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Smoothness of the law of some one-dimensional jumping S.D.E.s with non-constant rate of jump

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Abstract

We consider a one-dimensional jumping Markov process $\{X_t^x\}_{t\geq 0}$, solving a Poisson-driven stochastic differential equation. We prove that the law of X_t^x admits a smooth density for $t > 0$, under some regularity and non-degeneracy assumptions on the coefficients of the S.D.E. To our knowledge, our result is the first one including the important case of a non-constant rate of jump. The main difficulty is that in such a case, the map $x \mapsto X_t^x$ is not smooth. This seems to make impossible the use of Malliavin calculus techniques. To overcome this problem, we introduce a new method, in which the propagation of the smoothness of the density is obtained by analytic arguments.

Key words: Stochastic differential equations, Jump processes, Regularity of the density.

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1 Introduction

Consider a $\mathbb{R}$-valued jumping Markov process $\{X_t^x\}_{t\geq 0}$ with finite variations, starting from $x \in \mathbb{R}$, with generator $\mathcal{L}$, defined for $\phi : \mathbb{R} \mapsto \mathbb{R}$ sufficiently smooth and $y \in \mathbb{R}$, by

$$\mathcal{L}\phi(y) = b(y)\phi'(y) + \gamma(y) \int_G [\phi(y + h(y,z)) - \phi(y)]\, q(dz), \tag{1}$$

for some functions $\gamma, b : \mathbb{R} \mapsto \mathbb{R}$ with γ nonnegative, for some measurable space G endowed with a nonnegative measure q, and some function $h : \mathbb{R} \times G \mapsto \mathbb{R}$.

Roughly, $b(y)$ is the *drift* term: between t and $t + dt$, X_t^x moves from y to $y + b(y)dt$. Next, $\gamma(y)q(dz)$ stands for the *rate* at which X_t^x jumps from y to $y + h(y,z)$.

We aim to investigate the smoothness of the law of X_t^x for $t > 0$. Most of the known results are based on the use of some Malliavin calculus, i.e. on a sort of *differential calculus* with respect to the stochastic variable ω.

The first results in this direction were obtained by Bismut [4], see also Léandre [12]. Important results are due Bichteler et al. [2]. We refer to Graham-Méléard [9], Fournier [6] and Fournier-Giet [8] for relevant applications to physic integro-differential equations such as the Boltzmann and the coagulation-fragmentation equations. These results concern the case where $q(dz)$ is sufficiently smooth.

When q is singular, Picard [14] obtained some results using some fine arguments relying on the affluence of small (possibly irregular) jumps. Denis [5] and more recently Bally [1] and Kulik [10; 11] also obtained some regularity results when q is singular, using the drift and the density of the jump instants, see also Nourdin-Simon [13].

All the previously cited works apply only to the case where the *rate of jump* $\gamma(y)$ is constant. The case where γ is non constant is much more delicate. The main reason for this is that in such a case, the map $x \mapsto X_t^x$ cannot be regular (and even continuous). Indeed, if $\gamma(x) < \gamma(y)$, and if $q(G) = \infty$, then it is clear that for all small $t > 0$, X^y jumps infinitely more often than X^x before t. The only available results with γ not constant seem to be those of [7; 8], where only the existence of a density was proved. Bally [1] considers the case where $\gamma(y)q(dz)$ is replaced by something like $\gamma(y,z)q(dz)$, with $\sup_y |\gamma(y,z) - 1| \in L^1(q)$: the rate of jump is not constant, but this concerns only finitely many jumps.

From a physical point of view, the situation where γ is constant is quite particular. For example in the (nonlinear) Boltzmann equation, which describes the distribution of velocities in a gas, the rate of collision between two particles heavily depends on their relative velocity (except in the so-called Maxwellian case treated in [9; 6]). In a fragmentation equation, describing the distribution of masses in a system of particles subjected to breakage, the rate at which a particle splits into smaller ones will clearly almost always depend on its mass...

We will show here that when q is smooth enough, it is possible to obtain some regularity results in the spirit of [2]. Compared to [2], our result is

• stronger, since we allow γ to be non-constant;

• weaker, since we are not able, at the moment, to study the case of processes with infinite variations, and since we treat only the one-dimensional case (our method could also apply to multidimensional processes, but our non-degeneracy conditions would be very strong).

Our method relies on the following simple ideas:

(a) we consider, for $n \geq 1$, the first jump instant τ_n of the Poisson measure driving X^x, such that the corresponding *mark* Z_n falls in a subset $G_n \subset G$ with $q(G_n) \simeq n$;

(b) using some smoothness assumptions on q and h, we deduce that $X^x_{\tau_n}$ has a smooth density (less and less smooth as n tends to infinity);

(c) we also show that smoothness propagates with time in some sense, so that X^x_t has a smooth density conditionnally to $\{t \geq \tau_n\}$;

(d) we conclude by choosing carefully n very large in such a way that $\{t \geq \tau_n\}$ occurs with sufficiently great probability.

As a conclusion, we obtain the smoothness of the density using only the regularizing property of *one* (well-chosen) jump. On the contrary, Bichteler et al. [2] were using the regularization of infinitely many jumps, which was possible using a sort of Malliavin calculus. Surprisingly, our non-degeneracy condition does not seem to be stronger, see Subsection 2.4 for a detailed comparison in a particular (but quite typical) example.

Our method should extend directly to any dimension $d \geq 2$, but under some very stringent assumptions: first, one would have to assume that for each z, $y \mapsto y + h(y, z)$ is invertible and very smooth (see Section 4), which is not so easy in dimension $d \geq 2$. Secondly, and this is the most important, one would have to assume a strong non-degeneracy condition, to obtain the smooth density using one jump (see Section 3). Thus the jump measure of the process would need to be bounded below by a smooth density with respect to the Lebesgue measure on $\mathbb{R}^d$.

In [2] (and also in [11]), more subtile conditions are assumed: very roughly, the jump measure has to be bounded below by a sum of measures, and something like the convolution of these measures needs to have a smooth density with respect to the Lebesgue measure on $\mathbb{R}^d$. (The main idea is that two successive jumps with law supported by one-dimensional curves may produce a density for a 2-dimensional process, provided the two curves are not too much colinear). We hope that it might be possible to treat such a situation using our ideas, but still much work is needed.

Finally, let us mention that we may write our process as $X^x_t = Y^x_{\tau_t}$, where $(Y^x_t)_{t \geq 0}$ is a Markov process with generator $\gamma^{-1}\mathcal{L}$, and $(\tau_t)_{t \geq 0}$ is a time-change involving γ and $(Y^x_t)_{t \geq 0}$. Of course, the rate of jump of $(Y^x_t)_{t \geq 0}$ is constant, so that the results of [2] apply: under some reasonnable conditions, Y^x_t has a smooth law as soon as $t > 0$. It thus seems natural to start from this to study the smoothness of the law of X^x_t. However, the change of time $(\tau_t)_{t \geq 0}$ is random, and its corelation with $(Y^x_t)_{t \geq 0}$ is complicated. We have not been able to obtain any result from that point of view.

We present our results in Section 2, and we give the proofs in Sections 3 and 4. An Appendix lies at the end of the paper.

2 Results

In the whole paper, $\mathbb{N} = \{1, 2, ...\}$. Consider the one-dimensional S.D.E.

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \int_0^\infty \int_G h(X_{s-}^x, z)\mathbf{1}_{\{u \le \gamma(X_{s-}^x)\}} N(ds, du, dz), \tag{2}$$

where

Assumption (I)**:** The Poisson measure $N(ds, du, dz)$ on $[0, \infty) \times [0, \infty) \times G$ has the intensity measure $dsduq(dz)$, for some measurable space $(G, \mathcal{G})$ endowed with a nonnegative measure q. For each $t \ge 0$ we set $\mathcal{F}_t := \sigma\{N(A), A \in \mathcal{B}([0, t]) \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{G}\}$.

Observe that the role of the variable u in (2) is to control the *rate of jump*, using a sort of acceptance-rejection procedure: when a mark u of the Poisson measure satisfies $u \le \gamma(X_{s-}^x)$, the jump occurs, else it does not. This implies roughly that at time s, our process jumps with a rate proportionnal to $\gamma(X_{s-}^x)$.

We will require some smoothness of the coefficients. For $f(y) : \mathbb{R} \mapsto \mathbb{R}$ (and $h(y, z) : \mathbb{R} \times G \mapsto \mathbb{R}$), we will denote by $f^{(l)}$ (and $h^{(l)}$) the l-th derivative of f (resp. of h with respect to y). Below, $k \in \mathbb{N}$ and $p \in [1, \infty)$ are fixed.

Assumption $(A_{k,p})$**:** The functions $b : \mathbb{R} \mapsto \mathbb{R}$ and $\gamma : \mathbb{R} \mapsto \mathbb{R}_+$ are of class C^k, with all their derivatives of order 0 to k bounded.

The function $h : \mathbb{R} \times G \mapsto \mathbb{R}$ is measurable, and for each $z \in G$, $y \mapsto h(y, z)$ is of class C^k on $\mathbb{R}$. There exists $\eta \in (L^1 \cap L^p)(G, q)$ such that for all $y \in \mathbb{R}$, all $z \in G$, all $l \in \{0, ..., k\}$, $|h^{(l)}(y, z)| \le \eta(z)$.

Under $(A_{1,1})$, $\mathcal{L}\phi$, introduced in (1), is well-defined for all $\phi \in C^1(\mathbb{R})$ with a bounded derivative. The following result classically holds, see e.g. [7, Section 2] for the proof of a similar statement.

