Electronic Journal of Probability

Vol. 12 (2007), Paper no. 47, pages 1276–1298.

Journal URL
http://www.math.washington.edu/~ejpecp/

Uniformly Accurate Quantile Bounds Via The Truncated Moment Generating Function: The Symmetric Case

Michael J. Klass*
University of California
Departments of Mathematics and Statistics
Berkeley, CA 94720-3840, USA
klass@stat.berkeley.edu

Krzysztof Nowicki
Lund University
Department of Statistics
Box 743, S-220 07 Lund, Sweden
krzysztof.nowicki@stat.lu.se

Abstract

Let $X_1, X_2, \ldots$ be independent and symmetric random variables such that $S_n = X_1 + \ldots + X_n$ converges to a finite valued random variable S a.s. and let $S^* = \sup_{1 \le n < \infty} S_n$ (which is finite a.s.). We construct upper and lower bounds for s_y and s_y^*, the upper $\frac{1}{y}^{\text{th}}$ quantile of S_y and S^*, respectively. Our approximations rely on an explicitly computable quantity $\underline{q}_y$ for which we prove that

$$\frac{1}{2}\underline{q}_{y/2} < s_y^* < 2\underline{q}_{2y} \quad \text{and} \quad \frac{1}{2}\underline{q}_{\frac{y}{4}(1+\sqrt{1-8/y})} < s_y < 2\underline{q}_{2y}.$$

The RHS's hold for $y \ge 2$ and the LHS's for $y \ge 94$ and $y \ge 97$, respectively. Although our results are derived primarily for symmetric random variables, they apply to non-negative variates and extend to an absolute value of a sum of independent but otherwise arbitrary random variables.

Key words: Sum of independent random variables, tail distributions, tail probabilities, quantile approximation, Hoffmann-Jørgensen/Klass-Nowicki Inequality.

*Supported by NSF Grant DMS-0205054.

AMS 2000 Subject Classification: Primary 60G50, 60E15, 46E30; Secondary: 46B09.

Submitted to EJP on September 21, 2006, final version accepted September 24, 2007.

1 Introduction

The classical problem of approximating the tail probability of a sum of independent random variables dates back at least to the famous central limit theorem of de Moivre (1730–33) for sums of independent identically distributed (*i.i.d.*) Bernoulli random variables. As time has passed, increasingly general central limit theorems have been established and stable limit theorems proved. All of these results were asymptotic and approximated the distribution of the sum only sufficiently near its center.

With the advent of the upper and lower exponential inequalities of Kolmogoroff (1929), the possibility materialized of approximating the tail probabilities of a sum having a fixed, finite number of mean zero uniformly bounded random summands, over a much broader range of values.

In the interest of more exact approximation, Esscher (1932) recovered the error in the exponential bound by introducing a change of measure, thereby expressing the exact value of the tail probability in terms of an exponential upper-bound times an expectation factor (which is a kind of Laplace transform). Cramér (1938) brought it to the attention of the probability community, showing that it can be effectively used to reduce the relative error in approximation of small tail probabilities.

However, in the absence of special conditions, the expectation factor is not readily tractable, involving a function of the original sum transformed in such a way that the level to be exceeded by the original sum for the tail probability in question has been made equal or at least close to the mean of the sum of the transformed variable(s).

Over the years various approaches have been used to contend with this expectation term. By slightly re-shifting the mean of the sum of i.i.d. non-negative random variables and applying the generalized mean value theorem, Jain and Pruitt (1987) obtained a quite explicit tail probability approximation. Given any $n \geq 1$, any exceedance level z and any non-degenerate i.i.d. random variables $X_1, ..., X_n$, Hahn and Klass (1997) showed that for some unknown, universal constant $\Delta > 0$

$$\Delta B^2 \leq P(\sum_{j=1}^{n} X_j \geq z) \leq 2B. \tag{1}$$

The quantity B was determined by constructing a certain common truncation level and applying the usual exponential upper bound to the probability that the sum of the underlying variables each conditioned to remain below this truncation level had sum at least z. The level was chosen so that the chance of any X_j reaching or exceeding that height roughly equalled the exponential upper bound then obtained. Employing a local probability approximation theorem of Hahn and Klass (1995) it was found that the expectation factor in the Esscher transform could be as small as the exponential bound term itself but of no smaller order of magnitude as indicated by the LHS of (1). By separately considering the contributions to S_n of those X_j at or below some truncation level t_n and those above it, the method of Hahn and Klass bears some resemblance to work of Nagaev (1965) and Fuk and Nagaev (1971). Other methods required more restrictive distributional assumptions and so are of less concern to us here.

The general (non-i.i.d.) case has been elusive, we now believe, because there is no limit to how much smaller the reduction factor may be compared to the exponential bound (of even suitably truncated random variables).

To see this, let

$$X_i = \begin{cases} 1 & \text{wp } p_i \\ 0 & \text{wp } 1-2p_i \\ -1 & \text{wp } p_i \end{cases}$$

and suppose $p_1 >> p_2 >> ... >> p_n > p_{n+1} \equiv 0$. Then, for $1 \leq k \leq n$

$$P(\sum_{j=1}^{n} X_j \geq k) \approx p_1 p_2 ... p_k \prod_{j=k+1}^{n} (1-2p_j)$$

Using exponential bounds, the bound for $P(\sum_{j=1}^{n} X_j \geq k)$ is

$$\inf_{t>0} e^{-tk} \prod_{j=1}^{n} Ee^{tX_j}$$

and for $P(\sum_{j=1}^{n} X_j > k)$ it is

$$\lim_{\delta \searrow 0} \inf_{t>0} e^{-t(k+\delta)} \prod_{j=1}^{n} Ee^{tX_j},$$

which is, in fact, the same exponential factor.

Therefore in this second case the reduction factor is even smaller than p_{k+1}, which could be smaller than $P^{\alpha}(\sum_{j=1}^{n} X_j \geq k)$ for any $\alpha > 0$ prescribed before construction of p_{k+1}.

It is this additional reduction factor which the Esscher transform provides and which we now see can sometimes be far more accurate in identifying the order of magnitude of a tail probability than the exponential bound which it was thought to merely adjust a bit.

Nor does the direct approach to tail probability approximation offer much hope because calculation of convolutions becomes unwieldy with great rapidity.

As a lingering alternative one could try to employ characteristic functions. They have three principle virtues:

- They exist for all random variables.
- They retain all the distributional information.
- They readily handle sums of independent variables by converting a convolution to a product of marginal characteristic functions.

Thus if $S_n = X_1 + ... + X_n$ where the X_j are independent rv's, for any amount a with $P(S_n = a) = 0$ the inversion formula is

$$P(S_n \geq a) = \lim_{b \to \infty} \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \prod_{j=1}^{n} Ee^{itX_j} dt \tag{2}$$

Clearly, the method becomes troublesome when applied to marginal distributions $X_1, ..., X_n$ for which at least one of the successive limits converges arbitrary slowly. In addition, formula (2) does not hold nor is it continuous at any atom a of S_n. Moreover, as we have already witnessed, the percentage drop in order of magnitude of the tail probability as a moves just above a given atom can be arbitrarily close to 100%.

The very issues posed by both characteristic function inversion and change of measure have given rise to families of results: asymptotic results, large deviation results, moderate deviation results, steepest descent results, etc. Lacking are results which are explicit and non-asymptotic, applying to all fixed n sums of independent real-valued random variables without restrictions and covering essentially the entire range of the sum distribution without limiting or confining the order of magnitude of the tail probabilities.

