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A Hölderian FCLT for some multiparameter summation process of independent non-identically distributed random variables

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Abstract

Let $\{X_{\boldsymbol{n},\boldsymbol{k}},\ \boldsymbol{k} \leq \boldsymbol{k_n}\ \ \boldsymbol{n}, \boldsymbol{k_n} \in \mathbb{N}^d\}$ be a triangular array of independent non-identically distributed random variables. Using a non uniform grid adapted to the variances of the $X_{\boldsymbol{n},\boldsymbol{k}}$'s, we introduce a new construction of a summation process $\xi_{\boldsymbol{n}}$ of the triangular array based on the collection of sets $[0, t_1] \times \cdots \times [0, t_d]$, $0 \leq t_i \leq 1$, $i = 1, \ldots, d$. We investigate its convergence in distribution in some Hölder spaces. It turns out that for $d \geq 2$ the limiting process is not necessarily the standard Brownian sheet. This contrasts with a classical result of Prokhorov for the case $d = 1$.

Key words: Brownian sheet, functional central limit theorem, Hölder space, invariance principle, triangular array, summation process.

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1 Introduction

Convergence of stochastic processes to some Brownian motion or related process is an important topic in probability theory and mathematical statistics. The first functional central limit theorem by Donsker and Prokhorov states the $C[0,1]$-weak convergence of $n^{-1/2}\xi_n$ to the standard Brownian motion W. Here ξ_n denotes the random polygonal line process indexed by $[0,1]$ with vertices $(k/n, S_k)$, $k = 0, 1, \ldots, n$ and $S_0 := 0$, $S_k := X_1 + \cdots + X_k$, $k \geq 1$, are the partial sums of a sequence $(X_i)_{i\geq 1}$ of i.i.d. random variables such that $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 = 1$.

If we use the same construction for triangular array $\{X_{n,k}, k = 1, .., k_n, n \in \mathbb{N}\}$, where for each $n \in \mathbb{N}$ $X_{n,k}$ are independent but non-identically distributed, then polygonal line process will have vertices $(k/k_n, S_n(k))$ with $S_n(k) = X_{n,1} + \cdots + X_{n,k}$. The variance at fixed time t of such process is

$$\mathbf{E}\xi_n(t)^2 = \sum_{k\leq[nt]} \mathbf{E}X_{n,k}^2.$$

It is not hard to construct the example of triangular array, such that the left hand side of this expression does not converge to t. Thus the limiting process for this construction is not necessarily the Brownian motion, which does not compare with case of i.i.d. variables, where for any i.i.d. sequence, the limiting process is always the Brownian motion. To solve this problem Prokhorov [5] introduced random polygonal line process Ξ_n indexed by $[0,1]$ with vertices $(b_n(k), S_n(k))$, where $b_n(k) = \mathbf{E}X_{n,1}^2 + \cdots + \mathbf{E}X_{n,k}^2$, with assumption that $b_n(k_n) = 1$. Prokhorov proved that Ξ_n converges to a standard Brownian motion if the triangular array satisfies the conditions of the central limit theorem. Note that this process coincides with $n^{-1/2}\xi_n$ in the special case where $X_{n,k} = n^{-1/2}X_k$, with i.i.d. X_k's.

The functional central limit theorem implies via continuous mapping the convergence in distribution of $f(\Xi_n)$ to $f(W)$ for any continuous functional $f : C[0,1] \to \mathbb{R}$. This yields many statistical applications. On the other hand, considering that the paths of Ξ_n are piecewise linear and that W has roughly speaking, an α-Hölder regularity for any exponent $\alpha < 1/2$, it is tempting to look for a stronger topological framework for the weak convergence of Ξ_n to W. In addition to the satisfaction of mathematical curiosity, the practical interest of such an investigation is to obtain a richer set of continuous functionals of the paths. For instance, Hölder norms of Ξ_n (in i.i.d. case) are closely related to some test statistics to detect short "epidemic" changes in the distribution of the X_i's, see [9; 10].

In 2003, Račkauskas and Suquet [7] obtained functional central limit theorem in the separable Banach spaces H_α^o, $0 < \alpha < 1/2$, of functions $x : [0,1] \to \mathbb{R}$ such that

$$\|x\|_\alpha := |x(0)| + \omega_\alpha(x,1) < \infty,$$

with

$$\omega_\alpha(x,\delta) := \sup_{0<|t-s|\leq\delta} \frac{|x(t) - x(s)|}{|t-s|^\alpha} \xrightarrow[\delta\to 0]{} 0.$$

Assuming infinitesimal negligibility of triangular array and moment condition

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (\mathbf{E}X_{n,k}^2)^{-\alpha q} \mathbf{E}(|X_{n,k}|^q) = 0 \qquad (1)$$

for $q > 1/(1/2-\alpha)$, they proved the weak convergence of Ξ_n to W in the Hölder space H_α^o for any $\alpha < 1/2 - 1/q$.

For general summation processes, the case of non-identically distributed variables was investigated by Goldie and Greenwood [2; 3]. General summation process $\{\xi_{\boldsymbol{n}}(A);\ A \in \mathscr{A}\}$ is defined for a collection $\mathscr{A}$ of Borel subsets of $[0,1]^d$ by

$$\xi_{\boldsymbol{n}}(A) = \sum_{\boldsymbol{1}\le \boldsymbol{j}\le \boldsymbol{n}} |R_{\boldsymbol{n},\boldsymbol{j}}|^{-1}|R_{\boldsymbol{n},\boldsymbol{j}} \cap A| X_{\boldsymbol{j}}, \tag{2}$$

with $\{X_{\boldsymbol{j}},\ \boldsymbol{j} \in \mathbb{N}^d\}$ independent but not identically distributed mean-zero random variables and where $\boldsymbol{j} = (j_1, \dots, j_d)$, $\boldsymbol{n} = (n_1, \dots, n_d)$ and $R_{\boldsymbol{n},\boldsymbol{j}}$ is the "rectangle"

$$R_{\boldsymbol{n},\boldsymbol{j}} := \left[\frac{j_1 - 1}{n_1}, \frac{j_1}{n_1}\right) \times \dots \times \left[\frac{j_d - 1}{n_d}, \frac{j_d}{n_d}\right). \tag{3}$$

Here by $|A|$ we denoted Lebesgue measure of the set A and the condition "$\boldsymbol{1} \le \boldsymbol{j} \le \boldsymbol{n}$" is understood componentwise : $1 \le j_1 \le n_1, \dots, 1 \le j_d \le n_d$.

Goldie and Greenwood investigated the conditions when this process converges to standard Wiener process indexed by $\mathscr{A}$, which is defined as a mean zero Gaussian process W with covariance

$$\mathbf{E}\, W(A)W(B) = |A \cap B|, \quad A, B \in \mathscr{A}.$$

They proved the functional central limit theorem in $C(\mathscr{A})$ (space of continuous functions $f : \mathscr{A} \to \mathbb{R}$ with supremum norm) basically requiring that the variance of process $\xi_n(A)$ converge to the variance of $W(A)$ and that the collection $\mathscr{A}$ satisfies certain entropy condition. An important class of sets is $\mathscr{A} = \mathscr{Q}_d$ where

$$\mathscr{Q}_d := \{\, [0, t_1] \times \dots \times [0, t_d];\ \boldsymbol{t} = (t_1, \dots, t_d) \in [0,1]^d \,\}. \tag{4}$$

Note that when $d = 1$ the partial sum process ξ_n based on $\mathscr{Q}_d$ is the random polygonal line of Donsker-Prokhorov's theorem. Thus it is obvious that for case $d = 1$, $\xi_{\boldsymbol{n}}$ does not coincide with Ξ_n.

The attempt to introduce adaptive construction for general summation processes was made by Bickel and Wichura [1]. However they put some restrictions on variance of random variables in triangular array. For zero mean independent random variables $\{X_{n,ij}, 1 \le i \le I_n, 1 \le j \le J_n\}$ with variances $\mathbf{E} X_{n,ij} = a_{n,i} b_{n,j}$ satisfying $\sum a_{n,i} = 1 = \sum b_{n,j}$, they defined the summation process as

$$\zeta_n(t_1, t_2) = \sum_{i \le A_n(t_1)} \sum_{j \le B_n(t_1)} X_{n,ij},$$

where

$$A_n(t_1) = \max\{k : \sum_{i \le k} a_{n,i} < t_1\}, \quad B_n(t_2) = \max\{l : \sum_{j \le l} b_{n,j} < t_2\}.$$

It is easy to see that this construction is two-dimensional time generalisation of the jump version of Prokhorov construction. Bickel and Wichura proved that the process ζ_n converges in the space $D([0,1]^2)$ to a Brownian sheet, if $a_{n,i}$ and $b_{n,j}$ are infinitesimally small and random variables $\{X_{n,ij}\}$ satisfy Lindeberg condition.

In this paper we introduce new summation process $\{\Xi_{\boldsymbol{n}}(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^d\}$ which coincides with the process $\{\Xi_n(t), t \in [0,1]\}$, for $d = 1$. Sufficient conditions for weak convergence in Hölder spaces are given. For the case $d = 1$ they coincide with conditions given by Račkauskas and Suquet [7]. The limiting process in general case is not Brownian sheet. It is a mean zero Gaussian process with covariance depending on the limit of $\mathbf{E}\,\Xi_{\boldsymbol{n}}(\boldsymbol{t})^2$. Examples of possible limiting processes are given. In case of special variance structure of the triangular array as in Bickel and Wichura it is shown that the limiting process is a standard Brownian sheet.

2 Notations and results

In this paper vectors $\boldsymbol{t} = (t_1, \dots, t_d)$ of $\mathbb{R}^d$, $d \geq 2$, are typeset in italic bold. In particular,

$$\mathbf{1} := (1, \dots, 1).$$

For $1 \leq k < l \leq d$, $\boldsymbol{t}_{k:l}$ denotes the "subvector"

$$\boldsymbol{t}_{k:l} := (t_k, t_{k+1}, \dots, t_l). \tag{5}$$

The set $\mathbb{R}^d$ is equipped with the partial order

$$\boldsymbol{s} \leq \boldsymbol{t} \quad \text{if and only if} \quad s_k \leq t_k, \text{ for all } k = 1, \dots, d.$$

As a vector space $\mathbb{R}^d$, is endowed with the norm

$$|\boldsymbol{t}| = \max(|t_1|, \dots, |t_d|), \quad \boldsymbol{t} = (t_1, \dots, t_d) \in \mathbb{R}^d.$$

Together with the usual addition of vectors and multiplication by a scalar, we use also the componentwise multiplication and division of vectors $\boldsymbol{s} = (s_1, \dots, s_d)$, $\boldsymbol{t} = (t_1, \dots, t_d)$ in $\mathbb{R}^d$ defined whenever it makes sense by

$$\boldsymbol{s}\boldsymbol{t} := (s_1 t_1, \dots, s_d t_d), \quad \boldsymbol{s}/\boldsymbol{t} := (s_1/t_1, \dots, s_d/t_d).$$

Also we will use componentwise minimum

$$\boldsymbol{t} \wedge \boldsymbol{s} := (t_1 \wedge s_1, \dots, t_d \wedge s_d)$$

Partial order as well as all these operations are also intended componentwise when one of the two involved vectors is replaced by a scalar. So for $c \in \mathbb{R}$ and $\boldsymbol{t} \in \mathbb{R}^d$, $c \leq \boldsymbol{t}$ means $c \leq t_k$ for $k = 1, \dots, d$, $\boldsymbol{t} + c := (t_1 + c, \dots, t_d + c)$, $c/\boldsymbol{t} := (c/t_1, \dots, c/t_d)$.

