$

This implies the following non-orthogonal decomposition of the quadratic form $\mathbb{A}$,

$$(\boldsymbol{y}, \mathbb{A}\boldsymbol{x}) = \sum_{i=1}^{n} \hat{\gamma}_i (\boldsymbol{y}, \boldsymbol{v}^{(i)})(\boldsymbol{x}, \boldsymbol{v}^{(i)}). \tag{4.31}$$

A consequence of the computation in the proof of Lemma 4.2, on which we shall rely in the sequel, is the following:

Lemma 4.3. *There exists a positive constant $\delta > 0$ such that independently of n,*

$$\delta \leq \min_k \boldsymbol{v}_k \leq \max_k \boldsymbol{v}_k \leq \frac{1}{\delta}. \tag{4.32}$$

Proof. Due to our explicit computations,

$$r_k \hat{\lambda}_k = \frac{1}{2}\left(1 - \frac{\boldsymbol{z}_k^*}{\rho_k}\right) \left[\beta \frac{1}{|\Lambda_k|} \sum_{i \in \Lambda_k} \left(1 - \tanh^2\left(\beta(z^* + h_i)\right)\right)\right]^{-1} e^{-2\beta\left[z^* + \bar{h}_k\right]_+}. \tag{4.33}$$

Recall that random fields h_i have bounded support and that $\hat{\gamma}_1$ in (4.65) satisfies $\hat{\gamma}_1 \in (-\infty, 0)$. Consequently, in view of (3.20), the relation (4.33) implies that the quantities $\phi_k \equiv r_k \hat{\lambda}_k - \hat{\gamma}_1(N, n)$ are bounded away from zero and infinity, uniformly in N, n and $k = 1, \ldots, n$. Since by (4.27) and (4.24) the entries of $\boldsymbol{v}$ are given by

$$\boldsymbol{v}_k = \frac{1}{\phi_k} \left\{\sum_\ell \frac{r_\ell}{\phi_\ell^2}\right\}^{-1/2}, \tag{4.34}$$

the assertion of the lemma follows. □

Finally, define the function $f : \mathbb{R} \to \mathbb{R}_+$ by

$$\begin{aligned} f(a) &= \frac{\int_{-\infty}^{a} e^{-\beta N |\hat{\gamma}_1| u^2/2} du}{\int_{-\infty}^{\infty} e^{-\beta N |\hat{\gamma}_1| u^2/2} du} \\ &= \sqrt{\frac{\beta N |\hat{\gamma}_1|}{2\pi}} \int_{-\infty}^{a} e^{-\beta N |\hat{\gamma}_1| u^2/2} du. \end{aligned} \tag{4.35}$$

We claim that the function

$$g(\boldsymbol{x}) \equiv f((\boldsymbol{v},\boldsymbol{x})) \tag{4.36}$$

is the desired approximated harmonic function.

The reason behind such a choice of the test function g should be very clear: A formal continuum limit for the Euler equation for (4.17) gives,

$$\sum_{\ell} r_{\ell} \frac{\partial^2}{\partial \boldsymbol{x}_{\ell}^2} g(\boldsymbol{x}) - \sum_{k,\ell} r_{\ell} A_{k,\ell} \boldsymbol{x}_k \frac{\partial}{\partial \boldsymbol{x}_{\ell}} g(\boldsymbol{x}) = 0. \tag{4.37}$$

If one tries to find a vector $\boldsymbol{v}$ which would give rise to a solution of the form $g(\boldsymbol{x}) = f((\boldsymbol{v},\boldsymbol{x}))$, then, in order to have things in a closed form one readily arrives to the following eigenvalue-type constraint on $\boldsymbol{v}$: There exists γ, such that

$$\boldsymbol{v}_k = \gamma \sum_{\ell} A_{k,\ell} r_{\ell} \boldsymbol{v}_{\ell}, \quad k = 1,\dots,n. \tag{4.38}$$

Which means that one has to look among $\boldsymbol{v}$-s of the form (4.27). The choice of the negative eigenvalue just traces the natural geometry of the saddle point on a rout from one minimum to another.

Back to our problem: In order to verify that g in (4.36) is the required approximated harmonic function, notice first that $g(\boldsymbol{x}) = \mathrm{o}(1)$ for all $\boldsymbol{x} \in W_1 \cap D_N(\rho)$, while $g(\boldsymbol{x}) = 1 - \mathrm{o}(1)$ for all $\boldsymbol{x} \in W_2 \cap D_N(\rho)$. Moreover, the following holds:

Lemma 4.4. *Let g be defined in (4.36). Then, for all $\boldsymbol{x} \in D_N(\rho)$, there exists a constant $c < \infty$ such that*

$$\left| \left(\widetilde{L}_N g\right)(\boldsymbol{x}) \right| \leq \left(\sqrt{\frac{\beta|\hat{\gamma}_1|}{2\pi N}} e^{-\beta N |\hat{\gamma}_1| (\boldsymbol{x},\boldsymbol{v})^2/2} \sum_{\ell=1}^{n} r_{\ell} \boldsymbol{v}_{\ell} \right) c\rho^2. \tag{4.39}$$

Remark. The point of the estimate (4.39) is that it is by a factor ρ^2 smaller than what we would get for an arbitrary choice of the parameters $\boldsymbol{v}$ and γ_1. We will actually use this estimate in the proof of the *lower bound*.

Proof. To simplify the notation we will assume throughout the proof that coordinates are chosen such that $\boldsymbol{z}^* = 0$. We also set $\mathbb{A} \equiv \mathbb{A}(\boldsymbol{z}^*)$. Using the detailed balance condition, we get

$$\widetilde{r}(\boldsymbol{x}, \boldsymbol{x} - \boldsymbol{e}_{\ell}) = \frac{\widetilde{\mathscr{Q}}_{\beta,N}(\boldsymbol{x} - \boldsymbol{e}_{\ell})}{\widetilde{\mathscr{Q}}_{\beta,N}(\boldsymbol{x})} \widetilde{r}(\boldsymbol{x} - \boldsymbol{e}_{\ell}, \boldsymbol{x}) = \frac{\widetilde{\mathscr{Q}}_{\beta,N}(\boldsymbol{x} - \boldsymbol{e}_{\ell})}{\widetilde{\mathscr{Q}}_{\beta,N}(\boldsymbol{x})} r_{\ell}. \tag{4.40}$$

Moreover, from the definition of $\widetilde{\mathscr{Q}}_{\beta,N}$ and using that we are near a critical point, we have that

$$\begin{aligned} \frac{\widetilde{\mathscr{Q}}_{\beta,N}(\boldsymbol{x} - \boldsymbol{e}_{\ell})}{\widetilde{\mathscr{Q}}_{\beta,N}(\boldsymbol{x})} &= \exp\left(-\tfrac{\beta N}{2}\left[(\boldsymbol{x}, \mathbb{A}\boldsymbol{x}) - ((\boldsymbol{x} - \boldsymbol{e}_{\ell}), \mathbb{A}(\boldsymbol{x} - \boldsymbol{e}_{\ell}))\right]\right) \\ &= \exp\left(-\tfrac{\beta N}{2}(\boldsymbol{e}_{\ell}, \mathbb{A}\boldsymbol{x})\right)\left(1 + O\left(N^{-1}\right)\right). \end{aligned} \tag{4.41}$$

From (4.40) and (4.41), the generator can be written as

$$\left(\widetilde{L}_N g\right)(\boldsymbol{x}) = \sum_{\ell=1}^{n} r_\ell \left(g(\boldsymbol{x}+\boldsymbol{e}_\ell)-g(\boldsymbol{x})\right) \times \left(1-\exp\left(-\tfrac{\beta N}{2}(\boldsymbol{e}_\ell,\mathbb{A}\boldsymbol{x})\right)\frac{g(\boldsymbol{x})-g(\boldsymbol{x}-\boldsymbol{e}_\ell)}{g(\boldsymbol{x}+\boldsymbol{e}_\ell)-g(\boldsymbol{x})}\left(1+O(N^{-1})\right)\right). \tag{4.42}$$

Now we use the explicit form of g to obtain

$$\begin{aligned} g(\boldsymbol{x}+\boldsymbol{e}_\ell)-g(\boldsymbol{x}) &= f((\boldsymbol{x},\boldsymbol{v})+\boldsymbol{v}_\ell/N)-f((\boldsymbol{x},\boldsymbol{v}) \\ &= f'((\boldsymbol{x},\boldsymbol{v}))\boldsymbol{v}_\ell/N+\boldsymbol{v}_\ell^2N^{-2}f''(\boldsymbol{x},\boldsymbol{v})/2+\boldsymbol{v}_\ell^3N^{-3}f'''((\tilde{\boldsymbol{x}},\boldsymbol{v}))/6 \\ &= \boldsymbol{v}_\ell\sqrt{\frac{\beta|\hat{\gamma}_1|}{2\pi N}}e^{-\beta N|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})^2/2}\left(1-\boldsymbol{v}_\ell\beta|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})/2+O\left(\rho^2\right)\right). \end{aligned} \tag{4.43}$$

In particular, we get from here that

$$\begin{aligned} \frac{g(\boldsymbol{x})-g(\boldsymbol{x}-\boldsymbol{e}_\ell)}{g(\boldsymbol{x}+\boldsymbol{e}_\ell)-g(\boldsymbol{x})} &= \exp\left(-\beta N|\hat{\gamma}_1|\left[(\boldsymbol{x}-\boldsymbol{e}_\ell,\boldsymbol{v})^2-(\boldsymbol{x},\boldsymbol{v})^2\right]/2\right) \\ &\quad\times\frac{1-\boldsymbol{v}_\ell\beta|\hat{\gamma}_1|[(\boldsymbol{x},\boldsymbol{v})-\boldsymbol{v}_\ell/N]/2+O\left(\rho^2\right)}{1-\boldsymbol{v}_\ell\beta|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})/2+O\left(\rho^2\right)} \\ &= \exp\left(-\beta|\hat{\gamma}_1|\boldsymbol{v}_\ell(\boldsymbol{x},\boldsymbol{v})\right)\left(1+\frac{\boldsymbol{v}_\ell^2\beta|\hat{\gamma}_1|/2N+O\left(\rho^2\right)}{1-\boldsymbol{v}_\ell\beta|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})+O\left(\rho^2\right)}\right) \\ &= \exp\left(-\beta|\hat{\gamma}_1|\boldsymbol{v}_\ell(\boldsymbol{x},\boldsymbol{v})\right)\left(1+O(\rho^2)\right) \end{aligned} \tag{4.44}$$

Let us now insert these equations into (4.42):

$$\begin{aligned} \left(\widetilde{L}_N g\right)(\boldsymbol{x}) &= \sqrt{\frac{\beta|\hat{\gamma}_1|}{2\pi N}}e^{-\beta N|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})^2/2}\sum_{\ell=1}^{n} r_\ell\boldsymbol{v}_\ell\left(1-\boldsymbol{v}_\ell\beta|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})/2+O\left(\rho^2\right)\right). \\ &\quad\times\left(1-\exp\left\{-\tfrac{\beta N}{2}(\boldsymbol{e}_\ell,\mathbb{A}\boldsymbol{x})-\beta|\hat{\gamma}_1|\boldsymbol{v}_\ell(\boldsymbol{x},\boldsymbol{v})\right\}\left(1+O(\rho^2)\right)\right). \end{aligned} \tag{4.45}$$

Now

$$\begin{aligned} &1-\exp\left(-\tfrac{\beta N}{2}(\boldsymbol{e}_\ell,\mathbb{A}\boldsymbol{x})-\beta|\hat{\gamma}_1|\boldsymbol{v}_\ell(\boldsymbol{x},\boldsymbol{v})\right)\left(1+O(\rho^2)\right) \\ &\qquad= \tfrac{\beta N}{2}(\boldsymbol{e}_\ell,\mathbb{A}\boldsymbol{x})+\beta|\hat{\gamma}_1|\boldsymbol{v}_\ell(\boldsymbol{x},\boldsymbol{v})+O(\rho^2). \end{aligned} \tag{4.46}$$

Using this fact, and collecting the leading order terms, we get

$$\begin{aligned} \left(\widetilde{L}_N g\right)(\boldsymbol{x}) &= \sqrt{\frac{\beta|\hat{\gamma}_1|}{2\pi N}}e^{-\beta N|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})^2/2} \\ &\quad\times\sum_{\ell=1}^{n} r_\ell\boldsymbol{v}_\ell\left[\left(\tfrac{\beta N}{2}(\boldsymbol{e}_\ell,\mathbb{A}\boldsymbol{x})+\beta|\hat{\gamma}_1|\boldsymbol{v}_\ell(\boldsymbol{x},\boldsymbol{v})\right)+O(\rho^2)\right]. \end{aligned} \tag{4.47}$$

Thus we will have proved the lemma provided that

$$\sum_{\ell=1}^{n} r_\ell\boldsymbol{v}_\ell\left(\tfrac{N}{2}(\boldsymbol{e}_\ell,\mathbb{A}\boldsymbol{x})-\hat{\gamma}_1\boldsymbol{v}_\ell(\boldsymbol{x},\boldsymbol{v})\right)=0. \tag{4.48}$$

But note that from (4.31), and recalling that $\boldsymbol{e}_\ell$ denotes the lattice vector with length $2/N$, we get

$$\tfrac{N}{2}(\boldsymbol{e}_\ell, \mathbb{A}\boldsymbol{x}) - \hat{\gamma}_1 \boldsymbol{v}_\ell(\boldsymbol{x}, \boldsymbol{v}) = \sum_{j=2}^{n} \hat{\gamma}_j \boldsymbol{v}_\ell^{(j)}(\boldsymbol{x}, \boldsymbol{v}^{(j)}). \tag{4.49}$$

Hence, using that by (4.28) $r_\ell \boldsymbol{v}_\ell = \check{\boldsymbol{v}}_\ell$ and that by (4.30) $\check{\boldsymbol{v}}$ is orthogonal to $\boldsymbol{v}^{(j)}$ with $j \geq 2$, (4.48) follows and the lemma is proven. □

Having established that g is a good approximation of the equilibrium potential in a neighborhood of $\boldsymbol{z}^*$, we can now use it to compute a good upper bound for the capacity. Fix now $\rho = C\sqrt{\ln N/N}$.

Proposition 4.5. *With the notation introduced above and for every $n \in \mathbb{N}$, we get*

$$\operatorname{cap}(A,B) \leq \mathscr{Q}_{\beta,N}(\boldsymbol{z}^*) \frac{\beta|\hat{\gamma}_1|}{2\pi N} \left(\frac{\pi N}{2\beta}\right)^{n/2} \prod_{\ell=1}^{n} \sqrt{\frac{r_\ell}{|\hat{\gamma}_j|}} \left(1 + O(\varepsilon + \sqrt{(\ln N)^3/N})\right). \tag{4.50}$$

Proof. The upper bound on $\operatorname{cap}(A,B)$ is inherited from the upper bound on the mesoscopic capacity $\mathbb{C}\mathrm{ap}_N^n(\boldsymbol{A},\boldsymbol{B})$. As for the latter, we first estimate the energy of the mesoscopic neighborhood $D_N \equiv D_N(\rho)$ of the saddle point $\boldsymbol{z}^*$. By Lemma 4.1, this can be controlled in terms of the modified Dirichlet form $\widetilde{\Phi}_{D_N}$ in (4.17). Thus, let g the function defined in (4.36) and choose coordinates such that $\boldsymbol{z}^* = 0$. Then

$$\begin{aligned}
\widetilde{\Phi}_{D_N}(g) \;&\equiv\; \widetilde{\mathscr{Q}}_{\beta,N}(0) \sum_{\boldsymbol{x}\in D_N} \sum_{\ell=1}^{n} e^{-\beta N((\boldsymbol{x},\mathbb{A}\boldsymbol{x}))/2} r_\ell \left(g(\boldsymbol{x}+\boldsymbol{e}_\ell) - g(\boldsymbol{x})\right)^2 && (4.51)\\
&=\; \widetilde{\mathscr{Q}}_{\beta,N}(0) \frac{\beta|\hat{\gamma}_1|}{2\pi N} \sum_{\boldsymbol{x}\in D_N} e^{-\beta N|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})^2} e^{-\beta N((\boldsymbol{x},\mathbb{A}\boldsymbol{x}))/2} \sum_{\ell=1}^{n} r_\ell \boldsymbol{v}_\ell^2 \\
&\qquad \times \left(1 - \boldsymbol{v}_\ell \beta |\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v}) + O\left(N^{-1}\ln N\right)\right)^2 \\
&=\; \widetilde{\mathscr{Q}}_{\beta,N}(0) \frac{\beta|\hat{\gamma}_1|}{2\pi N} \sum_{\boldsymbol{x}\in D_N} e^{-\beta N|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})^2} e^{-\beta N((\boldsymbol{x},\mathbb{A}\boldsymbol{x}))/2} \left(1 + O\left(\sqrt{\ln N/N}\right)\right).
\end{aligned}$$

Here we used that $\sum_\ell r_\ell \boldsymbol{v}_\ell^2 = \sum_\ell \hat{\boldsymbol{v}}_\ell^2 = 1$. It remains to compute the sum over $\boldsymbol{x}$. By a standard approximation of the sum by an integral we get

$$\begin{aligned}
&\sum_{\boldsymbol{x}\in D_N} e^{-\beta N|\hat\gamma_1|(\boldsymbol{x},\boldsymbol{v})^2} e^{-\beta N((\boldsymbol{x},\mathbb{A}\boldsymbol{x}))/2} \qquad (4.52)\\
&= \left(\frac{N}{2}\right)^n \int d^n\boldsymbol{x}\, e^{-\beta N|\hat\gamma_1|(\boldsymbol{x},\boldsymbol{v})^2} e^{-\beta N((\boldsymbol{x},\mathbb{A}\boldsymbol{x}))/2}\left(1+O(\sqrt{\ln N/N})\right)\\
&= \left(\frac{N}{2}\right)^n \left(\prod_{\ell=1}^n \sqrt{r_\ell}\right)\int d^n y e^{-\beta N|\hat\gamma_1|(y,\hat v)^2} e^{-\beta N((y,\mathbb{B}y))/2}\left(1+O(\sqrt{\ln N/N})\right)\\
&= \left(\frac{N}{2}\right)^n \left(\prod_{\ell=1}^n \sqrt{r_\ell}\right)\int d^n y e^{-\beta N\sum_{j=1}^n |\hat\gamma_j|(\hat v^{(j)},y)^2/2}\left(1+O(\sqrt{\ln N/N})\right)\\
&= \left(\frac{N}{2}\right)^n \left(\prod_{\ell=1}^n \sqrt{r_\ell}\right)\left(\frac{2\pi}{\beta N}\right)^{n/2}\frac{1}{\sqrt{\prod_{j=1}^n|\hat\gamma_j|}}\left(1+O(\sqrt{\ln N/N})\right)\\
&= \left(\frac{\pi N}{2\beta}\right)^{n/2}\prod_{\ell=1}^n\sqrt{\frac{r_\ell}{|\hat\gamma_\ell|}}\left(1+O(\sqrt{\ln N/N})\right).
\end{aligned}$$

Inserting (4.52) into (4.51) we see that the left-hand side of (4.51) is equal to the right-hand side of (4.50) up to error terms.

It remains to show that the contributions from the sum outside D_N in the Dirichlet form do not contribute significantly to the capacity. To do this, we define a global test function $\widetilde{g}$ given by

$$\widetilde{g}(\boldsymbol{x}) \equiv \begin{cases} 0, & \boldsymbol{x}\in W_1\\ 1, & \boldsymbol{x}\in W_2\\ g(\boldsymbol{x}), & \boldsymbol{x}\in W_0\end{cases} \qquad (4.53)$$

Clearly, the only non-zero contributions to the Dirichlet form $\Phi_N(\widetilde{g})$ come from $\overline{W}_0 \equiv W_0\cup\partial W_0$, where ∂W_0 denotes the boundary of W_0. Let us thus consider the sets $W_0^{in} = W_0\cap D_N$ and $W_0^{out} = W_0\cap D_N^c$ (see Figure 2.).