In 2001 Hitczenko and Montgomery-Smith, inspired by a reinvigorating approach of Latała (1997) to uniformly accurate p-norm approximation, showed that for any integer $n \geq 1$, any exceedance level z, and any n independent random variables satisfying what they defined to be a Levy condition, there exists a constructable function $f(z)$ and a universal positive constant $c > 1$ whose magnitude depends only upon the Levy-condition parameters such that

$$c^{-1} f(cz) \leq P(\mid \sum_{j=1}^{n} X_j \mid > z) \leq c f(c^{-1} z). \tag{3}$$

To obtain their results they employed an inequality due to Klass and Nowicki (2000) for sums of independent, symmetric random variables which they extended to sums of arbitrary random variables at some cost of precision. Using a common truncation level for the $|X_j|$ and a norm of the sum of truncated random variables, Hitczenko and Montgomery-Smith (1999) previously obtained approximations as in (3) for sums of independent random variables all of which were either symmetric or non-negative.

In this paper we show that the constant c in (3) can be chosen to be at most 2 for a slightly different function f if $|\sum_{j=1}^{n} X_j|$ is replaced by $S_n^* = \max_{1 \leq k \leq n} S_n$ and if the X_j are symmetric. A slightly weaker result is obtained for S_n itself. We obtain our function as the solution to a functional equation involving the moment generating function of the sum of truncated random variables.

The accuracy of our results depends primarily upon an inequality pertaining to the number of event recurrences, as given in Klass and Nowicki (2003), a corollary of which extends the tail probability inequality of Klass and Nowicki (2000) in an optimal way from independent symmetric random elements to arbitrary independent random elements.

In effect our approach has involved the recognition that uniformly good approximation of the tail probability of S_n was impossible. However, if we switched attention to upper-quantile approximation, then uniformly good approximation could become feasible at least for sums of independent symmetric random variables. In this endeavor we have been able to obtain adequate precision from moment generating function information for sums of truncated rv's obviating the need to entertain the added complexity of transform methods despite their potentially greater exactitude.

Although our results are derived for symmetric random variables, they apply to non-negative variates and have some extension to a sum of independent but otherwise arbitrary random variables via the concentration function $\mathcal{C}_Z(y)$, where

$$\mathcal{C}_Z(y) = \inf\{c : P(\mid Z - b \mid \leq c) \geq 1 - \frac{1}{y} \text{ for some } b \in \mathcal{R}\} \text{ (for } y > 1), \tag{4}$$

by making use of the fact that the concentration function for any random variable Z possesses a natural approximation based on the symmetric random variable $Z-\tilde{Z}$ where $\tilde{Z}$ is an independent copy of Z, due to the inequalities

$$\frac{1}{2}s_{|Z-\tilde{Z}|,\frac{y}{2}} \leq \mathcal{C}_Z(y) \leq s_{|Z-\tilde{Z}|,y}. \tag{5}$$

Thus, to obtain reasonably accurate concentration function and tail probability approximations of sums of independent random variables, it is sufficient to be able to obtain explicit upper and lower bounds of $s_{\sum_{j=1}^n X_j}$ if the X_j are independent and symmetric.

These results can be extended to Banach space settings.

2 Tail probability approximations

Let $X_1, X_2, ...$ be independent, symmetric random variables such that letting $S_n = X_1 + ... + X_n$ $\lim_{n\to\infty} S_n = S$ a.s. and $\sup_{n\to\infty} S_n = S^* < \infty$ a.s. We introduce the upper $\frac{1}{y}^{\underline{\text{th}}}$ quantile s_y satisfying

$$s_y = \sup\{s : P(S \geq s) \geq \frac{1}{y}\}. \tag{6}$$

For sums of random variables that behave like a normal random variable, s_y is slowly varying. For sums which behave like a stable random variable with parametern $\alpha < 2$, s_y grows at most polynomially fast. However, if an X_j has a sufficiently heavy tail, then s_y can grow arbitrary rapidly.

Analogously we define

$$s^*_y = \sup\{s : P(S^* \geq s) \geq \frac{1}{y}\}. \tag{7}$$

To approximate s_y and s^*_y in the symmetric case, to which this paper is restricted, we utilize the magnitude t_y, the upper $\frac{1}{y}^{\underline{\text{th}}}$ quantile of the maximum of the X_j, defined as

$$t_y = \sup\{t : P(\bigcup_{j=1}^{\infty}\{X_j \geq t\}) \geq \frac{1}{y}\}. \tag{8}$$

This quantity allows us to rewrite each X_j in terms of the sum of two quantities: a quantity of relatively large absolute value, $(| X_j | -t_y)^+ \text{sgn } X_j$, and one of more typical size, $(| X_j | \wedge t_y) \text{sgn } X_j$.

Take any $y > 1$ and let

$$\underline{X}_{j,y} = (| X_j | \wedge t_y) \text{sgn } X_j \tag{9}$$

$$\underline{S}_{j,y} = \underline{X}_{1,y} + ... + \underline{X}_{j,y} \tag{10}$$

$$\underline{S}_y = \lim_{n\to\infty} \underline{S}_{n,y} \text{ a.s.} \tag{11}$$

$$\underline{S}^*_y = \sup_{1\leq n<\infty} \underline{S}_{n,y} \tag{12}$$

$$\underline{s}_y = \sup\{s : P(\underline{S}_y \geq s) \geq \frac{1}{y}\} \tag{13}$$

$$\underline{s}_y^* = \sup\{s : P(\underline{S}_y^* \geq s) \geq \frac{1}{y}\} \tag{14}$$

Since t_y can be computed directly from the marginal distributions of the X_j's, we regard t_y as computable. At the very least, the tail probability functions defined in (6), (7), (13) and (14) require knowledge of convolutions. Generally speaking we regard such quantities as inherently difficult to compute. It is the object of this paper to construct good approximations to them. To simplify this task it is useful to compare them with one another.

Notice that for $y > 1$

$$s_y \leq s_y^*, \qquad \underline{s}_y \leq \underline{s}_y^* \qquad \text{and} \qquad s_y^* \leq s_{2y}. \tag{15}$$

The first two inequalities are trivial; the last one follows from Levy's inequality.

Due to the effect of truncation, one may think that $\underline{s}_y^*$ never exceeds s_y^*. The example below provides a case to the contrary. In fact, $\underline{s}_y^*/s_y^*$ can equal $+\infty$.

Example 2.1. Let, for $y > 2$,

$$X_1 = \begin{cases} 2 & \text{w p } 1 - \sqrt{1 - 1/y} \\ 0 & \text{w p } 2\sqrt{1 - 1/y} - 1 \\ -2 & \text{w p } 1 - \sqrt{1 - 1/y} \end{cases}$$

and

$$X_2 = \begin{cases} 1 & \text{w p } 1 - \sqrt{1 - 1/y} \\ 0 & \text{w p } 2\sqrt{1 - 1/y} - 1 \\ -1 & \text{w p } 1 - \sqrt{1 - 1/y} \end{cases}$$

and, for $0 < \epsilon < \frac{1}{2}$

$$X_3 = \begin{cases} \epsilon & \text{w p } (1 - \delta)/2 \\ 0 & \text{w p } \delta \\ -\epsilon & \text{w p } (1 - \delta)/2. \end{cases}$$

Then $t_y = 1$ and $P(S^* \geq 0) > \frac{1}{2}$. For simplicity set $p = 2\sqrt{1 - 1/y} - 1$. Then

$$\begin{aligned} & P(S^* > 0) = P(S^* \geq \epsilon) \\ & = P(X_1 = 2) + P(X_1 = 0, X_2 = 1) + P(X_1 = X_2 = 0, X_3 = \epsilon) \\ & = \frac{1-p}{2} + p\frac{1-p}{2} + p^2\frac{1-\delta}{2} = \frac{1-p^2\delta}{2}. \end{aligned}$$

There exists $0 < \delta^* < 1$ such that $\frac{1}{2}(1 - p^2\delta^*) = \frac{1}{y}$. To see this note that when $\delta = 0$, $\frac{1}{2} > \frac{1}{y}$ and when $\delta = 1$, $\frac{1}{2}(1 - p^2) < \frac{1}{y}$. Hence, when $\delta^* < \delta < 1$, $s_y^* = 0$.

