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Brownian Motion on Compact Manifolds: Cover Time and Late Points

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Abstract: Let M be a smooth, compact, connected Riemannian manifold of dimension $d \geq 3$ and without boundary. Denote by $\mathcal{T}(x,\epsilon)$ the hitting time of the ball of radius ϵ centered at x by Brownian motion on M. Then, $\mathcal{C}_\epsilon(M) = \sup_{x\in M}\mathcal{T}(x,\epsilon)$ is the time it takes Brownian motion to come within r of all points in M. We prove that $\mathcal{C}_\epsilon(M)/\epsilon^{2-d}|\log\epsilon| \to \gamma_d V(M)$ almost surely as $\epsilon \to 0$, where $V(M)$ is the Riemannian volume of M. We also obtain the "multi-fractal spectrum" $f(\alpha)$ for "late points", i.e., the dimension of the set of α-late points x in M for which $\limsup_{\epsilon\to 0}\mathcal{T}(x,\epsilon)/(\epsilon^{2-d}|\log\epsilon|) = \alpha > 0$.

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1 Introduction

Let M be a smooth, compact connected d-dimensional Riemannian manifold without boundary, and let $\{X_t\}_{t\geq 0}$ denote Brownian motion on M. $\{X_t\}_{t\geq 0}$ is a strongly symmetric Markov process with reference measure given by the Riemannian measure dV and infinitesimal generator 1/2 the Laplace-Beltrami operator Δ_M. We use $d(x,y)$ to denote the Riemannian distance between $x, y \in M$, using it whenever computing the diameter or dimension of subsets of M. Let $B(x,r)$ denote the open ball in M of radius r centered at x. For x in M we have the ϵ-hitting time

$$\mathcal{T}(x,\varepsilon) = \inf\{t > 0 \,|\, X_t \in B(x,\epsilon)\}.$$

Then

$$\mathcal{C}_\epsilon(M) = \sup_{x\in M} \mathcal{T}(x,\varepsilon)$$

is the ϵ-covering time of M, i.e. the amount of time needed for the Brownian motion X_t to come within ϵ of each point in M. Equivalently, $\mathcal{C}_\epsilon(M)$ is the amount of time needed for the Wiener sausage of radius ϵ to completely cover M. This quantity is closely related to the asymptotics of the spectrum of the manifold M with a small ball removed.

In [7] we considered $\mathcal{C}_\epsilon(M)$ for compact manifolds of dimension $d = 2$. By the use of isothermal coordinate systems this problem is reduced to the ϵ-covering time of the two-dimensional (flat) torus by a (standard) Brownian motion. In this paper we deal with manifolds of dimension $d \geq 3$, for which in general there is no direct reduction to the Euclidean case. Consequently, we work directly on the manifold, taking advantage of the fact that the Brownian motion is "locally transient" on such manifolds, in sharp contrast with the situation for $d = 2$.

To describe our results, let $V(M)$ denote the Riemannian volume of M and define

$$\kappa_M := \frac{2V(M)}{(d-2)V(S^{d-1})} = \frac{\Gamma(d/2)}{(d-2)\pi^{d/2}}V(M), \qquad h_d(\varepsilon) := \varepsilon^{2-d}\log\frac{1}{\varepsilon}$$

(where $V(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the unit sphere of dimension $d-1$).

Our first result provides the asymptotics of $\mathcal{C}_\epsilon(M)$ as $\epsilon \to 0$.

Theorem 1.1. *For Brownian motion in M,*

$$\lim_{\epsilon\to 0} \frac{\mathcal{C}_\epsilon(M)}{h_d(\varepsilon)} = d\kappa_M \qquad a.s. \tag{1.1}$$

Since $V(S^d) = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$, our theorem implies that

$$\lim_{\epsilon\to 0} \frac{\mathcal{C}_\epsilon(S^d)}{h_d(\varepsilon)} = \frac{2\sqrt{\pi}d}{(d-2)}\frac{\Gamma(d/2)}{\Gamma((d+1)/2)} \qquad \text{a. s.} \tag{1.2}$$

The asymptotics of $\mathbb{E}(\mathcal{C}_\epsilon(S^d))$ have been described in [12] (with $\lambda = d$ there).

Here is the heuristic leading to (1.1): The ε-hitting time $\mathcal{T}(x,\varepsilon)$ grows with decreasing ε like 1 over the minimal eigenvalue of $\frac{1}{2}\Delta_M$ on $M \setminus B(x,\varepsilon)$. The latter is known to be $\kappa_M^{-1}\varepsilon^{d-2}(1+o(1))$ (c.f. [3] and the references therein to earlier works by Ozawa and others). Since $B(x,\varepsilon)$ and $B(y,\varepsilon)$ have a substantial overlap whenever $d(x,y) \ll \varepsilon$, the value of $\mathcal{C}_\varepsilon(M)$ is roughly the maximum of $O(\varepsilon^{-d})$ random variables, corresponding to $\mathcal{T}(x,\varepsilon)$ for x in the centers of an $O(\varepsilon)$-cover of M. Assuming these variables are only weakly dependent, such maximum scales like $d\log(1/\varepsilon)$ times the scaling of each of the variables, leading to (1.1).

We next relate the almost sure asymptotics of the ε-covering time of subsets $E \subseteq M$ to their upper Minkowski dimension $\overline{\dim}_m(E)$ and packing dimension $\dim_p(E)$.

Theorem 1.2. *For Brownian motion in M and any set $E \subseteq M$ we have*

$$\limsup_{\epsilon\to 0}\sup_{x\in E}\frac{\mathcal{T}(x,\varepsilon)}{h_d(\varepsilon)} = \overline{\dim}_m(E)\kappa_M \qquad a.\ s. \tag{1.3}$$

Furthermore, for any analytic set $E \subseteq M$ we have

$$\sup_{x\in E}\limsup_{\epsilon\to 0}\frac{\mathcal{T}(x,\varepsilon)}{h_d(\varepsilon)} = \dim_p(E)\kappa_M \qquad a.\ s. \tag{1.4}$$

The next theorem describes the multi-fractal structure of $\{\mathcal{T}(x,\varepsilon)\}$ for Brownian motion in M and those points $x \in M$ for which $\mathcal{T}(x,\epsilon)$ is comparable with $\mathcal{C}_\epsilon(M)$ as $\epsilon \to 0$.