For $\boldsymbol{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ write

$$\pi(\boldsymbol{n}) := n_1 \dots n_d,$$

and for $\boldsymbol{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$,

$$\mathrm{m}(\boldsymbol{t}) := \min(t_1, \dots, t_d).$$

We define the Hölder space $\mathrm{H}_\alpha^o([0,1]^d)$ as the vector space of functions $x : [0,1]^d \to \mathbb{R}$ such that

$$\|x\|_\alpha := |x(0)| + \omega_\alpha(x, 1) < \infty,$$

with

$$\omega_\alpha(x, \delta) := \sup_{0 < |\boldsymbol{t} - \boldsymbol{s}| \leq \delta} \frac{|x(\boldsymbol{t}) - x(\boldsymbol{s})|}{|\boldsymbol{t} - \boldsymbol{s}|^\alpha} \xrightarrow[\delta \to 0]{} 0.$$

Endowed with the norm $\|.\|_\alpha$, $\mathrm{H}_\alpha^o([0,1]^d)$ is a separable Banach space.

As we are mainly dealing in this paper with weak convergence in some function spaces, it is convenient to introduce the following notations. Let B be some separable Banach space and $(Y_n)_{n\geq 1}$ and $(Z_{\boldsymbol{n}})_{\boldsymbol{n}\geq \boldsymbol{1}}$ be respectively a sequence and a random field of random elements in B. We write

$$Y_n \xrightarrow[n\to\infty]{B} Y, \qquad Z_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{B} Z,$$

for their weak convergence in the space B to the random elements Y or Z, i.e. $\mathbf{E}f(Y_n)\to\mathbf{E}f(Y)$ for any continuous and bounded $f: B\to\mathbb{R}$ and similarly with $Z_{\boldsymbol{n}}$, the weak convergence of $Z_{\boldsymbol{n}}$ to Z being understood in the net sense.

Define triangular array with multidimensional index as

$$(X_{\boldsymbol{n},\boldsymbol{k}},\ \boldsymbol{1}\leq\boldsymbol{k}\leq\boldsymbol{k_n}),\ \boldsymbol{n}\in\mathbb{N}^d,$$

where for each $\boldsymbol{n}$ the random variables $X_{\boldsymbol{n},\boldsymbol{k}}$ are independent. The expression $\boldsymbol{k_n}$ is the element from $\mathbb{N}^d$ with multidimensional index: $\boldsymbol{k_n}=(k_{\boldsymbol{n}}^1,\dots,k_{\boldsymbol{n}}^d)$. Assume that $\mathbf{E}X_{\boldsymbol{n},\boldsymbol{k}}=0$ and that $\sigma^2_{\boldsymbol{n},\boldsymbol{k}}:=\mathbf{E}X^2_{\boldsymbol{n},\boldsymbol{k}}<\infty$, for $\boldsymbol{1}\leq\boldsymbol{k}\leq\boldsymbol{k_n}$, $\boldsymbol{n}\in\mathbb{N}^d$. Define for each $\boldsymbol{1}\leq\boldsymbol{k}\leq\boldsymbol{k_n}$

$$S_{\boldsymbol{n}}(\boldsymbol{k}):=\sum_{\boldsymbol{j}\leq\boldsymbol{k}}X_{\boldsymbol{n},\boldsymbol{j}},\quad b_{\boldsymbol{n}}(\boldsymbol{k}):=\sum_{\boldsymbol{j}\leq\boldsymbol{k}}\sigma^2_{\boldsymbol{n},\boldsymbol{j}}.$$

We will require that the sum of all variances is one, i.e. $b_{\boldsymbol{n}}(\boldsymbol{k_n})=1$ and that $\mathrm{m}(\boldsymbol{k_n})\to\infty$, as $\mathrm{m}(\boldsymbol{n})\to\infty$. If $\pi(\boldsymbol{k})=0$, let $S_{\boldsymbol{n}}(\boldsymbol{k})=0$, $b_{\boldsymbol{n}}(\boldsymbol{k})=0$. For $i=1,\dots,d$ introduce the notations

$$\begin{aligned} b_i(k) &:= b_{\boldsymbol{n}}(k_{\boldsymbol{n}}^1,\dots,k_{\boldsymbol{n}}^{i-1},k,k_{\boldsymbol{n}}^{i+1},\dots,k_{\boldsymbol{n}}^d),\\ \Delta b_i(k) &:= b_i(k)-b_i(k-1). \end{aligned} \tag{6}$$

Note that $b_i(k)$ and $\Delta b_i(k)$ depend on $\boldsymbol{n}$ and $\boldsymbol{k_n}$. Now we can define our non-uniform grid. For $\boldsymbol{1}\leq\boldsymbol{j}\leq\boldsymbol{k_n}$, let

$$R_{\boldsymbol{n},\boldsymbol{j}}:=\Big[b_1(j_1-1),\ b_1(j_1)\Big)\times\cdots\times\Big[b_d(j_d-1),\ b_d(j_d)\Big). \tag{7}$$

Due to definition of $b_i(k)$ we get $R_{\boldsymbol{n},\boldsymbol{j}}\cap R_{\boldsymbol{n},\boldsymbol{k}}=\varnothing$, if $\boldsymbol{k}\neq\boldsymbol{j}$, $\cup_{\boldsymbol{j}\leq\boldsymbol{k_n}}R_{\boldsymbol{n},\boldsymbol{j}}=[0,1)^d$ and $\sum_{\boldsymbol{j}\leq\boldsymbol{k_n}}|R_{\boldsymbol{n},\boldsymbol{j}}|=1$.

Remark 1. *Thus defined, our non-uniform grid becomes the usual uniform grid in the case of i.i.d. variables. The triangular array then is defined as* $X_{\boldsymbol{n},\boldsymbol{k}}=\pi(\boldsymbol{n})^{-1/2}X_{\boldsymbol{k}}$*, where* $\{X_{\boldsymbol{k}},\boldsymbol{1}\leq\boldsymbol{k}\leq\boldsymbol{k_n}\}$ *is an i.i.d. random field. In this case we have* $b_i(k)=k/n_i$*.*

Now define the summation process on such grid as

$$\Xi_{\boldsymbol{n}}(\boldsymbol{t})=\sum_{\boldsymbol{1}\leq\boldsymbol{j}\leq\boldsymbol{k_n}}|R_{\boldsymbol{n},\boldsymbol{j}}|^{-1}|R_{\boldsymbol{n},\boldsymbol{j}}\cap[0,\boldsymbol{t}]|X_{\boldsymbol{n},\boldsymbol{j}}. \tag{8}$$

In section 3.2 we discuss in detail the construction of the random field $\Xi_{\boldsymbol{n}}$ and propose some useful representations.

Remark 2. *In case* $d=1$*, the process* $\Xi_{\boldsymbol{n}}$ *coincides with polygonal line process proposed by Prokhorov.*

Theorem 3. *For $0 < \alpha < 1/2$, set $p(\alpha) := 1/(1/2 - \alpha)$. If*

$$\max_{1\le l\le d} \max_{1\le k_l\le k_{\boldsymbol{n}}^l} \Delta b_l(k_l) \to 0, \text{ as } \mathrm{m}(\boldsymbol{n}) \to \infty \tag{9}$$

and for some $q > p(\alpha)$

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k}_n} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{k}}|^q) = 0, \tag{10}$$

then the net $\{\Xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}\}$ is asymptotically tight in the space $\mathrm{H}_\alpha^o([0,1]^d)$.

Remark 4. *For $d = 1$, condition* (9) *ensures infinitesimal negligibility and* (10) *reduces to* (1).

For each $\boldsymbol{k} \le \boldsymbol{n}$, define

$$\boldsymbol{B}(\boldsymbol{k}) = (b_1(k_1), \dots, b_d(k_d)).$$

Note that $\boldsymbol{B}(\boldsymbol{k}) \in [0,1]^d$. Now for each $\boldsymbol{t} \in [0,1]^d$, define

$$\mu_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k}_n} \mathbf{1}\{\boldsymbol{B}(\boldsymbol{k}) \in [0,\boldsymbol{t}]\}\sigma_{\boldsymbol{n},\boldsymbol{k}}^2.$$

Assumption 5. *There exists a function $\mu : [0,1]^d \to \mathbb{R}$ such that*

$$\forall \boldsymbol{t} \in [0,1]^d, \quad \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \mu_{\boldsymbol{n}}(\boldsymbol{t}) = \mu(\boldsymbol{t}). \tag{11}$$

Proposition 6. *Define $g : [0,1]^d \times [0,1]^d \to \mathbb{R}$ as $g(\boldsymbol{t},\boldsymbol{s}) := \mu(\boldsymbol{t} \wedge \boldsymbol{s})$. Then g is symmetric and positive definite.*

In this paper positive definiteness is to be understood as in [4], g is positive definite if for every integer $n \ge 1$, every n-tuple of reals $x_1, \dots, x_n$ and every n-tuple of vectors $\boldsymbol{t}_1, \dots, \boldsymbol{t}_n$ of $[0,1]^d$, $\sum_{i,j=1}^n x_i g(\boldsymbol{t}_i, \boldsymbol{t}_j) x_j$ is non negative (in other words, the quadratic form with matrix $[g(\boldsymbol{t}_i, \boldsymbol{t}_j)]$ is *positive semi-definite*). When g is positive definite, the existence of mean-zero Gaussian process $\{G(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^d\}$ with covariance function $\mathbf{E}\, G(\boldsymbol{t})G(\boldsymbol{s}) = \mu(\boldsymbol{t} \wedge \boldsymbol{s})$ is a classical result (see for example [4] theorem 1.2.1 in chapter 5).

Remark 7. *For the standard Brownian sheet $\mathbf{E}\, W(\boldsymbol{t})W(\boldsymbol{s}) = \pi(\boldsymbol{t} \wedge \boldsymbol{s})$.*

Theorem 8. *If* (9) *and* (11) *holds and for every $\varepsilon > 0$*

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k}_n} \mathbf{E} X_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}| \ge \varepsilon\} = 0, \tag{12}$$

then the finite dimensional distributions of process $\{\Xi_{\boldsymbol{n}}(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^d\}$ converges to the finite dimensional distributions of the process $\{G(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^d\}$.

Then the functional central limit theorem is stated as a corollary.

Corollary 9. *If* (9)*,* (10) *and* (11) *hold, then*

$$\Xi_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathrm{H}^o_\alpha([0,1]^d)} G. \tag{13}$$

The following examples may be useful to discuss conditions (9), (10) and (11).

Example 10. *For $\boldsymbol{n} = (n,n)$ and $\boldsymbol{k_n} = (2n,2n)$ take $X_{\boldsymbol{n},\boldsymbol{k}} = a_{\boldsymbol{n},\boldsymbol{k}} Y_k$, with $\{Y_{\boldsymbol{k}}, \boldsymbol{k} \le \boldsymbol{k_n}\}$ i.i.d. random variables with standard normal distribution, and*

$$a^2_{\boldsymbol{n},\boldsymbol{k}} = \begin{cases} \frac{1}{10n^2}, \text{ for } \boldsymbol{k} \le (n,n) \\ \frac{3}{10n^2}, \text{ otherwise.} \end{cases}$$

Thus defined triangular array satisfies conditions (9)*,* (10) *and* (11)*. Furthermore*

$$\mu_{\boldsymbol{n}}(\boldsymbol{t}) \to \nu(\boldsymbol{t}) := \frac{1}{10}\left(\frac{5}{2}t_1 \wedge 1\right)\left(\frac{5}{2}t_2 \wedge 1\right) + \frac{(5t_1-2)\vee 0}{10}\left(\frac{5}{2}t_2 \vee 1\right)$$
$$+ \frac{(5t_2-2)\vee 0}{10}\left(\frac{5}{2}t_1 \vee 1\right) + \frac{\big((5t_1-2)\vee 0\big)\big((5t_2-2)\vee 0\big)}{30}$$

Example 11. *For $\boldsymbol{n} = (n,n)$ and $\boldsymbol{k_n} = (n,n)$ take $X_{\boldsymbol{n},\boldsymbol{k}} = a_{\boldsymbol{n},\boldsymbol{k}} Y_k$, with $\{Y_{\boldsymbol{k}}, \boldsymbol{k} \le \boldsymbol{k_n}\}$ i.i.d. random variables with standard normal distribution, and*

$$b^2_{\boldsymbol{n},\boldsymbol{k}} = \begin{cases} \pi(\boldsymbol{n}), \text{ for } \boldsymbol{n} = (2l-1, 2l-1),\ l \in \mathbb{N} \\ a^2_{\boldsymbol{n},\boldsymbol{k}}, \text{ for } \boldsymbol{n} = (2l, 2l),\ l \in \mathbb{N} \end{cases}$$

Thus defined triangular array satisfies conditions (9)*,* (10) *but not* (11)*.*

From these examples we see that the weak limit of $\Xi_{\boldsymbol{n}}$ is not necessarily Brownian sheet and though process $\Xi_{\boldsymbol{n}}$ can be tight, this does not ensure that the finite dimensional distributions converge. This contrasts with the one dimensional case. It should be noted that both examples violate the conditions for $\xi_{\boldsymbol{n}}$ in Goldie and Greenwood. For the triangular arrays satisfying similar conditions as in Bickel and Wichura [1], condition (11) is always satisfied.

