Electronic Journal of Probability

Vol. 9 (2004), Paper no. 9, pages 230–254.

Journal URL
http://www.math.washington.edu/~ejpecp/

Stochastic Differential Equations with Boundary Conditions Driven by a Poisson Noise

Aureli Alabert [1]
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Catalonia
e-mail: alabert@mat.uab.es

and

Miguel Ángel Marmolejo [2]
Departamento de Matemáticas
Universidad del Valle
Cali, Colombia
e-mail: mimarmol@univalle.edu.co

Abstract. We consider one-dimensional stochastic differential equations with a boundary condition, driven by a Poisson process. We study existence and uniqueness of solutions and the absolute continuity of the law of the solution. In the case when the coefficients are linear, we give an explicit form of the solution and study the reciprocal process property.

Keywords and phrases: Stochastic differential equations, boundary conditions, reciprocal processes, Poisson noise

AMS subject classification (2000): 60H10, 60J25, 34F05

Submitted to EJP on September 9, 2003. Final version accepted on March 18, 2004.

[1]Supported by grants SGR99-87 of CIRIT and BFM2003-0026 of DGI (MCYT)
[2]Supported by a grant of Universidad del Valle, Cali-Colombia

1 Introduction

Stochastic differential equations (s.d.e.) with boundary conditions driven by a Wiener process have been extensively studied in the last fifteen years, both in the ordinary and the partial differential cases. We highlight the papers of Ocone and Pardoux [13], Nualart and Pardoux [10], Donati-Martin [6], Buckdahn and Nualart [4], and Alabert, Ferrante and Nualart [1]. These equations arise from the usual ones when we replace the customary initial condition by a functional relation $h(X_0, X_1) = 0$ between two variables of the solution process X, which is only considered in the bounded time interval $[0,1]$. Features that have been considered include existence and uniqueness, absolute continuity of the laws, numerical approximations, and Markovian-type properties.

Recently, in our work [2], we have considered boundary value problems where the stochastic integral with respect to the Wiener process is replaced by an additive Poisson perturbation N_t

$$\begin{cases} X_t = X_0 + \int_0^t f(r, X_r)\, dr + N_t \ , \quad t \in [0,1] \ , \\ X_0 = \psi(X_1) \ , \end{cases}$$

where the boundary condition is written in a more manageable form. We established an existence and uniqueness result, studied the absolutely continuity of the laws, and characterized several classes of coefficients f which lead to the so-called reciprocal property of the solution.

Let us recall that $X = \{X_t,\ t \in [0,1]\}$ is a *reciprocal process* if for all times $0 \le s < t \le 1$, the families or random variables $\{X_u,\ u \in [s,t]\}$ and $\{X_u,\ u \in [0,1] - [s,t]\}$ are conditionally independent given X_s and X_t. This property is weaker than the usual Markov property.

Interest in reciprocal processes dates back to Bernstein [3] (they are also called Bernstein processes by physicists) because of their role in the probabilistic interpretation of quantum mechanics. It is by far not true that all s.d.e. with boundary conditions give rise to reciprocal processes and it is also false that a general reciprocal process could be represented as the solution of some sort of first order boundary value problem, no matter which type of driving process is taken. Nevertheless, it is interesting to try to find in which cases the probabilistic dynamic representation given by a first order s.d.e., together with a suitable boundary relation, is indeed able to represent a reciprocal process.

In this paper, we develop the same program as in our previous paper [2], but with a multiplicative Poisson perturbation. Specifically, we consider the equation

$$\begin{cases} X_t = X_0 + \int_0^t f(r, X_{r^-})\, dr + \int_0^t F(r, X_{r^-})\, dN_r \ , \quad t \in [0,1] \ , \\ X_0 = \psi(X_1) \ , \end{cases} \tag{1.1}$$

where $f, F \colon [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\psi \colon \mathbb{R} \to \mathbb{R}$ are measurable functions satisfying certain hypotheses, and $N = \{N_t,\ t \ge 0\}$ is a Poisson process with intensity 1.

Due to the boundary condition, the solution will anticipate any filtration to which N is adapted, and therefore the stochastic integral appearing in the equation is, strictly speaking, an anticipating integral. However, the bounded variation character of the Poisson process permits to avoid most of the technical difficulties of the anticipating stochastic integrals with respect to the Wiener process.

Equation (1.1) is a "forward equation". One can also consider the "backward equation"

$$\text{(1.2)} \qquad \begin{cases} X_t = X_0 + \int_0^t f(r, X_r)\,dr + \int_0^t F(r, X_r)\,dN_r\ , & t \in [0,1]\ , \\ X_0 = \psi(X_1)\ , \end{cases}$$

and the Skorohod-type equation

$$\text{(1.3)} \qquad \begin{cases} X_t = X_0 + \int_0^t b(r, X_r)\,dr + \int_0^t B(r, X_r)\,\delta\tilde{N}_r\ , & t \in [0,1]\ , \\ X_0 = \psi(X_1)\ , \end{cases}$$

where $\delta\tilde{N}_r$ denotes the Skorohod integral with respect to the compensated Poisson process. While the stochastic integrals in (1.1) and (1.2) are no more than Stieljes integrals, the Skorohod integral operator is defined by means of the chaos decomposition on the canonical Poisson space. We refer the reader to [8], [12] or [11] for an introduction to the canonical Poisson space, the chaos decomposition and the Skorohod integral.