Theorem 1.3. *For Brownian motion in M and for any $a \le d$,*

$$\dim\left\{x\in M:\ \limsup_{\varepsilon\to 0}\frac{\mathcal{T}(x,\varepsilon)}{h_d(\varepsilon)} = a\kappa_M\right\} = d-a \qquad a.s. \tag{1.5}$$

We call a point $x \in M$ a *late point* if x is in the set considered in (1.5) for some $a > 0$. This theorem may be compared with our results on the multi-fractal structure of *thick points* for Brownian motion, [4, 6]. The first result of this type was the determination by Orey and Taylor [14] of the dimension of sets of *fast points* for Brownian motion.

Analytic tools provide in Section 2 simple uniform estimates on excursion times and exit probabilities for the annuli $B(m,R)\setminus B(m,r)$. Using these estimates we obtain in Section 3 upper bounds, first on the tail probability of $\mathcal{T}(x,\varepsilon)$, then on the limits considered in Theorem 1.2 and the dimensions of the sets considered in Theorem 1.3. The complementary lower bounds are derived in Section 4 by an adaptation of the methods of [5, 11] and of [12, 13].

While outside the scope of this work, it is interesting to find the structure of *consistently late* points, where the lim sup in (1.4) and (1.5) is replaced by lim inf (or lim), changing, if needed, the scaling function.

2 Excursion time estimates

We start with a uniform estimate on the mean time to exit a small ball $B(m,R)$, allowing us in the sequel to neglect the contribution of such times.

Lemma 2.1 (Interior mean exit time). *Let $\bar\tau_R = \inf\{t \ge 0\,|\,X_t \in B^c(m,R)\}$. For R sufficiently small, and any m, $x \in M$ with $d(m,x) < R$*

$$\mathbb{E}^x\left(\bar\tau_R\right) = \frac{R^2 - d(m,x)^2}{d} + O(R^3). \tag{2.1}$$

Proof of Lemma 2.1: We use geodesic polar coordinates r, θ centered at m. When applied to radial functions $f(r)$ ($r = \mathrm{dist}(m,x)$), the Laplace-Beltrami operator takes in these coordinates the form

$$\Delta_M f = \frac{\partial^2 f}{\partial r^2} + \Big(\frac{d-1}{r} + O(r)\Big)\frac{\partial f}{\partial r}, \tag{2.2}$$

which differs from the Euclidean Laplacian in the term denoted $O(r)$ (see [2, page 106]). The left hand side of (2.1) satisfies

$$(1/2)\Delta_M F = -1$$

in $B(m,R)$ with 0 boundary conditions on $\partial B(m,R)$. Set

$$u_\pm(r) = \frac{1}{d}(R^2 - r^2) \pm R^2(R-r)$$

Using (2.2) we see that for all R, hence r, sufficiently small, uniformly in $m \in M$ and $x \in B(m,R)$,

$$\Delta_M\{F - u_\pm\} = \pm R^2\Big(\frac{d-1}{r} + O(1)\Big).$$

Consequently, $F - u_\pm$ are sub(super)-harmonic in $B(m,R)$ with 0 boundary conditions on $\partial B(m,R)$. Since 0 is the only function harmonic in $B(m,R)$ with 0 boundary conditions on $\partial B(m,R)$, we obtain (2.1). $\square$

The next lemma provides an estimate on the mean hitting time of a small ball $B(m,r)$ starting at distance $R > r$ from its center m. It is this estimate that give rise to the constant κ_M.

Lemma 2.2 (Exterior mean exit time). *Let $\tau_r = \inf\{t \geq 0 \,|\, X_t \in B(m,r)\}$. There exists finite $\eta(R) \to 0$ as $R \to 0$, such that for all $r \leq R/2$ and $m \in M$,*

$$\begin{aligned}\kappa_M(r^{2-d}(1-\eta(r)) - R^{2-d}(1+\eta(R))) &\leq \inf_{x\in\partial B(m,R)} \mathbb{E}^x(\tau_r) \\ \leq \sup_{x\in\partial B(m,R)} \mathbb{E}^x(\tau_r) &\leq \kappa_M(r^{2-d}(1+\eta(r)) - R^{2-d}(1-\eta(R))).\end{aligned} \tag{2.3}$$

Further, for some $c < \infty$, $r_0 > 0$, all $0 < r \leq r_0$ and all $m \in M$,

$$\|\tau_r\| := \sup_{x\in M} \mathbb{E}^x(\tau_r) \leq c r^{2-d}. \tag{2.4}$$

Proof of Lemma 2.2: We use the fact that for any smooth compact $d \geq 3$ dimensional manifold there exists a function $G(x,y)$, (the Green's function), defined for $x \neq y$ with the following properties: $G(x,y)$ satisfies $\Delta_{M,x}G(x,y) = \frac{1}{V(M)}$ where $\Delta_{M,x}$ denotes the Laplace-Beltrami operator with respect to the variable x and, $G(x,y) = a_d(d(x,y))^{2-d}(1 + F(x,y))$ for $a_d = ((d-2)V(S^{d-1}))^{-1}$ and some function $F(x,y)$ which is continuous for $x \neq y$, and such that $\eta(r) := \sup\{|F(x,y)| : d(x,y) = r\} \to 0$ as $r \to 0$ (c.f. [2, page 108], see also [8]).

Now let $e(x) = \mathbb{E}^x(\tau_r)$. We have that $\frac{1}{2}\Delta_M e(x) = -1$ on $M \setminus B(m,r)$ and $e(x) = 0$ on $\partial B(m,r)$. Hence, with m fixed

$$\Delta_M\left(G(x,m) + \frac{1}{2V(M)}e(x)\right) = 0 \quad \text{on} \quad M \setminus B(m,r).$$

Thus by the maximum principle for all $x \in M \setminus B(m,r)$

$$\inf_{y \in \partial B(m,r)} G(y,m) \le G(x,m) + \frac{1}{2V(M)} e(x) \le \sup_{y \in \partial B(m,r)} G(y,m).$$

Our lemma follows immediately (using (2.1) to provide the bound (2.4) also for $x \in B(m,r)$ and r sufficiently small). □

The next lemma shows that the probability of hitting a small ball $B(m,\epsilon)$ upon exit of a small annuli $B(m,R) \setminus B(m,\epsilon)$ is (uniformly) comparable to that for $M = \mathbb{R}^d$.