We denote by $\Phi^{||}_{W_0^{in}}(\widetilde{g})$ the Dirichlet form of $\widetilde{g}$ restricted to W_0^{in} and to the part of its boundary contained in D_N, i.e. to $\overline{W}_0^{in}\cap D_N$, and by $\Phi^{\square}_{W_0^{out}}(\widetilde{g})$ the Dirichlet form of $\widetilde{g}$ restricted to $\overline{W}_0^{out}$. With this notation, we have

$$\begin{aligned}
\Phi_N(\widetilde{g}) &= \Phi^{||}_{W_0^{in}}(\widetilde{g}) + \Phi^{\square}_{W_0^{out}}(\widetilde{g}) \qquad (4.54)\\
&= \widetilde{\Phi}^{||}_{W_0^{in}}(\widetilde{g})\left(1+O\left(\sqrt{\ln N/N}\right)\right) + \Phi^{\square}_{W_0^{out}}(\widetilde{g})\\
&= \left(\widetilde{\Phi}^{||}_{W_0^{in}}(g) - \left(\widetilde{\Phi}^{||}_{W_0^{in}}(g) - \widetilde{\Phi}^{||}_{W_0^{in}}(\widetilde{g})\right)\right)\left(1+O\left(\sqrt{\ln N/N}\right)\right) + \Phi^{\square}_{W_0^{out}}(\widetilde{g}).
\end{aligned}$$

The first term in (4.54) satisfies trivially the bound

$$\widetilde{\Phi}_{D'_N}(g) \le \widetilde{\Phi}^{||}_{W_0^{in}}(g) \le \widetilde{\Phi}_{D_N}(g), \qquad (4.55)$$

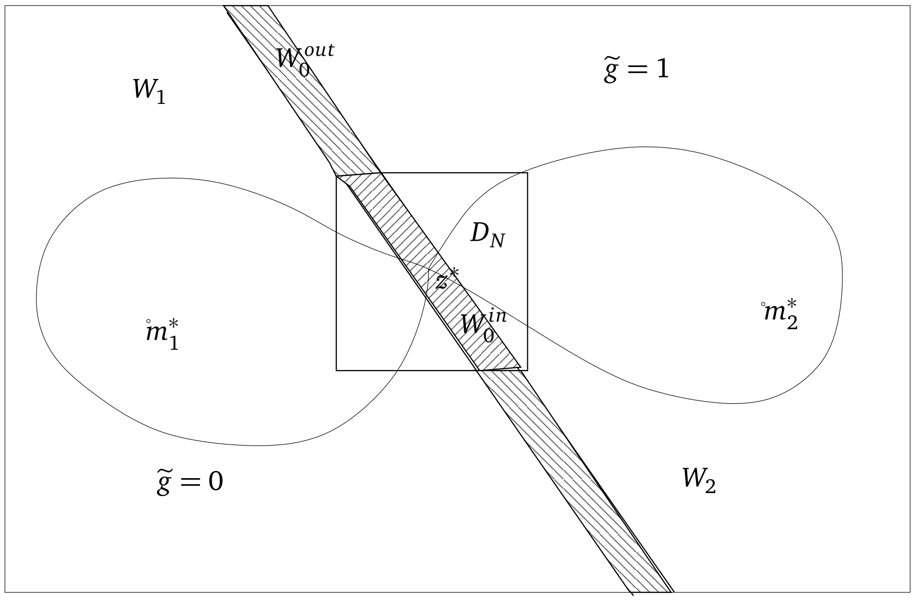

Figure 2: Domains for the construction of the test function in the upper bound

where $D_N' \equiv D_N(\rho')$ is defined as in (4.55) but with constant $\rho' = C'\sqrt{\ln N/N}$ such that $D_N' \subset W_0^{in}$. Performing the same computations as in (4.51) and (4.52) it is easy to show that $\widetilde{\Phi}_{D_N'}(g) = \widetilde{\Phi}_{D_N}(g)(1+o(1))$, and then from (4.54) it follows that

$$\widetilde{\Phi}^{||}_{W_0^{in}}(g) = \widetilde{\Phi}_{D_N}(g)(1-o(1)). \tag{4.56}$$

Consider now the second term in (4.54). Since $\widetilde{g} \equiv g$ on W_0, we get

$$\begin{aligned}\widetilde{\Phi}^{||}_{W_0^{in}}(g) - \widetilde{\Phi}^{||}_{W_0^{in}}(\widetilde{g}) = &\sum_{\boldsymbol{x}\in\partial W_0^{in}\cap W_1} \sum_{\ell=1}^{n} \widetilde{\mathcal{Q}}(\boldsymbol{x}) r_\ell \left[\left(g(\boldsymbol{x}+\boldsymbol{e}_\ell) - g(\boldsymbol{x})\right)^2 - g(\boldsymbol{x})^2 \right] \\ &+ \sum_{\boldsymbol{x}\in\partial W_0^{in}\cap W_2} \sum_{\ell=1}^{n} \widetilde{\mathcal{Q}}(\boldsymbol{x}) r_\ell \left[\left(g(\boldsymbol{x}+\boldsymbol{e}_\ell) - g(\boldsymbol{x})\right)^2 - \left(1-g(\boldsymbol{x})\right)^2 \right],\end{aligned} \tag{4.57}$$

where we also used that the function $\widetilde{g}$ has boundary conditions zero and one respectively on W_1 and W_2. By symmetry, let us just consider the first sum in the r.h.s. of (4.57). For $\boldsymbol{x} \in \partial W_0^{in} \cap W_1$ it holds that $(\boldsymbol{x}, \boldsymbol{v}) \le -\rho = -C\sqrt{\ln N/N}$, and hence

$$g(\boldsymbol{x})^2 \le \frac{1}{\sqrt{2\pi\beta|\hat{\gamma}_1|C\sqrt{\ln N}}} e^{-\beta N|\hat{\gamma}_1|\rho^2}. \tag{4.58}$$

Using this bound together with inequality (4.43) to control $\big(g(\boldsymbol{x}+\boldsymbol{e}_\ell)-g(\boldsymbol{x})\big)^2$, we get

$$
\begin{aligned}
&\sum_{\boldsymbol{x}\in\partial W_0^{in}\cap W_1}\sum_{\ell=1}^{n}\widetilde{\mathscr{Q}}(\boldsymbol{x})r_\ell\left[\big(g(\boldsymbol{x}+\boldsymbol{e}_\ell)-g(\boldsymbol{x})\big)^2-g(\boldsymbol{x})^2\right]\\
&\quad\le\frac{\beta|\hat\gamma_1|}{2\pi N}e^{-\beta N|\hat\gamma_1|\rho^2}\sum_{\boldsymbol{x}\in\partial W_0^{in}\cap W_1}\widetilde{\mathscr{Q}}(\boldsymbol{x})\left(1+\frac{cN}{\sqrt{\ln N}}\right)\\
&\quad\le\widetilde{\mathscr{Q}}_{\beta,N}(0)\frac{\beta|\hat\gamma_1|}{2\pi N}e^{-\beta N|\hat\gamma_1|\rho^2}\sum_{\boldsymbol{x}\in\partial W_0^{in}\cap W_1}e^{-\beta N((\boldsymbol{x},\mathbb{A}\boldsymbol{x}))/2}\left(1+c\frac{N}{\sqrt{\ln N}}\right)
\end{aligned}
\tag{4.59}
$$

for some constant c independent on N. The sum over $\boldsymbol{x}\in\partial W_0^{in}\cap W_1$ in the last term can then be computed as in (4.52). However, in this case the integration runs over the $(n-1)$-dimensional hyperplane orthogonal to v and thus we have

$$
\begin{aligned}
\sum_{\boldsymbol{x}\in\partial W_0^{in}\cap W_1}e^{-\beta N((\boldsymbol{x},\mathbb{A}\boldsymbol{x}))/2}
&=\left(\frac{N}{2}\right)^{n-1}\int d^{n-1}\boldsymbol{x}\,e^{-\beta N((\boldsymbol{x},\mathbb{A}\boldsymbol{x}))/2}\\
&=\left(\frac{N}{2}\right)^{n-1}\left(\prod_{\ell=2}^{n}\sqrt{r_\ell}\right)\int d^{n-1}y\,e^{-\beta N((y,\mathbb{B}y))/2}\\
&\le\left(\frac{N}{2}\right)^{n-1}\left(\prod_{\ell=2}^{n}\sqrt{r_\ell}\right)e^{-\beta N\hat\gamma_1\rho^2/2}\int d^{n-1}y\,e^{-\beta N\left(\sum_{j=2}^{n}\hat\gamma_j(\hat{\boldsymbol{v}}^{(j)},y)^2/2\right)}\\
&=\left(\frac{\pi N}{2\beta}\right)^{\frac{n-1}{2}}\prod_{\ell=2}^{n}\sqrt{\frac{r_\ell}{|\hat\gamma_\ell|}}e^{-\beta N\hat\gamma_1\rho^2/2}.
\end{aligned}
\tag{4.60}
$$

Inserting (4.60) in (4.59), and comparing the result with $\widetilde{\Phi}_{D_N}(g)$, we get that the l.h.s of (4.59) is bounded as

$$
\left(1+c\frac{N}{\ln N}\right)\sqrt{N}e^{-\beta N|\hat\gamma_1|\rho^2/2}\widetilde{\Phi}_{D_N}(g)=\mathrm{o}(N^{-K})\widetilde{\Phi}_{D_N}(g),\tag{4.61}
$$

with $K=\frac{\beta|\hat\gamma_1|C-1}{2}$, which is positive if C is large enough. A similar bound can be obtained for the second sum in (4.57), so that we finally get

$$
\left|\widetilde{\Phi}^{||}_{W_0^{in}}(g)-\widetilde{\Phi}^{||}_{W_0^{in}}(\widetilde{g})\right|\le\mathrm{o}(N^{-K})\widetilde{\Phi}_{D_N}(g).\tag{4.62}
$$

The last term to analyze is the Dirichlet form $\Phi^{\square}_{W_0^{out}}(\widetilde{g})$. But it is easy to realize that this is negligible with respect to the leading term. Indeed, since for all $\boldsymbol{x}\in D_N^c$ it holds that $F_{\beta,N}(\boldsymbol{x})\ge F_{\beta,N}(\boldsymbol{z}^*)+K'\ln N/N$, for some positive $K'<\infty$ depending on C, we get

$$
\Phi^{\square}_{W_0^{out}}(\widetilde{g})\le Z_{\beta,N}^{-1}e^{-\beta NF_{\beta,N}(\boldsymbol{z}^*)}N^{-(K'-n)}=\mathrm{o}(N^{-K''})\widetilde{\Phi}_{D_N}(g).\tag{4.63}
$$

From (4.54) and the estimates given in (4.56), (4.61) and (4.63), we get that $\Phi_N(\widetilde{g})=\widetilde{\Phi}_{D_N}(g)(1+\mathrm{o}(1))$ provides the claimed upper bound. $\square$

Combining this proposition with Proposition 3.1, yields, after some computations, the following more explicit representation of the upper bound.

Corollary 4.6. *With the same notation of Proposition 4.5,*

$$Z_{\beta,N}\mathrm{cap}(A,B) \leq \frac{\beta|\bar{\gamma}_1|}{2\pi N}\frac{\exp\left(-\beta N F_{\beta,N}(z^*)\right)(1+o(1))}{\sqrt{\beta N \mathbb{E}_h\left(1-\tanh^2\left(\beta\left(z^*+h\right)\right)\right)-1}}, \tag{4.64}$$

where $\bar{\gamma}_1$ is defined through Eq. (4.22).

Proof. First, we want to show that

$$|\det(\mathbb{A}(\boldsymbol{z}^*))| = \left(\prod_{\ell=1}^{n} r_\ell\right)^{-1} \prod_{\ell=1}^{n} \hat{\gamma}_\ell. \tag{4.65}$$

To see this, note that

$$\mathbb{B} = R\mathbb{A}(\boldsymbol{z}^*)R,$$

where R is the diagonal matrix with elements $R_{\ell,k} = \delta_{k,\ell}\sqrt{r_\ell}$. Thus

$$\prod_{\ell=1}^{n} |\hat{\gamma}_\ell| = |\det(\mathbb{B})| = \left|\det(R\mathbb{A}(\boldsymbol{z}^*)R)\right| = |\det(\mathbb{A}(\boldsymbol{z}^*))|\det(R^2) = \left|\det(\mathbb{A}(\boldsymbol{z}^*))\right|\prod_{\ell=1}^{n} r_\ell. \tag{4.66}$$

as desired. Substituting in (4.50) the expression of $\mathcal{Q}_{\beta,N}(\boldsymbol{z}^*)$ given in Proposition (3.1), and after the cancelation due to (4.65), we obtain an upper bound which is almost in the form we want. The only n-dependent quantity is the eigenvalue $\hat{\gamma}_1$ of the matrix $\mathbb{B}$. Taking the limit of $n \to \infty$ and using the second part of Lemma 4.2, we recover the assertion (4.64) of the corollary. □

This corollary concludes the first part of the proof of Theorem 1.3. The second part, namely the construction of a matching lower bound, will be discussed in the next section.

5 Lower bounds on capacities

In this section we will exploit the variational principle form Proposition 2.24 to derive lower bounds on capacities. Our task is to construct a suitable non-negative unit flow. This will be done in two steps. First we construct a good flow for the coarse grained Dirichlet form in the mesoscopic variables and then we use this to construct a flow on the microscopic variables.

5.1 Mesoscopic lower bound: The strategy

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be mesoscopic neighborhoods of two minima $\boldsymbol{m_A}$ and $\boldsymbol{m_B}$ of $F_{\beta,N}$, exactly as in the preceding section, and let $\boldsymbol{z}^*$ be the highest critical point of $F_{\beta,N}$ which lies between $\boldsymbol{m_A}$ and $\boldsymbol{m_B}$. It would be convenient to pretend that $\boldsymbol{m_A}, \boldsymbol{z}^*, \boldsymbol{m_B} \in \Gamma_N^n$: In general we should substitute critical points by their closest approximations on the latter grid, but the proofs will not be sensitive to the corresponding corrections. Recall that the energy landscape around $\boldsymbol{z}^*$ has been described in Subsection 3.2.

Recall that the *mesoscopic capacity*, $\mathbb{C}\mathrm{ap}_N^n(\boldsymbol{A},\boldsymbol{B})$, is defined in (4.1). We will construct a unit flow, $\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}$, from $\boldsymbol{A}$ to $\boldsymbol{B}$ of the form

$$\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x},\boldsymbol{x}') = \frac{\mathcal{Q}_{\beta,N}(\boldsymbol{x})r_N(\boldsymbol{x},\boldsymbol{x}')}{\Phi_N(\widetilde{g})}\phi_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x},\boldsymbol{x}'), \tag{5.1}$$

such that the associated Markov chain, $\left(\mathbb{P}_N^{\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}}, \mathscr{X}_{\boldsymbol{A},\boldsymbol{B}}\right)$, satisfies

$$\mathbb{P}_N^{\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}}\left(\sum_{\boldsymbol{e}\in\mathscr{X}_{\boldsymbol{A},\boldsymbol{B}}} \phi_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{e}) = 1+\mathrm{o}(1)\right) = 1-\mathrm{o}(1). \tag{5.2}$$

In view of the general lower bound (2.22), Eq. (5.2) implies that the mesoscopic capacities satisfy

$$\mathbb{C}\mathrm{ap}_N^n(\boldsymbol{A},\boldsymbol{B}) \geq \mathbb{E}_N^{\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}}\left\{\sum_{\boldsymbol{e}=(\boldsymbol{x},\boldsymbol{x}')\in\mathscr{X}} \frac{\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{e})}{\mathcal{Q}_{\beta,N}(\boldsymbol{x})r_N(\boldsymbol{e})}\right\}^{-1} \geq \Phi_N(\widetilde{g})(1-\mathrm{o}(1)), \tag{5.3}$$

which is the lower bound we want to achieve on the mesoscopic level.

We shall channel all of the flow $\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}$ through a certain (mesoscopic) neighborhood G_N of $\boldsymbol{z}^*$. Namely, our global flow, $\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}$, in (5.1) will consist of three (matching) parts, $\mathfrak{f}_{\boldsymbol{A}}, \mathfrak{f}$ and $\mathfrak{f}_{\boldsymbol{B}}$, where $\mathfrak{f}_{\boldsymbol{A}}$ will be a flow from $\boldsymbol{A}$ to ∂G_N, $\mathfrak{f}$ will be a flow through G_N, and $\mathfrak{f}_{\boldsymbol{B}}$ will be a flow from ∂G_N to $\boldsymbol{B}$. We will recover (5.2) as a consequence of the three estimates

$$\mathbb{P}_N^{\mathfrak{f}}\left(\sum_{\boldsymbol{e}\in\mathscr{X}} \phi(\boldsymbol{e}) = 1+\mathrm{o}(1)\right) = 1-\mathrm{o}(1), \tag{5.4}$$

whereas,

$$\mathbb{P}_N^{\mathfrak{f}_{\boldsymbol{A}}}\left(\sum_{\boldsymbol{e}\in\mathscr{X}_{\boldsymbol{A}}} \phi_{\boldsymbol{A}}(\boldsymbol{e}) = \mathrm{o}(1)\right) = 1-\mathrm{o}(1) \quad \text{and} \quad \mathbb{P}_N^{\mathfrak{f}_{\boldsymbol{B}}}\left(\sum_{\boldsymbol{e}\in\mathscr{X}_{\boldsymbol{B}}} \phi_{\boldsymbol{B}}(\boldsymbol{e}) = \mathrm{o}(1)\right) = 1-\mathrm{o}(1). \tag{5.5}$$

The construction of $\mathfrak{f}$ through G_N will be by far the most difficult part. It will rely crucially on Lemma 4.4.

5.2 Neighborhood G_N

We chose again mesoscopic coordinates in such a way that $\boldsymbol{z}^* = 0$. Set $\rho = N^{-1/2+\delta}$ and fix a (small) positive number, $\nu > 0$. Define

$$G_N \equiv G_N(\rho,\nu) \equiv D_N(\rho)\cap\{\boldsymbol{x} \,:\, (\boldsymbol{x},\check{\boldsymbol{v}})\in(-\nu\rho,\nu\rho)\}, \tag{5.6}$$

where $\check{\boldsymbol{v}} \equiv \check{\boldsymbol{v}}^{(1)}$ is defined in (4.28), and D_N is the same as in (4.5). Note that in view of the discussion in Section 4, within the region G_N we may work with the modified quantities, $\widetilde{\mathcal{Q}}_{\beta,N}$ and r_ℓ; $\ell = 1,\ldots,n$, defined in (4.11) and (4.17).

The boundary ∂G_N of G_N consists of three disjoint pieces, $\partial G_N = \partial_{\boldsymbol{A}} G_N \cup \partial_{\boldsymbol{B}} G_N \cup \partial_r G_N$, where

$$\partial_{\boldsymbol{A}} G_N = \left\{ \boldsymbol{x} \in \partial G_N : (\boldsymbol{x}, \check{\boldsymbol{v}}) \leq -\nu\rho \right\} \quad \text{and} \quad \partial_{\boldsymbol{B}} G_N = \left\{ \boldsymbol{x} \in \partial G_N : (\boldsymbol{x}, \check{\boldsymbol{v}}) \geq \nu\rho \right\}. \tag{5.7}$$

We choose ν in (5.6) to be so small that there exists $K > 0$, such that

$$F_{\beta,N}(\boldsymbol{x}) > F_{\beta,N}(0) + K\rho^2, \tag{5.8}$$

uniformly over the remaining part of the boundary $\boldsymbol{x} \in \partial_r G_N$.

Let $\widetilde{g}$ be the approximately harmonic function defined in (4.36) and (4.53). Proceeding along the lines of (4.51) and (4.52) we infer that,

$$\Phi_N(\widetilde{g})(1 + \mathrm{o}(1)) = \sum_{\boldsymbol{x} \in G_N \cup \partial_{\boldsymbol{A}} G_N} \widetilde{\mathcal{Q}}_{\beta,N}(\boldsymbol{x}) \sum_{\ell \in I_{G_N}(\boldsymbol{x})} r_\ell \left(\widetilde{g}(\boldsymbol{x} + \boldsymbol{e}_\ell) - \widetilde{g}(\boldsymbol{x}) \right)^2, \tag{5.9}$$

where $I_{G_N}(\boldsymbol{x}) \equiv \left\{ \ell \ : \ \boldsymbol{x} + \boldsymbol{e}_\ell \in G_N \right\}$. For functions, ϕ, on oriented edges, $(\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{e}_\ell)$, of D_N, we use the notation $\phi_\ell(\boldsymbol{x}) = \phi(\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{e}_\ell)$, and set

$$\begin{aligned} \mathscr{F}_\ell[\phi](\boldsymbol{x}) &\equiv \widetilde{\mathcal{Q}}_{\beta,N}(\boldsymbol{x}) r_\ell \phi_\ell(\boldsymbol{x}), \\ \mathrm{d}\mathscr{F}[\phi](\boldsymbol{x}) &\equiv \sum_{\ell=1}^{n} \left(\mathscr{F}_\ell[\phi](\boldsymbol{x}) - \mathscr{F}_\ell[\phi](\boldsymbol{x} - \boldsymbol{e}_\ell) \right). \end{aligned}$$

In particular, the left hand side of (4.39) can be written as $|\mathrm{d}\mathscr{F}[\nabla\widetilde{g}]| / \widetilde{\mathcal{Q}}_{\beta,N}(\boldsymbol{x})$.

Let us sum by parts in (5.9). By (5.8) the contribution coming from $\partial_r G_N$ is negligible and, consequently, we have, up to a factor of order $(1 + \mathrm{o}(1))$,

$$\sum_{\boldsymbol{x} \in G_N} \widetilde{g}(\boldsymbol{x}) \mathrm{d}\mathscr{F}[\nabla\widetilde{g}](\boldsymbol{x}) + \sum_{\boldsymbol{x} \in \partial_{\boldsymbol{A}} G_N} \sum_{\ell \in I_{G_N}(\boldsymbol{x})} \mathscr{F}_\ell[\nabla g](\boldsymbol{x}). \tag{5.10}$$

Furthermore, comparison between the claim of Lemma 4.4 and (4.51) (recall that $\rho^2 = N^{2\delta-1} \ll N^{-1/2}$) shows that the first term above is also negligible with respect to $\Phi_N(\widetilde{g})$. Hence,

$$\Phi_N(\widetilde{g})(1 + \mathrm{o}(1)) = \sum_{\boldsymbol{x} \in \partial_{\boldsymbol{A}} G_N} \sum_{\ell \in I_{G_N}(\boldsymbol{x})} \mathscr{F}_\ell[\nabla\widetilde{g}](\boldsymbol{x}). \tag{5.11}$$

5.3 Flow through G_N

The relation (5.11) is the starting point for our construction of a unit flow of the form

$$\mathfrak{f}_\ell(\boldsymbol{x}) = \frac{c}{\Phi_N(\widetilde{g})} \mathscr{F}_\ell[\phi](\boldsymbol{x}) \tag{5.12}$$

through G_N. Above $c = 1 + \mathrm{o}(1)$ is a normalization constant. Let us fix $0 < \nu_0 \ll \nu$ small enough and define,

$$G_N^0 = G_N \cap \left\{ \boldsymbol{x} \ : \ \left| \boldsymbol{x} - \frac{(\boldsymbol{x}, \check{\boldsymbol{v}})\check{\boldsymbol{v}}}{\|\check{\boldsymbol{v}}\|^2} \right| < \nu_0 \rho \right\}. \tag{5.13}$$

Thus, G_N^0 is a narrow tube along the principal $\check{\boldsymbol{v}}$-direction (Figure 3.). We want to construct ϕ in (5.12) such that the following properties holds:

P1: $\mathfrak{f}$ is confined to G_N, it runs from $\partial_{\boldsymbol{A}} G_N$ to $\partial_{\boldsymbol{B}} G_N$ and it is a unit flow. That is,

$$\forall \boldsymbol{x} \in G_N, d\mathscr{F}[\phi](\boldsymbol{x}) = 0 \quad \text{and} \quad \sum_{\boldsymbol{x}\in\partial_A G_N} \sum_{\ell\in I_{G_N}(\boldsymbol{x})} \mathfrak{f}_\ell[\phi](\boldsymbol{x}) = 1. \tag{5.14}$$

P2: ϕ is a small distortion of $\nabla\widetilde{g}$ inside G_N^0,

$$\phi_\ell(\boldsymbol{x}) = \nabla_\ell \widetilde{g}(\boldsymbol{x})(1+\mathrm{o}(1)), \tag{5.15}$$

uniformly in $\boldsymbol{x} \in G_N^0$ and $\ell = 1, \dots, n$.

P3: The flow $\mathfrak{f}$ is negligible outside G_N^0 in the following sense: For some $\kappa > 0$,

$$\max_{\boldsymbol{x}\in G_N\setminus G_N^0} \max_\ell \mathfrak{f}_\ell(\boldsymbol{x}) \le \frac{1}{N^\kappa}. \tag{5.16}$$

Once we are able to construct $\mathfrak{f}$ which satisfies **P1**-**P3** above, the associated Markov chain $\left(\mathbb{P}_N^{\mathfrak{f}}, \mathscr{X}\right)$ obviously satisfies (5.4).

