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Laws of the iterated logarithm for α-time Brownian motion

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Abstract

We introduce a class of iterated processes called α-time Brownian motion for $0 < \alpha \leq 2$. These are obtained by taking Brownian motion and replacing the time parameter with a symmetric α-stable process. We prove a Chung-type law of the iterated logarithm (LIL) for these processes which is a generalization of LIL proved in [14] for iterated Brownian motion. When $\alpha = 1$ it takes the following form

$$\liminf_{T\to\infty} T^{-1/2}(\log\log T) \sup_{0\leq t\leq T} |Z_t| = \pi^2\sqrt{\lambda_1} \quad a.s.$$

where λ_1 is the first eigenvalue for the Cauchy process in the interval $[-1, 1]$. We also define the local time $L^*(x, t)$ and range $R^*(t) = |\{x : Z(s) = x \text{ for some } s \leq t\}|$ for these processes for $1 < \alpha < 2$. We prove that there are universal constants $c_R, c_L \in (0, \infty)$ such that

$$\limsup_{t\to\infty} \frac{R^*(t)}{(t/\log\log t)^{1/2\alpha}\log\log t} = c_R \quad a.s.$$

$$\liminf_{t\to\infty} \frac{\sup_{x\in\mathbb{R}} L^*(x, t)}{(t/\log\log t)^{1-1/2\alpha}} = c_L \quad a.s.$$

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1 Introduction

In recent years, several iterated processes received much research interest from many mathematicians, see [1, 4, 7, 9, 20, 21, 25, 29] and references there in. Inspired by these results, we introduce a new class of iterated processes called α-time Brownian motion for $0 < \alpha \leq 2$. These are obtained by taking Brownian motion and replacing the time parameter with a symmetric α-stable process. For $\alpha = 2$, this is the iterated Brownian motion of Burdzy [4]. One of the main differences of these iterated processes and Brownian motion is that they are not Markov or Gaussian. However, for $\alpha = 1, 2$ these processes have connections with partial differential operators as described in [1, 22].

To define α-time Brownian motion, let X_t be a two-sided Brownian motion on $\mathbb{R}$. That is, $\{X_t : t \geq 0\}$ and $\{X_{-t} : t \leq 0\}$ are two independent copies of Brownian motion starting from 0. Let Y_t be a real-valued symmetric α-stable process, $0 < \alpha \leq 2$, starting from 0 and independent of X_t. Then α-time Brownian motion Z_t is defined by

$$Z_t \equiv X(Y_t), \quad t \geq 0. \tag{1.1}$$

It is easy to verify that Z_t has stationary increments and is a self-similar process of index $1/2\alpha$. That is, for every $k > 0$, $\{Z_t : t \geq 0\}$ and $\{k^{-1/2\alpha} Z_{kt} : t \geq 0\}$ have the same finite-dimensional distributions. We refer to Taqqu [27] for relations of self-similar stable processes to physical quantities. The α-time Brownian motion is an example of nonstable self-similar processes.

Our aim in this paper is two-fold. Firstly, we will be interested in the path properties of the process defined in (1.1). Since this process is not Markov or Gaussian, it is of interest to see how the lack of independence of increments affect the asymptotic behavior. Secondly, we will define the local time $L^*(x, t)$ for this process for $1 < \alpha < 2$. We will prove the joint continuity of the local time and extend LIL of Kesten [16] to these processes. We also obtain an LIL for the range of these processes.

In the first part of the paper, we will be interested in proving a "liminf" law of the iterated logarithm of the Chung-type for Z_t. The study of this type of LIL's was initiated by Chung [8] for Brownian motion W_t. He proved that

$$\liminf_{T\to\infty} (T^{-1} \log\log T)^{1/2} \sup_{0\leq t\leq T} |W_t| = \frac{\pi}{8^{1/2}} \quad a.s.$$

This LIL was extended to several other processes later including symmetric α-stable processes Y_t by Taylor [28] in the following form

$$\liminf_{T\to\infty} (T^{-1} \log\log T)^{1/\alpha} \sup_{0\leq t\leq T} |Y_t| = (\lambda_\alpha)^{1/\alpha} \quad a.s.$$

where λ_α is the first eigenvalue of the fractional Laplacian $(-\Delta)^{\alpha/2}$ in $[-1, 1]$.

One then wonders if a Chung-type LIL holds for the composition of symmetric stable processes. Although these processes are not Markov or Gaussian, this has been achieved for the composition of two Brownian motions, the so called iterated Brownian motion, which is the case of $\alpha = 2$ proved by Hu, Pierre-Loti-Viaud, and Shi in [14]. They showed that

$$\liminf_{T\to\infty} T^{-1/4} (\log\log T)^{3/4} \sup_{0\leq t\leq T} |S_t^1| = \left(\frac{3\pi^2}{8}\right)^{3/4} \quad a.s. \tag{1.2}$$

with $S_t^1 = X(W_t)$ denoting iterated Brownian motion, where X is a two-sided Brownian motion and W is another Brownian motion independent of X. This is the definition of iterated Brownian motion used by Burdzy [4].

Inspired by the above mentioned extensions of the Chung's LIL we extend the above results to composition of a Brownian motion and a symmetric α-stable process.

Theorem 1.1. *Let $\alpha \in (0, 2]$ and let Z_t be the α-time Brownian motion as defined in (1.1). Then we have*

$$\liminf_{T\to\infty} T^{-1/2\alpha}(\log\log T)^{(1+\alpha)/(2\alpha)} \sup_{0\le t\le T} |Z_t| = D_\alpha \quad a.s. \tag{1.3}$$

where $D_\alpha = C_\alpha^{(1+\alpha)/2\alpha}$, $C_\alpha = (\pi^2/8)^{\alpha/(1+\alpha)}(1+\alpha)(2^\alpha\lambda_\alpha)^{1/(1+\alpha)}(\alpha)^{-\alpha/(1+\alpha)}$.

A Chung-type LIL has also been established for other versions of iterated Brownian motion (see [7], [17]) as follows:

$$\liminf_{T\to\infty} T^{-1/4}(\log\log T)^{3/4} \sup_{0\le t\le T} |S_t| = \frac{3^{3/4}\pi^{3/2}}{2^{11/4}} \quad a.s. \tag{1.4}$$

with $S_t \equiv W(|\hat{W}_t|)$ denoting another version of iterated Brownian motion, where W and $\hat{W}$ are independent real-valued standard Brownian motions, each starting from 0. For a generalization of this result to α-time Brownian motions we define the process

$$Z_t^1 \equiv X(|Y_t|), \quad t \ge 0. \tag{1.5}$$

for Brownian motion X_t and symmetric α-stable process Y_t independent of X, each starting from 0, $0 < \alpha \le 2$. For this process we have

Theorem 1.2. *Let $\alpha \in (0, 2]$ and let Z_t^1 be the α-time Brownian motion as defined in (1.5). Then we have*

$$\liminf_{T\to\infty} T^{-1/2\alpha}(\log\log T)^{(1+\alpha)/(2\alpha)} \sup_{0\le t\le T} |Z_t^1| = D_\alpha^1 \quad a.s. \tag{1.6}$$

where $D_\alpha^1 = (C_\alpha^1)^{(1+\alpha)/2\alpha}$, $C_\alpha^1 = (\pi^2/8)^{\alpha/(1+\alpha)}(1+\alpha)(\lambda_\alpha)^{1/(1+\alpha)}(\alpha)^{-\alpha/(1+\alpha)}$.

We note that the constants appearing in (1.3) and (1.6) are different. The main reason for this is that the process Z_t have three independent processes $\{X_t : t \ge 0\}$, $\{X_{-t} : t \le 0\}$ and Y, while the process Z_t^1 does not have a contribution from $\{X_{-t} : t \le 0\}$. The proof of Theorem 1.2 follows the same line of proof of Theorem 1.1, except for the small deviation probability estimates for Z_t^1 we use Theorem 2.4.

The motivation for the study of local times of α-time Brownian motion came from the results of Csáki, Csörgö, Földes, and Révész [11] and Shi and Yor [25] about Kesten–type laws of iterated logarithm for iterated Brownian motion. The study of this type of LIL's was initiated by Kesten [16]. Let B_t be a Brownian motion, $L(x,t)$ its local time at x. Then Kesten showed

$$\limsup_{t\to\infty} \frac{L(0,t)}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \frac{\sup_{x\in\mathbb{R}} L(x,t)}{\sqrt{2t\log\log t}} = 1 \quad a.s. \tag{1.7}$$

and

$$\liminf_{t\to\infty} \frac{\sup_{x\in\mathbb{R}} L(x,t)}{\sqrt{t/\log\log t}} = c \quad a.s. \quad 0 < c < \infty. \tag{1.8}$$

These types of laws were generalized later to symmetric stable processes of index $\alpha \in (1,2)$. More specifically, Donsker and Varadhan [12] generalized (1.7) and Griffin [13] generalized (1.8) to symmetric stable processes.