The paper is organised as follows. Section 2 is devoted to the study of the stochastic flow (initial condition problem) associated with s.d.e. (1.1), which will give us the preliminary results needed for the boundary condition case. In Section 3 we study existence, uniqueness, regularity and absolute continuity of the solution to the problem (1.1). In both Sections 2 and 3, the case of linear equations is studied as a special example. In Section 4, we find some sufficient conditions for the solution of the linear equation to enjoy the reciprocal property. In the final Section 5, the relation of the forward equation (1.1) with the backward equation (1.2) and the Skorohod equation (1.3) is established. The linear equation is again considered with particular attention, and the chaos decomposition of the solution is computed in two very simple special cases.

We will use the notation $\partial_i g$ for the derivative of a function g with respect to the i-th coordinate, $g(s^-)$ and $g(s^+)$ for $\lim_{t\uparrow s} g(t)$ and $\lim_{t\downarrow s} g(t)$ respectively, and the acronym *càdlàg* for "right continuous with left limits". Throughout the paper, we employ the usual convention that a summation and a product with an empty set of indices are equal to zero and one, respectively.

2 Stochastic flows induced by Poisson equations

Let $N = \{N_t,\ t \geq 0\}$ be a standard Poisson process with intensity 1 defined on some probability space $(\Omega, \mathfrak{F}, P)$; that means, N has independent increments, $N_t - N_s$ has a Poisson law with parameter $t-s$, $N_0 \equiv 0$, and all its paths are integer-valued, non-decreasing, càdlàg, with jumps of size 1.

Throughout the paper, S_n will denote the n-th jump time of N:

$$S_n(\omega) := \inf\{t \geq 0 :\ N_t(\omega) \geq n\}\ .$$

The sequence S_n is strictly increasing to infinity, and $\{N_t = n\} = \{S_n \le t < S_{n+1}\}$.

Let us consider the pathwise equation

$$\varphi_{st}(x) = x + \int_s^t f(r, \varphi_{sr^-}(x))\, dr + \int_{(s,t]} F(r, \varphi_{sr^-}(x))\, dN_r \ , \quad 0 \le s \le t \le 1 \ , \tag{2.1}$$

where $x \in \mathbb{R}$, and assume that $f, F\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ are measurable functions such that f satisfies

(H_1) $\exists K_1 > 0 :\ \forall t \in [0,1],\ \forall x, y \in \mathbb{R},\ |f(t,x) - f(t,y)| \le K_1 |x - y|$,

(H_2) $M_1 := \sup_{t \in [0,1]} |f(t,0)| < \infty$.

For every $x \in \mathbb{R}$, denote by $\Phi(s,t;x)$ the solution to the deterministic equation

$$\Phi(s,t;x) = x + \int_s^t f(r, \Phi(s,r;x))\, dr \ , \quad 0 \le s \le t \le 1 \ . \tag{2.2}$$

All conclusions of the following lemma are well known or easy to show:

Lemma 2.1 *Under hypotheses (H_1) and (H_2), there exists a unique solution $\Phi(s,t;x)$ of equation (2.2). Moreover:*

1) *For every $0 \le s \le t \le 1$, and every $x \in \mathbb{R}$, $|\Phi(s,t;x)| \le (|x| + M_1) e^{K_1(t-s)}$.*
2) *For every $0 \le s \le r \le t \le 1$, and every $x \in \mathbb{R}$, $\Phi(r,t;\Phi(s,r;x)) = \Phi(s,t;x)$.*
3) *For every $0 \le s \le t \le 1$, and every $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$,*

$$(x_2 - x_1) e^{-K_1(t-s)} \le \Phi(s,t;x_2) - \Phi(s,t;x_1) \le (x_2 - x_1) e^{K_1(t-s)} \ .$$

   In particular, for every s, t, the function $x \mapsto \Phi(s,t;x)$ is a homeomorphism from $\mathbb{R}$ into $\mathbb{R}$.