Corollary 12. *Let $\sigma^2_{\boldsymbol{n},\boldsymbol{k}} = \pi(\boldsymbol{a}_{\boldsymbol{n},\boldsymbol{k}})$, where $\{\boldsymbol{a}_{\boldsymbol{n},\boldsymbol{k}} = (a^1_{\boldsymbol{n},k_1}, \dots, a^d_{\boldsymbol{n},k_d})\}$ is a triangular array of real vectors satisfying the following conditions for each $i = 1, \dots, d$ and for all $\boldsymbol{k} \le \boldsymbol{k_n}$.*

i) $\sum_{k=1}^{k^i_n} a^i_{\boldsymbol{n},k} = 1$ *with* $a^i_{\boldsymbol{n},k_i} > 0$.

ii)

$$\max_{1\le k\le k^i_n} a^i_{\boldsymbol{n},k} \to 0, \text{ as } \mathrm{m}(\boldsymbol{n}) \to \infty.$$

Then condition (10) *is sufficient for convergence* (13) *and $G(\boldsymbol{t})$ is simply $W(\boldsymbol{t})$.*

From this corollary it clearly follows that our result is a generalisation of i.i.d. case, since in i.i.d. case with triangular array defined as in Remark 1 we have $\sigma^2_{\boldsymbol{n},\boldsymbol{k}} = \pi(\boldsymbol{n})$.

The moment conditions for tightness can be relaxed. Introduce for every $\tau > 0$ truncated random variables:

$$X_{\boldsymbol{n},\boldsymbol{k},\tau} := X_{\boldsymbol{n},\boldsymbol{k}} \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}| \le \tau\sigma^{2\alpha}_{\boldsymbol{n},\boldsymbol{k}}\}$$

Theorem 13. *Suppose condition* (9) *and following conditions hold:*

1. *For every* $\varepsilon > 0$*,*

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\le \boldsymbol{k}\le \boldsymbol{k_n}} P(|X_{\boldsymbol{n},\boldsymbol{k}}| \ge \varepsilon \sigma_{\boldsymbol{n},\boldsymbol{k}}^{2\alpha}) = 0. \tag{14}$$

2. *For every* $\varepsilon > 0$*,*

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\le \boldsymbol{k}\le \boldsymbol{k_n}} \mathbf{E} X_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}| \ge \varepsilon\} = 0. \tag{15}$$

3. *For some* $q > 1/(1/2-\alpha)$*,*

$$\lim_{\tau\to 0}\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\le \boldsymbol{k}\le \boldsymbol{k_n}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q) = 0. \tag{16}$$

Then the net $\{\Xi_{\boldsymbol{n}}, \boldsymbol{n}\in\mathbb{N}\}$ *is tight in the space* $\mathrm{H}_\alpha^o([0,1]^d)$*.*

3 Background and tools

3.1 Hölder spaces and tightness criteria

We present briefly here some structure property of $\mathrm{H}_\alpha^o([0,1]^d)$ which is needed to obtain a tightness criterion. For more details, the reader is referred to [6] and [8]. Set

$$W_j = \{k2^{-j};\ 0 \le k \le 2^j\}^d, \quad j = 0,1,2,\dots$$

and

$$V_0 := W_0, \qquad V_j := W_j \setminus W_{j-1}, \quad j \ge 1,$$

so V_j is the set of dyadic points $\boldsymbol{v} = (k_1 2^{-j},\dots,k_d 2^{-j})$ in W_j with at least one k_i odd. Define

$$\begin{aligned} \lambda_{0,\boldsymbol{v}}(x) &= x(\boldsymbol{v}), \quad \boldsymbol{v}\in V_0;\\ \lambda_{j,\boldsymbol{v}}(x) &= x(\boldsymbol{v}) - \frac{1}{2}\big(x(\boldsymbol{v}^-) + x(\boldsymbol{v}^+)\big), \quad \boldsymbol{v}\in V_j,\ j\ge 1, \end{aligned}$$

where $\boldsymbol{v}^-$ and $\boldsymbol{v}^+$ are defined as follows. Each $\boldsymbol{v}\in V_j$ admits a unique representation $\boldsymbol{v} = (v_1,\dots,v_d)$ with $v_i = k_i/2^j$, $(1\le i\le d)$. The points $\boldsymbol{v}^- = (v_1^-,\dots,v_d^-)$ and $\boldsymbol{v}^+ = (v_1^+,\dots,v_d^+)$ are defined by

$$v_i^- = \begin{cases} v_i - 2^{-j}, & \text{if } k_i \text{ is odd;}\\ v_i, & \text{if } k_i \text{ is even} \end{cases} \qquad v_i^+ = \begin{cases} v_i + 2^{-j}, & \text{if } k_i \text{ is odd;}\\ v_i, & \text{if } k_i \text{ is even,} \end{cases}$$

We will use the following tightness criteria.

Theorem 14. *Let* $\{\zeta_{\boldsymbol{n}}, \boldsymbol{n}\in\mathbb{N}^d\}$ *be a net of random elements with values in the space* $\mathrm{H}_\alpha^o([0,1]^d)$*. Assume that the following conditions are satisfied.*

i) $\lim_{a\to\infty}\sup_{\boldsymbol{n}} P(\sup_{\boldsymbol{t}\in[0,1]^d}|\zeta_{\boldsymbol{n}}(\boldsymbol{t})|>a)=0.$

ii) For each $\varepsilon>0$

$$\lim_{J\to\infty}\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(\sup_{j\geq J} 2^{\alpha j}\max_{\boldsymbol{v}\in V_j}|\lambda_{j,\boldsymbol{v}}(\zeta_{\boldsymbol{n}})|>\varepsilon)=0.$$

Then the net $\{\zeta_{\boldsymbol{n}},\boldsymbol{n}\in\mathbb{N}^d\}$ *is assymptoticaly tight in the space* $\mathrm{H}^o_\alpha([0,1]^d)$.

Proof. The proof is the same as in theorem 2 in [12].

3.2 Summation process

For $t\in[0,1]$ and $\boldsymbol{t}\in[0,1]^d$, write

$$u_i(t):=\max\{j\geq 0: b_i(j)\leq t\},\quad \boldsymbol{U}(\boldsymbol{t}):=(u_1(t_1),\dots,u_d(t_d)).$$

Note that $\boldsymbol{U}(\boldsymbol{t})\in\mathbb{N}^d$. Recalling (6), write

$$\Delta B(\boldsymbol{k}):=(\Delta b_1(k_1),\dots,\Delta b_d(k_d)).$$

In [11] it was shown that process $\xi_{\boldsymbol{n}}$ defined by (2) has a certain barycentric representation. We will prove that similar representation exists for process $\Xi_{\boldsymbol{n}}$.

Proposition 15. *Let us write any* $\boldsymbol{t}\in[0,1]^d$ *as the barycenter of the* 2^d *vertices*

$$V(\boldsymbol{u}):=\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t}))+\boldsymbol{u}\Delta\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{1}),\quad \boldsymbol{u}\in\{0,1\}^d, \tag{17}$$

of the rectangle $R_{\boldsymbol{n},\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{1}}$ *with some weights* $w(\boldsymbol{u})\geq 0$ *depending on* $\boldsymbol{t}$*, i.e.,*

$$\boldsymbol{t}=\sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})V(\boldsymbol{u}),\quad \textit{where}\quad \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})=1. \tag{18}$$

Using this representation, define the random field Ξ^*_n *by*

$$\Xi^*_{\boldsymbol{n}}(\boldsymbol{t})=\sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{u}),\quad \boldsymbol{t}\in[0,1]^d.$$

Then $\Xi^*_{\boldsymbol{n}}$ *coincides with the summation process defined by* (8).

Proof. For fixed $\boldsymbol{n}\geq\boldsymbol{1}\in\mathbb{N}^d$, any $\boldsymbol{t}\neq\boldsymbol{1}\in[0,1]^d$ belongs to a unique rectangle $R_{\boldsymbol{n},\boldsymbol{j}}$, defined by (7), namely $R_{\boldsymbol{n},\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{1}}$. Then the 2^d vertices of this rectangle are clearly the points $V(\boldsymbol{u})$ given by (17). Put

$$\boldsymbol{s}=\frac{\boldsymbol{t}-\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t}))}{\Delta\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{1})},\quad \text{whence}\quad \boldsymbol{t}=\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t}))+\boldsymbol{s}\Delta\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{1}), \tag{19}$$

recalling that in this formula the division of vector is componentwise.

For any non empty subset L of $\{1,\dots,d\}$, we denote by $\{0,1\}^L$ the set of binary vectors indexed by L. In particular $\{0,1\}^d$ is an abridged notation for $\{0,1\}^{\{1,\dots,d\}}$. Now define the non negative weights

$$w_L(\boldsymbol{u}):=\prod_{l\in L} s_l^{u_l}(1-s_l)^{1-u_l},\quad \boldsymbol{u}\in\{0,1\}^L$$

and when $L = \{1,\dots,d\}$, simplify this notation in $w(\boldsymbol{u})$. For fixed L, the sum of all these weights is one since

$$\sum_{\boldsymbol{u}\in\{0,1\}^L} w_L(\boldsymbol{u}) = \prod_{l\in L}\big(s_l + (1-s_l)\big) = 1. \tag{20}$$

The special case $L = \{1,\dots,d\}$ gives the second equality in (18). From (20) we immediately deduce that for any K non empty and strictly included in $\{1,\dots,d\}$, with $L := \{1,\dots,d\}\setminus K$,

$$\sum_{\substack{\boldsymbol{u}\in\{0,1\}^d,\\ \forall k\in K, u_k=1}} w(\boldsymbol{u}) = \prod_{k\in K} s_k \sum_{\boldsymbol{u}\in\{0,1\}^L} s_l^{u_l}(1-s_l)^{1-u_l} = \prod_{k\in K} s_k. \tag{21}$$

Formula (21) remains obviously valid in the case where $K = \{1,\dots,d\}$.