The most natural candidate for ϕ would seem to be $\nabla\widetilde{g}$. However, since $\widetilde{g}$ is not strictly harmonic, this choice does not satisfies Kirchoff's law, and we would need to correct this by adding a (hopefully) small perturbation, which in principle can be constructed recursively. It turns out, however, to be more convenient to use as a starting choice

$$\phi_\ell^{(0)}(\boldsymbol{x}) \equiv \boldsymbol{v}_\ell \sqrt{\frac{\beta|\hat{\gamma}_1|}{2\pi N}} \exp\left(-\beta N|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})^2/2\right), \tag{5.17}$$

which, by (4.43), satisfies

$$\phi_\ell^{(0)}(\boldsymbol{x}) = \left(\widetilde{g}(\boldsymbol{x}+\boldsymbol{e}_\ell) - \widetilde{g}(\boldsymbol{x})\right)\left(1+O(\rho)\right), \tag{5.18}$$

uniformly in G_N. Notice that, by (5.12), this choice corresponds to the Markov chain with transition probabilities

$$q(\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_\ell) = \frac{\check{\boldsymbol{v}}_\ell}{\sum_k \check{\boldsymbol{v}}_k}(1+\mathrm{o}(1)) \equiv q_\ell(1+\mathrm{o}(1)). \tag{5.19}$$

From (3.16) and the decomposition (4.31) we see that

$$\begin{aligned}
\frac{1+O(\rho)}{\widetilde{\mathscr{Q}}_{N,\beta}(0)}\mathscr{F}_\ell[\phi^{(0)}] &= r_\ell \boldsymbol{v}_\ell \sqrt{\frac{\beta|\hat{\gamma}_1|}{2\pi N}} \exp\left(-\frac{\beta N}{2}\left(|\hat{\gamma}_1|(\boldsymbol{x},\boldsymbol{v})^2 + (\boldsymbol{x},\mathbb{A}\boldsymbol{x})\right)\right) \\
&= \check{\boldsymbol{v}}_\ell \sqrt{\frac{\beta|\hat{\gamma}_1|}{2\pi N}} \exp\left(-\frac{\beta N}{2}\left(\sum_{j=2}^n \hat{\gamma}_j(\boldsymbol{x},\boldsymbol{v}^{(j)})^2\right)\right).
\end{aligned}$$

In particular, there exists a constant $\chi_1 > 0$ such that

$$\frac{\mathscr{F}_\ell[\phi^{(0)}](x)}{\widetilde{\mathscr{Q}}_{N,\beta}(0)} \le \exp\left(-\chi_1 N^{2\delta}\right), \tag{5.20}$$

uniformly in $\boldsymbol{x} \in G_N \setminus G_N^0$ and $l = 1, \dots, n$.

Next, by inspection of the proof of Lemma 4.4, we see that there exists χ_2, such that,

$$\left| \mathrm{d}\mathscr{F}[\phi^{(0)}](\boldsymbol{x}) \right| \leq \chi_2 \rho^2 \mathscr{F}_\ell[\phi^{(0)}](\boldsymbol{x}), \tag{5.21}$$

uniformly in $x \in G_N$ and $\ell = 1, \ldots, n$. Notice that we are relying on the strict uniform (in n) positivity of the entries $\boldsymbol{\nu}_\ell$, as stated in Lemma 4.3

Truncation of ∇g, confinement of $\mathfrak{f}$ and property P1. Let $\mathscr{C}_+$ be the positive cone spanned by the axis directions $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. Note that the vector $\check{\boldsymbol{\nu}}$ lies in the interior of $\mathscr{C}_+$. Define (see Figure 3.)

$$G_N^1 = \mathrm{int}\left(\partial_B G_N^0 - \mathscr{C}_+\right) \cap G_N \quad \text{and} \quad G_N^2 = \left(\partial_A G_N^1 + \mathscr{C}_+\right) \cap G_N. \tag{5.22}$$

We assume that the constants ν and ν_0 in the definition of G_N and, respectively, in the definition of G_N^0 are tuned in such a way that $G_N^2 \cap \partial_r G_N = \emptyset$.

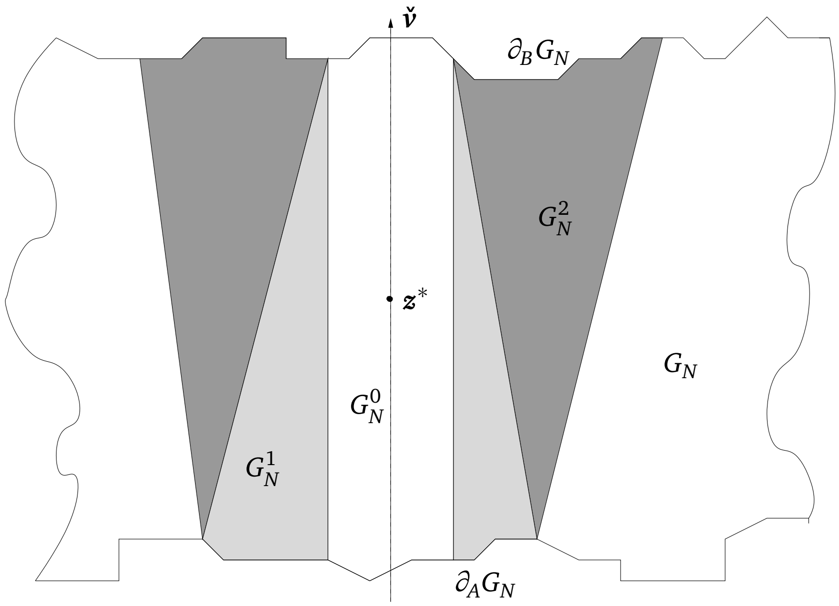

Figure 3: Narrow tube G_N^0 and sets G_N^1 and G_N^2

Let $\widetilde{\phi}^{(0)}$ be the restriction of $\phi^{(0)}$ to G_N^1,

$$\widetilde{\phi}_\ell^{(0)}(\boldsymbol{x}) \equiv \phi_\ell^{(0)}(\boldsymbol{x}) \mathbb{1}_{\{x \in G_N^1\}}. \tag{5.23}$$

Now we turn to the construction of the full flow. To this end we start by setting the values of ϕ_ℓ on $\partial_{\boldsymbol{A}} G_N$ equal to $\widetilde{\phi}_{(0)}$ if $\ell \in I_{G_N}(\boldsymbol{x})$ and zero otherwise. By (5.11) and the bound (5.20), the second of the relations in (5.14) is satisfied.

In order to satisfy Kirchoff's law inside G_N, we write ϕ as $\phi = \widetilde{\phi}^{(0)} + u$ with u satisfying the recursion,

$$\sum_{\ell=1}^{n} \mathscr{F}_\ell[u](\boldsymbol{x}) = \sum_{\ell=1}^{n} \mathscr{F}_\ell[u](\boldsymbol{x} - \boldsymbol{e}_\ell) - \mathrm{d}\mathscr{F}[\widetilde{\phi}^{(0)}](\boldsymbol{x}). \tag{5.24}$$

Since $\widetilde{\phi}^{(0)} \equiv 0$ on $G_N \setminus G_N^1$, we may trivially take $u \equiv 0$ on $G_N \setminus G_N^2$ and then solve (5.24) on G_N^2 using the latter as an insulated boundary condition on $\partial G_N^2 \cap G_N$.

Interpolation of the flow inside G_N^2. We first solve (5.24) inside G_N^1. By construction, if $\boldsymbol{x} \in G_N^1$ then $\boldsymbol{x} - \boldsymbol{e}_\ell \in G_N^1 \cup \partial_A G_N^1$, for every $\ell = 1, \dots, n$. Accordingly, let us slice G_N^1 into layers $\mathbb{L}_k$ as follows: Set

$$\mathbb{L}_0 = \partial_A G_N^1, \tag{5.25}$$

and, for $k = 0, 1, \dots,$

$$\mathbb{L}_{k+1} = \left\{ \boldsymbol{x} \in G_N \;:\; \boldsymbol{x} - \boldsymbol{e}_\ell \in \bigcup_{j=0}^{k} \mathbb{L}_j \text{ for all } \ell = 1, \dots, n \right\}. \tag{5.26}$$

Since all entries of $\boldsymbol{v}$ are positive, there exists $\chi_3 = c_3(n)$ and $M \le \chi_3/\rho$, such that

$$G_N^1 = \bigcup_{j=0}^{M} \mathbb{L}_j. \tag{5.27}$$

Now define recursively, for each $\boldsymbol{x} \in \mathbb{L}_{k+1}$,

$$\mathscr{F}_\ell[u](\boldsymbol{x}) = q_\ell \left(\sum_{j=1}^{n} \mathscr{F}_j[u](\boldsymbol{x} - \boldsymbol{e}_j) - \mathrm{d}\mathscr{F}[\widetilde{\phi}^{(0)}](\boldsymbol{x}) \right), \tag{5.28}$$

where the probability distribution, $q_1, \dots, q_n$, is defined as in (5.19). Obviously, this produces a solution of (5.24). The particular choice of the constants q_ℓ in (5.19) leads to a rather miraculous looking cancelation we will encounter below.

Properties P2 and P3. We now prove recursively a bound on u that will imply that Properties **P2** and **P3** hold. Let c_k be constants such that, for all $\boldsymbol{y} \in \mathbb{L}_k$,

$$\left| \mathscr{F}_\ell[u](\boldsymbol{y}) \right| \le c_k \rho^2 \mathscr{F}_\ell[\nabla \widetilde{g}](\boldsymbol{y}). \tag{5.29}$$

Then, for $\boldsymbol{x} \in \mathbb{L}_{k+1}$, we get by construction (5.28) and in view of (5.21) that

$$\begin{aligned} \frac{|\mathscr{F}_\ell[u](\boldsymbol{x})|}{\mathscr{F}_\ell[\widetilde{\phi}^{(0)}](\boldsymbol{x})} &\le q_\ell \sum_j \frac{|\mathscr{F}_j[u](\boldsymbol{x} - \boldsymbol{e}_j)|}{\mathscr{F}_\ell[\widetilde{\phi}^{(0)}](\boldsymbol{x})} + \chi_2 \rho^2 \\ &\le \rho^2 \left(c_k q_\ell \sum_j \frac{\mathscr{F}_j[\widetilde{\phi}^{(0)}](\boldsymbol{x} - \boldsymbol{e}_j)}{\mathscr{F}_\ell[\widetilde{\phi}^{(0)}](\boldsymbol{x})} + \chi_2 \right). \end{aligned} \tag{5.30}$$

By our choice of $\phi^{(0)}$ in (5.23),

$$\begin{aligned} \frac{\mathscr{F}_j[\widetilde{\phi}^{(0)}](\boldsymbol{x} - \boldsymbol{e}_j)}{\mathscr{F}_\ell[\widetilde{\phi}^{(0)}](\boldsymbol{x})} &= \frac{\check{\boldsymbol{v}}_j}{\check{\boldsymbol{v}}_\ell} \exp\left\{ \frac{\beta N}{2} \sum_{i=2}^{n} \hat{\gamma}_i \left((\boldsymbol{x}, \boldsymbol{v}^{(i)})^2 - (\boldsymbol{x} - \boldsymbol{e}_j, \boldsymbol{v}^{(i)})^2 \right) \right\} \\ &= \frac{\check{\boldsymbol{v}}_j}{\check{\boldsymbol{v}}_\ell} \exp\left\{ \beta N \sum_{i=2}^{n} \hat{\gamma}_i (\boldsymbol{x}, \boldsymbol{v}^{(i)})(\boldsymbol{e}_j, \boldsymbol{v}^{(i)}) \right\} (1 + O(1/N)) \\ &= \frac{\check{\boldsymbol{v}}_j + 2\beta(\boldsymbol{e}_j, \hat{\boldsymbol{v}}) \sum_{i=2}^{n} (\boldsymbol{e}_j, \hat{\boldsymbol{v}}^{(i)})(\boldsymbol{x}, \boldsymbol{v}^{(i)})}{\check{\boldsymbol{v}}_\ell} \left(1 + O(\rho^2) \right). \end{aligned} \tag{5.31}$$

However, for each $i = 2, \ldots, n$,

$$\sum_{j=1}^{n} (\boldsymbol{e}_j, \hat{\boldsymbol{v}})(\boldsymbol{e}_j, \hat{\boldsymbol{v}}^{(i)}) = 0. \tag{5.32}$$

Therefore, with the choice $q_\ell = \frac{\check{\boldsymbol{v}}_\ell}{\sum_k \check{\boldsymbol{v}}_k}(1 + \mathrm{o}(1))$, we get

$$q_\ell \sum_j \frac{\mathscr{F}_j[\widetilde{\phi}^{(0)}](\boldsymbol{x} - \boldsymbol{e}_j)}{\mathscr{F}_\ell[\widetilde{\phi}^{(0)}](\boldsymbol{x})} = 1 + O(\rho^2), \tag{5.33}$$

uniformly in $\boldsymbol{x} \in G_N^1$ and $l = 1, \ldots, n$. Thus, the coefficients c_k satisfy the recursive bound

$$c_{k+1} \leq c_k \left(1 + O(\rho^2)\right) + \chi_2 \rho^2, \tag{5.34}$$

with $c_0 = 0$. Consequently, there exists a constant, c, such that

$$c_k \leq k\rho^2 c e^{kc\rho^2}, \tag{5.35}$$

and hence, since $M \leq \chi_3/\rho$, $c_M = O(\rho)$. As a result, we have constructed u on G_N^1 such that

$$\left|\mathscr{F}_\ell[u](\boldsymbol{x})\right| = O\left(\rho\right) \mathscr{F}_\ell[\nabla g](\boldsymbol{x}), \tag{5.36}$$

uniformly in $\boldsymbol{x} \in G_N^1$ and $\ell = 1, \ldots, n$. In particular, (5.15) holds uniformly in $\boldsymbol{x} \in G_N^1$ and hence, by (5.20), **P3** is satisfied on $G_N^1 \setminus G_N^0$. Moreover, since by construction $\phi \equiv 0$ on $G_N \setminus G_N^2$, **P3** is trivially satisfied in the latter domain. Hence both **P2** and **P3** hold on $G_N^1 \cup \left(G_N \setminus G_N^2\right)$.

It remains to reconstruct u on $G_N^2 \setminus G_N^1$. Since we have truncated ∇g outside G_N^1, Kirchoff's equation (5.24), for $\boldsymbol{x} \in G_N^2 \setminus G_N^1$, takes the form $\mathscr{F}[u](\boldsymbol{x}) = 0$. Therefore, whatever we do in order to reconstruct ϕ, the total flow through $G_N^2 \setminus G_N^1$ equals

$$\frac{1 + \mathrm{o}(1)}{\Phi_N(\widetilde{g})} \sum_{\boldsymbol{x} \in G_N^1} \sum_{\ell=1}^{n} \mathscr{F}_\ell[\phi](\boldsymbol{x}) \mathbb{1}_{\{\boldsymbol{x}+\boldsymbol{e}_\ell \notin G_N^1\}}. \tag{5.37}$$

By (5.36) and (5.20), the latter is of the order $O\left(\rho^{1-n} e^{-\chi_1 N^{2\delta}}\right)$. Thus, **P3** is established.

5.4 Flows from $\boldsymbol{A}$ to $\partial_{\boldsymbol{A}} G_N$ and from $\partial_{\boldsymbol{B}} G_N$ to $\boldsymbol{B}$

Let $\mathfrak{f}$ be the unit flow through G_N constructed above. We need to construct a flow

$$\mathfrak{f}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) = (1 + \mathrm{o}(1)) \frac{\mathscr{Q}_{\beta,N}(\boldsymbol{x}) r_N(\boldsymbol{x}, \boldsymbol{y})}{\Phi_N(\widetilde{g})} \phi_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) \tag{5.38}$$

from $\boldsymbol{A}$ to $\partial_{\boldsymbol{A}} G_N$ and, respectively, a flow

$$\mathfrak{f}_{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y}) = (1 + \mathrm{o}(1)) \frac{\mathscr{Q}_{\beta,N}(\boldsymbol{x}) r_N(\boldsymbol{x}, \boldsymbol{y})}{\Phi_N(\widetilde{g})} \phi_{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y}) \tag{5.39}$$

from $\partial_{\boldsymbol{B}} G_N$ to $\boldsymbol{B}$, such that (5.5) holds and, of course, such that the concatenation $\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}} = \{\mathfrak{f}_{\boldsymbol{A}}, \mathfrak{f}, \mathfrak{f}_{\boldsymbol{B}}\}$ complies with Kirchoff's law. We shall work out only the $\mathfrak{f}_{\boldsymbol{A}}$-case, the $\mathfrak{f}_{\boldsymbol{B}}$-case is completely analogous.

The expressions for $\Phi_N(\widetilde{g})$ and $\mathcal{Q}_{\beta,N}(\boldsymbol{x})$ appear on the right-hand sides of (4.50) and (3.13). For the rest we need only rough bounds: There exists a constant $L = L(n)$, such that we are able to rewrite (5.38) as,

$$\phi_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) = \frac{(1+\mathrm{o}(1))\Phi_N(\widetilde{g})\mathfrak{f}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})}{\mathcal{Q}_{\beta,N}(\boldsymbol{x})r_N(\boldsymbol{x},\boldsymbol{y})} \leq LN^{n/2+1}e^{-N(F_{\beta,N}(\boldsymbol{z}^*)-F_{\beta,N}(\boldsymbol{x}))}. \tag{5.40}$$

This would imply a uniform stretched exponentially small upper bound on $\phi_{\boldsymbol{A}}$ at points $\boldsymbol{x}$ which are mesoscopically away from $\boldsymbol{z}^*$ in the direction of $\nabla F_{\beta,N}$, for example for $\boldsymbol{x}$ satisfying

$$F_{\beta,N}(\boldsymbol{z}^*) - F_{\beta,N}(\boldsymbol{x}) > cN^{2\delta-1}. \tag{5.41}$$

With the above discussion in mind let us try to construct $\mathfrak{f}_{\boldsymbol{A}}$ in such a way that it charges only bonds $(\boldsymbol{x},\boldsymbol{y})$ for which (5.41) is satisfied. Actually we shall do much better and give a more or less explicit construction of the part of $\mathfrak{f}_{\boldsymbol{A}}$ which flows through G_N^0: Namely, with each point $\boldsymbol{x} \in \partial_{\boldsymbol{A}} G_N^0$ we shall associate a nearest neighbor path $\gamma^{\boldsymbol{x}} = (\gamma^{\boldsymbol{x}}(-k_A(\boldsymbol{x})),\dots,\gamma^{\boldsymbol{x}}(0))$ on Γ_N^n such that (5.41) holds for all $\boldsymbol{y} \in \gamma^{\boldsymbol{x}}$ and,

$$\gamma^{\boldsymbol{x}}(-k_A(\boldsymbol{x})) \in \boldsymbol{A},\ \gamma^{\boldsymbol{x}}(0) = \boldsymbol{x} \quad \text{and} \quad m(\gamma^{\boldsymbol{x}}(\cdot+1)) = m(\gamma^{\boldsymbol{x}}(\cdot)) + 2/N. \tag{5.42}$$

The flow from $\boldsymbol{A}$ to $\partial_{\boldsymbol{A}} G_N^0$ will be then defined as

$$\mathfrak{f}_{\boldsymbol{A}}(\boldsymbol{e}) = \sum_{\boldsymbol{x}\in\partial_{\boldsymbol{A}} G_N^0} \mathbb{1}_{\{\boldsymbol{e}\in\gamma^{\boldsymbol{x}}\}} \sum_{\ell\in I_{G_N}(\boldsymbol{x})} \mathfrak{f}_\ell(\boldsymbol{x}). \tag{5.43}$$

By construction $\mathfrak{f}_{\boldsymbol{A}}$ above satisfies the Kirchoff's law and matches with the flow $\mathfrak{f}$ through G_N on $\partial_{\boldsymbol{A}} G_N^0$. Strictly speaking, we should also specify how one extends $\mathfrak{f}$ on the remaining part $\partial_{\boldsymbol{A}} G_N \setminus \partial_{\boldsymbol{A}} G_N^0$. But this is irrelevant: Whatever we do the $\mathbb{P}_N^{\mathfrak{f}_{A,B}}$-probability of passing through $\partial_{\boldsymbol{A}} G_N \setminus \partial_{\boldsymbol{A}} G_N^0$ is equal to

$$\sum_{\boldsymbol{x}\in\partial_{\boldsymbol{A}} G_N\setminus\partial_{\boldsymbol{A}} G_N^0} \sum_\ell \mathfrak{f}_\ell(\boldsymbol{x}) = \mathrm{o}(1). \tag{5.44}$$

It remains, therefore, to construct the family of paths $\{\gamma^{\boldsymbol{x}}\}$ such that (5.41) holds.

Each such path $\gamma^{\boldsymbol{x}}$ will be constructed as a concatenation $\gamma^{\boldsymbol{x}} = \hat{\gamma} \cup \eta^{\boldsymbol{x}}$.

STEP 1 Construction of $\hat{\gamma}$. Pick δ such that $\delta - 1 < m_A = m(\boldsymbol{m}_{\boldsymbol{A}})$ and consider the part $\hat{\boldsymbol{x}}[\delta-1, z^*]$ of the minimal energy curve as described in (3.31). Let γ be a nearest neighbor Γ_N^n-approximation of $\hat{\boldsymbol{x}}[\delta-1, z^*]$, which in addition satisfies $m(\hat{\gamma}(\cdot+1)) = m(\hat{\gamma}(\cdot)) + 2/N$. Since by (3.34) the curve $\hat{\boldsymbol{x}}[\delta-1, z^*]$ is coordinate-wise increasing, the Hausdorff distance between $\hat{\gamma}$ and $\hat{\boldsymbol{x}}[\delta-1, z^*]$ is at most $2\sqrt{n}/N$. Let $\boldsymbol{x}^{\boldsymbol{A}}$ be the first point where γ hits the set $D_N(\rho)$, and let $\boldsymbol{u}^{\boldsymbol{A}}$ be the last point where γ hits $\boldsymbol{A}$ (we assume now that the neighborhood $\boldsymbol{A}$ is sufficiently large so that $\boldsymbol{u}^{\boldsymbol{A}}$ is well defined). Then $\hat{\gamma}$ is just the portion of γ from $\boldsymbol{u}^{\boldsymbol{A}}$ to $\boldsymbol{x}^{\boldsymbol{A}}$.

STEP 2 Construction of $\eta^{\boldsymbol{x}}$. At this stage we assume that the parameter ν in (5.6) is so small that G_N lies deeply inside $D_N(\rho)$. In particular, we may assume that

$$F_{\beta,N}(\boldsymbol{x}^{\boldsymbol{A}}) < \min\left\{F_{\beta,N}(\boldsymbol{x}) \,:\, \boldsymbol{x} \in \partial_A G_N^0\right\},$$

and, in view of (3.34), we may also assume that

$$\boldsymbol{x}_\ell^{\boldsymbol{A}} < \boldsymbol{x}_\ell \quad \forall \boldsymbol{x} \in \partial_A G_N^0 \text{ and } \ell = 1, \ldots, n. \tag{5.45}$$

Therefore, $\boldsymbol{x} - \boldsymbol{x}^{\boldsymbol{A}}$ has strictly positive entries and, as it now follows from (4.29),

$$\left(\mathbb{A}\check{\boldsymbol{v}}, \boldsymbol{x} - \boldsymbol{x}^{\boldsymbol{A}}\right) = \left(\boldsymbol{v}, \boldsymbol{x} - \boldsymbol{x}^{\boldsymbol{A}}\right) > 0.$$

By construction G_N^0 is a small tube in the direction of $\check{\boldsymbol{v}}$. Accordingly, we may assume that $\left(\mathbb{A}\boldsymbol{x}, \boldsymbol{x} - \boldsymbol{x}^{\boldsymbol{A}}\right) > 0$ uniformly on $\partial_A G_N^0$. But this means that the function

$$t : [0,1] \mapsto \left(\mathbb{A}(\boldsymbol{x}^{\boldsymbol{A}} + t(\boldsymbol{x} - \boldsymbol{x}^{\boldsymbol{A}}), (\boldsymbol{x}^{\boldsymbol{A}} + t(\boldsymbol{x} - \boldsymbol{x}^{\boldsymbol{A}})\right)$$

is strictly increasing. Therefore, $F_{\beta,N}$ is, up to negligible corrections, increasing on the straight line segment, $[\boldsymbol{x}^{\boldsymbol{A}}, \boldsymbol{x}] \subset \mathbb{R}^n$ which connects $\boldsymbol{x}^{\boldsymbol{A}}$ and $\boldsymbol{x}$. Then, our target path $\eta^{\boldsymbol{x}}$ is a nearest neighbor Γ_N^n-approximation of $[\boldsymbol{x}^{\boldsymbol{A}}, \boldsymbol{x}]$ which runs from $\boldsymbol{x}^{\boldsymbol{A}}$ to $\boldsymbol{x}$. In view of the preceding discussion it is possible to prepare $\eta^{\boldsymbol{x}}$ in such a way that $F_{\beta,N}(\boldsymbol{z}^*) - F_{\beta,N}(\cdot) > cN^{2\delta-1}$ along $\eta^{\boldsymbol{x}}$. Moreover, by (5.45) it is possible to ensure that the total magnetization is increasing along $\eta^{\boldsymbol{x}}$.