More recently, Kesten–type LIL's were extended to iterated Brownian motion S. Let $L_S(x,t)$ be the local time of $S_t = W_1(|W_2(t)|)$, with W_1 and W_2 independent standard real-valued Brownian motions. (1.7) was extended to the IBM case by Csáki, Csörgö, Földes, and Révész [11] and Xiao [29]. This result asserts that there exist (finite) universal constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \leq \limsup_{t\to\infty} \frac{L_S(0,t)}{t^{3/4}(\log\log t)^{3/4}} \leq c_2 \quad a.s. \tag{1.9}$$

(1.8) was extended to IBM case by Csáki, Csörgö, Földes, and Révész [11] and Shi and Yor [25]. This result asserts that there exist universal constants $c_3 > 0$ and $c_4 > 0$ such that

$$c_3 \leq \liminf_{t\to\infty} t^{-3/4}(\log\log t)^{3/4} \sup_{x\in\mathbb{R}} L_S(x,t) \leq c_4 \quad a.s. \tag{1.10}$$

Inspired by the definition of local time of IBM in [11], we define the local time of α-time Brownian motion defined in (1.5) as follows:

$$\begin{aligned} L^*(x,t) &= \int_0^\infty \bar{L}_2(u,t)d_uL_1(x,u) \\ &= \int_0^\infty (L_2(u,t) + L_2(-u,t))d_uL_1(x,u), \end{aligned} \tag{1.11}$$

where L_1, L_2 and $\bar{L}_2$ denote, respectively, the local times of X, Y and $|Y|$. A similar definition can be given for α-time Brownian motion defined in (1.1) using the ideas in [5].

In §3, we will extend (1.10) to α-time Brownian motion. We will also obtain partial results towards extension of (1.7). However, our results does not imply an extension of LIL in (1.7) and are far from optimal yet, and leave many problems open. These results will follow from the study of Lévy classes for the local time of Z and Z^1. We extend (1.10) as follows.

Theorem 1.3. *There exists a universal constant $c_L \in (0,\infty)$ such that*

$$\liminf_{t\to\infty} \frac{\sup_{x\in\mathbb{R}} L^*(x,t)}{(t/\log\log t)^{1-1/2\alpha}} = c_L \quad a.s.$$

A similar result holds also for the local time of the process defined in (1.1). The usual LIL or Kolmogorov's LIL for Brownian motion which replaces the time parameter was used essentially in the results in [11] and [25] to prove Kesten's LIL for iterated Brownian motion. However, there does not exist an LIL of this type for symmetric α-stable process which replaces the time parameter in the definition of α-time Brownian motion. To overcome this difficulty we show in Lemma 3.1 that the LIL for the range process of symmetric α-stable process suffices to prove Theorem 1.3.

We also obtain usual LIL for the range of α-time Brownian motion. Then, we use it with a particular case of occupation times formula to obtain Kesten's LIL for these processes. This is also essential in the study of some of Lévy classes of local time of Z^1.

Theorem 1.4. *There exists a universal constant $c_R \in (0,\infty)$ such that*

$$\limsup_{t\to\infty} \frac{R^*(t)}{(t/\log\log t)^{1/2\alpha}\log\log t} = c_R \quad a.s.$$

where $R^*(t) = |\{x: \ Z^1(s) = x \text{ for some } \ s \le t\}|$.

A similar result holds also for range of the process defined in (1.1).

Our proofs of Theorems 1.1 and 1.2 in this paper follow the proofs in [14] making necessary changes at crucial points. In studying the local time of α-time Brownian motion we use the ideas we learned from Csáki, Csörgö, Földes, and Révész [11] and Shi and Yor [25] with necessary changes in the use of usual LIL of the range processes. Our proofs differ from theirs since there does not exist usual LIL for symmetric α-stable process. To overcome this difficulty we show that the usual LIL for the range of symmetric α-stable process suffice for our results, see Lemma 3.1. We also adapt the arguments of Griffin [13] to our case to prove the usual LIL for the range of α-time Brownian motion. Our proofs differs from his in that α-time Brownian motion does not have independent increments. So we decompose the range of symmetric α-stable process to get independent increments, see Lemma 3.6. One of the main difficulties arise from the fact that the inner process Y_t is a stable process instead of Brownian motion, which is not continuous. This makes the arguments more difficult than the previous results on iterared Brownian motion. The paper is organized as follows. We give the proof of Theorem 1.1 in §2. The local time and the range of α-time Brownian motion are studied in §3.

2 Chung's LIL for α-time Brownian motion

We will prove Theorem 1.1 in this section. Section 2.1 is devoted to the preliminary lemmas about the small deviation probabilities. In section 2.2 we prove the lower bound in Theorem 1.1. Upper bound is proved in section 2.3.

2.1 Preliminaries

In this section we give some definitions and preliminary lemmas which will be used in the proof of the main result.

A real-valued symmetric stable process Y_t with index $\alpha \in (0,2]$ is the process with stationary independent increments whose transition density

$$p_t^\alpha(x,y) = p^\alpha(t, x-y), \qquad (t,x,y) \in (0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n,$$

is characterized by the Fourier transform

$$\int_{\mathbb{R}^n} e^{iy\cdot\xi} p^\alpha(t,y)dy = \exp(-t|\xi|^\alpha), \qquad t>0, \xi\in\mathbb{R}^n.$$

The process has right continuous paths, it is rotation and translation invariant.

The following lemma gives the small ball probabilities for the process $\sup_{0\le t\le 1}|Y_t|$.

Lemma 2.1 (Mogul'skii, 1974, [19]). *Let $0 < \alpha \le 2$ and let Y_t be a symmetric α-stable process. Then*

$$\lim_{\epsilon \to 0^+} \epsilon^\alpha \log P[\sup_{0 \le t \le 1} |Y_t| \le \epsilon\ ] = -\lambda_\alpha,$$

where λ_α is the first eigenvalue of the fractional Laplacian operator in the interval $[-1, 1]$.

This is an equivalent statement of the fact that

$$\lim_{t \to \infty} \frac{1}{t} \log P[\tau > t] = -\lambda_\alpha,$$

due to scaling property of $\sup_{0 \le t \le 1} |Y_t|$, where $\tau = \inf\{s : |Y_s| \ge 1\}$ is the first exit time of the interval $[-1, 1]$

Let $R = \sup_{0 \le t \le 1} Y_t - \inf_{0 \le t \le 1} Y_t$ be the range of Y_t. The following is a special case of Theorem 2.1 in [6].

Theorem 2.1 (Mogul'skii, 1974, [19]).

$$\lim_{\epsilon \to 0^+} \epsilon^\alpha \log P[\ R \le \epsilon\ ] = -2^\alpha \lambda_\alpha.$$

We use the following theorem (Kasahara [15, Theorem 3] and Bingham, Goldie and Teugels [3, p. 254]) to find the asymptotics of the Laplace transform of R below.

Theorem 2.2 (de Bruijn's Tauberian Theorem). *Let X be a positive random variable such that for some positive B_1, B_2 and p,*

$$-B_1 \le \liminf_{x \to 0} x^p \log P[X \le x] \le \limsup_{x \to 0} x^p \log P[X \le x] \le -B_2.$$

Then

$$\begin{aligned} -(p+1)(B_1)^{1/(p+1)} p^{-p/(p+1)} &\le \liminf_{\lambda \to \infty} \lambda^{-p/(p+1)} \log E e^{-\lambda X} \\ \le \limsup_{\lambda \to \infty} \lambda^{-p/(p+1)} \log E e^{-\lambda X} &\le -(p+1)(B_2)^{1/(p+1)} p^{-p/(p+1)}. \end{aligned}$$

From de Bruijn's Tauberian theorem and Theorem 2.1 we have

Lemma 2.2.

$$\lim_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} \log E[e^{-\lambda R}] = -(1+\alpha)(2^\alpha \lambda_\alpha)^{1/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)}.$$

The following theorem gives the small ball deviation probabilities for the process Z_t defined in (1.1).

Theorem 2.3. *We have*

$$\lim_{u \to 0} u^{2\alpha/(1+\alpha)} \log P[\sup_{0 \le t \le 1} |Z_t| \le u\ ] = -(\pi^2/8)^{\alpha/(1+\alpha)} A_\alpha,$$

where $A_\alpha = (1+\alpha)(2^\alpha \lambda_\alpha)^{1/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)}$.