Now let us observe that

$$\begin{aligned}\sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})V(\boldsymbol{u}) &= \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\Big(\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})) + \boldsymbol{u}\Delta\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{1})\Big)\\ &= \boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})) + \Delta\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{1}) \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\boldsymbol{u}.\end{aligned}$$

Comparing with the expression of $\boldsymbol{t}$ given by (19), we see that the first equality in (18) will be established if we check that

$$\boldsymbol{s}' := \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\boldsymbol{u} = \boldsymbol{s}. \tag{22}$$

This is easily seen componentwise using (21) because for any fixed $l \in \{1,\dots,d\}$,

$$s_l' = \sum_{\substack{\boldsymbol{u}\in\{0,1\}^d,\\ u_l=1}} w(\boldsymbol{u}) = \prod_{k\in\{l\}} s_k = s_l.$$

Next we check that $\Xi_{\boldsymbol{n}}(\boldsymbol{t}) = \Xi_{\boldsymbol{n}}^*(\boldsymbol{t})$ for every $\boldsymbol{t} \in [0,1]^d$. Introduce the notation

$$D_{\boldsymbol{t},\boldsymbol{u}} := \mathbb{N}^d \cap \big(\big[0, \boldsymbol{U}(\boldsymbol{t})+\boldsymbol{u}\big] \setminus \big[0, \boldsymbol{U}(\boldsymbol{t})\big]\big).$$

Then we have

$$\begin{aligned}\Xi_{\boldsymbol{n}}^*(\boldsymbol{t}) &= \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\big(S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})) + (S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{u}) - S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t}))\big)\\ &= S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})) + \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u}) \sum_{\boldsymbol{i}\in D_{\boldsymbol{t},\boldsymbol{u}}} X_{\boldsymbol{n},\boldsymbol{i}}.\end{aligned}$$

Now in view of (8), the proof of $\Xi_{\boldsymbol{n}}(\boldsymbol{t}) = \Xi_{\boldsymbol{n}}^*(\boldsymbol{t})$ reduces clearly to that of

$$\sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u}) \sum_{\boldsymbol{i}\in D_{\boldsymbol{t},\boldsymbol{u}}} X_{\boldsymbol{n},\boldsymbol{i}} = \sum_{\boldsymbol{i}\in I} |R_{\boldsymbol{n},\boldsymbol{i}}|^{-1} |R_{\boldsymbol{n},\boldsymbol{i}} \cap [0,\boldsymbol{t}]| X_{\boldsymbol{n},\boldsymbol{i}}, \tag{23}$$

where

$$\begin{aligned}I := \{\boldsymbol{i} \le \boldsymbol{k_n};\ &\forall k \in \{1,\dots,d\},\ i_k \le u_k(t_k)+1 \text{ and}\\ &\exists l \in \{1,\dots,d\},\ i_l = u_k(t_k)+1\}.\end{aligned} \tag{24}$$

Clearly I is the union of all $D_{\boldsymbol{t},\boldsymbol{u}}$, $\boldsymbol{u}\in\{0,1\}^d$, so we can rewrite the left hand side of (23) under the form $\sum_{\boldsymbol{i}\in I} a_{\boldsymbol{i}}X_{\boldsymbol{i}}$. For $\boldsymbol{i}\in I$, put

$$K(\boldsymbol{i}) := \{k\in\{1,\dots,d\};\ \boldsymbol{i}_k = u_k(t_k)+1\}. \tag{25}$$

Then observe that for $\boldsymbol{i}\in I$, the $\boldsymbol{u}$'s such that $\boldsymbol{i}\in D_{\boldsymbol{t},\boldsymbol{u}}$ are exactly those which satisfy $u_k = 1$ for every $k\in K(\boldsymbol{i})$. Using (21), this gives

$$\forall \boldsymbol{i}\in I,\qquad a_{\boldsymbol{i}} = \sum_{\substack{\boldsymbol{u}\in\{0,1\}^d,\\ \forall k\in K(\boldsymbol{i}), u_k=1}} w(\boldsymbol{u}) = \prod_{k\in K(\boldsymbol{i})} s_k. \tag{26}$$

On the other hand we have for every $\boldsymbol{i}\in I$,

$$\begin{aligned}|R_{\boldsymbol{n},\boldsymbol{i}}\cap[0,\boldsymbol{t}]| &= \prod_{k\in K(\boldsymbol{i})}\big(t_k - b_k(u_k(t_k))\big)\prod_{k\notin K(\boldsymbol{i})}\Delta b_k(u_k(t_k)+1) = \\ &\pi(\Delta\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{1}))\prod_{k\in K(\boldsymbol{i})} s_k = a_{\boldsymbol{i}}\pi(\Delta\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{1}))\end{aligned} \tag{27}$$

As $|R_{\boldsymbol{n},\boldsymbol{i}}|^{-1} = \pi(\Delta\boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})+\boldsymbol{1}))$, (23) follows and the proof is complete.

For proving tightness of the process $\Xi_{\boldsymbol{n}}$ it is convenient to get yet another representation of it. For this introduce the notations similar to [11]:

$$\Delta_k^{(j)} S_{\boldsymbol{n}}(\boldsymbol{i}) = S_{\boldsymbol{n}}((i_1,\dots,i_{j-1},k,i_{j+1}\dots,i_d)) - S_{\boldsymbol{n}}(i_1,\dots,i_{j-1},k-1,i_{j+1}\dots,i_d) \tag{28}$$

Clearly the operators $\Delta_k^{(j)}$'s commute for different j's. Note that when applied to $S_{\boldsymbol{n}}(\boldsymbol{i})$, $\Delta_k^{(j)}$ is really a difference operator acting on the j-th argument of a function with d arguments. Also since k defines the differencing, $\Delta_k^{(j)} S_{\boldsymbol{n}}(\boldsymbol{i})$ does not depend on i_j, and the following useful representation holds for $\boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{k_n}$,

$$X_{\boldsymbol{n},\boldsymbol{i}} = \Delta_{i_1}^{(1)}\dots\Delta_{i_d}^{(d)} S_{\boldsymbol{n}}(\boldsymbol{i}). \tag{29}$$

Proposition 16. *The process $\Xi_{\boldsymbol{n}}$ admits the representation*

$$\begin{aligned}\Xi_{\boldsymbol{n}}(\boldsymbol{t}) &= S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})) + \\ &\sum_{l=1}^{d}\ \sum_{1\le i_1<i_2<\dots<i_l\le d}\left(\prod_{k=1}^{l}\frac{t_{i_k}-b_{i_k}(u_{i_k})}{\Delta b_{i_k}(u_{i_k}(t_{i_k})+1)}\right)\left(\prod_{k=1}^{l}\Delta^{(i_k)}_{u_{i_k}(t_{i_k})+1}\right)S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})).\end{aligned} \tag{30}$$

Proof. Recalling the notations (24), (25) and formula (27), we have

$$\begin{aligned}\Xi_{\boldsymbol{n}}(\boldsymbol{t}) &= S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})) + \sum_{\boldsymbol{i}\in I}|R_{\boldsymbol{n},\boldsymbol{i}}|^{-1}|R_{\boldsymbol{n},\boldsymbol{i}}\cap[0,\boldsymbol{t}]|X_{\boldsymbol{n},\boldsymbol{i}} \\ &= S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})) + \sum_{\boldsymbol{i}\in I}\left(\prod_{k\in K(\boldsymbol{i})} s_k\right)X_{\boldsymbol{n},\boldsymbol{i}}.\end{aligned}$$

This can be recast as

$$\Xi_{\boldsymbol{n}}(\boldsymbol{t}) = S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})) + \sum_{l=1}^{d} T_l(\boldsymbol{t}) \tag{31}$$

with

$$T_l(\boldsymbol{t}) := \sum_{\substack{\boldsymbol{i}\in I \\ \mathrm{card}(K)(\boldsymbol{i})=l}} \Big(\prod_{k\in K(\boldsymbol{i})} s_k\Big) X_{\boldsymbol{n},\boldsymbol{i}}, \tag{32}$$

here by card(K) we denote the cardinality of the set K. The expression (32) can be further recast as

$$T_l(\boldsymbol{t}) = \sum_{\substack{K\subset\{1,\dots,d\} \\ \mathrm{card}(K)=l}} \sum_{\substack{\boldsymbol{i}\in I \\ K(\boldsymbol{i})=K}} \Big(\prod_{k\in K} s_k\Big) X_{\boldsymbol{n},\boldsymbol{i}} = \sum_{\substack{K\subset\{1,\dots,d\} \\ \mathrm{card}(K)=l}} \Big(\prod_{k\in K} s_k\Big) \sum_{\substack{\boldsymbol{i}\in I \\ K(\boldsymbol{i})=K}} X_{\boldsymbol{n},\boldsymbol{i}}.$$

It should be clear that

$$\sum_{\substack{\boldsymbol{i}\in I \\ K(\boldsymbol{i})=K}} X_{\boldsymbol{n},\boldsymbol{i}} = \Big(\prod_{k\in K} \Delta^{(k)}_{u_k(t_k)+1}\Big) S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})),$$

where the symbol Π is intended as the composition product of differences operators. Recalling that $s_k = (t_k - b_k(u_k(t_k)))/\Delta b_k(u_k(t_k)+1)$, this leads to

$$T_l(\boldsymbol{t}) = \sum_{\substack{K\subset\{1,\dots,d\} \\ \mathrm{card}(K)=l}} \Big(\prod_{k\in K} \frac{t_k - b_k(u_k)}{\Delta b_k(u_k(t_k)+1)}\Big) \Big(\prod_{k\in K} \Delta^{(k)}_{u_k(t_k)+1}\Big) S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})). \tag{33}$$

To complete the proof report this expression to the equation (31).

3.3 Rosenthal and Doob inequalities

When applied to our triangular array, Rosenthal inequality for independent non-identically distributed random variables reads

$$\mathbf{E}\Big|\sum_{\boldsymbol{1}\le\boldsymbol{j}\le\boldsymbol{n}} X_{\boldsymbol{n},\boldsymbol{j}}\Big|^q \le c\Bigg(\Big(\sum_{\boldsymbol{1}\le\boldsymbol{j}\le\boldsymbol{n}} \sigma^2_{\boldsymbol{n},\boldsymbol{j}}\Big)^{q/2} + \sum_{\boldsymbol{1}\le\boldsymbol{j}\le\boldsymbol{n}} \mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{j}}|^q)\Bigg), \tag{34}$$

for every $q \ge 2$, with a constant c depending on q only.

As in [11] we can also extend Doob inequality for independent non-identicaly distributed variables

$$\mathbf{E} \max_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k}_n} |S_{\boldsymbol{n}}(\boldsymbol{k})|^q \le \Big(\frac{p}{p-1}\Big)^{dq} \mathbf{E}(|S_{\boldsymbol{n}}(\boldsymbol{k}_{\boldsymbol{n}})|^q), \tag{35}$$

for $q > 1$.

4 Finite-dimensional distributions

4.1 Proof of the proposition 6

We have

$$g(t,s) = \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \mu_{\boldsymbol{n}}(\boldsymbol{t}\wedge\boldsymbol{s}).$$

Take $p \in \mathbb{N}$, $v_1, \dots, v_p \in \mathbb{R}$ and $\boldsymbol{t}_1, \dots, \boldsymbol{t}_p \in [0,1]^d$. Note that for any $\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{r} \in [0,1]^d$ we have

$$\mathbf{1}\{\boldsymbol{r} \in [0, \boldsymbol{t} \wedge \boldsymbol{s}]\} = \mathbf{1}\{\boldsymbol{r} \in [0, \boldsymbol{t}] \cap [0, \boldsymbol{s}]\} = \mathbf{1}\{\boldsymbol{r} \in [0, \boldsymbol{t}]\}\mathbf{1}\{\boldsymbol{r} \in [0, \boldsymbol{s}]\}. \tag{36}$$

Then

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=1}^{p} v_i \mu_{\boldsymbol{n}}(\boldsymbol{t}_i \wedge \boldsymbol{t}_j) v_j &= \sum_{i=1}^{p}\sum_{j=1}^{p} v_i v_j \sum_{\boldsymbol{k} \le \boldsymbol{k}_n} \mathbf{1}\{B_{\boldsymbol{n}}(\boldsymbol{k}) \in [0, \boldsymbol{t}_i \wedge \boldsymbol{t}_j]\}\sigma^2_{\boldsymbol{n},\boldsymbol{k}} \\ &= \sum_{\boldsymbol{k} \le \boldsymbol{k}_n} \sigma^2_{\boldsymbol{n},\boldsymbol{k}} \left(\sum_{i=1}^{p} v_i \mathbf{1}\{B_{\boldsymbol{n}}(\boldsymbol{k}) \in [0, \boldsymbol{t}_i]\} \right)^2 \ge 0.\end{aligned}$$

Since this holds for each $\boldsymbol{n}$, taking the limit as $\mathrm{m}(\boldsymbol{n}) \to \infty$ gives the positive definiteness of $g(\boldsymbol{t}, \boldsymbol{s})$.