Lemma 2.3 (Hitting probabilities). *Let $\bar{\tau}_{\varepsilon,R} = \inf\{t \ge 0 \,|\, X_t \in B^c(m,R) \cup B(m,\varepsilon)\}$ and $p^x_{\varepsilon,R} = \mathbf{P}^x(d(m, X_{\bar{\tau}_{\varepsilon,R}}) = \varepsilon)$. Define $p_{\varepsilon,R}(r) = (r^{2-d} - R^{2-d})/(\varepsilon^{2-d} - R^{2-d})$. For any $\delta > 0$ there exists $R_0(\delta) > 0$ such that if $R_0 \ge R \ge 2r \ge 4\varepsilon$, then for all $m \in M$,*

$$(1+\delta) p_{\varepsilon,R}(r) \ge \sup_{x \in \partial B(m,r)} p^x_{\varepsilon,R} \ge \inf_{x \in \partial B(m,r)} p^x_{\varepsilon,R} \ge (1-\delta) p_{\varepsilon,R}(r) \tag{2.5}$$

Proof of Lemma 2.3: We follow an argument similar to that used in proving Lemma 2.1. Let $g(x) = p^x_{\varepsilon,R}$, noting that $\Delta_M g = 0$ for all $x \in B(m,R) \setminus B(m,\varepsilon)$, with boundary conditions $g = 0$ on $\partial B(m,R)$ and $g = 1$ on $\partial B(m,\varepsilon)$. Set

$$u_\pm(r) = \frac{r^{2-d} - R^{2-d} \pm (r^{2.5-d} - R^{2.5-d})}{\varepsilon^{2-d} - R^{2-d} \pm (\varepsilon^{2.5-d} - R^{2.5-d})}\,.$$

It is easy to check that for R small enough and $R \ge 2r \ge 4\varepsilon$,

$$(1 - O(\sqrt{R})) p_{\varepsilon,R}(r) \le u_\pm(r) \le p_{\varepsilon,R}(r)(1 + O(\sqrt{R}))\,.$$

Note that $g - u_\pm$ is 0 at the boundary of the annulus $B(m,R) \setminus B(m,\varepsilon)$ and by (2.2), for all R small enough, uniformly in $m \in M$ and x in this annulus,

$$\Delta_M \{g - u_\pm\} = \pm C_{\varepsilon,R} r^{0.5-d}(1 + o(R))\,,$$

where $C_{\varepsilon,R} = 0.5(d-2.5)/(\varepsilon^{2-d} - R^{2-d} \pm (\varepsilon^{2.5-d} - R^{2.5-d})) > 0$. Consequently, $g - u_\pm$ are sub(super)-harmonic in this annulus with 0 boundary conditions. Since 0 is the only function harmonic in this annulus with 0 boundary conditions, we obtain (2.5). □

Fixing $x \in M$ and constants $0 < r < R$ let

$$\tau^{(0)} = \inf\{t \ge 0 \,|\, X_t \in \partial B(x,R)\} \tag{2.6}$$

$$\sigma^{(1)} = \inf\{t \ge 0 \,|\, X_{t+\tau^{(0)}} \in \partial B(x,r)\} \tag{2.7}$$

and define inductively for $j = 1, 2, \ldots$

$$\tau^{(j)} = \inf\{t \ge \sigma^{(j)} \,|\, X_{t+\mathfrak{T}_{j-1}} \in \partial B(x,R)\}, \tag{2.8}$$

$$\sigma^{(j+1)} = \inf\{t \ge 0 \,|\, X_{t+\mathfrak{T}_j} \in \partial B(x,r)\}, \tag{2.9}$$

where $\mathfrak{T}_j = \sum_{i=0}^j \tau^{(i)}$ for $j = 0, 1, 2, \ldots$. Thus, $\tau^{(j)}$ is the length of the j'th excursion $\mathcal{E}_j$ from $\partial B(x,R)$ to itself via $\partial B(x,r)$, and $\sigma^{(j)}$ is the amount of time it takes to hit $\partial B(x,r)$ during the j'th excursion $\mathcal{E}_j$.

The next lemma which shows that excursion times are concentrated around their mean, will be used to relate excursions to hitting times.

Lemma 2.4. *With the above notation, for any $N \geq N_0$, $\delta_0 > 0$ small enough, $0 < \delta < \delta_0$, $0 < 2r < R < R_0(\delta)$, and $x, x_0 \in M$,*

$$\mathbf{P}^{x_0}\left(\sum_{j=0}^{N} \tau^{(j)} \leq (1-\delta)\kappa_M (r^{2-d} - R^{2-d})N\right) \leq e^{-C\delta^2 N} \tag{2.10}$$

and

$$\mathbf{P}^{x_0}\left(\sum_{j=0}^{N} \tau^{(j)} \geq (1+\delta)\kappa_M (r^{2-d} - R^{2-d})N\right) \leq e^{-C\delta^2 N} \tag{2.11}$$

Moreover, $C = C(R,r) > 0$ depends only upon δ_0 as soon as $R > r^{1-\delta_0}$.