Proof. Let X_t be a Brownian motion. From a well-known formula (see Chung [8]):

$$P[\sup_{0\le t\le 1}|X_t|\le u\,]=\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\exp\left[-\frac{(2k-1)^2\pi^2}{8u^2}\right],$$

we get that, for all $u>0$,

$$\frac{2}{\pi}\exp\left(-\frac{\pi^2}{8u^2}\right)\le P[\sup_{0\le t\le 1}|X_t|\le u\,]\le\frac{4}{\pi}\exp\left(-\frac{\pi^2}{8u^2}\right). \tag{2.1}$$

Let $Z_t=X(Y_t)$ be the α-time Brownian motion and let

$$S(t)\equiv\sup_{0\le s\le t}Y_s,\quad I(t)\equiv\inf_{0\le s\le t}Y_s, \tag{2.2}$$

then

$$\begin{aligned}
&P[\,\sup_{0\le t\le 1}|Z_t|\le u\,]\\
&= P[\sup_{0\le t\le S(1)}|X_t|\le u,\ \sup_{I(1)\le t\le 0}|X_t|\le u\,]\\
&= E\left[P\left(\sup_{0\le t\le 1}|X_t|\le\frac{u}{\sqrt{S(1)}}|Y\right)P\left(\sup_{0\le t\le 1}|X_t|\le\frac{u}{\sqrt{-I(1)}}|Y\right)\right]\\
&\le \frac{16}{\pi^2}E\exp\left(-\frac{\pi^2(S(1)-I(1))}{8u^2}\right).
\end{aligned} \tag{2.3}$$

This last inequality follows from the second part of (2.1). Similarly the first part of (2.1) gives us a lower bound, with $4\pi^{-2}$ instead of $16\pi^{-2}$. Now the proof follows from the given inequalities and Lemma 2.2 □

The following theorem gives the small ball deviation probabilities for the process Z_t^1 defined in (1.5).

Theorem 2.4. *We have*

$$\lim_{u\to 0}u^{2\alpha/(1+\alpha)}\log P[\sup_{0\le t\le 1}|Z_t^1|\le u\,]=-(\pi^2/8)^{\alpha/(1+\alpha)}A_\alpha^1,$$

where $A_\alpha^1=(1+\alpha)(\lambda_\alpha)^{1/(1+\alpha)}\alpha^{-\alpha/(1+\alpha)}$.

Proof. Let $M(t)\equiv\sup_{0\le s\le t}|Y_s|$. The proof follows the same line of the proof of Theorem 2.3, except at the end we have

$$\begin{aligned}
P[\sup_{0\le t\le 1}|Z_t^1|\le u\,] &= P[\sup_{0\le t\le M(1)}|X_t|\le u,]\\
&= E\left[P\left(\sup_{0\le t\le 1}|X_t|\le\frac{u}{\sqrt{M(1)}}|Y\right)\right]\\
&\le \frac{4}{\pi}E\exp\left(-\frac{\pi^2M(1)}{8u^2}\right),
\end{aligned} \tag{2.4}$$

and similarly a lower bound with $2/\pi$ instead of $4/\pi$. Then we use Lemma 2.1 and de Bruijn's Tauberian theorem. □

Lemma 2.3. *For all $0 < a \leq b$, $u > 0$ we have*

$$P[\ a < \sup_{0 \leq t \leq u} |Z_t| < b\ ] \leq (b/a - 1)^2.$$

Proof. The proof follows from the proof of Lemma 4.1 in [14]. □

The following proposition is the combination of two propositions in [2], which are stated as Proposition 2 on page 219 and Proposition 4 on page 221.

Proposition 2.1. *Let Y_t be a symmetric α-stable process. Let*

$$S(t) = \sup_{0 \leq s \leq t} Y_s.$$

Then there exists $k_1, k_2 > 0$ such that

$$\lim_{x \to \infty} x^{\alpha} P[Y_1 > x] = \lim_{x \to \infty} x^{\alpha} P[S(1) > x] = k_1,$$

and

$$\lim_{x \to 0^+} x^{-\alpha/2} P[S(1) < x] = k_2.$$

We will use following versions of Borel-Cantelli lemmas in our proofs.

Lemma 2.4 (Borel-Cantelli Lemma 1)**.** *Let $E_1, E_2, \cdots$ be a sequence of events (sets) for which $\sum_{n=1}^{\infty} P[E_n] < \infty$. Then*

$$P[E_n\ i.o] = P[\cap_{n=1}^{\infty} \cup_{i=n}^{\infty} E_i] = 0,$$

i.e with probability 1 only a finite number of events E_n occur simultaneously.

Since the process Z_t does not have independent increments we have to use another version of the Borel-Cantelli lemma which is due to Spitzer [26].

Lemma 2.5 (Borel-Cantelli Lemma 2, p. 28 in [23])**.** *Let $E_1, E_2, \cdots$ be a sequence of events (sets) for which*

$$\sum_{n=1}^{\infty} P[E_n] = \infty \quad \text{and} \ \liminf_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} P[E_i E_j] \bigg/ \left(\sum_{i=1}^{n} P[E_i] \right)^2 \leq c \quad (c \geq 1).$$

Then

$$P[\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k] = P[E_n\ i.o] \geq 1/c.$$

2.2 Proof of the lower bound

The lower bound is easier as always. We use Theorem 2.3. Let

$$C_{\alpha} = (\pi^2/8)^{\alpha/(1+\alpha)} (1+\alpha)(2^{\alpha}\lambda_{\alpha})^{1/(1+\alpha)} (\alpha)^{-\alpha/(1+\alpha)},$$

be the small deviation probability limit for $\sup_{0 \leq t \leq T} |Z_t|$ given in Theorem 2.3. For every fixed $\epsilon > 0$, it follows from Theorem 2.3 that, for T sufficiently large, we have

$$P\left[T^{-1/2\alpha}(\log\log T)^{(1+\alpha)/2\alpha}\sup_{0\le t\le T}|Z_t| \le (1-\epsilon)^{(1+\alpha)/\alpha}C_\alpha^{(1+\alpha)/2\alpha}\right]$$

$$\begin{aligned} &= P\left[\sup_{0\le t\le 1}|Z_t| \le (1-\epsilon)^{(1+\alpha)/\alpha}C_\alpha^{(1+\alpha)/2\alpha}(\log\log T)^{-(1+\alpha)/2\alpha}\right] \\ &\le \exp\left[-(1-\epsilon)(1-\epsilon)^{-2}C_\alpha C_\alpha^{-1}\log\log T\right] \\ &= \exp\left[-\frac{1}{(1-\epsilon)}\log\log T\right]. \end{aligned}$$

Taking a fixed rational number $a>1$ and $T_k = a^k$ gives that

$$\sum_{k\ge 1} P[T_k^{-1/2\alpha}(\log\log T_k)^{(1+\alpha)/2\alpha}\sup_{0\le t\le T_k}|Z_t| \le (1-\epsilon)^{(1+\alpha)/\alpha}C_\alpha^{(1+\alpha)/2\alpha}] < +\infty.$$

It follows from Borel-Cantelli lemma, by letting $\epsilon\to 0$, that

$$\liminf_{k\to+\infty} T_k^{-1/2\alpha}(\log\log T_k)^{(1+\alpha)/2\alpha}\sup_{0\le t\le T_k}|Z_t| \ge C_\alpha^{(1+\alpha)/2\alpha} = D_\alpha \ a.s. \tag{2.5}$$

Since for every $T>0$, there exists $k\ge 0$ such that $T_k \le T < T_{k+1}$, we have

$$\begin{aligned} T^{-1/2\alpha}(\log\log T)^{(1+\alpha)/2\alpha}\sup_{0\le t\le T}|Z_t| &\ge T_{k+1}^{-1/2\alpha}(\log\log T_k)^{(1+\alpha)/2\alpha}\sup_{0\le t\le T_k}|Z_t| \\ &= a^{-1/2\alpha}T_k^{-1/2\alpha}(\log\log T_k)^{(1+\alpha)/2\alpha}\sup_{0\le t\le T_k}|Z_t|, \end{aligned}$$

which together with (2.5), yields the lower bound, as the rational number $a>1$ can be arbitrarily close to 1.

