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CONVERGENCE IN FRACTIONAL MODELS AND APPLICATIONS[1]

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Abstract: We consider a fractional Brownian motion with Hurst parameter strictly between 0 and 1. We are interested in the asymptotic behaviour of functionals of the increments of this and related processes and we propose several probabilistic and statistical applications.

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1 Introduction

Let $\{b_\alpha(t), t \in \mathbb{R}\}$ be the fractional Brownian motion with parameter $0 < \alpha < 1$. Consider φ, a positive kernel with L^1 norm equal to one, and let $\varphi_\varepsilon(\cdot) = \frac{1}{\varepsilon}\varphi(\frac{\cdot}{\varepsilon})$ and then define $b_\alpha^\varepsilon(t) = \varphi_\varepsilon * b_\alpha(t)$, the regularized fractional Brownian motion.

Recently, there has been some interest in modeling a stock price $X(t)$ by a fractional version of Black-Scholes model (see Black and Scholes (1973)), say:

$$\mathrm{d}X(t) = X(t)(\sigma\,\mathrm{d}b_\alpha(t) + \mu\,\mathrm{d}t),$$

with $X(0) = c$ and $\alpha > 1/2$ (see also Cutland *et al.* (1993)). More generally, let $X(t)$ be the solution of

$$\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}b_\alpha(t) + \mu(X(t))\,\mathrm{d}t,$$

with $X(0) = c$.

First, assume that $\mu = 0$. The model becomes

$$\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}b_\alpha(t),$$

with $X(0) = c$.

Lin (1995) proved that such a solution can be written as $X(t) = K(b_\alpha(t))$ where K is solution of the ordinary differential equation $\dot{K} = \sigma(K)$ with $K(0) = c$.

We shall consider the statistical problem that consists in observing, instead of $X(t)$, the regularization $X_\varepsilon(t) := \frac{1}{\varepsilon}\int_{-\infty}^{+\infty}\varphi((t-x)/\varepsilon)X(x)\,\mathrm{d}x$ with φ as before and to make inference about $\sigma(\cdot)$. To achieve this purpose we establish first in section 4.1.1 a convergence result for the number of crossings of $X_\varepsilon(\cdot)$, using the following theorem (Azaïs and Wschebor (1996))

Theorem 1.1 *Let $\{b_\alpha(t), t \in \mathbb{R}\}$ be the fractional Brownian motion with parameter $0 < \alpha < 1$. Then, for every continuous function h*

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}&\int_{-\infty}^{+\infty} h(x)N_\varepsilon^{b_\alpha}(x)\,\mathrm{d}x \\ &= \sqrt{\frac{\pi}{2}}\int_0^1 h(b_\alpha^\varepsilon(u))|Z_\varepsilon(u)|\,\mathrm{d}u \xrightarrow{a.s.} \int_0^1 h(b_\alpha(u))\,\mathrm{d}u \\ &= \int_{-\infty}^{+\infty} h(x)\ell^{b_\alpha}(x)\,\mathrm{d}x,\end{aligned}$$

where $\xrightarrow{a.s.}$ denotes the almost-sure convergence, $N_\varepsilon^{b_\alpha}(x)$ the number of times the regularized process $b_\alpha^\varepsilon(\cdot)$ crosses level x before time 1, $Z_\varepsilon(u) = \varepsilon^{(1-\alpha)}\dot{b}_\alpha^\varepsilon(u)/\sigma_{2\alpha}$ with $\sigma_{2\alpha}^2 = \mathbb{V}\left[\varepsilon^{(1-\alpha)}\dot{b}_\alpha^\varepsilon(u)\right]$, and process $\ell^{b_\alpha}(x)$ is the local time in $[0,1]$ of $b_\alpha(\cdot)$ at level x.

To show the result quoted above for $X_\varepsilon(u)$, we shall use the fact that $X_\varepsilon(u)$ is close to $K(b_\alpha^\varepsilon(u))$ and $\dot{X}_\varepsilon(u)$ is close to $\dot{K}(b_\alpha^\varepsilon(u))\dot{b}_\alpha^\varepsilon(u) = \sigma(K(b_\alpha^\varepsilon(u)))\dot{b}_\alpha^\varepsilon(u)$, and this enables us to prove that

$$\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_{-\infty}^{+\infty} h(x)N_\varepsilon^X(x)\,\mathrm{d}x \simeq \sqrt{\frac{\pi}{2}}\int_0^1 h(K(b_\alpha^\varepsilon(u)))\sigma(K(b_\alpha^\varepsilon(u)))|Z_\varepsilon(u)|\,\mathrm{d}u,$$

converges almost surely to

$$\int_0^1 h(K(b_\alpha(u)))\sigma(K(b_\alpha(u)))\,\mathrm{d}u = \int_0^1 h(X(u))\sigma(X(u))\,\mathrm{d}u = \int_{-\infty}^{+\infty} h(x)\sigma(x)\ell^X(x)\,\mathrm{d}x,$$

where $\ell^X(\cdot)$ is the local time for X in $[0,1]$.

Now suppose that $\frac{1}{4} < \alpha < \frac{3}{4}$, we have the following result about the rates of convergence in Theorem 1.1 proved in section 3.4: there exists a Brownian motion $\widehat{W}(\cdot)$ independent of $b_\alpha(\cdot)$ and a constant $C_{\alpha,\varphi}$ such that,

$$\frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_{-\infty}^{\infty} h(x)N_\varepsilon^{b_\alpha}(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} h(x)\ell^{b_\alpha}(x)\,\mathrm{d}x\right] \xrightarrow{\mathcal{D}} C_{\alpha,\varphi}\int_0^1 h(b_\alpha(u))\,\mathrm{d}\widehat{W}(u).$$

Using this last result for $\frac{1}{2} < \alpha < \frac{3}{4}$, we can get the same one for the number of crossings of the process $X_\varepsilon(\cdot)$ and we obtain in section 4.1.1

$$\frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_{-\infty}^{\infty} h(x)N_\varepsilon^{X}(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} h(x)\sigma(x)\ell^{X}(x)\,\mathrm{d}x\right] \xrightarrow{\mathcal{D}} C_{\alpha,\varphi}\int_0^1 h(X(u))\sigma(X(u))\,\mathrm{d}\widehat{W}(u).$$

A similar result can be obtained under contiguous alternatives for $\sigma(\cdot)$ and provides in section 4.1.2 a test of hypothesis for such a function.

We study also the rate of convergence in the following result proved by Azaïs and Wschebor (1996) concerning the increments of the fractional Brownian motion given here as Theorem 1.2.

Theorem 1.2 *Let $\{b_\alpha(t), t \in \mathbb{R}\}$ be the fractional Brownian motion with parameter $0 < \alpha < 1$. Then, for all $x \in \mathbb{R}$ and $t \geq 0$*

$$\lambda\left\{0 \leq u \leq t : \frac{b_\alpha(u+\varepsilon) - b_\alpha(u)}{\varepsilon^\alpha v_{2\alpha}} \leq x\right\} \xrightarrow{a.s.} t\Pr\{N^* \leq x\},$$

where $v_{2\alpha}^2 = \mathbb{V}[b_\alpha(1)]$, λ is the Lebesgue measure and N^* is a standard Gaussian random variable.

This result also implies that for a smooth function f, we have for all $t \geq 0$

$$\int_0^t f\left(\frac{b_\alpha(u+\varepsilon) - b_\alpha(u)}{\varepsilon^\alpha v_{2\alpha}}\right)\mathrm{d}u \xrightarrow{a.s.} t\mathbb{E}\left[f(N^*)\right]. \tag{1}$$

It also can be shown for regularizations $b_\alpha^\varepsilon(u) = \varphi_\varepsilon * b_\alpha(u)$, where $\varphi_\varepsilon(\cdot) = \frac{1}{\varepsilon}\varphi(\frac{\cdot}{\varepsilon})$, φ defined before. In the special case where $\varphi = 1_{[-1,0]}$, we have $\varepsilon\dot{b}_\alpha^\varepsilon(u) = b_\alpha(u+\varepsilon) - b_\alpha(u)$.

Hence, we can write (1) in an other form say, for all $t \geq 0$

$$\int_0^t f\left(\frac{\varepsilon^{(1-\alpha)}\dot{b}_\alpha^\varepsilon(u)}{\sigma_{2\alpha}}\right) \mathrm{d}u \xrightarrow{a.s.} t\mathbb{E}\left[f(N^*)\right]. \tag{2}$$

We find the convergence rate in (2) for a function $f \in L^2(\phi(x)\,\mathrm{d}x)$ where $\phi(x)\,\mathrm{d}x$ stands for the standard Gaussian measure. We can formulate the problem in the following way:

Suppose that $g^{(N)}(x) = f(x) - \mathbb{E}\left[f(N^*)\right]$ is a function in $L^2(\phi(x)\,\mathrm{d}x)$, whose first non-zero coefficient in the Hermite expansion is a_N, i.e. $g^{(N)}(x) = \sum_{n=N}^\infty a_n H_n(x)$, ($N \geq 1$) for which $\|g^{(N)}\|_{2,\phi}^2 = \sum_{n=N}^\infty a_n^2 n!$. The index N is called the Hermite's rank. Find an exponent $a(\alpha, N)$ and a process $X_{g^{(N)}}(\cdot)$ such that the functional defined by $S_\varepsilon^{(N)}(t) = \varepsilon^{-a(\alpha,N)} \int_0^t g^{(N)}(Z_\varepsilon(u))\,\mathrm{d}u$ converges in distribution to $X_{g^{(N)}}(t)$.

Note that similar problems have been studied by Breuer and Major (1983), Ho and Sun (1990) and Taqqu (1977) for summations instead of integrals.

The limit depends on the value of α, and as stated in Section 3.1, $\alpha = 1 - 1/(2N)$ is a breaking point. As pointed out in Section 3.3, if instead of considering the first order increments, we take the second ones, then there is no more breaking points and the convergence is reached for any value of α in $(0, 1)$.

As applications of the previous results, we get in Section 4.2 the following:

$$\varepsilon^{-a(\alpha,2)} \int_0^t \left(|Z_\varepsilon(u)|^\beta - \mathbb{E}\left[|N^*|^\beta\right]\right) \mathrm{d}u \xrightarrow{\mathcal{D}} X_{g^{(2)}}(t),$$

and in Section 4.3 we get the rate of convergence in Theorem 1.2 and we obtain that for all $x \in \mathbb{R}$

$$\varepsilon^{-a(\alpha,1)} \left[\lambda\{0 \leq u \leq t, Z_\varepsilon(u) \leq x\} - \int_0^t \Pr(N^* \leq x)\,\mathrm{d}u\right] \xrightarrow{\mathcal{D}} X_{g^{(1)}}(t),$$

giving the form of the limit, depending also of x, and suggesting the convergence rate in the case where $1/2 < \alpha < 1$.

We observe that all the results quoted above for the fractional Brownian motion, have been considered in Berzin-Joseph and León (1997) for the Wiener process (corresponding to the case where $\alpha = 1/2$), in Berzin *et al.* (2001) for the F-Brownian motion, in Berzin *et al.* (1998) for a class of stationary Gaussian processes and in Perera and Wschebor (1998) for semimartingales.

It is worth noticing that in the case of stationary Gaussian processes the results are quite similar to those obtained in the present article for $N = 2$.

The paper is organized as follows. In this section we introduced the problems and their applications. In section 2 we state some notations and the hypotheses under which we work. Section 3 is devoted to establish the main results. The applications are developed in section 4. Section 5 contains the proofs.

2 Hypotheses and notations

Let $\{b_\alpha(t), t \in \mathbb{R}\}$ be the fractional Brownian motion with parameter $0 < \alpha < 1$ (see for instance Mandelbrot and Van Ness (1968)), *i.e.* $b_\alpha(\cdot)$ is a centered Gaussian process with the covariance function

$$\mathbb{E}\left[b_\alpha(t)b_\alpha(s)\right] = \frac{1}{2}\, v_{2\alpha}^2 \Big[|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\Big],$$

$$\text{with } \; v_{2\alpha}^2 = \frac{1}{\Gamma(2\alpha+1)\sin(\pi\alpha)}.$$

For each $t \geq 0$ and $\varepsilon > 0$, we define the regularized processes

$$b_\alpha^\varepsilon(t) = \varepsilon^{-1}\int_t^{t+\varepsilon} b_\alpha(u)\,\mathrm{d}u \quad \text{and} \quad Z_\varepsilon(t) = \frac{b_\alpha(t+\varepsilon) - b_\alpha(t)}{\varepsilon^\alpha v_{2\alpha}}.$$

We also define, for a C^1 density φ with compact support included in $[-1,1]$ satisfying $\int_{-\infty}^{\infty}\varphi(x)\,\mathrm{d}x = 1$,

$$b_\alpha^\varepsilon(t) = \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi\left(\frac{t-x}{\varepsilon}\right) b_\alpha(x)\,\mathrm{d}x \quad \text{and} \quad Z_\varepsilon(t) = \frac{\varepsilon^{(1-\alpha)}\dot{b}_\alpha^\varepsilon(t)}{\sigma_{2\alpha}},$$

with

$$\sigma_{2\alpha}^2 := \mathbb{V}\left[\varepsilon^{(1-\alpha)}\dot{b}_\alpha^\varepsilon(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|x|^{1-2\alpha}|\hat{\varphi}(-x)|^2\,\mathrm{d}x.$$

(Note that for $\alpha = 1/2$, $\sigma_{2\alpha}^2 = ||\varphi||_2^2 := \int_{-\infty}^{+\infty}\varphi^2(x)\,\mathrm{d}x$.)

We shall use the Hermite polynomials, which can be defined by $\exp(tx - t^2/2) = \sum_{n=0}^{\infty} H_n(x)t^n/n!$. They form an orthogonal system for the standard Gaussian measure $\phi(x)\,\mathrm{d}x$ and, if $h \in L^2(\phi(x)\,\mathrm{d}x)$, $h(x) = \sum_{n=0}^{\infty}\hat{h}_n H_n(x)$ and $||h||_{2,\phi}^2 = \sum_{n=0}^{\infty}\hat{h}_n^2 n!$. Mehler's formula (see Breuer and Major (1983)) gives a simple form to compute the covariance between two L^2 functions of Gaussian random variables. Actually, if $k \in L^2(\phi(x)\,\mathrm{d}x)$, $k(x) = \sum_{n=0}^{\infty}\hat{k}_n H_n(x)$ and if (X,Y) is a Gaussian random vector such that X and Y are standard Gaussian random variables with correlation ρ then

$$\mathbb{E}\left[h(X)k(Y)\right] = \sum_{n=0}^{\infty}\hat{h}_n\hat{k}_n n!\rho^n. \tag{3}$$

We will also use the following well-known property

$$\int_{-\infty}^{z} H_k(y)\phi(y)dy = -H_{k-1}(z)\phi(z),\; z \in \mathbb{R},\; k \geq 1. \tag{4}$$

Let $g^{(N)}$ be a function in $L^2(\phi(x)\,\mathrm{d}x)$ such that $g^{(N)}(x) = \sum_{n=N}^{\infty} a_n H_n(x)$, $N \geq 1$, with $||g^{(N)}||_{2,\phi}^2 = \sum_{n=N}^{\infty} a_n^2 n! < +\infty$.

For $0 < \alpha < 1 - 1/(2N)$ or $\alpha = 1/2$ and $N = 1$, we shall write

$$(\sigma_a^{(N)})^2 = 2 \sum_{l=N}^{\infty} a_l^2 l! \int_0^{+\infty} \rho_\alpha^l(x)\,\mathrm{d}x,$$

where we define $\rho_\alpha^{(\varepsilon)}(v) = \mathbb{E}\,[Z_\varepsilon(v+u)Z_\varepsilon(u)]$ and

$$\begin{aligned}
\rho_\alpha(x) &= \rho_\alpha^{(\varepsilon)}(\varepsilon x) = \frac{-v_{2\alpha}^2}{2\sigma_{2\alpha}^2} \int_{-\infty}^{\infty} \dot{\varphi} * \widetilde{\dot{\varphi}}(y)|x-y|^{2\alpha}\,\mathrm{d}y \\
&= \frac{1}{2\pi\sigma_{2\alpha}^2} \int_{-\infty}^{\infty} |y|^{1-2\alpha} e^{ixy} |\widehat{\varphi}(-y)|^2\,\mathrm{d}y,
\end{aligned}$$

where $\widetilde{\dot{\varphi}}(y) = \dot{\varphi}(-y)$. (If $\alpha = 1/2$, $\rho_\alpha(x) = \varphi * \widetilde{\varphi}(x)/||\varphi||_2^2$.) For $\varphi = 1_{[-1,0]}$, it is easy to show that

$$\rho_\alpha(x) = \frac{1}{2}[|x+1|^{2\alpha} - 2|x|^{2\alpha} + |x-1|^{2\alpha}] \text{ and } \frac{v_{2\alpha}^2}{\sigma_{2\alpha}^2} = 1.$$

Note that for $N = 1$ and $0 < \alpha < 1/2$, since $\int_0^{+\infty} \rho_\alpha(x)\,\mathrm{d}x = 0$, $(\sigma_a^{(1)})^2 = (\sigma_a^{(2)})^2$, and for $\alpha = 1/2$ with $N = 1$, $(\sigma_a^{(1)})^2 = a_1^2/||\varphi||_2^2 + (\sigma_a^{(2)})^2$.

For $N \geq 1$ and $0 \leq t \leq 1$, define

$$S_\varepsilon^{(N)}(t) = \varepsilon^{-a(\alpha,N)} \int_0^t g^{(N)}(Z_\varepsilon(u))\,\mathrm{d}u,$$

$a(\alpha, N)$ will be defined later.

Throughout the paper, **C** shall stand for a generic constant, whose value may change during a proof. N^* will denote a standard Gaussian random variable.

3 Results

3.1 Convergence for $S_\varepsilon^{(N)}(t)$

3.1.1 Case $0 < \alpha < 1 - 1/(2N)$ or $\alpha = 1/2$ and $N = 1$

If $N = 1$, let us define $A := \{k : k \geq 2 \text{ and } a_k \neq 0\}$. We suppose $A \neq \emptyset$ and we define $N_0 = \inf\{k : k \in A\}$.

Theorem 3.1 *$a(\alpha, N) = 1/2$ and*

1)

$$S_\varepsilon^{(N)}(\cdot) \xrightarrow{\mathcal{D}} \sigma_a^{(N)} \widehat{W}(\cdot),$$

where $\widehat{W}(\cdot)$ is a Brownian motion.

2) Furthermore,

(a) If $1/(2N) < \alpha < 1 - 1/(2N)$

$$(b^{\varepsilon}_{\alpha}(\cdot), S^{(N)}_{\varepsilon}(\cdot)) \xrightarrow{\mathcal{D}} (b_{\alpha}(\cdot), \sigma^{(N)}_{a}\widehat{W}(\cdot));$$

(b) If $1/(2N_0) < \alpha < 1/2$ *and* $N = 1$

$$(b^{\varepsilon}_{\alpha}(\cdot), S^{(1)}_{\varepsilon}(\cdot)) \xrightarrow{\mathcal{D}} (b_{\alpha}(\cdot), \sigma^{(N_0)}_{a}\widehat{W}(\cdot));$$

(c) If $\alpha = 1/2$ *and* $N = 1$

$$(b^{\varepsilon}_{\alpha}(\cdot), S^{(1)}_{\varepsilon}(\cdot)) \xrightarrow{\mathcal{D}} (b_{\alpha}(\cdot), \frac{a_1}{||\varphi||_2}b_{\alpha}(\cdot) + \sigma^{(N_0)}_{a}\widehat{W}(\cdot)).$$

The processes $b_{\alpha}(\cdot)$ *and* $\widehat{W}(\cdot)$ *are independent. The convergence taking place in 1) and 2) is in finite-dimensional distributions.*

Remark 1: If the cœfficients of the function $g^{(N)}$ verify the condition

$$\sum_{k=N}^{\infty} 3^{k/2}\sqrt{k!}|a_k| < \infty,$$

(*cf.* Chambers and Slud (1989), p.328), the sequence $S^{(N)}_{\varepsilon}(\cdot)$ is tight and the convergence takes place in $C[0,1]$ for 1) and in $C[0,1] \times C[0,1]$ for 2). This will be the case for $g^{(N)}$ a polynomial.