Proposition 2.1. *Assume* (I) *and* $(A_{k,p})$ *for some* $p \ge 1$*, some* $k \ge 1$*. For any* $x \in \mathbb{R}$*, there exists a unique càdlàg* $(\mathcal{F}_t)_{t\ge0}$*-adapted process* $(X_t^x)_{t\ge0}$ *solution to (2) such that for all all* $T \in [0, \infty)$*,* $E[\sup_{s\in[0,T]} |X_s^x|^p] < \infty$*.*

The process $(X_t^x)_{t\ge0,x\in\mathbb{R}}$ *is a strong Markov process with generator* $\mathcal{L}$ *defined by (1). We will denote by* $p(t, x, dy) := \mathcal{L}(X_t^x)$ *its semi-group.*

2.1 Propagation of smoothness

We consider the space $\mathcal{M}(\mathbb{R})$ of finite (signed) measures on $\mathbb{R}$, and we abusively write $||f||_{L^1(\mathbb{R})} := ||f||_{TV} = \int_\mathbb{R} |f|(dy)$ for $f \in \mathcal{M}(\mathbb{R})$. We denote by $C_b^k(\mathbb{R})$ (resp. $C_c^k(\mathbb{R})$) the set of C^k-functions with all their derivatives bounded (resp. compactly supported). We introduce, for $k \ge 1$, the space $\bar{W}^{k,1}(\mathbb{R})$ of measures $f \in \mathcal{M}(\mathbb{R})$ such that for all $l \in \{1, ..., k\}$, there exists $g_l \in \mathcal{M}(\mathbb{R})$ such that for all $\phi \in C_c^k(\mathbb{R})$ (and thus for all $\phi \in C_b^k(\mathbb{R})$),

$$\int_\mathbb{R} f(dy)\phi^{(l)}(y) = (-1)^l \int_\mathbb{R} g_l(dy)\phi(y).$$

If so, we set $f^{(l)} = g_l$. Classically, for $f \in \mathcal{M}(\mathbb{R})$, $f \in \bar{W}^{k,1}(\mathbb{R})$ if and only if

$$||f||_{\bar{W}^{k,1}(\mathbb{R})} := \sum_{l=0}^{k} \sup \left\{ \int_{\mathbb{R}} f(dy)\phi^{(l)}(y), \quad \phi \in C_b^k(\mathbb{R}), \ ||\phi||_\infty \leq 1 \right\} \tag{3}$$

is finite (here C_b^k could be replaced by C_c^k, C_b^∞, or C_c^∞), and in such a case,

$$||f||_{\bar{W}^{k,1}(\mathbb{R})} = \sum_{l=0}^{k} ||f^{(l)}||_{L^1(\mathbb{R})}.$$

Let us finally recall that

- for $f \in C^k(\mathbb{R})$, $f(y)dy$ belongs to $\bar{W}^{k,1}(\mathbb{R})$ if and only if $\sum_0^k |f^{(l)}| \in L^1(\mathbb{R})$;
- if $f \in \bar{W}^{k,1}(\mathbb{R})$, with $k \geq 2$, then $f(dy)$ has a density of class $C^{k-2}(\mathbb{R})$ (due to the classical Sobolev Lemmas).

We now introduce a first non-degeneracy assumption (here $h'(y,z) = \partial_y h(y,z)$).

Assumption (S)**:** There exists $c_0 > 0$ such that for all $z \in G$, all $y \in \mathbb{R}$, it holds $1 + h'(y,z) \geq c_0$.

Notation 2.2. *For $t \geq 0$ and a probability measure f on $\mathbb{R}$, we define $p(t,f,dy)$ on $\mathbb{R}$ by $p(t,f,A) = \int_{\mathbb{R}} f(dx)p(t,x,A)$, where $p(t,x,dy)$ was defined in Proposition 2.1.*

Observe that $p(t,f,dy)$ is the law of $X_t^{X_0}$ where $(X_t^x)_{t\geq 0, x\in\mathbb{R}}$ solves (2) and where $X_0 \sim f(dy)$ is independent of N.

Proposition 2.3. *Let $p \geq k+1 \geq 2$ be fixed, assume (I), $(A_{k+1,p})$, and (S). There is $C_k > 0$ such that for all probability measures $f \in \bar{W}^{k,1}(\mathbb{R})$, all $t \geq 0$,*

$$||p(t,f,.)||_{\bar{W}^{k,1}(\mathbb{R})} \leq ||f||_{\bar{W}^{k,1}(\mathbb{R})} e^{C_k t}.$$

The proof of this proposition, see Section 4, is purely analytic. It simply consists in writing rigorously the following idea: consider the integro-differential equation satisfied by $p(t,f,dy)$, differentiate formally k times this equation with repect to y, integrate its absolute value over $\mathbb{R}$, and try to obtain a Gronwall-like inequality. Such a scheme of proof is completely standard from the analytic point of view. However, we have not found any reference concerning the kind of equation under study.

Assumption (S) is probably far from optimal, but something in this spirit is needed: take $b \equiv 0$, $\gamma \equiv 1$ and $h(y,z) = -y\mathbf{1}_A(z) + y\eta(z)$ for some $A \subset G$ with $q(A) < \infty$ and some $\eta \in L^1(G,q)$. Of course, (S) is not satisfied, and one easily checks that there exists τ_A exponentially distributed (with parameter $q(A)$) such that a.s., for all $t \geq \tau_A$, all $x \in \mathbb{R}$, $X_t^x = 0$. This forbids the propagation of smoothness, since then $p(t,f,dy) \geq (1 - e^{-q(A)t})\delta_0(dy)$, even if f is smooth.

2.2 Regularization

We now give the non-degeneracy condition that will provide a smooth density to our process. A generic example of application (in the spirit of [2]) will be given below. For two nonnegative measures $\nu, \tilde{\nu}$ on G, we say that $\nu \leq \tilde{\nu}$ if for all $A \in \mathcal{G}$, $\nu(A) \leq \tilde{\nu}(A)$. Here $k \in \mathbb{N}$, $p \in [0, \infty)$ and $\theta > 0$.

Assumption $(H_{k,p,\theta})$**:** Consider the jump kernel $\mu(y, du)$ associated to our process, defined by $\mu(y, A) = \gamma(y) \int_G \mathbf{1}_A(h(y, z))q(dz)$ (which may be infinite) for all $A \in \mathcal{B}(\mathbb{R})$.

There exists a (measurable) family $(\mu_n(y, du))_{n \geq 1, y \in \mathbb{R}}$ of measures on $\mathbb{R}$ meeting the following points:

(i) for $n \geq 1$, $y \in G$, $0 \leq \mu_n(y, du) \leq \mu(y, du)$ and $\mu_n(y, \mathbb{R}) \geq n$;

(ii) for all $r > 0$, $n \geq 1$, $\sup_{|y| \leq r} \mu_n(y, \mathbb{R}) < \infty$;

(iii) there exists $C > 0$ such that for all $n \in \mathbb{N}$, $y \in \mathbb{R}$,

$$\frac{1}{\mu_n(y, \mathbb{R})} ||\mu_n(y, .)||_{\bar{W}^{k,1}(\mathbb{R})} \leq C(1 + |y|^p)e^{\theta n}.$$

The principle of this assumption is quite natural: it says that at any position y, our process will have sufficiently many jumps with a sufficiently smooth density. When possible, it is better to choose μ_n in such a way that $\mu_n(y, \mathbb{R}) \simeq n$: indeed, (i) says that we need to have $\mu_n(y, \mathbb{R}) \geq n$. But the more $\mu_n(y, \mathbb{R})$ is large, the less (iii) will be easily satisfied. Indeed, choosing $\mu_n(y, \mathbb{R})$ large implies that $\mu_n(y, dz)/\mu_n(y, \mathbb{R})$ gives a large weight to a neighborhood of $z \simeq 0$, and thus is close to a Dirac mass at 0, which of course makes (iii) difficult to hold.

Our main result is the following.

Theorem 2.4. *Let $p \geq k + 1 \geq 3$ and $\theta > 0$ be fixed. Assume (I), $(A_{k+1,p})$, (S) and $(H_{k,p,\theta})$. Consider the law $p(t, x, dy)$ at time $t \geq 0$ of the solution $(X_t^x)_{t \geq 0}$ to (2).*

(a) Let $t > \theta/(k-1)$. For any $x \in \mathbb{R}$, $p(t, x, dy)$ has a density $y \mapsto p(t, x, y)$ of class $C_b^l(\mathbb{R})$ as soon as $0 \leq l < kt/(\theta + t) - 1$.

(b) In particular, if $(H_{k,p,\theta})$ holds for all $\theta > 0$, then for all $t > 0$, all $x \in \mathbb{R}$, $y \mapsto p(t, x, y)$ is of class $C_b^{k-2}(\mathbb{R})$.

Observe that for t large enough, say $t \geq 1$, and if k is large enough, then the first condition $t > \theta/(k-1)$ in (a) will be neglected.

2.3 Another assumption

It might seem strange to state our regularity assumptions with the help of γ, h, q, and to our nondegeneracy conditions with the help of the jump kernel μ. However, it seems to us to be the best way to give understandable assumptions.

Let us give some conditions on γ, h, q, in the spirit of [2], which imply $(H_{k,p,\theta})$.

Assumption $(B_{k,p,\theta})$**:** $G = \mathbb{R}$, and for all $y \in \mathbb{R}$, $\gamma(y) > 0$ and there exists $I(y) = (a(y), \infty)$ (or $(-\infty, a(y))$) with $a(y) \in \mathbb{R}$, with $y \mapsto a(y)$ measurable, such that $q(dz) \geq \mathbf{1}_{I(y)}(z)dz$ and such that the following conditions are fulfilled:

(a) for all $y \in \mathbb{R}$, $z \mapsto h(y, z)$ is of class C^{k+1} on $I(y)$. The derivatives $h_z^{(l)}$ (w.r.t. z) for $l = 1, ..., k+1$ are uniformly bounded on $\{(y, z);\ y \in \mathbb{R}, z \in I(y)\}$;

(b) for all $y \in \mathbb{R}$, all $z \in I(y)$, $h'_z(y,z) \neq 0$, and with $I_n(y) = [a(y), a(y) + n/\gamma(y)]$ (or $[a(y) - n/\gamma(y), a(y)]$),

$$\frac{\gamma(y)}{n} \int_{I_n(y)} |h'_z(y,z)|^{-2k} dz \leq C(1 + |y|^p) e^{\theta n}. \tag{4}$$

Remark 2.5. *$(B_{k,p,\theta})$ and $(A_{1,1})$ imply $(H_{k,p,\theta})$.*

This lemma is proved in the Appendix. Let us give some examples for (4).

Examples: Assume that $|h'_z(y,z)| \geq \epsilon(1 + |y|)^{-\alpha}\zeta(z)$, for all $y \in \mathbb{R}$, all $z \in I(y)$, for some $\alpha \geq 0$, $\epsilon > 0$.

• If $\zeta(z) = (1 + |z|)^{-\delta}$, for some $\delta \geq 0$, and $\gamma(y) \geq c(1 + |y|)^{-\beta}$ for some $c > 0$, $\beta \geq 0$, then (4) holds for all $k \geq 1$, all $\theta > 0$ and all $p \geq 2k(\alpha + \beta\delta)$.

• If $\zeta(z) = e^{-d|z|^\delta}$, for some $d > 0$, $\delta \in (0,1)$, and if $\gamma(y) \geq c[\log(2 + |y|)]^{-\beta}$, with $c > 0$, $\beta \in [0, (1-\delta)/\delta)$, then (4) holds for all $k \geq 1$, all $\theta > 0$, all $p > 2k\alpha$.