Proof of Lemma 2.4: Applying Kac's moment formula for the first hitting time τ_r of the strong Markov process X_t (see [9, Equation (6)]), it follows by (2.4) that for any integer k, all $m \in M$ and $r \leq r_0 \leq 1$,

$$\sup_y \mathbb{E}^y(\tau_r^k) \leq k! \|\tau_r\|^k \leq k! c^k r^{k(2-d)}. \tag{2.12}$$

Hence, for some $\lambda > 0$,

$$\sup_{0<r\leq r_0} \sup_{m,y} \mathbb{E}^y(e^{\lambda \tau_r r^{d-2}}) < \infty.$$

Reducing r_0 as needed, by the same argument, Lemma 2.1 implies that

$$\sup_{R\leq r_0} \sup_m \sup_{z\in B(m,R)} \mathbb{E}^z(e^{\lambda \bar{\tau}_R}) < \infty.$$

By the strong Markov property of X_t at $\tau^{(0)}$ and at $\tau^{(0)} + \sigma^{(1)}$ we then deduce that

$$\sup_{0<2r\leq R<r_0} \sup_{x,y} \mathbb{E}^y(e^{\lambda \mathfrak{T}_1 r^{d-2}}) < \infty. \tag{2.13}$$

Fixing $\delta > 0$, let $0 < R_0(\delta) \leq r_0$ be such that for all $R \leq R_0$ both $\eta(R) < \eta := \delta/6$ and

$$\|\bar{\tau}_R\| := \sup_m \sup_{z\in B(m,R)} \mathbb{E}^z(\bar{\tau}_R) \leq \eta \kappa_M R^{2-d} \tag{2.14}$$

(see (2.1)). Fixing $x \in M$ and $0 < 2r \leq R \leq r_0$, let $\tau = \tau^{(1)}$ and $v = \kappa_M(r^{2-d} - R^{2-d})$. It follows from (2.3) and (2.12) that there exists a universal constant $c_4 < \infty$ such that for $\rho = c_4 r^{2(2-d)}$ and all $\theta \geq 0$,

$$\begin{aligned}
&\sup_x \sup_{y\in\partial B(x,R)} \mathbb{E}^y(e^{-\theta\tau}) \leq \sup_x \sup_{y\in\partial B(x,R)} \mathbb{E}^y(e^{-\theta\tau_r}) \\
\leq\ & 1 - \theta \inf_x \inf_{y\in\partial B(x,R)} \mathbb{E}^y(\tau_r) + \frac{\theta^2}{2} \sup_x \sup_{y\in\partial B(x,R)} \mathbb{E}^y(\tau_r^2) \\
\leq\ & 1 - \theta(1-\eta)v + \rho\theta^2 \leq \exp(\rho\theta^2 - \theta(1-\eta)v).
\end{aligned} \tag{2.15}$$

Since $\tau^{(0)} \geq 0$, using Chebycheff's inequality we bound the left hand side of (2.10) by

$$\begin{aligned}
\mathbf{P}^{x_0}\Big(\sum_{j=1}^{N} \tau^{(j)} \leq (1-\eta-\delta/2)vN\Big) &\leq e^{\theta(1-\eta-\delta/2)vN}\, \mathbb{E}^{x_0}\left(e^{-\theta\sum_{j=1}^{N}\tau^{(j)}}\right) \\
&\leq e^{-\theta v N\delta/2}\left[e^{\theta(1-\eta)v} \sup_{y\in\partial B(x,R)} \mathbb{E}^y(e^{-\theta\tau})\right]^N,
\end{aligned} \tag{2.16}$$

where the last inequality follows by the strong Markov property of X_t at $\{\mathfrak{T}_j\}$. Combining (2.15) and (2.16) for $\theta = \delta v/(4\rho)$, results with (2.10), where $C = v^2/16\rho > 0$ is bounded below by $(1-8^{-\delta_0})^2\kappa_M^2/(16c_4)$ if $r^{1-\delta_0} < R$.

To prove (2.11) we first note that for $\lambda > 0$ as in (2.13), it follows that

$$\mathbf{P}^{x_0}\left(\tau^{(0)} \geq \frac{\delta}{6}vN\right) \leq e^{-\lambda r^{d-2}v(\delta/6)N}\mathbb{E}^{x_0}(e^{\lambda\tau^{(0)}r^{d-2}}) \leq c_5 e^{-c_6\delta N},$$

where $c_5 < \infty$ is a universal constant and $c_6 = c_6(R,r) > 0$ does not depend upon N, δ, x_0 or x, and is bounded below by some $c_7(\delta_0) > 0$ when $r^{1-\delta_0} < R$. Thus, the proof of (2.11), in analogy to that of (2.10), comes down to bounding

$$\mathbf{P}^{x_0}\Big(\sum_{j=1}^N \tau^{(j)} \geq (1+2\eta+\delta/2)vN\Big) \leq e^{-\theta\delta vN/2}\Big(e^{-\theta(1+2\eta)v}\sup_{y\in\partial B(x,R)}\mathbb{E}^y(e^{\theta\tau})\Big)^N$$

Noting that, by (2.3), (2.14) and (2.13), there exists a universal constant $c_8 < \infty$ such that for $\rho = c_8 r^{2(2-d)}$ and all $0 < \theta < (\lambda/2)r^{d-2}$,

$$\begin{aligned}\sup_x \sup_{y\in\partial B(x,R)}\mathbb{E}^y(e^{\theta\tau}) &\leq 1+\theta(1+2\eta)v + \sup_x\sup_{y\in\partial B(x,R)}\sum_{n=2}^\infty \frac{\theta^n}{n!}\mathbb{E}^y(\tau^n)\\ &\leq 1+\theta(1+2\eta)v+\rho\theta^2 \leq \exp(\theta(1+2\eta)v+\rho\theta^2),\end{aligned}$$

the proof of (2.11) now follows as in the proof of (2.10). □

3 Hitting time estimates and upper bounds

The first step in getting upper bounds is to control the tail probabilities of $\mathcal{T}(x,\epsilon)$, uniformly in x and the initial position x_0.

Lemma 3.1. *For any $\delta > 0$ we can find $c < \infty$ and $\varepsilon_0 > 0$ so that for all $\varepsilon \leq \varepsilon_0$ and $y \geq 0$*

$$\mathbf{P}^{x_0}(\mathcal{T}(x,\varepsilon) \geq yh_d(\varepsilon)) \leq c\varepsilon^{(1-\delta)^2\kappa_M^{-1}y} \tag{3.1}$$

for all $x, x_0 \in M$.