4) *If $G\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ has continuous partial derivatives, then for every $0 \le s \le t \le 1$,*

$$G(t, \Phi(s,t;x)) = G(s,x) + \int_s^t \Big[\partial_1 G(r, \Phi(s,r;x)) + \partial_2 G(r, \Phi(s,r;x)) f(r, \Phi(s,r;x)) \Big]\, dr \ .$$

Using Lemma 2.1 one can prove easily the following analogous properties for equation (2.1):

Proposition 2.2 *Assume that f satisfies hypotheses (H_1) and (H_2) with constants K_1, M_1. Then, for each $x \in \mathbb{R}$, there exists a unique process $\varphi(x) = \{\varphi_{st}(x),\ 0 \le s \le t \le 1\}$ that solves (2.1). Moreover:*

(1) *If F satisfies hypotheses (H_1) and (H_2) with constants K_2 and M_2, then for every $0 \le s \le t \le 1$ and every $x \in \mathbb{R}$:*

$$|\varphi_{st}(x)| \le \Big[|x| + (M_1 + M_2)(N_t - N_s + 1) \Big] (1 + K_2)^{(N_t - N_s)} e^{K_1} \ .$$

(2) *For every* $0 \le s \le r \le t \le 1$ *and every* $x \in \mathbb{R}$, $\varphi_{rt}(\varphi_{sr}(x)) = \varphi_{st}(x)$.

(3) *If there exist constants* $-1 \le k_2 \le K_2$ *such that*

$$k_2(x-y) \le F(t,x) - F(t,y) \le K_2(x-y) \ , \quad t \in [0,1] \ , \quad x > y \ ,$$

then for all $0 \le s \le t \le 1$, *and all* $x_1, x_2 \in \mathbb{R}$ *with* $x_1 < x_2$,

$$(1+k_2)^{N_t - N_s} e^{-K_1(t-s)} \le \frac{\varphi_{st}(x_2) - \varphi_{st}(x_1)}{x_2 - x_1} \le (1+K_2)^{N_t - N_s} e^{K_1(t-s)} \ ,$$

with the convention $0^0 = 1$. *In particular, if* $k_2 > -1$, *then for each* $0 \le s \le t \le 1$ *the function* $x \mapsto \varphi_{st}(x)$ *is a random homeomorphism from* $\mathbb{R}$ *into* $\mathbb{R}$.

(4) *Suppose that* $G : [0,1] \times \mathbb{R} \to \mathbb{R}$ *has continuous partial derivatives. Then, for all* $0 \le s \le t \le 1$,

$$\begin{aligned} G(t, \varphi_{st}(x)) = G(s,x) &+ \int_s^t \Big[\partial_1 G(r, \varphi_{sr^-}(x)) + \partial_2 G(r, \varphi_{sr^-}(x)) f(r, \varphi_{sr^-}(x)) \Big] \, dr \\ &+ \int_{(s,t]} \Big[G(r, \varphi_{sr^-}(x) + F(r, \varphi_{sr^-}(x))) - G(r, \varphi_{sr^-}(x)) \Big] \, dN_r \ . \end{aligned}$$

By solving equation (2.2) between jumps, the value $\varphi_{st}(\omega, x)$ can be found recursively in terms of Φ: If $s_1 = S_1(\omega), \ldots, s_n = S_n(\omega)$ are the jump times of the path $N(\omega)$ on $(s,1]$, then

$$\begin{aligned} \varphi_{st}(x) = \Phi(s,t;x) \mathbf{1}_{[s,s_1)}(t) &+ \sum_{i=1}^{n-1} \Phi(s_i, t; \varphi_{ss_i^-}(x) + F(s_i, \varphi_{ss_i^-}(x))) \mathbf{1}_{[s_i, s_{i+1})}(t) \\ &+ \Phi(s_n, t; \varphi_{ss_n^-}(x) + F(s_n, \varphi_{ss_n^-}(x))) \mathbf{1}_{[s_n,1]}(t) \ . \end{aligned} \tag{2.3}$$

Notice that the paths $t \mapsto \varphi_{st}(x)$ $(t \ge s)$ are càdlàg and $\varphi_{st}(x) - \varphi_{st^-}(x) = F(t, \varphi_{st^-})(N_t - N_t^-)$. In the sequel, when $s = 0$, we will write $\varphi_t(x)$ in place of $\varphi_{0t}(x)$.

Example 2.3 *(Linear equation).* Let $f_1, f_2, F_1, F_2 \colon [0,1] \to \mathbb{R}$ be continuous functions, and $x \in \mathbb{R}$. Consider equation (2.1) with $s = 0$ and linear coefficients:

$$\varphi_t(x) = x + \int_0^t [f_1(r) + f_2(r)\varphi_{r^-}(x)] \, dr + \int_0^t [F_1(r) + F_2(r)\varphi_{r^-}(x)] \, dN_r \ , \quad 0 \le t \le 1 \ . \tag{2.4}$$

We can describe the solution of this equation as follows: Set $S_0 := 0$ and let $0 < S_1 < S_2 < \ldots$ be the jumps of Poisson process. For $t \in [S_i, S_{i+1})$, $i = 0, 1, 2, \ldots$,

$$\varphi_t(x) = \varphi_{S_i}(x) + \int_{S_i}^t [f_1(r) + f_2(r)\varphi_{r^-}(x)] \, dr \ .$$