Remark 2: In case 1), when $0 < \alpha < 1/2$, $(N = 1)$, note that $\sigma^{(N)}_{a} = \sigma^{(N_0)}_{a}$.

Remark 3: In case 1), when $0 < \alpha < 1/2$, $(N = 1)$ and $a_k \equiv 0$ for $k \geq 2$, (*i.e.* $A = \emptyset$), the limit gives zero, so the normalization must be changed; in fact in this case, $a(\alpha, 1) = 1 - \alpha$ is the convenient normalization and $S^{(1)}_{\varepsilon}(\cdot)$ converges in L^2 towards $a_1 b_{\alpha}(\cdot)/\sigma_{2\alpha}$; this last result is also true when $\alpha \geq 1/2$.
Furthermore with this normalization, for $0 < \alpha < 1$, $(b^{\varepsilon}_{\alpha}(\cdot), S^{(1)}_{\varepsilon}(\cdot)) \xrightarrow{\mathcal{D}} (b_{\alpha}(\cdot), \frac{a_1}{\sigma_{2\alpha}}b_{\alpha}(\cdot))$.

3.1.2 Case $\alpha = 1 - 1/(2N)$ and $N > 1$

Theorem 3.2 $a(\alpha, N) = \frac{1}{2}$ *and*
$(b^{\varepsilon}_{1-1/(2N)}(\cdot), [ln(\varepsilon^{-1})]^{-\frac{1}{2}}S^{(N)}_{\varepsilon}(\cdot)) \xrightarrow{\mathcal{D}} (b_{1-1/(2N)}(\cdot), \sqrt{2N!}[(1-\frac{1}{2N})(1-\frac{1}{N})]^{N/2}(\frac{v_{2-1/N}}{\sigma_{2-1/N}})^{N}a_N\widehat{W}(\cdot))$,
where $\widehat{W}(\cdot)$ *is a Brownian motion and the processes* $b_{1-1/(2N)}(\cdot)$ *and* $\widehat{W}(\cdot)$ *are independent.*
The convergence taking place is in finite-dimensional distributions.

3.1.3 Case $1 - 1/(2N) < \alpha < 1$

Theorem 3.3 *$a(\alpha, N) = N(1-\alpha)$ and for fixed t in $[0,1]$*

$$S_\varepsilon^{(N)}(t) \xrightarrow{\mathcal{L}^2} \sqrt{N!}a_N \left(\frac{i}{\sqrt{2\pi}\sigma_{2\alpha}}\right)^N \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} K_t\left(\sum_{i=1}^N \lambda_i\right) \times \prod_{i=1}^N \left(\lambda_i |\lambda_i|^{-\frac{1}{2}-\alpha}\right) \mathrm{d}W(\lambda_1) \dots \mathrm{d}W(\lambda_N),$$

where

$$K_t(\lambda) = \frac{\exp(it\lambda) - 1}{i\lambda}.$$

Remark: Note that for $N = 1$ ($\alpha > \frac{1}{2}$) the limit is $\frac{a_1}{\sigma_{2\alpha}} b_\alpha(t)$.

3.2 Rate of convergence for $S_\varepsilon^{(1)}(t)$ and $\frac{1}{2} < \alpha < 1$

Remember that for $\frac{1}{2} < \alpha < 1$, $S_\varepsilon^{(1)}(t) = \varepsilon^{\alpha-1} \sum_{k=1}^\infty a_k \int_0^t H_k(Z_\varepsilon(u))\,\mathrm{d}u$. We shall prove that for fixed $t \in [0,1]$, $S_\varepsilon^{(1)}(t)$ converges in L^2 towards $\dfrac{a_1}{\sigma_{2\alpha}} b_\alpha(t)$ (see the remark below Theorem 3.3). We can also give the rate of this convergence using the three last theorems. Let consider $A = \{k : k \geq 2 \text{ and } a_k \neq 0\}$. If $A \neq \emptyset$, we define $N_0 = \inf\{k : k \in A\}$ and for $0 \leq t \leq 1$,

$$V_\varepsilon(t) = \begin{cases} \varepsilon^{-d(\alpha,N_0)} \left(S_\varepsilon^{(1)}(t) - \dfrac{a_1}{\sigma_{2\alpha}} b_\alpha(t)\right), & \text{if } A \neq \emptyset \\ \varepsilon^{-\alpha} \left(S_\varepsilon^{(1)}(t) - \dfrac{a_1}{\sigma_{2\alpha}} b_\alpha(t)\right), & \text{otherwise.} \end{cases}$$

The exponent $d(\alpha, N_0)$ will be defined later.

We have the following corollary.

Corollary 3.1 *1. If $A \neq \emptyset$,*
(i) For $\frac{1}{2} < \alpha < 1 - \frac{1}{2N_0}$,

$$d(\alpha, N_0) = \alpha - \frac{1}{2} \quad \text{and} \quad V_\varepsilon(\cdot) \xrightarrow{\mathcal{D}} \sigma_a^{(N_0)} \widehat{W}(\cdot),$$

where $(\sigma_a^{(N_0)})^2 = 2\sum_{l=N_0}^\infty a_l^2 l! \int_0^{+\infty} \rho_\alpha^l(x)\,\mathrm{d}x$.
(ii) For $\alpha = 1 - \frac{1}{2N_0}$, $d(\alpha, N_0) = \frac{1}{2} - \frac{1}{2N_0}$ and

$$[ln(\varepsilon^{-1})]^{-\frac{1}{2}} V_\varepsilon(\cdot) \xrightarrow{\mathcal{D}} \sqrt{2N_0!}\left[\left(1 - \frac{1}{2N_0}\right)\left(1 - \frac{1}{N_0}\right)\right]^{N_0/2} \left(\frac{v_{2-1/N_0}}{\sigma_{2-1/N_0}}\right)^{N_0} a_{N_0} \widehat{W}(\cdot).$$

(iii) For $1 - \frac{1}{2N_0} < \alpha < 1$, $d(\alpha, N_0) = (N_0 - 1)(1-\alpha)$ *and for fixed* t *in* $[0,1]$,

$$V_\varepsilon(t) \xrightarrow{\mathcal{L}^2} \sqrt{N_0!}a_{N_0}\left(\frac{i}{\sqrt{2\pi}\sigma_{2\alpha}}\right)^{N_0}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}K_t\left(\sum_{i=1}^{N_0}\lambda_i\right)\times$$
$$\prod_{i=1}^{N_0}\left(\lambda_i|\lambda_i|^{-\frac{1}{2}-\alpha}\right)\mathrm{d}W(\lambda_1)\ldots\mathrm{d}W(\lambda_{N_0}).$$

2. If $A = \emptyset$, *for* $\frac{1}{2} < \alpha < 1$,

$$V_\varepsilon(\cdot) \xrightarrow{\mathcal{D}} \frac{a_1}{\sigma_{2\alpha}}\sqrt{c_{\alpha,\varphi}}\,[B(\cdot) - B(0)],$$

where $c_{\alpha,\varphi} = \mathbb{V}\left[\int_{-\infty}^{\infty}\varphi(x)b_\alpha(x)\,\mathrm{d}x\right]$ *and* $B(\cdot)$ *is a cylindrical standard Gaussian process with zero correlation independent of* $b_\alpha(\cdot)$. *The symbol* $\xrightarrow{\mathcal{D}}$ *means weak convergence in finite-dimensional distributions.*

Remark: Statement 2. is also true for $0 < \alpha \leq \frac{1}{2}$ with the same definition for $S_\varepsilon^{(1)}(\cdot)$.

3.3 2^{nd}-order increments

Instead of considering the first order increments of $b_\alpha(\cdot)$, we study the asymptotic behaviour of the second order increments. We also get convergence for the corresponding functionals to a Brownian motion for all the values of α in $(0,1)$.

Thus, if φ is now in C^2 instead of in C^1, we define

$$\widetilde{Z}_\varepsilon(u) = \frac{\varepsilon^{2-\alpha}\ddot{b}_\alpha^\varepsilon(u)}{\widetilde{\sigma}_{2\alpha}},$$

with

$$\widetilde{\sigma}_{2\alpha}^2 = \mathbb{V}\left[\varepsilon^{2-\alpha}\ddot{b}_\alpha^\varepsilon(u)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|x|^{3-2\alpha}|\hat{\varphi}(-x)|^2\,\mathrm{d}x.$$

Note that if $\varphi = \varphi_1 * \varphi_2$ with $\varphi_1 = \mathbf{1}_{[-1,0]}$ and $\varphi_2 = \mathbf{1}_{[0,1]}$ then $\varepsilon^2\ddot{b}_\alpha^\varepsilon(u) = b_\alpha(u+\varepsilon) - 2b_\alpha(u) + b_\alpha(u-\varepsilon)$.

Now for $N \geq 1$ and $0 \leq t \leq 1$, define

$$\tilde{S}_\varepsilon^{(N)}(t) := \frac{1}{\sqrt{\varepsilon}}\int_0^t g^{(N)}(\widetilde{Z}_\varepsilon(u))\,\mathrm{d}u,$$

where we suppose as in Theorem 3.1 that $A := \{k : k \geq 2 \text{ and } a_k \neq 0\}$ is not an empty set and we define $N_0 = inf\{k : k \in A\}$. With the technics used in Theorem 3.1, we can prove Corollary 3.2.

Corollary 3.2 *For $0 < \alpha < 1$,*

(i)
$$\tilde{S}_\varepsilon^{(N)}(\cdot) \xrightarrow{\mathcal{D}} \widetilde{\sigma}_a^{(N_0)} \widehat{W}(\cdot).$$

Furthermore,

(ii) If $1/(2N_0) < \alpha < 1$,

$$(b_\alpha^\varepsilon(\cdot), \tilde{S}_\varepsilon^{(N)}(\cdot)) \xrightarrow{\mathcal{D}} (b_\alpha(\cdot), \widetilde{\sigma}_a^{(N_0)} \widehat{W}(\cdot)),$$

where we have noted $(\widetilde{\sigma}_a^{(N_0)})^2 = 2\sum_{l=N_0}^{\infty} a_l^2 l! \int_0^{+\infty} \widetilde{\rho}_\alpha^l(x)\,\mathrm{d}x$, $\widetilde{\rho}_\alpha(x) = \mathbb{E}\left[\widetilde{Z}_\varepsilon(\varepsilon x + u)\widetilde{Z}_\varepsilon(u)\right]$.

Remark: If $A = \emptyset$ $(N = 1)$, the convenient normalization for $\tilde{S}_\varepsilon^{(1)}(t)$ is ε^{-1} and in this case for $0 < \alpha < 1$, $(b_\alpha^\varepsilon(\cdot), \tilde{S}_\varepsilon^{(1)}(\cdot)) \xrightarrow{\mathcal{D}} (b_\alpha(\cdot), a_1\sigma_{2\alpha}/\widetilde{\sigma}_{2\alpha}(B(\cdot) - B(0)))$ where $B(\cdot)$ is again a cylindrical standard Gaussian process with zero correlation independent of $b_\alpha(\cdot)$.

3.4 Crossings and Local time

Let us define the following random variable

$$\Sigma_\varepsilon(h) = \varepsilon^{-e(\alpha,2)} \int_{-\infty}^{+\infty} h(x)\Big[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}} N_\varepsilon^{b_\alpha}(x) - \ell^{b_\alpha}(x)\Big]\,\mathrm{d}x,$$

where $N_\varepsilon^{b_\alpha}(x)$ is the number of times that the process $b_\alpha^\varepsilon(\cdot)$ crosses level x before time 1 and $\ell^{b_\alpha}(\cdot)$ is the local time for the fractional Brownian motion in $[0, 1]$ (see Berman (1970)) that satisfies, for every continuous function h

$$\int_{-\infty}^{+\infty} h(x)\ell^{b_\alpha}(x)\,\mathrm{d}x = \int_0^1 h(b_\alpha(u))\,\mathrm{d}u,$$

and then by Banach-Kac (Banach (1925) and Kac (1943)), $\Sigma_\varepsilon(h)$ can be expressed as

$$\Sigma_\varepsilon(h) = \varepsilon^{-e(\alpha,2)}\Big[\int_0^1 h(b_\alpha^\varepsilon(u))\sqrt{\frac{\pi}{2}}|Z_\varepsilon(u)|\,\mathrm{d}u - \int_0^1 h(b_\alpha(u))\,\mathrm{d}u\Big].$$

Using Theorem 3.1 2)(a) we can get Theorem 3.4.

Theorem 3.4 *Let h be C^3 such that $|h^{(3)}(x)| \le P(|x|)$, where P is a polynomial.*

(i) If $0 < \alpha < \frac{1}{4}$, then $e(\alpha, 2) = 2\alpha$ and

$$\Sigma_\varepsilon(h) \xrightarrow{\mathcal{L}^2} K_{\alpha,\varphi}\int_{-\infty}^{+\infty} \ddot{h}(x)\ell^{b_\alpha}(x)\,\mathrm{d}x = K_{\alpha,\varphi}\int_0^1 \ddot{h}(b_\alpha(u))\,\mathrm{d}u,$$

where $K_{\alpha,\varphi} = \frac{-v_{2\alpha}^2}{4}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x)\varphi(y)|x-y|^{2\alpha}\,\mathrm{d}x\mathrm{d}y\Big)$.

(ii) If $\frac{1}{4} < \alpha < \frac{3}{4}$ and furthermore h is C^4 such that $|h^{(4)}(x)| \le P(|x|)$, then $e(\alpha, 2) = \frac{1}{2}$ and $\Sigma_\varepsilon(h)$ converges stably towards a random variable $Y(h)$

$$Y(h) := C_{\alpha,\varphi}\int_0^1 h(b_\alpha(u))d\widehat{W}(u),$$

where $C^2_{\alpha,\varphi} = 2\sum_{l=1}^{+\infty} a_{2l}^2 (2l)! \int_0^{+\infty} \rho_\alpha^{2l}(x)\,\mathrm{d}x$ *and* a_{2l} *is defined by* $\sqrt{\frac{\pi}{2}}|x| - 1 = \sum_{l=1}^{+\infty} a_{2l} H_{2l}(x) := g^{(2)}(x)$.

Remark 1: If $0 < \alpha \leq \frac{1}{2}$, Theorem 3.4 is true under weaker hypotheses. Indeed, if $0 < \alpha < \frac{1}{2}$, it is enough to ask for $h \in C^3$ with $|h^{(3)}(x)| \leq P(|x|)$, and if $\alpha = \frac{1}{2}$, for $h \in C^2$ with $|\ddot{h}(x)| \leq P(|x|)$.

Remark 2: It can be proved that, under the same hypotheses as in (ii) and for general f, with $(2+\delta)$-moments with respect to the standard Gaussian measure, $\delta > 0$, even, or odd with Hermite's rank greater than or equal to three,

$$\frac{1}{\sqrt{\varepsilon}}\left[\int_0^1 f(Z_\varepsilon(u))h(b_\alpha^\varepsilon(u))\,\mathrm{d}u - \mathbb{E}\left[f(N^*)\right]\int_0^1 h(b_\alpha(u))\,\mathrm{d}u\right] \xrightarrow{\mathcal{D}} C_{\alpha,\varphi}(f)\int_0^1 h(b_\alpha(u))d\widehat{W}(u),$$

where $C_{\alpha,\varphi}(f)$ is similar to $C_{\alpha,\varphi}$, but now a_{2l} are the Hermite's coefficients of $f - E(f(N^*))$. So taking, $f(x) = \sqrt{\frac{\pi}{2}}|x|$, we can see the last convergence as a generalization of (ii).

Remark 3: If $h \equiv 1$, the convenient normalization for $\Sigma_\varepsilon(h)$ is $e(\alpha, 2) = \frac{1}{2}$ but for all $0 < \alpha < \frac{3}{4}$ and we can prove that $\frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{+\infty}[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}} N_\varepsilon^{b_\alpha}(x) - \ell^{b_\alpha}(x)]\,\mathrm{d}x$ converges in distribution towards $C_{\alpha,\varphi}\widehat{W}(1)$ by Theorem 3.1 1).

4 Applications

4.1 Pseudo-diffusion

4.1.1 Estimation of the variance of a pseudo-diffusion.

As is well-known, the process $b_\alpha(\cdot)$ is not a semimartingale. Thus we cannot, in general, integrate $\int_0^t a(u)\,\mathrm{d}b_\alpha(u)$ for a predictable process $a(\cdot)$. However if the coefficient α is greater than $\frac{1}{2}$, the integral with respect to $b_\alpha(\cdot)$ can be defined pathwise as the limit of Riemann sums (see for example the works of Lin (1995) and Lyons (1994)). This allows us to consider, under certain regularity conditions for μ and σ, the "pseudo-diffusion" equations with respect to $b_\alpha(\cdot)$

$$X(t) = c + \int_0^t \sigma(X(u))\,\mathrm{d}b_\alpha(u) + \int_0^t \mu(X(u))\,\mathrm{d}u,$$

for $t \geq 0$, $\alpha > \frac{1}{2}$ and positive σ. We consider the problem of estimating σ when $\mu \equiv 0$. Suppose we observe instead of $X(t)$ a regularization $X_\varepsilon(t) = \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi(\frac{t-x}{\varepsilon})X(x)\,\mathrm{d}x$, with φ as in section 2, where we have extended $X(\cdot)$ by means of $X(t) = c$, if $t < 0$. It is easy to see that the process $X(t)$ has a local time $\ell^X(x)$ in $[0,1]$ for every level x, in fact we have $\ell^X(x) = \ell^{b_\alpha}(K^{-1}(x))/\sigma(x)$ where K is solution of the ordinary differential equation (ODE), $\dot{K} = \sigma(K)$ with $K(0) = c$. Considering $N_\varepsilon^X(x)$ the number of times that the process $X_\varepsilon(\cdot)$ crosses level x before time 1 and using Theorem 1.1 we can prove:

Proposition 4.1 *Let $\frac{1}{2} < \alpha < 1$, if $h \in C^0$ and $\sigma \in C^1$ then*

$$\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_{-\infty}^{\infty} h(x)N_\varepsilon^X(x)\,\mathrm{d}x \xrightarrow{a.s.} \int_{-\infty}^{\infty} h(x)\sigma(x)\ell^X(x)\,\mathrm{d}x.$$

Moreover, using Theorem 3.4 (ii) we can also obtain the following theorem.

Theorem 4.1 *Let us suppose that $\frac{1}{2} < \alpha < \frac{3}{4}$, $h \in C^4$, $\sigma \in C^4$, σ is bounded and $\sup\{|\sigma^{(4)}(x)|, |h^{(4)}(x)|\} \leq P(|x|)$, where P is a polynomial, then*

$$\frac{1}{\sqrt{\varepsilon}}[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_{-\infty}^{\infty} h(x)N_\varepsilon^X(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} h(x)\sigma(x)\ell^X(x)\,\mathrm{d}x],$$

converges stably towards

$$C_{\alpha,\varphi}\int_0^1 h(X(u))\sigma(X(u))d\widehat{W}(u).$$

Here, $\widehat{W}(\cdot)$ is still a standard Brownian motion independent of $b_\alpha(\cdot)$, $C_{\alpha,\varphi}^2$ given by

$$C_{\alpha,\varphi}^2 = 2\sum_{l=1}^{+\infty} a_{2l}^2(2l)!\int_0^{+\infty}\rho_\alpha^{2l}(v)\,\mathrm{d}v \quad \textit{and} \quad g^{(2)}(x) = \sqrt{\frac{\pi}{2}}|x| - 1 = \sum_{l=1}^{+\infty} a_{2l}H_{2l}(x).$$

Remark: This type of result was obtained for a class of semimartingales, and in particular for diffusions, in Perera and Wschebor (1998).