This concludes the construction of a flow $\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}$ satisfying 5.3. □

In the sequel we shall index vertices of $\gamma^{\boldsymbol{x}} = \hat{\gamma} \cup \eta^{\boldsymbol{x}}$ as,

$$\gamma^{\boldsymbol{x}} = \left(\hat{\gamma}^{\boldsymbol{x}}(-k_{\boldsymbol{A}}), \ldots \hat{\gamma}^{\boldsymbol{x}}(0)\right). \tag{5.46}$$

Since,

$$F_{\beta,N}(\boldsymbol{y}) \leq F_{\beta,N}(\boldsymbol{z}^*) - c_1 \left(\boldsymbol{y} - \boldsymbol{z}^*, \boldsymbol{v}\right)^2, \tag{5.47}$$

for every $\boldsymbol{y}$ lying on the minimal energy curve $\hat{\boldsymbol{x}}[\delta - 1, z^*]$ and since the Hessian of $F_{\beta,N}$ is uniformly bounded on $\hat{\boldsymbol{x}}[\delta - 1, z^*]$, we conclude that if v_0 is chosen small enough, then there exists $c_2 > 0$ such that

$$F_{\beta,N}(\gamma^{\boldsymbol{x}}(\cdot)) \leq F_{\beta,N}(\boldsymbol{z}^*) - c_2 \left(\gamma^{\boldsymbol{x}}(\cdot) - \boldsymbol{z}^*, \boldsymbol{v}\right)^2, \tag{5.48}$$

uniformly in $\boldsymbol{x} \in \partial_{\boldsymbol{A}} G_N^0$. Finally, since the entries of $\boldsymbol{v}$ are uniformly strictly positive, it follows from (5.48) that,

$$F_{\beta,N}(\gamma^{\boldsymbol{x}}(-k)) \leq F_{\beta,N}(\boldsymbol{z}^*) - c_3 \frac{(N^{1/2+\delta} + k)^2}{N^2}, \tag{5.49}$$

uniformly in $\boldsymbol{x} \in \partial_{\boldsymbol{A}}$ and $k \in \{0, \ldots, k_{\boldsymbol{A}}(\boldsymbol{x})\}$.

5.5 Lower bound on $\mathrm{cap}(A,B)$ via microscopic flows

Recall that $\boldsymbol{A}$ and $\boldsymbol{B}$ are mesoscopic neighborhoods of two minima of $F_{\beta,N}$, $\boldsymbol{z}^*$ is the corresponding saddle point, and $A = \mathscr{S}_N[\boldsymbol{A}]$, $B = \mathscr{S}_N[\boldsymbol{B}]$ are the microscopic counterparts of $\boldsymbol{A}$ and $\boldsymbol{B}$. Let $\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}} = \{\mathfrak{f}_{\boldsymbol{A}}, \mathfrak{f}, \mathfrak{f}_{\boldsymbol{B}}\}$ be the mesoscopic flow from $\boldsymbol{A}$ to $\boldsymbol{B}$ constructed above. In this section we are going to construct a subordinate microscopic flow, $f_{A,B}$, from A to B. In the sequel, given a microscopic bond, $b = (\sigma, \sigma')$, we use $\boldsymbol{e}(b) = (\boldsymbol{m}(\sigma), \boldsymbol{m}(\sigma'))$ for its mesoscopic pre-image. Our subordinate flow will satisfy

$$\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{e}) = \sum_{b:\boldsymbol{e}(b)=\boldsymbol{e}} f_{A,B}(b). \tag{5.50}$$

In fact, we are going to employ a much more stringent notion of subordination on the level of induced Markov chains: Let us label the realizations of the mesoscopic chain $\mathscr{X}_{\boldsymbol{A},\boldsymbol{B}}$ as $\underline{\boldsymbol{x}} = \left(\boldsymbol{x}_{-\ell_A}, \dots, \boldsymbol{x}_{\ell_B}\right)$, in such a way that $\boldsymbol{x}_{-\ell_A} \in \boldsymbol{A}$, $\boldsymbol{x}_{\ell_B} \in \boldsymbol{B}$, and $m(\boldsymbol{x}_0) = m(\boldsymbol{z}^*)$. If $\boldsymbol{e}$ is a mesoscopic bond, we write $\boldsymbol{e} \in \underline{\boldsymbol{x}}$ if $\boldsymbol{e} = (\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1})$ for some $\ell = -\ell_A, \dots, \ell_B - 1$. To each path, $\underline{\boldsymbol{x}}$, of positive probability, we associate a subordinate microscopic *unit flow*, $f^{\underline{\boldsymbol{x}}}$, such that

$$f^{\underline{\boldsymbol{x}}}(b) > 0 \text{ if and only if } \boldsymbol{e}(b) \in \underline{\boldsymbol{x}}. \tag{5.51}$$

Then the total microscopic flow, $f_{A,B}$, can be decomposed as

$$f_{A,B} = \sum_{\underline{\boldsymbol{x}}} \mathbb{P}_N^{\mathfrak{f}_{A,B}} \left(\mathscr{X}_{\boldsymbol{A},\boldsymbol{B}} = \underline{\boldsymbol{x}}\right) f^{\underline{\boldsymbol{x}}}. \tag{5.52}$$

Evidently, (5.50) is satisfied: By construction,

$$\sum_{b:\boldsymbol{e}(b)=\boldsymbol{e}} f^{\underline{\boldsymbol{x}}}(b) = 1 \text{ for every } \underline{\boldsymbol{x}} \text{ and each } \boldsymbol{e} \in \underline{\boldsymbol{x}}. \tag{5.53}$$

On the other hand, $\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{e}) = \sum_{\underline{\boldsymbol{x}}} \mathbb{P}_N^{\mathfrak{f}_{A,B}} \left(\mathscr{X}_{\boldsymbol{A},\boldsymbol{B}} = \underline{\boldsymbol{x}}\right) \mathbb{1}_{\{\boldsymbol{e} \in \underline{\boldsymbol{x}}\}}$.
Therefore, (5.52) gives rise to the following decomposition of unity,

$$\mathbb{1}_{\{f_{A,B}(b)>0\}} = \sum_{\underline{\boldsymbol{x}} \ni \boldsymbol{e}(b)} \sum_{\underline{\sigma} \ni b} \frac{\mathbb{P}_N^{\mathfrak{f}_{A,B}} \left(\mathscr{X}_{\boldsymbol{A},\boldsymbol{B}} = \underline{\boldsymbol{x}}\right) \mathbb{P}^{\underline{\boldsymbol{x}}} \left(\Sigma = \underline{\sigma}\right)}{\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{e}(b)) f^{\underline{\boldsymbol{x}}}(b)}, \tag{5.54}$$

where $(\mathbb{P}^{\underline{\boldsymbol{x}}}, \Sigma)$ is the *microscopic* Markov chain from A to B which is associated to the flow $f^{\underline{\boldsymbol{x}}}$.
Consequently, our general lower bound (2.24) implies that

$$\begin{aligned} \operatorname{cap}(A,B) &\geq \sum_{\underline{\boldsymbol{x}}} \mathbb{P}_N^{\mathfrak{f}_{A,B}} \left(\mathscr{X}_{\boldsymbol{A},\boldsymbol{B}} = \underline{\boldsymbol{x}}\right) \mathbb{E}^{\underline{\boldsymbol{x}}} \left\{ \sum_{\ell=-\ell_A}^{\ell_B-1} \frac{\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1}) f^{\underline{\boldsymbol{x}}}(\sigma_\ell, \sigma_{\ell+1})}{\mu_{\beta,N}(\sigma_\ell) p_N(\sigma_\ell, \sigma_{\ell+1})} \right\}^{-1} \\ &\geq \sum_{\underline{\boldsymbol{x}}} \mathbb{P}_N^{\mathfrak{f}_{A,B}} \left(\mathscr{X}_{\boldsymbol{A},\boldsymbol{B}} = \underline{\boldsymbol{x}}\right) \left\{ \mathbb{E}^{\underline{\boldsymbol{x}}} \sum_{\ell=-\ell_A}^{\ell_B-1} \frac{\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1}) f^{\underline{\boldsymbol{x}}}(\sigma_\ell, \sigma_{\ell+1})}{\mu_{\beta,N}(\sigma_\ell) p_N(\sigma_\ell, \sigma_{\ell+1})} \right\}^{-1} \end{aligned} \tag{5.55}$$

We need to recover $\Phi_N(\widetilde{g})$ from the latter expression. In view of (5.1), write,

$$\begin{aligned} \frac{\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1}) f^{\underline{\boldsymbol{x}}}(\sigma_\ell, \sigma_{\ell+1})}{\mu_{\beta,N}(\sigma_\ell) p_N(\sigma_\ell, \sigma_{\ell+1})} &= \frac{\phi_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1})}{\Phi_N(\widetilde{g})} \\ &\times \frac{\mathscr{Q}_{\beta,N}(\boldsymbol{x}_\ell) r_N(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1}) f^{\underline{\boldsymbol{x}}}(\sigma_\ell, \sigma_{\ell+1})}{\mu_{\beta,N}(\sigma_\ell) p_N(\sigma_\ell, \sigma_{\ell+1})}. \end{aligned} \tag{5.56}$$

Since we prove lower bounds, we may restrict attention to a subset of *good* realizations $\underline{\boldsymbol{x}}$ of the mesoscopic chain $\mathscr{X}_{\boldsymbol{A},\boldsymbol{B}}$ whose $\mathbb{P}_N^{\mathfrak{f}_{A,B}}$ -probability is close to one. In particular, (5.4) and (5.5) insure that the first term in the above product is precisely what we need. The remaining effort, therefore, is to find a judicious choice of $f^{\underline{\boldsymbol{x}}}$ such that the second factor in (5.56) is close to one. To this end we need some additional notation: Given a mesoscopic trajectory $\underline{\boldsymbol{x}} = (\boldsymbol{x}_{-\ell_A}, \ldots, \boldsymbol{x}_{\ell_B})$, define $k = k(\ell)$ as the direction of the increment of ℓ-th jump. That is, $\boldsymbol{x}_{\ell+1} = \boldsymbol{x}_\ell + \boldsymbol{e}_k$. On the microscopic level such a transition corresponds to a flip of a spin from the Λ_k slot. Thus, recalling the notation $\Lambda_k^{\pm}(\sigma) \equiv \{i \in \Lambda_k \ : \ \sigma(i) = \pm 1\}$, we have that, if $\sigma_\ell \in \mathscr{S}_N[\boldsymbol{x}_\ell]$ and $\sigma_{\ell+1} \in \mathscr{S}_N[\boldsymbol{x}_{\ell+1}]$, then $\sigma_{\ell+1} = \theta_i^+ \sigma_\ell$ for some $i \in \Lambda^-_{k(\ell)}(\sigma_\ell)$. By our choice of transition probabilities, p_N, and their mesoscopic counterparts, r_N, in (4.2),

$$\frac{r_N(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1})}{p_N(\sigma_\ell, \sigma_{\ell+1})} = \left|\Lambda^-_{k(\ell)}(\sigma_\ell)\right| (1 + O(\epsilon)), \tag{5.57}$$

uniformly in ℓ and in all pairs of neighbors $\sigma_\ell, \sigma_{\ell+1}$. Note that the cardinality, $\left|\Lambda^-_{k(\ell)}(\sigma_\ell)\right|$, is the same for all $\sigma_\ell \in \mathscr{S}_N[\boldsymbol{x}_\ell]$.

For $\boldsymbol{x} \in \Gamma_N^n$, define the canonical measure,

$$\mu^{\boldsymbol{x}}_{\beta,N}(\sigma) = \frac{\mathbb{1}_{\{\sigma \in \mathscr{S}_N[\boldsymbol{x}]\}} \mu_{\beta,N}(\sigma)}{\mathscr{Q}_{\beta,N}(\boldsymbol{x})}. \tag{5.58}$$

The second term in (5.56) is equal to

$$\frac{f^{\underline{\boldsymbol{x}}}(\sigma_\ell, \sigma_{\ell+1})}{\mu^{\boldsymbol{x}_\ell}_{\beta,N}(\sigma_\ell) \cdot 1/\left|\Lambda^-_{k(\ell)}(\sigma_\ell)\right|} (1 + O(\epsilon)). \tag{5.59}$$

If the magnetic fields, h, were constant on each set I_k, then we could chose the flow $f^{\underline{\boldsymbol{x}}}(\sigma_\ell, \sigma_{\ell+1}) = \mu^{\boldsymbol{x}_\ell}_{\beta,N}(\sigma_\ell) \cdot 1/\left|\Lambda^-_{k(\ell)}(\sigma_\ell)\right|$, and consequently we would be done. In the general case of continuous distribution of h, this is not the case. However, since the fluctuations of h are bounded by $1/n$, we can hope to construct $f^{\underline{\boldsymbol{x}}}$ in such a way that the ratio in (5.59) is kept very close to one.

Construction of $f^{\underline{\boldsymbol{x}}}$. We construct now a Markov chain, $\mathbb{P}^{\underline{\boldsymbol{x}}}$, on microscopic trajectories, $\Sigma = \{\sigma_0, \ldots, \sigma_{\ell_B}\}$, from $\mathscr{S}[\boldsymbol{x}_0]$ to B, such that $\sigma_\ell \in \mathscr{S}[\boldsymbol{x}_\ell]$, for all $\ell = 0, \ldots, \ell_B$. The microscopic flow, $f^{\underline{\boldsymbol{x}}}$, is then defined through the identity $\mathbb{P}^{\underline{\boldsymbol{x}}}(b \in \Sigma) = f^{\underline{\boldsymbol{x}}}(b)$.

The construction of a microscopic flow from A to $\mathscr{S}[\boldsymbol{x}_0]$ is completely similar (it is just the reversal of the above) and we will omit it.

We now construct $\mathbb{P}^{\underline{\boldsymbol{x}}}$.

STEP 1. Marginal distributions: For each $\ell = 0, \ldots, \ell_B$ we use $\nu^{\underline{\boldsymbol{x}}}_\ell$ to denote the marginal distribution of σ_ℓ under $\mathbb{P}^{\underline{\boldsymbol{x}}}$. The measures $\nu^{\underline{\boldsymbol{x}}}_\ell$ are concentrated on $\mathscr{S}[\boldsymbol{x}_\ell]$. The initial measure, $\nu^{\underline{\boldsymbol{x}}}_0$, is just the canonical measure $\mu^{\boldsymbol{x}_0}_{\beta,N}$. The measures $\nu^{\underline{\boldsymbol{x}}}_{\ell+1}$ are then defined through the recursive equations

$$\nu^{\underline{\boldsymbol{x}}}_{\ell+1}(\sigma_{\ell+1}) = \sum_{\sigma_\ell \in \mathscr{S}[\boldsymbol{x}_\ell]} \nu^{\underline{\boldsymbol{x}}}_\ell(\sigma) q_\ell(\sigma_\ell, \sigma_{\ell+1}). \tag{5.60}$$

STEP 2. Transition probabilities. The transition probabilities, $q_\ell(\sigma_\ell, \sigma_{\ell+1})$, in (5.60) are defined in the following way: As we have already remarked, all the microscopic jumps are of the form $\sigma_\ell \mapsto \theta_j^+ \sigma_\ell$, for some $j \in \Lambda^-_{k(\ell)}(\sigma)$, where θ_j^+ flips the j-th spin from -1 to 1. For such a flip define

$$q_\ell(\sigma_\ell, \theta_j^+ \sigma_\ell) = \frac{e^{2\beta \tilde{h}_j}}{\sum_{i \in \Lambda_k^-(\sigma_\ell)} e^{2\beta \tilde{h}_i}}. \tag{5.61}$$

Then the microscopic flow through an admissible bound, $b = (\sigma_\ell, \sigma_{\ell+1})$, is equal to

$$f^{\underline{x}}(\sigma_\ell, \sigma_{\ell+1}) = \mathbb{P}^{\underline{x}}(b \in \Sigma) = \nu_\ell^{\underline{x}}(\sigma_\ell) q_\ell(\sigma_\ell, \sigma_{\ell+1}) = \frac{\nu_\ell^{\underline{x}}(\sigma_\ell)}{\left|\Lambda_{k(\ell)}^-(\sigma_\ell)\right|}(1 + O(\epsilon)). \tag{5.62}$$

Consequently, the expression in (5.59), and hence the second term in (5.56), is equal to

$$\frac{\nu_\ell^{\underline{x}}(\sigma_\ell)}{\mu_{\beta,N}^{\boldsymbol{x}_\ell}(\sigma_\ell)}(1 + O(\epsilon)) \equiv \Psi_\ell(\sigma_\ell)(1 + O(\epsilon)). \tag{5.63}$$

Main result. We claim that there exists a set, $\mathscr{T}_{\boldsymbol{A},\boldsymbol{B}}$, of *good* mesoscopic trajectories from $\boldsymbol{A}$ to $\boldsymbol{B}$, such that

$$\mathbb{P}_N^{\mathfrak{f}_{A,B}}\left(\mathscr{X}_{\boldsymbol{A},\boldsymbol{B}} \in \mathscr{T}_{\boldsymbol{A},\boldsymbol{B}}\right) = 1 - \mathrm{o}(1), \tag{5.64}$$

and, uniformly in $\underline{\boldsymbol{x}} \in \mathscr{T}_{\boldsymbol{A},\boldsymbol{B}}$,

$$\mathbb{E}^{\underline{x}}\left(\sum_{\ell=-\ell_A}^{\ell_B-1} \Psi_\ell(\sigma_\ell)\phi_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1})\right) \le 1 + O(\epsilon). \tag{5.65}$$

This will imply that,

$$\mathrm{cap}(A,B) \ge \Phi_N(\widetilde{g})(1 - O(\epsilon)), \tag{5.66}$$

which is the lower bound necessary to prove Theorem 1.3.

The rest of the Section is devoted to the proof of (5.65). First of all we derive recursive estimates on Ψ_ℓ for a given realization, $\underline{\boldsymbol{x}}$, of the mesoscopic chain. After that it will be obvious how to define $\mathscr{T}_{\boldsymbol{A},\boldsymbol{B}}$.

5.6 Propagation of errors along microscopic paths

Let $\underline{\boldsymbol{x}}$ be given. Notice that $\mu_{\beta,N}^{\boldsymbol{x}_\ell}$ is the product measure,

$$\mu_{\beta,N}^{\boldsymbol{x}_\ell} = \bigotimes_{j=1}^{n} \mu_{\beta,N}^{\boldsymbol{x}_\ell(j)}, \tag{5.67}$$

where $\mu_{\beta,N}^{\boldsymbol{x}_\ell(j)}$ is the corresponding canonical measure on the mesoscopic slot $\mathscr{S}_N^{(j)} = \{-1,1\}^{\Lambda_j}$. On the other hand, according to (5.61), the *big* microscopic chain Σ splits into a direct product of *n small* microscopic chains, $\Sigma^{(1)}, \dots, \Sigma^{(n)}$, which independently evolve on $\mathscr{S}_N^{(1)}, \dots, \mathscr{S}_N^{(n)}$. Thus, $k(\ell) = k$ means that the ℓ-th step of the mesoscopic chain induces a step of the k-th small microscopic chain $\Sigma^{(k)}$. Let $\tau_1[\ell], \dots, \tau_n[\ell]$ be the numbers of steps performed by each of the small microscopic chains after ℓ steps of the mesoscopic chain or, equivalently, after ℓ steps of the big microscopic chain Σ. Then the corrector, Ψ_ℓ, in (5.63) equals

$$\Psi_\ell\left(\sigma_\ell\right) = \prod_{j=1}^{n} \psi_{\tau_j[\ell]}^{(j)}(\sigma_\ell^{(j)}), \tag{5.68}$$

where $\sigma_\ell^{(j)}$ is the projection of σ_ℓ on $\mathscr{S}_N^{(j)}$. Therefore we are left with two separate tasks: On the microscopic level we need to control the propagation of errors along *small* chains and, on the mesoscopic level, we need to control the statistics of $\tau_1[\ell],\dots,\tau_n[\ell]$. The latter task is related to characterizing the set, $\mathscr{T}_{\mathbf{A},\mathbf{B}}$, of *good* mesoscopic trajectories and it is relegated to Subsection 5.7

Small microscopic chains. It would be convenient to study the propagation of errors along small microscopic chains in the following slightly more general context: Fix $1 \ll M \in \mathbb{N}$ and $0 \le \epsilon \ll 1$. Let $g_1,\dots,g_M \in [-1,1]$. Consider spin configurations, $\xi \in \mathscr{S}_M = \{-1,1\}^M$, with product weights

$$w(\xi) = \mathrm{e}^{\epsilon \sum_i g_i \xi(i)}. \tag{5.69}$$

As before, let $\Lambda^{\pm}(\xi) = \{i : \xi(i) = \pm 1\}$. Define layers of fixed magnetization, $\mathscr{S}_M[K] = \left\{\xi \in \mathscr{S}_M \;:\; \left|\Lambda^+(\xi)\right| = K\right\}$. Finally, fix $\delta_0,\delta_1 \in (0,1)$, such that $\delta_0 < \delta_1$.

Set $K_0 = \lfloor \delta_0 M \rfloor$ and $r = \lfloor (\delta_1 - \delta_0) M \rfloor$. We consider a Markov chain, $\Xi = \left\{\Xi_0,\dots,\Xi_r\right\}$ on $\mathscr{S}_M$, such that $\Xi_\tau \in \mathscr{S}_M[K_0+\tau] \equiv \mathscr{S}_M^\tau$ for $\tau = 0,1,\dots,r$. Let μ_τ be the canonical measure,

$$\mu_\tau(\xi) = \frac{w(\xi)\mathbb{1}_{\{\xi \in \mathscr{S}_M^\tau\}}}{Z_\tau}. \tag{5.70}$$

We take $\nu_0 = \mu_0$ as the initial distribution of Ξ_0 and, following (5.61), we define transition rates,

$$q_\tau(\xi_\tau, \theta_j^+ \xi_\tau) = \frac{\mathrm{e}^{2\epsilon g_j}}{\sum_{i \in \Lambda^-(\xi_\tau)} \mathrm{e}^{2\epsilon g_i}}. \tag{5.71}$$

We denote by $\mathbb{P}$ the law of this Markov chain and let ν_τ be the distribution of Ξ_τ (which is concentrated on $\mathscr{S}_M^\tau$), that is, $\nu_\tau(\xi) = \mathbb{P}\left(\Xi_\tau = \xi\right)$. The propagation of errors along paths of our chain is then quantified in terms of $\psi_\tau(\cdot) \equiv \nu_\tau(\cdot)/\mu_\tau(\cdot)$.