Proof of Lemma 3.1: We use the notation of the last lemma and its proof, with $2r < R < R_0(\delta)$. Let $n_\varepsilon := (1-\delta)\kappa_M^{-1}(r^{2-d}-R^{2-d})^{-1}yh_d(\varepsilon)$. It is easy to see that

$$\mathbf{P}^{x_0}\Big(\mathcal{T}(x,\varepsilon) \geq yh_d(\varepsilon)\Big) \leq \mathbf{P}^{x_0}\Big(\mathcal{T}(x,\varepsilon) \geq \sum_{j=0}^{n_\varepsilon}\tau^{(j)}\Big) + \mathbf{P}^{x_0}\Big(\sum_{j=0}^{n_\varepsilon}\tau^{(j)} \geq yh_d(\varepsilon)\Big) \tag{3.2}$$

It follows from Lemma 2.4 that

$$\mathbf{P}^{x_0}\Big(\sum_{j=0}^{n_\varepsilon}\tau^{(j)} \geq yh_d(\varepsilon)\Big) \leq e^{-C'yh_d(\varepsilon)}$$

for some $C' = C'(\delta) > 0$. On the other hand, the first probability in the right hand side of (3.2) is bounded above by the probability of not hitting the ball $B(x, \epsilon)$ during n_ε excursions, each starting at $\partial B(x,r)$ and ending at $\partial B(x,R)$, so that by Lemma 2.3, for all $\varepsilon > 0$, small enough

$$\mathbf{P}^{x_0}\Big(\mathcal{T}(x,\varepsilon) \geq \sum_{j=0}^{n_\varepsilon} \tau^{(j)}\Big) \leq \Big(1 - \varepsilon^{d-2}(r^{2-d} - R^{2-d})(1-\delta)\Big)^{n_\varepsilon} \leq e^{-(1-\delta)^2 \kappa_M^{-1} y |\log \varepsilon|}$$

and (3.1) follows. □

We next provide the required upper bound in Theorem 1.3. Namely, with the notation

$$\mathsf{Late}_{\geq a} = \Big\{x \in M : \limsup_{\varepsilon \to 0} \frac{\mathcal{T}(x,\varepsilon)}{h_d(\varepsilon)} \geq a\kappa_M\Big\}\,, \tag{3.3}$$

we will show that for any $a \in (0, d]$,

$$\dim(\mathsf{Late}_{\geq a}) \leq d - a\,, \quad a.s. \tag{3.4}$$

Fix $\delta > 0$, $\tilde{\epsilon}_0$, and define set $\tilde{\epsilon}_n$ inductively so that

$$h_d(\tilde{\epsilon}_{n+1}) = (1+\delta) h_d(\tilde{\epsilon}_n).$$

Since, for $\tilde{\epsilon}_{n+1} \leq \epsilon \leq \tilde{\epsilon}_n$ we have

$$\frac{\mathcal{T}(x,\tilde{\epsilon}_{n+1})}{h_d(\tilde{\epsilon}_{n+1})} = \frac{h_d(\tilde{\epsilon}_n)}{h_d(\tilde{\epsilon}_{n+1})} \frac{\mathcal{T}(x,\tilde{\epsilon}_{n+1})}{h_d(\tilde{\epsilon}_n)} \geq (1+\delta)^{-1} \frac{\mathcal{T}(x,\epsilon)}{h_d(\epsilon)}\,, \tag{3.5}$$

it is easy to see that for any $a > 0$,

$$\mathsf{Late}_{\geq a} \subseteq D_a := \{x \in M \,\Big|\, \limsup_{n\to\infty} \frac{\mathcal{T}(x,\tilde{\epsilon}_n)}{h_d(\tilde{\epsilon}_n)} \geq (1-\delta) a \kappa_M\}.$$

Fix $x_0 \in M$ and let $\{x_j : j = 1, \ldots, \bar{K}_n\}$, denote a maximal collection of points in M, such that $\inf_{\ell \neq j} d(x_\ell, x_j) \geq \delta \tilde{\epsilon}_n$. Let $\mathcal{A}_n$ be the set of $1 \leq j \leq \bar{K}_n$, such that

$$\mathcal{T}(x_j, (1-\delta)\tilde{\epsilon}_n) \geq (1-2\delta) a h_d(\tilde{\epsilon}_n) \kappa_M.$$

It follows by Lemma 3.1 that

$$\mathbf{P}^{x_0}(\mathcal{T}(x, (1-\delta)\tilde{\epsilon}_n) \geq (1-2\delta) a h_d(\tilde{\epsilon}_n)\kappa_M) \leq c\, \tilde{\epsilon}_n^{(1-10\delta)a}\,,$$

for some $c = c(\delta) < \infty$, all sufficiently large n and any $x \in M$. Thus, for all sufficiently large n, any j and $a > 0$,

$$\mathbf{P}^{x_0}(j \in \mathcal{A}_n) \leq c\, \tilde{\epsilon}_n^{(1-10\delta)a}\,, \tag{3.6}$$

implying that

$$\mathbb{E}^{x_0}|\mathcal{A}_n| \leq c'\, \tilde{\epsilon}_n^{(1-10\delta)a-d}. \tag{3.7}$$

Let $\mathcal{V}_{n,j} = B(x_j, \delta\tilde{\epsilon}_n)$. For any $x \in M$ there exists $j \in \{1, \ldots, \bar{K}_n\}$ such that $x \in \mathcal{V}_{n,j}$, hence $B(x, \tilde{\epsilon}_n) \supseteq B(x_j, (1-\delta)\tilde{\epsilon}_n)$. Consequently, $\cup_{n \geq m} \cup_{j \in \mathcal{A}_n} \mathcal{V}_{n,j}$ forms a cover of D_a by sets of maximal diameter $2\delta\tilde{\epsilon}_m$. Fix $a \in (0,2]$. Let $d(B)$ denote the diameter of a set $B \in M$. Since $d(\mathcal{V}_{n,j}) = 2\delta\tilde{\epsilon}_n$, it follows from (3.6) that for $\gamma = d - (1-11\delta)a > 0$,

$$\mathbb{E}^{x_0} \sum_{n=m}^{\infty} \sum_{j \in \mathcal{A}_n} d(\mathcal{V}_{n,j})^{\gamma} \leq c' \, (2\delta)^{\gamma} \sum_{n=m}^{\infty} \tilde{\epsilon}_n^{\,\delta a} < \infty \, .$$

Thus, $\sum_{n=m}^{\infty} \sum_{j \in \mathcal{A}_n} d(\mathcal{V}_{n,j})^{\gamma}$ is finite a.s. implying that $\dim(\mathsf{Late}_{\geq a}) \leq \dim(D_a) \leq \gamma$ a.s. Taking $\delta \downarrow 0$ completes the proof of the upper bound (3.4).