It also follows that there exists a unique $\delta^* < \delta^{**} < 1$ such that

$$\frac{1}{2}(1 - p^2\delta^{**}) + \frac{(1-p)^2}{4}\frac{(1 - \delta^{**})}{2} = \frac{1}{y}.$$

Hence for $\delta^* < \delta < \delta^{**}$ and when the random variables are truncated at $t_y = 1$:

$$\begin{aligned}
P(\underline{S}_y^* \geq \epsilon) &= P(\underline{X}_1 = 1) + P(\underline{X}_1 = 0, \underline{X}_2 = 1) + P(\underline{X}_1 = \underline{X}_2 = 0, \underline{X}_3 = \epsilon) \\
&+ P(\underline{X}_1 = -1, \underline{X}_2 = 1, \underline{X}_3 = \epsilon) \\
&= \frac{1-p}{2} + p\frac{1-p}{2} + p^2\frac{1-\delta}{2} + \frac{(1-p)^2}{4}(\frac{1-\delta}{2}) \\
&= \frac{1-p^2\delta}{2} + \frac{(1-p)^2}{4}(\frac{1-\delta}{2}) \\
&\geq \frac{1}{y}.
\end{aligned}$$

So $\underline{s}_y^* \geq \epsilon$ for $\delta^* < \delta < \delta^{**}$. Consequently, $\underline{s}_y^*/s_y^* = \infty$.

Nevertheless, reducing y by a factor u_y allows us to compare these quantities, as the following lemma describes.

Lemma 2.1. *Suppose $y \geq 4$ and let $u_y = \frac{y}{2}(1 - \sqrt{1 - \frac{4}{y}})$. Then for independent symmetric X_j's,*

$$\underline{s}_{y/u_y}^* \leq s_y^* \leq \underline{s}_{2y}^*. \tag{16}$$

Thus, for $y \geq 2$

$$\underline{s}_y^* \leq s_{y^2/(y-1)}^*. \tag{17}$$

Proof: To verify the RHS of (16), suppose there exists $\underline{s}_{2y}^* < s < s_y^*$. We have

$$\begin{aligned}
\frac{1}{y} &\leq P(S^* \geq s) \leq P(\underline{S}_{2y}^* \geq s, \bigcap_{j=1}^{\infty}\{X_j \leq t_{2y}\}) \\
&+ P(\bigcup_{j=1}^{\infty}\{X_j > t_{2y}\}) < \frac{1}{2y} + \frac{1}{2y} = \frac{1}{y}.
\end{aligned}$$

which gives a contradiction. Hence $s_y^* \leq \underline{s}_{2y}^*$.

To prove the LHS of (16) by contradiction, suppose there exists $s_y^* < s < \underline{s}_{y/u_y}^*$. Let

$$A_y = \bigcap_{j=1}^{\infty}\{X_j \geq -t_{y/u_y}\}.$$

We have

$$\begin{aligned}
&\frac{u_y}{y} \le P(\underline{S}^*_{y/u_y} \ge s) \\
&\le P(\underline{S}^*_{y/u_y} \ge s \mid A_y) \\
&\text{(because each of the marginal distributions} \\
&(\mid X_j \mid \wedge t_{y/u_y})\mathrm{sgn}(X_j) \text{ is made stochastically} \\
&\text{larger by conditioning on } A_y \text{ while preserving independence)} \\
&\le P(\sup_n \sum_{j=1}^n (X_j \vee (-t_{y/u_y}) \mid A_y) = P(S^* \ge s \mid A_y) \\
&= \frac{P(S^* \ge s, A_y)}{P(A_y)} \le \frac{P(S^* \ge s)}{1 - P(A_y^c)} < \frac{\frac{1}{y}}{1 - u_y/y}.
\end{aligned}$$

However, by the quadratic formula,

$$\frac{u_y}{y} - (\frac{u_y}{y})^2 = \frac{1}{y},$$

which gives a contradiction.

Reparametrizing the middle of (16) gives (17)

$\square$

Remark 2.1. Removing the stars in the proof of (16) and (17) establishes two new chains of inequalities

$$\underline{s}_{y/u_y} \le s_y \le \underline{s}_{2y}, \quad y \ge 4 \tag{18}$$

and

$$\underline{s}_y \le s_{y^2/(y-1)}, \quad y \ge 2. \tag{19}$$

$\square$

Lemma 2.2. *Take any $y > 1$. Then*

$$t_y \le s^*_{2y} \wedge \underline{s}^*_{2y} \tag{20}$$

and

$$t_y \le s_{4y^2/(2y-1)} \wedge \underline{s}_{4y^2/(2y-1)} \tag{21}$$

These inequalities are essentially best possible, as will be shown by examples below.

Proof: To prove (20) we first show that $t_y \le s^*_{2y}$. Let

$$\tau = \begin{cases} \text{last } 1 \le k < \infty : X_k \ge t_y \\ \infty \text{ if no such } k \text{ exists.} \end{cases}$$

Conditional on $\tau = k$, S_{k-1} is a symmetric random variable. Hence

$$\begin{aligned}
P(S^* \geq t_y) &\geq \sum_{k=1}^{\infty} P(S^* \geq t_y, \tau = k) \\
&\geq \sum_{k=1}^{\infty} P(S_k \geq t_y, \tau = k) \\
&\geq \sum_{k=1}^{\infty} P(S_{k-1} \geq 0, \tau = k) \\
&\geq \sum_{k=1}^{\infty} \frac{1}{2} P(\tau = k) \quad \text{(by conditional symmetry)} \\
&= \frac{1}{2} P(\bigcup_{k=1}^{\infty} \{X_k \geq t_y\}) \geq (2y)^{-1}.
\end{aligned}$$

Thus $s^*_{2y} \geq t_y$.

The other inequality is proved similarly.

To prove (21) let

$$\tau = \begin{cases} \text{first } 1 \leq k < \infty : |X_k| \geq t_y \\ \infty \text{ if no such } k \text{ exists.} \end{cases}$$

The RHS of (21) is non-negative. Hence we may suppose $t_y > 0$.

$$\begin{aligned}
P(S \geq t_y) &\geq \sum_{k=1}^{\infty} P(S \geq t_y, \tau = k) \\
&\geq \sum_{k=1}^{\infty} P(S \geq t_y, \tau = k, X_k \geq t_y) \\
&\geq \sum_{k=1}^{\infty} P(\sum_{j=1}^{k-1} X_j I(|X_j| < t_y) + \sum_{j=k+1}^{\infty} X_j \geq 0, \tau = k, X_k \geq t_y) \\
&\geq \sum_{k=1}^{\infty} \frac{1}{2} P(\tau = k, X_k \geq t_y) \\
&= \frac{1}{4} P(\tau < \infty).
\end{aligned}$$

To lower-bound $P(\tau < \infty)$ let $P^+_{j,y} = P(X_j \geq t_y)$. Notice that since X is symmetric and $t_y > 0$ we must have $P^+_{j,y} \leq \frac{1}{2}$. Given $P(\bigcup_{j=1}^{\infty} \{X_j \geq t_y\}) \geq \frac{1}{y}$ we have

$$1 - \prod_{j=1}^{\infty} (1 - P^+_{j,y}) = P(\bigcup_{j=1}^{\infty} \{X_j \geq t_y\}) \geq \frac{1}{y}$$

so

$$\prod_{j=1}^{\infty}(1-P^+_{j,y}) \le 1-\frac{1}{y}.$$

Since $0 \le 1-2P^+_{j,y} \le (1-P^+_{j,y})^2$

$$\prod_{j=1}^{\infty}(1-2P^+_{j,y}) \le\le \big(\prod_{j=1}^{\infty}(1-P^+_{j,n})\big)^2 \le (1-\frac{1}{y})^2.$$

Consequently,

$$P(\tau<\infty) = (1-\prod_{j=1}^{\infty}(1-2P^+_{j,y})) \ge 1-(1-\frac{1}{y})^2 = \frac{2}{y}-\frac{1}{y^2}$$

so that $P(S \ge t_y) \ge \frac{1}{2y}-\frac{1}{4y^2}$. Therefore $t_y \le s_{4y^2/(2y-1)}$.