4.2 Proof of theorem 8

Consider the jump process defined as

$$\zeta_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{1} \le \boldsymbol{k} \le \boldsymbol{k}_n} \mathbf{1}\{\boldsymbol{B}(\boldsymbol{k}) \in [0, \boldsymbol{t}]\} X_{\boldsymbol{n},\boldsymbol{k}}.$$

Now for each $\boldsymbol{t}$

$$|\Xi_{\boldsymbol{n}}(\boldsymbol{t}) - \zeta_{\boldsymbol{n}}(\boldsymbol{t})| = \sum_{\boldsymbol{1} \le \boldsymbol{k} \le \boldsymbol{k}_n} \alpha_{\boldsymbol{n},\boldsymbol{k}} X_{\boldsymbol{n},\boldsymbol{k}},$$

where

$$\alpha_{\boldsymbol{n},\boldsymbol{k}} = |R_{\boldsymbol{n},\boldsymbol{k}}|^{-1}|R_{\boldsymbol{n},\boldsymbol{k}}| - \mathbf{1}\{\boldsymbol{B}(\boldsymbol{k}) \in [0, \boldsymbol{t}]\}.$$

Now $|\alpha_{\boldsymbol{n},\boldsymbol{k}}| < 1$, and vanishes if $R_{\boldsymbol{n},\boldsymbol{k}} \subset [0, \boldsymbol{t}]$, or $R_{\boldsymbol{n},\boldsymbol{k}} \cap [0, \boldsymbol{t}] = \varnothing$. Actually $\alpha_{\boldsymbol{n},\boldsymbol{k}} \ne 0$ if and only if $\boldsymbol{k} \in I$, where I is defined by (24). Thus

$$\mathbf{E}\,|\Xi_{\boldsymbol{n}}(\boldsymbol{t}) - \zeta_{\boldsymbol{n}}(\boldsymbol{t})|^2 = \sum_{\boldsymbol{k} \in I} \alpha_{\boldsymbol{n},\boldsymbol{k}} \sigma^2_{\boldsymbol{n},\boldsymbol{k}} \le \sum_{\boldsymbol{k} \in I} \sigma^2_{\boldsymbol{n},\boldsymbol{k}} \le \sum_{l=1}^{d} \Delta b_l(u_l(t_l) + 1).$$

Using (9) we get

$$|\Xi_{\boldsymbol{n}}(\boldsymbol{t}) - \zeta_{\boldsymbol{n}}(\boldsymbol{t})| \xrightarrow{P} 0, \text{ as } \mathrm{m}(\boldsymbol{n}) \to \infty.$$

We will concentrate now on finite-dimensional distributions of $\zeta_{\boldsymbol{n}}$.
Fix $\boldsymbol{t}_1, \dots, \boldsymbol{t}_r \in [0,1]^d$ and $v_1, \dots, v_r$ real, set

$$V_{\boldsymbol{n}} = \sum_{p=1}^{r} v_j \zeta_{\boldsymbol{n}}(\boldsymbol{t}_p) = \sum_{\boldsymbol{1} \le \boldsymbol{k} \le \boldsymbol{k}_n} \alpha_{\boldsymbol{n},\boldsymbol{k}} X_{\boldsymbol{n},\boldsymbol{k}},$$

where

$$\alpha_{\boldsymbol{n},\boldsymbol{k}} = \sum_{p=1}^{r} v_p \mathbf{1}\{\boldsymbol{B}(\boldsymbol{k}) \in [0, \boldsymbol{t}_p]\}.$$

Now using (36) we get

$$\begin{aligned}
b_{\boldsymbol{n}} := \mathbf{E} V_{\boldsymbol{n}}^2 &= \sum_{\boldsymbol{k}\le \boldsymbol{k}_n} \alpha_{\boldsymbol{n},\boldsymbol{k}}^2 \sigma_{\boldsymbol{n},\boldsymbol{k}}^2 \\
&= \sum_{\boldsymbol{k}\le \boldsymbol{k}_n} \sum_p \sum_q v_p v_q \mathbf{1}\{\boldsymbol{B}(\boldsymbol{k}) \in [0, \boldsymbol{t}_p]\}\mathbf{1}\{\boldsymbol{B}(\boldsymbol{k}) \in [0, \boldsymbol{t}_q]\}\sigma_{\boldsymbol{n},\boldsymbol{k}}^2 \\
&= \sum_p \sum_q v_p v_q \mu_{\boldsymbol{n}}(\boldsymbol{t}_p \wedge \boldsymbol{t}_q).
\end{aligned}$$

Letting $\mathrm{m}(\boldsymbol{n})$ to to infinity and using assumption 5, we obtain

$$b_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{} \sum_p \sum_q v_p v_q \mu(\boldsymbol{t}_p \wedge \boldsymbol{t}_q) = \mathbf{E}\left(\sum v_p G(\boldsymbol{t}_p)\right)^2 =: b.$$

If $b = 0$, then V_n converges to zero in distribution since $\mathbf{E} V_n^2$ tends to zero. In this special case we also have $\sum_p v_p G(\boldsymbol{t}_p) = 0$ almost surely, thus the convergence of finite dimensional distributions holds.

Assume now, that $b > 0$. For convenience put $Y_{\boldsymbol{n},\boldsymbol{k}} = \alpha_{\boldsymbol{n},\boldsymbol{k}} X_{\boldsymbol{n},\boldsymbol{k}}$ and $v = \sum_p \sum_q v_p v_q$. Since

$$Y_{\boldsymbol{n},\boldsymbol{k}}^2 \le v X_{\boldsymbol{n},\boldsymbol{k}}^2,$$

$Y_{\boldsymbol{n},\boldsymbol{k}}$ satisfies the condition of infinitesimal negligibility. For $\mathrm{m}(\boldsymbol{n})$ large enough to have $b_{\boldsymbol{n}} > b/2$, we get

$$\frac{1}{\mathbf{E} V_{\boldsymbol{n}}^2} \sum_{1\le \boldsymbol{k}\le \boldsymbol{k}_n} \mathbf{E}\left(Y_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\{|Y_{\boldsymbol{n},\boldsymbol{k}}|^2 > \varepsilon^2 \mathbf{E} V_{\boldsymbol{n}}^2\}\right) \le \frac{2v}{b} \sum_{1\le \boldsymbol{k}\le \boldsymbol{k}_n} \mathbf{E}\left(X_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}|^2 > \frac{b\varepsilon^2}{2v}\}\right).$$

Thus Lindeberg condition for $V_{\boldsymbol{n}}$ is satisfied and that gives us the convergence of finite dimensional distributions.

5 Tightness results

5.1 Proof of theorem 3

We will use theorem 14. Using Doob inequality we have

$$\begin{aligned}
P(\sup_{\boldsymbol{t}\in[0,1]^d} |\Xi_{\boldsymbol{n}}(\boldsymbol{t})| > a) &= P(\max_{\boldsymbol{k}\le \boldsymbol{k}_n} |S_{\boldsymbol{n}}(k)| > a) \\
&\le a^{-2} \mathbf{E} S_{\boldsymbol{n}}(\boldsymbol{k}_{\boldsymbol{n}})^2 = a^{-2} \to 0, \text{ as } a \to \infty,
\end{aligned}$$

thus condition *(i)* is satisfied. For proving *(ii)* note that due to the definitions of $\boldsymbol{v}$, $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$ we can write

$$\begin{aligned}
\Xi_{\boldsymbol{n}}(\boldsymbol{v}) - \Xi_{\boldsymbol{n}}(\boldsymbol{v}^+) &= \sum_{i=1}^{l} \left(\Xi_{\boldsymbol{n}}(\boldsymbol{v} + \boldsymbol{w}_{i-1}) - \Xi_{\boldsymbol{n}}(\boldsymbol{v} + \boldsymbol{w}_i)\right) \\
\Xi_{\boldsymbol{n}}(\boldsymbol{v}) - \Xi_{\boldsymbol{n}}(\boldsymbol{v}^-) &= \sum_{i=1}^{l} \left(\Xi_{\boldsymbol{n}}(\boldsymbol{v} - \boldsymbol{w}_{i-1}) - \Xi_{\boldsymbol{n}}(\boldsymbol{v} - \boldsymbol{w}_i)\right)
\end{aligned}$$

where l is the number of odd coordinates in $2^j \boldsymbol{v}$, $\boldsymbol{w}_0 = \boldsymbol{0}$, $\boldsymbol{w}_i$ has 2^{-j} in the first i odd coordinates of $2^j \boldsymbol{v}$, and zero in other coordinates. So the condition (ii) holds provided one proves for every $\varepsilon > 0$

$$\lim_{J\to\infty} \limsup_{\boldsymbol{n}\to\infty} \Pi^-(J,\boldsymbol{n};\varepsilon) = 0, \tag{37}$$

$$\lim_{J\to\infty} \limsup_{\boldsymbol{n}\to\infty} \Pi^+(J,\boldsymbol{n};\varepsilon) = 0, \tag{38}$$

where

$$\Pi^-(J,\boldsymbol{n};\varepsilon) := P\Bigg(\sup_{j\geq J} 2^{\alpha j} \max_{\substack{r\in D_j \\ \boldsymbol{0}\leq \boldsymbol{\ell}\leq \boldsymbol{2}^j}} |\Xi_{\boldsymbol{n}}(r,s_{\boldsymbol{\ell}}) - \Xi_{\boldsymbol{n}}(r^-,s_{\boldsymbol{\ell}})| > \varepsilon\Bigg), \tag{39}$$

$$\Pi^+(J,\boldsymbol{n};\varepsilon) := P\Bigg(\sup_{j\geq J} 2^{\alpha j} \max_{\substack{r\in D_j \\ \boldsymbol{0}\leq \boldsymbol{\ell}\leq \boldsymbol{2}^j}} |\Xi_{\boldsymbol{n}}(r^+,s_{\boldsymbol{\ell}}) - \Xi_{\boldsymbol{n}}(r,s_{\boldsymbol{\ell}})| > \varepsilon\Bigg), \tag{40}$$

with $D_j = \{2(l-1)2^{-j}; 1 \leq l \leq 2^{j-1}\}$, $r^- = r - 2^{-j}$, $\boldsymbol{\ell} = (l_2,\dots,l_d)$, $\boldsymbol{2}^j = (2^j,\dots,2^j)$ (vector of dimension $d-1$) and $\boldsymbol{s}_\ell = \boldsymbol{\ell}\boldsymbol{2}^{-j}$.