4.1.2 Proofs of hypothesis

Now, we observe $X_\varepsilon(\cdot)$, solution of the stochastic differential equation, for $t \geq 0$,

$$\mathrm{d}X_\varepsilon(t) = \sigma_\varepsilon(X_\varepsilon(t))\,\mathrm{d}b_\alpha(t) \text{ with } X_\varepsilon(0) = c,$$

$X_\varepsilon(t) = c$, for $t < 0$ and we consider testing the hypothesis

$$H_0 : \sigma_\varepsilon(\cdot) = \sigma_0(\cdot),$$

against the sequence of alternatives

$$H_\varepsilon : \sigma_\varepsilon(\cdot) = \sigma_0(\cdot) + \sqrt{\varepsilon}d(\cdot) + \sqrt{\varepsilon}F(\cdot, \sqrt{\varepsilon}),$$

where $F(\cdot, 0) = 0$, σ_0, d and F are C^1.
Let us define the observed process $Y_\varepsilon(\cdot) := \frac{1}{\varepsilon}\int_{-\infty}^{+\infty}\varphi(\frac{\cdot - x}{\varepsilon})X_\varepsilon(x)\,\mathrm{d}x$ with φ as in section 2. We are interested in observing the following functionals

$$T_\varepsilon(h) := \frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_{-\infty}^{\infty} h(x)N_\varepsilon^Y(x)\,\mathrm{d}x - \int_0^1 h(X_\varepsilon(u))\sigma_0(X_\varepsilon(u))\,\mathrm{d}u\right].$$

Using Theorem 3.1 2)(a) we can prove Theorem 4.2.

Theorem 4.2 *Let us suppose that* $\frac{1}{2} < \alpha < \frac{3}{4}$*,* $h \in C^4$*,* $\sigma_0 \in C^4$*,* $d \in C^2$*,* $F \in C^1$*,* σ_0 *is bounded and* $\sup\{|\sigma_0^{(4)}(x)|, |h^{(4)}(x)|, |d^{(2)}(x)|\} \leq P(|x|)$ *where* P *is a polynomial then* $T_\varepsilon(h)$ *converges stably towards*

$$C_{\alpha,\varphi} \int_0^1 h(X(u))\sigma_0(X(u))d\widehat{W}(u) + \int_0^1 h(X(u))d(X(u))\, \mathrm{d}u,$$

where $X(\cdot) = K(b_\alpha(\cdot)) \overset{a.s.}{=} \lim_{\varepsilon\to 0} X_\varepsilon(\cdot)$*, and* K *is solution of the ODE,* $\dot{K} = \sigma_0(K)$ *with* $K(0) = c$ *and* $\widehat{W}(\cdot)$ *is a standard Brownian motion independent of* $X(\cdot)$*.*

Remark 1: There is a random asymptotic bias, and the larger the bias the easier it is to discriminate between the two hypotheses.

Remark 2: We can consider the very special case $h \equiv 1$ and σ_0 constant. The limit random variable is

$$C_{\alpha,\varphi}\sigma_0 N^* + \int_0^1 d(\sigma_0 b_\alpha(u) + c)\, \mathrm{d}u.$$

Recall that the two terms in the sum are independent.

4.2 β-increments

Let

$$S_\varepsilon^\beta(t) = \varepsilon^{-a(\alpha,2)} \int_0^t \left\{|Z_\varepsilon(u)|^\beta - \mathbb{E}\left[|N^*|^\beta\right]\right\} \mathrm{d}u, \quad \text{for } \beta > 0 \text{ and } 0 \leq t \leq 1.$$

As an application of Theorems 3.1 1), 3.2 (i) and 3.3, we obtain the following corollary.

Corollary 4.1 *(i) If* $0 < \alpha < \frac{3}{4}$*,*

$$a(\alpha, 2) = \frac{1}{2} \quad \text{and} \quad S_\varepsilon^\beta(\cdot) \xrightarrow{\mathcal{D}} \sigma_\beta^{(2)}\widehat{W}(\cdot),$$

where

$$(\sigma_\beta^{(2)})^2 = \frac{2^{\beta+1}}{\pi}\left[\sum_{l=1}^{+\infty}(2l)!\left(\sum_{p=0}^{l}\frac{(-1)^{l-p}}{(2p)!(l-p)!2^{l-p}}2^p\Gamma(p+\frac{\beta+1}{2})\right)^2\int_0^{+\infty}\rho_\alpha^{2l}(x)\,\mathrm{d}x\right].$$

(ii) If $\alpha = \frac{3}{4}$*,*

$$a(\alpha, 2) = \frac{1}{2} \quad \text{and} \quad [ln(\varepsilon^{-1})]^{-\frac{1}{2}} S_\varepsilon^\beta(\cdot) \xrightarrow{\mathcal{D}} \frac{3\beta 2^{\beta/2-1}}{4\sqrt{\pi}}\Gamma((\beta+1)/2)\frac{v_{3/2}^2}{\sigma_{3/2}^2}\widehat{W}(\cdot).$$

(iii) If $\frac{3}{4} < \alpha < 1$,

$$a(\alpha,2) = 2(1-\alpha) \text{ and for fixed } t \text{ in } [0,1],\ S_\varepsilon^\beta(t) \xrightarrow{\mathcal{L}^2} -\beta\frac{2^{(\beta-1)/2}}{\sqrt{\pi}}\Gamma\left(\frac{\beta+1}{2}\right)\left(\frac{1}{\sqrt{2\pi}\sigma_{2\alpha}}\right)^2$$

$$\times\Big(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_t(\lambda+\mu)\lambda|\lambda|^{-\alpha-\frac{1}{2}}\mu|\mu|^{-\alpha-\frac{1}{2}}\,\mathrm{d}W(\lambda)\,\mathrm{d}W(\mu)\Big).$$

4.3 Lebesgue measure

Let

$$S_\varepsilon^\lambda(t) = \varepsilon^{-a(\alpha,1)}\Big[\lambda\{0 \le u \le t, Z_\varepsilon(u) \le x\} - \int_0^t \Pr(N^* \le x)\,\mathrm{d}u\Big] \text{ for } \ x \in \mathbb{R} \text{ and } 0 \le t \le 1.$$

Thanks to Theorems 3.1 1) and 3.3 we have the following corollary.

Corollary 4.2 *(i) If $0 < \alpha < \frac{1}{2}$, $a(\alpha,1) = \frac{1}{2}$ and*

$$S_\varepsilon^\lambda(\cdot) \xrightarrow{\mathcal{D}} \sigma_\lambda^{(2)}\widehat{W}(\cdot),$$

where $(\sigma_\lambda^{(2)})^2 = 2\sum_{l=2}^{+\infty}\frac{1}{l!}H_{l-1}^2(x)\phi^2(x)\Big[\int_0^{+\infty}\rho_\alpha^l(y)\,\mathrm{d}y\Big]$.
(ii) If $\alpha = \frac{1}{2}$, $a(\alpha,1) = \frac{1}{2}$ and

$$S_\varepsilon^\lambda(\cdot) \xrightarrow{\mathcal{D}} \sigma_\lambda^{(2)}\widehat{W}(\cdot) - \frac{\phi(x)}{||\varphi||_2}b_\alpha(\cdot).$$

(iii) If $\frac{1}{2} < \alpha < 1$, $a(\alpha,1) = 1-\alpha$ and for fixed t in $[0,1]$

$$S_\varepsilon^\lambda(t) \xrightarrow{\mathcal{L}^2} \frac{-\phi(x)}{\sigma_{2\alpha}}b_\alpha(t).$$

Remark: In case (ii), if $\varphi = \mathbf{1}_{[-1,0]}$, then $(\sigma_\lambda^{(2)})^2 = 2\sum_{l=2}^{+\infty}\frac{1}{(l+1)!}H_{l-1}^2(x)\phi^2(x)$.

Thanks to Corollary 3.1.1 we can give the rate of convergence when $\frac{1}{2} < \alpha < 1$. Indeed for $0 \le t \le 1$ and $x \in \mathbb{R}^*$, let

$$V_\varepsilon^\lambda(t) = \varepsilon^{-d(\alpha,2)}\left(S_\varepsilon^\lambda(t) + \frac{\phi(x)}{\sigma_{2\alpha}}b_\alpha(t)\right).$$

We have the following corollary.

Corollary 4.3 *(i) If $\frac{1}{2} < \alpha < \frac{3}{4}$, $d(\alpha,2) = \alpha - \frac{1}{2}$ and*

$$V_\varepsilon^\lambda(\cdot) \xrightarrow{\mathcal{D}} \sigma_\lambda^{(2)}\widehat{W}(\cdot),$$

where $\sigma_\lambda^{(2)}$ *is the same as previous corollary.*
(ii) If $\alpha = \frac{3}{4}$, $d(\alpha,2) = \frac{1}{4}$ *and*

$$[ln(\varepsilon^{-1})]^{-\frac{1}{2}} V_\varepsilon^\lambda(\cdot) \xrightarrow{\mathcal{D}} -\frac{3}{8} x\phi(x) \frac{v_{3/2}^2}{\sigma_{3/2}^2} \widehat{W}(\cdot).$$

(iii) If $\alpha > \frac{3}{4}$, $d(\alpha,2) = 1-\alpha$ *and for fixed* t *in* $[0,1]$

$$V_\varepsilon^\lambda(t) \xrightarrow{\mathcal{L}^2} \frac{1}{\sqrt{2}} x\phi(x) \Big(\frac{1}{\sqrt{2\pi}\sigma_{2\alpha}}\Big)^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_t(\lambda+\mu)\lambda|\lambda|^{-\alpha-\frac{1}{2}}\mu|\mu|^{-\alpha-\frac{1}{2}}\,\mathrm{d}W(\lambda)\,\mathrm{d}W(\mu).$$

5 Proofs of the results

5.1 Asymptotic variance of $S_\varepsilon^{(N)}(t)$

5.1.1 Case where $0 < \alpha < 1 - 1/(2N)$ or $\alpha = 1/2$ and $N = 1$

Proposition 5.1 $a(\alpha, N) = 1/2$ *and* $\mathbb{E}\left[S_\varepsilon^{(N)}(t)\right]^2 \underset{\varepsilon\to 0}{\longrightarrow} t(\sigma_a^{(N)})^2$.

Proof of Proposition 5.1. By Mehler's formula (see Equation (3))

$$\mathbb{E}\left[S_\varepsilon^{(N)}(t)\right]^2 = \frac{2}{\varepsilon}\sum_{l=N}^{+\infty} a_l^2 l! \int_0^t (t-u)(\rho_\alpha^{(\varepsilon)})^l(u)\,\mathrm{d}u.$$

If we let $u = \varepsilon x$, we get

$$\mathbb{E}\left[S_\varepsilon^{(N)}(t)\right]^2 = 2\sum_{l=N}^{+\infty} a_l^2 l! \int_0^{\frac{t}{\varepsilon}} (t-\varepsilon x)\rho_\alpha^l(x)\,\mathrm{d}x.$$

But $|\rho_\alpha(x)|$ is equivalent to $x^{2\alpha-2}\alpha|2\alpha-1|v_{2\alpha}^2/\sigma_{2\alpha}^2$ when x tends to infinity and is bounded from above by $\mathbf{C}\,x^{2\alpha-2}$.
Since $\alpha < 1-1/(2N)$ or $\alpha = 1/2$ and $N = 1$, $||g^{(N)}||_{2,\phi}^2 < +\infty$ and $|\rho_\alpha(x)| \le 1$, we can use the Lebesgue's dominated convergence theorem to get the result. □

Proof of Theorem 3.1. 1) We give the proof for the special case where $N = 2$ $(0 < \alpha < 3/4)$ to propose a demonstration rather different than in 2)(a). Using the Chaos representation for the increments of the fractional Brownian motion (see Hunt (1951)), we can write

$$b_\alpha(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\exp(i\lambda t) - 1]|\lambda|^{-\alpha-\frac{1}{2}}\,\mathrm{d}W(\lambda),$$

thus

$$Z_\varepsilon(t) = \frac{1}{\sqrt{2\pi}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}} \int_{-\infty}^{\infty} \exp(i\lambda t) i\lambda\hat{\varphi}(-\lambda\varepsilon)|\lambda|^{-\alpha-\frac{1}{2}}\,\mathrm{d}W(\lambda),$$

making the change of variable $x = \varepsilon\lambda$ in the stochastic integral, we get

$$Z_\varepsilon(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_{2\alpha}} \int_{-\infty}^{\infty} \exp\left(i\frac{xt}{\varepsilon}\right) ix\hat{\varphi}(-x)|x|^{-\alpha-\frac{1}{2}} \,\mathrm{d}W(x).$$

We shall consider the following functional

$$S_\varepsilon^{(2)}(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t g^{(2)}(Z_\varepsilon(u)) \,\mathrm{d}u = \sqrt{\varepsilon} \int_0^{\frac{t}{\varepsilon}} g^{(2)}(Z_\varepsilon(\varepsilon x)) \,\mathrm{d}x,$$

where the function $g^{(2)}$ verifies $\mathbb{E}\left[g^{(2)}(N^*)\right] = 0$ and $\mathbb{E}\left[N^* g^{(2)}(N^*)\right] = 0$. Notice that $Z(x) := Z_\varepsilon(\varepsilon x)$ is a stationary Gaussian process having spectral density

$$f_\alpha(x) = \frac{x^2|\hat{\varphi}(x)|^2}{2\pi\sigma_{2\alpha}^2|x|^{2\alpha+1}}.$$

The function f_α belongs to L^2 only if $0 < \alpha < \frac{3}{4}$. The correlation function is

$$\rho_\alpha(x) = \int_{-\infty}^{\infty} \exp(iyx) f_\alpha(y) \,\mathrm{d}y.$$

Now, for $k \in \mathbb{N}^*$ and $0 = t_0 < t_1 < t_2 < \cdots < t_k$, let

$$S_\varepsilon^{(2)}(\mathbf{t}) = \sum_{i=1}^{k} \alpha_i [S_\varepsilon^{(2)}(t_i) - S_\varepsilon^{(2)}(t_{i-1})],$$

where $\mathbf{t} := (t_0, \ldots, t_k)$ and α_i, $i = 1, \ldots, k$, are defined by

$$\alpha_i = \frac{c_i}{[\sum_{i=1}^{k} c_i^2 (t_i - t_{i-1})]^{\frac{1}{2}}},$$

while c_j, $j = 1, \ldots, k$, are real constants. We want to prove that

$$S_\varepsilon^{(2)}(\mathbf{t}) \xrightarrow[\varepsilon \to 0]{\mathcal{D}} \mathcal{N}(0; (\sigma_a^{(2)})^2),$$

where

$$(\sigma_a^{(2)})^2 = 2\sum_{l=2}^{\infty} a_l^2 l! \int_0^{+\infty} \rho_\alpha^l(x) \,\mathrm{d}x.$$

Let

$$S_{\varepsilon,M}^{(2)}(\mathbf{t}) = \sum_{i=1}^{k} \alpha_i [S_{\varepsilon,M}^{(2)}(t_i) - S_{\varepsilon,M}^{(2)}(t_{i-1})],$$

where

$$S_{\varepsilon,M}^{(2)}(t) = \sqrt{\varepsilon} \int_0^{\frac{t}{\varepsilon}} g_M^{(2)}(Z(x)) \,\mathrm{d}x \quad \text{and} \quad g_M^{(2)}(y) = \sum_{l=2}^{M} a_l H_l(y).$$

First, let us prove the following lemma

Lemma 5.1

$$S^{(2)}_{\varepsilon,M}(\mathbf{t}) \xrightarrow[\varepsilon\to 0]{\mathcal{D}} \mathcal{N}(0;(\sigma^{(2)}_{a,M})^2),$$

where $(\sigma^{(2)}_{a,M})^2 = 2\sum_{l=2}^{M} a_l^2 l! \int_0^{+\infty} \rho_\alpha^l(x)\,\mathrm{d}x.$

Proof of Lemma 5.1. Let n be the integer part of $\frac{1}{\varepsilon}$, i.e. $n := \lfloor 1/\varepsilon \rfloor$. To study the weak convergence of $S^{(2)}_{\varepsilon,M}(\mathbf{t})$ it is sufficient to consider that of $S^{(2)}_{n,M}(\mathbf{t})$ where

$$S^{(2)}_{n,M}(\mathbf{t}) = \sum_{i=1}^{k} \alpha_i \frac{\sqrt{t_i - t_{i-1}}}{\sqrt{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}} \int_{\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} g^{(2)}_M(Z(x))\,\mathrm{d}x.$$

We consider the following functional

$$S^{(2,m)}_{n,M}(\mathbf{t}) = \sum_{i=1}^{k} \alpha_i \frac{\sqrt{t_i - t_{i-1}}}{\sqrt{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}} \int_{\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} g^{(2)}_M(Z^{(m)}(x))\,\mathrm{d}x,$$

where $Z^{(m)}(\cdot)$ is an approximation of $Z(\cdot)$ defined as follows, let ψ defined by

$$\psi(x) = \begin{cases} 2(1-2|x|), & -\frac{1}{2} \le x \le \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\int \psi(x)\,\mathrm{d}x = 1$. Let $\Psi(x) = \psi * \psi(x)$ and $\xi(\lambda) = \frac{1}{2\pi}\Psi(\lambda)/\Psi(0)$, then $\Psi \ge 0$, $\operatorname{supp}\Psi \subset [-1,1]$ and $\int \hat{\xi}(\lambda) d\lambda = 1$. We define $\hat{\xi}^{(m)}(\lambda) = m\hat{\xi}(m\lambda)$ and

$$Z^{(m)}(x) := \int_{-\infty}^{+\infty} \exp(ixy)[f_\alpha * \hat{\xi}^{(m)}]^{\frac{1}{2}}(y)\,\mathrm{d}W(y).$$

Then $(Z(x), Z^{(m)}(x))$ is a mean zero Gaussian vector verifying

$$\mathbb{E}\left[Z(0)Z(x)\right] = \rho_\alpha(x), \qquad \mathbb{E}\left[Z^{(m)}(0)Z^{(m)}(x)\right] = \rho_\alpha(x)\frac{\Psi(x/m)}{\Psi(0)},$$

and

$$\mathbb{E}\left[Z(x)Z^{(m)}(0)\right] = r^{(m)}(x) = \int_{-\infty}^{\infty} \exp(ixy)[f_\alpha]^{\frac{1}{2}}(y)[f_\alpha * \hat{\xi}^{(m)}]^{\frac{1}{2}}(y)\,\mathrm{d}y.$$

The covariance for $Z^{(m)}(\cdot)$ has support in $[-m,m]$ and thus $Z^{(m)}(\cdot)$ is m-dependent. Lemma 5.2 gives the asymptotic value of $\mathbb{E}\left[S^{(2)}_{n,M}(\mathbf{t}) - S^{(2,m)}_{n,M}(\mathbf{t})\right]^2$.