By essentially the same reasoning $t_y \le \underline{s}_{4y^2/(2y-1)}$.

$\square$

(20) is best possible for $y > \frac{4}{3}$ as the following example demonstrates.

Example 2.2. For any $y > \frac{4}{3}$ and $y < z < 2y$ we can have $s^*_z < t_y$ and $\underline{s}^*_z < t_y$. Let X_1 be uniform on $(-a,a)$ for any $0 < a \le 1$. For $j = 2,3$ let

$$X_j = \begin{cases} 1 & \text{w p } 1-\sqrt{1-\frac{1}{y}} \\ 0 & \text{w p } 2\sqrt{1-\frac{1}{y}}-1 \\ -1 & \text{w p } 1-\sqrt{1-\frac{1}{y}} \end{cases}$$

and $X_j = 0$ for $j \ge 4$.
Then $P(\bigcup_{j=1}^{\infty}\{X_j \ge 1\}) = \frac{1}{y}$ so $t_y = 1$. If $s^*_z \ge t_y$ or $\underline{s}^*_z \ge t_y$, then

$$\begin{aligned}
\frac{1}{z} &\le P(\max_{1\le k\le 3}\sum_{j=1}^{k} X_j \ge 1) \\
&= P(X_1 \ge 0, X_2 = 1) + P(X_1 \ge 0, X_2 = 0, X_3 = 1) \\
&+ P(X_1 < 0, X_2 = 1, X_3 = 1) \\
&= \frac{1}{2}(1-\sqrt{1-\frac{1}{y}})(1+(2\sqrt{1-\frac{1}{y}}-1)+1-\sqrt{1-\frac{1}{y}}) \\
&= \frac{1}{2}(1-\sqrt{1-\frac{1}{y}})(1+\sqrt{1-\frac{1}{y}}) \\
&= \frac{1}{2y}
\end{aligned}$$

which gives a contradiction.

Next we show that

Example 2.3. For any fixed $0 < \epsilon < 1$ there exists $1 << y_\epsilon < \infty$ such that for each $y \geq y_\epsilon$ we may have

$$s_{4y^2/(2y-1+\epsilon)} < t_y.$$

Fix n large. Let X_1 be uniform on $(-a, a)$ for any $0 < a \leq 1$. For $j = 2, 3, ..., n+1$ let

$$X_j = \begin{cases} 1 & \text{w p } 1 - (1 - \frac{1}{y})^{1/n} \\ 0 & \text{w p } 2(1 - \frac{1}{y})^{1/n} - 1 \\ -1 & \text{w p } 1 - (1 - \frac{1}{y})^{1/n} \end{cases}$$

and let $X_j = 0$ for $j > n+1$.
Then

$$P(\bigcup_{j=1}^{\infty}\{X_j \geq 1\}) = 1 - P^n(X_2 < 1) = 1 - (1 - \frac{1}{y}) = \frac{1}{y}$$

so $t_z = 1$ for $z \geq y$.
Fix any $\epsilon > 0$ and suppose that $s_{4y^2/(2y-1+\epsilon)} \geq 1$. Then

$$\begin{aligned}
\frac{2y-1+\epsilon}{4y^2} &\leq P(\sum_{j=1}^{n+1} X_j \geq 1) \\
&= P(X_1 \geq 0, \sum_{j=2}^{n+1} X_j = 1) + P(\sum_{j=2}^{n+1} X_j \geq 2) \\
&\leq \frac{n}{2} P(X_2 = 1) P(\bigcap_{j=3}^{n+1}\{X_j = 0\}) \\
&+ \binom{n}{2} P^2(X_2 = 1) P(\bigcap_{j=4}^{n+1}\{X_j = 0\}) + P(\sum_{j=2}^{n+1} \mid X_j \mid \geq 3) \\
&\underset{n\to\infty}{\longrightarrow} (-\frac{1}{2}\ln(1 - \frac{1}{y}))(1 - \frac{1}{y})^2 + \frac{1}{2}(\ln(1 - \frac{1}{y}))^2(1 - \frac{1}{y})^2 + O(\frac{1}{y^3}).
\end{aligned}$$

As $y \to \infty$ the latter quantity equals

$$\frac{1}{2y} - \frac{1}{4y^2} + O(\frac{1}{y^3}) < \frac{2y - 1 + \epsilon/2}{4y^2},$$

for all y sufficiently large, which gives a contradiction.

Moreover, as the next example demonstrates, t_y divided by the upper $\frac{1}{y}^{\text{th}}$-quantile of each of the above variables may be unbounded. It also shows that s_y^* and $\underline{s}_y^*$ (as well as s_y and $\underline{s}_y$) can be zero for arbitrary large values of y. (Note: These quantities are certainly non-negative for $y \geq 2$.)

Example 2.4. Take any $y \geq \frac{4}{3}$. Let X_1 and X_2 be i.i.d. with

$$X_1 = \begin{cases} 1 & \text{w p } 1 - \sqrt{1 - 1/y} \\ 0 & \text{w p } 2\sqrt{1 - 1/y} - 1 \\ -1 & \text{w p } 1 - \sqrt{1 - 1/y} \end{cases}$$

Also, set $0 = X_3 = X_4 = ...$ Then,

$$0 = P(\bigcup_{j=1}^{\infty}\{X_j > 1)\}) < P(\bigcup_{j=1}^{\infty}\{X_j \geq 1\}) = 1 - P^2(X_1 < 1) = \frac{1}{y}.$$

Therefore $t_y = 1$. Observe that

$$\begin{aligned} P(S^* > 0) = P(S^* \geq 1) &= P(X_1 = 1) + P(X_1 = 0, X_2 = 1) \\ &= 2(1 - \sqrt{1 - 1/y})\sqrt{1 - 1/y}. \end{aligned}$$

This quantity is less than $1/y$. Therefore $s_y^* = \underline{s}_y^* = 0$. Also, since $P(\underline{S}_y < 0) = P(S < 0) < \frac{1}{y}$, $\underline{s}_y = s_y = 0$.

□

We want to approximate s_y^* and s_y based on the behavior of the marginal distributions of variables whose sum is S.

Suppose we attempt to construct our approximation by means of a quantity $\underline{q}_y$ which involves some reparametrization of the moment generating function of a truncated version of S. Inspired by Luxemburg's approach to constructing norms for functions in Orlicz space and affording ourselves as much latitude as possible, we temporarily introduce arbitrary positive functions $f_1(y)$ and $f_2(y)$, defining $\underline{q}_y$ as the real satisfying

$$\underline{q}_y = \sup\{q > 0 : E(f_1(y))^{\underline{S}_y/q} \geq f_2(y)\}. \tag{22}$$

To avoid triviality we also require that $\underline{S}_y$ be non-constant. Since $\underline{S}_y$ is symmetric, $E(f_1(y))^{\underline{S}_y/q} = E(f_1(y))^{-\underline{S}_y/q} > 1$. Therefore we may assume that $f_1(y) > 1$ and $f_2(y) > 1$. Given $f_1(y)$ and constant $0 < c^* < \infty$, we want to choose $f_2(y)$ so that $\underline{s}_y^* < c^*\underline{q}_y$ and $\underline{q}_y$ is as small as possible.