2.3 Proof of the upper bound

We follow the steps in the proof of Lemma 4.2 in [14]. Let $\epsilon>0$ be fixed. For notational simplicity, we use the following in the sequel

$$\begin{aligned} T_k &= \exp(k\log k) \\ a_k &= (1+3\epsilon)^{(1+\alpha)/\alpha}C_\alpha^{(1+\alpha)/2\alpha}T_k^{1/2\alpha}(\log\log T_k)^{-(1+\alpha)/2\alpha} \\ B_k &= \{\sup_{0\le t\le T_k}|Z_t| \le a_k\}. \end{aligned}$$

It follows from Theorem 2.3 that there exists $k_o(\epsilon)$, depending only on ϵ, such that for every $k>k_o(\epsilon)$, we have

$$\begin{aligned} P(B_k) &\ge \exp\left(-(1+3\epsilon)C_\alpha(1+3\epsilon)^{-2}C_\alpha^{-1}\log\log T_k\right) \\ &= \exp\left(-\frac{1}{1+3\epsilon}\log\log T_k\right) \\ &\ge k^{-1/(1+2\epsilon)}, \end{aligned}$$

which yields existence of positive constants $C = C(\epsilon)$ and $N = N(\epsilon)$ such that for every $n > N$,

$$\sum_{k=1}^{n} P(B_k) \geq C n^{2\epsilon/(1+2\epsilon)}. \tag{2.6}$$

We now establish the following

Lemma 2.6. *We have*

$$\liminf_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} P[B_iB_j] \bigg/ \left(\sum_{i=1}^{n} P[B_i]\right)^2 \leq 1. \tag{2.7}$$

Proof of Lemma 2.6. Let $K > 0$ be a constant such that

$$K \geq 1/\alpha(3(1+2\epsilon)/\epsilon - 2),$$

and let $n_\epsilon = [n^{\epsilon/(1+2\epsilon)}]$ (with $[x]$ denoting the integer part of x). We set furthermore

$$E_1 = \{(i,j) : \ 1 \leq i,j \leq n, \ |i-j| \leq 19\}$$

$$E_2 = \{(i,j) : \ n_\epsilon \leq i,j \leq n, \ |i-j| \geq 20\}.$$

Since by (2.6),

$$\sum_{i=1}^{n_\epsilon}\sum_{j=1}^{n} P[B_iB_j] \bigg/ \left(\sum_{i=1}^{n} P[B_i]\right)^2 \leq n_\epsilon \bigg/ \sum_{i=1}^{n} P[B_i] \leq C^{-1} n^{-\epsilon/(1+2\epsilon)},$$

and

$$\sum \sum_{(i,j)\in E_1} P[B_iB_j] \bigg/ \left(\sum_{i=1}^{n} P[B_i]\right)^2 \leq 38 \bigg/ \sum_{i=1}^{n} P[B_i] \leq 38 C^{-1} n^{-2\epsilon/(1+2\epsilon)},$$

it suffices to prove that

$$\liminf_{n\to\infty} \sum \sum_{(i,j)\in E_2} P[B_iB_j] \bigg/ \left(\sum_{i=1}^{n} P[B_i]\right)^2 \leq 1. \tag{2.8}$$

Let

$$S(t) = \sup_{0\leq s\leq t} Y_s, \quad I(t) = \inf_{0\leq s\leq t} Y_s,$$

$$F(x) = P[\sup_{0\leq t\leq 1} |X_t| \leq x], \qquad \forall x > 0.$$

Then for all $i < j$ and all positive numbers $p_1 < p_2, \ q_1 < q_2$,

$$\begin{aligned}
&P[B_iB_j | S(T_i) = p_1, S(T_j) = p_2, I(T_i) = q_1, I(T_j) = q_2] \\
&= P[\sup_{-q_1\leq t\leq p_1} |X_t| \leq a_i, \ \sup_{p_1\leq t\leq p_2} |X_t| \leq a_j, \ \sup_{-q_2\leq t\leq -q_1} |X_t| \leq a_j] \\
&= P[\sup_{0\leq t\leq p_1} |X_t| \leq a_i, \ \sup_{p_1\leq t\leq p_2} |X_t| \leq a_j] \\
&\quad \times P[\sup_{0\leq t\leq q_1} |X_t| \leq a_i, \ \sup_{q_1\leq t\leq q_2} |X_t| \leq a_j].
\end{aligned}$$

Notice that for all $x > 0$ and $y > 0$,
$P[\sup_{0\le t\le p_1} |X_t| \le x,\ \sup_{p_1\le t\le p_2} |X_t| \le y]$

$$\begin{aligned}
&\le P[\sup_{0\le t\le p_1} |X_t| \le x] \sup_{|u|\le x} P[\sup_{0\le t\le p_2-p_1} |X_t + u| \le y] && (2.9)\\
&= P[\sup_{0\le t\le p_1} |X_t| \le x] P[\sup_{0\le t\le p_2-p_1} |X_t| \le y] && (2.10)\\
&= F(xp_1^{-1/2}) F(y(p_2-p_1)^{-1/2}).
\end{aligned}$$

The equation (2.9) is from the Markov property of Wiener processes, and equation (2.10) is due to a general property of Gaussian measures (see, e.g. Ledoux and Talagrand [18, p.73])
It follows that

$$\begin{aligned}
P[B_i] &= E[F(a_i S^{-1/2}(T_i)) F(a_i(-I(T_i))^{-1/2})],\\
P[B_iB_j] &\le E\left[F\left(\frac{a_i}{\sqrt{S(T_i)}}\right) F\left(\frac{a_i}{\sqrt{-I(T_i)}}\right)\right.\\
&\qquad \left.\times F\left(\frac{a_j}{\sqrt{S(T_j)-S(T_i)}}\right) F\left(\frac{a_j}{\sqrt{I(T_i)-I(T_j)}}\right)\right].
\end{aligned}$$

Let $\mathcal{F}_t \equiv \{Y_s, s \le t\}$ and $M(t) \equiv \sup_{0\le s\le t} |Y_s|$. Let $f(t) = \exp(Kt)$. By noticing that $S(T_j) - S(T_i)$ (resp. $I(T_i) - I(T_j)$) is bounded below by the positive part of σ^+ (resp. τ^+) of

$$\begin{aligned}
\sigma &\equiv \sup_{T_i\le t\le T_j} (Y_t - Y_{T_i}) - 2M(T_i)\\
&\qquad (resp.\ \ \tau \equiv -\inf_{T_i\le t\le T_j} (Y_t - Y_{T_i}) - 2M(T_i)), && (2.11)
\end{aligned}$$

$$\begin{aligned}
(i.e.\ \sup_{T_i\le t\le T_j} (Y_t - Y_{T_i}) - 2M(T_i)^+ &\le (\sup_{T_i\le t\le T_j} (Y_t) + |Y_{T_i}| - 2M(T_i))^+\\
&\le M(T_j) - M(T_i)),
\end{aligned}$$

we obtain that

$$\begin{aligned}
&E[F(a_j(S(T_j)-S(T_i))^{-1/2}) F(a_j(I(T_j)-I(T_i))^{-1/2}) | \mathcal{F}_{T_i}]\\
&\quad \le E[F(a_j(\sigma^+)^{-1/2}) F(a_j(\tau^+)^{-1/2}) | \mathcal{F}_{T_i}]\\
&\quad \le \sup_{0\le x\le 2M(T_i)} E[F(a_j((S(T_j-T_i)-x)^+)^{-1/2}) && (2.12)\\
&\qquad \times F(a_j((I(T_j-T_i)-x)^+)^{-1/2})]\\
&\quad \le 1_{\{M(T_i) > T_i^{1/\alpha} f(\log\log T_i)\}} + E[F(a_j(\mu^+)^{-1/2}) F(a_j(\nu^+)^{-1/2})],
\end{aligned}$$

with

$$\mu = S(T_j - T_i) - 2T_i^{1/\alpha} f(\log\log T_i)\ and\ \nu = -I(T_j - T_i) - 2T_i^{1/\alpha} f(\log\log T_i).$$

In the inequality (2.12) we use the fact that α-stable process has stationary independent increments.

As F is always between 0 and 1, we get that

$$
\begin{aligned}
P[B_iB_j] &\le P[M(T_i) > T_i^{1/\alpha} f(\log\log T_i)] \\
&\quad + P[B_i]E[F(a_j(\mu^+)^{-1/2})F(a_j(\nu^+)^{-1/2})] \\
&= P[M(T_i) > T_i^{1/\alpha} f(\log\log T_i)] \\
&\quad + P[B_i]E\left[F\left(\frac{a_j}{\sqrt{(S(1)-\theta)^+(T_j-T_i)^{1/\alpha}}}\right)\right. \\
&\quad \left. \times F\left(\frac{a_j}{\sqrt{(-I(1)-\theta)^+(T_j-T_i)^{1/\alpha}}}\right)\right]. \qquad (2.13)
\end{aligned}
$$