Lemma 5.2

$$\mathbb{E}\left[S^{(2)}_{n,M}(\mathbf{t}) - S^{(2,m)}_{n,M}(\mathbf{t})\right]^2 \le k\, c^{(m)}_M \xrightarrow[m\to+\infty]{} 0,$$

where

$$c_M^{(m)} = 2\sqrt{2}(\sum_{l=2}^{M} a_l^2 l!l)\Big(\int_0^{+\infty} (\rho_\alpha^2(x) + (r^{(m)})^2(x))\,\mathrm{d}x\Big)^{\frac{1}{2}} \times$$
$$\Big[2\Big(\int_0^{+\infty} (r^{(m)}(x) - \rho_\alpha(x))^2\,\mathrm{d}x\Big)^{\frac{1}{2}} + \Big(\int_0^{+\infty} \rho_\alpha^2(x)\left[1 - \frac{\Psi(x/m)}{\Psi(0)}\right]^2 \mathrm{d}x\Big)^{\frac{1}{2}}\Big].$$

Proof of Lemma 5.2. Let

$$X_i = \alpha_i \frac{\sqrt{t_i - t_{i-1}}}{\sqrt{\lfloor nt_i\rfloor - \lfloor nt_{i-1}\rfloor}} \int_{\lfloor nt_{i-1}\rfloor}^{\lfloor nt_i\rfloor} [g_M^{(2)}(Z^{(m)}(x)) - g_M^{(2)}(Z(x))]\,\mathrm{d}x = \alpha_i\sqrt{t_i - t_{i-1}}Y_i.$$

Applying the Schwarz inequality $\Big(\sum_{i=1}^k a_i\Big)^2 \le k\Big(\sum_{i=1}^k a_i^2\Big)$ to $a_i = \alpha_i\sqrt{t_i - t_{i-1}}\,Y_i$ and since $\sum_i^k \alpha_i^2(t_i - t_{i-1}) = 1$, it is enough to prove that $\mathbb{E}\,[Y_i]^2 \le c_M^{(m)}$. Notice that $\mathbb{E}\,[Y_i]^2 = \mathbb{E}\left[S^{(2)}_{\lfloor nt_i\rfloor - \lfloor nt_{i-1}\rfloor, M} - S^{(2,m)}_{\lfloor nt_i\rfloor - \lfloor nt_{i-1}\rfloor, M}\right]^2$ where

$$S_{n,M}^{(2)} = \frac{1}{\sqrt{n}}\int_0^n g_M^{(2)}(Z(x))\,\mathrm{d}x \text{ and } S_{n,M}^{(2,m)} = \frac{1}{\sqrt{n}}\int_0^n g_M^{(2)}(Z^{(m)}(x))\,\mathrm{d}x.$$

Applying Lemma 4.1 of Berman (1992), which gives the required inequality not exactly for $g_M^{(2)}$ but for an Hermite polynomial H_l, and Mehler's formula (see Equation (3)), we get Lemma 5.2. □

Now, we write $S_{n,M}^{(2,m)}(\mathbf{t})$ as

$$S_{n,M}^{(2,m)}(\mathbf{t}) = \sum_{i=1}^{j_n} b_{i,n}\xi_i,$$

where $j_n = \lfloor nt_k\rfloor$, $b_{i,n} = \frac{\alpha_j\sqrt{t_j - t_{j-1}}}{\sqrt{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor}}$ for $\lfloor nt_{j-1}\rfloor + 1 \le i \le \lfloor nt_j\rfloor$, $j \in [1, k]$ and

$$\xi_i = \int_{i-1}^{i} g_M^{(2)}(Z^{(m)}(x))\,\mathrm{d}x.$$

$\{\xi_i\}_{i\in\mathbb{N}^*}$ is a strictly stationary m-dependent sequence (and then strongly mixing sequence) of real-valued random variables with mean zero and strong mixing coefficients $(\beta_n)_{n\ge 0}$. Furthermore, $\sum_{i=1}^{j_n} b_{i,n}^2 = 1$ and $\lim_{n\to+\infty}\max_{i\in[1,j_n]}|b_{i,n}| = 0$.
On the other hand, as in Rio (1995), defining

$$M_{2,\alpha}(Q_{\xi_1}) = \int_0^{\frac{1}{2}} [\beta^{-1}(t/2)Q_{\xi_1}(t)]^2 \frac{\mathrm{d}t}{\beta^{-1}(t/2)},$$

where Q_{ξ_1} is the inverse function of $t \to \Pr(|\xi_1| > t)$, $\beta(t) = \beta_{\lfloor t \rfloor}$ the cadlag rate function, β^{-1} the inverse function of this rate function β.
We have

$$M_{2,\alpha}(Q_{\xi_1}) \leq \int_0^1 \beta^{-1}(t) Q_{\xi_1}^2(t)\,\mathrm{d}t.$$

This last integral is finite if, and only if, $E(\xi_1^2) < \infty$ (see Doukhan *et al.* (1994)). But $E(\xi_1^2) \leq \sum_{l=2}^{M} a_l^2 l! < +\infty$, so $M_{2,\alpha}(Q_{\xi_1}) < +\infty$.
Moreover,

$$\sum_{l=2}^{M} a_l^2 l! \int_0^{+\infty} \rho_\alpha^l(x)\,\mathrm{d}x > 0,$$

for $M \geq 2$, because all the terms are limits of variances hence greater or equal to zero, and for $l = 2$ we have, by Plancherel's theorem

$$\int_0^\infty \rho_\alpha^2(x)\,\mathrm{d}x = \pi \int_{-\infty}^\infty f_\alpha^2(x)\,\mathrm{d}x > 0.$$

Then,

$$\lim_{n\to+\infty} \mathbb{E}\left[S_{n,M}^{(2,m)}(\mathbf{t})\right]^2 = A_M^{(m)} = 2\sum_{l=2}^{M} a_l^2 l! \int_0^{+\infty} \Big(\frac{\rho_\alpha(x)\Psi(\frac{x}{m})}{\Psi(0)}\Big)^l \mathrm{d}x > 0,$$

for $M \geq 2$ and $m \geq m_M$ and then applying Application 1 (Corollary 1, p.39 of Rio (1995)), we finally get that

$$S_{n,M}^{(2,m)}(\mathbf{t}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0; A_M^{(m)}),$$

for $M \geq 2$ and $m \geq m_M$. Also,

$$\mathcal{N}(0; A_M^{(m)}) \xrightarrow[m\to\infty]{\mathcal{D}} \mathcal{N}(0; (\sigma_{a,M}^{(2)})^2),$$

and by Lemma 5.2, $\lim_{m\to+\infty} \sup_n \mathbb{E}\left[S_{n,M}^{(2)}(\mathbf{t}) - S_{n,M}^{(2,m)}(\mathbf{t})\right]^2 = 0$. Applying Lemma 1.1 of Dynkin (1988), we proved that $S_{n,M}^{(2)}(\mathbf{t}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0; (\sigma_{a,M}^{(2)})^2)$ for $M \geq 2$ and then Lemma 5.1 follows. □

Now since

$$\lim_{M\to\infty} \sup_{\varepsilon>0} \mathbb{E}\left[S_{\varepsilon,M}^{(2)}(\mathbf{t}) - S_\varepsilon^{(2)}(\mathbf{t})\right]^2 = 0,$$

applying the Dynkin's result, the proof is completed for the case where $N = 2$ and $0 < \alpha < \frac{3}{4}$. Note that this demonstration uses the crucial fact that ρ_α belongs to $L^2([0,\infty[)$ and so can not be implemented for the other cases. For those cases, Theorem 3.1 1) can be proved using the diagram formula, going in the same way as in Chambers and Slud (1989); indeed for this it is sufficient to adapt the following proof of 2)(a).

2)(a). The following result heavily depends on the N value, known in the literature as the Hermite's rank.
Suppose that $1/(2N) < \alpha < 1 - 1/(2N)$. As before, it is enough to prove that

$$A_{\varepsilon,M}(t) = (b_\alpha^\varepsilon(t_0) = b_\alpha^\varepsilon(0), \ldots, b_\alpha^\varepsilon(t_k), S_{\varepsilon,M}^{(N)}(t_1), \ldots, S_{\varepsilon,M}^{(N)}(t_k) - S_{\varepsilon,M}^{(N)}(t_{k-1})),$$

converges weakly when $\varepsilon \to 0$ to

$$A_M(t) = (b_\alpha(t_0) = b_\alpha(0) = 0, \ldots, b_\alpha(t_k), \sigma_{a,M}^{(N)} \widehat{W}(t_1), \ldots, \sigma_{a,M}^{(N)}(\widehat{W}(t_k) - \widehat{W}(t_{k-1}))),$$

where

$$S_{\varepsilon,M}^{(N)}(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t g_M^{(N)}(Z_\varepsilon(u))\,\mathrm{d}u \quad \text{with} \quad g_M^{(N)}(x) = \sum_{l=N}^{M} a_l H_l(x),$$

and

$$(\sigma_{a,M}^{(N)})^2 = 2 \sum_{l=N}^{M} a_l^2 l! \int_0^{+\infty} \rho_\alpha^l(x)\,\mathrm{d}x.$$

Furthermore $(b_\alpha(t_1), \ldots, b_\alpha(t_k))$ and $(\widehat{W}(t_1), \ldots, \widehat{W}(t_k))$ are independent Gaussian vectors. We shall follow closely the arguments of Ho and Sun (1990) with necessary modifications due to the fact that we are considering a non-ergodic situation.

Let $c_0, \ldots, c_k, d_1, \ldots, d_k$, be real constants, we are interested in the limit distribution of

$$\sum_{j=0}^{k} c_j b_\alpha^\varepsilon(t_j) \; + \; \sum_{j=1}^{k} d_j \left[S_{\varepsilon,M}^{(N)}(t_j) - S_{\varepsilon,M}^{(N)}(t_{j-1}) \right].$$

To simplify the notation we shall write

$$\Gamma_\varepsilon(\mathbf{t}) = \sum_{j=0}^{k} c_j b_\alpha^\varepsilon(t_j) \quad \text{and} \quad U_\varepsilon(\mathbf{t}) = \sum_{j=1}^{k} d_j \left[S_{\varepsilon,M}^{(N)}(t_j) - S_{\varepsilon,M}^{(N)}(t_{j-1}) \right],$$

then $\Gamma_\varepsilon(\mathbf{t})$ is a mean zero Gaussian random variable and

$$a_\varepsilon^2(\mathbf{t}) \equiv \mathbb{V}\left[\Gamma_\varepsilon(\mathbf{t})\right] = \sum_{i,j=0,\cdots,k} c_i c_j \gamma_\varepsilon(t_i, t_j),$$

where $\gamma_\varepsilon(s,t) \equiv \mathbb{E}\left[b_\alpha^\varepsilon(s) b_\alpha^\varepsilon(t)\right]$ is given by Lemma 5.3 whose proof is an easy computation.

Lemma 5.3

$$\gamma_\varepsilon(s,t) = \frac{v_{2\alpha}^2}{2} \left[\int_{-\infty}^{\infty} \varphi(x) |s - \varepsilon x|^{2\alpha}\,\mathrm{d}x - \right.$$

$$\left. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x)\varphi(y) |s - t - \varepsilon(x-y)|^{2\alpha}\,\mathrm{d}x\,\mathrm{d}y + \int_{-\infty}^{\infty} \varphi(x) |t - \varepsilon x|^{2\alpha}\,\mathrm{d}x \right].$$

We normalize $\Gamma_\varepsilon(\mathbf{t})$ defining $\Gamma'_\varepsilon(\mathbf{t}) = \Gamma_\varepsilon(\mathbf{t})/a_\varepsilon(\mathbf{t})$. The correlation between $\Gamma'_\varepsilon(\mathbf{t})$ and $Z_\varepsilon(s)$ is denoted by $\nu_\varepsilon(s,\mathbf{t})$ and

$$\nu_\varepsilon(s,\mathbf{t}) \;=\; \sum_{j=0}^{k} c_j \alpha_\varepsilon(s,t_j)/a_\varepsilon(\mathbf{t}),$$

where $\alpha_\varepsilon(s,t) \;\equiv\; \mathbb{E}\left[b^\varepsilon_\alpha(t) Z_\varepsilon(s)\right]$ is given by the following lemma, whose proof is a straightforward calculation.

Lemma 5.4

$$\alpha_\varepsilon(s,t) = \frac{\alpha v^2_{2\alpha}\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\Big[\int_{-\infty}^{\infty}\varphi(x)|s-\varepsilon x|^{2\alpha-1}\mathrm{sign}(s-\varepsilon x)\,\mathrm{d}x -$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x)\varphi(y)|t-s-\varepsilon(y-x)|^{2\alpha-1}\mathrm{sign}(s-t+\varepsilon y-\varepsilon x)\,\mathrm{d}y\,\mathrm{d}x\Big].$$

Thus we deduce the following Lemma 5.5.

Lemma 5.5 *Let* $\beta = \min\{\alpha, 1-\alpha\}$,

$$|\nu_\varepsilon(s,\mathbf{t})| \le \mathbf{C}\,\varepsilon^\beta.$$

The proof is a direct consequence of Lemma 5.4, according to $|t_j - s| > 2\varepsilon$ or $|t_j - s| \le 2\varepsilon$ and to $s > \varepsilon$ or $s \le \varepsilon$.

We want to study the asymptotic behaviour of

$$\begin{aligned} D_\varepsilon \;\;=\;\; & \mathbb{E}\left[\Gamma^m_\varepsilon(\mathbf{t})U^r_\varepsilon(\mathbf{t})\right] \;=[a_\varepsilon(\mathbf{t})]^m \sum_{l_1,\ldots,l_r=N,\cdots,M} a_{l_1}a_{l_2}\cdots a_{l_r} \sum_{j_1,\ldots,j_r=1,\cdots,k} d_{j_1}d_{j_2}\cdots d_{j_r} \\ \times\;\; & \frac{1}{\varepsilon^{r/2}}\int_{t_{j_1-1}}^{t_{j_1}}\int_{t_{j_2-1}}^{t_{j_2}}\cdots\int_{t_{j_r-1}}^{t_{j_r}} \mathbb{E}\left[H^m_1(\Gamma'_\varepsilon(\mathbf{t}))H_{l_1}(Z_\varepsilon(s_1))H_{l_2}(Z_\varepsilon(s_2))\cdots H_{l_r}(Z_\varepsilon(s_r))\right]d\mathbf{s}, \end{aligned}$$

where $d\mathbf{s} = \mathrm{d}s_1\,\mathrm{d}s_2\ldots\mathrm{d}s_r$. We use the diagram formula. In this case:

$$\mathbb{E}\left[H^m_1(\Gamma'_\varepsilon(\mathbf{t}))H_{l_1}(Z_\varepsilon(s_1))H_{l_2}(Z_\varepsilon(s_2))\cdots H_{l_r}(Z_\varepsilon(s_r))\right]$$

$$=\sum_{G\in\Gamma}\;\prod_{w\in G(V),d_1(w)<d_2(w)} \hat\rho\big(s_{d_1(w)}, s_{d_2(w)}\big),$$

where G is an undirected graph with $l_1+l_2+\cdots+l_r+m$ vertices and $r+m$ levels (for definitions, see Breuer and Major (1983), p.431), $\Gamma = \Gamma(1,1,\ldots,1,l_1,\ldots,l_r)$ denotes the set of diagrams having these properties, $G(V)$ denotes the set of edges of G; the edges w are oriented, beginning in $d_1(w)$ and finishing in $d_2(w)$.

To the set Γ belong the diagrams such that the first m levels correspond to the $\Gamma'_\varepsilon(\mathbf{t})$ variables, $\hat{\rho}$ is defined as

$$\begin{cases} \rho_\alpha^{(\varepsilon)}(s_i - s_j), & \text{if } i \text{ and } j \text{ are in the last } r \text{ levels,} \\ \nu_\varepsilon(s_j, \mathbf{t}), & \text{if the edge } w \text{ joins the first } m \text{ levels with the last } r \text{ levels,} \\ 1, & \text{otherwise.} \end{cases}$$

We say that an edge belongs to the first group if it links two among the first m levels, and to the second if not.

We shall classify the diagrams in $\Gamma(1, 1, \ldots, 1, l_1, l_2, \ldots, l_r)$ as in Ho and Sun (1990), p. 1166, calling R the set of the regular graphs and R^c the rest. We start by considering R.

In a regular graph, since $N > 1$, the levels are paired in such a way that it is not possible for a level of the first group to link with one of the second, yielding a factorization into two graphs, both regular, and then m and r are both even. We can show as in Berzin *et al.* (1998) that the contribution of such graphs tends to

$$\left(\frac{v_{2\alpha}^2}{2} \sum_{i,j=0,\cdots,k} c_i c_j (|t_i|^{2\alpha} + |t_j|^{2\alpha} - |t_i - t_j|^{2\alpha}) \right)^{\frac{m}{2}}$$
$$\times (m-1)!!\,(r-1)!! \left(\sum_{j=1}^{k} d_j^2 (t_j - t_{j-1}) \right)^{\frac{r}{2}} \left(2 \sum_{l=N}^{M} l!\; a_l^2 \int_0^\infty \rho_\alpha^l(x)\, \mathrm{d}x \right)^{\frac{r}{2}}.$$

Using the notations of Ho and Sun (1990), p. 1167, and calling D_ε/R^c the contribution of the irregular graphs in D_ε:

$$D_\varepsilon/R^c \;=\; \sum_{G \in R^c} A_1^\varepsilon \;\times\; A_2^\varepsilon \;\times\; A_3^\varepsilon \;\times\; \varepsilon^{-\frac{r}{2}}.$$

Any diagram $G \in R^c$ can be partitioned into three disjoint subdiagrams $V_{G,1}$, $V_{G,2}$ and $V_{G,3}$ which are defined as follows. $V_{G,1}$ is the maximal subdiagram of G which is regular within itself and all its edges satisfy $1 \le d_1(w) < d_2(w) \le m$ or $m + 1 \le d_1(w) < d_2(w) \le m + r$. Define

$$\begin{aligned} V_{G,1}^*(1) &= \{j \in V_{G,1}^* \mid 1 \le j \le m\}, \\ V_{G,1}^*(2) &= \{j \in V_{G,1}^* \mid m + 1 \le j \le m + r\}, \end{aligned}$$

where $V_{G,1}^*$ are the levels of $V_{G,1}$.

A_i^ε is the factor of the product corresponding to the edges of $V_{G,i}, i = 1, 2, 3$. The normalization for A_1^ε is therefore $\varepsilon^{-|V_{G,1}^*(2)|/2}$ and as shown in Berzin *et al.* (1998), $\varepsilon^{-|V_{G,1}^*(2)|/2} A_1^\varepsilon$ tends to

$$\left(\frac{v_{2\alpha}^2}{2} \sum_{i,j=0,\ldots,k} c_i c_j (|t_i|^{2\alpha} + |t_j|^{2\alpha} - |t_i - t_j|^{2\alpha}) \right)^{|V_{G,1}^*(1)|/2} (|V_{G,1}^*(1)| - 1)!!(2q - 1)!!$$
$$\times \left(\sum_{j=1}^{k} d_j^2 (t_j - t_{j-1}) \right)^{q} \left(2 \sum_{l=N}^{M} a_l^2 l! \int_0^\infty \rho_\alpha^l(x)\; \mathrm{d}x \right)^{q},$$

as $\varepsilon \to 0$, where $q = |V^*_{G,1}(2)|/2$. The limit is then O(1).

Consider now A^ε_2 and define $V_{G,2}$ to be the maximal subdiagram of $G - V_{G,1}$, whose edges satisfy $m+1 \leq d_1(w) < d_2(w) \leq m+r$. The normalization for A^ε_2 is $\varepsilon^{-|V^*_{G,2}(2)|/2}$, where $V^*_{G,2}(2)$ are the levels of $V_{G,2}$. A graph in $V_{G,2}$ is necessarily irregular, otherwise, it would have been taken into account in A^ε_1. As in Berzin *et al.* (1998), $\varepsilon^{-|V^*_{G,2}(2)|/2} A^\varepsilon_2$ tends to zero as ε goes to zero. For A^ε_3 define

$$\begin{aligned} V_{G,3} &= G - (V_{G,1} \cup V_{G,2}), \\ V^*_{G,3}(1) &= \{j \in V^*_{G,3} \mid 1 \leq j \leq m\}, \\ V^*_{G,3}(2) &= \{j \in V^*_{G,3} \mid m+1 \leq j \leq m+r\}, \end{aligned}$$

where $V^*_{G,3}$ are the levels of $V_{G,3}$. The normalization for A^ε_3 is $\varepsilon^{-|V^*_{G,3}(2)|/2}$.