• If $\zeta(z) = e^{-d|z|}$, for some $d > 0$, if $\gamma(y) \geq c > 0$, then (4) holds for all $k \geq 1$, all $\theta \geq 2kd/c$ and all $p \geq 2k\alpha$.

• With our assumption that γ is bounded, (4) does never hold if $\zeta(z) = e^{-d|z|^\delta}$ for some $d > 0$, $\delta > 1$.

Observe on these examples that there is a balance between the *rate* of jump γ and the *regularization power* of jumps (given, in some sense, by lowerbounds of $|h'_z|$). The more the power of regularization is small, the more the rate of jump has to be bounded from below. This is quite natural and satisfying.

2.4 Comments

In this subsection, we compare our result with existing results. Recall that the main contribution of our method is that it allows to treat the case where γ is not constant: to our knowledge, all the previous results were dealing with a constant rate of jump.

The works of Denis [5], Nourdin-Simon [13], Bally [1], Kulik [10; 11] treat the difficult case of a possibly singular jump measure. They obtain some regularity results assuming that the drift is non-degenerated. Thus, this can not really be compared to our work: we need much more regularity of the jump measure, but we can take $b \equiv 0$. On the contrary, they assume much less on the jump measure, but some non-degeneracy conditions are supposed about b.

After the pionneering papers of Bismut [4], Bichteler-Jacod [3], Léandre [12], the first systematical study of regularizing properties for jump processes is the one of Bichteler-Gravereaux-Jacod [2]. This work has certainly be refined, see in particular the remarkable results by Picard [14]. However, the method of [14] is quite complicated, and it seems difficult to extend it to our case.

The aim of this section is thus to compare precisely our result to that of [2]. Let us recall that when γ is constant, the result of [2] (restricted to the dimension 1), is essentially the following.

Roughly, they also assume something like $q(dz) \geq \mathbf{1}_{(a,\infty)}(z)dz$ (they actually consider the case where $q(dz) \geq \mathbf{1}_O(z)dz$ for some infinite open subset O of $\mathbb{R}$).

They assume more integrability on the coefficients (something like $(A_{k,p})$ for all $p > 1$). They assume (S), and much more joint regularity (in y, z) of h (see Assumption $(A - r)$ page 9 in [2]), the uniform boundedness of $\partial_{z^\alpha}\partial_{y^\beta} h$ as soon as $\alpha \geq 1$.

Their non-degeneracy condition (see Assumption $(SB - (\zeta, \theta))$ page 14 in [2]) is of the form $|h'_z(y,z)|^2 \geq \epsilon(1 + |x|)^{-\delta}\zeta(z)$, for some $\delta \geq 0$, some $\epsilon > 0$, and some *broad* function ζ (see Definition 2-20 and example 2-35 pages 13 and 17 in [2]). This notion is probably not exactly comparable to (4). Roughly,

- when $\zeta(z) = e^{-\alpha|z|^\delta}$ with $\delta > 1$, their result does not apply (as ours);
- when $\zeta(z) = e^{-\alpha|z|^\delta}$ with $\delta < 1$, or when $\zeta(z) = (1 + |z|)^{-\beta}$ with $\beta > 0$, their result applies for all times $t > 0$ (as ours);
- when $\zeta(z) = e^{-\alpha|z|}$, their result applies for sufficiently large times (as ours).

As a conclusion, we have slightly less technical assumptions. About the nondegeneracy assumption, it seems that the condition in [2] and ours are very similar (when $\gamma \equiv 1$). Let us insist on the fact that this is quite surprising: one could think that since we use only the regularization of *one* jump, our nondegeneracy condition should be much stronger than that of [2].

We could probably state an assumption as $(B_{k,p,\theta})$ for a general lowerbound of the form $q(dz) \geq \mathbf{1}_O(z)\varphi(z)dz$, for some open subset O of $\mathbb{R}$ and some C^∞ function $\varphi : O \mapsto \mathbb{R}$, but this would be very technical.

Finally, it seems highly probable that one may assume, instead of (S), that $0 < 1/(1+h'(x,z)) \leq \alpha(z) \in L^1 \cap L^r(G,q)$ (with r large enough); and that the assumptions b, γ bounded and $|h(x,z)| \leq \eta(z)$ (in $(A_{k,p})$) could be replaced by $|b(x)| \leq C(1+|x|)$ and $\gamma(x)|h(x,z)| \leq (1+|x|)\eta(z)$, with $\eta \in L^1 \cap L^p(G,q)$. However, the paper is technical enough.

We prove Theorem 2.4 in Section 3 and Proposition 2.3 in Section 4.

3 Smoothness of the density

In this section, granting Proposition 2.3 for the moment, we give the proof of our main result. Proposition 2.3 is proved in the next section. We refer to the introduction for the main ideas of the proof.

Proof of Theorem 2.4. We consider here $x \in \mathbb{R}$, the associated process $(X_t^x)_{t\geq 0}$. We assume (I), (S), $(A_{k+1,p})$, and $(H_{k,p,\theta})$ for some $p \geq k+1 \geq 3$, some $\theta > 0$. Due to Proposition 2.1,

$$\forall\, t > 0, \quad C_t := E\left[\sup_{[0,t]} |X_s^x|^p\right] < \infty. \tag{5}$$

Recall $(H_{k,p,\theta})$, and denote by $f_n(y,u)$ the density (bounded by 1) of $\mu_n(y,du)$ with respect to $\mu(y,du)$. We now write μ_n in terms of γ, h, and f_n. First, we set $d_n(y,z) := f_n(y, h(y,z))$ (which is bounded by 1). Then for $q_n(y,dz) := d_n(y,z)q(dz)$, one easily checks that for all $A \in \mathcal{B}(\mathbb{R})$,

$$\mu_n(y,A) = \gamma(y)\int_G \mathbf{1}_A(h(y,z))q_n(y,dz).$$

In words, $\mu_n(y,.)$ can be seen as the image measure of $\gamma(y)q_n(y,.)$ by the map $z \mapsto h(y,z)$.

As a consequence, still using $(H_{k,p,\theta})$,

(i) $0 \le q_n(y,dz) \le q(dz)$, and $\gamma(y)q_n(y,G) = \mu_n(y,\mathbb{R}) \ge n$;

(ii) for all $r > 0$, $n \in \mathbb{N}$, $\sup_{|y|\le r} \gamma(y)q_n(y,G) < \infty$.

This second point asserts that the total mass of our jump measure is locally bounded for each n.

We now divide the proof into four parts.

Step 1. We first introduce some well-chosen instants of jump that will provide a density to our process. To this end, we write $N = \sum_{i\ge1}\delta_{(t_i,u_i,z_i)}$, we consider a family of i.i.d. random variables $(v_i)_{i\ge1}$ uniformly distributed on $[0,1]$, independent of N. We introduce the Poisson measure $M = \sum_{i\ge1}\delta_{(t_i,u_i,z_i,v_i)}$ on $[0,\infty)\times[0,\infty)\times G\times[0,1]$ with intensity measure $dsduq(dz)dv$. Then we observe that $N(ds,du,dz) = M(ds,du,dz,[0,1])$. Let $\mathcal{H}_t = \sigma\{M(A),\ A \in \mathcal{B}([0,t])\otimes\mathcal{B}([0,\infty))\otimes\mathcal{G}\otimes\mathcal{B}([0,1])\}$.

Next, we observe, using point (ii) above and (5), that a.s., for all $t \ge 0$,

$$\begin{aligned}\sup_{[0,t]}\textstyle\int_0^\infty\int_G\int_0^1 \mathbf{1}_{\{u\le\gamma(X^x_{s-}),v\le d_n(X^x_{s-},z)\}}duq(dz)dv\\ = \sup_{[0,t]}\gamma(X^x_{s-})q_n(X^x_{s-},G) < \infty.\end{aligned}$$

We thus may consider, for each $n \ge 1$, the a.s. positive $(\mathcal{H}_t)_{t\ge0}$-stopping time

$$\tau_n = \inf\left\{t\ge0; \int_0^t\int_0^\infty\int_G\int_0^1 \mathbf{1}_{\{u\le\gamma(X^x_{s-}),v\le d_n(X^x_{s-},z)\}}M(ds,du,dz,dv) > 0\right\},$$

and the associated *mark* (U_n,Z_n,V_n) of M. Then one easily checks that

(a) for $t \ge 0$, $P[\tau_n \ge t] \le e^{-nt}$, since due to point (i), a.s., for all $s \ge 0$,

$$\begin{aligned}\int_0^\infty\int_G\int_0^1 \mathbf{1}_{\{u\le\gamma(X^x_{s-}),v\le d_n(X^x_{s-},z)\}}duq(dz)dv &= \gamma(X^x_{s-})\int_G d_n(X^x_{s-})q(dz)\\ &= \gamma(X^x_{s-})q_n(X^x_{s-},G) \ge n;\end{aligned}$$

(b) $U_n \le \gamma(X^x_{\tau_n-})$ a.s. by construction;

(c) conditionnally to $\mathcal{H}_{\tau_n-}$, $Z_n \sim q_n(X^x_{\tau_n-},dz)/q_n(X^x_{\tau_n-},G)$. Indeed, the triple (U_n,Z_n,V_n) classically follows, conditionnally to $\mathcal{H}_{\tau_n-}$, the distribution

$$\frac{1}{\gamma(X^x_{\tau_n-})q_n(X^x_{\tau_n-},G)}\mathbf{1}_{\{u\le\gamma(X^x_{\tau_n-}),v\le d_n(X^x_{\tau_n-},z)\}}duq(dz)dv,$$

and it then suffices to integrate over $u \in [0,\infty)$ and $v \in [0,1]$ and to use that $d_n(y,z)q(dz) = q_n(y,dz)$.