Proposition 5.1. *For every $\tau = 1,\dots,r$ and each $\xi \in \mathscr{S}_M^\tau$ define*

$$\mathscr{B}_\tau(\xi) \equiv \sum_{i=1}^M \mathrm{e}^{2\epsilon g_i} \mathbb{1}_{\{i \in \Lambda^-(\xi)\}} \quad \text{and} \quad \mathscr{A}_\tau = \mu_\tau\left(\mathscr{B}_\tau(\cdot)\right) = \sum_{i=1}^M \mathrm{e}^{2\epsilon g_i} \mu_\tau\left(i \in \Lambda^-(\cdot)\right). \tag{5.72}$$

Then there exists $c = c(\delta_0,\delta_1)$ such that the following holds: For any trajectory, $\underline{\xi} = (\xi_0,\dots,\xi_r)$, of positive probability under $\mathbb{P}$, it holds that

$$\psi_\tau(\xi_\tau) \le \left[\frac{\mathscr{A}_0}{\mathscr{B}_0(\xi_0)}\right]^\tau \mathrm{e}^{c\epsilon\tau^2/M}, \tag{5.73}$$

for all $\tau = 0,1,\dots,r$.

Proof. By construction, $\psi_0 \equiv 1$. Let $\xi_{\tau+1} \in \mathscr{S}_M^{\tau+1}$. Since ν_τ satisfies the recursion

$$\nu_{\tau+1}(\xi_{\tau+1}) = \sum_{j \in \Lambda^+(\xi_{\tau+1})} \nu_\tau(\theta_j^- \xi_{\tau+1}) q_\tau(\theta_j^- \xi_{\tau+1}, \xi_{\tau+1}), \tag{5.74}$$

it follows that ψ_τ satisfies

$$\begin{aligned}\psi_{\tau+1}(\xi_{\tau+1}) &= \sum_{j\in\Lambda^+(\xi_{\tau+1})} \frac{\nu_\tau(\theta_j^-\xi_{\tau+1})q_\tau(\theta_j^-\xi_{\tau+1},\xi_{\tau+1})}{\mu_{\tau+1}(\xi_{\tau+1})} \\ &= \sum_{j\in\Lambda^+(\xi_{\tau+1})} \frac{\mu_\tau(\theta_j^-\xi_{\tau+1})q_\tau(\theta_j^-\xi_{\tau+1},\xi_{\tau+1})}{\mu_{\tau+1}(\xi_{\tau+1})}\psi_\tau(\theta_j^-\xi_{\tau+1}).\end{aligned}$$

By our choice of transition probabilities in (5.71),

$$\frac{\mu_\tau(\theta_j^-\xi_{\tau+1})q_\tau(\theta_j^-\xi_{\tau+1},\xi_{\tau+1})}{\mu_{\tau+1}(\xi_{\tau+1})} = \frac{Z_{\tau+1}}{Z_\tau}\left\{\sum_{i\in\Lambda^-(\theta_j^-\xi_{\tau+1})} \mathrm{e}^{2\epsilon g_i}\right\}^{-1}. \tag{5.75}$$

Recalling that $\left|\Lambda^+(\xi_\tau)\right| \equiv \left|\Lambda_\tau^+\right| = K_0+\tau$ does not depend on the particular value of ξ_τ,

$$\begin{aligned}\frac{Z_{\tau+1}}{Z_\tau} &= \frac{1}{Z_\tau}\sum_{\xi\in\mathscr{S}_M^{\tau+1}} w(\xi) = \frac{1}{Z_\tau}\sum_{\xi\in\mathscr{S}_M^{\tau+1}} \frac{1}{\left|\Lambda^+(\xi)\right|}\sum_{j\in\Lambda^+(\xi)} w(\theta_j^-\xi)\mathrm{e}^{2\epsilon g_j} \\ &= \frac{1}{Z_\tau}\sum_{\xi\in\mathscr{S}_M^{\tau}} w(\xi)\cdot\frac{1}{\left|\Lambda_{\tau+1}^+\right|}\sum_{j\in\Lambda^-(\xi)}\mathrm{e}^{2\epsilon g_j} = \mu_\tau\left(\frac{1}{\left|\Lambda^+(\xi_{\tau+1})\right|}\sum_{j\in\Lambda^-(\cdot)}\mathrm{e}^{2\epsilon g_j}\right).\end{aligned}$$

We conclude that the right hand side of (5.75) equals

$$\frac{1}{\left|\Lambda^+(\xi_{\tau+1})\right|}\cdot\frac{\mu_\tau\left(\sum_{i\in\Lambda^-(\cdot)}\mathrm{e}^{2\epsilon g_i}\right)}{\sum_{i\in\Lambda^-(\theta_j^-\xi_{\tau+1})}\mathrm{e}^{2\epsilon g_i}} = \frac{1}{\left|\Lambda^+(\xi_{\tau+1})\right|}\cdot\frac{\mathscr{A}_\tau}{\mathscr{B}_\tau(\theta_j^-\xi_{\tau+1})}. \tag{5.76}$$

As a result,

$$\psi_{\tau+1}(\xi_{\tau+1}) = \frac{1}{\left|\Lambda_+(\xi_{\tau+1})\right|}\sum_{j\in\Lambda_+(\xi_{\tau+1})}\frac{\mathscr{A}_\tau}{\mathscr{B}_\tau(\theta_j^-\xi_{\tau+1})}\psi_\tau(\theta_j^-\xi_{\tau+1}). \tag{5.77}$$

Iterating the above procedure we arrive to the following conclusion: Consider the set, $\mathscr{D}(\xi_{\tau+1})$, of all paths, $\underline{\xi} = (\xi_0,\dots,\xi_\tau,\xi_{\tau+1})$, of positive probability from $\mathscr{S}_M^0$ to $\mathscr{S}_M^{\tau+1}$ to $\xi_{\tau+1}$. The number, $D_{\tau+1} \equiv \left|\mathscr{D}(\xi_{\tau+1})\right|$, of such paths does not depend on $\xi_{\tau+1}$. Then, since $\psi_0 \equiv 1$,

$$\psi_{\tau+1}(\xi_{\tau+1}) = \frac{1}{D_{\tau+1}}\sum_{\underline{\xi}\in\mathscr{D}(\xi_{\tau+1})}\prod_{s=0}^{\tau}\frac{\mathscr{A}_s}{\mathscr{B}_s(\xi_s)}. \tag{5.78}$$

We claim that

$$\frac{\mathscr{A}_s}{\mathscr{B}_s(\xi_s)} = \left(1+\frac{O(\epsilon)}{M}\right)\frac{\mathscr{A}_{s-1}}{\mathscr{B}_{s-1}(\xi_{s-1})}, \tag{5.79}$$

uniformly in all the quantities under consideration. Once (5.79) is verified,

$$\psi_\tau(\xi_\tau) \le \mathrm{e}^{O(\epsilon)\tau^2/M}\max_{\xi_0\sim\xi_\tau}\left[\frac{\mathscr{A}_0}{\mathscr{B}_0(\xi_0)}\right]^\tau, \tag{5.80}$$

where for $\xi_0 \in \mathscr{S}_M^0$, the relation $\xi_0 \sim \xi_\tau$ means that there is a path of positive probability from ξ_0 to ξ_τ. But all such ξ_0's differ at most in 2τ coordinates. It is then straightforward to see that if $\xi_0 \sim \xi_\tau$ and $\xi_0' \sim \xi_\tau$, then

$$\frac{\mathscr{B}_0(\xi_0)}{\mathscr{B}_0(\xi_0')} \leq \mathrm{e}^{O(\epsilon)\tau/M}, \tag{5.81}$$

and (5.73) follows.

It remains to prove (5.79). Let $\xi \in \mathscr{S}_M^s$ and $\xi' = \theta_j^- \xi \in \mathscr{S}_M^{s-1}$. Notice, first of all, that

$$\mathscr{B}_{s-1}(\xi') - \mathscr{B}_s(\xi) = \mathrm{e}^{2\epsilon g_j} = 1 + O(\epsilon). \tag{5.82}$$

Similarly,

$$\begin{aligned}\mathscr{A}_{s-1} - \mathscr{A}_s &= \sum_{i=1}^{M} \mathrm{e}^{2\epsilon g_i} \left\{\mu_{s-1}(i \in \Lambda^-) - \mu_s(i \in \Lambda^-)\right\} \\ &= 1 + \sum_{i=1}^{M} \left(\mathrm{e}^{2\epsilon g_i} - 1\right) \left\{\mu_{s-1}(i \in \Lambda^-) - \mu_s(i \in \Lambda^-)\right\}.\end{aligned}$$

By usual local limit results for independent Bernoulli variables,

$$\mu_{s-1}(i \in \Lambda^-) - \mu_s(i \in \Lambda^-) = O\left(\frac{1}{M}\right), \tag{5.83}$$

uniformly in $s = 1, \dots, r-1$ and $i = 1, \dots, M$. Hence, $\mathscr{A}_{s-1} - \mathscr{A}_s = 1 + O(\epsilon)$.

Finally, both $\mathscr{A}_{s-1}$ and $\mathscr{B}_{s-1}(\xi')$ are (uniformly) $O(M)$, whereas,

$$\mathscr{A}_{s-1} - \mathscr{B}_{s-1}(\xi') = \sum_{i=1}^{M} \left(\mathrm{e}^{2\epsilon g_i} - 1\right) \left\{\mu_{s-1}(i \in \Lambda^-) - \mathbb{1}_{\{i \in \Lambda^-(\xi')\}}\right\} = O(\epsilon) M. \tag{5.84}$$

Hence,

$$\frac{\mathscr{A}_s}{\mathscr{B}_s(\xi)} = \frac{\mathscr{A}_{s-1} - 1 + O(\epsilon)}{\mathscr{B}_{s-1}(\xi') - 1 + O(\epsilon)} = \frac{\mathscr{A}_{s-1}}{\mathscr{B}_{s-1}(\xi')} \left(1 + \frac{O(\epsilon)}{M}\right), \tag{5.85}$$

which is (5.79). □

Back to the big microscopic chain. Going back to (5.68) we infer that the corrector of the *big* chain Σ satisfies the following upper bound: Let $\underline{\sigma} = (\sigma_0, \sigma_1, \dots)$ be a trajectory of Σ (as sampled from $\mathbb{P}_{\underline{x}}$). Then, for every $\ell = 0, 1, \dots, \ell_B - 1$,

$$\Psi_\ell(\sigma_\ell) \leq \exp\left\{c\epsilon \sum_{j=1}^{n} \frac{\tau_j[\ell]^2}{M_j}\right\} \prod_{j=1}^{n} \left[\frac{\mathscr{A}_0^{(j)}}{\mathscr{B}_0^{(j)}(\sigma_0^{(j)})}\right]^{\tau_j[\ell]}, \tag{5.86}$$

where $M_j = \left|\Lambda_j\right| = \rho_j N$,

$$\mathscr{A}_0^{(j)} = \sum_{i \in \Lambda_j} \mathrm{e}^{2\tilde{h}_i} \mu_{\beta,N}^{\boldsymbol{x}_0(j)}\left(i \in \Lambda_j^-\right), \quad \text{and} \quad \mathscr{B}_0^{(j)}(\sigma_0^{(j)}) = \sum_{i \in \Lambda_j} \mathrm{e}^{2\tilde{h}_i} \mathbb{1}_{\left\{i \in \Lambda_j^-(\sigma_0^{(j)})\right\}}. \tag{5.87}$$

Of course, $\mathscr{A}_0^{(j)} = \mu_{\beta,N}^{\boldsymbol{x}_0(j)}\left(\mathscr{B}_0^{(j)}\right)$. It is enough to control the first order approximation,

$$\left[\frac{\mathscr{A}_0^{(j)}}{\mathscr{B}_0^{(j)}(\sigma_0^{(j)})}\right]^{\tau_j[\ell]} \approx \exp\left\{-\tau_j[\ell]\frac{\mathscr{B}_0^{(j)}(\sigma_0^{(j)}) - \mathscr{A}_0^{(j)}}{\mathscr{B}_0^{(j)}(\sigma_0^{(j)})}\right\} \equiv \exp\left(\tau_j[\ell]Y_j\right). \tag{5.88}$$

The variables $Y_1, \dots, Y_n$ are independent once $\boldsymbol{x}_0$ is fixed. Thus, in view of our target, (5.65), we need to derive an upper bound of order $(1 + O(\epsilon))$ for

$$\begin{aligned}
&\mathbb{E}^{\underline{\boldsymbol{x}}} \sum_{\ell=0}^{\ell_B-1} \exp\left\{c\epsilon \sum_{j=1}^n \frac{\tau_j[\ell]^2}{M_j} + \sum_{j=1}^n \tau_j[\ell]Y_j\right\} \phi_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1}) \\
&\quad = \sum_{\ell=0}^{\ell_B-1} \exp\left\{c\epsilon \sum_{j=1}^n \frac{\tau_j[\ell]^2}{M_j}\right\} \prod_1^n \mu_{\beta,N}^{\boldsymbol{x}_0(j)}\left(\mathrm{e}^{\tau_j[\ell]Y_j}\right) \phi_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1}),
\end{aligned} \tag{5.89}$$

which holds with $\mathbb{P}_N^{\mathfrak{f}_{A,B}}$-probability of order $1 - O(\epsilon)$.

5.7 Good mesoscopic trajectories

A look at (5.89) reveals what is to be expected from *good* mesoscopic trajectories. First of all, we may assume that it passes through the tube G_N^0 (see (5.13)) of $\boldsymbol{z}^*$. In particular, $\boldsymbol{x}_0 \in G_N^0$. Next, by our construction of the mesoscopic chain $\mathbb{P}_N^{\mathfrak{f}_{A,B}}$, and in view of (3.20) and (3.21), the step frequencies, $\tau_j[\ell]/\ell$, are, on average, proportional to ρ_j. Therefore, there exists a constant, C_1, such that, up to exponentially negligible $\mathbb{P}_N^{\mathfrak{f}_{A,B}}$-probabilities,

$$\max_j \frac{\tau_j[\ell_B]}{M_j} \le C_1 \tag{5.90}$$

holds.

A bound on microscopic moment-generating functions. We will now use the estimate (5.90) to obtain an upper bound on the product terms in (5.89). Clearly, $\mathscr{B}_0^{(j)}(\sigma_0^{(j)}) = (1+O(\epsilon))M_j$, uniformly in j and $\sigma_0^{(j)}$. Thus, by (5.88),

$$Y_j(1 + O(\epsilon)) = \frac{1}{M_j}\sum_{i\in\Lambda_j}\left(1 - \mathrm{e}^{2\widetilde{h}_i}\right)\left(\mathbb{1}_{\{\sigma(i)=-1\}} - \mu_{\beta,N}^{\boldsymbol{x}_0(j)}(\sigma(i) = -1)\right) \equiv \widetilde{Y}_j. \tag{5.91}$$

Now, for any $t \ge 0$,

$$\ln \mu_{\beta,N}^{\boldsymbol{x}_0(j)}\left(\mathrm{e}^{t\widetilde{Y}_j}\right) \le \frac{t^2}{2M_j^2}\max_{s\le t}\mathbb{V}_{\beta,N}^{\boldsymbol{x}_0(j),s}\left(\sum_{i\in\Lambda_j}\left(1 - \mathrm{e}^{2\widetilde{h}_i}\right)\mathbb{1}_{\{\sigma(i)=-1\}}\right), \tag{5.92}$$

where $\mathbb{V}_{\beta,N}^{\boldsymbol{x}_0(j),s}$ is the variance with respect to the tilted conditional measure, $\mu_{\beta,N}^{\boldsymbol{x}_0(j),s}$, defined through

$$\mu_{\beta,N}^{\boldsymbol{x}_0(j),s}(f) \equiv \frac{\mu_{\beta,N}^{\boldsymbol{x}_0(j)}\left(f\,\mathrm{e}^{s\widetilde{Y}_j}\right)}{\mu_{\beta,N}^{\boldsymbol{x}_0(j)}\left(\mathrm{e}^{s\widetilde{Y}_j}\right)}. \tag{5.93}$$

However, $\mu_{\beta,N}^{\boldsymbol{x}_0(j),s}(\cdot)$ is again a conditional product Bernoulli measure on $\mathcal{S}_N^{(j)}$, i.e.,

$$\mu_{\beta,N}^{\boldsymbol{x}_0(j),s}(\cdot) = \bigotimes_{i\in\Lambda_j} \mathbb{B}_{p_i(\epsilon,s)}\left(\ \cdot\ \Big|\sum_{i\in\Lambda_j}\sigma(i) = N\boldsymbol{x}_0(j)\right), \tag{5.94}$$

where

$$p_i(\epsilon,s) = \frac{\mathrm{e}^{\widetilde{h}_i}}{\mathrm{e}^{\widetilde{h}_i} + \mathrm{e}^{-\widetilde{h}_i + \frac{s}{M_j}(1-\mathrm{e}^{2\widetilde{h}_i})}}. \tag{5.95}$$

By (5.90) we need to consider only the case $s/M_j \le C_1$. Evidently, there exists $\delta_1 > 0$, such that,

$$\delta_1 \le \min_j \min_{s\le C_1 M_j} \min_{i\in\Lambda_j} p_i(\epsilon,s) \le \max_j \max_{s\le C_1 M_j} \max_{i\in\Lambda_j} p_i(\epsilon,s) \le 1-\delta_1. \tag{5.96}$$

On the other hand, since $\boldsymbol{x}_0 \in G_N^0$, there exists $\delta_2 > 0$, such that

$$\delta_2 \le \min_j \frac{N\boldsymbol{x}_0(j)}{M_j} \le \max_j \frac{N\boldsymbol{x}_0(j)}{M_j} \le 1-\delta_2. \tag{5.97}$$

We use the following general covariance bound for product of Bernoulli measures, which can be derived from local limit results in a straightforward, albeit painful manner.

Lemma 5.2. *Let $\delta_1 > 0$ and $\delta_2 > 0$ be fixed. Then, there exists a constant, $C = C(\delta_1,\delta_2) < \infty$, such that, for all conditional Bernoulli product measures on $\mathcal{S}_M$, $M \in \mathbb{N}$, of the form*

$$\bigotimes_{i=1}^{M} \mathbb{B}_{p_i}\left(\ \cdot\ \Big|\sum_{k=1}^{M}\xi_k = 2M_0\right), \tag{5.98}$$

with $p_1,\dots,p_M \in (\delta_1, 1-\delta_1)$ and $2M_0 \in (-M(1-\delta_2), M(1-\delta_2))$, and for all $1 \le k < l \le M$, it holds that

$$\left|\mathbb{C}\mathrm{ov}\left(\mathbb{1}_{\{\xi_k=-1\}};\mathbb{1}_{\{\xi_l=-1\}}\right)\right| \le \frac{C}{M}. \tag{5.99}$$

Going back to (5.92) we infer from this that

$$\prod_1^n \mu_{\beta,N}^{\boldsymbol{x}_0(j)}\left(\mathrm{e}^{\tau_j[\ell]Y_j}\right) \le \exp\left\{O(\epsilon^2)\sum_{j=1}^{n}\frac{\tau_j[\ell]^2}{M_j}\right\}, \tag{5.100}$$

uniformly in $\ell = 0,\dots,\ell_B$.

Statistics of mesoscopic trajectories. (5.89) together with the bound (5.100) suggests the following notion of *goodness* of mesoscopic trajectories $\underline{\boldsymbol{x}}$:

Definition 5.3. We say that a mesoscopic trajectory $\underline{\boldsymbol{x}} = (\boldsymbol{x}_{-\ell_A}, \ldots, \boldsymbol{x}_{\ell_B})$ is good, and write $\underline{\boldsymbol{x}} \in \mathscr{T}_{\boldsymbol{A},\boldsymbol{B}}$, if it passes through G_N^0, satisfies (5.90) (and its analog for the reversed chain) and, in addition, it satisfies

$$\sum_{\ell=-\ell_A}^{\ell_B-1} \exp\left\{ O(\epsilon) \sum_{j=1}^{n} \frac{\tau_j[\ell]^2}{M_j} \right\} \phi_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1}) \le 1 + O(\epsilon). \tag{5.101}$$

By construction (5.65) automatically holds for any $\underline{\boldsymbol{x}} \in \mathscr{T}_{\boldsymbol{A},\boldsymbol{B}}$. Therefore, our target lower bound (5.66) on microscopic capacities will follow from

Proposition 5.4. *Let $\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}$ be the mesoscopic flow constructed in Subsections 5.3 and 5.4, and let the set of mesoscopic trajectories $\mathscr{T}_{\boldsymbol{A},\boldsymbol{B}}$ be as in Definition 5.3. Then* (5.64) *holds.*

Proof. By (5.49) we may assume that there exists $C > 0$ such that, for all $\underline{\boldsymbol{x}}$ under consideration and for all $\ell = -\ell_A, \ldots, \ell_B - 1$,

$$\phi_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1}) \le \mathrm{e}^{-C\ell^2/N}. \tag{5.102}$$

In view of (5.2) it is enough to check that

$$\sum_{\ell=0}^{\ell_B-1} \left(\exp\left\{ O(\epsilon) \sum_{j=1}^{n} \frac{\tau_j[\ell]^2}{M_j} \right\} - 1 \right) \phi_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{x}_\ell, \boldsymbol{x}_{\ell+1}) = O(\epsilon), \tag{5.103}$$

with $\mathbb{P}_N^{\mathfrak{f}_{A,B}}$-probabilities of order $1 - \mathrm{o}(1)$. Fix $\delta > 0$ small and split the sum on the left hand side of (5.103) into two sums corresponding to the terms with $\ell \le N^{1/2-\delta}$ and $\ell > N^{1/2-\delta}$ respectively. Clearly,

$$\sum_{j=1}^{n} \frac{\tau_j[\ell]^2}{M_j} = \mathrm{o}(1), \tag{5.104}$$

uniformly in $0 \le \ell \le N^{1/2-\delta}$. On the other hand, from our construction of the mesoscopic flow $\mathfrak{f}_{\boldsymbol{A},\boldsymbol{B}}$, namely from the choice (5.19) of transition rates inside G_N^0, and from the property (3.34) of the minimizing curve $\hat{\boldsymbol{x}}(\cdot)$, it follows that there exists a universal (ϵ-independent) constant, $K < \infty$, such that

$$\mathbb{P}_N^{\mathfrak{f}_{A,B}} \left(\max_j \max_{\ell > N^{1/2-\delta}} \frac{\tau_j[\ell]}{\ell \rho_j} > K \right) = \mathrm{o}(1). \tag{5.105}$$

Therefore, up to $\mathbb{P}_N^{\mathfrak{f}_{A,B}}$-probabilities of order $\mathrm{o}(1)$, the inequality

$$O(\epsilon) \sum_{j=1}^{n} \frac{\tau_j^2[\ell]}{M_j} \le O(\epsilon) K^2 \ell^2 \sum_{j=1}^{n} \frac{\rho_j^2}{M_j} = K^2 O(\epsilon) \frac{\ell^2}{N}, \tag{5.106}$$

holds uniformly in $\ell > N^{1/2-\delta}$. A comparison with (5.102) yields (5.103). $\square$

The last proposition leads to the inequality (5.66), which, together the upper bound given in (4.64), concludes the proof of Theorem 1.3.