Notice that a sum of independent, symmetric, uniformly bounded rv's converges a.s. iff its variance is finite. Consequently, by Lemma 3.1 to follow, $h(q) < \infty$ where $h(q) = E(f_1(y))^{\underline{S}_y/q}$. Clearly, $h(q)$ is a strictly decreasing continuous function of $q > 0$ with range $(1, \infty)$, (see Remark 3.1). Hence, there is a unique $\underline{q}_y$ such that

$$E(f_1(y))^{\underline{S}_y/\underline{q}_y} = f_2(y). \tag{23}$$

Ideally, we would like to choose $f_2(y)$ so that $s_y^* = c^*\underline{q}_y$. But since we can not directly compare $\underline{s}_y^*$ and $\underline{q}_y$, we must content ourselves with selecting a value for $f_2(y)$ in $(1, y^{-1}(f_1(y))^{c^*}]$ because for these values we can demonstrate that $\underline{s}_y^* < c^*\underline{q}_y$. Since $\underline{q}_y$ decreases as $f_2(y)$ increases, we will set $f_2(y) = y^{-1}(f_1(y))^{c^*}$. This gives the sharpest inequality our method of proof can provide.

Lemma 2.3. *Fix any $y > 1$, any $f_1(y) > 1$, and any $c^* > 0$ such that $(f_1(y))^{c^*}/y > 1$. Define $\underline{q}_y$ according to (22) with $f_2(y) = y^{-1}(f_1(y))^{c^*}$. Then, if $t_y \neq 0$,*

$$\underline{s}_y^* < c^* \underline{q}_y. \tag{24}$$

Proof: Since $t_y \neq 0$, $\underline{S}_y$ is non-constant and so $\underline{q}_y > 0$. To prove (24) suppose $\underline{s}_y^* \geq c^* \underline{q}_y$. Let

$$\tau = \begin{cases} 1^{\text{st}} n : \underline{S}_{y,n} \geq c^* \underline{q}_y & \\ \infty & \text{if such } n \text{ doesn't exist.} \end{cases}$$

We have

$$\begin{aligned} \frac{1}{y} &\leq P(\underline{S}_y^* \geq c^* \underline{q}_y) = P(\tau < \infty) \\ &\leq \sum_{n=1}^{\infty} E[(f_1(y))^{(\underline{S}_{y,n}/\underline{q}_y)-c^*} I(\tau = n)] \\ &\leq \sum_{n=1}^{\infty} E[(f_1(y))^{(\underline{S}_y/\underline{q}_y)-c^*} I(\tau = n)] \\ &= E[(f_1(y))^{(\underline{S}_y/\underline{q}_y)-c^*} I(\tau < \infty)] \\ &< E(f_1(y))^{(\underline{S}_y/\underline{q}_y)-c^*} \\ &\text{(since } P(\tau = \infty) > 0 \text{ and } \underline{S}_y \text{ is finite)} \\ &\leq \frac{1}{y} - E[(f_1(y))^{(\underline{S}_y/\underline{q}_y)-c^*} I(\tau = \infty)] \\ &\leq \frac{1}{y}, \end{aligned} \tag{25}$$

which gives a contradiction.

□

We can extend this inequality to the following theorem:

Theorem 2.4. *Let $f(y) > 1$ and $0 < c^* < \infty$ be such that $f^{c^*}(y) > y \geq 2$. Suppose $t_y > 0$. Let*

$$\underline{q}_y = \sup\{q > 0 : E(f(y))^{\underline{S}_y/q} \geq y^{-1} f^{c^*}(y)\}. \tag{26}$$

Then

$$\max\{\underline{s}_y^*, t_y\} < c^* \underline{q}_y. \tag{27}$$

Proof: We already know that $\underline{s}_y^* < c^* \underline{q}_y$. Suppose $t_y = c^{**} \underline{q}_y$. Let $\hat{X}_j = t_y(I(X_j \geq t_y) - I(X_j \leq -t_y))$. There exist $0-1$ valued random variables δ_j such that $\{X_i, \delta_j : 1 \leq i, j < \infty\}$ are independent and $P(\bigcup_{j=1}^{\infty}\{\delta_j \hat{X}_j = t_y\}) = \frac{1}{y}$. Setting $p_j = P(\delta_j \hat{X}_j = t_y)(= P(\delta_j \hat{X}_j > 0))$ we obtain $1 - \frac{1}{y} = \prod_{j=1}^{\infty}(1 - p_j)$.

Then let $a = (f(y))^{c^{**}} - 2 + (f(y))^{-c^{**}} > 0$ and notice that, for all $y \geq 2$, $1 + a/y > f^{c^{**}}(y)/y$. Moreover, $\prod_{j=1}^{\infty}(1-p_j) \geq 1 - \sum_{j=1}^{\infty} p_j$. Consequently, $\sum_{j=1}^{\infty} p_j \geq \frac{1}{y}$.
Clearly,

$$\begin{aligned}
& y^{-1}(f(y))^{c^*} = E(f(y))^{\underline{S}_y/\underline{q}_y} = E(f(y))^{c^{**}\underline{S}_y/t_y} = \prod_{j=1}^{\infty} E(f(y))^{c^{**}\underline{X}_{j,y}/t_y} \\
& \geq \prod_{j=1}^{\infty} E(f(y))^{c^{**}\delta_j \hat{X}_j/t_y} = \prod_{j=1}^{\infty}(1 + p_j((f(y))^{c^{**}} - 2 + (f(y))^{-c^{**}})) \\
& = \prod_{j=1}^{\infty}(1 + ap_j) \geq 1 + a\sum_{j=1}^{\infty} p_j \geq 1 + \frac{a}{y} \\
& > \frac{f(y)^{c^{**}}}{y} \quad \text{for all } y \geq 2.
\end{aligned} \tag{28}$$

This gives a contradiction whenever $c^{**} \geq c^*$.

$\square$

Since
$$E(f(y))^{\underline{S}_y/\underline{q}_y} = E((f(y))^{c^*})^{\underline{S}_y/(c^*\underline{q}_y)},$$
by redefining $f(y)$ as $(f(y))^{c^*}$, q_y is redefined in a way which makes $c^* = 1$. Hence Theorem 2.4 can be restated as

Corollary 2.5. *Take any $y \geq 2$ such that $\underline{S}_y$ is non-constant. Then*

$$\max\{\underline{s}_y^*, t_y\} \leq \inf_{z>y}\{q_y(z) : Ez^{\underline{S}_y/q_y(z)} = \frac{z}{y}\} \tag{29}$$

with equality iff $\max\{\underline{s}_y^*, t_y\} = \mathit{ess}\sup \underline{S}_y$.

Proof: (29) follows from (23) and (27). To get the details correct, note first that for any such $y > 1$

$$\lim_{z\to\infty} Ez^{(\underline{S}_y/q)-1} = \begin{cases} \infty & \text{if } 0 < q < \text{ess sup}\,\underline{S}_y \\ P(\underline{S}_y = \text{ess sup}\,\underline{S}_y) & \text{if } q = \text{ess sup}\,\underline{S}_y \\ 0 & \text{if } q > \text{ess sup}\,\underline{S}_y \end{cases}$$

Hence $q_y(z) \to \operatorname{ess\,sup} \underline{S}_y$ as $z \to \infty$ and so

$$\inf_{z>y}\{q_y(z) : Ez^{\underline{S}_y/q_y(z)} = \frac{z}{y}\} \leq \operatorname{ess\,sup} \underline{S}_y. \tag{30}$$

Therefore, if $\max\{\underline{s}_y^*, t_y\} = \operatorname{ess\,sup} \underline{S}_y$ we must have equality in (29). W.l.o.g. we may assume that

$$\max\{\underline{s}_y^*, t_y\} < \operatorname{ess\,sup} \underline{S}_y. \tag{31}$$

There exist monotonic $z_n \in (y, \infty)$ such that

$$q_y(z_n) \longrightarrow \inf_{z>y}\{q_y(z) : Ez^{\underline{S}_y/q_y(z)} = \frac{z}{y}\} < \infty. \tag{32}$$

Let $z_\infty = \lim_{n\to\infty} z_n$. If $z_\infty = y$ then $q_y(z_n) \to \infty$ which is impossible as indicated in (32).