Step 2. By construction and due to Step 1-(b),

$$X^x_{\tau_n} = X^x_{\tau_n -} + h(X^x_{\tau_n -}, Z_n)\mathbf{1}_{\{U_n \le \gamma(X^x_{\tau_n -})\}} = X^x_{\tau_n -} + h(X^x_{\tau_n -}, Z_n).$$

Hence conditionnally to $\mathcal{H}_{\tau_n -}$, the law of $X^x_{\tau_n}$ is $g_n(\omega, dy) := \mu_n(X^x_{\tau_n -}, dy - X^x_{\tau_n -})/\mu_n(X^x_{\tau_n -}, \mathbb{R})$. Indeed, for any bounded measurable function $\phi : \mathbb{R} \mapsto \mathbb{R}$, using Step 1-(c) and that $\mu_n(y, A) = \gamma(y)\int_G \mathbf{1}_A(h(y,z))q_n(y, dz)$,

$$\begin{aligned} E\left[\phi(X^x_{\tau_n})|\mathcal{H}_{\tau_n -}\right] &= \int_G \phi[X^x_{\tau_n -} + h(X^x_{\tau_n -}, z)]\frac{q_n(X^x_{\tau_n -}, dz)}{q_n(X^x_{\tau_n -}, G)} \\ &= \int_{\mathbb{R}} \phi(X^x_{\tau_n -} + y)\frac{\mu_n(X^x_{\tau_n -}, dy)}{\mu_n(X^x_{\tau_n -}, \mathbb{R})} = \int_{\mathbb{R}} \phi(y) g_n(dy). \end{aligned}$$

Due to assumption $(H_{k,p,\theta})$, we know that for some constant C, a.s.,

$$||g_n||_{\bar{W}^{k,1}(\mathbb{R})} = \frac{1}{\mu_n(X^x_{\tau_n -}, \mathbb{R})}||\mu_n(X^x_{\tau_n -}, .)||_{\bar{W}^{k,1}(\mathbb{R})} \le C(1 + |X^x_{\tau_n -}|^p)e^{\theta n}. \tag{6}$$

Step 3. We now use the strong Markov property. For $t \ge 0$ and $n \ge 1$, for $\phi : \mathbb{R} \mapsto \mathbb{R}$, using Notation 2.2, since $\{t \ge \tau_n\} \in \mathcal{H}_{\tau_n -}$,

$$E[\phi(X^x_t)] = E[\phi(X^x_t)\mathbf{1}_{\{t < \tau_n\}}] + E\left[\mathbf{1}_{\{t \ge \tau_n\}}\int_{\mathbb{R}} \phi(y) p(t - \tau_n, g_n, dy)\right]. \tag{7}$$

But from Proposition 2.3 and (6), there exists a constant $C_{t,k}$ such that a.s.

$$\mathbf{1}_{\{t \ge \tau_n\}}||p(t - \tau_n, g_n, .)||_{\bar{W}^{k,1}(\mathbb{R})} \quad \le \quad C_{t,k}\mathbf{1}_{\{t \ge \tau_n\}}\sup_{[0,t]}(1 + |X^x_s|^p)e^{\theta n}. \tag{8}$$

Step 4. Consider finally the application $\psi(\xi, y) = e^{i\xi y}$. Then the Fourier transform of the law $p(t, x, dy)$ of X^x_t is given by $\hat{p}_{t,x}(\xi) := E[\psi(\xi, X^x_t)]$. We apply (7) with the choice $\phi(y) = \psi^{(k)}(\xi, y) = (i\xi)^k\psi(\xi, y)$. We get, for $n \ge 1$, $\xi \in \mathbb{R}$,

$$|\xi|^k|\hat{p}_{t,x}(\xi)| \le |\xi|^k P[\tau_n > t] + E\left[\mathbf{1}_{\{t \ge \tau_n\}}\left|\int_{\mathbb{R}} \psi^{(k)}(\xi, y) p(t - \tau_n, g_n, dy)\right|\right]. \tag{9}$$

But on $\{t \ge \tau_n\}$, an integration by parts and then (8) leads us to

$$\begin{aligned} &\left|\int_{\mathbb{R}} \psi^{(k)}(\xi, y) p(t - \tau_n, g_n, dy)\right| = \left|\int_{\mathbb{R}} \psi(\xi, y) p^{(k)}(t - \tau_n, g_n, dy)\right| \\ &\le ||\psi(\xi, .)||_\infty ||p(t - \tau_n, g_n, .)||_{\bar{W}^{k,1}(\mathbb{R})} \le C_{t,k}e^{\theta n}\sup_{[0,t]}(1 + |X^x_s|^p). \end{aligned}$$

Hence (9) becomes, using Step 1-(a) and (5), for all $t \ge 0$, all $\xi \in \mathbb{R}$, all $n \ge 1$,

$$|\xi|^k|\hat{p}_{t,x}(\xi)| \le |\xi|^k e^{-nt} + C_{t,k}(1 + C_t)e^{\theta n}.$$

We now fix ξ, and choose $n = n(\xi)$ the integer part of $\frac{k}{\theta + t}\log|\xi|$. We obtain, for some constant A_t, for all $\xi \in \mathbb{R}$,

$$|\xi|^k|\hat{p}_{t,x}(\xi)| \le (e^t + C_{t,k}(1 + C_t))|\xi|^{k\theta/(\theta + t)} =: A_t|\xi|^{k\theta/(\theta + t)}.$$

Since on the other hand $|\hat{p}_{t,x}(\xi)|$ is clearly bounded by 1, we deduce that for all $t \geq 0$, all $\xi \in \mathbb{R}$,

$$|\hat{p}_{t,x}(\xi)| \leq 1 \wedge A_t |\xi|^{-kt/(\theta+t)}. \tag{10}$$

Let finally $l \geq 0$ such that $l < \frac{kt}{\theta+t} - 1$, which is possible if $t > \frac{\theta}{k-1}$. Then (10) ensures us that $|\xi|^l |\hat{p}_{t,x}(\xi)|$ belongs to $L^1(\mathbb{R}, d\xi)$, which classically implies that $p(t,x,dy)$ has a density of class $C_b^l(\mathbb{R})$. □

4 Propagation of smoothness

It remains to prove Proposition 2.3. It is very technical, but the principle is quite simple: we study the Fokker-Planck integro-partial-differential equation associated with our process, and show that if the initial condition is smooth, so is the solution for all times, in the sense of $\bar{W}^{k,1}(\mathbb{R})$ spaces.

In the whole section, K is a constant whose value may change from line to line, and which depends only on k and on the bounds of the coefficients assumed in assumptions $(A_{k+1,p})$ and (S).

For functions $f(y) : \mathbb{R} \mapsto \mathbb{R}$, $g(t,y) : [0,\infty) \times \mathbb{R} \mapsto \mathbb{R}$, $h(y,z) : \mathbb{R} \times G \mapsto \mathbb{R}$, we will always denote by $f^{(l)}$, $g^{(l)}$, and $h^{(l)}$ the l-th derivative of f, g, h with respect to the variable y.

A map $(t,y) \mapsto f(t,y)$ is of class $C_b^{1,k}([0,T] \times \mathbb{R})$ if the derivatives $f^{(l)}(t,y)$ and $\partial_t f^{(l)}(t,y)$ exist, are continuous and bounded, for all $l \in \{0,...,k\}$.

We consider for $i \geq 1$ the approximation $\mathcal{L}^i$ of $\mathcal{L}$, recall (1), defined for all bounded and measurable $\phi : \mathbb{R} \mapsto \mathbb{R}$ by

$$\mathcal{L}^i \phi(y) = i \left[\phi\left(y + \frac{b(y)}{i} \right) - \phi(y) \right] + \gamma(y) \int_{G_i} q(dz) \left[\phi(y + h(y,z)) - \phi(y) \right].$$

Here, $(G_i)_{i \geq 1}$ is an increasing sequence of subsets of G such that $\cup_{i \geq 1} G_i = G$ and such that for each $i \geq 1$, $q(G_i) < \infty$.

Lemma 4.1. *Assume (I) and $(A_{1,1})$.*

(i) For any $i \geq 1$, any probability measure $f_i(dy)$ on $\mathbb{R}$, there exists a unique family of (possibly signed) bounded measures $(f_i(t,dy))_{t \geq 0}$ on $\mathbb{R}$ such that for all $T > 0$, $\sup_{[0,T]} \int_{\mathbb{R}} |f_i(t)|(dy) < \infty$, and for all bounded measurable $\phi : \mathbb{R} \mapsto \mathbb{R}$,

$$\int_{\mathbb{R}} \phi(y) f_i(t,dy) = \int_{\mathbb{R}} \phi(y) f_i(dy) + \int_0^t ds \int_{\mathbb{R}} \mathcal{L}^i \phi(y) f_i(s,dy). \tag{11}$$

Furthermore, $f_i(t)$ is a probability measure for all $t \geq 0$.

(ii) Assume now that $f_i(dy)$ goes weakly to some probability measure $f(dy)$ as i tends to infinity. Then for all $t \geq 0$, $f_i(t,dy)$ tends weakly to $p(t,f,dy)$ as i tends to infinity, where we use Notation 2.2.

Proof. Let us first prove the uniqueness part. We observe that for ϕ bounded and measurable, $\mathcal{L}^i\phi$ is also measurable and satisfies $||\mathcal{L}^i\phi||_\infty \le C_i||\phi||_\infty$, where $C_i := 2i + 2||\gamma||_\infty q(G_i)$. Hence for two solutions $f_i(t,dy)$ and $\tilde{f}_i(t,dy)$ to (11), an immediate computation leads us to

$$||f_i(t) - \tilde{f}_i(t)||_{TV} \le C_i \int_0^t ds ||f_i(s) - \tilde{f}_i(s)||_{TV},$$

since the total variation norm satisfies $||\nu||_{TV} := \sup_{||\phi||_\infty \le 1} | \int_{\mathbb{R}} \phi(y)\nu(dy)|$. The uniqueness of the solution to (11) follows from the Gronwall Lemma.

Let us consider $X_0 \sim f$ independent of N, and $(X_t^x)_{t\ge0,x\in\mathbb{R}}$ the solution to (2), associated to the Poisson measure N. Recall that $p(t,f,dy) = \mathcal{L}(X_t^{X_0})(dy)$.

We introduce another Poisson measure $M^i(ds)$ on $[0,\infty)$ with intensity measure ids, independent of N, and $X_0^i \sim f_i$, independent of (M^i, N). Let $(X_t^i)_{t\ge0}$ be the (clearly unique) solution to

$$X_t^i = X_0^i + \int_0^t \frac{b(X_{s-}^i)}{i} M^i(ds) + \int_0^t\int_0^\infty\int_{G_i} h(X_{s-}^i, z)\mathbf{1}_{\{u\le\gamma(X_{s-}^i)\}} N(ds,du,dz).$$

Then one immediately checks that $f_i(t,dy) = \mathcal{L}(X_t^i)(dy)$ solves (11). This shows the existence of a solution to (11), and that this solution consists of a family of probability measures. Finally, we use the Skorokhod representation Theorem: we build $X_0^i \sim f_i$ in such a way that X_0^i tends a.s. to X_0. Then one easily proves that $\sup_{[0,t]} |X_s^i - X_s^{X_0}|$ tends to 0 in probability, for all $t \ge 0$, using repeatedly $(A_{1,1})$. We refer to [8, Step 1 page 653] for a similar proof. This of course implies that for all $t \ge 0$, $f_i(t,dy) = \mathcal{L}(X_t^i)$ tends weakly to $p(t,f,dy) = \mathcal{L}(X_t^{X_0})$. □

We now introduce some inverse functions in order to write (11) in a strong form.