Applying Proposition 2.2(4) with $G(t,x) = A(t)^{-1}x$, where

$$A(t) = \exp\left\{ \int_0^t f_2(r) \, dr \right\} \ ,$$

we obtain

$$\frac{\varphi_t(x)}{A(t)} = \frac{\varphi_{S_i}(x)}{A(S_i)} + \int_{S_i}^t \frac{f_1(r)}{A(r)}\,dr \ . \tag{2.5}$$

On the other hand, for $i = 1, 2, 3, \dots$,

$$\begin{aligned} \varphi_{S_i}(x) &= \varphi_{S_i^-}(x) + F_1(S_i) + F_2(S_i)\varphi_{S_i^-}(x) \\ &= F_1(S_i) + [1 + F_2(S_i)]\varphi_{S_i^-}(x) \ . \end{aligned} \tag{2.6}$$

¿From (2.5) and (2.6), it follows that, for $t \in [0, S_1)$,

$$\frac{\varphi_t(x)}{A(t)} = x + \int_0^t \frac{f_1(r)}{A(r)}\,dr \ ,$$

and that for $t \in [S_i, S_{i+1})$, $i = 1, 2, \dots$,

$$\begin{aligned} \frac{\varphi_t(x)}{A(t)} =& \Big[x + \int_0^{S_1} \frac{f_1(r)}{A(r)}\,dr\Big] \prod_{j=1}^{i}(1 + F_2(S_j)) + \Big[\frac{F_1(S_1)}{A(S_1)} + \int_{S_1}^{S_2} \frac{f_1(r)}{A(r)}\,dr\Big] \prod_{j=2}^{i}(1 + F_2(S_j)) + \cdots \\ & \cdots + \Big[\frac{F_1(S_{i-1})}{A(S_{i-1})} + \int_{S_{i-1}}^{S_i} \frac{f_1(r)}{A(r)}\,dr\Big] \prod_{j=i}^{i}(1 + F_2(S_j)) + \Big[\frac{F_1(S_i)}{A(S_i)} + \int_{S_i}^{t} \frac{f_1(r)}{A(r)}\,dr\Big] \ . \end{aligned}$$

When $F_2(t) \neq -1$ for almost all $t \in [0, 1]$ with respect to Lebesgue measure, we can also write the solution as follows:

$$\varphi_t(x) = \eta_t\Big[x + \int_0^t \frac{f_1(r)}{\eta_r}\,dr + \int_0^t \frac{F_1(r)}{\eta_r}\,dN_r\Big] \ , \quad \text{a.s.,}$$

where

$$\eta_t = A(t) \prod_{0<S_i\leq t} [1 + F_2(S_i)] \ . \quad \square$$

Under differentiability assumptions on f and F we obtain differentiability properties of the solution to (2.1):

Proposition 2.4 *Assume that f satisfies the stronger hypotheses:*

(H_1') *f, $\partial_2 f$ are continuous functions.*

(H_2') $\exists K > 0 :\ |\partial_2 f| \leq K$.

Then

(1) *For every $\omega \in \Omega$ and every $x \in \mathbb{R}$, the function $t \mapsto \varphi_{st}(\omega, x)$ is differentiable on $[s, 1] - \{s_1, s_2, \dots\}$, where $s_1, s_2, \dots$ are the jump times of $N(\omega)$ on $(s, 1]$, and*

$$\frac{d\varphi_{st}(x)}{dt} = f(t, \varphi_{st}(x)) \ .$$

(2) *If moreover F and $\partial_2 F$ are continuous functions, then for every $\omega \in \Omega$ and every $0 \le s \le t \le 1$, the function $x \mapsto \varphi_{st}(\omega, x)$ is continuously differentiable and*

$$\frac{d\varphi_{st}(\omega,x)}{dx} = \exp\Big\{ \int_s^t \partial_2 f(r, \varphi_{sr}(\omega,x))\, dr \Big\} \prod_{s<s_i\le t} [1 + \partial_2 F(s_i, \varphi_{ss_i^-}(\omega,x))] \ .$$

In particular, when $\partial_2 F > -1$, $x \mapsto \varphi_{st}(x)$ is a random diffeomorphism from $\mathbb{R}$ into $\mathbb{R}$.

(3) *Assume moreover that F, $\partial_1 F$ and $\partial_2 F$ are continuous functions. Fix $0 \le s < t \le 1$ and $n \in \{1, 2, \dots\}$. On the set $\{N_t - N_s = n\}$, the mapping $\omega \mapsto \varphi_{st}(\omega, x)$ regarded as a function $\varphi_{st}(s_1, s_2, \dots, s_n; x)$ defined on $\{s < s_1 < s_2 < \cdots < s_n \le t\}$ (where $s_j = S_j(\omega)$ are the jump times of $N(\omega)$ on $(s,t]$), is continuously differentiable and, for every $j \in \{1, \dots, n\}$,*

$$\begin{aligned}\frac{\partial\varphi_{st}(x)}{\partial s_j} =& \exp\Big\{ \int_{s_j}^t \partial_2 f(r, \varphi_{sr}(x))\, dr \Big\} \prod_{i=j+1}^{n} [1 + \partial_2 F(s_i, \varphi_{ss_i^-}(x))] \\ &\times \Big[- f(s_j, \varphi_{ss_j}(x)) + \partial_1 F(s_j, \varphi_{ss_j^-}(x)) + f(s_j, \varphi_{ss_j^-}(x))[1 + \partial_2 F(s_j, \varphi_{ss_j^-}(x))] \Big] \ .\end{aligned}$$