We conclude this section with the derivation of the upper bound for (1.3), that is, for any $E \subseteq M$

$$\limsup_{\varepsilon \to 0} \sup_{x \in E} \frac{\mathcal{T}(x,\varepsilon)}{h_d(\varepsilon)} \leq \overline{\dim}_m(E)\kappa_M \, , \quad a.s. \tag{3.8}$$

Fix $x_0 \in M$ and let $\{x_j : j = 1, \ldots, k_n\}$, denote a maximal collection of points in E, such that $\inf_{\ell \neq j} d(x_\ell, x_j) \geq \delta\tilde{\epsilon}_n$. Let $\mathcal{A}_n(E)$ be the set of $1 \leq j \leq k_n$, such that

$$\mathcal{T}(x_j, (1-\delta)\tilde{\epsilon}_n) \geq (1-2\delta)a h_d(\tilde{\epsilon}_n)\kappa_M.$$

If $\overline{\dim}_m(E) = \gamma$, we have that for any $\delta > 0$

$$k_n \tilde{\epsilon}_n^{(\gamma+\delta)} \to 0.$$

Thus, as in (3.7) we have

$$\mathbb{E}^{x_0}|\mathcal{A}_n(E)| \leq c' \, \tilde{\epsilon}_n^{(1-10\delta)a-(\gamma+\delta)}.$$

Therefore with $a = (\gamma + 2\delta)/(1-10\delta)$

$$\sum_{n=1}^{\infty} \mathbf{P}^{x_0}(|\mathcal{A}_n(E)| \geq 1) \leq \sum_{n=1}^{\infty} \mathbb{E}^{x_0}|\mathcal{A}_n(E)| \leq c' \sum_{n=1}^{\infty} \tilde{\epsilon}_n^{\delta} < \infty \, .$$

By Borel-Cantelli, it follows that $\mathcal{A}_n(E)$ is empty a.s. for all $n > n_0(\omega)$ and some $n_0(\omega) < \infty$. For any $x \in E$ there exists $j \in \{1, \ldots, k_n\}$ such that $x \in B(x_j, \delta\tilde{\epsilon}_n)$, hence $B(x, \tilde{\epsilon}_n) \supseteq B(x_j, (1-\delta)\tilde{\epsilon}_n)$. We then see from (3.5) that for some $C = C(\gamma, d) < \infty$, all $\delta > 0$ small enough and $n > n_1(\delta, \omega)$

$$\sup_{0 < \epsilon \leq \epsilon_{n_1}} \sup_{x \in E} \frac{\mathcal{T}(x,\epsilon)}{h_d(\epsilon)} \leq (\gamma + C\delta)\kappa_M \, ,$$

and (3.8) follows by taking $\delta \downarrow 0$. $\square$

4 Lower bounds

For any $E \subseteq M$ we define

$$\mathcal{C}_\epsilon(E) = \sup_{x \in E} \mathcal{T}(x,\epsilon). \tag{4.1}$$

The following is a restatement of (1.3).

Lemma 4.1. *Let E be a subset of M. Then*

$$\limsup_{\epsilon\to 0}\frac{\mathcal{C}_\epsilon(E)}{h_d(\epsilon)}=\overline{\dim}_m(E)\kappa_M \qquad a.\ s. \tag{4.2}$$

Proof of Lemma 4.1: The almost sure upper bound in (4.2) was established in Section 3. Hence it suffices to show that

$$\mathbb{E}^{x_0}\left(\limsup_{\epsilon\to 0}\frac{\mathcal{C}_\epsilon(E)}{h_d(\epsilon)}\right)\geq \overline{\dim}_m(E)\kappa_M, \tag{4.3}$$

when $\overline{\dim}_m(E)=\gamma>0$. For any $1>\delta>0$ we can find a sequence $\epsilon_n\downarrow 0$ and a collection of points $\{x_{n,j}: j=1,\dots,k_n\}$ in $E\setminus B(x_0,\varepsilon_n^{1-\delta})$, such that $\inf_{\ell\neq j}d(x_{n,\ell},x_{n,j})\geq\epsilon_n^{1-\delta}$ and $k_n\geq\epsilon_n^{-\gamma(1-2\delta)}$. Let $\tau_{n,j}=\inf\{t\geq 0\,|\,X_t\in B(x_{n,j},\epsilon_n)\}$. By (2.3) we have for all n sufficiently large

$$\inf_{\ell\neq j}\ \inf_{x\in B(x_{n,\ell},\epsilon_n)\bigcup\{x_0\}}\mathbb{E}^x(\tau_{n,j})\geq(1-\delta)\kappa_M\epsilon_n^{2-d}.$$

Recall that $h_d(\epsilon)=\epsilon^{2-d}\log\frac{1}{\epsilon}$, so by Theorem 2.6 of [12] this implies that for all $n\geq N$

$$\mathbb{E}^{x_0}\left(\mathcal{C}_{\epsilon_n}(E)\right)\geq(1-4\delta)\gamma\kappa_M h_d(\epsilon_n). \tag{4.4}$$

Let $\widetilde{\tau}_{n,j}=\inf\{t\geq 0\,|\,X_t\in B(y_{n,j},\epsilon_n)\}$, where $y_{n,j}$ are the centers of a minimal cover of M by balls of radius ϵ_n (in particular, having at most $\epsilon_n^{-(d+1)}$ such points for n large enough). Since $\mathbb{E}^x(\widetilde{\tau}_{n,j})\leq c\epsilon_n^{2-d}$ for all n sufficiently large and $x\in M$ (c.f. (2.4)), it follows from Theorem 2.6 of [12] that for all $n\geq N$

$$\sup_{x\in M}\mathbb{E}^x\left(\mathcal{C}_{\epsilon_n}(E)\right)\leq(d+1)ch_d(\epsilon_n). \tag{4.5}$$

Combining (4.4) and (4.5) we see that for some $c_1<\infty$ and all n,

$$\xi_n:=\sup_{x\in M}\ \text{Median}\ [\mathcal{C}_{\epsilon_n}(E)]\leq 2\sup_{x\in M}\mathbb{E}^x\left(\mathcal{C}_{\epsilon_n}(E)\right)\leq c_1\mathbb{E}^{x_0}\left(\mathcal{C}_{\epsilon_n}(E)\right). \tag{4.6}$$