We assume now that $l_1, l_2, \ldots, l_r$, are fixed by the graph. Let $L = |V^*_{G,3}(2)|$,

$$\begin{aligned} \varepsilon^{-|V^*_{G,3}(2)|/2} A^\varepsilon_3 \leq\ & \varepsilon^{-|V^*_{G,3}(2)|/2} \\ & \times \sum_{j_{\xi(1)},\ldots,j_{\xi(L)}=1,\ldots,k} \prod_{i=1}^{L} |d_{j_{\xi(i)}}| \int_{t_{j_{\xi(i)}-1}}^{t_{j_{\xi(i)}}} \prod_{e\in E(V_{G,3}),d_1(e)\in V^*_{G,3}(1)} |\nu_\varepsilon(s_{d_2(e)},\mathbf{t})| \\ & \times \prod_{w\in E(V_{G,3}),d_1(w)\in V^*_{G,3}(2)} |\rho^{(\varepsilon)}_\alpha(s_{d_1(w)} - s_{d_2(w)})| \ \mathrm{d}s_{\xi(i)}, \end{aligned} \tag{5}$$

where $E(V_{G,3})$ are the edges of $V_{G,3}$ and $\nu_\varepsilon(s,\mathbf{t}) = \frac{1}{a_\varepsilon(\mathbf{t})} \sum_{j=0}^k c_j \alpha_\varepsilon(s,t_j)$ where $\alpha_\varepsilon(s,t)$ is given by Lemma 5.4. $V^*_{G,3}(2)$ can be decomposed in two parts,

$$\begin{aligned} B_G &= \{i \in V^*_{G,3}(2) :\ g(i)(2-2\alpha) \leq 1\}, \\ C_G &= \{i \in V^*_{G,3}(2) :\ g(i)(2-2\alpha) > 1\}, \end{aligned}$$

where $g(i)$ is the number of edges in the i-th level not connected by edges to any of the first levels. Furthermore we note $B^*_G = \{i \in B_G :\ k(i)(2-2\alpha) = 1\}$ where $k(i)$ is the number of edges such that $d_1(w) = i$. As in Ho and Sun (1990), p. 1169, we can rearrange the levels in $V^*_{G,3}(2)$ in such a way that the levels of B_G are followed by the levels of C_G. Within B_G and C_G, the levels are also rearranged so that those with smaller $g(i)$ come first. We have $|V^*_{G,3}(2)| = |B_G| + |C_G|$.

If $i \in V^*_{G,3}(2)$, we have $(l_i - g(i))$ edges coming from levels in the first group and thanks to Lemma 5.5 their contribution to A^ε_3 is bounded by $\mathbf{C}\ \varepsilon^{\beta(l_i - g(i))}$ and in total for these levels we get the bound $\mathbf{C}\ \varepsilon^{\beta\sum_{i\in B_G}(l_i-g(i)) + \beta\sum_{i\in C_G}(l_i-g(i))}$; now the other terms are of the form: $\rho^{(\varepsilon)}_\alpha(s_{d_1(w)} - s_i)$ which are bounded by 1, or of that one:

$$\int_{t_{j_i-1}}^{t_{j_i}} \prod_{l=1}^{k(i)} \rho^{(\varepsilon)}_\alpha(s_i - s_{j_l})\,\mathrm{d}s_i.$$

This last integral can be bounded by

$$\mathbf{C}\ \left(\varepsilon^{\frac{k(i)}{g(i)}}\ \mathbf{1}_{i\in C_G}\ +\ \varepsilon^{(2-2\alpha)k(i)}\left(\mathbf{1}_{i\in B_G/B_G^*}\ +\ ln(\frac{1}{\varepsilon})\ \mathbf{1}_{i\in B_G^*}\right)\right),$$

hence, by (5)

$$\begin{aligned}\varepsilon^{-|V_{G,3}^*(2)|/2}A_3^\varepsilon\ &=\ O\Big(\varepsilon^{-(|B_G|+|C_G|)/2}\varepsilon^{\beta\sum_{i\in B_G}(l_i-g(i))}\varepsilon^{\sum_{i\in C_G}\beta(l_i-g(i))}\\&\qquad\times\ \varepsilon^{\left((2-2\alpha)\sum_{i\in B_G}k(i)+\sum_{i\in C_G}\frac{k(i)}{g(i)}\right)}(ln(\frac{1}{\varepsilon}))^{|B_G^*|}\Big),\end{aligned}$$

and since $l_i\geq N$, then we have

$$\begin{aligned}\varepsilon^{-|V_{G,3}^*(2)|/2}A_3^\varepsilon\ &=\ O\Big(\varepsilon^{\left((2-2\alpha)\sum_{i\in B_G}k(i)+\sum_{i\in C_G}\frac{k(i)}{g(i)}-(1-\alpha)\sum_{i\in B_G}g(i)-|C_G|/2\right)}\\&\qquad\times\ \varepsilon^{(1-\alpha-\beta)\sum_{i\in B_G}g(i)}\varepsilon^{(\beta N-\frac{1}{2})|B_G|}\varepsilon^{\beta\sum_{i\in C_G}(l_i-g(i))}(ln(\frac{1}{\varepsilon}))^{|B_G^*|}\Big).\end{aligned}\qquad(6)$$

We have the following bounds

$$\begin{aligned}(1-\alpha-\beta)\sum_{i\in B_G}g(i)\ &\geq\ 0,\\(\beta N-\frac{1}{2})|B_G|\ &\geq\ 0,\\\beta\sum_{i\in C_G}(l_i-g(i))\ &\geq\ 0,\\(2-2\alpha)\sum_{i\in B_G}k(i)+\sum_{i\in C_G}\frac{k(i)}{g(i)}\ &\geq\ (1-\alpha)\sum_{i\in B_G}g(i)+\frac{1}{2}|C_G|.\end{aligned}$$

The last inequality is obtained by the same argument for showing (27) in Ho and Sun (1990), p. 1170.

Three cases can occur: $|B_G|\neq 0$, $|B_G|=0$ and $|C_G|=0$, or $|B_G|=0$ and $|C_G|\neq 0$.
First case: $|B_G|\neq 0$.
Since $\beta>1/(2N)$, one has $(\beta N-1/2)|B_G|>0$ and then $\varepsilon^{(\beta N-\frac{1}{2})|B_G|}(ln(\frac{1}{\varepsilon}))^{|B_G^*|}=o(1)$ thus (6) tends to zero with ε.

Second case: $|B_G|=0$ (then $|B_G^*|=0$) and $|C_G|=0$.
In this case $V_{G,3}=\emptyset$ (otherwise it would have been taken in account before in $V_{G,1}$) and then $V_{G,2}\neq\emptyset$ thus $\varepsilon^{-|V_{G,2}^*(2)|/2}A_2^\varepsilon$ tends to zero with ε and this gives the required limit.

Third case: $|B_G| = 0$ (then $|B^*_G| = 0$) and $|C_G| \neq 0$.
In this case $\varepsilon^{\beta \sum_{i\in C_G}(l_i - g(i))} = \varepsilon^{\beta\ |V^*_{G,3}(1)|}$ with $|V^*_{G,3}(1)| > 0$ (otherwise it would have been taken in account before in $V_{G,2}$) and (6) tends to zero with ε.

(b). Suppose that $1/(2N_0) < \alpha < \frac{1}{2}$ and $N = 1$, since $Z_\varepsilon(u) = \frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}} \dot{b}^\varepsilon_\alpha(u)$, and $H_1(x) = x$, then $S^{(1)}_\varepsilon(t) = \varepsilon^{\frac{1}{2}-\alpha} \frac{a_1}{\sigma_{2\alpha}}(b^\varepsilon_\alpha(t) - b^\varepsilon_\alpha(0)) + S^{(N_0)}_\varepsilon(t)$ and the result follows by 2)(a).

(c). To conclude the proof, suppose that $\alpha = \frac{1}{2}$ and $N = 1$. As in (b) and since in this case $\sigma_{2\alpha} = ||\varphi||_2$, $S^{(1)}_\varepsilon(t) = \frac{a_1}{||\varphi||_2}(b^\varepsilon_\alpha(t) - b^\varepsilon_\alpha(0)) + S^{(N_0)}_\varepsilon(t)$ and we get the result by using 2)(a).

Remark 3 also follows from the fact that $\frac{a_1}{\varepsilon^{1-\alpha}} \int_0^t H_1(Z_\varepsilon(u))\,\mathrm{d}u = \frac{a_1}{\sigma_{2\alpha}}(b^\varepsilon_\alpha(t) - b^\varepsilon_\alpha(0))$.
□

5.1.2 Case where $\alpha = 1 - 1/(2N)$ and $N > 1$

Proposition 5.2

$$[ln(\varepsilon^{-1})]^{-1}\mathbb{E}\left[S^{(N)}_\varepsilon(t)\right]^2 \underset{\varepsilon\to 0}{\longrightarrow} 2N! \left[\left(1 - \frac{1}{2N}\right)\left(1 - \frac{1}{N}\right)\right]^N t a_N^2 \left(\frac{\upsilon_{2-1/N}}{\sigma_{2-1/N}}\right)^{2N}.$$

Proof of Proposition 5.2. We suppose $t > 0$. As in Proposition 5.1, we use Mehler's formula and we make the change of variable $u = \varepsilon x$ to get

$$[ln(\varepsilon^{-1})]^{-1}\mathbb{E}\left[S^{(N)}_\varepsilon(t)\right]^2 = \frac{-2}{\ln(\varepsilon)} \sum_{l=N}^{+\infty} a_l^2 l! \int_0^{\frac{t}{\varepsilon}} (t - \varepsilon x)\rho^l_{1-1/(2N)}(x)\,\mathrm{d}x.$$

Now, since $|\rho_{1-1/(2N)}(x)|$ is equivalent to $(1 - 1/(2N))(1 - 1/N)x^{-1/N}\upsilon^2_{2-1/N}/\sigma^2_{2-1/N}$, when x tends to infinity, and since $||g^{(N)}||^2_{2,\phi} < +\infty$, we have

$$[ln(\varepsilon^{-1})]^{-1}\mathbb{E}\left[S^{(N)}_\varepsilon(t)\right]^2 = \frac{-2}{\ln(\varepsilon)} a_N^2 N! \int_0^{\frac{t}{\varepsilon}} (t - \varepsilon x)\rho^N_{1-1/(2N)}(x)\,\mathrm{d}x + O(-1/\ln(\varepsilon)).$$

Since

$$\rho^N_{1-\frac{1}{(2N)}}(x) = \left(1 - \frac{1}{2N}\right)^N \left(1 - \frac{1}{N}\right)^N x^{-1} \frac{\upsilon^{2N}_{2-1/N}}{\sigma^{2N}_{2-1/N}} + \frac{1}{x^2}\varepsilon\left(\frac{1}{x}\right),$$

with x large enough, the result follows. □

Proof of Theorem 3.2. From Proposition 5.2 we prove that

$$[ln(\varepsilon^{-1})]^{-\frac{1}{2}} S^{(N)}_\varepsilon(t) \simeq \frac{a_N}{\sqrt{\varepsilon \ln(\frac{1}{\varepsilon})}} \int_0^t H_N(Z_\varepsilon(u))\,\mathrm{d}u := [ln(\varepsilon^{-1})]^{-\frac{1}{2}} F^{(N)}_\varepsilon(t),$$

i.e. $\lim_{\varepsilon\to 0}[ln(\varepsilon^{-1})]^{-1}\mathbb{E}\left[S_\varepsilon^{(N)}(t)-F_\varepsilon^{(N)}(t)\right]^2=0$ and the result is an adaptation of Theorem 3.1 2)(a). □

5.1.3 Case where $1-1/(2N)<\alpha<1$

Proposition 5.3 *$a(\alpha,N)=N(1-\alpha)$ and*

$$\mathbb{E}\left[S_\varepsilon^{(N)}(t)\right]^2 \underset{\varepsilon\to 0}{\longrightarrow} \left[\Big((2\alpha-2)N+1\Big)\Big((\alpha-1)N+1\Big)\right]^{-1}N! \\ \times a_N^2\Big(\alpha(2\alpha-1)\frac{v_{2\alpha}^2}{\sigma_{2\alpha}^2}\Big)^N t^{(2\alpha-2)N+2}.$$

Proof of Proposition 5.3. We suppose $t>0$ and then $t\geq 4\varepsilon$. As in Propositions 5.1 and 5.2, we use Mehler's formula and we break the integration domain into two intervals: $[0,4\varepsilon]$ and $[4\varepsilon,t]$.
For the first one, making the change of variable $u=\varepsilon v$, we get

$$2\varepsilon^{2N[\alpha-(1-1/(2N))]}\sum_{l=N}^{+\infty}a_l^2 l!\int_0^4(t-\varepsilon v)\rho_\alpha^l(v)\,\mathrm{d}v.$$

Since $|\rho_\alpha(v)|\leq 1$ and $||g^{(N)}||_{2,\phi}^2<+\infty$ this term is $O(\varepsilon^{2N[\alpha-(1-1/(2N))]})=o(1)$.

Now, let us have a closer look to the second interval

$$\frac{2}{\varepsilon^{2N(1-\alpha)}}\sum_{l=N}^{+\infty}a_l^2 l!\int_{4\varepsilon}^t(t-u)\Big[-\frac{v_{2\alpha}^2}{2\sigma_{2\alpha}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dot\varphi(z-x)\dot\varphi(z)\left|\frac{u}{\varepsilon}-x\right|^{2\alpha}\mathrm{d}x\,\mathrm{d}z\Big]^l\mathrm{d}u.$$

Using a second order Taylor's expansion of $(u-\varepsilon x)^{2\alpha}$ in the neighborhood of $x=0$, it becomes

$$2\sum_{l=N}^{+\infty}a_l^2 l!\int_{4\varepsilon}^t(t-u)\left[\frac{\alpha(2\alpha-1)}{2}\Big(\frac{-v_{2\alpha}^2}{\sigma_{2\alpha}^2}\Big)\right. \\ \left.\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dot\varphi(z-x)\dot\varphi(z)x^2(u-\theta\varepsilon x)^{2\alpha-2}\,\mathrm{d}x\,\mathrm{d}z\right]^l\varepsilon^{2(1-\alpha)(l-N)}\,\mathrm{d}u,$$

with $0\leq\theta<1$.
Then, since $1-1/(2N)<\alpha$ and $||g^{(N)}||_{2,\phi}^2<\infty$, we can apply the Lebesgue's dominated convergence theorem and the limit is given by the first term in the sum. □

Proof of Theorem 3.3. Define

$$G_\varepsilon^{(N)}(t)=\frac{a_N}{\varepsilon^{N(1-\alpha)}}\int_0^t H_N(Z_\varepsilon(u))\,\mathrm{d}u.$$

A straightforward calculation shows that $\mathbb{E}\left[S_\varepsilon^{(N)}(t) - G_\varepsilon^{(N)}(t)\right]^2 \to 0$ as $\varepsilon \to 0$: the proof is similar to the one of Proposition 5.3.
Thus studying the asymptotic behaviour of $G_\varepsilon^{(N)}(t)$ allows us to obtain the same for $S_\varepsilon^{(N)}(t)$. We have seen in the proof of Theorem 3.1 1) that

$$Z_\varepsilon(u) = \varepsilon^{1-\alpha} \int_{-\infty}^{+\infty} i\lambda \exp(i\lambda u)\, \mathrm{d}W_\varepsilon(\lambda),$$

where the stochastic measure $\mathrm{d}W_\varepsilon(\lambda)$ is defined as

$$\mathrm{d}W_\varepsilon(\lambda) = \frac{1}{\sqrt{2\pi}\sigma_{2\alpha}} \hat{\varphi}(-\varepsilon\lambda)|\lambda|^{-\alpha-\frac{1}{2}}\, \mathrm{d}W(\lambda),$$

and then

$$G_\varepsilon^{(N)}(t) = \frac{a_N}{\varepsilon^{N(1-\alpha)}} \int_0^t H_N\Big(\varepsilon^{1-\alpha} \int_{-\infty}^{+\infty} i\lambda \exp(i\lambda u)\, \mathrm{d}W_\varepsilon(\lambda)\Big)\, \mathrm{d}u.$$

Using Itô's formula for the Wiener-Itô integral (see Dobrushin and Major (1979)), we obtain

$$G_\varepsilon^{(N)}(t) = \sqrt{N!}a_N \int_0^t \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} w(\lambda_1, u) \cdots w(\lambda_N, u)\, \mathrm{d}W_\varepsilon(\lambda_1) \cdots \mathrm{d}W_\varepsilon(\lambda_N)\, \mathrm{d}u,$$

where

$$w(\lambda, u) = i\lambda \exp(i\lambda u).$$

As in Chambers and Slud (1989) p. 330, integrating this expression with respect to u, we get

$$G_\varepsilon^{(N)}(t) = \sqrt{N!}i^N a_N \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} K_t(\lambda_1 + \cdots + \lambda_N)\lambda_1 \cdots \lambda_N\, \mathrm{d}W_\varepsilon(\lambda_1) \cdots \mathrm{d}W_\varepsilon(\lambda_N),$$

where $K_t(\lambda) = (\exp(i\lambda t) - 1)/(i\lambda)$.

So

$$\mathbb{E}\left[G_\varepsilon^{(N)}(t)\right]^2 = a_N^2 N! \Big(\tfrac{1}{\sqrt{2\pi}\sigma_{2\alpha}}\Big)^{2N} \int_{\mathbb{R}^N} |K_t\Big(\sum_{i=1}^N \lambda_i\Big)|^2\ \prod_{i=1}^N \Big(|\hat{\varphi}(-\varepsilon\lambda_i)|^2 |\lambda_i|^{1-2\alpha}\, \mathrm{d}\lambda_i\Big).$$

On one hand the inner integrand converges to $|K_t\Big(\sum_{i=1}^N \lambda_i\Big)|^2 \Big(\prod_{i=1}^N |\lambda_i|^{1-2\alpha}\Big)$ when ε tends to zero.
On the other hand, we can bound this integrand by $|K_t\Big(\sum_{i=1}^N \lambda_i\Big)|^2 \Big(\prod_{i=1}^N |\lambda_i|^{1-2\alpha}\Big)$.
Thus if we prove that $\int_{\mathbb{R}^N} |K_t\Big(\sum_{i=1}^N \lambda_i\Big)|^2 \prod_{i=1}^N \Big(|\lambda_i|^{1-2\alpha} d\lambda_i\Big) < +\infty$ then $\mathbb{E}\left[G_\varepsilon^{(N)}(t)\right]^2 \to$

$\mathbb{E}\left[G^{(N)}(t)\right]^2$ when $\varepsilon \to 0$ where

$$G^{(N)}(t) = \sqrt{N!}a_N \left(\frac{i}{\sqrt{2\pi}\sigma_{2\alpha}}\right)^N \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} K_t\left(\sum_{i=1}^N \lambda_i\right) \times \prod_{i=1}^N \left(\lambda_i |\lambda_i|^{-\frac{1}{2}-\alpha}\right) \mathrm{d}W(\lambda_1)\dots \mathrm{d}W(\lambda_N),$$

which is well defined.
Let us define

$$I_t := \int_{\mathbb{R}^N} |K_t\Big(\sum_{i=1}^N \lambda_i\Big)|^2 \prod_{i=1}^N \Big(|\lambda_i|^{1-2\alpha}\,\mathrm{d}\lambda_i\Big) = \int_{\mathbb{R}^N} \frac{4\sin^2\Big(t(\sum_{i=1}^N \lambda_i)/2\Big)}{\Big(\sum_{i=1}^N \lambda_i\Big)^2} \prod_{i=1}^N \Big(|\lambda_i|^{1-2\alpha}\,\mathrm{d}\lambda_i\Big).$$

I_t is always well defined with the possible value $+\infty$. Making the change of variables: $\lambda_1 = 2y_1(1-y_2-\ldots-y_N)/t$, and $\lambda_i = 2y_1 y_i/t$, for $i=2,\ldots,N$, we get

$$I_t = \left(\frac{2}{t}\right)^{2N(1-\alpha)} t^2 \left[\int_{-\infty}^{+\infty} \sin^2(y_1)|y_1|^{(2N-3-2N\alpha)}\,\mathrm{d}y_1\right] \times \left[\int_{\mathbb{R}^{N-1}} |1-y_2-\ldots-y_N|^{1-2\alpha}|y_2|^{1-2\alpha}\cdots|y_N|^{1-2\alpha}\,\mathrm{d}y_2\cdots\mathrm{d}y_N\right].$$

Now, let the following change of variables $y_2+y_3+\ldots+y_N = w_2$, $y_3 = w_2w_3,\ldots$, $y_N = w_2w_N$.
Then $y_2 = w_2(1-w_3-\ldots-w_N)$ and the Jacobian is $w_2^{(N-2)}$. Thus

$$\int_{\mathbb{R}^{N-1}} |1-y_2-\ldots-y_N|^{1-2\alpha}|y_2|^{1-2\alpha}\cdots|y_N|^{1-2\alpha}\,\mathrm{d}y_2\cdots\mathrm{d}y_N = \Big(\int_{-\infty}^{\infty} |1-w_2|^{1-2\alpha}|w_2|^{(N-1)(1-2\alpha)+(N-2)}\,\mathrm{d}w_2\Big) \times \Big(\int_{\mathbb{R}^{N-2}} |1-w_3-\cdots-w_N|^{1-2\alpha}\prod_{i=3}^N\Big(|w_i|^{1-2\alpha}\,\mathrm{d}w_i\Big)\Big).$$