Proof: It is easy to see for the solution Φ of (2.2) that

$$\begin{aligned}\partial_1\Phi(s,t;x) &= -f(s,x)\exp\Big\{ \int_s^t \partial_2 f(r, \Phi(s,r;x))\, dr \Big\} \ , \\ \partial_2\Phi(s,t;x) &= f(t, \Phi(s,t;x)) \ , \\ \partial_3\Phi(s,t;x) &= \exp\Big\{ \int_s^t \partial_2 f(r, \Phi(s,r;x))\, dr \Big\} \ ,\end{aligned}$$

and that these derivatives are continuous on $\{0 \le s \le t \le 1\} \times \mathbb{R}$. Claims (1) and (2) follow from here and representation (2.3).

The existence and regularity of the function $\varphi_{st}(s_1, \dots, s_n; x)$ of (3) are also clear from (2.3). We compute now its derivative with respect to s_j. For $n = 1$, we get

$$\begin{aligned}\frac{d\varphi_{st}(x)}{ds_1} &= \partial_1\Phi(s_1, t; \varphi_{ss_1}(x)) + \partial_3\Phi(s_1, t; \varphi_{ss_1}(x))\frac{d\varphi_{ss_1}(x)}{ds_1} \\ &= \exp\Big\{ \int_{s_1}^t \partial_2 f(r, \varphi_{sr}(x))\, dr \Big\} \\ &\quad\times \Big[- f(s_1, \varphi_{ss_1}(x)) + \partial_1 F(s_1, \varphi_{ss_1^-}(x)) + f(s_1, \varphi_{ss_1^-}(x))[1 + \partial_2 F(s_1, \varphi_{ss_1^-}(x))] \Big] \ .\end{aligned}$$

Suppose that (3) holds for $n = k$. Then, for $n = k+1$ and $j = 1, \dots, k$,

$$\begin{aligned}\frac{\partial\varphi_{st}(x)}{\partial s_j} &= \partial_3\Phi(s_{k+1}, t; \varphi_{ss_{k+1}}(x))\frac{\partial\varphi_{ss_{k+1}}(x)}{\partial s_j} \\ &= \exp\Big\{ \int_{s_j}^t \partial_2 f(r, \varphi_{sr}(x))\, dr \Big\} \prod_{i=j+1}^{k+1} [1 + \partial_2 F(s_i, \varphi_{ss_i^-}(x))] \\ &\quad\times \Big[- f(s_j, \varphi_{ss_j}(x)) + \partial_1 F(s_j, \varphi_{ss_j^-}(x)) + f(s_j, \varphi_{ss_j^-}(x))[1 + \partial_2 F(s_j, \varphi_{ss_j^-}(x))] \Big] \ .\end{aligned}$$

Taking into account that

$$\varphi_{ss_{k+1}}(x) = \varphi_{ss_{k+1}^-}(x) + F(s_{k+1}, \varphi_{ss_{k+1}^-}(x)) \ ,$$
$$\frac{\partial \varphi_{ss_{k+1}}(x)}{\partial s_{k+1}} = [1 + \partial_2 F(s_{k+1}, \varphi_{ss_{k+1}^-}(x))] f(s_{k+1}, \varphi_{ss_{k+1}^-}(x)) + \partial_1 F(s_{k+1}, \varphi_{ss_{k+1}^-}(x)) \ ,$$

we obtain, for $j = k+1$,

$$\begin{aligned}\frac{\partial \varphi_{st}(x)}{\partial s_{k+1}} &= \partial_1 \Phi(s_{k+1}, t; \varphi_{ss_{k+1}}(x)) + \partial_3 \Phi(s_{k+1}, t; \varphi_{ss_{k+1}}(x)) \frac{\partial \varphi_{ss_{k+1}}(x)}{\partial s_{k+1}} \\ &= \exp\Big\{ \int_{s_{k+1}}^t \partial_2 f(r, \varphi_{sr}(x))\, dr \Big\} \\ &\quad \times \Big[- f(s_{k+1}, \varphi_{ss_{k+1}}(x)) + \partial_1 F(s_{k+1}, \varphi_{ss_{k+1}^-}(x)) \\ &\qquad + f(s_{k+1}, \varphi_{ss_{k+1}^-}(x))[1 + \partial_2 F(s_{k+1}, \varphi_{ss_{k+1}^-}(x))] \Big] \ . \quad \square\end{aligned}$$

In the next proposition we find that under the regularity hypotheses of Proposition 2.4 and an additional condition relating f and F, the law of $\varphi_t(x)$ is a weighted sum of a Dirac-δ and an absolutely continuous probability.