Lemma 4.2. *Assume (S) and $(A_{k+1,p})$ for some $p \ge k+1 \ge 2$.*

(i) For each fixed $z \in G$, the map $y \mapsto y + h(y,z)$ is an increasing C^{k+1}-diffeomorphism from $\mathbb{R}$ into itself. We thus may introduce its inverse function $\tau(y,z) : \mathbb{R}\times G \mapsto \mathbb{R}$ defined by $\tau(y,z) + h(\tau(y,z),z) = y$. For each $z \in G$, $y \mapsto \tau(y,z)$ is of class $C^{k+1}(\mathbb{R})$. One may find a function $\alpha \in L^1(G,q)$ such that all the following points hold:

there exists $K > 0$ such that

$$|\tau(y,z) - y| + |\tau'(y,z) - 1| + \tfrac{|\tau'(y,z)-1|}{\tau'(y,z)} \le \alpha(z), \tag{12}$$

$$0 < \tau'(y,z) \le K; \tag{13}$$

for all $l \in \{0,...,k\}$, there exist some functions $\alpha_{l,r} : \mathbb{R}\times G \mapsto \mathbb{R}$ with

$$\left(1 + \frac{1}{\tau'(y,z)}\right)\sum_{r=0}^{l} |\alpha_{l,r}(y,z)| \le \alpha(z) \tag{14}$$

such that for all $\phi \in C^l(\mathbb{R})$,

$$\left[\phi(\tau(y,z))\tau'(y,z)\right]^{(l)} = \phi^{(l)}(\tau(y,z)) + \sum_{r=0}^{l} \alpha_{l,r}(y,z)\phi^{(r)}(\tau(y,z)). \tag{15}$$

(ii) Let $i_0 := 2||b'||_\infty$. For all $i \geq i_0$, the map $y \mapsto y + b(y)/i$ is an increasing C^{k+1}-diffeomorphism from $\mathbb{R}$ into itself. Let its inverse $\tau_i : \mathbb{R} \mapsto \mathbb{R}$ be defined by $\tau_i(y) + b(\tau_i(y))/i = y$. Then $\tau_i \in C^{k+1}(\mathbb{R})$. There exists $c > 0, K > 0$ such that

$$|\tau_i(y) - y| \leq K/i, \quad |\tau_i'(y) - 1| \leq K/i, \quad c < \tau_i'(y) \leq K. \tag{16}$$

For all $l \in \{0, ..., k\}$, there exist $\beta^i_{l,r} : \mathbb{R} \mapsto \mathbb{R}$ with

$$\sum_{r=0}^{l} i|\beta^i_{l,r}(y)| \leq K \tag{17}$$

such that for all $\phi \in C^l(\mathbb{R})$,

$$\big[\phi(\tau_i(y))\tau_i'(y)\big]^{(l)} = \phi^{(l)}(\tau_i(y)) + \sum_{r=0}^{l} \beta^i_{l,r}(y)\phi^{(r)}(\tau_i(y)). \tag{18}$$

(iii) For all $i \geq i_0$, all bounded measurable $\phi : \mathbb{R} \mapsto \mathbb{R}$ and all $g \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} g(y)\mathcal{L}^i\phi(y)dy = \int_{\mathbb{R}} \phi(y)\mathcal{L}^{i*}g(y)dy,$$

where

$$\begin{aligned}\mathcal{L}^{i*}g(y) \quad &= \quad i\big[g(\tau_i(y))\tau_i'(y) - g(y)\big] \\ &\quad + \int_{G_i} q(dz)\big[\gamma(\tau(y,z))g(\tau(y,z))\tau'(y,z) - \gamma(y)g(y)\big].\end{aligned} \tag{19}$$

Proof. We start with

Point (i). The fact that for each $z \in G$, $y + h(y,z)$ is an increasing C^{k+1}-diffeomorphism follows immediately from $(A_{k+1,p})$ and (S). Thus its inverse function $y \mapsto \tau(y,z)$ is of class C^{k+1}. Next, $\tau'(y,z) = 1/(1 + h'(\tau(y,z),z))$, and thus is positive and bounded by $1/c_0$ due to (S). This shows (13). Of course, $\sup_y |\tau(y,z) - y| = \sup_y |y + h(y,z) - y| \leq \eta(z) \in L^1(G,q)$ due to $(A_{k+1,p})$. Next, $|\tau'(y,z) - 1| = |h'(\tau(y,z),z)|/(1 + h'(\tau(y,z),z)) \leq \eta(z)/c_0 \in L^1(G,q)$, due to (S) and $(A_{k+1,p})$. Finally, $|\tau'(y,z) - 1|/\tau'(y,z) = |h'(\tau(y,z),z)| \leq \eta(z) \in L^1(G,q)$, due to $(A_{k+1,p})$. Thus (12) holds.

We next show that for $l = 1, ..., k+1$,

$$|\tau^{(l)}(y,z)| \leq K(\eta(z) + \eta^{l-1}(z)). \tag{20}$$

When $l = 1$, it suffices to use that $|\tau'(y,z) - 1| \leq K\eta(z)$, which was already proved. For $l \geq 2$, we use (30) (with $f(y) = y + h(y,z)$), the fact that $f'(y) = 1 + h'(y,z) \geq c_0$ due to (S), and that for all $n = 2, ..., k+1$, $f^{(n)}(y) = h^{(n)}(y,z) \leq \eta(z)$ (due to $(A_{k+1,p})$): this yields, setting $I_{l,r} := \{q \in \mathbb{N},\ i_1, ..., i_q \in \{2, ..., l\};\ i_1 + ... + i_q = r - 1\}$,

$$\begin{aligned}|\tau^{(l)}(y,z)| \quad &\leq \quad K \sum_{r=l+1}^{2l-1} \sum_{I_{l,r}} \prod_{j=1}^{q} |h^{(i_j)}(\tau(y,z),z)| \leq K \sum_{r=l+1}^{2l-1} \sum_{I_{l,r}} \eta^q(z) \\ &\leq \quad K \sum_{q=1}^{l-1} \eta^q(z) \leq K(\eta(z) + \eta^{l-1}(z)).\end{aligned}$$

We now consider $\phi \in C^k(\mathbb{R})$. Due to (29), for $n = 1, ..., k$,

$$[\phi(\tau(y,z))]^{(n)} = [\tau'(y,z)]^n \phi^{(n)}(\tau(y,z)) + \sum_{r=1}^{n-1} \delta_{n,r}(y,z)\phi^{(r)}(\tau(y,z)) \tag{21}$$

with $\delta_{n,r}(y,z) = \sum_{J_{n,r}} a^n_{i_1,...,i_r} \prod_1^r \tau^{(i_j)}(y,z)$, where $J_{n,r} := \{i_1 \geq 1, ..., i_r \geq 1,\ i_1 + ... + i_r = n\}$. Using (20), we get, for $r = 1, ..., n-1$,

$$\begin{aligned} |\delta_{n,r}(y,z)| &\leq K \sum_{J_{n,r}} \prod_1^r (\eta(z) + \eta^{i_j - 1}(z)) \leq K \sum_{m=1}^{n-1} \eta^m(z) \\ &\leq K(\eta(z) + \eta^{n-1}(z)). \end{aligned} \tag{22}$$

To obtain the second inequality, we used that since $i_1 + ... + i_r = n > r$, there is at least one j with $i_j \geq 2$, and that $\sum_{j=1}^r (i_j - 1) \vee 1 = \sum_{j=1}^r (i_j - 1) + \sum_{j=1}^r \mathbf{1}_{\{i_j = 1\}} \leq n - r + r - 1 = n - 1$.

Applying now the Leibniz formula and then (21), we get, for $l = 0, ..., k$,

$$\begin{aligned} [\phi(\tau)\tau']^{(l)} &= \tau'[\phi(\tau)]^{(l)} + \sum_{n=0}^{l-1} \binom{l}{n} \tau^{(l+1-n)} [\phi(\tau)]^{(n)} \\ &= (\tau')^{l+1}\phi^{(l)}(\tau) + \sum_{r=0}^{l-1} \phi^{(r)}(\tau)\alpha_{l,r} = \phi^{(l)}(\tau) + \sum_{r=0}^{l} \phi^{(r)}(\tau)\alpha_{l,r}, \end{aligned}$$

where $\alpha_{l,0} = \tau^{(l+1)}$, $\alpha_{l,l} = (\tau')^{l+1} - 1$, and for $r = 1, ..., l-1$,

$$\alpha_{l,r} = \binom{l}{r} \tau^{(l+1-r)} (\tau')^r + \sum_{j=r+1}^{l} \binom{l}{j} \tau^{(l+1-j)} \delta_{j,r}.$$

It only remains to prove (14). First, since τ' is bounded, we deduce that $|\alpha_{l,l}(y,z)| \leq K|\tau'(y,z) - 1| \leq K\eta(z)$. Next, using (20), (22) and that τ' is bounded, we get, for $l = 1, ..., k$, (with the convention $\sum_1^0 = 0$),

$$\begin{aligned} \sum_{r=0}^{l} |\alpha_{l,r}(y,z)| &\leq K\eta(z) + K(\eta(z) + \eta^l(z)) + K \sum_{r=1}^{l-1} (\eta(z) + \eta^{l-r}(z)) \\ &\quad + K \sum_{r=1}^{l-1} \sum_{j=r+1}^{l} (\eta(z) + \eta^{l-j}(z))(\eta(z) + \eta^{j-1}(z)) \\ &\leq K(\eta(z) + \eta^l(z)) \leq K(\eta(z) + \eta^k(z)) \end{aligned}$$

Finally, $(1 + 1/\tau'(y,z)) = (1 + 1 + h'(\tau(y,z),z)) \leq 2 + \eta(z)$ by $(A_{k+1,p})$. We conclude that for $l = 1, ..., k$,

$$\left(1 + \frac{1}{\tau'(y,z)}\right) \sum_{r=0}^{l} |\alpha_{l,r}(y,z)| \leq K(1 + \eta(z))(\eta(z) + \eta^k(z)) =: \alpha(z),$$

and $\alpha \in L^1(G,q)$, since by assumption, $\eta \in L^1 \cap L^p(G,q)$ with $p \geq k + 1 \geq 2$.