6 Sharp estimates on the mean hitting times

In this section we conclude the proof of Theorem 1.2. To do this we will use Equation (2.12) with $A = \mathscr{S}[m_0^*]$ and $B = \mathscr{S}[M]$, where m_0^* is a local minimum of $F_{\beta,N}$ and M is the set of minima deeper than m_0^*. The denominator on the right-hand side of (2.12), the capacity, is controlled by Theorem 1.3. What we want to prove now is that the equilibrium potential, $h_{A,B}(\sigma)$, is close to one in the neighborhood of the starting set A, and so small elsewhere that the contributions from the sum over σ away from the valley containing the set A can be neglected. Note that this is not generally true but depends on the choice of sets A and B: the condition that all minima m of $F_{\beta,N}$ such that $F_{\beta,N}(m) < F_{\beta,N}(m_0^*)$ belong to the target set B is crucial.

In earlier work (see, e.g., [5]) the standard way to estimate the equilibrium potential $h_{A,B}(\sigma)$ was to use the renewal inequality $h_{A,B}(\sigma) \leq \frac{\text{cap}(A,\sigma)}{\text{cap}(B,\sigma)}$ and bounds on capacities.

This bound cannot be used here, since the capacities of single points are too small. In other words, hitting times of fixed microscopic configurations are separated on exponential scales from hitting times of the corresponding mesoscopic neighborhoods. We will therefore use another method to cope with this problem.

6.1 Mean hitting time and equilibrium potential

Let us start by considering a local minimum m_0^* of the one-dimensional function $F_{\beta,N}$, and denote by M the set of minima m such that $F_{\beta,N}(m) < F_{\beta,N}(m_0^*)$. We then consider the disjoint subsets $A \equiv \mathscr{S}[m_0^*]$ and $B \equiv \mathscr{S}[M]$, and write Eq. (2.12) as

$$\sum_{\sigma \in A} \nu_{A,B}(\sigma) \mathbb{E}_\sigma \tau_B = \frac{1}{\text{cap}(A,B)} \sum_{m \in [-1,1]} \sum_{\sigma \in \mathscr{S}[m]} \mu_{\beta,N}(\sigma) h_{A,B}(\sigma). \tag{6.1}$$

We want to estimate the right-hand side of (6.1). This is expected to be of order $\mathscr{Q}_{\beta,N}(m_0^*)$, thus we can readily do away with all contributions where $\mathscr{Q}_{\beta,N}$ is much smaller. More precisely, we choose $\delta > 0$ in such a way that, for all N large enough, there is no critical point z of $F_{\beta,N}$ with $F_{\beta,N}(z) \in \left[F_{\beta,N}(m_0^*), F_{\beta,N}(m_0^*) + \delta\right]$, and define

$$\mathscr{U}_\delta \equiv \{m : F_{\beta,N}(m) \leq F_{\beta,N}(m_0^*) + \delta\}. \tag{6.2}$$

Denoting by $\mathscr{U}_\delta^c$ the complement of $\mathscr{U}_\delta$, we obviously have

Lemma 6.1.

$$\sum_{m \in \mathscr{U}_\delta^c} \sum_{\sigma \in \mathscr{S}[m]} \mu_{\beta,N}(\sigma) h_{A,B}(\sigma) \leq N e^{-\beta N \delta} \mathscr{Q}_{\beta,N}(m_0^*). \tag{6.3}$$

The main problem is to control the equilibrium potential $h_{A,B}(\sigma)$ for configurations $\sigma \in \mathscr{S}[\mathscr{U}_\delta]$. To do that, first notice that

$$\mathscr{U}_\delta = \mathscr{U}_\delta(m_0^*) \bigcup_{m \in M} \mathscr{U}_\delta(m), \tag{6.4}$$

where $\mathscr{U}_\delta(m)$ is the connected component of $\mathscr{U}_\delta$ containing m (see Figure 4.). Note that it can happen that $\mathscr{U}_\delta(m) = \mathscr{U}_\delta(m')$ for two different minima $m, m' \in M$.

With this notation we have the following lemma.

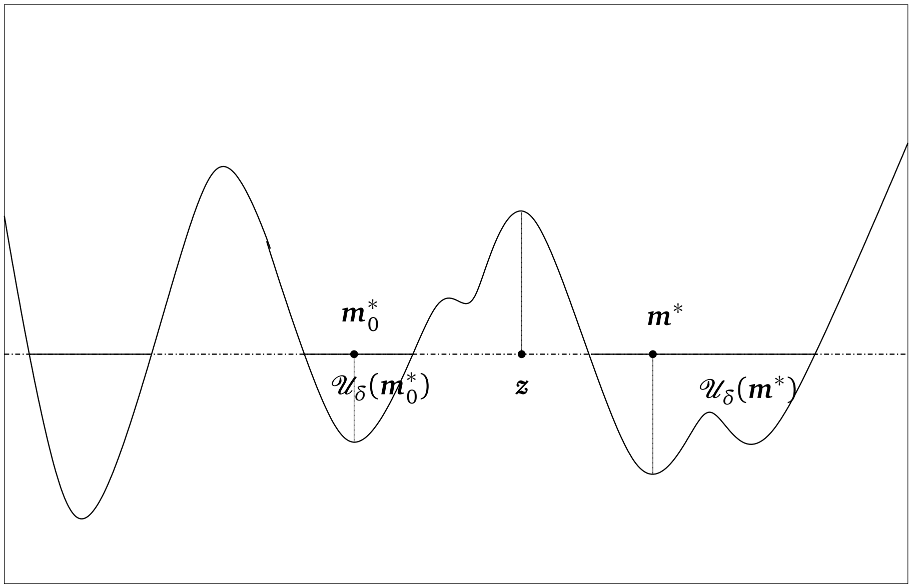

Figure 4: Decomposition of the magnetization space $[-1,1]$: The dotted lines and the continuous lines correspond respectively to $\mathscr{U}_\delta^c$ and $\mathscr{U}_\delta = \mathscr{U}_\delta(m_0^*)\bigcup_{m\in M}\mathscr{U}_\delta(m)$.

Lemma 6.2. *There exists a constant, $c > 0$, such that*

(i) for every $m \in M$,

$$\sum_{\sigma\in\mathscr{S}[\mathscr{U}_\delta(m)]} \mu_{\beta,N}(\sigma)h_{A,B}(\sigma) \le e^{-\beta N c}\mathscr{Q}_{\beta,N}(m_0^*) \tag{6.5}$$

and

(ii)

$$\sum_{\sigma\in\mathscr{S}[\mathscr{U}_\delta(m_0^*)]} \mu_{\beta,N}(\sigma)\left[1-h_{A,B}(\sigma)\right] \le e^{-\beta N c}\mathscr{Q}_{\beta,N}(m_0^*). \tag{6.6}$$

The treatment of points (i) and (ii) is completely similar, as both rely on a rough estimate of the probabilities to leave the starting well before visiting its minimum, and it will be discussed in the next section.

Assuming Lemma 6.2, we can readily conclude the proof of Theorem 1.2. Indeed, using (6.5) together with (6.3), we obtain the upper bound

$$\begin{aligned}\sum_{\sigma\in S_N}\mu_{\beta,N}(\sigma)h_{A,B}(\sigma) &\le \sum_{m\in\mathscr{U}_\delta(m_0^*)}\mathscr{Q}_{\beta,N}(m)+O\left(\mathscr{Q}_{\beta,N}(m_0^*)e^{-\beta N c}\right)\\ &= \mathscr{Q}_{\beta,N}(m_0^*)\sqrt{\frac{\pi N}{2\beta a(m_0^*)}}(1+o(1)),\end{aligned} \tag{6.7}$$

where $a(m_0^*)$ is given in (1.19). On the other hand, using (6.6), we get the corresponding lower bound

$$\begin{aligned}
\sum_{\sigma\in S_N}\mu_{\beta,N}(\sigma)h_{A,B}(\sigma) &\geq \sum_{m\in\mathscr{U}_\delta(m_0^*)}\sum_{\sigma\in\mathscr{S}[m]}\mu_{\beta,N}(\sigma)\left[1-(1-h_{A,B}(\sigma))\right]\\
&\geq \sum_{m\in\mathscr{U}_\delta(m_0^*)}\mathscr{Q}_{\beta,N}(m)-O(\mathscr{Q}_{\beta,N}(m_0^*)e^{-\beta Nc})\\
&= \mathscr{Q}_{\beta,N}(m_0^*)\sqrt{\frac{\pi N}{2\beta a(m_0^*)}}(1+o(1)).
\end{aligned}\tag{6.8}$$

From Equation (1.12) for $\mathscr{Q}_{\beta,N}(m_0^*)$ and Equation (1.32) for $\mathrm{cap}(A,B)$, we finally obtain

$$\begin{aligned}
\mathbb{E}_{\nu_{A,B}}\tau_B &= \sum_{\sigma\in S_N}\frac{\mu_{\beta,N}(\sigma)h_{A,B}(\sigma)}{\mathrm{cap}(A,B)}\\
&= \exp\left(\beta N\left(F_{\beta,N}(z^*)-F_{\beta,N}(m_0^*)\right)\right)\\
&\times\ \frac{2\pi N}{\beta|\hat{\gamma}_1|}\sqrt{\frac{\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(z^*+h)\right)\right)-1}{1-\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(m_0^*+h)\right)\right)}}(1+o(1)),
\end{aligned}\tag{6.9}$$

which proves Theorem 1.2.

6.2 Upper bounds on harmonic functions.

We now prove Lemma 6.2 giving a detailed proof only for (i), the proof of (ii) being completely analogous. This requires, for the first time in this paper, to get an estimate on the minimizer of the Dirichlet form, the harmonic function $h_{A,B}(\sigma)$.

First note that, since $h_{A,B}(\sigma)\equiv\mathbb{P}_\sigma(\tau_A<\tau_B)$ for all $\sigma\notin A\cup B$, the only non zero contributions to the sum in (i) come from those sets $\mathscr{U}_\delta(m)$ (at most two) whose corresponding m is such that there are no minima of M between m_0^* and m. By symmetry we can just analyze one of these two sets, denoted by $\mathscr{U}_\delta(m^*)$, assuming for definiteness that $m_0^*<m^*$. Note also that since $h_{A,B}(\sigma)=0$ for all σ such that $m^*\leq m(\sigma)$, the problem can be reduced further on to the set

$$\mathscr{U}_\delta^-\equiv\mathscr{U}_\delta(m^*)\cap\{m:m<m^*\}.\tag{6.10}$$

Define the mesoscopic counterpart of $\mathscr{U}_\delta^-$, namely, for fixed $m^*\in M$ and $n\in\mathbb{N}$, let $\boldsymbol{m}^*\in\Gamma_N^n$ be the minimum of $F_{\beta,N}(\boldsymbol{x})$ correspondent to m^*, and define

$$\boldsymbol{U}_\delta\equiv\boldsymbol{U}_\delta(\boldsymbol{m}^*)\equiv\{\boldsymbol{x}\in\Gamma_N^n:m(\boldsymbol{x})\in\mathscr{U}_\delta^-\}.\tag{6.11}$$

We write the boundary of $\boldsymbol{U}_\delta$ as $\partial\boldsymbol{U}_\delta=\partial_A\boldsymbol{U}_\delta\sqcup\partial_B\boldsymbol{U}_\delta$, where $\partial_B\boldsymbol{U}_\delta=\partial\boldsymbol{U}_\delta\cap\boldsymbol{B}$, and observe that, for all $\sigma\in\mathscr{S}[\boldsymbol{U}_\delta]$

$$h_{A,B}(\sigma)=\mathbb{P}_\sigma[\tau_A<\tau_B]\leq\mathbb{P}_\sigma[\tau_{S[\partial_A\boldsymbol{U}_\delta]}<\tau_{S[\partial_B\boldsymbol{U}_\delta]}].\tag{6.12}$$

Let $\max_\ell\rho_\ell\ll\theta(\varepsilon)\ll 1$, and for $\theta\equiv\theta(\varepsilon)$ define

$$\boldsymbol{G}_\theta\equiv\left\{\boldsymbol{m}\in\boldsymbol{U}_\delta:\sum_{\ell=1}^n\frac{(\boldsymbol{m}_\ell-\boldsymbol{m}_\ell^*)^2}{\rho_\ell}\leq\frac{\varepsilon^2}{\theta}\right\}.\tag{6.13}$$

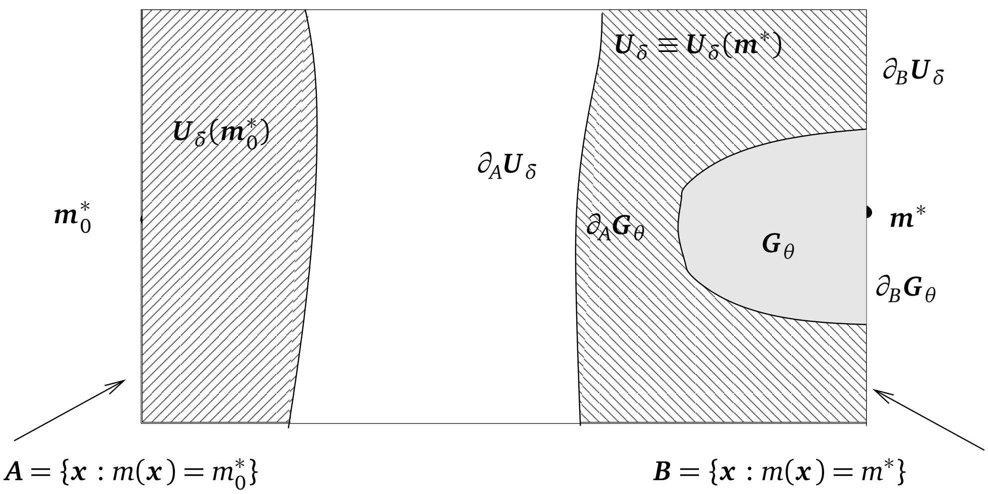

Figure 5: Neighborhoods of $\boldsymbol{m}_0^*$ and $\boldsymbol{m}^*$ in the space Γ_N^n. Here we denoted by $\boldsymbol{U}_\delta(\boldsymbol{m}_0^*)$ the mesoscopic counterpart of $\mathscr{U}(m_0^*)$.

As before, we denote by $\partial \boldsymbol{G}_\theta$ the boundary of $\boldsymbol{G}_\theta$, and write $\partial \boldsymbol{G}_\theta = \partial_A \boldsymbol{G}_\theta \sqcup \partial_B \boldsymbol{G}_\theta$, where $\partial_B \boldsymbol{G}_\theta = \partial \boldsymbol{G}_\theta \cap \boldsymbol{B}$ (see Figure 5.).

The strategy to control the equilibrium potential, $\mathbb{P}_\sigma(\tau_A < \tau_B)$, consists in estimating the probabilities $\mathbb{P}_\sigma[\tau_A < \tau_{S[\partial_A \boldsymbol{G}_\theta] \cup B}]$, for $\sigma \in \mathscr{S}[\boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta]$, and $\mathbb{P}_\sigma[\tau_{S[\partial_A \boldsymbol{G}_\theta]} < \tau_B]$, for $\sigma \in \boldsymbol{G}_\theta$, in order to apply a renewal argument and to get from these estimates a bound on the probability of the original event.

Proceeding on this line, we state the following:

Proposition 6.3. *For any $\alpha \in (0,1)$, there exists $n_0 \in \mathbb{N}$, such that the inequality*

$$\mathbb{P}_\sigma(\tau_A < \tau_{S[\partial_A \boldsymbol{G}_\theta] \cup B}) \le e^{-(1-\alpha)\beta N \left[F_{\beta,N}(m_0^*) + \delta - F_{\beta,N}(\boldsymbol{m}(\sigma))\right]} \tag{6.14}$$

holds for all $\sigma \in \mathscr{S}[\boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta]$, $n \ge n_0$, and for all N sufficiently large.

Proof of Proposition 6.3: Super-harmonic barrier functions. Throughout the next computations, c, c' and c'' will denote positive constants which are independent on n but may depend on β and on the distribution of h. The particular value of c and c' may change from line to line as the discussion progresses.

We first observe that, for all $\sigma \in \mathscr{S}[\boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta]$,

$$\mathbb{P}_\sigma[\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta] \cup B}] \le \mathbb{P}_\sigma[\tau_{S[\partial_A \boldsymbol{U}_\delta]} < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta] \cup B}]. \tag{6.15}$$

The probability in the r.h.s. of (6.15) is the main object of investigation here.

The idea which is beyond the proof of bound (6.14) is quite simple. Suppose that ψ is a bounded super-harmonic function defined on $\mathscr{S}[\boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta]$, i.e.

$$(L_N \psi)(\sigma) \le 0 \qquad \text{for all } \sigma \in \mathscr{S}[\boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta]. \tag{6.16}$$

Then $\psi(\sigma_t)$ is a supermartingale, and $T \equiv \tau_{S[\partial_A \boldsymbol{U}_\delta]} \wedge \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup B}$ is an integrable stopping time, so that, by Doob's optional stopping theorem, $\forall\, \sigma \in \mathscr{S}[\boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta]$,

$$\mathbb{E}_\sigma \psi(\sigma_T) \leq \psi(\sigma). \tag{6.17}$$

On the other hand,

$$\mathbb{E}_\sigma \psi(\sigma_T) \geq \min_{\sigma' \in \mathscr{S}[\partial_A \boldsymbol{U}_\delta]} \psi(\sigma') \mathbb{P}_\sigma(\tau_{\mathscr{S}[\partial_A \boldsymbol{U}_\delta]} < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup B}), \tag{6.18}$$

and hence

$$\mathbb{P}_\sigma(\tau_{\mathscr{S}[\partial_A \boldsymbol{U}_\delta]} < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup B}) \leq \max_{\sigma' \in \mathscr{S}[\partial_A \boldsymbol{U}_\delta]} \frac{\psi(\sigma)}{\psi(\sigma')}. \tag{6.19}$$

The problem is to find a super-harmonic function in order to get a suitable bound in (6.19).

Proposition 6.4. *For any $\alpha \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that the function $\psi(\sigma) \equiv \phi(\boldsymbol{m}(\sigma))$, with $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ defined as*

$$\phi(\boldsymbol{x}) \equiv e^{(1-\alpha)\beta N F_{\beta,N}(\boldsymbol{x})}, \tag{6.20}$$

is super-harmonic in $\mathscr{S}[\boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta]$ for all $n \geq n_0$ and N sufficiently large.

The proof of Proposition 6.4 will involve computations with differences of the functions $F_{\beta,N}$. We therefore first collect some elementary properties that we will use later. First we need some control on the second derivative of this function. From (3.14) we infer that

$$\frac{\partial^2 F_{\beta,N}(\boldsymbol{x})}{\partial \boldsymbol{x}_\ell^2} = \frac{2}{N}\left(-1 + \frac{1}{\beta\rho_\ell} I''_{N,\ell}(\boldsymbol{x}_\ell/\rho_\ell)\right). \tag{6.21}$$

Thus all the potential problems come from the function $I_{N,\ell}$.

Lemma 6.5. *For any $y \in (-1,1)$,*

$$\tanh^{-1}(y) - \beta\varepsilon \leq I'_{N,\ell}(y) \leq \tanh^{-1}(y) + \beta\varepsilon, \tag{6.22}$$

In particular, as $y \to \pm 1$, $I'_{N,\ell}(y) \to \pm\infty$.

Proof. Recall that $I'_{N,\ell}(y) = U'^{-1}_{N,\ell}(y)$. Set $I'_{N,\ell}(y) \equiv t$. Then

$$y = \frac{1}{|\Lambda_\ell|} \sum_{i \in \Lambda_\ell} \tanh(t + \beta \tilde{h}_i) \tag{6.23}$$

and hence

$$\tanh(t - \beta\varepsilon) \leq y \leq \tanh(t + \beta\varepsilon), \tag{6.24}$$

or, equivalently, (6.22), which proves the lemma. □

Lemma 6.6. *For any $y \in (-1,1)$ we have that*

$$0 \leq I''_{N,\ell}(y) \leq \frac{1}{1 - \left(|y| + \varepsilon\beta(1-y^2)\right)^2}. \tag{6.25}$$

In particular, for all $y \in [-1+\nu, 1-\nu]$, *with* $\nu \in (0,1/2)$,

$$0 \le I''_{N,\ell}(y) \le \frac{1}{2\nu + \nu^2 + O(\varepsilon)} \le c, \tag{6.26}$$

and, for all $y \in (-1,-1+\nu] \cup [1-\nu, 1)$,

$$0 \le I''_{N,\ell}(y) \le \frac{1}{1-|y|}. \tag{6.27}$$

Proof. We consider only the case $y \ge 0$, the case $y < 0$ is completely analogous. Using the relation $I''_{N,\ell}(x) = \left(U''_{N,\ell}(I'_{N,\ell}(x))\right)^{-1}$ and setting $t_\ell \equiv I'_{N,\ell}(y)\mathrm{arctanh}(y)$, and using Lemma 6.5, we obtain

$$\begin{aligned} I''_{N,\ell}(y) &= \frac{1}{\frac{1}{|\Lambda_\ell(\boldsymbol{x})|}\sum_{i\in\Lambda_\ell(\boldsymbol{x})}(1-\tanh^2(\beta\tilde{h}_i + t_\ell))} \\ &\le \frac{1}{1-\tanh^2(\varepsilon\beta + t_\ell)} \\ &\le \frac{1}{1-\tanh^2(\tanh^{-1}(y) + 2\varepsilon\beta)} \\ &\le \frac{1}{1-\left(y + 2\varepsilon\beta\tanh'(\tanh^{-1}(y))\right)^2} \\ &= \frac{1}{1-\left(y+2\varepsilon\beta(1-y^2)\right)^2}, \end{aligned} \tag{6.28}$$

where we used that tanh is monotone increasing. The remainder of the proof is elementary algebra. □

Let us define, for all $\boldsymbol{m}$ such that $\boldsymbol{x}_\ell/\rho_\ell \in [-1, 1-2/N]$,

$$g_\ell(\boldsymbol{x}) \equiv \tfrac{N}{2}\left(F_{N,\beta}(\boldsymbol{x}+\boldsymbol{e}_\ell) - F_{N,\beta}(\boldsymbol{x})\right). \tag{6.29}$$

Lemma 6.6 has the following corollary.

Corollary 6.7. *(i) If* $\boldsymbol{x}_\ell/\rho_\ell \in [-1+\nu, 1-\nu]$, *with* $\nu > 0$, *then*

$$g_\ell(\boldsymbol{x}) = -x - \bar{h}_\ell + \tfrac{1}{\beta} I'_{N,\ell}(\boldsymbol{x}_\ell/\rho_\ell) + O(1/N). \tag{6.30}$$

(ii) If $\boldsymbol{x}_\ell/\rho_\ell \in [-1,-1+\nu] \cup [1-\nu, 1-2/N]$, *then*

$$g_\ell(\boldsymbol{x}) = -x - \bar{h}_\ell + \tfrac{1}{\beta} I'_{N,\ell}(\boldsymbol{x}_\ell/\rho_\ell) + O(1), \tag{6.31}$$

where $O(1)$ *is independent of* N, n, *and* ν.