Proposition 2.5 *Let f satisfy hypotheses (H_1') and (H_2') of Proposition 2.4, and assume that F, $\partial_1 F$ and $\partial_2 F$ are continuous functions. Assume moreover that*

$$(2.7) \qquad |f(t, x + F(t,x)) - f(t,x)[1 + \partial_2 F(t,x)] - \partial_1 F(t,x)| > 0 \ , \quad \forall t \in [0,1], \ \forall x \in \mathbb{R} \ .$$

Let $\varphi(x) = \{\varphi_t(x), \ t \in [0,1]\}$ be the solution to (2.1) for $s = 0$. Then, for all $t > 0$, the distribution function L of $\varphi_t(x)$ can be written as

$$L(y) = e^{-t} L^D(y) + (1 - e^{-t}) L^C(y) \ ,$$

with

$$L^D(y) = \mathbf{1}_{[\Phi(0,t;x), \infty)}(y) \ ,$$

and

$$L^C(y) = (e^t - 1)^{-1} \int_{-\infty}^{y} \sum_{n=1}^{\infty} \frac{t^n}{n!} h_n(r)\, dr \ ,$$

where h_n is the density function of the law of $\varphi_t(x)$ conditioned to $N_t = n$.

Proof: Let $S_1, S_2, \dots$ be the jump times of $\{N_t, \ t \in [0,1]\}$. From Proposition 2.4(3), on the set $\{N_t = n\}$ $(n = 1, 2, \dots)$ we have $\varphi_t(x) = G(S_1, \dots, S_n)$ for some continuously differentiable function G, and that

$$\begin{aligned}\partial_n G(s_1, ..., s_n) = &\exp\Big\{ \int_{s_n}^t \partial_2 f(r, \varphi_r(x))\, dr \Big\} \\ &\times \Big[- f(s_n, \varphi_{s_n}(x)) + \partial_1 F(s_n, \varphi_{s_n^-}(x)) + f(s_n, \varphi_{s_n^-}(x))[1 + \partial_2 F(s_n, \varphi_{s_n^-}(x))] \Big] \ .\end{aligned}$$

Using $\varphi_{s_n}(x) = \varphi_{s_n^-}(x) + F(s_n, \varphi_{s_n^-}(x))$ and condition (2.7), we obtain $|\partial_n G| > 0$.

It is known that, conditionally to $\{N_t = n\}$, $(S_1, \dots, S_n)$ follows the uniform distribution on $D_n = \{0 < s_1 < \cdots < s_n < t\}$. If we define $T(s_1, \dots, s_n) = (z_1, \dots, z_n)$, with $z_i = s_i$, $1 \le i \le n-1$, and $z_n = G(s_1, \dots, s_n)$, then $(Z_1, \dots, Z_n) = T(S_1, \dots, S_n)$ is a random vector with density

$$h(z_1, \dots, z_n) = n!\, t^{-n} \big|\partial_n s_n(z_1, \dots, z_n)\big| \mathbf{1}_{T(D_n)}(z_1, \dots, z_n) \ ,$$

and therefore $\varphi_t(x)$ is absolutely continuous on $\{N_t = n\}$, for every $n \ge 1$, with conditional density

$$h_n(y) = \mathbf{1}_{G(D_n)}(y) \int \int \cdots \int h(z_1, \dots, z_{n-1}, y)\, dz_1 \ \dots \ dz_{n-1} \ .$$

Now,

$$\begin{aligned} L(y) &= \sum_{n=0}^{\infty} P\big\{\varphi_t(x) \le y /_{N_t} = n\big\} P\{N_t = n\} \\ &= e^{-t} P\big\{\varphi_t(x) \le y /_{N_t} = 0\big\} + e^{-t} \sum_{n=1}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{y} h_n(r)\, dr \\ &= e^{-t} \mathbf{1}_{[\Phi(0,t;x),\infty)}(y) + e^{-t} \int_{-\infty}^{y} \sum_{n=1}^{\infty} \frac{t^n}{n!} h_n(r)\, dr \ , \end{aligned}$$

and the result follows. □

Remark 2.6 When $f(t,x) \equiv f(x)$, $F(t,x) \equiv F(x)$ and f'' is continuous, condition (2.7) is satisfied if

$$|f'F - fF'| > \frac{1}{2} \|f''\|_\infty \|F\|_\infty^2 \ ,$$

which is the hypothesis used by Carlen and Pardoux in [5] (Theorem 4.3) to prove that, in the autonomous case, the law of $\varphi_1(x)$ is absolutely continuous on the set $\{N_1 \ge 1\}$.