Point (ii). The proof is the similar (but simpler) to that of Point (i). We observe that for $i \geq i_0$, $(y + b(y)/i)' \geq 1/2$, so that under $(A_{k+1,p})$, $y + b(y)/i$ is clearly a C^{k+1}-diffeomorphism. Next, (16) is easily obtained, and we prove as in Point (i) that

$$|\tau_i^{(l)}(z)| \leq K(1/i + (1/i)^{l-1}) \leq K/i, \quad l = 2, ..., k+1,$$

using that for all $n = 2, ..., k+1$, $(y + b(y)/i)^{(n)} \leq K/i$ thanks to $(A_{k+1,p})$. Then (17)-(18) are obtained as (14)-(15).

Point (iii). Let thus ϕ and g as in the statement. Then

$$\begin{aligned}
\int_{\mathbb{R}} g(y)\mathcal{L}^i\phi(y)dy &= i \int_{\mathbb{R}} \phi\left(y + b(y)/i\right) g(y)dy - i \int_{\mathbb{R}} \phi\left(y\right) g(y)dy \\
&+ \int_{G_i} q(dz) \int_{\mathbb{R}} \gamma(y)\phi(y + h[y,z])g(y)dy - \int_{G_i} q(dz) \int_{\mathbb{R}} \gamma(y)\phi(y)g(y)dy \\
&= i \int_{\mathbb{R}} \phi\left(y\right) g(\tau_i(y))\tau_i'(y)dy - i \int_{\mathbb{R}} \phi\left(y\right) g(y)dy \\
&+ \int_{G_i} q(dz) \int_{\mathbb{R}} \gamma(\tau(y,z))\phi(y)g(\tau(y,z))\tau'(y,z)dy \\
&- \int_{G_i} q(dz) \int_{\mathbb{R}} \gamma(y)\phi(y)g(y)dy = \int_{\mathbb{R}} \phi(y)\mathcal{L}^{i*}g(y)dy,
\end{aligned}$$

where we used the substitution $y \mapsto \tau_i(y)$ (resp. $y \mapsto \tau(y,z)$) in the first (resp. third) integral. □

The following technical lemma shows that when starting with a smooth initial condition, the solution of (11) remains smooth for all times (not uniformly in i). This will enable us to handle rigorous computations.

Lemma 4.3. *Assume (I), $(A_{k+1,p})$ for some $p \geq k+1 \geq 2$, and (S). Let $i \geq i_0$ be fixed. Consider a probability measure $f_i(dy)$ admitting a density $f_i(y)$ of class $C_b^k(\mathbb{R})$, and the associated solution $f_i(t,dy)$ to (11). Then for all $t \geq 0$, $f_i(t,dy)$ has a density $f_i(t,y)$, and $(t,y) \mapsto f_i(t,y)$ belongs to $C_b^{1,k}([0,T] \times \mathbb{R})$ for all $T \geq 0$. For all $t \geq 0$, all $y \in \mathbb{R}$, all $l = 0, ..., k$,*

$$\begin{aligned}
\partial_t f_i^{(l)}(t,y) &= \left[\mathcal{L}^{i*} f_i(t,y)\right]^{(l)} \\
&= i\left[f_i(t,\tau_i(y))\tau_i'(y) - f_i(t,y)\right]^{(l)} \\
&\quad + \int_{G_i} q(dz)\left[\gamma(\tau(y,z))f_i(t,\tau(y,z))\tau'(y,z) - \gamma(y)f_i(t,y)\right]^{(l)}.
\end{aligned} \tag{23}$$

Proof. We will prove, using a Picard iteration, that (23) (with $l = 0$) admits a solution, which also solves (11), which is regular, and of which the derivatives solve (23). We omit the fixed subscript $i \geq i_0$ in this part of the proof, and the initial probability measure $f(dy) = f(y)dy$ with $f \in C^k(\mathbb{R})$ is fixed.

Step 1. Consider the function $f^0(t,y) := f(y)$, and define, for $n \geq 0$,

$$f^{n+1}(t,y) = f(y) + \int_0^t \mathcal{L}^{i*} f^n(s,y)ds. \tag{24}$$

Then one easily checks by induction (on n), using Lemma 4.2, $(A_{k+1,p})$ and the fact that $q(G_i) < \infty$, that for all $n \geq 0$, $f^n(t,y)$ is of class $C_b^{0,k}([0,\infty) \times \mathbb{R})$, and that for all $l \in \{0,...,k\}$,

$$(f^{n+1})^{(l)}(t,y) = f^{(l)}(y) + \int_0^t [\mathcal{L}^{i*} f^n]^{(l)}(s,y)ds. \tag{25}$$

Step 2. We now show that there exists $C_{k,i} > 0$ such that for $n \geq 1$, $t \geq 0$,

$$\sum_{l=0}^{k} ||(\delta^{n+1})^{(l)}(t,.)||_\infty \leq C_{k,i} \int_0^t ds \sum_{l=0}^{k} ||(\delta^n)^{(l)}(s,.)||_\infty,$$

where $\delta^{n+1}(t,y) = f^{n+1}(t,y) - f^n(t,y)$. Due to (25), for $l = 0,...,k$,

$$(\delta^{n+1})^{(l)}(t,y) = \int_0^t i\big[\delta^n(s,\tau_i(y))\tau_i'(y) - \delta^n(s,y)\big]^{(l)} ds$$
$$+ \int_0^t ds \int_{G_i} q(dz)\big[\gamma(\tau(y,z))\delta^n(s,\tau(y,z))\tau'(y,z) - \gamma(y)\delta^n(s,y)\big]^{(l)}.$$

We now use (18) (with $\phi = \delta^n(s,.)$) and (15) (with $\phi = \gamma\delta^n(s,.)$), and we easily obtain, since $q(G_i) < \infty$, for some constant $C_{k,i}$, for all $y \in \mathbb{R}$,

$$\begin{aligned} |(\delta^{n+1})^{(l)}(t,y)| &\leq C_{k,i} \int_0^t ds \sum_{r=0}^{l} \Big(||(\delta^n)^{(r)}(s)||_\infty + ||(\gamma\delta^n)^{(r)}(s)||_\infty \Big) \\ &\leq C_{k,i} \int_0^t ds \sum_{r=0}^{l} ||(\delta^n)^{(r)}(s)||_\infty, \end{aligned}$$

the last inequality holding since $l \leq k$ and $\gamma \in C_b^k(\mathbb{R})$. Taking now the supremum over $y \in \mathbb{R}$ and suming for $l = 0,...,k$, we get the desired inequality.

Step 3. We classically deduce from Step 2 that the sequence f^n tends to a function $f(t,y) \in C_b^{0,k}([0,T] \times \mathbb{R})$ (for all $T > 0$), and that for $l = 0,...,k$,

$$f^{(l)}(t,y) = f^{(l)}(y) + \int_0^t [\mathcal{L}^{i*} f]^{(l)}(s,y)ds. \tag{26}$$

But one can check, using arguments as in Step 1, that since $f(t,y) \in C_b^{0,k}([0,T] \times \mathbb{R})$, so does $[\mathcal{L}^{i*} f](t,y)$. Hence (26) can be differentiated with respect to time, we obtain (23), and thus also that $f(t,y) \in C_b^{1,k}([0,T] \times \mathbb{R})$.

Step 4. It only remains to show that $f(t,y)dy$ is indeed the solution of (11) defined in Lemma 4.1-(i). First, using (24) and rough estimates, we have $||f^{n+1}(t)||_{L^1} \leq ||f||_{L^1} + C_i \int_0^t ds ||f^n(s)||_{L^1}$, where $C_i = 2i + 2||\gamma||_\infty q(G_i)$. This classically ensures that $||f(t)||_{L^1} \leq \limsup_n ||f^n(t)||_{L^1} \leq ||f||_{L^1} e^{C_i t}$. Thus $\sup_{[0,T]} \int_{\mathbb{R}} |f(t,y)|dy < \infty$ for all $T > 0$.

Next, we multiply (26) (with $l = 0$) by $\phi(y)$, for a bounded measurable $\phi : \mathbb{R} \mapsto \mathbb{R}$, we integrate over $y \in \mathbb{R}$, and we use the duality proved in Lemma 4.2-(iii). This yields (11). $\square$

The central part of this section consists of the following result.

Lemma 4.4. *Assume (I), (S) and $(A_{k+1,p})$ for some $p \geq k+1 \geq 2$. For $i \geq i_0$, let $f_i(dy) \in \bar{W}^{k,1}(\mathbb{R})$ be a probability measure with a density $f_i(y) \in C^k(\mathbb{R})$, and consider the unique solution $f_i(t, dy)$ to (11). There exists a constant C_k (not depending on $i \geq i_0$) such that for all $t \geq 0$,*

$$||f_i(t,.)||_{\bar{W}^{k,1}(\mathbb{R})} \leq ||f_i||_{\bar{W}^{k,1}(\mathbb{R})} e^{C_k t}.$$

Proof. We know from Lemma 4.3 that $f_i(t,y)$ is of class $C_b^{1,k}([0,T] \times \mathbb{R})$, and that (23) holds for $l = 0, ..., k$.