The identity (2.13) is due to the scaling property of α-stable process, with

$$\theta = 2(T_j - T_i)^{-1/\alpha} T_i^{1/\alpha} f(\log\log T_i).$$

Now by using Proposition 2.1, $\lim_{x\to\infty} x^\alpha P[S(1) > x] = k_1$, we get that, if $i \ge n_\epsilon = [n^{\epsilon/(1+2\epsilon)}]$, then since $f(\log\log T_i)$ is large

$$
\begin{aligned}
P[M(T_i) > T_i^{1/\alpha} f(\log\log T_i)] &\le 2P[S(1) > f(\log\log T_i)] \\
&\le 4k_1(f(\log\log T_i))^{-\alpha} \\
&= 4k_1(i\log i)^{-\alpha K} \\
&\le 4k_1 n^{-\alpha K\epsilon/(1+2\epsilon)}, \qquad (2.14)
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\sum_{i=n_\epsilon}^{n}\sum_{j=1}^{n} P[M(T_i) > T_i^{1/\alpha} f(\log\log T_i)] \Bigg/ \left(\sum_{i=1}^{n} P[B_i]\right)^2 \\
&\le 4k_1 C^{-2} n^{2-(\alpha K+2)\epsilon/(1+2\epsilon)} \\
&\le 4k_1 C^{-2} n^{-1}, \qquad (2.15)
\end{aligned}
$$

as $K \ge 1/\alpha(3(1+2\epsilon)/\epsilon - 2)$. On the other hand, for $j \ge i + 20$

$$\theta(T_j/T_i)^{1/2\alpha} \le 2(i\log i)^K/(j^{(j-i)/2}-1)^{1/\alpha} \le 2C_0 j^{-(j-i)/5\alpha} \le 2C_0 j^{-20/5\alpha},$$

which is small for the range of j we consider (if needed we can take $j \ge i + 20 + C(K)$, where $C(K)$ is a constant multiple of K). Since from Proposition 2.1, for x close to 0,

$$P[S(1) < x] \le (1+\epsilon)k_2 x^{\alpha/2},$$

we have

$$
\begin{aligned}
P[S(1) - \theta < (1 - (T_j/T_i)^{1/2\alpha})S(1)] &= P[S(1) < \theta(T_j/T_i)^{1/2\alpha}] \\
&\le 2k_2(\theta(T_j/T_i)^{1/2\alpha})^{\alpha/2} \\
&\le 2k_2(2C_0 j^{-20/5\alpha})^{\alpha/2} \\
&\le C_1 j^{-2},
\end{aligned}
$$

with some universal constant C_1. This inequality holds for $-I(1)$ instead of $S(1)$ as well, since symmetric α-stable process is symmetric. Therefore for all $(i,j) \in E_2$, C_2 being a universal constant, we have

$$\begin{aligned}
&E\left[F\left(\frac{a_j}{\sqrt{(S(1)-\theta)^+(T_j-T_i)^{1/\alpha}}}\right)F\left(\frac{a_j}{\sqrt{(-I(1)-\theta)^+(T_j-T_i)^{1/\alpha}}}\right)\right] \\
&\le 2C_1 j^{-2} + E\left[F\left(\frac{a_j}{\sqrt{(S(1)G_{ij}}}\right)F\left(\frac{a_j}{\sqrt{(-I(1)G_{ij}}}\right)\right] \\
&\qquad (G_{ij} \equiv (T_j-T_i)^{1/\alpha}(1-(T_i/T_j)^{1/2\alpha})) \\
&\le 2C_1 j^{-2} + E\left[F\left(\frac{a_j(1+C_2 j^{-(j-i)/5\alpha})}{\sqrt{S(1)T_j^{1/\alpha}}}\right)F\left(\frac{a_j(1+C_2 j^{-(j-i)/5\alpha})}{\sqrt{-I(1)T_j^{1/\alpha}}}\right)\right] \\
&\qquad (since \ \sqrt{T_j^{1/\alpha}/G_{ij}} \le 1 + C_2 j^{-(j-i)/5\alpha}) \\
&\le 2C_1 j^{-2} + P[\sup_{0\le t\le T_j}|Z_t| \le a_j(1+C_2 j^{-2})] \\
&\le 2C_1 j^{-2} + P[B_j] + P[a_j \le \sup_{0\le t\le T_j}|Z_t| \le a_j(1+C_2 j^{-2})] \\
&\le 2C_1 j^{-2} + P[B_j] + C_2^2 j^{-4},
\end{aligned}$$

where the last inequality follows from Lemma 2.3. Combining this with (2.13), (2.15) and (2.6) gives that

$$\begin{aligned}
&\sum\sum_{(i,j)\in E_2} P[B_iB_j] \Big/ \left(\sum_{i=1}^n P[B_i]\right)^2 \\
&\quad \le 4k_1 C^{-2} n^{-1} + 1 + (2C_1 + C_2^2)\sum_{i=1}^n P[B_i] \sum_{j=1}^n j^{-2} \Big/ \left(\sum_{i=1}^n P[B_i]\right)^2 \\
&\quad \le 1 + 4k_1 C^{-2} n^{-1} + \pi^2(2C_1 + C_2^2)(6C)^{-1} n^{-2\epsilon/(1+2\epsilon)},
\end{aligned}$$

which yields (2.8). □

Since $\sum_{k=1}^\infty P[B_k] = \infty$, it follows from (2.7) and a well-known version of Borel-Cantelli lemma (Lemma 2.5 above) that $P[\limsup_{k\to\infty} B_k] = 1$ which implies the upper bound in Theorem 1.1.

3 Local time of α-time Brownian motion

In this section we give the definition of the local time of α-time Brownian motion and prove its joint continuity. In section 3.0.1 we prove a lemma which is crucial in the proofs of the main theorems. Sections 3.1-3.3 and section 3.5 give a study of the Lévy classes for the local time. In section 3.4 we prove an LIL for the range of α-time Brownian motion.

Let $L_1(x,t)$ be the local time of Brownian motion, and $L_2(x,t)$ be the local time of symmetric α-stable process for $1 < \alpha \le 2$ (see [23] for the properties of the local time of Brownian motion and see [13] and references there in for the properties of the local time of symmetric α-stable processes). Let f, $x \in \mathbb{R}$, be a locally integrable real-valued function. Then

$$\int_0^t f(W_i(s))ds = \int_{-\infty}^{\infty} f(x)L_i(x,t)dx, \quad i = 1, 2, \tag{3.1}$$

for W_1 a standard Brownian motion and W_2 a symmetric stable process.
Then we define the local time of the α-time Brownian motion as

$$L^*(x,t) := \int_0^{\infty} \bar{L}_2(s,t)d_s L_1(x,s), \quad x \in \mathbb{R}, \quad t \geq 0, \tag{3.2}$$

where $\bar{L}_2(x,t) := L_2(x,t) + L_2(-x,t)$, $x \geq 0$.
We prove next the joint continuity of $L^*(x,t)$ and establish the occupation times formula for Z_t^1.

Proposition 3.1. *There exists an almost surely jointly continuous family of "local times", $\{L^*(x,t) : \ t \geq 0, \ x \in \mathbb{R}\}$, such that for all Borel measurable integrable functions, $f : \mathbb{R} \to \mathbb{R}$ and all $t \geq 0$,*

$$\int_0^t f(Z^1(s))ds = \int_0^t f(X(|Y(s)|))ds = \int_{-\infty}^{\infty} f(x)L^*(x,t)dx. \tag{3.3}$$

Proof. By equations (3.1) and (3.2)

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)L^*(x,t)dx &= \int_{-\infty}^{\infty} f(x) \int_0^{\infty} \bar{L}_2(s,t)d_s L_1(x,s)dx \\
&= \int_0^{\infty} \bar{L}_2(s,t)d_s \int_{-\infty}^{\infty} f(x)L_1(x,s)dx \\
&= \int_0^{\infty} \bar{L}_2(s,t)d_s \int_0^s f(X(u))du \\
&= \int_0^{\infty} \bar{L}_2(s,t)f(X(s))ds \\
&= \int_0^t f(X(|Y(s)|))ds. \qquad (3.4)
\end{aligned}$$

Hence we have the equation (3.3). The joint continuity of $L^*(x,t)$ follows from the joint continuity of the local times of Brownian motion and of symmetric stable process. $\square$

We now give the scaling property of local time of Z^1.

Theorem 3.1.

$$L^*(xt^{1/2\alpha},t)/t^{1-1/2\alpha} \stackrel{(d)}{=} \int_0^{\infty} \bar{L}_2(s,1)d_s L_1(x,s)dx = L^*(x,1), \quad x \in \mathbb{R}. \tag{3.5}$$

Corollary 3.1. *For each fixed $t \geq 0$, we have*

$$L^*(0,t)/t^{1-1/2\alpha} \stackrel{(d)}{=} L^*(0,1). \tag{3.6}$$

Proof of Theorem 3.1. The following scaling properties of the Brownian local time and stable local time are well-known.