(iii) If $\boldsymbol{x}_\ell/\rho_\ell \in [-1+\nu, 1-\nu]$, *with* $\nu > 0$, *then there exists* $c < \infty$, *independent of* N, *such that*

$$|g_\ell(\boldsymbol{x}) - g_\ell(\boldsymbol{x}-\boldsymbol{e}_\ell)| \le \frac{c}{N}. \tag{6.32}$$

(iv) *If* $\boldsymbol{x}_\ell/\rho_\ell \in [-1,-1+\nu] \cup [1-\nu, 1-2/N]$, *then*

$$|g_\ell(\boldsymbol{x}) - g_\ell(\boldsymbol{x}-\boldsymbol{e}_\ell)| \le C, \tag{6.33}$$

where C *is a numerical constant independent of* N, n, *and* ν.

The proof of this corollary is elementary and will not be detailed.

The usefulness of (ii) results from the fact that $|I'_{N,\ell}|$ is large on that domain. More precisely, we have the following lemma.

Lemma 6.8. *There exists* $\nu > 0$, *independent of* N *and* n, *such that, if* $\boldsymbol{x}_\ell/\rho_\ell > 1-\nu$, *then* $g_\ell(\boldsymbol{x})$ *is strictly increasing in* $\boldsymbol{x}_\ell$ *and tends to* $+\infty$ *as* $\boldsymbol{x}_\ell/\rho_\ell \uparrow +1$; *similarly if* $\boldsymbol{x}_\ell/\rho_\ell < -1+\nu$, *then* $g_\ell(\boldsymbol{x})$ *is strictly decreasing in* $\boldsymbol{x}_\ell$ *and tends to* $-\infty$ *as* $\boldsymbol{x}_\ell/\rho_\ell \downarrow -1$.

Proof. Combine (ii) of Corollary 6.7 with Lemma 6.5 and note that $\bar{h}_\ell$ is bounded by hypothesis. ☐

The next step towards the proof of Proposition 6.4 is the following lemma.

Lemma 6.9. *Let* $\boldsymbol{m} \in \boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta$ *and denote by* $S(\boldsymbol{m}) = \{\ell : \boldsymbol{m}_\ell/\rho_\ell \neq 1\}$. *Then there exists a constant* $c \equiv c(\beta, h) > 0$, *independent of* N *and* n, *such that the following holds. If*

$$\sum_{\ell \notin S(\boldsymbol{m})} \rho_\ell \le \frac{\varepsilon^2}{8\theta}, \tag{6.34}$$

then

$$\sum_{\ell \in S(\boldsymbol{m})} \rho_\ell \left(g_\ell(\boldsymbol{m})\right)^2 \ge c\frac{\varepsilon^2}{\theta}, \tag{6.35}$$

Proof. From the relation $I'_{N,\ell}(x) = U'^{-1}_{N,\ell}(x)$, we get that, for all $\ell \in S(\boldsymbol{m})$,

$$\boldsymbol{m}_\ell = \frac{1}{N}\sum_{i \in \Lambda_\ell} \tanh\left(\beta\left(g_\ell(\boldsymbol{m})(1+o(1)) + m + h_i\right)\right). \tag{6.36}$$

Here $o(1)$ tends to zero as $N \to \infty$.

We are concerned about small $g_\ell(\boldsymbol{m})$. Subtracting $\frac{1}{N}\sum_{i\in\Lambda_\ell} \tanh\left(\beta\left(m+h_i\right)\right)$ on both sides of (6.36) and expanding the right-hand side to first order in $g_\ell(\boldsymbol{m})$, and then summing over $\ell \in S(\boldsymbol{m})$, we obtain

$$\begin{aligned}&\left| m - \frac{1}{N}\sum_{i=1}^{N} \tanh\left(\beta\left(m+h_i\right)\right) - \sum_{\ell \notin S(\boldsymbol{m})}\left(\boldsymbol{m}_\ell - \frac{1}{N}\sum_{i\in\Lambda_\ell}\tanh\left(\beta(m+h_i)\right)\right)\right| \\ &\le c\sum_{\ell\in S(\boldsymbol{m})} \rho_\ell \left|g_\ell(\boldsymbol{m})\right| \le c\left(\sum_{\ell \in S(\boldsymbol{m})} \rho_\ell g_\ell^2(\boldsymbol{m})\right)^{1/2}.\end{aligned} \tag{6.37}$$

Notice that the function $m \mapsto m - \frac{1}{N}\sum_{i=1}^{N} \tanh\left(\beta\left(m+h_i\right)\right)$ has, by (1.20), non-zero derivative at m^*. Moreover, by construction, m^* is the only zero of this function in $\mathscr{U}_\delta^-(m^*)$. From this observations, together with (6.37), we conclude that

$$\left(\sum_{\ell=1}^{n} \rho_\ell g_\ell^2(\boldsymbol{m})\right)^{1/2} \geq c|m-m^*| - 2\sum_{\ell \notin S(\boldsymbol{m})} \rho_\ell, \tag{6.38}$$

for some constant $c < \infty$. Here we used the triangle inequality and the fact that $\left|\boldsymbol{m}_\ell - \frac{1}{N}\sum_{i\in\Lambda_\ell} \tanh\left(\beta(m+h_i)\right)\right| \leq 2\rho_\ell$. Under the hypothesis of the lemma, this gives the desired bound if $|m-m^*| \geq c''\varepsilon/\sqrt{\theta}$ for some constant $c'' < \infty$. On the other hand, we can write, for $\ell \in S(\boldsymbol{m})$,

$$\begin{aligned}
\left|\boldsymbol{m}_\ell - \boldsymbol{m}_\ell^*\right| &\leq \frac{1}{N}\sum_{i\in\Lambda_\ell}\left|\tanh\left(\beta\left(g_\ell(\boldsymbol{m})(1+o(1))+m+h_i\right)\right) - \tanh\left(\beta\left(m+h_i\right)\right)\right| \\
&+ \frac{1}{N}\sum_{i\in\Lambda_\ell}\left|\tanh\left(\beta\left(m+h_i\right)\right) - \tanh\left(\beta\left(m^*+h_i\right)\right)\right| \\
&\leq c\rho_\ell|m-m^*| + c'\rho_\ell|g_\ell(\boldsymbol{m})|.
\end{aligned} \tag{6.39}$$

Hence we get the bound

$$\begin{aligned}
\left(\sum_{\ell\in S(\boldsymbol{m})} \rho_\ell g_\ell^2(\boldsymbol{m})\right)^{1/2} &\geq c\left(\sum_{\ell\in S(\boldsymbol{m})} \frac{(\boldsymbol{m}_\ell-\boldsymbol{m}_\ell^*)^2}{\rho_\ell}\right)^{1/2} - c'|m-m^*| \\
&= c\left(\sum_{\ell=1}^{n} \frac{(\boldsymbol{m}_\ell-\boldsymbol{m}_\ell^*)^2}{\rho_\ell} - \sum_{\ell\notin S(\boldsymbol{m})} \frac{(\boldsymbol{m}_\ell-\boldsymbol{m}_\ell^*)^2}{\rho_\ell}\right)^{1/2} - c'|m-m^*| \\
&\geq c\left(\varepsilon^2/\theta - 4\sum_{\ell\notin S(\boldsymbol{m})} \rho_\ell\right)^{1/2} - c'|m-m^*| \\
&\geq c\varepsilon/\sqrt{2\theta} - c'|m-m^*|
\end{aligned} \tag{6.40}$$

where in the last line we just used that $\boldsymbol{m} \notin \boldsymbol{G}_\theta$. The inequalities (6.38) and (6.40) now yield (6.35), concluding the proof of the lemma. $\square$

Proof of Proposition 6.4. Let $\sigma \in \mathscr{S}[\boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta]$ and set $\boldsymbol{x} \equiv \boldsymbol{m}(\sigma)$, so that, for ψ as in Proposition 6.4, $L_N\psi(\sigma) = L_N\phi(\boldsymbol{x})$. Let σ^i be the configuration obtained from σ after a spin-flip at i, and introduce the notation

$$L_N\phi(\boldsymbol{x}) = \sum_{\ell=1}^{n} L_\ell\phi(\boldsymbol{x}), \tag{6.41}$$

where

$$L_\ell\phi(\boldsymbol{x}) = \sum_{i\in\Lambda_\ell^-(\boldsymbol{x})} p_N(\sigma,\sigma^i)[\phi(\boldsymbol{x}+\boldsymbol{e}_\ell)-\phi(\boldsymbol{x})] + \sum_{i\in\Lambda_\ell^+(\boldsymbol{x})} p_N(\sigma,\sigma^i)[\phi(\boldsymbol{x}-\boldsymbol{e}_\ell)-\phi(\boldsymbol{x})]. \tag{6.42}$$

Notice that when $\boldsymbol{x}_\ell/\rho_\ell = \pm 1$, then $\Lambda_\ell^\pm(\boldsymbol{x}) = \emptyset$ and the summation over $\Lambda_\ell^\pm(\boldsymbol{x})$ in (6.42) disappears. We define the probabilities

$$\mathbb{P}^\sigma_{\pm,\ell} \equiv \sum_{i \in \Lambda_\ell^\mp(\boldsymbol{x})} p_N(\sigma, \sigma^i), \tag{6.43}$$

and observe that they are uniformly close to the mesoscopic rates defined in (4.2), namely

$$e^{-c\varepsilon} \le \frac{\mathbb{P}^\sigma_{\pm,\ell}}{r_N(\boldsymbol{x}, \boldsymbol{x} \pm \boldsymbol{e}_\ell)} \le e^{c\varepsilon}, \tag{6.44}$$

for some $c > 0$ and $\varepsilon = 1/n$. Notice also that

$$c\rho_\ell \le \mathbb{P}^\sigma_{+,\ell} + \mathbb{P}^\sigma_{-,\ell} \le c'\rho_\ell. \tag{6.45}$$

With the above notation and using the convention $0/0 = 0$, we get

$$\begin{aligned} L_\ell \phi(\boldsymbol{x}) &= \phi(\boldsymbol{x})\mathbb{P}^\sigma_{+,\ell}\left[\exp\left(2\beta(1-\alpha)g_\ell(\boldsymbol{x})\right) - 1\right] \\ &\quad + \phi(\boldsymbol{x})\mathbb{P}^\sigma_{-,\ell}\left[\exp\left(-2\beta(1-\alpha)g_\ell(\boldsymbol{x} - \boldsymbol{e}_\ell)\right) - 1\right] \\ &= \phi(\boldsymbol{x})\left(\mathbb{1}_{\{\mathbb{P}^\sigma_{+,\ell} \ge \mathbb{P}^\sigma_{-,\ell}\}}\mathbb{P}^\sigma_{+,\ell} G_\ell^+(\boldsymbol{x}) + \mathbb{1}_{\{\mathbb{P}^\sigma_{-,\ell} > \mathbb{P}^\sigma_{+,\ell}\}}\mathbb{P}^\sigma_{-,\ell} G_\ell^-(\boldsymbol{x})\right) \end{aligned} \tag{6.46}$$

where we introduced the functions

$$G_\ell^+(\boldsymbol{x}) = \exp\left(2\beta(1-\alpha)g_\ell(\boldsymbol{x})\right) - 1 + \tfrac{\mathbb{P}^\sigma_{-,\ell}}{\mathbb{P}^\sigma_{+,\ell}}\left(\exp\left(-2\beta(1-\alpha)g_\ell(\boldsymbol{x} - \boldsymbol{e}_\ell)\right) - 1\right) \tag{6.47}$$

$$G_\ell^-(\boldsymbol{x}) = \exp\left(-2\beta(1-\alpha)g_\ell(\boldsymbol{x} - \boldsymbol{e}_\ell)\right) - 1 + \tfrac{\mathbb{P}^\sigma_{+,\ell}}{\mathbb{P}^\sigma_{-,\ell}}\left(\exp\left(2\beta(1-\alpha)g_\ell(\boldsymbol{x})\right) - 1\right) \tag{6.48}$$

When $\boldsymbol{x}_\ell/\rho_\ell = \pm 1$, the local generator takes the simpler form

$$L_\ell \phi(\boldsymbol{x}) = \begin{cases} \phi(\boldsymbol{x})\mathbb{P}^\sigma_{-,\ell}\left[\exp\left(-2\beta(1-\alpha)g_\ell(\boldsymbol{x} - \boldsymbol{e}_\ell)\right) - 1\right] & \text{if } \boldsymbol{x}_\ell/\rho_\ell = 1 \\ \phi(\boldsymbol{x})\mathbb{P}^\sigma_{+,\ell}\left[\exp\left(2\beta(1-\alpha)g_\ell(\boldsymbol{x})\right) - 1\right] & \text{if } \boldsymbol{x}_\ell/\rho_\ell = -1 \end{cases} \tag{6.49}$$

Thus, from Lemma 6.8 and inequalities (6.45), we get immediately that for all ℓ such that $\boldsymbol{x}_\ell/\rho_\ell = \pm 1$

$$L_\ell \phi(\boldsymbol{x}) \le -(1 + o(1))\rho_\ell \phi(\boldsymbol{x}). \tag{6.50}$$

Let us now return to the case when $\boldsymbol{x}$ is not a boundary point. By the detailed balance conditions, we get

$$\begin{aligned} r_N(\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{e}_\ell) &= \exp\left(-2\beta g_\ell(\boldsymbol{x})\right) r_N(\boldsymbol{x} + \boldsymbol{e}_\ell, \boldsymbol{x}) \\ r_N(\boldsymbol{x}, \boldsymbol{x} - \boldsymbol{e}_\ell) &= \exp\left(2\beta g_\ell(\boldsymbol{x} - \boldsymbol{e}_\ell)\right) r_N(\boldsymbol{x} - \boldsymbol{e}_\ell, \boldsymbol{x}), \end{aligned} \tag{6.51}$$

and thus, together with (6.44),

$$\begin{aligned} \exp\left(-2\beta g_\ell(\boldsymbol{x}) - c\varepsilon\right) &\le \tfrac{\mathbb{P}^\sigma_{+,\ell}}{\mathbb{P}^\sigma_{-,\ell}} \le \exp\left(-2\beta g_\ell(\boldsymbol{x}) + c\varepsilon\right) \\ \exp\left(2\beta g_\ell(\boldsymbol{x} - \boldsymbol{e}_\ell) - c\varepsilon\right) &\le \tfrac{\mathbb{P}^\sigma_{-,\ell}}{\mathbb{P}^\sigma_{+,\ell}} \le \exp\left(2\beta g_\ell(\boldsymbol{x} - \boldsymbol{e}_\ell) + c\varepsilon\right) \end{aligned} \tag{6.52}$$

Inserting the last bounds in (6.47) and (6.48), and with some computations, we obtain

$$\begin{aligned} G_\ell^+(\boldsymbol{x}) \;\le\; & \left(\exp\left(2\beta(1-\alpha)g_\ell(\boldsymbol{x})\right) - 1\right)\left(1 - \exp\left(2\beta\alpha g_\ell(\boldsymbol{x} - \boldsymbol{e}_\ell) \mp c\varepsilon\right)\right) \\ & + \exp\left(2\beta g_\ell(\boldsymbol{x} - \boldsymbol{e}_\ell) \mp c\varepsilon\right)\left(\exp 2\beta(1-\alpha)\left(g_\ell(\boldsymbol{x}) - g_\ell(\boldsymbol{x} - \boldsymbol{e}_\ell)\right) - 1\right) \end{aligned} \tag{6.53}$$

$$G_\ell^-(\boldsymbol{x}) \leq \left(\exp\left(-2\beta(1-\alpha)g_\ell(\boldsymbol{x}-\boldsymbol{e}_\ell)\right)-1\right)\left(1-\exp\left(-2\beta\alpha g_\ell(\boldsymbol{x})\mp c\varepsilon\right)\right) + \exp\left(-2\beta g_\ell(\boldsymbol{x})\mp c\varepsilon\right)\left(\exp 2\beta(1-\alpha)\left(g_\ell(\boldsymbol{x})-g_\ell(\boldsymbol{x}-\boldsymbol{e}_\ell)\right)-1\right) \tag{6.54}$$

where $\mp \equiv -\text{sign}\left(g_\ell(\boldsymbol{x})\right) = -\text{sign}\left(g_\ell(\boldsymbol{x}-\boldsymbol{e}_\ell)\right)$.

For all ℓ such that $\boldsymbol{x}_\ell/\rho_\ell \in [-1+\nu, 1-\nu]$, we can use (6.32) to get

$$G_\ell^+(\boldsymbol{x}) \leq \left(\exp\left(2\beta(1-\alpha)g_\ell(\boldsymbol{x})\right)-1\right)\left(1-\exp\left(2\alpha\beta g_\ell(\boldsymbol{x})\mp c\varepsilon\right)\right)+c/N \tag{6.55}$$

$$G_\ell^-(\boldsymbol{x}) \leq \left(\exp\left(-2\beta(1-\alpha)g_\ell(\boldsymbol{x})\right)-1\right)\left(1-\exp\left(-2\alpha\beta g_\ell(\boldsymbol{x})\mp c\varepsilon\right)\right)+c/N. \tag{6.56}$$

The right hand sides of both (6.55) and (6.56) are negative if and only if $\left|g_\ell\right| > \frac{c\varepsilon}{2\alpha\beta}$. Let us define the index sets

$$S^< \equiv \{\ell : \boldsymbol{x}_\ell/\rho_\ell \in [-1+\nu, 1-\nu], \left|g_\ell(\boldsymbol{x})\right| \leq \tfrac{c\varepsilon}{\alpha\beta}\} \tag{6.57}$$

$$S^> \equiv \{\ell : \boldsymbol{x}_\ell/\rho_\ell \in [-1+\nu, 1-\nu], \left|g_\ell(\boldsymbol{x})\right| > \tfrac{c\varepsilon}{\alpha\beta}\}. \tag{6.58}$$

If $\ell \in S^<$, it holds that

$$\max\{G_\ell^+(\boldsymbol{x}), G_\ell^-(\boldsymbol{x})\} \leq \tfrac{c}{\alpha}\varepsilon^2, \tag{6.59}$$

which implies, together with (6.46) and (6.45), that

$$L_\ell\phi(\boldsymbol{x}) \leq \tfrac{c'}{\alpha}\varepsilon^2\rho_\ell\phi(\boldsymbol{x}). \tag{6.60}$$

To control the r.h.s. of (6.55) and (6.56) when $\ell \in S^>$, set

$$y_\ell \equiv \min\left\{\beta\left|g_\ell(\boldsymbol{x})\right|, \tfrac{1}{2}\right\} \leq \beta\left|g_\ell(\boldsymbol{x})\right|. \tag{6.61}$$

If $g_\ell(\boldsymbol{x}) > \frac{c\varepsilon}{\alpha\beta}$, then

$$\exp\left(2\beta(1-\alpha)g_\ell(\boldsymbol{x})\right)-1 \geq \exp\left(2(1-\alpha)y_\ell\right)-1 \geq 2(1-\alpha)y_\ell \tag{6.62}$$

and

$$1-\exp\left(2\beta\alpha g_\ell(\boldsymbol{x})-c\varepsilon\right) \leq 1-\exp(\alpha y_\ell) \leq -\alpha y_\ell, \tag{6.63}$$

so that the product in the r.h.s. of (6.55) is bounded from above by $-2(1-\alpha)\alpha y_\ell^2$. On the other hand, when $g_\ell(\boldsymbol{x}) < -\frac{c\varepsilon}{\alpha\beta}$,

$$\exp\left(2\beta(1-\alpha)g_\ell(\boldsymbol{x})\right)-1 \leq \exp\left(-2(1-\alpha)y_\ell\right)-1 \leq -(1-\alpha)y_\ell \tag{6.64}$$

and

$$1-\exp\left(2\beta\alpha g_\ell(\boldsymbol{x})+c\varepsilon\right) \geq 1-\exp(-\alpha y_\ell) \geq \tfrac{3}{4}\alpha y_\ell, \tag{6.65}$$

and the product in the r.h.s. of (6.55) is bounded from above by $-\frac{3}{4}(1-\alpha)\alpha y_\ell^2$. Altogether, this proves that, for all $\ell \in S^>$,

$$G_\ell^+(\boldsymbol{x}) \leq -\tfrac{3}{4}(1-\alpha)\alpha y_\ell^2, \tag{6.66}$$

and with a similar computation that

$$G_\ell^-(\boldsymbol{x}) \leq -\tfrac{3}{4}(1-\alpha)\alpha y_\ell^2. \tag{6.67}$$

For all $\ell \in S^>$, we then have

$$L_\ell \phi(\boldsymbol{x}) \leq -c\alpha\rho_\ell y_\ell^2 \phi(\boldsymbol{x}). \tag{6.68}$$

It remains to control the case when $\boldsymbol{x}_\ell/\rho_\ell \in (-1,-1+\nu] \cup [1-\nu,1)$. But from Lemma 6.8 it follows that, while the positive contribution to $G_\ell^+(\boldsymbol{x})$ and $G_\ell^-(\boldsymbol{x})$ remains bounded by a constant, the negative contribution becomes very large as soon as ν is small enough. More explicitly, for all ν small enough, we have

$$\begin{aligned} G_\ell^+(\boldsymbol{x}) &\leq -(\exp(\pm C') - 1)^2 + \exp(\pm C')(\exp(2\beta(1-\alpha)c) - 1) \leq -(1+\mathrm{o}(1)) \\ G_\ell^-(\boldsymbol{x}) &\leq -(1-\exp(\mp C'))^2 + \exp(\mp C'')(\exp(2\beta(1-\alpha)c) - 1) \leq -(1+\mathrm{o}(1)) \end{aligned} \tag{6.69}$$

where C' and C'' are positive constants tending to $+\infty$ as $\nu \downarrow 0$, and the sign $\pm$ is equal to the sign of $\boldsymbol{x}_\ell$.

From (6.45) and (6.46), we finally get

$$L_\ell \phi(\boldsymbol{x}) \leq -(1+\mathrm{o}(1))\rho_\ell \phi(\boldsymbol{x}). \tag{6.70}$$

From (6.50), (6.60), (6.68) and (6.70), we conclude that the positive contribution to the generator $L_N\phi(\boldsymbol{x}) = \sum_{\ell=1}^n L_\ell \phi(\boldsymbol{x})$, comes at most from the ℓ's in $S^<$, and can be estimated by

$$\tfrac{c'}{\alpha}\varepsilon^2 \sum_{\ell \in S^<} \rho_\ell \leq \tfrac{c'}{\alpha}\varepsilon^2. \tag{6.71}$$

Now we distinguish two cases, according to whether the hypothesis of Lemma 6.9 are satisfied or not.