Therefore we can apply the iteration and we have

$$I_t = \left(\frac{2}{t}\right)^{2N(1-\alpha)} t^2 \left(\int_{-\infty}^{+\infty} \sin^2(y_1)|y_1|^{(2N-3-2N\alpha)}\,\mathrm{d}y_1\right) \times \prod_{k=2}^N \left(\int_{-\infty}^{\infty} |1-w_k|^{1-2\alpha}|w_k|^{(k-1)(1-2\alpha)+(k-2)}\,\mathrm{d}w_k\right) < +\infty,$$

since $1-1/(2N) < \alpha < 1$.
Consider now

$$D^{(N)}(\varepsilon) = \sqrt{N!}a_N \left(\frac{i}{\sqrt{2\pi}\sigma_{2\alpha}}\right)^N K_t\Big(\sum_{i=1}^N \lambda_i\Big) \prod_{i=1}^N \Big(\hat{\varphi}(-\varepsilon\lambda_i)\lambda_i|\lambda_i|^{-\frac{1}{2}-\alpha}\Big).$$

$D^{(N)}(\varepsilon)$ converges pointwise to

$$D^{(N)}(0) = \sqrt{N!}a_N \left(\frac{i}{\sqrt{2\pi}\sigma_{2\alpha}}\right)^N K_t\Big(\sum_{i=1}^N \lambda_i\Big) \prod_{i=1}^N \Big(\lambda_i |\lambda_i|^{-\frac{1}{2}-\alpha}\Big),$$

and from the previous calculations and Lebesgue's theorem $D^{(N)}(\varepsilon) \to D^{(N)}(0)$ as $\varepsilon \to 0$ in the L^2-norm with respect to Lebesgue's measure and Theorem 3.3 follows. □

Proof of Corollary 3.1. Since $Z_\varepsilon(u) = \frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\dot{b}^\varepsilon_\alpha(u)$, and $H_1(x) = x$, if $A \neq \emptyset$,

$$V_\varepsilon(t) = \frac{a_1}{\sigma_{2\alpha}}[b^\varepsilon_\alpha(t) - b^\varepsilon_\alpha(0) - b_\alpha(t)]\ \varepsilon^{-d(\alpha,N_0)} + S^{(N_0)}_\varepsilon(t).$$

A straightforward calculation shows that the second order moment of the first term above is $O(\varepsilon^{2(\alpha-d(\alpha,N_0))}) = O(\varepsilon)$. So 1. (i), (ii) and (iii) follows by Theorem 3.1 1), 3.2 (i) and 3.3.
Now if $A = \emptyset$, $V_\varepsilon(t) = \frac{a_1}{\sigma_{2\alpha}}\sqrt{c_{\alpha,\varphi}}[B_\varepsilon(t) - B_\varepsilon(0)]$ where $B_\varepsilon(t) = \dfrac{b^\varepsilon_\alpha(t) - b_\alpha(t)}{\varepsilon^\alpha\sqrt{c_{\alpha,\varphi}}}$ and when $0 < \alpha < 1$, $B_\varepsilon(t) \xrightarrow{\mathcal{D}} B(t)$ and this concludes the proof of the corollary. □

5.1.4 Asymptotic behaviour of the second order increments

Proof of Corollary 3.2. The proof of corollary follows by using the technics developed in the proof of Theorem 3.1 1) and 2)(a). We just give a sketch of this proof.
We can show that

$$\widetilde{\rho}_\alpha(x) = \frac{1}{2\pi\widetilde{\sigma}^2_{2\alpha}}\int_{-\infty}^{+\infty} |y|^{3-2\alpha}e^{ixy}|\hat{\varphi}(-y)|^2\,\mathrm{d}y = \frac{-v^2_{2\alpha}}{2\widetilde{\sigma}^2_{2\alpha}}\int_{-\infty}^{+\infty} \ddot{\varphi} * \widetilde{\ddot{\varphi}}(y)|x-y|^{2\alpha}\,\mathrm{d}y,$$

and $\widetilde{\rho}_\alpha(x) \simeq -v^2_{2\alpha}2\alpha(2\alpha-1)(\alpha-1)(2\alpha-3)x^{2\alpha-4}/\widetilde{\sigma}^2_{2\alpha}$, when x tends to infinity, so it holds that $\widetilde{\rho}_\alpha \in L^1([0,\infty[)$ and furthermore $\int_0^{+\infty}\widetilde{\rho}_\alpha(x)\,\mathrm{d}x = 0$, for all $\alpha \in (0,1)$, so (i) follows. In case of the first order increments we required $\alpha < 1 - 1/(2N)$ or $\alpha = 1/2$ and $N = 1$ to ensure that $\rho_\alpha \in L^N([0,\infty[)$.
Furthermore, we can show that $|\mathbb{E}\,[b^\varepsilon_\alpha(t)\widetilde{Z}_\varepsilon(u)]| \leq \mathbf{C}\,\varepsilon^\alpha$ and that the first coefficient of $\tilde{S}^{(1)}_\varepsilon(t)$, that is, $(\sqrt{\varepsilon}a_1/\widetilde{\sigma}_{2\alpha})\varepsilon^{1-\alpha}(\dot{b}^\varepsilon_\alpha(t) - \dot{b}^\varepsilon_\alpha(0))$, does not contribute to the limit because tending to zero in L^2, so (ii) follows. For the first order increments the bound was ε^β with $\beta = \inf\{\alpha, 1-\alpha\}$ (see Lemma 5.5 in the proof of Theorem 3.1) and we required $\beta > 1/(2N)$ and $N > 1$ to obtain independence between the limit processes. □

5.2 Some particular functionals

5.2.1 Crossings and Local time

We have to prove the result corresponding to crossings. Recall that

$$\Sigma_\varepsilon(h) = \varepsilon^{-e(\alpha,2)} \int_{-\infty}^{+\infty} h(x)\Big[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}} N_\varepsilon^{b_\alpha}(x) - \ell^{b_\alpha}(x)\Big]\,\mathrm{d}x.$$

Proof of Theorem 3.4. The proof follows along the lines of Berzin *et al.* (1998) and uses Theorem 3.1 2)(a). By Banach-Kac (Banach (1925) and Kac (1943)), $\Sigma_\varepsilon(h)$ can be expressed as

$$\begin{aligned}\Sigma_\varepsilon(h) &= \varepsilon^{-e(\alpha,2)}\int_0^1 h(b_\alpha^\varepsilon(u))g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u + \varepsilon^{-e(\alpha,2)}\int_0^1 \Big(h(b_\alpha^\varepsilon(u)) - h(b_\alpha(u))\Big)\,\mathrm{d}u \\ &= \varepsilon^{-e(\alpha,2)}S_1 + \varepsilon^{-e(\alpha,2)}S_2,\end{aligned}$$

where $g^{(2)}(x) = \sqrt{\frac{\pi}{2}}|x| - 1 = \sum_{l=1}^{+\infty} a_{2l}H_{2l}(x)$.
We will show on the one hand, that under hypotheses of (ii) and if $0 < \alpha < \frac{3}{4}$ (not only for $\frac{1}{4} < \alpha < \frac{3}{4}$ as is required for the theorem), $\mathbb{E}\,[S_1^2] = O(\varepsilon)$ and that

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}\left[S_1^2\right] = C_{\alpha,\varphi}^2\int_0^1 \mathbb{E}\left[h^2(b_\alpha(u))\right]\,\mathrm{d}u, \tag{7}$$

and on the other hand, if $\alpha \in (0,1)$ (instead of $0 < \alpha < \frac{1}{4}$), $\mathbb{E}\,[S_2^2] = O(\varepsilon^{4\alpha}) + o(\varepsilon)$. Moreover we can show that equality (7) is true under the less restrictive hypotheses: $h \in C^2$ with $|\ddot{h}(x)| \le P(|x|)$ when $0 < \alpha \le \frac{1}{2}$ and furthermore in case where $\alpha \ge \frac{1}{2}$, we will show that $\mathbb{E}\,[S_2^2] = o(\varepsilon)$. Thus the term S_2 only matters when $\alpha < \frac{1}{4}$ and we will prove in this case that, $\lim_{\varepsilon\to 0}\varepsilon^{-4\alpha}\mathbb{E}\,[S_2^2] = K_{\alpha,\varphi}^2\mathbb{E}\left[\int_0^1 \ddot{h}(b_\alpha(u))\,\mathrm{d}u\right]^2$.
Let us look more closely at S_1.

We decompose S_1 into two terms

$$S_1 = \int_{M\varepsilon}^1 h(b_\alpha^\varepsilon(u))g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u + \int_0^{M\varepsilon} h(b_\alpha^\varepsilon(u))g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u := J_1 + J_2,$$

with M big enough.
Using Hölder's inequality it's easy to see that

$$\mathbb{E}\left[J_2^2\right] \le \mathbf{C}\,\varepsilon^2 = o(\varepsilon). \tag{8}$$

Let $D_\varepsilon = \{(u,v) \in [0,1]^2/\, u \ge M\varepsilon,\, v \ge M\varepsilon,\, |u-v| < M\varepsilon\}$ and $C_\varepsilon = \{(u,v) \in [0,1]^2/\, u \ge M\varepsilon,\, v \ge M\varepsilon,\, |u-v| \ge M\varepsilon\}$.
We decompose $\mathbb{E}\,[J_1^2]$ into two terms

$$\mathbb{E}\left[J_1^2\right] = \int_{D_\varepsilon} + \int_{C_\varepsilon},$$

where
$$\int_{D_\varepsilon} = \int_{D_\varepsilon} \mathbb{E}\left[h(b_\alpha^\varepsilon(u))h(b_\alpha^\varepsilon(v))g^{(2)}(Z_\varepsilon(u))g^{(2)}(Z_\varepsilon(v))\right]\,du\,dv.$$
Let us see $\displaystyle\int_{D_\varepsilon}$.
Applying the change of variable $v = u + \varepsilon x$, one has
$$\int_{D_\varepsilon} = \varepsilon \int_{D'_\varepsilon} \mathbb{E}\left[h(b_\alpha^\varepsilon(u))h(b_\alpha^\varepsilon(u+\varepsilon x))g^{(2)}(Z_\varepsilon(u))g^{(2)}(Z_\varepsilon(u+\varepsilon x))\right]\,du\,dx,$$
where $D'_\varepsilon = \{(u,x)/\, M\varepsilon \le u \le 1,\, M\varepsilon \le u + \varepsilon x \le 1,\, |x| < M\}$.
A straightforward calculation shows that $\Big(b_\alpha^\varepsilon(u), b_\alpha^\varepsilon(u+\varepsilon x), Z_\varepsilon(u), Z_\varepsilon(u+\varepsilon x)\Big)$ converges weakly when ε goes to zero towards $\Big(b_\alpha(u), b_\alpha(u), Y(u), Z_x(u)\Big)$ where $Y(u)$ and $Z_x(u)$ are standard Gaussian variables with correlation $\rho_\alpha(x)$; furthermore the Gaussian vector $\Big(Y(u), Z_x(u)\Big)$ is independent of $b_\alpha(u)$.

Using the Lebesgue's dominated convergence theorem one obtains
$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{D_\varepsilon} = \sum_{l=1}^{\infty} a_{2l}^2\,(2l)! \int_{|x|<M} \rho_\alpha^{2l}(x)\,dx \int_0^1 \mathbb{E}\left[h^2(b_\alpha(u))\right]\,du. \tag{9}$$
Let us look at $\displaystyle\int_{C_\varepsilon}$.
We now fix $(u,v) \in C_\varepsilon$ and we consider the change of variables
$$\begin{aligned} b_\alpha^\varepsilon(u) &= Z_{1,\varepsilon}(u,v) + A_{1,\varepsilon}(u,v)Z_\varepsilon(u) + A_{2,\varepsilon}(u,v)Z_\varepsilon(v),\\ b_\alpha^\varepsilon(v) &= Z_{2,\varepsilon}(u,v) + B_{1,\varepsilon}(u,v)Z_\varepsilon(u) + B_{2,\varepsilon}(u,v)Z_\varepsilon(v), \end{aligned}$$
with $(Z_{1,\varepsilon}(u,v), Z_{2,\varepsilon}(u,v))$ a mean zero Gaussian vector independent of $(Z_\varepsilon(u), Z_\varepsilon(v))$ and
$$\begin{aligned} A_{1,\varepsilon}(u,v) &= \frac{\alpha_\varepsilon(u,u) - \rho_\alpha^\varepsilon(v-u)\,\alpha_\varepsilon(v,u)}{\Delta_\varepsilon(u,v)},\\ A_{2,\varepsilon}(u,v) &= \frac{\alpha_\varepsilon(v,u) - \rho_\alpha^\varepsilon(v-u)\,\alpha_\varepsilon(u,u)}{\Delta_\varepsilon(u,v)}, \end{aligned}$$
where $\alpha_\varepsilon(u,v)$ is given by Lemma 5.4, $\rho_\alpha^{(\varepsilon)}(v-u) = \mathbb{E}\left[Z_\varepsilon(u)Z_\varepsilon(v)\right] = \rho_\alpha(\dfrac{v-u}{\varepsilon})$ and $\Delta_\varepsilon(u,v) = 1 - (\rho_\alpha^{(\varepsilon)})^2(v-u)$.
Two similar formulas hold for $B_{1,\varepsilon}(u,v)$ and $B_{2,\varepsilon}(u,v)$.
A straightforward computation shows that for M big enough, $\varepsilon \le \varepsilon(M)$ and $(u,v) \in C_\varepsilon$,
$$\max_{i=1,2} |A_{i,\varepsilon}(u,v), B_{i,\varepsilon}(u,v)| \le \mathbf{C}\,\varepsilon^{1-\alpha}\,[u^{2\alpha-1} + v^{2\alpha-1} + |u-v|^{2\alpha-1}], \tag{10}$$

and

$$|\rho_\alpha^{(\varepsilon)}(v-u)| \leq \mathbf{C}\,\varepsilon^{2-2\alpha}\,|v-u|^{2\alpha-2}. \tag{11}$$

Writing the Taylor development of h one has,

$$\begin{aligned} h(b_\alpha^\varepsilon(u)) &= \sum_{j=0}^{3}\frac{1}{j!}\,h^{(j)}(Z_{1,\varepsilon}(u,v))[A_{1,\varepsilon}(u,v)Z_\varepsilon(u)+A_{2,\varepsilon}(u,v)Z_\varepsilon(v)]^j \\ &+ \frac{1}{4!}\,h^{(4)}(\theta_{1,\varepsilon}(u,v))[A_{1,\varepsilon}(u,v)Z_\varepsilon(u)+A_{2,\varepsilon}(u,v)Z_\varepsilon(v)]^4, \end{aligned}$$

with $\theta_{1,\varepsilon}(u,v)$ between $b_\alpha^\varepsilon(u)$ and $Z_{1,\varepsilon}(u,v)$.
A similar formula holds for $h(b_\alpha^\varepsilon(v))$.
We can decompose $\int_{C_\varepsilon}$ as the sum of twenty five terms. We use the notations J_{j_1,j_2} for the corresponding integrals, where $j_1, j_2 = 0, \ldots, 4$ are the subscripts involving $h^{(j_1)}$ and $h^{(j_2)}$. We only consider J_{j_1,j_2} with $j_1 \leq j_2$. Then we obtain the followings

(A) One term of the form

$$J_{0,0} = \int_{C_\varepsilon} \mathbb{E}\,[h(Z_{1,\varepsilon}(u,v))h(Z_{2,\varepsilon}(u,v))]\;\mathbb{E}\left[g^{(2)}(Z_\varepsilon(u))g^{(2)}(Z_\varepsilon(v))\right] du\,dv.$$

Making the change of variable $u-v=\varepsilon x$ and applying the Lebesgue's dominated convergence theorem we get

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\,J_{0,0} = \Big[\sum_{l=1}^{\infty} a_{2l}^2\,(2l)!\int_{|x|\geq M}\rho_\alpha^{2l}(x)\,dx\Big]\,\Big[\int_0^1 \mathbb{E}\left[h^2(b_\alpha(u))\right]\,du\Big]. \tag{12}$$

(B) Two terms of the form $J_{0,1}\equiv 0$ by a symmetry argument: if $\mathcal{L}(U,V)=N(0,\Sigma)$ then $\mathbb{E}\left[Ug^{(2)}(U)g^{(2)}(V)\right]=0$.

(C) Two terms of the form

$$\begin{aligned} J_{0,2} &= \frac{1}{2}\int_{C_\varepsilon}\mathbb{E}\Big[h(Z_{1,\varepsilon}(u,v))\ddot{h}(Z_{2,\varepsilon}(u,v))\Big]\times \\ &\qquad \mathbb{E}\left[g^{(2)}(Z_\varepsilon(u))g^{(2)}(Z_\varepsilon(v))[B_{1,\varepsilon}(u,v)Z_\varepsilon(u)+B_{2,\varepsilon}(u,v)Z_\varepsilon(v)]^2\right] du\,dv. \end{aligned}$$

Since $|\rho_\alpha^{(\varepsilon)}(u-v)| \leq 1$,

$$\begin{aligned} &\Big|\mathbb{E}\left[g^{(2)}(Z_\varepsilon(u))g^{(2)}(Z_\varepsilon(v))[B_{1,\varepsilon}(u,v)Z_\varepsilon(u)+B_{2,\varepsilon}(u,v)Z_\varepsilon(v)]^2\right]\Big| \\ &\qquad\qquad \leq \mathbf{C}\max_{i=1,2}|B_{i,\varepsilon}^2(u,v)|\,|\rho_\alpha^{(\varepsilon)}(u-v)|, \end{aligned}$$

using (10) and (11), we get

$$J_{0,2} = O(\varepsilon^{1+2\alpha})\mathbf{1}_{0<\alpha<\frac{1}{2}} + O(\varepsilon^2 ln(\frac{1}{\varepsilon}))\mathbf{1}_{\alpha=\frac{1}{2}} + O(\varepsilon^{4-4\alpha})\mathbf{1}_{\frac{1}{2}<\alpha<\frac{3}{4}} = o(\varepsilon).$$

(D) Two terms of the form $J_{0,3} \equiv 0$ by a symmetry argument: if $\mathcal{L}(U,V) = N(0,\Sigma)$ then $\mathbb{E}\left[(aU+bV)^3 g^{(2)}(U)g^{(2)}(V)\right] = 0$ for any two constants a and b .

(E) Two terms of the form

$$J_{0,4} = \frac{1}{4!}\int_{C_\varepsilon} \mathbb{E}[h(Z_{1,\varepsilon}(u,v))h^{(4)}(\theta_{2,\varepsilon}(u,v))g^{(2)}(Z_\varepsilon(u))g^{(2)}(Z_\varepsilon(v))$$

$$\times[B_{1,\varepsilon}(u,v)Z_\varepsilon(u) + B_{2,\varepsilon}(u,v)Z_\varepsilon(v)]^4]dudv.$$

Therefore

$$|J_{0,4}| \le \mathbf{C}\int_{C_\varepsilon} \max_{i=1,2}\left[A^4_{i,\varepsilon}(u,v), B^4_{i,\varepsilon}(u,v)\right] du\, dv.$$

Using (10), one obtains

$$J_{0,4} = O(\varepsilon^{1+4\alpha})\mathbf{1}_{0<\alpha<\frac{3}{8}} + O(\varepsilon^{\frac{5}{2}} ln(\frac{1}{\varepsilon}))\mathbf{1}_{\alpha=\frac{3}{8}} + O(\varepsilon^{4-4\alpha})\mathbf{1}_{\frac{3}{8}<\alpha<\frac{3}{4}} = o(\varepsilon).$$

Using the same type of arguments as for (C), (D), (E) we can prove that the other terms are all $o(\varepsilon)$.
Using (8), (9) and (12) we have shown that $\lim\limits_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[S_1^2\right] = C^2_{\alpha,\varphi}\int_{[0,1]^2}\mathbb{E}\left[h^2(b_\alpha(u))\right]\, du.$
Note that if $\alpha \le \frac{1}{2}$, we must only make the Taylor development of h until $\ddot{h}$ and a similar proof gives the result (for $\alpha = \frac{1}{2}$, we use furthermore the fact that for $i = 1, 2$, $\varepsilon^{-\frac{1}{2}}A_{i,\varepsilon}(u,v)$ and $\varepsilon^{-\frac{1}{2}}B_{i,\varepsilon}(u,v)$ have a limit when ε goes to zero, and then by the Lebesgue's dominated convergence theorem, $\varepsilon^{-3/2}J_{1,2}$, $\varepsilon^{-1}J_{0,2}$ and $\varepsilon^{-1}J_{1,1}$ have a limit that is zero since $\mathbb{E}\left[N^* g^{(2)}(N^*)\right] = \mathbb{E}\left[g^{(2)}(N^*)\right] = 0$).