By the Markov property of X_t at $k\xi_n$, for any $x\in M$ and $k=0,1,\dots,$

$$\begin{aligned}&\mathbf{P}^x[\mathcal{C}_{\epsilon_n}(E)>(k+1)\xi_n\,|\,\mathcal{C}_{\epsilon_n}(E)>k\xi_n]\\ \leq\ &\mathbb{E}^x\left[\mathbf{P}^{X_{k\xi_n}}(\mathcal{C}_{\epsilon_n}(E)>\xi_n\,|\,\mathcal{C}_{\epsilon_n}(E)>k\xi_n)\right]\leq\frac{1}{2}.\end{aligned}$$

Hence, $\mathcal{C}_{\epsilon_n}(E)/\xi_n$ is stochastically dominated by a Geometric(1/2) random variable (for any n). Considering (4.6), the sequence $\{\mathcal{C}_{\epsilon_n}(E)/\mathbb{E}^{x_0}(\mathcal{C}_{\epsilon_n}(E))\}_n$ is then uniformly integrable, implying by an extension of Fatou's lemma (c.f. [1, Theorem 7.5.2]), that,

$$\mathbb{E}^{x_0}\left(\limsup_{n\to\infty}\frac{\mathcal{C}_{\epsilon_n}(E)}{\mathbb{E}^{x_0}(\mathcal{C}_{\epsilon_n}(E))}\right)\geq 1$$

Thus, by (4.4)

$$\mathbb{E}^{x_0}\left(\limsup_{\epsilon\to 0}\frac{\mathcal{C}_\epsilon(E)}{h_d(\epsilon)}\right)\geq(1-4\delta)\gamma\kappa_M$$

and (4.3) follows by taking $\delta\downarrow 0$. $\square$

We claim that in fact

$$\limsup_{n\to\infty} \frac{\mathcal{C}_{1/n}(E)}{h_d(1/n)} = \overline{\dim}_m(E)\kappa_M \qquad a.s. \tag{4.7}$$

The upper bound is obvious. Now observe that for any $0 < \epsilon < \varepsilon_0$ we can find an n with $1/(n+1) < \epsilon < 1/n$. Then

$$\frac{\mathcal{C}_{1/(n+1)}(E)}{h_d(1/n)} \geq \frac{\mathcal{C}_{\epsilon}(E)}{h_d(\epsilon)}$$

and the lower bound follows since $h_d(n^{-1})/h_d((n+1)^{-1}) \to 1$.

Recall that

$$\mathsf{Late}_{\geq a} = \Big\{x \in M: \limsup_{\varepsilon\to 0} \frac{\mathcal{T}(x,\varepsilon)}{h_d(\epsilon)} \geq a\kappa_M\Big\}.$$

The statement (1.4) will follow immediately from the next result.

Lemma 4.2. *For any analytic set $E \subseteq M$*

$$\mathbf{P}^{x_0}(E \cap \mathsf{Late}_{\geq a} \neq \emptyset) = \begin{cases} 1 & \textit{if } \dim_p(E) > a \\ 0 & \textit{if } \dim_p(E) < a. \end{cases} \tag{4.8}$$

Proof of Lemma 4.2: If $\dim_p(E) < a$, then by regularization we can represent E as a countable union $E = \cup_n E_n$ with $\overline{\dim}_m(E_n) < a$. It follows immediately from (4.2) that $E_n \cap \mathsf{Late}_{\geq a} = \emptyset$ a.s., and therefore $E \cap \mathsf{Late}_{\geq a} = \emptyset$ a.s.

Now assume that $\dim_p(E) > a$. By [10], we can find a closed $E_\star \subset E$, such that for all open sets V, whenever $E_\star \cap V \neq \emptyset$, then for some $\delta > 0$

$$\overline{\dim}_{\mathrm{m}}\big(E_\star \cap V\big) = a + \delta. \tag{4.9}$$

Define the open sets

$$A_a(k) = \Big\{x \in M: \frac{\mathcal{T}(x,1/k)}{h_d(1/k)} > a\kappa_M\Big\}$$

and

$$A_a = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_a(k).$$

Since $A_a \subseteq \mathsf{Late}_{\geq a}$ it suffices to show that with probability one, $A_a \cap E_\star \neq \emptyset$. Define the open sets $B_a(n) := \cup_{k=n}^{\infty} A_a(k)$, $n \geq 1$. We claim that for all $n \geq 1$, the relatively open set $B_a(n) \cap E_\star$ is a.s. d-dense in (the complete metric space) $E_\star$. If so, Baire's category theorem implies that $E_\star \cap \bigcap_{n=1}^{\infty} B_a(n)$ is dense in $E_\star$ and in particular, nonempty. Since $A_a = \cap_n B_a(n)$, the result follows. Fix an open set V such that $V \cap E_\star \neq \emptyset$. Using (4.9) and (4.7) with E replaced by $V \cap E_\star$ we see that $A_a(n) \cap V \cap E_\star \neq \emptyset$ for infinitely many n, a.s. Thus $B_a(n) \cap V \cap E_\star \neq \emptyset$ for all n a.s.; by letting V run over a countable base for the open sets, we conclude that $B_a(n) \cap E_\star$ is a.s. d-dense in $E_\star$. $\square$

Proof of Theorem 1.3: The upper bound has already been proven in Section 3. Set $A_a^+ = \cap_{m=1}^{\infty} A_{a-1/m}$ and $B_a^+ = A_a^+ - \cup_{m=1}^{\infty}\mathsf{Late}_{\geq a+1/m}$. It is easy to see that

$$B_a^+ \subseteq \mathsf{Late}_a =: \Big\{x \in M: \limsup_{\varepsilon\to 0} \frac{\mathcal{T}(x,\varepsilon)}{h_d(\epsilon)} = a\kappa_M\Big\}.$$