If $z_\infty = \infty$, then $\lim_{n\to\infty} q_y(z_n) \geq \operatorname{ess\,sup} \underline{S}_y$ which gives strict inequality in (29) by application of (31).

Finally, suppose $z_\infty \in (y, \infty)$. By dominated convergence the equation $Ez_n^{\underline{S}_y/q_y(z_n)} = \frac{z_n}{y}$ converges to equation $Ez_\infty^{\underline{S}_y/q_y(z_\infty)} = \frac{z_\infty}{y}$. Hence $\max\{\underline{s}_y^*, t_y\} = \inf_{z>y}\{q_y(z) : Ez^{\underline{S}_y/q_y(z)} = \frac{z}{y}\} = q_y(z_\infty) > \max\{s_y^*, t_y\}$ by Theorem 2.4.

□

The example below verifies that inequality (29) is sharp.

Example 2.5. Consider a sequence of probability distributions such that, for each $n \geq 1$, $X_{n1}, X_{n2}, ..., X_{nn}$ are i.i.d. and $P(X_{nj} = 1/\sqrt{n}) = P(X_{nj} = -1/\sqrt{n}) = 0.5,\ j = 1, ..., n$. Letting n tend to ∞ we find from (29) that

$$s_y^* \leq \inf_{z>y}\{q_y(z) : Ez^{B(1)/q_y(z)} = \frac{z}{y}\} \tag{33}$$

where $B(t)$ is standard Brownian motion and

$$s_y^* = \sup\{s : P(\max_{1\leq t\leq 1} B(t) \geq s) \geq \frac{1}{y}\}. \tag{34}$$

As is well known,

$$s_y^* \sim \sqrt{2\ln y} \ \text{ as } y \to \infty. \tag{35}$$

Notice that

$$Ez^{B(1)/q_y(z)} = \exp(\frac{\ln^2(z)}{2q_y^2(z)}), \tag{36}$$

whence

$$q_y(z) = \frac{\ln(z)}{\sqrt{2\ln\frac{z}{y}}}. \tag{37}$$

Noting that $\inf_{z>y} q_y(z) = q_y(y^2) = \sqrt{2\ln y}$ we find that

$$\frac{q_y(y^2)}{s_y^*} \longrightarrow 1 \ \text{ as } y \to \infty, \tag{38}$$

which implies that (29) is best possible.

□

Proceeding, we seek a lower bound of $\max\{\underline{s}_y^*, t_y\}$ in terms of $\underline{q}_y$.

Theorem 2.6. *Take any $y \geq 47$ such that $\underline{S}_y$ is non-constant. Let $f(y) = -1.5/\ln(1-2/y)$ and let*

$$\underline{q}_y = \sup\{q > 0 : E(f(y))^{\underline{S}_y/q} \geq y^{-1}f^2(y))\}. \tag{39}$$

Then

$$\frac{1}{2}\underline{q}_y < \max\{\underline{s}_y^*, t_y\} < 2\underline{q}_y \tag{40}$$

with the RHS holding for $y \geq 2$. Moreover, if $\frac{t_y}{\underline{q}_y} \to 0$ as $y \to \infty$ then

$$\liminf_{y\to\infty} \frac{s^*_{y^2/(y-1)}}{\underline{q}_y} \geq \liminf_{y\to\infty} \frac{\underline{s}_y^*}{\underline{q}_y} \geq 1. \tag{41}$$

Proof: For the moment let $f(y) > 1$ denote any real such that $f^2(y) > y$. Then $0 < \underline{q}_y < \infty$ satisfies (39) with equality. The RHS of (40) is contained in Theorem 2.4.

Let $\tau =$ last j : there exists $i : 1 \leq i < j$ and $\underline{S}_{j,y} - \underline{S}_{i,y} > \underline{s}_y^*$. Then, let $\tau_0 =$ last $i < \tau$: $\underline{S}_{\tau,y} - \underline{S}_{i,y} > \underline{s}_y^*$. Notice that

$$\begin{aligned}
&P(\bigcup_{0\leq i<j<\infty}\{\underline{S}_{j,y} - \underline{S}_{i,y} > \underline{s}_y^*\}) = P(\bigcup_{j=2}^{\infty}\{\tau = j\}) \\
&= \sum_{j=2}^{\infty}\sum_{i=1}^{j-1} P(\tau_0 = i, \tau = j) \\
&\leq 2\sum_{j=2}^{\infty}\sum_{i=1}^{j-1} P(\tau_0 = i, \tau = j, \underline{S}_{i,y} \geq 0) \\
&\leq 2\sum_{j=2}^{\infty}\sum_{i=1}^{j-1} P(\tau_0 = i, \tau = j, \max_{1\leq k<\infty} \underline{S}_{k,y} > \underline{s}_y^*) \\
&\leq 2P(\max_{1\leq k<\infty} \underline{S}_{k,y} > \underline{s}_y^*) \leq \frac{2}{y}.
\end{aligned} \tag{42}$$

Therefore, by (39) and (73) of Theorem 3.2,

$$\begin{aligned}
&y^{-1}(f(y))^2 = E(f(y))^{\underline{S}_y/\underline{q}_y} \\
&\leq E(f(y))^{\underline{S}_y^*/\underline{q}_y} < (f(y))^{\underline{s}_y^*/\underline{q}_y}(1 - \frac{2}{y})^{1-(f(y))^{(\underline{s}_y^*+t_y)/\underline{q}_y}}.
\end{aligned} \tag{43}$$

Let c_1 and c_2 satisfy $t_y = c_1\underline{q}_y$ and $\underline{s}_y^* = c_2\underline{q}_y$. From (43) it follows that

$$(f(y))^{2-c_2}(1 - \frac{2}{y})^{(f(y))^{c_1+c_2}} < y - 2. \tag{44}$$

Suppose that the LHS of (40) fails. Then $0 < c_1 \vee c_2 \leq \frac{1}{2}$. Hence

$$(f(y))^{3/2}(1 - \frac{2}{y})^{f(y)} < y - 2. \tag{45}$$

Since (45) holds for all choices of $f(y) > \sqrt{y}$, putting

$$f(y) = \frac{3}{2\ln(1 + \frac{2}{y-2})}$$

we must have

$$(\frac{3}{2e\ln(1+\frac{2}{y-2})})^{3/2} < y-2. \tag{46}$$

This inequality is violated for $y \geq 47$. Therefore $c_1 \vee c_2 > \frac{1}{2}$ if $y \geq 47$.

As for (41) suppose $c_1 \to 0$ as $y \to \infty$. If there exist $\epsilon > 0$ and $y_n \to \infty$ such that c_2 for y_n (call it c_{2n} and similarly let c_{1n} denote c_1 for y_n) is at most $1-\epsilon$ then, since $f(y_n) \sim \frac{3y_n}{4}$, the LHS of (44) is asymptotic to

$$(\frac{3y_n}{4})^{2-c_{2n}}(1-\frac{2}{y_n})^{(3y_n/4)^{c_{1n}+c_{2n}}} \sim (\frac{3y_n}{4})^{2-c_{2n}} > (\frac{3y_n}{4})^{1+\epsilon} \tag{47}$$

thereby contradicting (44). Thus the RHS of (41) holds and its LHS follows by application of (17).