We prove only the (37), since the proof of (38) is the same. Denote by $\mathbf{v} = (r, \boldsymbol{s}_{\boldsymbol{\ell}})$, and $\mathbf{v}^- = (r^-, \boldsymbol{s}_{\boldsymbol{\ell}})$. From (32) we have

$$\Xi_{\boldsymbol{n}}(r,s_{\boldsymbol{\ell}}) - \Xi_{\boldsymbol{n}}(r^-,s_{\boldsymbol{\ell}}) = S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v})) - S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v}^-)) + \sum_{l=1}^{d}(T_l(\mathbf{v}) - T_l(\mathbf{v}^-)).$$

To estimate this increment we discuss according to following configurations

Case 1. $u_1(r) = u_1(r^-)$. Consider first the increment $T_1(\mathbf{v}) - T_1(\mathbf{v}^-)$ and note that by (33) with $l = 1$,

$$T_1(\mathbf{v}) = \sum_{1\leq k\leq d} \frac{\mathrm{v}_k - b_k(u_k(\mathrm{v}_k))}{\Delta b_k(u_k(\mathrm{v}_k)+1)} \Delta^{(k)}_{u_k(\mathrm{v}_k)+1} S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v})).$$

Recall the notation (5) and note that by definition $\mathbf{v}_{2:d} = \mathbf{v}^-_{2:d}$ with $\boldsymbol{U}(\mathbf{v}) = \boldsymbol{U}(\mathbf{v}^-)$. Thus all terms indexed by $k \geq 2$ disappear in difference $T_1(\mathbf{v}) - T_1(\mathbf{v}^-)$. This leads to the factorisation

$$T_1(\mathbf{v}) - T_1(\mathbf{v}^-) = \frac{r - r^-}{\Delta b_1(u_1(r)+1)} \Delta^{(1)}_{u_1(r)+1} S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v})). \tag{41}$$

For $l \geq 2$, $T_l(\mathbf{v})$ is expressed by (33) as

$$T_l(\mathbf{v}) = \sum_{1\leq i_1<\dots<i_l\leq d} \frac{\mathrm{v}_{i_1} - b_{i_1}(u_{i_1}(\mathrm{v}_{i_1}))}{\Delta b_{i_1}(u_{i_1}(\mathrm{v}_{i_1})+1)} \cdots \frac{\mathrm{v}_{i_l} - b_{i_l}(u_{i_l}(\mathrm{v}_{i_l}))}{\Delta b_{i_l}(u_{i_l}(\mathrm{v}_{i_l})+1)}$$
$$\Delta^{(i_1)}_{u_{i_1}(\mathrm{v}_{i_1})+1} \cdots \Delta^{(i_l)}_{u_{i_l}(\mathrm{v}_{i_l})+1} S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v})).$$

In the difference $T_1(\mathbf{v}) - T_1(\mathbf{v}^-)$ all the terms for which $i_1 \geq 2$ again disappear and we obtain

$$T_l(\mathbf{v}) - T_l(\mathbf{v}^-) = \frac{r - r^-}{\Delta b_1(u_1(r)+1)} \sum_{1<i_2<\dots<i_l\leq d} \frac{\mathrm{v}_{i_2} - b_{i_2}(u_{i_2}(\mathrm{v}_{i_2}))}{\Delta b_{i_1}(u_{i_1}(\mathrm{v}_{i_1})+1)} \cdots \frac{\mathrm{v}_{i_l} - b_{i_l}(u_{i_l}(\mathrm{v}_{i_l}))}{\Delta b_{i_l}(u_{i_l}(\mathrm{v}_{i_l})+1)}$$
$$\Delta^{(1)}_{u_1(r)+1} \Delta^{(i_2)}_{u_{i_2}(\mathrm{v}_{i_2})+1} \cdots \Delta^{(i_l)}_{u_{i_l}(\mathrm{v}_{i_l})+1} S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v})). \tag{42}$$

Since $u_1(r) = u_1(r^-)$, we have $b_1(u_1(r)) \le r < r^- < b_1(u_1(r)+1)$, thus

$$\frac{r - r^-}{\Delta b_1(u_1(r)+1)} \le \left(\frac{r - r^-}{\Delta b_1(u_1(r)+1)}\right)^\alpha.$$

Now

$$\frac{\mathrm{v}_{i_2} - b_{i_2}(u_{i_2}(\mathrm{v}_{i_2}))}{\Delta b_{i_1}(u_{i_1}(\mathrm{v}_{i_1})+1)} \cdots \frac{\mathrm{v}_{i_l} - b_{i_l}(u_{i_l}(\mathrm{v}_{i_l}))}{\Delta b_{i_l}(u_{i_l}(\mathrm{v}_{i_l})+1)} < 1$$

and

$$\begin{aligned} |\Delta^{(1)}_{u_1(r)+1} \Delta^{(i_2)}_{u_{i_2}(\mathrm{v}_{i_2})+1} \cdots \Delta^{(i_l)}_{u_{i_l}(\mathrm{v}_{i_l})+1} S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v}))| &= |\Delta^{(1)}_{u_1(r)+1} \sum_{\boldsymbol{i}\in I} \varepsilon_{\boldsymbol{i}} S_{\boldsymbol{n}}(\boldsymbol{i})| \\ &\le \sum_{\boldsymbol{i}\in I} |\Delta^{(1)}_{u_1(r)+1} S_{\boldsymbol{n}}(\boldsymbol{i})|, \end{aligned} \tag{43}$$

where $\varepsilon_{\boldsymbol{i}} = \pm 1$ and I is some appropriate subset of $[0, \boldsymbol{n}] \cap \mathbb{N}^d$ with 2^{l-1} elements. Denote for convenience

$$Z_{\boldsymbol{n}} = \max_{1 \le \boldsymbol{k} \le \boldsymbol{k}_{\boldsymbol{n}}} \frac{|\Delta^{(1)}_{k_1} S_{\boldsymbol{n}}(\boldsymbol{k})|}{(\Delta b_1(k_1))^\alpha}. \tag{44}$$

Now noting that $r - r^- = 2^{-j}$ and $\Delta b_1(k_1)$ depends only on k_1, we obtain for $l \ge 2$

$$|T_l(\boldsymbol{u}') - T_l(\boldsymbol{u})| \le 2^{-j\alpha} \binom{d-1}{l-1} 2^{l-1} Z_{\boldsymbol{n}}.$$

Thus

$$|\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)| \le \sum_{l=1}^{d} 2^{-j\alpha} \binom{d-1}{l-1} 2^{l-1} Z_{\boldsymbol{n}} = 3^{d-1} 2^{-j\alpha} Z_{\boldsymbol{n}} \tag{45}$$

Case 2. $u_1(r) = u_1(r^-) + 1$. In this case we have $b_1(u_1(r^-)) \le r^- < b_1(u_1(r)) \le r$. Using previous definitions we can write

$$|\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)| \le |\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(b_1(u_1(r)), \boldsymbol{s}_\ell)| + |\Xi_{\boldsymbol{n}}(b_1(u_1(r)), \boldsymbol{s}_\ell) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)|.$$

Now

$$\frac{r - b_1(u_1(r))}{\Delta b_1(u_1(r)+1)} \le \left(\frac{r - b_1(u_1(r))}{\Delta b_1(u_1(r)+1)}\right)^\alpha \le \frac{2^{-j\alpha}}{(\Delta b_1(u_1(r)+1))^\alpha}$$

and similarly

$$\frac{b_1(u_1(r)) - r^-}{\Delta b_1(u_1(r^-)+1)} \le \frac{2^{-j\alpha}}{(\Delta b_1(u_1(r^-)+1))^\alpha}.$$

Combining these inequalities with (41) and (42) we get as in(45)

$$|\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)| \le 2 \cdot 3^{d-1} 2^{-j\alpha} Z_{\boldsymbol{n}}.$$

Case 3. $u_1(r) > u_1(r^-) + 1$. Put

$$\mathbf{u} = (b_1(u_1(r)), \boldsymbol{s}_\ell), \quad \mathbf{u}^- = (b_1(u_1(r^-)) + 1, \boldsymbol{s}_\ell)$$

and

$$\psi_{\boldsymbol{n}}(r, r^-) = \max_{\boldsymbol{k}_{2:d} \leq \boldsymbol{k}_{\boldsymbol{n},2:d}} \left| \sum_{i=u_1(r^-)+2}^{u_1(r)} \Delta_i^{(1)} S_{\boldsymbol{n}}((i, \boldsymbol{k}_{2:d})) \right|. \tag{46}$$

Then

$$|\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)| \leq |\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{u})| + |\Xi_{\boldsymbol{n}}(\mathbf{u}) - \Xi_{\boldsymbol{n}}(\mathbf{u}^-)| + |\Xi_{\boldsymbol{n}}(\mathbf{u}^-) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)|$$

Recall notation (5) and note that $\boldsymbol{U}(\mathbf{u})_{2:d} = \boldsymbol{U}(\mathbf{u}^-)_{2:d} = \boldsymbol{U}(\mathbf{v})_{2:d}$. We have

$$\Xi_{\boldsymbol{n}}(\mathbf{u}) = S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{u})) + \sum_{l=1}^{d-1} \sum_{2 \leq i_1 < i_2 < \cdots < i_l \leq d} \left(\prod_{k=1}^{l} \frac{\mathrm{v}_{i_k} - b_{i_k}(u_{i_k}(\mathrm{v}_{i_k}))}{\Delta b_{i_k} u_{i_k}(\mathrm{v}_{i_k}) + 1)} \right) \left(\prod_{k=1}^{l} \Delta^{(i_k)}_{u_{i_k}(\mathrm{v}_{i_k})+1} \right) S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{u}))$$

and similar representation holds for $\Xi_n(\mathbf{u}^-)$. Since

$$S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{u})) - S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{u}^-)) = \sum_{i=u_1(r^-)+2}^{u_1(r)} \Delta_i^{(1)} S_{\boldsymbol{n}}((i, \boldsymbol{U}(\boldsymbol{s}_\ell))),$$

similar to (43) and (45) we get

$$|\Xi_{\boldsymbol{n}}(\mathbf{u}) - \Xi_{\boldsymbol{n}}(\mathbf{u}^-)| \leq \psi_{\boldsymbol{n}}(r, r^-) \sum_{l=0}^{d-1} 2^l \leq 3^{d-1} \psi_{\boldsymbol{n}}(r, r^-).$$

We can bound $|\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{u})|$ and $|\Xi_{\boldsymbol{n}}(\mathbf{u}^-) - \Xi_{\boldsymbol{n}}(\mathbf{v})|$ as in case 2. Thus we get

$$|\Xi_{\boldsymbol{n}}(r, \boldsymbol{s}_\ell) - \Xi_{\boldsymbol{n}}(r^-, \boldsymbol{s}_\ell)| \leq 3^{d-1} \psi_{\boldsymbol{n}}(r, r^-) + 2 \cdot 3^{d-1} 2^{-j\alpha} Z_{\boldsymbol{n}}. \tag{47}$$

Substituting this inequality into (39) we get that

$$\Pi^-(J, \boldsymbol{n}; \varepsilon) \leq \Pi_1(J, \boldsymbol{n}; \varepsilon/(2 \cdot 3^{d-1})) + \Pi_2(\boldsymbol{n}; \varepsilon/(4 \cdot 3^{d-1}))$$

where

$$\Pi_1(J, \boldsymbol{n}; \varepsilon) = P\left(Z_{\boldsymbol{n}} > \varepsilon \right) \tag{48}$$

and

$$\Pi_2(J, \boldsymbol{n}; \varepsilon) = P\left(\sup_{j \geq J} 2^{-j\alpha} \max_{r \in D_j} \psi_{\boldsymbol{n}}(r, r^-) > \varepsilon \right). \tag{49}$$

Thus (37) will hold if

$$\lim_{J\to\infty}\limsup_{\boldsymbol{n}\to\infty}\Pi_1(J,\boldsymbol{n};\varepsilon)=0, \tag{50}$$

$$\lim_{J\to\infty}\limsup_{\boldsymbol{n}\to\infty}\Pi_2(J,\boldsymbol{n};\varepsilon)=0. \tag{51}$$

Proof of (50). Using Markov and Doob inequalities for $q>1/(1/2-\alpha)$

$$\begin{aligned}
P\left(Z_{\boldsymbol{n}}>\varepsilon\right) &\le \sum_{k=1}^{k_{\boldsymbol{n}}^1} P\left(\max_{\boldsymbol{k}_{2:d}\le \boldsymbol{k}_{\boldsymbol{n},2:d}}|\Delta_k^{(1)}S_{\boldsymbol{n}}(\boldsymbol{k})|>\varepsilon(\Delta b_1(k))^{\alpha}\right)\\
&\le \sum_{k=1}^{k_{\boldsymbol{n}}^1}\varepsilon^{-q}(\Delta b_1(k))^{-q\alpha}\mathbf{E}\left(\max_{\boldsymbol{k}_{2:d}\le \boldsymbol{k}_{\boldsymbol{n},2:d}}|\Delta_k^{(1)}S_{\boldsymbol{n}}(\boldsymbol{k})|\right)^q\\
&\le \sum_{k=1}^{k_{\boldsymbol{n}}^1}\varepsilon^{-q}(\Delta b_1(k))^{-q\alpha}\mathbf{E}(|\Delta_k^{(1)}S_{\boldsymbol{n}}(\boldsymbol{k}_{\boldsymbol{n}})|^q).
\end{aligned}$$