<u>*Case 1:*</u> $\sum_{\ell \notin S(\boldsymbol{x})} \rho_\ell > \frac{\varepsilon^2}{8\theta}$. By (6.50), we obtain

$$\begin{aligned} \sum_{\ell=1}^n L_\ell \phi(\boldsymbol{x}) &\leq \sum_{\ell \notin S(\boldsymbol{x})} L_\ell \phi(\boldsymbol{x}) + \sum_{\ell \in S^<} L_\ell \phi(\boldsymbol{x}) \\ &\leq -\frac{\varepsilon^2}{8\theta}(1+\mathrm{o}(1))\phi(\boldsymbol{x}) + \tfrac{c'}{\alpha}\varepsilon^2, \end{aligned} \tag{6.72}$$

which is negative as desired if θ is small enough, that is, with our choice, if ε is small enough.

<u>*Case 2:*</u> $\sum_{\ell \notin S(\boldsymbol{x})} \rho_\ell \leq \frac{\varepsilon^2}{8\theta}$. In this case, the assertion of Lemma 6.9 holds.

By (6.50), (6.68), and (6.70), and for all $\ell \in S(\boldsymbol{x}) \setminus L^<$, we have

$$L_\ell \phi(\boldsymbol{x}) \leq -\rho_\ell \phi(\boldsymbol{x}) \min\{c\alpha y_\ell^2, 1\} \leq -c\alpha\rho_\ell y_\ell^2 \phi(\boldsymbol{x}), \tag{6.73}$$

where the last inequality holds for $\alpha < 4/c$. Now we use that

$$L_N \phi(\boldsymbol{x}) \leq \sum_{\ell \in S(\boldsymbol{x}) \setminus S^<} L_\ell \phi(\boldsymbol{x}) + \sum_{\ell \in S^<} L_\ell \phi(\boldsymbol{x}). \tag{6.74}$$

The first sum in (6.74) is bounded from above by

$$\begin{aligned} -c\alpha\phi(\boldsymbol{x}) \sum_{\ell \in S(\boldsymbol{x}) \setminus S^<} \rho_\ell y_\ell^2 &\leq -c\alpha\phi(\boldsymbol{x}) \sum_{\ell \in s(\boldsymbol{x}) \setminus S^<} \rho_\ell \min\left\{\beta^2 g_\ell^2(\boldsymbol{x}); \tfrac{1}{4}\right\} \\ &\leq -c\alpha\phi(\boldsymbol{x}) \min\left\{\beta^2 \sum_{\ell \in S(\boldsymbol{x}) \setminus S^<} \rho_\ell g_\ell^2(\boldsymbol{x}); \tfrac{1}{4}\right\}. \end{aligned} \tag{6.75}$$

But from Lemma 6.9 we know that, for all $\boldsymbol{x} \in \boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta$,

$$\sum_{\ell \in S(\boldsymbol{x}) \setminus S^<} \rho_\ell g_\ell^2(\boldsymbol{x}) \geq c\frac{\varepsilon^2}{\theta} - \frac{c'}{\alpha^2}\varepsilon^2 \geq c''\frac{\varepsilon^2}{\theta}, \tag{6.76}$$

where c'' is a positive constant provided that $\alpha \geq c\theta$. If n large enough, the inequality

$$\min\left\{\beta^2 \sum_{\ell \in s(\boldsymbol{x}) \setminus S^<} \rho_\ell g_\ell^2(\boldsymbol{x}); \tfrac{1}{4}\right\} \geq \min\left\{c''\frac{\varepsilon^2}{\theta}; \tfrac{1}{4}\right\} = c''\frac{\varepsilon^2}{\theta}, \tag{6.77}$$

holds, and from (6.71) and (6.75) we get

$$L_N\psi(\sigma) \leq -\varepsilon^2(1-\alpha)\phi(\boldsymbol{x})(c''\alpha\theta^{-1} - c'\alpha^{-1}). \tag{6.78}$$

By our choice of θ, the condition $c''\alpha\theta^{-1} - c'\alpha^{-1} > 0 \Longleftrightarrow \alpha > c\theta$ is satisfied for any $\alpha \in (0,1)$ as soon as n is large enough. Hence, for such n's and for N large enough, we get that $L_N\psi(\sigma) = L_N\phi(\boldsymbol{x}) \leq 0$, concluding the proof of Proposition 6.4. ☐

Substituting the expression of the super-harmonic function (6.20) in (6.19) and using with (6.15), we obtain that for all $\sigma \in \mathscr{S}[\boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta]$

$$\begin{aligned}\mathbb{P}_\sigma[\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta] \cup B}] &\leq \max_{\sigma' \in \mathscr{S}[\partial_A \boldsymbol{U}_\delta]} e^{-(1-\alpha)\beta N\left[F_{\beta,N}(\boldsymbol{m}(\sigma')) - F_{\beta,N}(\boldsymbol{m}(\sigma))\right]} \\ &\leq e^{-(1-\alpha)\beta N\left[F_{\beta,N}(m_0^*) + \delta - F_{\beta,N}(\boldsymbol{m}(\sigma))\right]},\end{aligned} \tag{6.79}$$

where the last inequality follows from the definition of $\boldsymbol{U}_\delta$ together with the bounds in (3.33). This concludes the proof of Proposition 6.3.

Renewal estimates on escape probabilities. Let us now come back to the proof of Lemma 6.2. An easy consequence of Eq. (6.14) is that, for all $\sigma \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]$,

$$\mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta] \cup B}) \leq e^{-(1-\alpha)\beta N\left(F_{\beta,N}(m_0^*) + \delta\right)} \max_{\boldsymbol{m} \in \partial_A \boldsymbol{G}_\theta} e^{(1-\alpha)\beta N F_{\beta,N}(\boldsymbol{m})}, \tag{6.80}$$

while obviously $\mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta] \cup B}) \equiv 0$ for all $\sigma \in \mathscr{S}[\boldsymbol{G}_\theta \setminus \partial_A \boldsymbol{G}_\theta]$. To control the r.h.s. of (6.80), we need the following lemma:

Lemma 6.10. *There exists a constant $c < \infty$, independent of n, such that, for all $\boldsymbol{m} \in \boldsymbol{G}_\theta$,*

$$F_{\beta,N}(\boldsymbol{m}) \leq F_{\beta,N}(\boldsymbol{m}^*) + c\varepsilon. \tag{6.81}$$

Proof. Fix $\boldsymbol{m} \in \boldsymbol{G}_\theta$ and set $\boldsymbol{m} - \boldsymbol{m}^* \equiv \boldsymbol{v}$. Notice that, from the definition of $\boldsymbol{G}_\theta$,

$$\|\boldsymbol{v}\|_2^2 \leq \max_\ell \rho_\ell \sum_{\ell=1}^n \frac{(\boldsymbol{m}_\ell - \boldsymbol{m}_\ell^*)^2}{\rho_\ell} \leq \varepsilon^2. \tag{6.82}$$

Using Taylor's formula, we have

$$F_{\beta,N}(\boldsymbol{m}) = F_{\beta,N}(\boldsymbol{m}^*) + \frac{1}{2}\left(\boldsymbol{v}, \mathbb{A}(\boldsymbol{m}^*)\boldsymbol{v}\right) + \frac{1}{6}D^3 F_{\beta,N}(\boldsymbol{x})\boldsymbol{v}^3, \tag{6.83}$$

where $\mathbb{A}(\boldsymbol{m}^*)$ is the positive-definite matrix described in Sect. 3.2 (see Eq. (3.16)) and $\boldsymbol{x}$ is a suitable element of the ball around $\boldsymbol{m}^*$. From the explicit representation of the eigenvalues of $\mathbb{A}(\boldsymbol{m}^*)$, we see that $\|\mathbb{A}(\boldsymbol{m}^*)\| \leq c\varepsilon^{-1}$, and hence

$$\big(\boldsymbol{v}, \mathbb{A}(\boldsymbol{m}^*)\boldsymbol{v}\big) \leq c\varepsilon^{-1}\|\boldsymbol{v}\|_2^2 \leq c\varepsilon. \tag{6.84}$$

The remainder is given in explicit form as

$$\begin{aligned} D^3F_{\beta,N}(\boldsymbol{x})\boldsymbol{v}^3 &= \sum_{\ell=1}^{n} \frac{\partial^3 F_{\beta,N}}{\partial \boldsymbol{x}_\ell^3}(\boldsymbol{x})\boldsymbol{v}_\ell^3 = \frac{1}{\beta}\sum_{\ell=1}^{n}\frac{1}{\rho_\ell^2} I'''_{N,\ell}(\boldsymbol{x}_\ell/\rho_\ell)\boldsymbol{v}_\ell^3 \\ &= -\frac{1}{\beta}\sum_{\ell=1}^{n}\frac{1}{\rho_\ell^2}\frac{U'''_{N,\ell}(t_\ell)}{\Big(U''_{N,\ell}(t_\ell)\Big)^3}\boldsymbol{v}_\ell^3 \\ &= -\frac{1}{\beta}\sum_{\ell=1}^{n}\frac{1}{\rho_\ell^2}\frac{|\Lambda_\ell|^{-1}\sum_{i\in\Lambda_\ell}\tanh(t_\ell+\beta\tilde{h}_i)(1-\tanh^2(t_\ell+\beta\tilde{h}_i))}{\Big(|\Lambda_\ell|^{-1}\sum_{i\in\Lambda_\ell}(1-\tanh^2(t_\ell+\beta\tilde{h}_i))\Big)^3}\boldsymbol{v}_\ell^3, \end{aligned} \tag{6.85}$$

where $t_\ell = I'_{N,\ell}(\boldsymbol{x}_\ell/\rho_\ell)$. Thus

$$\Big|D^3F_{\beta,N}(\boldsymbol{x})\boldsymbol{v}^3\Big| \leq c\sum_{\ell=1}^{n}\frac{1}{\rho_\ell^2}\boldsymbol{v}_\ell^3 \leq c'\varepsilon^{-1}\|\boldsymbol{v}\|_2^2 \leq c'\varepsilon, \tag{6.86}$$

where we used that $|\boldsymbol{v}_\ell/\rho_\ell| \leq 1$. Hence, for some $c < \infty$, independent of n,

$$F_{\beta,N}(\boldsymbol{m}) \quad\leq\quad F_{\beta,N}(\boldsymbol{m}^*) + c\varepsilon \tag{6.87}$$

which proves the lemma. □

From (6.80), applying inequality (6.81) and recalling that $F_{\beta,N}(\boldsymbol{m}^*) = F_{\beta,N}(m^*)$, we get that for all $\sigma \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]$

$$\mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_A\boldsymbol{G}_\theta]\cup B}) \leq e^{-(1-\alpha)\beta N\left(F_{\beta,N}(m_0^*)+\delta-F_{\beta,N}(m^*)-c\varepsilon\right)}. \tag{6.88}$$

The last needed ingredient in order to get a suitable estimate on $\mathbb{P}_\sigma(\tau_A < \tau_B)$, is stated in the following lemma.

Lemma 6.11. *For any $\delta_2 > 0$, there exists $n_0 \in \mathbb{N}$, such that, for all $n \geq n_0$, for all $\sigma \in \mathscr{S}[\partial_A\boldsymbol{G}_\theta]$, and for all N large enough,*

$$\mathbb{P}_\sigma(\tau_B < \tau_{\mathscr{S}[\partial_A\boldsymbol{G}_\theta]}) \geq e^{-N\beta\delta_2}. \tag{6.89}$$

Proof. Fix $\sigma \in \mathscr{S}[\partial_A\boldsymbol{G}_\theta]$ and set $\boldsymbol{m}(0) \equiv \boldsymbol{m}(\sigma)$. As pointed out in the proof of Lemma 6.10, every $\boldsymbol{m}(0) \in \partial_A\boldsymbol{G}_\theta$ can be written in the form $\boldsymbol{m}(0) = \boldsymbol{m}^* + \boldsymbol{v}$, with $\boldsymbol{v} \in \Gamma_N^n$ such that $\|\boldsymbol{v}\|_2 \leq \varepsilon$. Then, let $\underline{\boldsymbol{m}} = (\boldsymbol{m}(0), \boldsymbol{m}(1), \ldots, \boldsymbol{m}(\|\boldsymbol{v}\|_1 N) \equiv \boldsymbol{m}^*)$ be a nearest neighbor path in Γ_N^n from $\boldsymbol{m}(0)$ to $\boldsymbol{m}^*$, of length $N\|\boldsymbol{v}\|_1$, with the following property: Denoting by ℓ_t the unique index in $\{1, \ldots, n\}$ such that $\boldsymbol{m}_{\ell_t}(t) \neq \boldsymbol{m}_{\ell_t}(t-1)$, it holds that

$$\boldsymbol{m}_{\ell_t}(t) = \boldsymbol{m}_{\ell_t}(t-1) + \tfrac{2}{N}s_t, \quad \forall t \geq 1, \tag{6.90}$$

where we define

$$s_t \equiv \operatorname{sign}\left(\boldsymbol{m}^*_{\ell_t} - \boldsymbol{m}_{\ell_t}(t-1)\right). \tag{6.91}$$

Note that, by property (6.90), $\boldsymbol{m}(t) \in \boldsymbol{G}_\theta$ for all $t \geq 0$. Thus, all microscopic paths, $(\sigma(t))_{t\geq 0}$, such that $\sigma(0) = \sigma$ and $\boldsymbol{m}(\sigma(t)) = \boldsymbol{m}(t)$, for all $t \geq 1$, are contained in the event $\{\tau_B < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]}\}$. Thus we get that

$$\begin{aligned}
\mathbb{P}_\sigma(\tau_B < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]}) &\geq \mathbb{P}_\sigma(\boldsymbol{m}(\sigma(t)) = \boldsymbol{m}(t), \forall t = 1, \ldots, \|\boldsymbol{v}\|_1 N) \\
&= \prod_{t=1}^{\|\boldsymbol{v}\|_1 N} \mathbb{P}_\sigma(\boldsymbol{m}(\sigma(t)) = \boldsymbol{m}(t) \big| \boldsymbol{m}(\sigma(t-1)) = \boldsymbol{m}(t-1)) \\
&= \prod_{t=1}^{\|\boldsymbol{v}\|_1 N} \sum_{i \in \Lambda^{s_t}_{\ell_t}} p_N(\sigma(t-1), \sigma^i(t-1)).
\end{aligned} \tag{6.92}$$

Note that $\Lambda^{s_t}_{\ell_t}$ is the set of sites in which a spin-flip corresponds to a step from $\boldsymbol{m}(t-1)$ to $\boldsymbol{m}(t)$.

The sum of the probabilities in the r.h.s. of (6.92) corresponds to the quantity $\mathbb{P}^{\sigma(t-1)}_{s_t,\ell_t}$ defined in (6.43). From the inequalities (6.44) and (4.15), it follows that, for some constant $c > 0$ depending on β and on the distribution of the field,

$$\mathbb{P}^{\sigma(t-1)}_{s_t,\ell_t} \geq c|\Lambda^{s_t}_{\ell_t}(\boldsymbol{m}(t-1))|/N \geq c|\Lambda^{s_t}_{\ell_t}(\boldsymbol{m}^*)|/N, \tag{6.93}$$

where the second inequality follows by our choice of the path $\underline{\boldsymbol{m}}$.

Now, since $|\Lambda^\pm_\ell(\boldsymbol{m}^*)|/N = \frac{1}{2}\left(\rho_\ell \pm \boldsymbol{m}^*_\ell\right)$, using the expression (3.20) for $\boldsymbol{m}^*_{\ell_t}$ and continuing from (6.93), we obtain

$$\mathbb{P}^{\sigma(t-1)}_{s_t,\ell_t} \geq c' \rho_{\ell_t}. \tag{6.94}$$

Inserting the last inequality in (6.92), and using that, by definition of the path $\underline{\boldsymbol{m}}$, the number of steps corresponding to a spin-flip in Λ_ℓ is equal to $|\boldsymbol{v}_\ell|N$, for all $\ell = \{1, \ldots, n\}$, we get

$$\begin{aligned}
\mathbb{P}_\sigma(\tau_B < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]}) &\geq \prod_{t=1}^{\|\boldsymbol{v}\|_1 N} c' \rho_{\ell_t} \\
&= e^{\|\boldsymbol{v}\|_1 N \ln(c')} \prod_{\ell=1}^n \rho_\ell^{|\boldsymbol{v}_\ell| N} \\
&\geq e^{N\sqrt{\varepsilon}\ln(c')} e^{-N\sum_{\ell=1}^n \boldsymbol{v}_\ell \ln\left(1/\rho_\ell\right)} \\
&\geq e^{N\sqrt{\varepsilon}\ln(c')} e^{-N\sum_{\ell=1}^n \boldsymbol{v}_\ell/\sqrt{\rho_\ell}} \\
&\geq e^{N\varepsilon\ln(c')} e^{-N\left(\sum_{\ell=1}^n \boldsymbol{v}_\ell^2/\rho_\ell\right)^{1/2}\varepsilon^{-1/2}} \\
&\geq e^{-N\left(\sqrt{\frac{\varepsilon}{\theta}} - \sqrt{\varepsilon}\ln(c')\right)},
\end{aligned} \tag{6.95}$$

where in the third line we used the inequality $\|\boldsymbol{v}\|_1 \leq \varepsilon^{-1/2}\|\boldsymbol{v}\|_2 \leq \sqrt{\varepsilon}$, and in the last line we used that $\boldsymbol{m}(0) = \boldsymbol{m}^* + \boldsymbol{v} \in \boldsymbol{G}_\theta$. By our choice of $\theta \gg \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\sqrt{\frac{\varepsilon}{\theta}} - \sqrt{\varepsilon}\ln(c') \leq \beta\delta_2$. For such n's, inequality (6.95) yields the bound (6.89) and concludes the proof of the Lemma. $\square$

We finally state the following proposition:

Proposition 6.12. *For all $\sigma \in \mathscr{S}[\boldsymbol{U}_\delta]$ it holds that*

$$\mathbb{P}_\sigma(\tau_A < \tau_B) \leq e^{-\beta N\left((1-\alpha)\left(F_{\beta,N}(m_0^*)+\delta-F_{\beta,N}(m^*)-c\varepsilon\right)-\delta_2\right)}(1+o(1)) \tag{6.96}$$

Proof. Let us first consider a configuration $\sigma \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]$. Then it holds

$$\begin{aligned}
\mathbb{P}_\sigma(\tau_A < \tau_B) &\leq \mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup B}) + \sum_{\eta \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]} \mathbb{P}_\sigma(\tau_A < \tau_B,\, \tau_\eta \leq \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup A\cup B}) \\
&\leq \mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup B}) + \max_{\eta \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]} \mathbb{P}_\eta(\tau_A < \tau_B)\mathbb{P}_\sigma(\tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]} < \tau_B) \\
&\leq \mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup B}) + \max_{\eta \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]} \mathbb{P}_\eta(\tau_A < \tau_B)\left(1 - e^{-\beta N\delta_2}\right),
\end{aligned} \tag{6.97}$$

where in the second line we applied the Markov property, and in the last line we insert the result (6.12). Now, taking the maximum over $\sigma \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]$ on both sides of (6.97) and rearranging the summation, we get

$$\begin{aligned}
\max_{\sigma \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]} \mathbb{P}_\sigma(\tau_A < \tau_B) &\leq \max_{\sigma \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta \cup B]} \mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]})e^{\beta N\delta_2} \\
&\leq e^{-\beta N\left((1-\alpha)\left(F_{\beta,N}(m_0^*)+\delta-F_{\beta,N}(m^*)-c\varepsilon\right)-\delta_2\right)},
\end{aligned} \tag{6.98}$$

where in the last line we used the bound (6.88). This concludes the proof of (6.96) for $\sigma \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]$.

Then, let us consider $\sigma \in \mathscr{S}[\boldsymbol{U}_\delta \setminus \partial_A \boldsymbol{G}_\theta]$. As before, it holds

$$\begin{aligned}
\mathbb{P}_\sigma(\tau_A < \tau_B) &\leq \mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup B}) + \sum_{\eta \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]} \mathbb{P}_\sigma(\tau_A < \tau_B,\, \tau_\eta \leq \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup A\cup B}) \\
&\leq \mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup B}) + \max_{\eta \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]} \mathbb{P}_\eta(\tau_A < \tau_B)\mathbb{P}_\sigma(\tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]} < \tau_B) \\
&\leq \mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_A \boldsymbol{G}_\theta]\cup B}) + \max_{\eta \in \mathscr{S}[\partial_A \boldsymbol{G}_\theta]} \mathbb{P}_\eta(\tau_A < \tau_B),
\end{aligned} \tag{6.99}$$

where $\mathbb{P}_\sigma(\tau_A < \tau_{\mathscr{S}[\partial_a \boldsymbol{G}_\theta]\cup B})$ is 0 for all $\sigma \in \mathscr{S}[\boldsymbol{G}_\theta \setminus \partial_A \boldsymbol{G}_\theta]$, and exponentially small in N for all $\sigma \in \mathscr{S}[\boldsymbol{U}_\delta \setminus \boldsymbol{G}_\theta]$ (due to Proposition 6.3). Inserting the bound (6.98) in the last equation, provides Eq. (6.96) for $\sigma \in \mathscr{S}[\boldsymbol{U}_\delta \setminus \partial_A \boldsymbol{G}_\theta]$ and thus concludes the proof of proposition. □

The proof of formula (6.5) now follows straightforwardly. From (6.96), we get

$$\begin{aligned}
&\sum_{\sigma \in \mathscr{S}[\mathscr{U}_\delta(m^*)]} \mu_{\beta,N}(\sigma)\mathbb{P}_\sigma(\tau_A < \tau_B) \\
&\quad \leq e^{-\beta N\left[(1-\alpha)\left(F_{\beta,N}(m_0^*)+\delta-F_{\beta,N}(m^*)-c\varepsilon\right)-\delta_2\right]} \sum_{\boldsymbol{m} \in \boldsymbol{U}_\delta} \mathscr{Q}_{\beta,N}(\boldsymbol{m}) \\
&\quad = \mathscr{Q}_{\beta,N}(m_0^*)e^{\beta N\left[\alpha F_{\beta,N}(m_0^*)-(1-\alpha)(\delta-F_{\beta,N}(m^*)-c\varepsilon)+\delta_2\right]} \sum_{\boldsymbol{m} \in \boldsymbol{U}_\delta} e^{-\beta N F_{\beta,N}(\boldsymbol{m})} \\
&\quad \leq \mathscr{Q}_{\beta,N}(m_0^*)N^n e^{\beta N\left[\alpha\left(F_{\beta,N}(m_0^*)-F_{\beta,N}(m^*)\right)-(1-\alpha)(\delta-c\varepsilon)+\delta_2\right]},
\end{aligned} \tag{6.100}$$

where in the second inequality we used the expression (1.9) for $\mathscr{Q}_{\beta,N}(m_0^*)$, while in the last line we applied the bound $F_{\beta,N}(\boldsymbol{m}) \leq F_{\beta,N}(\boldsymbol{m}^*) = F_{\beta,N}(m^*)$, and then bounded the cardinality of $\boldsymbol{U}_\delta$ by N^n. Finally, choosing α small enough, namely

$$\alpha < \frac{\delta - c\varepsilon - \delta_2}{F_{\beta,N}(m_0^*) - F_{\beta,N}(m^*) + \delta - c\varepsilon}, \tag{6.101}$$

we can easily ensure that (6.100) implies (6.5).

In exactly the same way one proves (6.6). This concludes the proof of Lemma 6.2 and thus of Theorem 1.2.

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