$$\{L_1(x,t);\ x \in \mathbb{R}, t \geq 0\} \stackrel{(d)}{=} \{\frac{1}{c^{1/2}} L_1(c^{1/2}x, ct);\ x \in \mathbb{R}, t \geq 0\}, \tag{3.7}$$

and

$$\{L_2(x,t);\ x \in \mathbb{R}, t \geq 0\} \stackrel{(d)}{=} \{\frac{1}{c^{1-1/\alpha}} L_1(c^{1/\alpha}x, ct);\ x \in \mathbb{R}, t \geq 0\}, \tag{3.8}$$

where $c > 0$ is an arbitrary fixed number. Consequently we have

$$\begin{aligned}
L^*(x,t) &= \int_0^\infty \bar{L}_2(s,t) d_s L_1(x,s) \\
&\stackrel{(d)}{=} \frac{1}{c^{1-1/\alpha}} \int_0^\infty \bar{L}_2(c^{1/\alpha}s, ct) d_s L_1(x,s) \quad c > 0 \ \text{ fixed} \\
&\stackrel{(d)}{=} t^{1-1/\alpha} \int_0^\infty \bar{L}_2(\frac{s}{t^{1/\alpha}}, 1) d_s L_1(x,s), \quad c = 1/t, t > 0 \ \text{ fixed} \\
&\stackrel{(d)}{=} t^{1-1/\alpha} \int_0^\infty \bar{L}_2(u,1) d_u L_1(x, ut^{1/\alpha}), \quad u = s/t^{1/\alpha}, t > 0 \ \text{ fixed} \\
&\stackrel{(d)}{=} t^{1-1/\alpha} t^{1/2\alpha} \int_0^\infty \bar{L}_2(u,1) d_u L_1(\frac{x}{t^{1/\alpha}}, u), \quad t > 0 \ \text{ fixed } \ x \in \mathbb{R}.
\end{aligned}$$

Clearly, the last equation is equivalent to (3.5). □

3.0.1 Preliminaries

In this section we will prove a lemma which is crucial in the proof of the following theorems. The usual LIL or Kolmogorov's LIL for Brownian motion which replaces the time parameter was used essentially in the results in [11] and [25] to prove Kesten's LIL for iterated Brownian motion. However, there does not exist an LIL of this type for symmetric α-stable process. To overcome this difficulty with the use of the following lemma we show below that the LIL for the range process of symmetric α-stable process suffices to prove Theorem 1.3.

Lemma 3.1. *Let $A \subset \mathbb{R}_+$ be Lebesgue measurable. Let $L(x,A)$ be local time of Brownian motion over the set A. Then*

$$\sup_{x\in\mathbb{R}} L(x,A) \stackrel{(d)}{=} \sup_{x\in\mathbb{R}} L(x,|A|) = \sup_{x\in\mathbb{R}} L(x,[0,|A|]),$$

where $|.|$ denotes Lebesgue measure and $\stackrel{(d)}{=}$ means equality in distribution.

Proof. We use monotone class theorem from [24]. Define

$$\mathcal{S} = \{A \subset \mathbb{R}_+ : \quad \sup_{x\in\mathbb{R}} L(x,A) \stackrel{(d)}{=} \sup_{x\in\mathbb{R}} L(x,|A|)\}.$$

Obviously $\mathbb{R}_+ \in \mathcal{S}$. Let $A, B \in \mathcal{S}$ and $A \subset B$. Since $L(x,A)$ is an additive measure in the set variable

$$\sup_{x\in\mathbb{R}} L(x,B) = \sup_{x\in\mathbb{R}} L(x,A) + \sup_{x\in\mathbb{R}} L(x, B \setminus A),$$

so

$$\begin{aligned}\sup_{x\in\mathbb{R}} L(x,|B|) &\overset{(d)}{=} \sup_{x\in\mathbb{R}} L(x,B)\\ &= \sup_{x\in\mathbb{R}} L(x,A)+\sup_{x\in\mathbb{R}} L(x,B\setminus A).\end{aligned}$$

On the other hand,

$$\sup_{x\in\mathbb{R}} L(x,|B|)=\sup_{x\in\mathbb{R}} L(x,|A|)+\sup_{x\in\mathbb{R}} L(x,(|A|,|B|)),$$

hence

$$\begin{aligned}\sup_{x\in\mathbb{R}} L(x,|B|) &\overset{(d)}{=} \sup_{x\in\mathbb{R}} L(x,|A|)+\sup_{x\in\mathbb{R}} L(x,(|A|,|B|))\\ &\overset{(d)}{=} \sup_{x\in\mathbb{R}} L(x,A)+\sup_{x\in\mathbb{R}} L(x,B\setminus A).\end{aligned}$$

For $0<|A|<\infty$, the moment generating function of $\sup_{x\in\mathbb{R}} L(x,|A|)$ satisfies for some $\delta>0$, (see [16, Remark p.452])

$$0<MGF(t)=E[e^{t\sup_{x\in\mathbb{R}} L(x,|A|)}]<\infty, \quad for \quad |t|<\delta.$$

Since $\sup_{x\in\mathbb{R}} L(x,|A|)$ and $\sup_{x\in\mathbb{R}} L(x,(|A|,|B|))$ are independent and similarly $\sup_{x\in\mathbb{R}} L(x,A)$ and $\sup_{x\in\mathbb{R}} L(x,B\setminus A)$ are independent, considering moment generating functions (in case $|A|=0$ or $|B\setminus A|=0$, we do not need generating functions) which is the product of the moment generating functions, we get that $B\setminus A\in\mathcal{S}$.

Let $A_n\subset A_{n+1}$ be an increasing sequence of sets in $\mathcal{S}$. For $\lambda>0$,

$$\begin{aligned}P[\sup_{x\in\mathbb{R}} L(x,\cup_{n=1}^{\infty}A_n)\le\lambda] &= P[\sup_{x\in\mathbb{R}}\lim_{n\to\infty} L(x,A_n)\le\lambda]\\ &= \lim_{n\to\infty} P[\sup_{x\in\mathbb{R}} L(x,A_n)\le\lambda]\\ &= \lim_{n\to\infty} P[\sup_{x\in\mathbb{R}} L(x,|A_n|)\le\lambda]\\ &= P[\sup_{x\in\mathbb{R}} L(x,|\cup_{n=1}^{\infty}A_n|)\le\lambda].\end{aligned}$$

Hence $\cup_{n=1}^{\infty}A_n\in\mathcal{S}$.

Now to complete the proof we show that open intervals are in $\mathcal{S}$. Every interval is in $\mathcal{S}$, since the increments of Brownian motion are stationary. It is clear that sets of measure zero are also in $\mathcal{S}$. Hence $\mathcal{S}$ contains every Lebesgue measurable set by monotone class theorem.

□

3.1 On upper-upper classes

For further information on the Lévy classes we refer to Révész [23].

In this section we prove

Theorem 3.2. *There exists a $t_0 = t_0(w)$ and a universal constant $C_{uu} \in (0,\infty)$ such that for $t > t_0$ we have*

$$L^*(0,t) \le \sup_{x\in\mathbb{R}} L^*(x,t) \le C_{uu} t^{1-1/2\alpha} (\log\log t)^{1+1/2\alpha} \quad a.s. \tag{3.9}$$

Proof. By the LIL for the range $R(t) = |\{x;\ Y(s) = x \text{ for some } \ s \le t\}|$ given in Griffin [13]

$$\limsup_{t\to\infty} t^{-1/\alpha} (\log\log t)^{-(1-1/\alpha)} R(t) = c_1 \ a.s. \tag{3.10}$$

and the Kesten type LIL for Y, established by Donsker and Varadhan [12], for some finite constant c_2

$$\limsup_{t\to\infty} t^{-(1-1/\alpha)} (\log\log t)^{-1/\alpha} \sup_{x\in\mathbb{R}} L_2(x,t) = c_2 \quad a.s. \tag{3.11}$$

$$\begin{aligned}
\sup_{x\in\mathbb{R}} L^*(x,t) &= \sup_{x\in\mathbb{R}} \int_0^\infty \bar{L}_2(s,t) d_s L_1(x,s) \\
&= \sup_{x\in\mathbb{R}} \int_{Y[0,t]} \bar{L}_2(s,t) d_s L_1(x,s) \quad a.s \\
&= O(t^{1-1/\alpha} (\log\log t)^{1/\alpha} \sup_{x\in\mathbb{R}} L_1(x, \mathbb{R}_+ \cap Y[0,t])) \\
&= O(t^{1-1/\alpha} (\log\log t)^{1/\alpha} \sup_{x\in\mathbb{R}} L_1(x, |Y[0,t]|)) \ (\textit{by Lemma } 3.1) \\
&= O(t^{1-1/\alpha} (\log\log t)^{1/\alpha} \sup_{x\in\mathbb{R}} L_1(x, ct^{1/\alpha} (\log\log t)^{1-1/\alpha})) \quad a.s. \\
&= O(t^{1-1/2\alpha} (\log\log t)^{1+1/2\alpha}) \quad a.s.
\end{aligned} \tag{3.12}$$

with t_0 big enough, by using the LIL for the range of Y and LIL of Kesten for $\sup_{s\in\mathbb{R}} \bar{L}_2(s,t)$ from (3.10)-(3.11) respectively, and then applying the Kesten's LIL once again to $\sup_{x\in\mathbb{R}} L_1(x, ct^{1/\alpha}(\log\log t)^{1-1/\alpha})$ given in [16]. □

3.2 On upper-lower classes

In this section we prove

Theorem 3.3. *There exists a universal constant $C_{ul} \in (0,\infty)$ such that*

$$P[\sup_{x\in\mathbb{R}} L^*(x,t) \ge C_{ul} t^{1-1/2\alpha} (\log\log t)^{(1+\alpha)/2\alpha} \quad i.o] = 1. \tag{3.13}$$

Since the $\log\log$ powers do not match in equations (3.9) and (3.13), we cannot deduce an LIL for $\sup_{x\in\mathbb{R}} L^*(x,t)$.