□

Remark 2.2. Letting c^* be any real exceeding 1.5, the method of proof of Theorem 2.6 also shows that if we define $f(y) = -\frac{c^*-1/2}{\ln(1-\frac{2}{y})}$ and

$$\underline{q}_{y,c^*} = \sup\{q > 0 : E(f(y))^{\underline{S}_y/q} \geq y^{-1}f^{c^*}(y))\} \tag{48}$$

there exists $\underline{y}_{c^*}$ such that for $y \geq \underline{y}_{c^*}$ such that $t_y > 0$

$$\frac{1}{2}\underline{q}_{y,c^*} < \max\{\underline{s}^*_y, t_y\} < c^*\underline{q}_{y,c^*}. \tag{49}$$

Hence, as $y \to \infty$ our upper and lower bounds for $\max\{\underline{s}^*_y, t_y\}$ differ by a factor which can be made to converge to 3.

Theorem 2.6 implies the following bounds on s_v and s^*_v for v related to y.

Corollary 2.7. *Take any $y \geq 47$ such that $\underline{S}_y$ is non-constant. Then if $\underline{q}_y$ is as defined in* (39)

$$\frac{1}{2}s^*_{y/2} < \underline{q}_y < 2s^*_{2y} \tag{50}$$

and

$$\frac{1}{2}s_{y/2} < \underline{q}_y < 2s_{\frac{2y^2}{y-1}} \tag{51}$$

with the LHS's holding for $y \geq 2$.

Proof: To prove the LHS of (50) and (51) write

$$\begin{aligned} 2\underline{q}_y &> \underline{s}^*_y \quad \text{(by Theorem 2.6)} \\ &\geq s^*_{y/2} \quad \text{(by the RHS of (16) in Lemma 2.1)} \\ &\geq s_{y/2}. \end{aligned}$$

To prove the RHS of (50) and (51) we show that

$$\underline{q}_y < 2(s^*_{2y} \wedge s_{2y^2/(y-1)}). \tag{52}$$

To do so first recall that, by Theorem 2.6,

$$\underline{q}_y < 2(\underline{s}^*_y \vee t_y).$$

By Remark 2.1, $\underline{s}^*_y \leq s^*_{2y}$. Moreover,

$$\begin{aligned}\underline{s}^*_y &\leq s^*_{\frac{y^2}{y-1}} \quad \text{(by (17) in Lemma 2.2)}\\ &\leq s_{\frac{2y^2}{y-1}} \quad \text{(by (15))}\end{aligned}$$

so $\underline{s}^*_y \leq s^*_{2y} \wedge s_{2y^2/(y-1)}$. Finally,

$$\begin{aligned}t_y &\leq s^*_{2y} \wedge s_{4y^2/(2y-1)} \quad \text{(by combining results in Lemma 2.2)}\\ &\leq s^*_{2y} \wedge s_{2y^2/(y-1)} \quad \text{(since } s_y \text{ is non-decreasing in } y)\end{aligned}$$

so (52) and consequently the RHS of (50) and (51) hold.
The LHS's of (50) and (51) are best possible in that $cs_{y/2}$ may exceed $\underline{q}_y$ as $y \to \infty$ if $c > \frac{1}{2}$, since in Example 2.5 above, $s^*_y \sim 2\underline{q}_y$ and $s_{y/2} \sim s^*_y$ as $y \to \infty$. □

Corollary 2.7 can be restated as

Remark 2.3. Take any $y \geq 94$. Then

$$\frac{1}{2}\underline{q}_{y/2} < s^*_y < 2\underline{q}_{2y}. \tag{53}$$

The RHS of (53) is valid for $y \geq 2$. Moreover, take any $y \geq 97$. Then

$$\frac{1}{2}\underline{q}_{\frac{y}{4}(1+\sqrt{1-8/y})} < s_y < 2\underline{q}_{2y}. \tag{54}$$

The RHS of (54) is valid for $y \geq 2$. □

Remark 2.4. To approximate s^*_y we have set our truncation level at t_y. Somewhat more careful analysis might, especially in particular cases, use other truncation levels for upper and lower bounds of s^*_y and s_y.

□

Remark 2.2 suggests that there is a sharpening of Theorem 2.6 which can be obtained if we allow c^* to vary with y. Our next result gives one such refinement by identifying the greatest lower bound which our approach permits and then adjoining Corollary 2.7, which identifies the least such upper bound.

Theorem 2.8. *For any $y \geq 4$ there is a unique $w_y > \ln\frac{y}{y-2}$ such that*

$$\left(\frac{w_y}{e\ln\frac{y}{y-2}}\right)^{w_y} = y - 2. \tag{55}$$

Let γ_y satisfy

$$\frac{1-\gamma_y}{2\gamma_y} = w_y \tag{56}$$

and $z_y > 0$ satisfy

$$z_y^{2\gamma_y} = \frac{1-\gamma_y}{2\gamma_y \ln \frac{y}{y-2}}. \tag{57}$$

Then $z_y > y$ for $y \geq 4$. For $z > y \geq 4$ let $q_y(z)$ be the unique positive real satisfying

$$Ez^{\underline{S}_y/q_y(z)} = \frac{z}{y}. \tag{58}$$

Then

$$\gamma_y q_y(z_y) \leq \max\{\underline{s}_y^*, t_y\} \leq \inf_{z>y}\{q_y(z)\} \ (\leq q_y(z_y)) \tag{59}$$

and

$$\lim_{y\to\infty} \gamma_y = \frac{1}{3}. \tag{60}$$

Proof: For $y > 3$ there is exactly one solution to equation (55). To see this consider $(\frac{w}{ea})^w$ for fixed $a > 0$. The log of this is convex in w, strictly decreasing for $0 \leq w \leq a$ and stricly increasing for $w \geq a$. Hence, $\sup_{0<w\leq a}(\frac{w}{ea})^w = 1$ and $\sup_{w\geq a}(\frac{w}{ea})^w = \infty$. Consequently, for $b > 1$ there is a unique $w = w_b$ such that $(\frac{w}{ea})^w = b$. Now (55) holds with $a = \ln \frac{y}{y-2}$ and $b = y - 2 > 1$

The RHS of (59) follows from Corollary 2.7. As for the LHS of (59), we employ the same notation and approach as used in the proof of Theorem 2.6.

Next we show that $z_y > y$ for $y \geq 4$. To obtain a contradiction, suppose $z_y \leq y$. Then

$$y \geq (y-2)^{1/(1-\gamma_y)} \exp(\frac{1}{2\gamma_y}) \geq \inf_{0<\gamma<1} (y-2)^{1/(1-\gamma)} \exp(\frac{1}{2\gamma}). \tag{61}$$

To minimize $(y-2)^{1/(1-\gamma)} \exp(\frac{1}{2\gamma})$ set $\gamma = 1/(1+\sqrt{2\ln(y-2)})$. Then, for $y \geq 4$,

$$\begin{aligned} &\inf_{0<\gamma<1} (y-2)^{1/(1-\gamma)} \exp(\frac{1}{2\gamma}) = \exp(\frac{1}{2}(1+\sqrt{2\ln(y-2)})^2) \\ &\geq \exp(\frac{1}{2}) \exp(\sqrt{\ln 4})(y-2) > 3(y-2) > y, \end{aligned} \tag{62}$$

giving a contradiction.

Hence $q_y(z_y)$ exists. Put $c_1 = \frac{t_y}{q_y(z_y)}$ and $c_2 = \frac{\underline{s}_y^*}{q_y(z_y)}$. Using Theorem 3.2 and arguing as in (43) but incorporating (58),

$$\frac{z_y}{y} < (z_y)^{c_2}(1-\frac{2}{y})^{1-z_y^{c_1+c_2}} \tag{63}$$

and consequently

$$z_y^{1-c_2}(1-\frac{2}{y})^{z_y^{c_1+c_2}} < y-2. \tag{64}$$

To obtain a contradiction of (64) suppose

$$\max\{\underline{s}_y^*, t_y\} \leq \gamma_y q_y(z_y). \tag{65}$$

Then $\max\{c_1, c_2\} \leq \gamma_y$ and

$$z_y^{1-c_2}(1-\frac{2}{y})^{z_y^{c_1+c_2}} \geq z_y^{1-\gamma_y}(1-\frac{2}{y})^{z_y^{2\gamma_y}} = y-2, \tag{66}$$

where the last equality follows since, using (57), (56), and then (55),

$$z_y^{1-\gamma_y} = (\frac{1-\gamma_y}{2\gamma_y \ln\frac{y}{y-2}})^{\frac{1-\gamma_y}{2\gamma_y}} = (y-2)\exp(w_y) \tag{67}$$

and by (57) and (56),

$$(1-\frac{2}{y})^{z_y^{2\gamma_y}} = \exp(-w_y). \tag{68}$$

This gives the desired contradiction. (60) holds by direct calculation, using the definition of γ_y found in (56) and the fact that $w_y \to 1$ as $y \to \infty$.