By applying the Rosenthal inequality we get

$$P\left(Z_{\boldsymbol{n}}>\varepsilon\right)\le c\sum_{k=1}^{k_{\boldsymbol{n}}^1}\varepsilon^{-q}(\Delta b_1(k))^{-q\alpha}\left((\Delta b_1(k))^{q/2}+\sum_{k_2=1}^{k_{\boldsymbol{n}}^2}\cdots\sum_{k_d=1}^{k_{\boldsymbol{n}}^d}\mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{k}}|^q)\right). \tag{52}$$

We have

$$\begin{aligned}
\sum_{k=1}^{k_{\boldsymbol{n}}^1}(\Delta b_1(k))^{q(1/2-\alpha)} &\le \left(\max_{1\le k\le k_{\boldsymbol{n}}^1}\Delta b_1(k)\right)^{q(1/2-\alpha)-1}\sum_{k=1}^{k_{\boldsymbol{n}}^1}\Delta b_1(k)\\
&= \left(\max_{1\le k\le k_{\boldsymbol{n}}^1}\Delta b_1(k)\right)^{q(1/2-\alpha)-1}\to 0,\ \text{as } \mathrm{m}(\boldsymbol{n})\to\infty,
\end{aligned}$$

due to (9) and the fact that $q>(1/2-\alpha)$. Also

$$\begin{aligned}
\sum_{k=1}^{k_{\boldsymbol{n}}^1}(\Delta b_1(k))^{-q\alpha}\sum_{k_2=1}^{k_{\boldsymbol{n}}^2}\cdots\sum_{k_d=1}^{k_{\boldsymbol{n}}^d}\mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{k}}|^q) &= \sum_{\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}}(\Delta b_1(k_1))^{-q\alpha}\mathbf{E}|(X_{\boldsymbol{n},\boldsymbol{k}}|^q)\\
&\le \sum_{\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}}\sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha}\mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{k}}|^q)\to 0,\ \text{as } \mathrm{m}(\boldsymbol{n})\to\infty,
\end{aligned}$$

due to (10), since $(\Delta b_1(k_1))^{-q\alpha}\le\sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha}$ for all $\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}$. Reporting these estimates to (52) we see that (9) and (10) imply (50).

Proof of (51). We have

$$\begin{aligned}
\Pi_2(J,\boldsymbol{n},\varepsilon) &\le \sum_{j\ge J}P(2^{\alpha j}\max_{r\in D_j}\psi_{\boldsymbol{n}}(r,r^-)>\varepsilon)\\
&\le \sum_{j\ge J}\sum_{r\in D_j}\varepsilon^{-q}2^{\alpha jq}\mathbf{E}|\psi_{\boldsymbol{n}}(r,r^-)|^q.
\end{aligned}$$

Doob inequality together with (34) gives us

$$\mathbf{E}\,\psi_{\boldsymbol{n}}(r,r^-)^q \le \mathbf{E}\Big|\sum_{\boldsymbol{k}_{2:d}\le \boldsymbol{k}_{\boldsymbol{n},2:d}}\Big(\sum_{k_1=u_1(r^-)+2}^{u_1(r)} X_{\boldsymbol{n},\boldsymbol{k}}\Big)\Big|^q$$
$$\le c\Big(\Big(\sum_{k_1=u_1(r^-)+2}^{u_1(r)}\sum_{\boldsymbol{k}_{2:d}\le \boldsymbol{k}_{\boldsymbol{n},2:d}}\sigma^2_{\boldsymbol{n},\boldsymbol{k}}\Big)^{q/2}+\sum_{k_1=u_1(r^-)+2}^{u_1(r)}\sum_{\boldsymbol{k}_{2:d}\le \boldsymbol{k}_{\boldsymbol{n},2:d}}\mathbf{E}\,(|X_{\boldsymbol{n},\boldsymbol{k}}|^q)\Big).$$

Due to definition of $u_1(r)$

$$\sum_{k_1=u_1(r^-)+2}^{u_1(r)}\sum_{\mathbf{1}\le \boldsymbol{k}_{2:d}\le \boldsymbol{k}_{\boldsymbol{n},2:d}}\sigma^2_{\boldsymbol{n},\boldsymbol{k}}=\sum_{k_1=u_1(r^-)+2}^{u_1(r)}\Delta b_1(k_1)\le r-r^-=2^{-j},$$

thus

$$\Pi_1(J,\boldsymbol{n},\varepsilon)\le \frac{c}{\varepsilon^q}\sum_{j\ge J}2^{(q\alpha+1-q/2)j}+\frac{c}{\varepsilon^q}\sum_{j\ge J}\sum_{r\in D_j}2^{q\alpha j}\sum_{k_1=u_1(r^-)+2}^{u_1(r)}\sum_{\boldsymbol{k}_{2:d}\le \boldsymbol{k}_{\boldsymbol{n},2:d}}\mathbf{E}\,(|X_{\boldsymbol{n},\boldsymbol{k}}|^q). \tag{53}$$

Denote by $I(J,\boldsymbol{n},q)$ the second sum without the constant $c\varepsilon^{-q}$. By changing the order of summation we get

$$I(J,\boldsymbol{n},q)=\sum_{\mathbf{1}\le \boldsymbol{k}\le \boldsymbol{k}_{\boldsymbol{n}}}\mathbf{E}\,(|X_{\boldsymbol{n},\boldsymbol{k}}|^q)\sum_{j\ge J}2^{\alpha qj}\sum_{r\in D_j}\mathbf{1}\{u_1(r^-)+1<k_1\le u_1(r)\}. \tag{54}$$

The proof further proceeds as in one dimensional case [7]. Consider for fixed k_1 the condition

$$u_1(r^-)+1<k_1<u_1(r). \tag{55}$$

Suppose that there exists $r\in D_j$ satisfying (55) and take another $r'\in D_j$. Since u_1 is non decreasing, if $r'<r^-$ we have $u_1(r')<u_1(r^-)+1<k$, and thus r' cannot satisfy (55). If $r'>r$, we have that $r'^->r$, and we have that $k\le u_1(r)\le u_1(r'^-)<u_1(r'^-)+1$ and again if follows that r' cannot satisfy (55). Thus there will exists at most only one r satisfying (55). If such r exists we have

$$r^-\le \sum_{i=1}^{u_1(r^-)+1}\Delta b_1(i)<\sum_{i=1}^{k_1}\Delta b_1(i)\le \sum_{i=1}^{u_1(r)}\Delta b_1(i)\le r.$$

Thus $\Delta b_1(k_1)\le 2^{-j}$. So

$$\forall k_1=1,\dots,k^1_{\boldsymbol{n}},\sum_{r\in D_j}\mathbf{1}\{u_1(r^-)+1<k_1\le u_1(r)\}\le \mathbf{1}\{\Delta b_1(k_1)\le 2^{-j}\}$$

so

$$\sum_{j\ge J}2^{\alpha qj}\sum_{r\in D_j}\mathbf{1}\{u_1(r^-)+1<k_1\le u_1(r)\}\le \frac{2^{q\alpha}}{2^{q\alpha}-1}(\Delta b_1(k_1))^{-\alpha q} \tag{56}$$

(we can sum only those j, for which $\Delta b_1(k_1)\le 2^{-j}$, because for larger j, r and r^- will be closer together and will fall in the same $R_{\boldsymbol{n},\boldsymbol{k}}$).

Reporting estimate (56) to (54) we get

$$I(J,\boldsymbol{n},q) \le C \sum_{\boldsymbol{k}\le\boldsymbol{k_n}} (\Delta b_1(k_1))^{-q\alpha}\mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{k}}|^q) \le \sum_{\boldsymbol{k}\le\boldsymbol{k_n}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha}\mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{k}}|^q)$$

and substituting this to inequality (53) we get

$$\Pi_1(J,\boldsymbol{n};\varepsilon) \le C_1\varepsilon^{-q}2^{-Jq\alpha+1-q/2} + C_2 \sum_{\boldsymbol{k}\le\boldsymbol{k_n}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha}\mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{k}}|^q).$$

Thus (51) follows from (10), and condition *(ii)* holds.

5.2 Proof of the theorem 13

It suffices to check that (50) and (51) hold.

Proof of (50) Define:

$$S_{\boldsymbol{n},\tau}(\boldsymbol{k}) = \sum_{\boldsymbol{1}\le\boldsymbol{j}\le\boldsymbol{k}} X_{\boldsymbol{n},\boldsymbol{j},\tau}, \quad S'_{\boldsymbol{n},\tau}(\boldsymbol{k}) = \sum_{\boldsymbol{1}\le\boldsymbol{j}\le\boldsymbol{k}} (X_{\boldsymbol{n},\boldsymbol{j},\tau} - \mathbf{E}X_{\boldsymbol{n},\boldsymbol{j},\tau}). \tag{57}$$

and

$$A_{\boldsymbol{n}} = \left\{ \max_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} |X_k| \le \tau\sigma_{\boldsymbol{n},\boldsymbol{k}}^{2\alpha} \right\}.$$

Then the we can estimate the probability in (50) by

$$P\left(\max_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} \frac{|\Delta_{k_1}^{(1)} S_{\boldsymbol{n}}(\boldsymbol{k})|}{(\Delta b_1(k_1))^\alpha} > \varepsilon \right) =: \Pi_1(\boldsymbol{n},\varepsilon) \le \Pi_1(\boldsymbol{n},\varepsilon,\tau) + P(A_{\boldsymbol{n}}^c)$$

where

$$\Pi_1(\boldsymbol{n},\varepsilon,\tau) = P\left(\max_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} \frac{|\Delta_{k_1}^{(1)} S_{\boldsymbol{n},\tau}(\boldsymbol{k})|}{(\Delta b_1(k_1))^\alpha} > \varepsilon \right). \tag{58}$$

Due to (14) the probability $P(A_{\boldsymbol{n}}^c)$ tends to zero so we need only to study the asymptotics of $\Pi_1(\boldsymbol{n},\varepsilon,\tau)$.

Recall the definition (57). Using the splitting

$$\Delta_{k_1}^{(1)} S_{\boldsymbol{n},\tau}(\boldsymbol{k}) = \Delta_{k_1}^{(1)} S'_{\boldsymbol{n},\tau}(\boldsymbol{k}) + \mathbf{E}\,\Delta_{k_1}^{(1)} S_{\boldsymbol{n},\tau}(\boldsymbol{k}),$$

let us begin with some estimate of the expectation term, since $X_{\boldsymbol{n},\boldsymbol{j},\tau}$ are not centered.