Now for S_2, we write the Taylor development of h

$$h(b^\varepsilon_\alpha(u)) - h(b_\alpha(u)) \;=\; (b^\varepsilon_\alpha(u) - b_\alpha(u))\,\dot{h}(b_\alpha(u)) + \frac{1}{2}(b^\varepsilon_\alpha(u) - b_\alpha(u))^2\,\ddot{h}(\theta_\varepsilon(u)),$$

with $\theta_\varepsilon(u)$ between $b^\varepsilon_\alpha(u)$ and $b_\alpha(u)$.
To this development correspond two terms $S_{2,i}$, $i = 1, 2$.
Consider the first one, let $S_{2,2}$,

$$S_{2,2} = \frac{1}{2}\int_0^1 (b^\varepsilon_\alpha(u) - b_\alpha(u))^2\,\ddot{h}(\theta_\varepsilon(u))\, du.$$

$$\varepsilon^{-4\alpha}\,\mathbb{E}\left[S^2_{2,2}\right] = \frac{1}{4}\,c^2_{\alpha,\varphi}\int_0^1\int_0^1 \mathbb{E}\left[B^2_\varepsilon(u)B^2_\varepsilon(v)\ddot{h}(\theta_\varepsilon(u))\ddot{h}(\theta_\varepsilon(v))\right] du\, dv,$$

with

$$B_\varepsilon(u) = \frac{b^\varepsilon_\alpha(u) - b_\alpha(u)}{\varepsilon^\alpha\sqrt{c_{\alpha,\varphi}}} \to N(0,1) \text{ and } c_{\alpha,\varphi} = \mathbb{V}\left[\int_{-\infty}^{\infty}\varphi(x)\, b_\alpha(x)\, dx\right].$$

A computation shows that $(B_\varepsilon(u), B_\varepsilon(v), \theta_\varepsilon(u), \theta_\varepsilon(v))$ converges weakly when ε goes to zero towards $(B(u), B(v), b_\alpha(u), b_\alpha(v))$ where $B(u)$ and $B(v)$ are standard Gaussian independent variables; furthermore $(b_\alpha(u), b_\alpha(v))$ is independent of $(B(u), B(v))$.
Using the Lebesgue's dominated convergence theorem we get

$$\varepsilon^{-4\alpha}\,\mathbb{E}\left[S_{2,2}^2\right] \to \frac{1}{4}\,c_{\alpha,\varphi}^2\mathbb{E}\left[\int_0^1 \ddot{h}(b_\alpha(u))\,du\right]^2. \tag{13}$$

Now let us consider $S_{2,1}$.

$$S_{2,1} = \int_0^1 (b_\alpha^\varepsilon(u) - b_\alpha(u))\,\dot{h}(b_\alpha(u))\,du = \varepsilon^\alpha\sqrt{c_{\alpha,\varphi}}\int_0^1 B_\varepsilon(u)\,\dot{h}(b_\alpha(u))\,du,$$

so

$$\mathbb{E}\left[S_{2,1}^2\right] \;=\; \varepsilon^{2\alpha}c_{\alpha,\varphi}\int_0^1\int_0^1 \mathbb{E}\left[B_\varepsilon(u)B_\varepsilon(v)\dot{h}(b_\alpha(u))\dot{h}(b_\alpha(v))\right]\,du\,dv. \tag{14}$$

Applying the Lebesgue's dominated convergence theorem we get, with the same notations as before

$$\varepsilon^{-2\alpha}\,\mathbb{E}\left[S_{2,1}^2\right] \to \int_0^1\int_0^1 \mathbb{E}\left[B(u)\right]\mathbb{E}\left[B(v)\right]\mathbb{E}\left[\dot{h}(b_\alpha(u))\dot{h}(b_\alpha(v))\right]\;\mathrm{d}u\,\mathrm{d}v = 0,$$

and then

$$\mathbb{E}\left[S_{2,1}^2\right] = o(\varepsilon),\text{ if }\;\alpha \geq \frac{1}{2}. \tag{15}$$

Using (13) and (15), we have then proved that if $\alpha \geq \frac{1}{2}$, $h \in C^2$ and $|\ddot{h}(x)| \leq P(|x|)$, $\mathbb{E}\left[S_2^2\right] = o(\varepsilon)$.
Now let $\alpha < \frac{1}{2}$, using (14) one gets

$$\mathbb{E}\left[S_{2,1}^2\right] = \int_{C_\varepsilon} + \int_{C_\varepsilon^c} = K_1 + K_2,$$

where C_ε was defined before. It is obvious that

$$|K_2| \leq \mathbf{C}\,\varepsilon^{1+2\alpha}. \tag{16}$$

Now we look at K_1.
We fix u and v and consider the change of variables

$$\begin{aligned} b_\alpha(u) &= \alpha_1 B_\varepsilon(u) + \alpha_2 B_\varepsilon(v) + \alpha_3 Z_3, \\ b_\alpha(v) &= \beta_1 B_\varepsilon(u) + \beta_2 B_\varepsilon(v) + \beta_3 Z_3 + \beta_4 Z_4, \end{aligned}$$

with (Z_3, Z_4) standard Gaussian vector independent of $(B_\varepsilon(u), B_\varepsilon(v))$.
A simple calculus gives

$$\alpha_1 = \frac{a_\varepsilon(u) - \rho_\varepsilon(u,v)b_\varepsilon(u,v)}{\Delta_\varepsilon(u,v)},$$

$$\alpha_2 = \frac{b_\varepsilon(u,v) - \rho_\varepsilon(u,v)a_\varepsilon(u)}{\Delta_\varepsilon(u,v)},$$

$$\alpha_3^2 = \mathbb{E}\left[b_\alpha^2(u)\right] - \alpha_1^2 - \alpha_2^2 - 2\alpha_1\alpha_2\rho_\varepsilon(u,v),$$

$$\beta_1 = \frac{b_\varepsilon(v,u) - \rho_\varepsilon(u,v)a_\varepsilon(v)}{\Delta_\varepsilon(u,v)},$$

$$\beta_2 = \frac{a_\varepsilon(v) - \rho_\varepsilon(u,v)b_\varepsilon(v,u)}{\Delta_\varepsilon(u,v)},$$

$$\beta_3 = \frac{\mathbb{E}\left[b_\alpha(u)b_\alpha(v)\right] - \alpha_1\beta_1 - (\alpha_1\beta_2 + \alpha_2\beta_1)\rho_\varepsilon(u,v) - \alpha_2\beta_2}{\alpha_3},$$

and

$$\beta_4^2 = \mathbb{E}\left[b^2(v)\right] - \beta_1^2 - \beta_2^2 - \beta_3^2 - 2\beta_1\beta_2\rho_\varepsilon(u,v),$$

where $a_\varepsilon(u) = \mathbb{E}\left[b_\alpha(u)B_\varepsilon(u)\right]$, $\rho_\varepsilon(u,v) = \mathbb{E}\left[B_\varepsilon(u)B_\varepsilon(v)\right]$, $b_\varepsilon(u,v) = \mathbb{E}\left[b_\alpha(u)B_\varepsilon(v)\right]$ and $\Delta_\varepsilon(u,v) = 1 - \rho_\varepsilon^2(u,v)$.
We can show that for M big enough and $\varepsilon \leq \varepsilon(M)$,

$$|\rho_\varepsilon(u,v)| \leq \mathbf{C}\,\varepsilon^{2-2\alpha}\,|u-v|^{2\alpha-2}, \tag{17}$$

$$\max_{i=1,2}(|\alpha_i|,|\beta_i|) \leq \mathbf{C}\,\varepsilon^{\alpha} \qquad \text{and} \qquad \max_{i=3,4}(|\beta_i|,|\alpha_3|) \leq \mathbf{C}. \tag{18}$$

Furthermore we have the following limits

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\alpha_1}{\varepsilon^\alpha} = \lim_{\varepsilon\to 0}\frac{\beta_2}{\varepsilon^\alpha} = \frac{-v_{2\alpha}^2[\int_{-\infty}^{+\infty}\varphi(u)|u|^{2\alpha}du]}{2\sqrt{c_{\alpha,\varphi}}}, \\
\lim_{\varepsilon\to 0}\frac{\alpha_2}{\varepsilon^\alpha} = \lim_{\varepsilon\to 0}\frac{\beta_1}{\varepsilon^\alpha} = 0, \\
\lim_{\varepsilon\to 0}\alpha_3 = \sqrt{\mathbb{E}\left[b_\alpha^2(u)\right]}, \qquad \lim_{\varepsilon\to 0}\beta_3 = \frac{\mathbb{E}\left[b_\alpha(u)b_\alpha(v)\right]}{\sqrt{\mathbb{E}\left[b_\alpha^2(u)\right]}} \qquad \text{and} \\
\lim_{\varepsilon\to 0}\beta_4^2 = \frac{\mathbb{E}\left[b_\alpha^2(u)\right]\mathbb{E}\left[b_\alpha^2(v)\right] - (\mathbb{E}\left[b_\alpha(u)b_\alpha(v)\right])^2}{\mathbb{E}\left[b_\alpha^2(u)\right]}.
\end{aligned} \tag{19}$$

So

$$\begin{aligned}
K_1 &= \varepsilon^{2\alpha}c_{\alpha,\varphi}\int_{C_\varepsilon}\int_{\mathbb{R}^4} xy\,\dot{h}(\alpha_1 x + \alpha_2 y + \alpha_3 z)\,\dot{h}(\beta_1 x + \beta_2 y + \beta_3 z + \beta_4 w) \\
&\times\ p_{B_\varepsilon(u),B_\varepsilon(v)}(x,y)\,\phi(z)\,\phi(w)\,dz\,dw\,dx\,dy\,du\,dv,
\end{aligned}$$

where $p_{B_\varepsilon(u),B_\varepsilon(v)}(x,y)$ stands for the density of vector $(B_\varepsilon(u),B_\varepsilon(v))$ in (x,y).

Writing the third order Taylor development for h one has

$$\dot{h}(\alpha_1 x+\alpha_2 y+\alpha_3 z) = \dot{h}(\alpha_3 z)+(\alpha_1 x+\alpha_2 y)\,\ddot{h}(\alpha_3 z)+\frac{1}{2}(\alpha_1 x+\alpha_2 y)^2\, h^{(3)}(\theta_1),$$

and

$$\begin{aligned}\dot{h}(\beta_1 x+\beta_2 y+\beta_3 z+\beta_4 w) &= \dot{h}(\beta_3 z+\beta_4 w)\;+(\beta_1 x+\beta_2 y)\,\ddot{h}(\beta_3 z+\beta_4 w)\\ &+ \frac{1}{2}(\beta_1 x+\beta_2 y)^2\, h^{(3)}(\theta_2),\end{aligned}$$

with θ_1 between $\alpha_3 z$ and $(\alpha_1 x+\alpha_2 y+\alpha_3 z)$ and θ_2 between $(\beta_3 z+\beta_4 w)$ and $(\beta_1 x+\beta_2 y+\beta_3 z+\beta_4 w)$.

Therefore K_1 is decomposed as the sum of nine terms.

(A) One term of the type

$$\begin{aligned}K_{1,1} &= \varepsilon^{2\alpha} c_{\alpha,\varphi}\int_{C_\varepsilon}\int_{\mathbb{R}^4} xy\,\dot{h}(\alpha_3 z)\,\dot{h}(\beta_3 z+\beta_4 w) p_{B_\varepsilon(u),B_\varepsilon(v)}(x,y)\,\phi(z)\,\phi(w)\,dz\,dw\,dx\,dy\,du\,dv\\ &= \varepsilon^{2\alpha} c_{\alpha,\varphi}\int_{C_\varepsilon}\rho_\varepsilon(u,v)\int_{\mathbb{R}^2}\dot{h}(\alpha_3 z)\,\dot{h}(\beta_3 z+\beta_4 w)\,\phi(z)\,\phi(w)\,dz\,dw\,du\,dv.\end{aligned}$$

By (17) and (18), we get

$$\begin{aligned}|K_{1,1}| &\leq \mathbf{C}\,\varepsilon^{2\alpha}\int_{C_\varepsilon}|\rho_\varepsilon(u,v)|\,du\,dv\\ &\leq \mathbf{C}\,\varepsilon^{2}\int_{C_\varepsilon}|u-v|^{2\alpha-2}\,du\,dv,\end{aligned}$$

proving that

$$K_{1,1}=O(\varepsilon^{1+2\alpha})=o(\varepsilon).$$

(B) Two terms of the type

$$K_{1,2}=\varepsilon^{2\alpha} c_{\alpha,\varphi}\int_{C_\varepsilon}\int_{\mathbb{R}^4} xy\,(\beta_1 x+\beta_2 y)\,\dot{h}(\alpha_3 z)\,\ddot{h}(\beta_3 z+\beta_4 w)$$
$$\times p_{B_\varepsilon(u),B_\varepsilon(v)}(x,y)\,\phi(z)\,\phi(w)\,dz\,dw\,dx\,dy\,du\,dv,$$

and this term is zero by Mehler's formula (3).

(C) Two terms of the type

$$K_{1,3}=c_{\alpha,\varphi}\,\varepsilon^{2\alpha}\int_{C_\varepsilon}\int_{\mathbb{R}^4} xy\,\frac{1}{2}\,(\beta_1 x+\beta_2 y)^2\,\dot{h}(\alpha_3 z)\,h^{(3)}(\theta_2)$$
$$\times p_{B_\varepsilon(u),B_\varepsilon(v)}(x,y)\,\phi(z)\,\phi(w)\,dz\,dw\,dx\,dy\,du\,dv.$$

By (18) and (19), we can apply the Lebesgue's dominated convergence theorem getting

$$\lim_{\varepsilon\to 0}\varepsilon^{-4\alpha}K_{1,3}=\mathbf{C}\int_{[0,1]^2}\int_{\mathbb{R}^2}\frac{1}{2}\,xy^3\,\mathbb{E}\left[\dot{h}(b_\alpha(u))h^{(3)}(b_\alpha(v))\right]\,\phi(x)\,\phi(y)\,dx\,dy\,du\,dv=0.$$

(D) One term of the type

$$\begin{aligned} K_{1,4} &= \varepsilon^{2\alpha}c_{\alpha,\varphi}\int_{C_\varepsilon}\int_{\mathbb{R}^4} xy\,(\alpha_1 x+\alpha_2 y)\,(\beta_1 x+\beta_2 y)\,\ddot{h}(\alpha_3 z)\,\ddot{h}(\beta_3 z+\beta_4 w)\\ &\times\ p_{B_\varepsilon(u),B_\varepsilon(v)}(x,y)\,\phi(z)\,\phi(w)\,dz\,dw\,dx\,dy\,du\,dv. \end{aligned}$$

As for (C), we can apply the Lebesgue's dominated convergence theorem and

$$\lim_{\varepsilon\to 0}\varepsilon^{-4\alpha}K_{1,4}=\mathbf{C}\,\mathbb{E}\left[\int_0^1\ddot{h}(b_\alpha(u))\,du\right]^2, \tag{20}$$

with $\mathbf{C}=\left(\frac{-v_{2\alpha}^2}{2}\,(\int_{-\infty}^{\infty}\varphi(u)|u|^{2\alpha}\,du)\right)^2$.
(E) Two terms of the type

$$\begin{aligned} K_{1,5} &= \varepsilon^{2\alpha}\frac{1}{2}\,c_{\alpha,\varphi}\int_{C_\varepsilon}\int_{\mathbb{R}^4} xy\,(\alpha_1 x+\alpha_2 y)\,(\beta_1 x+\beta_2 y)^2\,\ddot{h}(\alpha_3 z)\,h^{(3)}(\theta_2)\\ &\times\ p_{B_\varepsilon(u),B_\varepsilon(v)}(x,y)\,\phi(z)\,\phi(w)\,dz\,dw\,dx\,dy\,du\,dv, \end{aligned}$$

and with the same arguments as before $K_{1,5}=O(\varepsilon^{5\alpha})$.
(F) One term of the type

$$\begin{aligned} K_{1,6} &= \varepsilon^{2\alpha}\frac{1}{4}\,c_{\alpha,\varphi}\int_{C_\varepsilon}\int_{\mathbb{R}^4} xy\,(\alpha_1 x+\alpha_2 y)^2\,(\beta_1 x+\beta_2 y)^2\,h^{(3)}(\theta_1)\,h^{(3)}(\theta_2)\\ &\times\ p_{B_\varepsilon(u),B_\varepsilon(v)}(x,y)\,\phi(z)\,\phi(w)\,dz\,dw\,dx\,dy\,du\,dv. \end{aligned}$$

As previous cases $K_{1,6}=O(\varepsilon^{6\alpha})$.
We have then proved that if $\alpha<\frac{1}{2}$, $K_1=O(\varepsilon^{4\alpha})+o(\varepsilon)$ and then using (13) and (16) that $\mathbb{E}\,[S_2]^2=O(\varepsilon^{4\alpha})+o(\varepsilon)$.