□

The following corollary follows from the previous results.

Corollary 2.9.

$$\gamma_y q_y(z_y) \leq s_{\frac{2y^2}{y-1}} \wedge s_{2y}^* \tag{69}$$

and

$$s_y^* \leq \inf_{z>2y} q_{2y}(z) \leq q_{2y}(z_{2y}). \tag{70}$$

3 Appendix

Lemma 3.1. *Let $Y_1, Y_2, \ldots$ be independent, symmetric, uniformly bounded rv's such that $\sigma^2 = \sum_{j=1}^{\infty} EY_j^2$ is finite. Then, for all real t,*

$$E\exp(t\sum_{j=1}^{\infty} Y_j) < \infty. \tag{71}$$

Proof: See, for example, Theorem 1 in Prokhorov (1959).

□

Remark 3.1. For all $p > 0$ the submartingale $\exp(t\sum_{j=1}^{n} Y_j)$ convergences almost surely and in $\mathcal{L}^p$ to $\exp(t\sum_{j=1}^{\infty} Y_j)$. Consequently, for all $t > 0$,
$E\sup_n \exp(t\sum_{j=1}^{n} Y_j) < \infty$. Therefore, by dominated convergence
$\exp(t\sum_{j=1}^{\infty} Y_j)$ is continuous and the submartingale has moments of all orders.

□

Theorem 3.2. *Let $X_1, X_2, \ldots$ be independent real valued random variables such that if $S_n = X_1 + \ldots + X_n$, then S_n converges to a finite valued random variable S almost surely. For $0 \leq i < j < \infty$ let $S_{(i,j]} \equiv S_j - S_i = X_{i+1} + \ldots + X_j$ and for any real $a_0 > 0$ let $\lambda = P(\bigcup_{0 \leq i < j < \infty} \{S_{(i,j]} > a_0\})$. Further, suppose that the X_j's take values not exceeding a_1. Then, for every integer $k \geq 1$ and all positive reals a_0 and a_1,*

$$P(\sup_{1 \leq j < \infty} S_j > ka_0 + (k-1)a_1) \leq P(\mathcal{N}_{-\ln(1-\lambda)} \geq k), \tag{72}$$

(with strict inequality for $k \geq 2$) where $\mathcal{N}_\gamma$ denotes a Poisson variable with parameter $\gamma > 0$. Then, for every $b > 0$

$$Ee^{b \sup_{1 \leq j < \infty} S_j} < e^{ba_0}(1-\lambda)^{1-\exp(b(a_0+a_1))}. \tag{73}$$

Proof: (72) follows from Theorem 1 proved in Klass and Nowicki (2003):
To prove (73) denote $W = \sup_{1 \leq j < \infty} S_j$. For $k \geq 2$,

$$\begin{aligned} &P(\mathcal{N}_{-\ln(1-\lambda)} \geq k) > P(W \geq ka_0 + (k-1)a_1) = P(\frac{W - a_0}{a_0 + a_1} > k - 1) \\ &= P(\lceil \frac{(W-a_0)^+}{a_0 + a_1} \rceil \geq k). \end{aligned}$$

For $k = 1$ we get equality. Letting

$$\underline{W} = \lceil \frac{(W-a_0)^+}{a_0+a_1} \rceil,$$

$\underline{W}$ is a non-negative integer-valued random variable which is stochastically smaller than $\mathcal{N}_{-\ln(1-\lambda)}$. Since $\exp(b(a_0+a_1)x)$ is an increasing function of x,

$$\begin{aligned} &Ee^{bW} = e^{ba_0} Ee^{b(W-a_0)} \leq e^{ba_0} Ee^{b(a_0+a_1)\underline{W}} \\ &< e^{ba_0} Ee^{b(a_0+a_1)\mathcal{N}_{-\ln(1-\lambda)}} = e^{ba_0} \exp(-\ln(1-\lambda)(e^{b(a_0+a_1)} - 1)). \end{aligned}$$

□

4 Acknowledgment

We would like to thank the referee for his thoughtful comments. They helped us to improve the presentation of the paper.

References

[1] H. Cramér, On a new limit in theory of probability, in *Colloquium on the Theory of Probability*, (1938), Hermann, Paris. Review number not available.

[2] F. Esscher, On the probability function in the collective theory of risk, Skand. Aktuarietidskr. **15** (1932), 175—195. Review number not available.

[3] M. G. Hahn and M. J. Klass, Uniform local probability approximations: improvements on Berry-Esseen, Ann. Probab. **23** (1995), no. 1, 446–463. MR1330778 (96d:60069) MR1330778

[4] M. G. Hahn and M. J. Klass, Approximation of partial sums of arbitrary i.i.d. random variables and the precision of the usual exponential upper bound, Ann. Probab. **25** (1997), no. 3, 1451–1470. MR1457626 (99c:62039) MR1457626

[5] Fuk, D. H.; Nagaev, S. V. Probabilistic inequalities for sums of independent random variables. (Russian) Teor. Verojatnost. i Primenen. 16 (1971), 660–675. MR0293695 (45 #2772) MR0293695

[6] P. Hitczenko and S. Montgomery-Smith, A note on sums of independent random variables, in *Advances in stochastic inequalities (Atlanta, GA, 1997)*, 69–73, Contemp. Math., 234, Amer. Math. Soc., Providence, RI. MR1694763 (2000d:60080) MR1694763

[7] P. Hitczenko and S. Montgomery-Smith, Measuring the magnitude of sums of independent random variables, Ann. Probab. **29** (2001), no. 1, 447–466. MR1825159 (2002b:60077) MR1825159

[8] N. C. Jain and W. E. Pruitt, Lower tail probability estimates for subordinators and nondecreasing random walks, Ann. Probab. **15** (1987), no. 1, 75–101. MR0877591 (88m:60075) MR0877591

[9] M. J. Klass, Toward a universal law of the iterated logarithm. I, Z. Wahrsch. Verw. Gebiete **36** (1976), no. 2, 165–178. MR0415742 (54 #3822) MR0415742

[10] M. J. Klass and K. Nowicki, An improvement of Hoffmann-Jørgensen's inequality, Ann. Probab. **28** (2000), no. 2, 851–862. MR1782275 (2001h:60029) MR1782275

[11] M. J. Klass and K. Nowicki, An optimal bound on the tail distribution of the number of recurrences of an event in product spaces, Probab. Theory Related Fields **126** (2003), no. 1, 51–60. MR1981632 (2004f:60038) MR1981632

[12] A. Kolmogoroff, Über das Gesetz des iterierten Logarithmus, Math. Ann. **101** (1929), no. 1, 126–135. MR1512520 MR1512520

[13] R. Latała, Estimation of moments of sums of independent real random variables, Ann. Probab. **25** (1997), no. 3, 1502–1513. MR1457628 (98h:60021) MR1457628

[14] S. V. Nagaev, Some limit theorems for large deviations, Teor. Verojatnost. i Primenen **10** (1965), 231–254. MR0185644 (32 #3106) MR0185644

[15] Yu. V. Prokhorov, An extremal problem in probability theory, Theor. Probability Appl. **4** (1959), 201–203. MR0121857 (22 #12587) MR0121857