Proof. Proof follows from Theorem 1.2 and the observation

$$t = \int_{x\in S(t)} L^*(x,t)dx \le R^*(t) \sup_{x\in\mathbb{R}} L^*(x,t), \quad a.s.$$

where $R^*(t) = |S(t)| = |\{x : \ Z^1(s) = x \text{ for some } \ s \le t\}|$ and the fact that $R^*(t) \le 2\sup_{0\le s\le t} |Z_s^1|$. □

3.3 On lower-upper classes

In this section we prove

Theorem 3.4. *There exists a universal constant* $C_{lu} \in (0, \infty)$ *such that*

$$P[L^*(0,t) \le \sup_{x\in\mathbb{R}} L^*(x,t) \le C_{lu}(\frac{t}{\log\log t})^{1-1/2\alpha} \quad i.o] = 1. \tag{3.14}$$

Proof. We have from Csáki and Földes [10]: for $0 \le a \le 1$

$$P[\sup_{x\in\mathbb{R}} L_1(x,1) \le a] \ge \exp(-\frac{c}{a^2}), \tag{3.15}$$

for some absolute constant $c > 0$. A similar result for the local time of Y is given in [13]: there exists $\theta > 0$ and $c_1 > 0$ such that for t large

$$P[\sup_{x\in\mathbb{R}} L_2(x,1) \le \theta/(\log\log t)^{1-1/\alpha}] \ge c_1\beta^{(\log\log t)}, \tag{3.16}$$

with $e^{-1} < \beta < 1$.

Define $c_2 = \sqrt{4c/d}$ with $C_\beta = d + \log\beta^{-1} < 1$ and $t_k = \exp(k^p)$, with $p \in (1, 1/C_\beta)$ for k large. Consider

$$\begin{aligned}
s_k &\stackrel{def}{=} 2c_4 t_k^{1/\alpha}(\log\log t_k)^{1-1/\alpha}, \quad c_4 \text{ constant in LIL of range of Y,}\\
D_k &\stackrel{def}{=} \left\{\sup_{x\in\mathbb{R}}(L_1(x,s_k) - L_1(x,s_{k-1})) \le \frac{c_2 t_k^{1/2\alpha}}{2(\log\log t_k)^{1/2\alpha}}\right\},\\
E_k &\stackrel{def}{=} \left\{\sup_{x\in\mathbb{R}}(L_2(x,t_k) - L_2(x,t_{k-1})) \le \frac{\theta t_k^{1-1/\alpha}}{(\log\log t_k)^{1-1/\alpha}}\right\},\\
F_k &\stackrel{def}{=} D_k \cap E_k.
\end{aligned}$$

We have by means of (3.15) and (3.16),

$$P[D_k] \ge P[\sup_{x\in\mathbb{R}} L_1(x,1) \le \frac{c_2}{2(\log\log t_k)^{1/2}}] \ge \exp(-d\log\log t_k),$$

$$P[E_k] \ge P[\sup_{x\in\mathbb{R}} L_2(x,1) \le \frac{\theta}{(\log\log t_k)^{1-1/\alpha}}] \ge c_1\beta^{(\log\log t_k)}.$$

Hence

$$P[F_k] = P[D_k]P[E_k] \ge \frac{c_1}{k^{pC_\beta}},$$

which implies $\sum_k P[F_k] = \infty$. Thanks to the independence of the $F_k's$, we can apply the Borel-Cantelli lemma to conclude that, almost surely there exists infinitely many k's for which F_k is realized. On the other hand, by the Kesten LIL, for X and Y for all large k,

$$\sup_{x\in\mathbb{R}} L_1(x,s_{k-1}) \le 2(s_{k-1}\log\log s_{k-1})^{1/2} \le \frac{c_2 t_k^{1/2\alpha}}{(\log\log t_k)^{1/2\alpha}},$$

$$\sup_{x\in\mathbb{R}} L_2(x,t_{k-1}) \le 2c_5 t_{k-1}^{1-1/\alpha}(\log\log t_{k-1})^{1/\alpha} \le \frac{c_6 t_k^{1-1/\alpha}}{(\log\log t_k)^{1-1/\alpha}}.$$

Therefore there exist infinitely many k's such that

$$\sup_{x\in\mathbb{R}} L_1(x,s_k) \le \frac{2c_2 t_k^{1/2\alpha}}{(\log\log t_k)^{1/2\alpha}}, \tag{3.17}$$

$$\sup_{x\in\mathbb{R}} L_2(x,t_k) \le \frac{(\theta+c_6) t_k^{1-1/\alpha}}{(\log\log t_k)^{1-1/\alpha}}. \tag{3.18}$$

For those k satisfying (3.17)-(3.18), we have, by the usual LIL for the range of Y given in (3.10),

$$\begin{aligned}
\sup_{x\in\mathbb{R}} L^*(x,t_k) &= \sup_{x\in\mathbb{R}} \int_0^\infty \bar{L}_2(u,t_k)d_u L_1(x,u) \\
&= \sup_{x\in\mathbb{R}} \int_{Y[0,t_k]} \bar{L}_2(u,t_k)d_u L_1(x,u) \\
&\le 2\sup_{y\in\mathbb{R}} L_2(y,t_k) \sup_{x\in\mathbb{R}} L_1(x,\mathbb{R}_+\cap Y[0,t_k]) \\
&\le 4c_2(\theta+c_6)\left(\frac{t_k}{\log\log t_k}\right)^{1-1/2\alpha} \quad (\textit{by Lemma } 3.1).
\end{aligned}$$

□

3.4 The range

In this section we will prove an LIL for the range $R^*(t) = |\{x: \ Z^1(s) = x \text{ for some } \ s \le t\}|$. The idea of the proof is to look at the large jumps of the symmetric stable process which replaces the time parameter in the process $Z^1(t)$. To prove LIL for the range of Z^1 we need several lemmas. We adapt the arguments of Griffin [13] to our case in the following lemmas.

If $Y(t)$ is a process and T is some, possibly random, time then $Y^*(T) = \sup\{|Y(r)|: \ 0 \le r \le T\}$. If $S < T$ then $(Y(T) - Y(S))^* = \sup\{|Y(r) - Y(S)| : S \le r \le T\}$.

Definition 3.1. $T_Y(a) = \inf\{s: \ |Y(s) - Y(s-)| > a\}$.

We will usually write $T_Y(a) = T(a)$ if it is clear which process we are referring to.

Lemma 3.2 (Griffin [13]). *The random variables $a^{-1/\alpha}Y^*(T(a^{1/\alpha}-))$ and $Y^*(T(1)-)$ have the same distribution.*

Using the scaling of Brownian motion it is easy to deduce

Lemma 3.3. *The random variables*

$$a^{-1/2\alpha}X^*(Y^*(T(a^{1/\alpha}-)))$$

and

$$X^*(Y^*(T(1)-))$$

have the same distribution.

As in Griffin [13], we can decompose Y as the sum of two independent Lévy processes

$$Y(t) = Y_1(t) + Y_2(t),$$

where

$$\begin{aligned} Y_2 &= \sum_{s\leq t}(Y(s) - Y(s-))1\{|Y(s) - Y(s-)| > 1\} \\ Y_1(t) &= Y(t) - Y_2(t). \end{aligned}$$

The Lévy measure of Y_1 is given by $1\{|x| \leq 1\}|x|^{-1-\alpha}dx$ and the moment generating function by

$$E[\exp(aY_1(t))] = \exp(t\psi(a)),$$

where

$$\psi(a) = \int_{-1}^{1}(e^{ax} - 1)\frac{dx}{|x|^{1+\alpha}}.$$

Observe that $\psi(a) \to 0$ as $a \to 0$.

Lemma 3.4. *[Griffin [13]] If a is small enough that $\psi(a) < 2\alpha^{-1}$, then*

$$E[\exp(aY^*(T(1)-))] \leq \frac{8\alpha^{-1}}{2\alpha^{-1} - \psi(a)}.$$

We deduce the following from the last lemma.