Furthermore using (16) and (20) we have proved that if $0<\alpha<\frac{1}{4}$,

$$\lim_{\varepsilon\to 0}\varepsilon^{-4\alpha}\mathbb{E}\left[S_{2,1}^2\right]=\left(\frac{-v_{2\alpha}^2}{2}\,(\int_{-\infty}^{\infty}\varphi(u)|u|^{2\alpha}\,du)\right)^2\mathbb{E}\left[\int_0^1\ddot{h}(b_\alpha(u))\,du\right]^2. \tag{21}$$

To obtain the asymptotic behaviour of $\mathbb{E}\,[S_2^2]$ when $0<\alpha<\frac{1}{4}$, we have, by (13) and (21), to compute $\mathbb{E}\,[S_{2,1}S_{2,2}]$.
With an argument similar to the one used before, we obtain for $0<\alpha<\frac{1}{4}$,

$$\lim_{\varepsilon\to 0}2\varepsilon^{-4\alpha}\mathbb{E}\,[S_{2,1}S_{2,2}]=\left(\frac{-v_{2\alpha}^2}{2}\,c_{\alpha,\varphi}\,(\int_{-\infty}^{\infty}\varphi(u)|u|^{2\alpha}\,du)\right)\mathbb{E}\left[\int_0^1\ddot{h}(b_\alpha(u))\,du\right]^2, \tag{22}$$

and then using (13), (21) and (22) we proved that

$$\lim_{\varepsilon\to 0}\varepsilon^{-4\alpha}\mathbb{E}\left[S_2^2\right]=K^2_{\alpha,\varphi}\mathbb{E}\left[\int_0^1\ddot{h}(b_\alpha(u))\,du\right]^2,\ \text{if}\ \ 0<\alpha<\frac{1}{4}.$$

Now let us achieve the proof of (i), proving that $\varepsilon^{-2\alpha}S_2\xrightarrow{L^2}K_{\alpha,\varphi}\int_0^1\ddot{h}(b_\alpha(u))\,du$. It is enough to show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[(\varepsilon^{-2\alpha}S_2)\int_0^1\ddot{h}(b_\alpha(u))\,du\right]=K_{\alpha,\varphi}\,\mathbb{E}\left[\int_0^1\ddot{h}(b_\alpha(u))\,du\right]^2$$

For this we write the second order Taylor development for h and we study the two inner corresponding integrals, and doing the same computations as before it yields (i).
Now, to achieve the proof of the theorem, we consider, for $\frac{1}{4}<\alpha<\frac{3}{4}$, a discrete version of

$$T_1:=\frac{1}{\sqrt{\varepsilon}}\int_0^1 h(b^\varepsilon_\alpha(u))g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u,$$

defining

$$\begin{aligned}Z^n_\varepsilon(h)&=\frac{1}{\sqrt{\varepsilon}}\sum_{i=1}^n h(b^\varepsilon_\alpha(\frac{i-1}{n}))\int_{\frac{i-1}{n}}^{\frac{i}{n}}g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u,\\ \text{and}\qquad Z^n(h)&=C_{\alpha,\varphi}\sum_{i=1}^n h(b_\alpha(\frac{i-1}{n}))[\hat{W}(\frac{i}{n})-\hat{W}(\frac{i-1}{n})].\end{aligned}$$

We know by Theorem 3.1 2)(a) that $Z^n_\varepsilon(h)\to Z^n(h)$, weakly as $\varepsilon\to 0$. On the other hand $Z^{2^n}(h)$ is a Cauchy sequence in $L^2(\Omega)$, this implies that there exists a r.v. $Y(h)\in L^2(\Omega)$ such that $Z^{2^n}(h)\to Y(h)$ in $L^2(\Omega)$ as $n\to\infty$; furthermore, we can characterize this variable using the asymptotic independence between $b_\alpha(\cdot)$ and $\hat{W}(\cdot)$, say

$$\mathcal{L}\Big(Y(h)\Big/b_\alpha(s),0\le s\le 1\Big)=N\Big(0;C^2_{\alpha,\varphi}\int_0^1 h^2(b_\alpha(u))\,\mathrm{d}u\Big).\tag{23}$$

To finish the proof it is enough to show

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\ \mathbb{E}\left[T_1-Z^n_\varepsilon(h)\right]^2=0.$$

Such a proof goes on using the same technics that we have implemented above, for the asymptotic of the second moment. □

5.2.2 Pseudo-diffusion

Estimation of the variance of a pseudo-diffusion.
Proof of Proposition 4.1. We just give an outline of the proof showing that it is enough to consider the fractional Brownian motion case. Because $b_\alpha(t)$ has zero quadratic variation when $\alpha > \frac{1}{2}$, it turns out that when $\sigma \in C^1$ and $\mu \equiv 0$ the solution for the stochastic differential equation can be expressed as $X(t) = K(b_\alpha(t))$, for $t \geq 0$, where $K(t)$ is the solution of the ordinary differential equation

$$\dot{K}(t) = \sigma(K(t)); \quad K(0) = c.$$

(for $t < 0$, $X(t) = c$). Using the Banach-Kac formula (Banach (1925) and Kac (1943)) we have

$$\begin{aligned}
\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_{-\infty}^{\infty} h(x)N_\varepsilon^X(x)\,\mathrm{d}x &= \sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_0^1 h(X_\varepsilon(u))|\dot{X}_\varepsilon(u)|\,\mathrm{d}u \\
&\approx \sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_0^1 h(K(b_\alpha^\varepsilon(u)))\dot{K}(b_\alpha^\varepsilon(u))|\dot{b}_\alpha^\varepsilon(u)|\,\mathrm{d}u \\
&= \sqrt{\frac{\pi}{2}}\int_0^1 h(K(b_\alpha^\varepsilon(u)))\sigma(K(b_\alpha^\varepsilon(u)))|Z_\varepsilon(u)|\,\mathrm{d}u.
\end{aligned}$$

We shall prove in Theorem 4.1 that $[\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_0^1 h(X_\varepsilon(u))|\dot{X}_\varepsilon(u)|\,\mathrm{d}u - \int_0^1 h(K(b_\alpha^\varepsilon(u)))\sigma(K(b_\alpha^\varepsilon(u)))|Z_\varepsilon(u)|\,\mathrm{d}u]$ is $o(\sqrt{\varepsilon})$, hence the proposition follows from Theorem 1.1. □

Proof of Theorem 4.1. For any $\delta > 0$, if $t \geq \varepsilon$,

$$\begin{aligned}
|X_\varepsilon(t) - K(b_\alpha^\varepsilon(t))| &\leq \int_{-\infty}^{\infty} \varphi(x)|K(b_\alpha(t-\varepsilon x)) - K(b_\alpha^\varepsilon(t))|\,\mathrm{d}x \\
&= \int_{-1}^{1} \varphi(v)|\dot{K}(\theta)(b_\alpha(t-\varepsilon v) - b_\alpha^\varepsilon(t))|\,\mathrm{d}v \\
&\leq \mathbf{C} \sup_{v\in[-1,1]} |b_\alpha(t-\varepsilon v) - b_\alpha^\varepsilon(t)| \\
&\leq \mathbf{C}\ \varepsilon^{\alpha-\delta},
\end{aligned}$$

where θ is a point between $b_\alpha(t-\varepsilon v)$ and $b_\alpha^\varepsilon(t)$. A similar proof can be done for $0 \leq t < \varepsilon$. Hence, we get $\varepsilon^{-\frac{1}{2}}|X_\varepsilon(t) - K(b_\alpha^\varepsilon(t))| = o(1)$ uniformly in t.
In a similar way, we can write for any $\delta > 0$ and $t \geq \varepsilon$
$\varepsilon(\dot{X}_\varepsilon(t) - \dot{K}(b_\alpha^\varepsilon(t))\dot{b}_\alpha^\varepsilon(t))$

$$= \int_{-\infty}^{\infty} \dot{\varphi}(x)[K(b_\alpha(t-\varepsilon x)) - K(b_\alpha^\varepsilon(t)) - \dot{K}(b_\alpha^\varepsilon(t))(b_\alpha(t-\varepsilon x) - b_\alpha^\varepsilon(t))]\,\mathrm{d}x,$$

we used the fact that $\int_{-\infty}^{\infty}\dot{\varphi}(x)\,\mathrm{d}x = 0$. Taking the second order Taylor's development for the function K, we obtain

$$|\varepsilon(\dot{X}_\varepsilon(t) - \dot{K}(b_\alpha^\varepsilon(t))\dot{b}_\alpha^\varepsilon(t))| \leq \frac{1}{2}\int_{-1}^{1} |\dot{\varphi}(v)||\ddot{K}(\theta)|(b_\alpha(t-\varepsilon v) - b_\alpha^\varepsilon(t))^2\,\mathrm{d}v \leq \mathbf{C}\ \varepsilon^{2\alpha-\delta}.$$

In a similar way, for $0 \leq t < \varepsilon$, we can prove that this expression is bounded by $\mathbf{C}\ \varepsilon^{\alpha-\delta}$. Thus, multiplying the last expression by $\varepsilon^{-(\alpha+\frac{1}{2})}$ it holds uniformly in t

$$\varepsilon^{(\frac{1}{2}-\alpha)}(\dot{X}_\varepsilon(t) - \dot{K}(b_\alpha^\varepsilon(t))\dot{b}_\alpha^\varepsilon(t)) = o(1)\ \mathbf{1}_{\varepsilon\leq t} + O(\varepsilon^{-\frac{1}{2}-\delta})\ \mathbf{1}_{0\leq t\leq\varepsilon}.$$

Now

$$\frac{1}{\sqrt{\varepsilon}}[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_0^1 h(X_\varepsilon(u))|\dot{X}_\varepsilon(u)|\,\mathrm{d}u - \sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_0^1 h(K(b_\alpha^\varepsilon(u)))\sigma(K(b_\alpha^\varepsilon(u)))|\dot{b}_\alpha^\varepsilon(u)|\,\mathrm{d}u]$$
$$= L_1 + L_2,$$

where

$$L_1 := \int_0^1 \frac{1}{\sqrt{\varepsilon}}\Big[h(X_\varepsilon(u)) - h(K(b_\alpha^\varepsilon(u)))\Big]\Big|\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\dot{X}_\varepsilon(u)\Big|\,\mathrm{d}u$$

and

$$L_2 := \frac{1}{\sqrt{\varepsilon}}\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_0^1 h(K(b_\alpha^\varepsilon(u)))\{|\dot{X}_\varepsilon(u)| - \sigma(K(b_\alpha^\varepsilon(u)))|\dot{b}_\alpha^\varepsilon(u)|\}\,\mathrm{d}u.$$

Now, let us study L_1 and L_2. For L_1, we have

$$L_1 = \int_0^1 \frac{1}{\sqrt{\varepsilon}}\dot{h}(\theta')\Big(X_\varepsilon(u) - K(b_\alpha^\varepsilon(u))\Big)\Big|\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\dot{X}_\varepsilon(u)\Big|\,\mathrm{d}u,$$

where θ' is a point between $X_\varepsilon(u)$ and $K(b_\alpha^\varepsilon(u))$ and then

$$|L_1| \leq \mathbf{C}\int_0^1 \varepsilon^{-\frac{1}{2}}\Big|X_\varepsilon(u) - K(b_\alpha^\varepsilon(u))\Big|\Big|\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\dot{X}_\varepsilon(u)\Big|\,\mathrm{d}u = o(1),$$

because of the boundness of $\displaystyle\int_0^1\Big|\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\dot{X}_\varepsilon(u)\Big|\,\mathrm{d}u$.

Note that this last remark and the fact that $h \in C^0$ imply that $\sqrt{\varepsilon}L_1$ tends to zero when ε goes to zero.

Moreover, for L_2 we have

$$|L_2| \leq \mathbf{C}\int_0^1 \varepsilon^{\frac{1}{2}-\alpha}|\dot{X}_\varepsilon(u) - \sigma(K(b_\alpha^\varepsilon(u)))\dot{b}_\alpha^\varepsilon(u)|\,\mathrm{d}u = o(1).$$

Therefore, we can conclude that the asymptotic behaviour of

$$\frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-\alpha}}{\sigma_{2\alpha}}\int_{-\infty}^{\infty} h(x)N_\varepsilon^X(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} h(x)\sigma(x)\ell^X(x)\,\mathrm{d}x\right],$$

is equivalent to the asymptotic behaviour of $\Sigma_\varepsilon\Big((h\circ K)\cdot(\sigma\circ K)\Big)$.

As an application of our result for the fractional Brownian motion in Theorem 3.4 (ii), this term converges stably towards

$$C_{\alpha,\varphi}\int_0^1 h(K(b_\alpha(u)))\sigma(K(b_\alpha(u)))d\widehat{W}(u) = C_{\alpha,\varphi}\int_0^1 h(X(u))\sigma(X(u))d\widehat{W}(u).$$

This equation completes the proof. □

Remark: We conjecture that the same type of result holds for $\mu \neq 0$, but for the moment we do not have a proof of this statement.

Proofs of hypothesis

Proof of Theorem4.2. Let $K(t,\delta)$ be the C^2-function, solution of the ordinary differential equation

$$\frac{\partial K}{\partial t}(t,\delta)=\sigma(K(t,\delta),\delta) \text{ where } \sigma(u,\delta)=\sigma_0(u)+\delta d(u)+\delta F(u,\delta) \text{ and } \ K(0,\delta)=c.$$

Then for $t\geq 0$, $X_\varepsilon(t)=K(b_\alpha(t),\sqrt{\varepsilon})$ (for $t<0$, $X_\varepsilon(t)=c$) and almost surely, uniformly for $t\geq 0$ in a compact (see Pontryagin (1962), section "local theorems of continuity and differentiability of solutions" p.170-180)

$$\lim_{\varepsilon\to 0} X_\varepsilon(t)=X(t) \text{ where } X(t)=K(b_\alpha(t),0),$$

and

$$\frac{\partial K}{\partial t}(t,0)=\sigma_0(K(t,0)) \text{ with } K(0,0)=c.$$

Furthermore, we can prove that almost surely

$$\lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{2}}(X_\varepsilon(t)-X(t))=\frac{\partial K}{\partial \delta}(b_\alpha(t),0) \text{ uniformly for } t\geq 0 \text{ in a compact.} \tag{24}$$

From now on, K will denote the function defined by $K(t):=K(t,0)$ and then $X(t)=K(b_\alpha(t))$ for $t\geq 0$.

Using that h, σ and σ_0 are in C^1, it holds uniformly for $t\in[0,1]$ that

$$\varepsilon^{-\frac{1}{2}}|Y_\varepsilon(t)-X_\varepsilon(t)|=o(1),$$

$$\varepsilon^{(\frac{1}{2}-\alpha)}|\dot{Y}_\varepsilon(t)-\sigma_\varepsilon(X_\varepsilon(t))\,\dot{b}^\varepsilon_\alpha(t)|=o(1)\ \mathbf{1}_{\varepsilon\leq t}+O(\varepsilon^{-\frac{1}{2}-\delta})\ \mathbf{1}_{0\leq t\leq\varepsilon}.$$

Thus as in in Theorem 4.1, we can show that the equivalent functional under the alternatives is

$$\begin{aligned}
\frac{1}{\sqrt{\varepsilon}}&\left[\int_0^1 h(X_\varepsilon(u))\sigma_\varepsilon(X_\varepsilon(u))\sqrt{\frac{\pi}{2}}|Z_\varepsilon(u)|\,\mathrm{d}u-\int_0^1 h(X_\varepsilon(u))\sigma_0(X_\varepsilon(u))\,\mathrm{d}u\right]\\
&= \frac{1}{\sqrt{\varepsilon}}\int_0^1 h(X_\varepsilon(u))\sigma_0(X_\varepsilon(u))g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u\\
&\quad+\int_0^1 h(X_\varepsilon(u))d(X_\varepsilon(u))\sqrt{\frac{\pi}{2}}|Z_\varepsilon(u)|\,\mathrm{d}u\\
&\quad+\int_0^1 h(X_\varepsilon(u))F(X_\varepsilon(u),\sqrt{\varepsilon})\sqrt{\frac{\pi}{2}}|Z_\varepsilon(u)|\,\mathrm{d}u\\
&= M_1+M_2+M_3,
\end{aligned}$$

with $g^{(2)}(x)=\sqrt{\frac{\pi}{2}}|x|-1$.
We are going to prove that

$$T_\varepsilon(h)\simeq\frac{1}{\sqrt{\varepsilon}}\int_0^1 h(X(u))\sigma_0(X(u))g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u+\int_0^1 h(X(u))d(X(u))\,\mathrm{d}u. \tag{25}$$

Since almost-surely $X_\varepsilon(u)$ converges uniformly for $u \in [0,1]$ to $X(u)$ when ε goes to zero, and h, F are in C^0, $F(\cdot,0)=0$ and moreover $\int_0^1 |Z_\varepsilon(u)|\,\mathrm{d}u$ is bounded, we get that M_3 goes almost-surely to zero with ε.

Let us look now at M_1. Since $(h\sigma_0)$ is in C^1, making a first order Taylor development for this function we get

$$J_1 = \frac{1}{\sqrt{\varepsilon}} \int_0^1 h(X(u))\,\sigma_0(X(u))\,g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u$$
$$+ \int_0^1 \Big[(h\sigma_0)'(\theta_\varepsilon(u))\Big(\frac{X_\epsilon(u)-X(u)}{\sqrt{\varepsilon}}\Big) - (h\sigma_0)'(X(u))\frac{\partial K}{\partial \delta}(b_\alpha(u),0)\Big]\,g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u$$
$$+ \int_0^1 (h\sigma_0)'(K(b_\alpha(u))\frac{\partial K}{\partial \delta}(b_\alpha(u),0)\,g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u,$$

where the symbol $'$ stands for the derivative and $\theta_\varepsilon(u)$ is a point between $X_\varepsilon(u)$ and $X(u)$.

Using the facts that $(h\sigma_0)'$ is in C^0, that almost-surely $\theta_\varepsilon(u)$ converges uniformly in u to $X(u)$ when ε goes to zero, (24) and that $\int_0^1 |g^{(2)}(Z_\varepsilon(u))|\,\mathrm{d}u$ is bounded, we can prove that the second integral almost-surely goes to zero when ε goes to zero.

Now, by using the fact that $(h\sigma_0)'\,K(\cdot)\,\dfrac{\partial K}{\partial \delta}(\cdot,0)$ is in C^0 and using a generalization of Theorem 1.1, we can prove that almost-surely as ε goes to zero the last integral goes to $\mathbb{E}\left[g^{(2)}(N^*)\right] \displaystyle\int_0^1 (h\sigma_0)'(K(b_\alpha(u))\frac{\partial K}{\partial \delta}(b_\alpha(u),0)\,\mathrm{d}u \equiv 0$.

To finish with the proof of (25) we look at M_2.

$$M_2 = \int_0^1 \Big(h(X_\varepsilon(u))\,d(X_\varepsilon(u)) - h(X(u))\,d(X(u))\Big)\sqrt{\frac{\pi}{2}}\,|Z_\varepsilon(u)|\,\mathrm{d}u$$
$$+ \int_0^1 h(K(b_\alpha(u)))\,d(K(b_\alpha(u)))\,g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u + \int_0^1 h(X(u))d(X(u))\,\mathrm{d}u.$$

As before, since almost-surely $X_\varepsilon(u)$ converges uniformly in u to $X(u)$ when ε goes to zero, h, d are in C^0 and that $\int_0^1 |Z_\varepsilon(u)|\,\mathrm{d}u$ is bounded, the first term tends almost-surely to zero with ε.

The second term tends to $\mathbb{E}\left[g^{(2)}(N^*)\right] \int_0^1 h(K(b_\alpha(u)))\,d(K(b_\alpha(u)))\,\mathrm{d}u \equiv 0$, since h, d and K are in C^0 and by using a generalization of Theorem 1.1.

Thus we have proved (25).

Now if we put $H := (h\circ K)\cdot(\sigma_0\circ K)$ and $G := (h\circ K)\cdot(d\circ K)$, we obtain

$$T_\varepsilon(h) \simeq \frac{1}{\sqrt{\varepsilon}} \int_0^1 H(b_\alpha(u))g^{(2)}(Z_\varepsilon(u))\,\mathrm{d}u + \int_0^1 G(b_\alpha(u))\,\mathrm{d}u.$$

The asymptotic behaviour of this functional can be treated in the same manner that we have done in Theorem 3.4 (ii). Note that in the argument of functions H and G it appears $b_\alpha(\cdot)$ instead of $b_\alpha^\varepsilon(\cdot)$. However the same type of proof can be done with small changes. $\square$

5.2.3 β-increments and Lebesgue measure

Proof of Corollary 4.1. This corollary is a consequence of Theorems 3.1 1), 3.2 (i) and 3.3. Indeed, since

$$\int_{-\infty}^{+\infty} H_{2l}(x)|x|^{\beta}\phi(x)\,\mathrm{d}x = \frac{2(2l)!}{\sqrt{2\pi}}\sum_{p=0}^{l}\frac{(-1)^{l-p}}{(2p)!(l-p)!2^{l-p}}2^{p+\frac{\beta-1}{2}}\Gamma(p+\frac{\beta+1}{2}),$$

(i) follows. To conclude the proof it's enough to compute a_2.
$a_2 = \frac{1}{2}\int_{-\infty}^{+\infty}|x|^{\beta}H_2(x)\phi(x)\,\mathrm{d}x = \beta\frac{2^{\beta/2-1}}{\sqrt{\pi}}\Gamma(\frac{\beta+1}{2})$ by the last calculation, thus (ii) and (iii) follow. □

Proof of Corollary 4.2. This corollary is a direct application of Theorem 3.1 1), 2)(c) and Theorem 3.3. In fact, using formula (4) we obtain that $a_l = \frac{-1}{l!}H_{l-1}(x)\phi(x)$ and the result follows.
The remark is a consequence of a straightforward calculation of $\int_0^{+\infty}\rho_{1/2}^{l}(v)\,\mathrm{d}v = \frac{1}{l+1}$.
□

Proof of Corollary 4.3. This corollary is a direct application of Corollary 3.1.1. □

6 Conclusion

It is interesting to pinpoint the main idea of our methods based on the Gaussian structure of the underlying processes and remark the similarity of the limits obtained in different models considered in Berzin-Joseph and León (1997), Berzin *et al.* (1998) or Berzin *et al.* (2001).
Also note that our technics allow us the parameters estimation and the setup of tests of hypothesis when the partition is finer.

Following the same approach, future investigations will be made in a more general setup where a drift is introduced in the model.
Other results are expected, related with the second order increments, to estimate the Hurst parameter α using variation technics.

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