We have

$$\mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{j},\tau}| \le \mathbf{E}^{1/2} X_{\boldsymbol{n},\boldsymbol{j}}^2 P^{1/2}(|X_{\boldsymbol{n},\boldsymbol{j}}| > \tau\sigma_{\boldsymbol{n},\boldsymbol{j}}^{2\alpha})$$

By applying Cauchy inequality we get

$$\begin{aligned}
\max_{\mathbf{1}\le \boldsymbol{k}\le \boldsymbol{k_n}} \frac{|\mathbf{E}\,\Delta^{(1)}_{k_1} S_{\boldsymbol{n},\tau}(\boldsymbol{k})|}{(\Delta b_1(k_1))^\alpha} &\le \max_{1\le k_1\le k^1_{\boldsymbol{n}}} \frac{\sum_{\boldsymbol{k}_{2:d}=1}^{\boldsymbol{k}_{\boldsymbol{n},2:d}} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{j},\tau}|}{(\Delta b_1(k_1))^\alpha}\\
&\le \max_{1\le k_1\le k^1_{\boldsymbol{n}}} \frac{(\Delta b_1(k_1))^{1/2}\left(\sum_{\boldsymbol{k}_{2:d}=1}^{\boldsymbol{k}_{\boldsymbol{n},2:d}} P(|X_{\boldsymbol{n},\boldsymbol{k}}|>\tau\sigma^{2\alpha}_{\boldsymbol{n},\boldsymbol{k}})\right)^{1/2}}{(\Delta b_1(k_1))^\alpha}\\
&\le \max_{1\le k_1\le k^1_{\boldsymbol{n}}} (\Delta b_1(k_1))^{1/2-\alpha}\left(\sum_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} P(|X_{\boldsymbol{n},\boldsymbol{k}}|>\tau\sigma^{2\alpha}_{\boldsymbol{n},\boldsymbol{k}})\right)^{1/2}.
\end{aligned}$$

Due to (9) and (14) the last expression is bounded by $\varepsilon/2$ for $\boldsymbol{n}\ge\boldsymbol{n}_0$, where $\boldsymbol{n}_0$ depends on ε and τ. Thus for $\boldsymbol{n}\ge\boldsymbol{n}_0$ we have $\Pi_1(\boldsymbol{n},\varepsilon,\tau)\le\Pi_1'(\boldsymbol{n},\varepsilon,\tau)$, where

$$\Pi_1'(\boldsymbol{n},\varepsilon,\tau)=P\left(\max_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k_n}}\frac{|\Delta^{(1)}_{k_1}S'_{\boldsymbol{n},\tau}(\boldsymbol{k})|}{(\Delta b_1(k_1))^\alpha}>\varepsilon/2\right) \tag{59}$$

Since

$$\operatorname{Var} X_{\boldsymbol{n},\boldsymbol{k},\tau}\le\mathbf{E}X^2_{\boldsymbol{n},\boldsymbol{k},\tau}\le\mathbf{E}X^2_{\boldsymbol{n},\boldsymbol{k}}=\sigma^2_{\boldsymbol{n},\boldsymbol{k}},$$

using Markov, Doob and Rosenthal inequalities for $q>1/(1/2-\alpha)$ we get

$$\begin{aligned}
\Pi_1'(\boldsymbol{n},\varepsilon,\tau)&\le\sum_{k=1}^{k^1_{\boldsymbol{n}}}(\varepsilon/2)^{-q}(\Delta b_1(k))^{-q\alpha}\mathbf{E}(|\Delta^{(1)}_k S'_{\boldsymbol{n},\tau}(\boldsymbol{k_n})|^q)\\
&\le c\sum_{k=1}^{k^1_{\boldsymbol{n}}}(\varepsilon/2)^{-q}(\Delta b_1(k))^{-q\alpha}\left((\Delta b_1(k))^{q/2}+\sum_{\boldsymbol{k}_{2:d}=1}^{\boldsymbol{k}_{\boldsymbol{n},2:d}}\mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q)\right)\\
&\le c(\varepsilon/2)^{-q}\left(\sum_{k=1}^{k^1_{\boldsymbol{n}}}\Delta b_1(k))^{q(1/2-\alpha)}+\sum_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k_n}}\sigma^{-2q\alpha}_{\boldsymbol{n},\boldsymbol{k}}\mathbf{E}(|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q)\right).
\end{aligned}$$

Note that this estimate holds for each $\tau>0$. Combining all the estimates we get

$$\forall\tau>0,\quad \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty}\Pi_1(\boldsymbol{n},\varepsilon)\le c\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty}\sum_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k_n}}\sigma^{-2q\alpha}_{\boldsymbol{n},\boldsymbol{k}}\mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q.$$

with the constant c depending only on q. By letting $\tau\to0$ due to (16), (50) follows.

Proof of (51) Introduce similar definitions $\psi_{\boldsymbol{n},\tau}(r,r^-)$ and $\psi'_{\boldsymbol{n},\tau}(r,r^-)$ by exchanging variables $X_{\boldsymbol{n},\boldsymbol{k}}$ with variables $X_{\boldsymbol{n},\boldsymbol{k},\tau}$ and $X'_{\boldsymbol{n},\boldsymbol{k},\tau}:=X_{\boldsymbol{n},\boldsymbol{k},\tau}-\mathbf{E}X_{\boldsymbol{n},\boldsymbol{k},\tau}$ respectively. Similar to the proof of (50) we get that we need only to deal with asymptotics of $\Pi_2(J,\boldsymbol{n},\varepsilon,\tau)$, where

$$\Pi_2(J,\boldsymbol{n},\varepsilon,\tau)=P\left(\sup_{j\ge J}2^{\alpha j}\max_{r\in D_j}\psi_{\boldsymbol{n},\tau}(r,r^-)>\varepsilon\right).$$

Again we need to estimate the expectation term. We have

$$\begin{aligned}
&\sup_{j\geq J} 2^{\alpha j}\max_{r\in D_j}\max_{\mathbf{1}_{2:d}\leq \mathbf{k}_{2:d}\leq \mathbf{k}_{\mathbf{n},2:d}}\left|\sum_{i=u_1(r^-)+2}^{u_1(r)}\Delta_i^{(1)}\mathbf{E}S_{\mathbf{n},\tau}((i,\mathbf{k}_{2:d}))\right|\\
&\quad\leq \sup_{j\geq J} 2^{\alpha j}\max_{r\in D_j}\left(\sum_{i=u_1(r^-)+2}^{u_1(r)}\Delta b_1(i)\right)^{1/2}\left(\sum_{i=u_1(r^-)+2}^{u_1(r)}\sum_{\mathbf{k}_{2:d}=1}^{\mathbf{k}_{\mathbf{n},2:d}}P(|X_{\mathbf{n},\mathbf{k}}|>\sigma_{\mathbf{n},\mathbf{k}}^{2\alpha})\right)^{1/2}\\
&\quad\leq 2^{J(\alpha-1/2)}\left(\sum_{\mathbf{1}\leq \mathbf{k}\leq \mathbf{k}_{\mathbf{n}}}P(|X_{\mathbf{n},\mathbf{k}}|>\sigma_{\mathbf{n},\mathbf{k}}^{2\alpha})\right)^{1/2}.
\end{aligned}$$

The last expression is bounded by $\varepsilon/2$ for $\mathbf{n}\geq \mathbf{n}_0$, due to (14) where $\mathbf{n}_0$ depends on ε and τ, but not on J. Thus $\Pi_2(J,\mathbf{n},\varepsilon,\tau)\leq \Pi_2'(J,\mathbf{n},\varepsilon,\tau)$, where

$$\Pi_2'(J,\mathbf{n},\varepsilon,\tau):=P\left(\sup_{j\geq J}2^{\alpha j}\max_{r\in D_j}\psi'_{\mathbf{n},\tau}(r,r^-)>\varepsilon/2\right). \tag{60}$$

Applying the same arguments as in proving (53) we get

$$\begin{aligned}
\Pi_2'(J,\mathbf{n},\varepsilon,\tau)\leq{}& \frac{c}{\varepsilon^q}\sum_{j\geq J}2^{(q\alpha+1-q/2)j}\\
&+\frac{c}{\varepsilon^q}\sum_{j\geq J}\sum_{r\in D_j}2^{q\alpha j}\sum_{k_1=u_1(r^-)+2}^{u_1(r)}\sum_{\mathbf{1}\leq \mathbf{k}_{2:d}\leq \mathbf{k}_{\mathbf{n},2:d}}\mathbf{E}\,|X_{\mathbf{n},\mathbf{k},\tau}|^q.
\end{aligned}$$

Now using estimate (56) we get

$$P\left(\sup_{j\geq J}2^{\alpha j}\max_{r\in D_j}\psi'_{\mathbf{n},\tau}(r,r^-)>\varepsilon/2\right)\leq C_1 2^{(q\alpha+1-q/2)J}+C_2\sum_{\mathbf{1}\leq \mathbf{k}\leq \mathbf{k}_{\mathbf{n}}}\sigma_{\mathbf{n},\mathbf{k}}^{-2q\alpha}\mathbf{E}\,(|X_{\mathbf{n},\mathbf{k},\tau}|^q).$$

where constants C_1 and C_2 depend on q, α and ε. Note that this inequality holds for each $\tau>0$. Combining all the estimates we get

$$\forall\tau>0,\quad \lim_{J\to\infty}\limsup_{\mathrm{m}(\mathbf{n})\to\infty}\Pi_2(J,\mathbf{n},\varepsilon)\leq C_2\limsup_{\mathrm{m}(\mathbf{n})\to\infty}\sum_{\mathbf{1}\leq \mathbf{k}\leq \mathbf{k}_{\mathbf{n}}}\sigma_{\mathbf{n},\mathbf{k}}^{-2q\alpha}\mathbf{E}\,|X_{\mathbf{n},\mathbf{k},\tau}|^q.$$

By letting $\tau\to 0$ due to (16), (51) follows.

6 Proof of corollaries

6.1 Proof of the corollary 9

We have

$$\sum_{\mathbf{1}\leq \mathbf{k}\leq \mathbf{k}_{\mathbf{n}}}\mathbf{E}X_{\mathbf{n},\mathbf{k}}^2\mathbf{1}\{|X_{\mathbf{n},\mathbf{k}}|\geq\varepsilon\}\leq\frac{1}{\varepsilon^{q-2}}\sum_{\mathbf{1}\leq \mathbf{k}\leq \mathbf{k}_{\mathbf{n}}}\mathbf{E}\,(|X_{\mathbf{n},\mathbf{k}}|^q).$$

Since $\sigma_{\mathbf{n},\mathbf{k}}^2\leq 1$ condition (10) ensures that $\sum_{\mathbf{1}\leq \mathbf{k}\leq \mathbf{k}_{\mathbf{n}}}\mathbf{E}\,(|X_{\mathbf{n},\mathbf{k}}|^q)$ converges to zero. So Lindeberg condition (12) is ensured by (10), and we get the convergence of finite dimensional distributions. Since the set $\mathbb{N}^d$ with the binary relation $\mathbf{j}\leq\mathbf{n}$ is directed, the summation process $\{\Xi_{\mathbf{n}}(\mathbf{t}),\mathbf{t}\in[0,1]^d\}$ is a net. So (13) follows due to Prokhorov's theorem for nets, see e.g. [13, th.1.3.9, p.21].

6.2 Proof of the corollary 12

It is sufficient to check that

$$\mu_{\boldsymbol{n}}(\boldsymbol{t}) \to \pi(\boldsymbol{t}) \tag{61}$$

We have

$$b_i(k_i) = \sum_{k=1}^{k_i} a^i_{\boldsymbol{n},k},$$

thus

$$\mathbf{1}\{\boldsymbol{B}(\boldsymbol{k}) \in [0,\boldsymbol{t}]\} = \prod_{i=1}^{d} \mathbf{1}\{b_i(k_i) \in [0,t_i]\},$$

so

$$\mu_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{k}\le \boldsymbol{k_n}} \mathbf{1}\{\boldsymbol{B}(\boldsymbol{t}) \in [0,\boldsymbol{t}]\}\sigma^2_{\boldsymbol{n},\boldsymbol{k}} = \prod_{i=1}^{d} \sum_{k_i=1}^{k^i_{\boldsymbol{n}}} \mathbf{1}\{b_i(k_i) \in [0,t_i]\} a^i_{\boldsymbol{n},k_i}.$$

But

$$\sum_{k_i=1}^{k^i_{\boldsymbol{n}}} \mathbf{1}\{b_i(k_i) \in [0,t_i]\} a^i_{\boldsymbol{n},k_i} = \sum_{k_i=1}^{u_i(t_i)} a^i_{\boldsymbol{n},k_i} \to t_i,$$

thus (61) holds, which together with (10) gives us (13).

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