Lemma 3.5. *If a is small enough that $\psi(a^2/2) \leq 2\alpha^{-1}$, then*

$$E[\exp(aX^*(Y^*(T(1)-)))] \leq \frac{32\alpha^{-1}}{2\alpha^{-1} - \psi(a^2/2)}.$$

Proof. The moments of Brownian motion X are given by $E[\exp(\theta X(t))] = \exp(\theta^2 t/2)$, so

$$E[\exp(aX^*(Y^*(T(1)-)))] = \int_0^\infty \int_0^\infty E[\exp(aX^*(l))]f_l(s)2\alpha^{-1}e^{-2\alpha^{-1}}dlds,$$

where $f_l(s)$ is the density of $Y^*(s)$.
Now $P[X^*(t) > x] \leq 2P[|X(t)| > x]$ for each $x > 0$ $t > 0$, hence

$$E[\exp(aX^*(l))] \leq 2E[\exp(a|X(l)|)] \leq 4E[\exp(aX(l))] = \exp(a^2 l/2),$$

therefore

$$E[\exp(aX^*(Y^*(T(1)-)))] \leq 4\int_0^\infty E[\exp(\frac{a^2}{2}Y^*(s))]2\alpha^{-1}e^{-2\alpha^{-1}}ds. \tag{3.19}$$

From Lemma 3.4 we deduce that

$$E[\exp(aX^*(Y^*(T(1)-)))] \leq \frac{32\alpha^{-1}}{2\alpha^{-1} - \psi(a^2/2)}.$$

□

Definition 3.2. $J(t,\gamma(t)) = \#\{s \le t : \ |Y(s) - Y(s-)| > \gamma(t)\}$ *where* $\gamma(t) = (t/\log\log t)^{1/\alpha}$.

We know from [13] that $J(t,\gamma(t))$ has Poisson distribution with parameter $2\log\log t/\alpha$.

Definition 3.3. *Fix* $t > 0$ *and define*

$$\begin{aligned}
T_1 &= \inf\{s : \ |Y(s) - Y(s-)| \ge \gamma(t)\} \\
T_{k+1} &= \inf\{s > T_k : \ |Y(s) - Y(s-)| \ge \gamma(t)\}. \\
V_k^1 &= (X(|Y(T_k-)|) - X(|Y(T_{k-1})|))^* \\
V_k &= X^*(((Y(T_k-)) - Y(T_{k-1}))^*) \\
W_k^1 &= (t/\log\log t)^{-1/2\alpha} V_k^1 \\
W_k &= (t/\log\log t)^{-1/2\alpha} V_k.
\end{aligned}$$

Observe that W_k, $k = 1, 2, \cdots$ are identically distributed as

$$X^*(Y^*(T(1)-))$$

by Lemma 3.3.

Observe also that W_k^1, $k = 1, 2, \cdots$ are identically distributed as

$$X^*((|Y(T_2-)| - |Y(T_1)|)^*).$$

Furthermore

$$X^*((|Y(T_2-)| - |Y(T_1)|)^*) \le X^*((Y(T_2-) - Y(T_1))^*).$$

Finally, observe that $X^*((Y(T_2-) - Y(T_1))^*)$ and $X^*(Y^*(T(1)-))$ are identically distributed by Lemma 3.3.

Since the paths of Y are not non-decreasing, the processes W_k, $k = 1, 2, \cdots$ are not independent. To get independent processes we have to decompose the image of Y. So we define

$$\begin{aligned}
V_1^* &= \sup_{0 \le s \le \sup_{0 \le r \le T_1-} |Y(r)|} |X(s)| \\
V_k^* &= \sup_{s,l \in A_k} |X(s) - X(l)|,
\end{aligned}$$

where $A_k = |Y|[T_{k-1}, T_k-] \cap (|Y|[0, T_{k-1}-])^C$, $k = 2, 3, \cdots$ Observe that given Y, V_k^* are independent for $k = 1, 2, \cdots$, and $V_k^* \le 2V_k^1$. Define

$$W_k^* = (t/\log\log t)^{-1/2\alpha} V_k^*.$$

Now let $\varphi(t)$ denote the function $(t/\log\log t)^{1/2\alpha}\log\log t$.

Lemma 3.6. *If* λ *is sufficiently large, then*

$$P[V_1^* + \cdots + V_{J(t,\gamma(t))+1}^* \ge \lambda\varphi(t)] \le (\log t)^{-2}.$$

Proof. We have by a lemma in Griffin [13], for β sufficiently large

$$P[J(t,\gamma(t)) \geq [\beta \log\log t]] \leq \frac{1}{2}\frac{1}{(\log t)^2}. \tag{3.20}$$

Since $V_k^* \geq 0$,

$$\begin{aligned} P[V_1^* + \cdots + V_{J(t,\gamma(t))+1}^* \geq \lambda\varphi(t)] &\leq P[V_1^* + \cdots + V_{[\beta\log\log t]}^* \geq \lambda\varphi(t)] \\ &+ P[J(t,\gamma(t)) \geq [\beta\log\log t]]. \end{aligned}$$

Let $\xi = 32\alpha^{-1}/(2\alpha^{-1} - \psi((2a)^2/2))$ and choosing β to satisfy (3.20), we see that by Lemma 3.5 for a sufficiently small and llt denoting $\log\log t$

$$\begin{aligned} P[V_1^* + \cdots + V_{[\beta llt]}^* &\geq \lambda\varphi(t)] = P[W_1^* + \cdots + W_{[\beta llt]}^* \geq \lambda llt] \\ &\leq \exp(-a\lambda llt)E[(E[\exp(aW_1^*)|Y] \\ &\quad \times E[\exp(aW_2^*)|Y] \cdots E[\exp(aW_{[\beta llt]}^*)|Y])] \\ &\leq \exp(-a\lambda llt)E[(E[\exp(2aW_1)|Y] \\ &\quad \times E[\exp(2aW_2)|Y] \cdots E[\exp(2aW_{[\beta llt]})|Y])] \\ &\leq \exp(-a\lambda llt)E[(4E[\exp(2a^2U_1)|Y] \\ &\quad \times 4E[\exp(2a^2U_2)|Y] \cdots 4E[\exp(2a^2U_{[\beta llt]})|Y])] \\ &\leq \exp(-a\lambda llt)(4E[\exp(2a^2U_1)])^{[\beta llt]} \qquad (3.21) \\ &\leq \exp(-llt(a\lambda - \beta\log\xi)) \\ &\leq \frac{1}{2(\log t)^2}, \end{aligned}$$

if λ is sufficiently large. Where

$$U_k = (t/\log\log t)^{-1/\alpha}(Y(T_k-) - Y(T_{k-1}))^*.$$

In the fourth line inequality we use equation (3.19). In equation (3.21) we use the fact that $U_k's$ are i.i.d. with common distribution $Y^*(T(1)-)$ and Lemma 3.4. □

Theorem 3.5. *There exits a t_0 such that for $t > t_0$, and for certain constants $C, K \in (0,\infty)$*

$$R^*(t) \geq Ct^{1/2\alpha}(\log\log t)^{-(1+1/2\alpha)} \ \ a.s.$$

and

$$P[R^*(t) \geq Kt^{1/2\alpha}(\log\log t)^{1-1/2\alpha} \quad i.o] = 1.$$

Proof. This follows from Theorem 3.2 and

$$t = \int_{x\in S(t)} L^*(x,t)dx \leq R^*(t)\sup_{x\in\mathbb{R}} L^*(x,t),$$

where $R^*(t) = |S(t)| = |\{x : \ Z^1(s) = x \text{ for some } \ s \leq t\}|$. The last probability follows similarly using Theorem 3.4. □

Proof of Theorem 1.4. We will prove only the upper bound in the light of Theorem 3.5. To prove the upper bound observe that

$$R^*(t) \le V_1^* + \cdots + V_{J(t,\gamma(t))+1}^*.$$

Thus by Lemma 3.6 for λ sufficiently large

$$P[R^*(t) \ge \lambda\varphi(t)] \le (\log t)^{-2}.$$

Hence for large n,

$$\begin{aligned} P[R^*(t) &\ge \lambda\varphi(t) \text{ for some } t \in [2^n, 2^{n+1})] \\ &\le P[R^*(2^{n+1}) \ge \lambda\varphi(2^n)] \\ &\le P[R^*(2^{n+1}) \ge (\lambda c)\varphi(2^{n+1})] \\ &\le \frac{C_{last}}{(n+1)^2}. \end{aligned}$$

if λ is sufficiently large. The result follows from Borel-Cantelli lemma. □

3.5 On lower-lower classes

In this section we prove

Theorem 3.6. *There exists a $t_0 = t_0(w)$ and a universal constant $C_{ll} \in (0, \infty)$ such that for $t > t_0$*

$$\sup_{x\in\mathbb{R}} L^*(x,t) \ge C_{ll}\left(\frac{t}{\log\log t}\right)^{1-1/2\alpha} \quad a.s. \tag{3.22}$$

Proof. The proof follows from Theorem 1.4 and the observation

$$t = \int_{x\in S(t)} L^*(x,t)dx \le R^*(t) \sup_{x\in\mathbb{R}} L^*(x,t),$$

where $R^*(t) = |S(t)| = |\{x : \; Z^1(s) = x \text{ for some } \; s \le t\}|$. □

Proof of Theorem 1.3. Theorems 3.4 and 3.6 imply the proof. □

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