It follows from the argument in the proof of Corollary 3.3 in [5] that for any analytic set $E \subseteq M$ with $\dim_p(E) > a$, we have $B_a^+ \cap E \neq \emptyset$ a.s. For the convenience of the reader we reproduce the short proof. Let $\Lambda_m(n) := \cup_{k=n}^{\infty} A_{a-1/m}(k)$. Since $\dim_p(E) > a$, by [10] there exists a closed $E_\star \subset E$ such that $\dim_p\left(E_\star \cap V\right) = a$ for any open set V such that $E_\star \cap V \neq \emptyset$, implying as in the last proof that $\Lambda_m(n) \cap E_\star$ is a.s. dense in the complete metric space $E_\star$. Consequently, by Baire's theorem it follows that $E_\star \cap (\cap_{n=1}^{\infty} \cap_{m=1}^{\infty} \Lambda_m(n))$ is dense in $E_\star$, a.s., and in particular is non-empty. Obviously, $\dim_p(E_\star) = a$, so by Lemma 4.2, $\mathtt{Late}_{\geq a+1/m} \cap E_\star = \emptyset$, a.s.. It follows that $B_a^+ \cap E \neq \emptyset$, a.s. as claimed.

We now wish to conclude the proof of the lower bound of Theorem 1.3 by appealing to Lemma 3.4 of [11]. However, the statement and proof of that Lemma are for sets in $[0,1]^d$, whereas our sets are in M. This can be easily remedied as follows. Let $\phi : M' \mapsto [0,1]^d$ be a diffeomorphism from some subset $M' \subseteq M$. We can choose ϕ, M' so that for some $\delta > 0$

$$B(\phi(x), (1-\delta)r) \subseteq \phi(B(x,r)) \subseteq B(\phi(x), (1+\delta)r)$$

for all $x \in M'$ and r sufficiently small. This is enough to show that for any $K \subseteq M'$ we have $\dim(\phi(K)) = \dim(K)$ where the first dimension is computed in $[0,1]^d$ using the Euclidean distance while the second is computed in M using the Riemannian distance. Our theorem now follows from the above cited Lemma 3.4 of [11]. $\square$

Proof of Theorem 1.1: In view of (1.3) it suffices to show that

$$\liminf_{\epsilon \to 0} \frac{\mathcal{C}_\varepsilon(M)}{h_d(\varepsilon)} \geq d\kappa_M \qquad \text{a. s.}$$

To this end, fix $\eta > 0$, $\delta > 0$ and $R_0 > 0$ small such that (2.3) holds with $\eta(R) \leq \eta/2$ for all $R \leq R_0$. Let $\epsilon_n = (1-\eta)^n$ and fix points $\{x_{n,j} : j = 0,1,\dots,K_n\}$ in $B(x_0, R_0/3)$ such that $x_{n,0} = x_0$, $\inf_{\ell \neq j} d(x_{n,\ell}, x_{n,j}) \geq \epsilon_n^{1-\delta}$, and $K_n \geq \epsilon_n^{-d(1-2\delta)}$ for all n large enough. Let $\tau_{n,j} = \inf\{t \geq 0 \,|\, X_t \in B(x_{n,j}, \epsilon_n)\}$ for $j = 1,\dots,K_n$, noting that $\mathcal{C}_{\epsilon_n}(M) \geq \max_j \tau_{n,j} := \widehat{\mathcal{C}}_n$. With $\mathcal{C}_\epsilon(M)$ non-increasing in ϵ and $h_d(\epsilon)$ non-decreasing in ϵ, such that

$$d\, h_d(\epsilon_{n+1}) \leq (1-\eta)^{1-d}(1-2\delta)\epsilon_n^{2-d} \log K_n \ ,$$

for all n large enough, it suffices to show that

$$\liminf_{n\to\infty} \frac{\widehat{\mathcal{C}}_n}{\epsilon_n^{2-d} \log K_n} \geq (1-\eta)(1-2\eta)\kappa_M \qquad \text{a. s.} \tag{4.10}$$

By (2.3) we have for all n large enough

$$\inf_{\ell \neq j} \inf_{x \in \partial B(x_{n,\ell}, \epsilon_n)} \mathbb{E}^x(\tau_{n,j}) \geq (1-\eta)\kappa_M \epsilon_n^{2-d} := v_n \,. \tag{4.11}$$

Combining (4.11) with (2.12) we get that for some constant $c_0 = c_0(\eta) < \infty$, all n large enough and all $\theta \geq 0$,

$$g_n(\theta) := \sup_{\ell \neq j} \sup_{x \in B(x_{n,\ell}, \epsilon_n)} \mathbb{E}^x(e^{-\theta \tau_{n,j}}) \leq 1 - \theta v_n + \frac{c_0}{2}\theta^2 v_n^2 \tag{4.12}$$

(see (2.15) for a similar derivation). Note that $g_n(\theta) \le 1$. Further, it is not hard to verify that Theorem 1.3 of [13] applies here, leading to the bound

$$\begin{aligned}\mathbb{E}^{x_0}\left[e^{-\theta\widehat{C}_n}\right] &\le \prod_{i=1}^{K_n}\frac{i}{i-1+1/g_n(\theta)} \le \prod_{i=1}^{K_n}\frac{i}{i+1-g_n(\theta)} \\ &\le 4e^{-(\theta v_n - \frac{c_0}{2}\theta^2 v_n^2)\log K_n}\,, \qquad (4.13)\end{aligned}$$

where the last inequality follows from (4.12) and the fact that

$$\prod_{i=1}^{k}\frac{i}{i+b} \le \exp\left(\int_1^k \log(\frac{x}{x+b})dx\right) \le 4e^{-b\log k}\,,$$

for all integer k and $b \in [0,1]$. Setting $\theta = 2\eta/(c_0 v_n) > 0$, it follows from Chebycheff's inequality and (4.13) that for some $c_2 = c_2(\eta,\delta) > 0$ and all n large,

$$\begin{aligned}\mathbf{P}^{x_0}(\widehat{C}_n \le (1-2\eta)v_n\log K_n) &\le e^{(1-2\eta)\theta v_n \log K_n}\mathbb{E}^{x_0}\left[e^{-\theta\widehat{C}_n}\right] \\ &\le 4e^{-\eta\theta v_n\log K_n} \le 4(1-\eta)^{-c_2 n}\,.\end{aligned}$$

By Borel-Cantelli we get (4.10), completing the proof of the theorem. □

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