Since for each $l = 0, ..., k$, each $y \in \mathbb{R}$, $t \mapsto f_i^{(l)}(t,y)$ is of class C^1, we classically deduce that $|f_i^{(l)}(t,y)| = |f_i^{(l)}(y)| + \int_0^t sg(f_i^{(l)}(s,y)) \partial_t f_i^{(l)}(s,y) ds$, where $sg(u) = \mathbf{1}_{(0,\infty)}(u) - \mathbf{1}_{(-\infty,0)}(u)$. Using thus (23) and integrating over $y \in \mathbb{R}$, we get

$$||f_i^{(l)}(t,.)||_{L^1} = ||f_i^{(l)}||_{L^1} + \int_0^t (A_i^l(s) + B_i^l(s)) ds, \tag{27}$$

for $l = 1, ..., k$, where, setting $\gamma f_i(t,y) = \gamma(y) f_i(t,y)$ for simplicity,

$$\begin{aligned} A_i^l(t) &= \int_{\mathbb{R}} dy \; i \big[f_i(t, \tau_i(y)) \tau_i'(y) - f_i(t,y) \big]^{(l)} sg(f_i^{(l)}(t,y)) \\ B_i^l(t) &= \int_{G_i} q(dz) \int_{\mathbb{R}} dy \; i \big[\gamma f_i(t, \tau(y,z)) \tau'(y,z) - \gamma f_i(t,y) \big]^{(l)} sg(f_i^{(l)}(t,y)). \end{aligned}$$

Using (18) (with $\phi = f_i(t,.)$) and then (17), we obtain

$$\begin{aligned} A_i^l(t) &\leq \int_{\mathbb{R}} dy \; i \big[f_i^{(l)}(t, \tau_i(y)) - f_i^{(l)}(t,y) \big] sg(f_i^{(l)}(t,y)) \\ &\quad + \int_{\mathbb{R}} dy \; \sum_{r=0}^{l} i |\beta_{l,r}^i(y)| . |f_i^{(r)}(t, \tau_i(y))| \\ &\leq \int_{\mathbb{R}} dy \; i \big[|f_i^{(l)}(t, \tau_i(y))| - |f_i^{(l)}(t,y)| \big] + K \int_{\mathbb{R}} dy \; \sum_{r=0}^{l} |f_i^{(r)}(t, \tau_i(y))| \\ &=: A_i^{l,1}(t) + A_i^{l,2}(t). \end{aligned}$$

First,

$$\begin{aligned} A_i^{l,1}(t) &\leq i \int_{\mathbb{R}} dy |f_i^{(l)}(t, \tau_i(y))| \tau_i'(y) - i \int_{\mathbb{R}} dy |f_i^{(l)}(t,y)| \\ &\quad + \int_{\mathbb{R}} dy |f_i^{(l)}(t, \tau_i(y))| \times i |\tau_i'(y) - 1|. \end{aligned}$$

Using the substitution $\tau_i(y) \mapsto y$ in the first integral, we deduce that the first and second integral are equal. Next, due to (16), we get

$$A_i^{l,1}(t) \leq 0 + K \int_{\mathbb{R}} dy |f_i^{(l)}(t, \tau_i(y))| \leq K ||f_i^{(l)}(t,.)||_{L^1}.$$

To obtain the last inequality, we used again the substitution $\tau_i(y) \mapsto y$ and the fact that τ_i' is bounded below (uniformly in $i \geq i_0$, see (16)). The same argument shows that

$$A_i^{l,2}(t) \leq K \sum_{r=0}^{l} ||f_i^{(r)}(t,.)||_{L^1}.$$

Using now (15) with $\phi = \gamma f_i(t,.)$, we get

$$\begin{aligned} B_i^l(t) &\leq \int_{G_i} q(dz) \int_{\mathbb{R}} dy \big[(\gamma f_i)^{(l)}(t,\tau(y,z)) - (\gamma f_i)^{(l)}(t,y)\big] sg(f_i^{(l)}(t,y)) \\ &+ \int_{G_i} q(dz) \int_{\mathbb{R}} dy \sum_{r=0}^{l} |\alpha_{l,r}(y,z)|.|(\gamma f_i(t,.))^{(r)}(\tau(y,z)) \end{aligned}$$

With the help of the Leibniz formula, we obtain

$$\begin{aligned} B_i^l(t) &\leq \int_{G_i} q(dz) \int_{\mathbb{R}} dy \big[\gamma f_i^{(l)}(t,\tau(y,z)) - \gamma f_i^{(l)}(t,y)\big] sg(f_i^{(l)}(t,y)) \\ &+ \int_{G_i} q(dz) \int_{\mathbb{R}} dy \sum_{r=0}^{l-1} \binom{l}{r} \big|\gamma^{(l-r)} f_i^{(r)}(t,\tau(y,z)) - \gamma^{(l-r)} f_i^{(r)}(t,y)\big| \\ &+ \int_{G_i} q(dz) \int_{\mathbb{R}} dy \sum_{r=0}^{l} |\alpha_{l,r}(y,z)|.|(\gamma f_i(t,.))^{(r)}(\tau(y,z))| \\ &=: B_i^{l,1}(t) + B_i^{l,2}(t) + B_i^{l,3}(t). \end{aligned}$$

First,

$$\begin{aligned} B_i^{l,1}(t) \leq &\int_{G_i} q(dz) \int_{\mathbb{R}} dy \big[(\gamma |f_i^{(l)}|)(t,\tau(y,z)).\tau'(y,z) - (\gamma |f_i^{(l)}|)(t,y)|\big] \\ &+ \int_{G_i} q(dz) \int_{\mathbb{R}} dy (\gamma |f_i^{(l)}|)(t,\tau(y,z)) \times |\tau'(y,z) - 1|. \end{aligned}$$

Using the substitution $\tau(y,z) \mapsto y$ is the first part of the first integral, we deduce that the first integral equals 0. Since γ is bounded, we get

$$\begin{aligned} B_i^{l,1}(t) &\leq 0 + K \int_{G_i} q(dz) \int_{\mathbb{R}} dy |f_i^{(l)}(t,\tau(y,z))| \times |\tau'(y,z) - 1| \\ &\leq K \int_{G_i} \alpha(z) q(dz) \int_{\mathbb{R}} dy |f_i^{(l)}(t,\tau(y,z))| \tau'(y,z) \end{aligned}$$

for some $\alpha \in L^1(G,q)$, where we used (12). But using again the subsitution $\tau(y,z) \mapsto y$, we find

$$B_i^{l,1}(t) \leq K \int_{G_i} \alpha(z) q(dz) \int_{\mathbb{R}} dy |f_i^{(l)}(t,y)| \leq K ||f_i^{(l)}(t,.)||_{L^1}.$$

Next, using (14), then the substitution $\tau(y,z) \mapsto y$ and that $\gamma \in C_b^k(\mathbb{R})$, we obtain, for some $\alpha \in L^1(G,q)$,

$$\begin{aligned} B_i^{l,3}(t) &\leq K \sum_{r=0}^{l} \int_{G_i} q(dz) \int_{\mathbb{R}} dy |(\gamma f_i(t,.))^{(r)}(\tau(y,z))| \tau'(y,z) \alpha(z) \\ &\leq \sum_{r=0}^{l} \left(\int_{G_i} \alpha(z) q(dz) \right) ||(\gamma f_i(t,.)^{(r)}||_{L^1} \leq K \sum_{r=0}^{l} ||f_i^{(r)}(t,.)||_{L^1} \end{aligned}$$

Finally, due to (12), there exists $\alpha \in L^1(G,q)$ such that $\sup_y |\tau(y,z)-y| \leq \alpha(z)$. Hence for any $\phi \in C^1(\mathbb{R})$,

$$\int_{G_i} q(dz)\int_{\mathbb{R}} dy|\phi(\tau(y,z))-\phi(y)| \leq \int_{G_i} q(dz)\int_{\mathbb{R}} dy \int_{y-\alpha(z)}^{y+\alpha(z)} du|\phi'(u)|$$
$$\leq 2\int_{G_i} \alpha(z)q(dz)||\phi'||_{L^1} \leq K||\phi'||_{L^1}.$$

As a consequence, using that $\gamma \in C_b^{k+1}$, we get, since $l \leq k$,

$$B_i^{l,2}(t) \leq K\sum_{r=0}^{l-1} ||(\gamma^{(l-r)}f_i^{(r)})'(t,.)||_{L^1} \leq K\sum_{r=0}^{l} ||f_i^{(r)}(t,.)||_{L^1}.$$

We finally have proved that for $l = 1,...,k$, for all $t \geq 0$,

$$||f_i^{(l)}(t,.)||_{L^1} \leq ||f_i^{(l)}||_{L^1} + K\sum_{r=0}^{l}\int_0^t ds||f_i^{(r)}(s,.)||_{L^1}.$$

Using that for all $t \geq 0$, $f_i(t,.)$ is a probability measure (so that $||f_i(t,.)||_{L^1} = 1$) and summing over $l = 0,...,k$, we immediately conclude that

$$||f_i(t,.)||_{\bar{W}^{k,1}(\mathbb{R})} \leq ||f_i||_{\bar{W}^{k,1}(\mathbb{R})} + K\int_0^t ds||f_i(s,.)||_{\bar{W}^{k,1}(\mathbb{R})}.$$

The Gronwall Lemma allows us to conclude the proof. □

We finally conclude the

Proof of Proposition 2.3. We thus assume (I), $(A_{k+1,p})$ for some $p \geq k+1 \geq 2$, and (S). Consider a probability measure $f \in \bar{W}^{k,1}(\mathbb{R})$, and a sequence of probability measures $f_i \in \bar{W}^{k,1}(\mathbb{R})$ with densities $f_i \in C_b^k(\mathbb{R})$, such that f_i goes weakly to f, and such that $\lim_i ||f_i||_{\bar{W}^{k,1}(\mathbb{R})} = ||f||_{\bar{W}^{k,1}(\mathbb{R})}$. Consider the unique solution $f_i(t,y)$ to (11). Then we deduce from Lemma 4.4 that for $t \geq 0$,

$$||f_i(t,.)||_{\bar{W}^{k,1}(\mathbb{R})} \leq ||f_i||_{\bar{W}^{k,1}(\mathbb{R})}e^{C_k t}. \tag{28}$$

On the other hand, Lemma 4.1 implies that for all $t \geq 0$, $f_i(t,dy)$ goes weakly to $p(t,f,dy)$ as i tends to infinity. Thus for any $\phi \in C_b^k(\mathbb{R})$, any $l \in \{0,...,k\}$, any $t \geq 0$,

$$\int_{\mathbb{R}} \phi^{(l)}(y)p(t,f,dy) = \lim_{i\to\infty}\int_{\mathbb{R}} \phi^{(l)}(y)f_i(t,dy).$$

We then immediately deduce from (28), recalling (3), that for any $t \geq 0$,

$$||p(t,f,.)||_{\bar{W}^{k,1}(\mathbb{R})} \leq ||f||_{\bar{W}^{k,1}(\mathbb{R})}e^{C_k t}.$$

The proof is finished. □

5 Appendix

We first gather some formulae about derivatives of composed and inverse functions from $\mathbb{R}$ into itself. Here $f^{(l)}$ stands for the l-th derivative of f.

Let us recall the Faa di Bruno formula. Let $l \geq 1$ be fixed. There exist some coefficients $a^{l,r}_{i_1,...,i_r} > 0$ such that for $\phi : \mathbb{R} \mapsto \mathbb{R}$ and $\tau : \mathbb{R} \mapsto \mathbb{R}$ of class $C^l(\mathbb{R})$,

$$[\phi(\tau)]^{(l)} = [\tau']^l \phi^{(l)}(\tau) + \sum_{r=1}^{l-1} \left(\sum_{i_1+...+i_r=l} a^{l,r}_{i_1,...,i_r} \prod_{j=1}^{r} \tau^{(i_j)} \right) \phi^{(r)}(\tau), \tag{29}$$

where the sum is taken over $i_1 \geq 1$, ..., $i_r \geq 1$ with $i_1 + ... + i_r = l$.

We carry on with another formula. For $l \geq 2$ fixed, there exist some coefficients $c^{l,r}_{i_1,..,i_q} \in \mathbb{R}$ such that for $f : \mathbb{R} \mapsto \mathbb{R}$ a C^l-diffeomorphism, and for τ its inverse function,

$$\tau^{(l)} = \sum_{r=l+1}^{2l-1} \frac{1}{(f'(\tau))^r} \sum_{i_1+...+i_q=r-1} c^{l,r}_{i_1,..,i_q} \prod_{j=1}^{q} f^{(i_j)}(\tau), \tag{30}$$

where the sum is taken over $q \in \mathbb{N}$, over $i_1, ..., i_q \in \{2, ..., l\}$ with $i_1 + ... + i_q = r - 1$. This (not optimal) formula can be checked by induction on $k \geq 2$.

We finally give the

Proof of Remark 2.5. In the whole proof, $y \in \mathbb{R}$ is fixed. We assume for example that $I(y) = (a(y), \infty)$, and we may suppose without loss of generality that $a(y) = 0$ (replacing if necessary $h[y, z]$ by $\bar{h}(y, z) := h[y, z + a(y)]$).

We introduce a family of C^∞ functions $\phi_n : \mathbb{R} \mapsto [0, 1]$, such that $\phi_n(z) = 0$ for $z \leq 1$ and $z \geq n + 3$, $\phi_n(z) = 1$ for $z \in [2, n + 2]$, and $\sup_n ||\phi_n^{(l)}||_\infty \leq C_l$ for all $l \in \mathbb{N}$. Then we set $q_n(y, dz) = \phi_n(\gamma(y).z)dz$, and we define μ_n by $\mu_n(y, A) = \gamma(y) \int_G \mathbf{1}_A(h(y, z))q_n(y, dz)$.

Clearly $0 \leq q_n(y, dz) \leq q(dz)$ so that $\mu_n(y, du) \leq \mu(y, du)$, and an immediate computation leads us to $\mu_n(y, \mathbb{R}) = \gamma(y)q_n(y, G) \in [n, n + 2]$. Thus points (i) and (ii) of assumption $(H_{k,p,\theta})$ are fulfilled.

Since $h'_z(y, z)$ does never vanish, $z \mapsto h(y, z)$ is either increasing or decreasing. We assume for example that we are in the latter case. We also necessarily have $\lim_{z\to\infty} h(y, z) = 0$, since $h(y, z) \in L^1((0, \infty), dz)$ (due to $(A_{1,1})$). As a conclusion, $z \mapsto h(y, z)$ is a decreasing C^{k+1}-diffeomorphism from $(0, \infty)$ into $(0, h(y, 0))$.

Let $\xi(y, .) : (0, h(y, 0)) \mapsto (0, \infty)$ be its inverse, that is $h(y, \xi(y, u)) = u$. Then by definition of μ_n and by using the subsitution $u = h(y, z)$, we get $\mu_n(y, du) = \mu_n(y, u)du$ with

$$\mu_n(y, u) = \gamma(y)\phi_n(\gamma(y)\xi(y, u))\xi'_u(y, u)\mathbf{1}_{\{u\in(0,h(y,0))\}}. \tag{31}$$

Since the properties of ϕ_n ensure us that $\mu_n(y, u) = 0$ for

$$u \notin (h(y, (n + 2)/\gamma(y)), h(y, 1/\gamma(y))),$$

it suffices to study the regularity of $\mu_n(y,.)$ on $(0,h(y,0))$. Since $\xi(y,.)$ is of class C^{k+1} and since ϕ_n is C^∞, we deduce that $\mu_n(y,.)$ is C^k on $(0,h(y,0))$ (and thus on $\mathbb{R}$).

Using (30) and that $h_z^{(l)}$ is uniformly bounded (for all $l=1,...,k+1$), we easily get, for $l=2,...,k+1$,

$$|\xi_u^{(l)}(y,u)| \leq K \sum_{r=l+1}^{2l-1} |h_z'(y,\xi(y,u))|^{-r}$$

Since h_z' is uniformly bounded, we get

$$|\xi_u^{(l)}(y,u)| \leq K|h_z'(y,\xi(y,u))|^{-2l+1}, \tag{32}$$

and the formula holds for $l=1,...,k+1$ (when $l=1$, it is obvious).

Applying now (29), using (32) and that γ is bounded, we get, for $l=1,...,k$,

$$|[\phi_n(\gamma(y)\xi(y,u))]_u^{(l)}| \quad \leq \quad K\sum_{r=1}^{l} |h_z'(y,\xi(y,u))|^{-2l+r}\phi_n^{(r)}(\gamma(y).\xi(y,u)).$$

We used here that for $i_1 \geq 1,...,i_r \geq 1$ with $i_1+...+i_r = l$, one has the inequality $\prod_{j=1}^r |\xi_u^{(i_j)}(y,u)| \leq K|h_z'(y,\xi(y,u))|^{\sum_1^r(-2i_j+1)} \leq K|h_z'(y,\xi(y,u))|^{-2l+r}$. Hence

$$|[\phi_n(\gamma(y)\xi(y,u))]_u^{(l)}| \quad \leq \quad K|h_z'(y,\xi(y,u))|^{-2l+1}\mathbf{1}_{\{\gamma(y)\xi(y,u)\leq n+3\}}, \tag{33}$$

since h_z' is uniformly bounded and $\phi_n(z)=0$ for $z\geq n+3$.

Applying finally the Leibniz formula, using (31), (32) and (33), we get, for $l=1,...,k$, for $u\in(0,h(y,0))$,

$$\begin{aligned}|(\mu_n)_u^{(l)}(y,u)| \quad &\leq \quad K\gamma(y)\sum_{r=0}^{l} |\xi_u^{(l+1-r)}(y,u)| \times |[\phi_n(\gamma(y)\xi(y,u))]_u^{(r)}| \\ &\leq \quad K\gamma(y)|h_z'(y,\xi(y,u))|^{-2l-1}\mathbf{1}_{\{\xi(y,u)\leq(n+3)/\gamma(y)\}}\end{aligned}$$

and the formula obviously holds for $l=0$. Finally, since h_z' is uniformly bounded, and performing the substitution $z=\xi(y,u)$, i.e. $u=h(y,z)$, we obtain, recalling that $\mu_n(y,\mathbb{R})\in[n,n+2]$,

$$\begin{aligned}&\frac{1}{\mu_n(y,\mathbb{R})}||\mu_n(y,.)||_{\bar{W}^{k,1}(\mathbb{R})} \\ &\quad\leq \frac{K\gamma(y)}{n}\int_{\mathbb{R}}\sum_{l=0}^{k}|h_z'(y,\xi(y,u))|^{-2l-1}\mathbf{1}_{\{0<\xi(y,u)\leq\frac{(n+3)}{\gamma(y)}\}}du \\ &\quad\leq \frac{K\gamma(y)}{n}\int_{\mathbb{R}}|h_z'(y,\xi(y,u))|^{-2k-1}\mathbf{1}_{\{0<\xi(y,u)\leq\frac{(n+3)}{\gamma(y)}\}}du \\ &\quad\leq \frac{K\gamma(y)}{n}\int_{\mathbb{R}}|h_z'(y,z)|^{-2k}\mathbf{1}_{\{0<z\leq\frac{(n+3)}{\gamma(y)}\}}dz \leq KCe^{3\theta}(1+|y|^p)e^{\theta n},\end{aligned}$$

where we finally used (4) (because $I_n(y)=[0,n/\gamma(y)]$ here). This proves that $(H_{k,p,\theta})$-(iii) holds. $\square$

References

[1] V. Bally, *Malliavin calculus for locally smooth laws and applications to diffusion processes with jumps*, preprint.

[2] K. Bichteler, J.B. Gravereaux, J. Jacod, *Malliavin calculus for processes with jumps*, Stochastics Monographs, 2. Gordon and Breach Science Publishers, New York, 1987. MR1008471

[3] K. Bichteler, J. Jacod, *Calcul de Malliavin pour les diffusions avec sauts: existence d'une densité dans le cas unidimensionel*, Séminaire de Probabilités XVII, L.N.M. 986, p 132-157, Springer, 1983. MR0770406

[4] J.M. Bismut, *Calcul des variations stochastiques et processus de sauts*, Z. Wahrsch. Verw. Gebiete 63, no. 2, 147-235, 1983. MR0701527

[5] L. Denis, *A criterion of density for solutions of Poisson-driven SDEs*, Probability Theory and Related Fields 118, no. 3, 406-426, 2000. MR1800539

[6] N. Fournier, *Existence and regularity study for two-dimensional Kac equation without cutoff by a probabilistic approach,* Ann. Appl. Probab. 10 no. 2, 434–462, 2000. MR1768239

[7] N. Fournier, *Jumping SDEs: absolute continuity using monotonicity*, Stochastic Process. Appl., 98 (2), pp 317-330, 2002. MR1887538

[8] N. Fournier, J.S. Giet, *On small particles in coagulation-fragmentation equations*, J. Statist. Phys., 111 (5/6), pp 1299-1329, 2003. MR1975930

[9] C. Graham, S. Méléard, *Existence and regularity of a solution to a Kac equation without cutoff using Malliavin Calculus*, Commun. Math. Phys., 205, pp 551-569, 1999. MR1711273

[10] A.M. Kulik, *Malliavin calculus for Levy processes with arbitrary Levy measures*, Theory Probab. Math. Statist. No. 72, 75–92, 2006. MR2168138

[11] A.M. Kulik, *Stochastic calculus of variations for general Levy processes and its applications to jump type SDEs with nondegenerated drift*, preprint, 2006.

[12] R. Léandre, *Régularité de processus de sauts dégénérés,* Ann. Inst. H. Poincaré Probab. Statist. 21, no. 2, 125–146, 1985. MR0798891

[13] I. Nourdin, T. Simon, *On the absolute continuity of Levy processes with drift,* Annals of Probability 34 (3), 1035-1051, 2006. MR2243878

[14] J. Picard, *On the existence of smooth densities for jump processes*, Probability Theory and Related Fields 105, no. 4, 481-511, 1996. MR1402654