3 Equations with boundary conditions

In this section we establish first an easy existence and uniqueness theorem, based on Proposition 2.2 above, when the initial condition is replaced by a boundary condition. Then we prove in this situation the analogue of Propositions 2.4(3) and 2.5 on the differentiability with respect to the jump times and the absolute continuity of the laws (Proposition 3.4 and Theorem 3.5 below, respectively).

Theorem 3.1 *Let $f, F\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be measurable functions such that f satisfies hypotheses (H_1) and (H_2) of Section 2, with constants K_1 and M_1 respectively, $F(t, \cdot)$ is continuous for each t, and there exists a constant $k_2 \ge -1$ such that $F(t,x) - F(t,y) \ge k_2(x-y)$, $\forall t \in [0,1]$, $x > y$. Assume that $\psi\colon \mathbb{R} \to \mathbb{R}$ satisfies*

(H_3) *ψ is a continuous and non-increasing function.*

Then

$$(3.1)\qquad \begin{cases} X_t = X_0 + \int_0^t f(r, X_{r^-})\, dr + \int_0^t F(r, X_{r^-})\, dN_r \ , & t \in [0,1] \ , \\ X_0 = \psi(X_1) \ , \end{cases}$$

admits a unique solution X, which is a càdlàg process.

Proof: By Proposition 2.2, for each $x \in \mathbb{R}$ there exists a unique càdlàg process $\varphi(x) = \{\varphi_t(x),\ t \in [0,1]\}$ that satisfies the equation

$$\varphi_t(x) = x + \int_0^t f(r, \varphi_{r^-}(x))\, dr + \int_0^t F(r, \varphi_{r^-}(x))\, dN_r \ , \quad t \in [0,1] \ .$$

¿From part (3) of the same proposition, for each $\omega \in \Omega$, the function $x \mapsto \varphi_1(\omega, x)$ is non-decreasing. Thus, by hypothesis (H_3), the function $x \mapsto \psi(\varphi_1(\omega, x))$ has a unique fixed point, that we define as $X_0(\omega)$. It follows that (3.1) has the unique solution $X_t(\omega) = \varphi_t(\omega, X_0(\omega))$. □

Remark 3.2 In general, the condition $k_2 \geq -1$ cannot be relaxed. For instance, the problem

$$\begin{cases} X_t = X_0 + \int_0^t X_{r^-}\, dr + \int_0^t -2X_{r^-}\, dN_r \ , \\ X_0 = 1 - \frac{1}{e} X_1 \ , \quad t \in [0,1] \ , \end{cases}$$

has no solutions. Indeed, the first equality implies $X_t = X_0 e^t (-1)^{N_t}$ (see Example 2.3), which gives $X_1 = -X_0 e$ for $N_1 \in \{1, 3, 5, \dots\}$, and this is incompatible with the boundary condition.

On the other hand, if we change the boundary condition to $X_0 = \frac{-1}{e} X_1$ the new problem has an infinite number of solutions:

$$X_t(\omega) = \begin{cases} 0 \ , & \text{if } N_1(\omega) = 0, 2, 4, \dots \\ x(\omega) e^t (-1)^{N_t(\omega)} \ , & \text{if } N_1(\omega) = 1, 3, 5, \dots \end{cases}$$

where $x(\omega)$ is an arbitrary real number.

Notice that the purpose of (H_3) is to ensure that $x \mapsto \psi(\varphi_1(x))$ has a unique fixed point. Alternative hypotheses that lead to the same consequence may be used instead for particular cases. See for instance the comments at the end of Example 3.3. □

Example 3.3 *(Linear equation).* Consider the linear equation

$$(3.2)\qquad \begin{cases} X_t = X_0 + \int_0^t [f_1(r) + f_2(r) X_{r^-}]\, dr + \int_0^t [F_1(r) + F_2(r) X_{r^-}]\, dN_r \ , \\ X_0 = \psi(X_1) \ , \quad t \in [0,1] \ , \end{cases}$$

where $f_1, f_2, F_1, F_2 \colon [0,1] \to \mathbb{R}$ are continuous functions, and $\psi \colon \mathbb{R} \to \mathbb{R}$ is a continuous and non-increasing function. Assume $F_2(t) \geq -1$ for all $t \in [0,1]$. By Theorem 3.1 there is a unique solution, and using Example 2.3 we can describe it as follows:

For $\omega \in \{S_1 > 1\}$,

$$X_t = A(t)\Big[x^* + \int_0^t \frac{f_1(r)}{A(r)}\,dr\Big]\ ,$$

where $A(t) = \exp\{\int_0^t f_2(r)\,dr\}$, and x^* solve

$$x = \psi\Big(A(1)\Big[x + \int_0^1 \frac{f_1(r)}{A(r)}\,dr\Big]\Big)\ .$$

For $\omega \in \{S_n < 1 < S_{n+1}\}$ $(n \geq 1)$ and $t \in [S_i, S_{i+1})$ we have

$$\begin{aligned}\frac{X_t}{A(t)} =& \Big[X_0 + \int_0^{S_1} \frac{f_1(r)}{A(r)}\,dr\Big]\prod_{j=1}^{i}(1+F_2(S_j)) + \Big[\frac{F_1(S_1)}{A(S_1)} + \int_{S_1}^{S_2} \frac{f_1(r)}{A(r)}\,dr\Big]\prod_{j=2}^{i}(1+F_2(S_j)) + \cdots \\ &\cdots + \Big[\frac{F_1(S_{i-1})}{A(S_{i-1})} + \int_{S_{i-1}}^{S_i} \frac{f_1(r)}{A(r)}\,dr\Big]\prod_{j=i}^{i}(1+F_2(S_j)) + \Big[\frac{F_1(S_i)}{A(S_i)} + \int_{S_i}^{t} \frac{f_1(r)}{A(r)}\,dr\Big]\ .\end{aligned}$$

where X_0 solve $x = \psi(\varphi_1(x))$, with

$$\begin{aligned}\frac{\varphi_1(x)}{A(1)} =& \Big[x + \int_0^{S_1} \frac{f_1(r)}{A(r)}\,dr\Big]\prod_{j=1}^{n}(1+F_2(S_j)) + \Big[\frac{F_1(S_1)}{A(S_1)} + \int_{S_1}^{S_2} \frac{f_1(r)}{A(r)}\,dr\Big]\prod_{j=2}^{n}(1+F_2(S_j)) + \cdots \\ &\cdots + \Big[\frac{F_1(S_{i-1})}{A(S_{i-1})} + \int_{S_{i-1}}^{S_i} \frac{f_1(r)}{A(r)}\,dr\Big]\prod_{j=i}^{n}(1+F_2(S_j)) + \Big[\frac{F_1(S_i)}{A(S_i)} + \int_{S_i}^{1} \frac{f_1(r)}{A(r)}\,dr\Big]\ .\end{aligned}$$

When $F_2(t) > -1$ for almost all $t \in [0,1]$ with respect to Lebesgue measure, we can also write the solution as follows:

$$X_t = \eta_t\Big[X_0 + \int_0^t \frac{f_1(r)}{\eta_r}\,dr + \int_0^t \frac{F_1(r)}{\eta_r}\,dN_r\Big]\ , \quad \text{a.s.,} \tag{3.3}$$

where

$$\eta_t = A(t)\prod_{0<S_i\leq t}[1+F_2(S_i)] = \exp\Big\{\int_0^t f_2(r)\,dr + \int_0^t \log(1+F_2(r))\,dN_r\Big\}\ .$$

Finally, we remark that if $-1 \leq F_2 \leq 0$, the monotonicity condition on ψ can be relaxed to

$$x > y \Rightarrow \psi(x) - \psi(y) \leq \alpha(x-y)\ ,$$

with $\alpha A(1) < 1$, because in this case the mapping $x \mapsto \psi(\varphi_1(\omega, x))$ has still a unique fixed point. □

Under differentiability assumptions on f, F and ψ, we will obtain differentiability properties of the solution to (3.1). Denote

$$A(s_j,t,X) := \exp\left\{\int_{s_j}^t \partial_2 f(r,X_r)\,dr\right\} \prod_{s_j<s_i\le t} [1+\partial_2 F(s_i,X_{s_i^-})]$$
$$\times \left[-f(s_j,X_{s_j}) + f(s_j,X_{s_j^-})[1+\partial_2 F(s_j,X_{s_j^-})] + \partial_1 F(s_j,X_{s_j^-})\right] ,$$

and

$$B(t,X) := \exp\left\{\int_0^t \partial_2 f(r,X_r)\,dr\right\} \prod_{0<s_i\le t} [1+\partial_2 F(s_i,X_{s_i^-})] .$$

Proposition 3.4 *Let $f,F\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ and $\psi\colon\mathbb{R}\to\mathbb{R}$ be measurable functions such that f satisfies hypotheses (H_1'), (H_2') of Section 2; F, $\partial_1 F$ and $\partial_2 F$ are continuous functions with $\partial_2 F \ge -1$, and*

(H_3') ψ is a continuously differentiable function with $\psi' \le 0$.

Let $X=\{X_t,\ t\in[0,1]\}$ be the solution to (3.1). Then:

(1) *Fix $n\in\{1,2,\dots\}$. On the set $\{N_1=n\}$, X_0 can be regarded as a function $X_0(s_1,s_2,...,s_n)$, defined on $\{0<s_1<\cdots<s_n<1\}$, where $s_j=S_j(\omega)$ are the jumps of $N(\omega)$ in $[0,1]$. This function is continuously differentiable, and for any $j=1,2,\dots,n$,*

$$\frac{\partial X_0}{\partial s_j} = \frac{\psi'(X_1)A(s_j,1,X)}{1-\psi'(X_1)B(1,X)} .$$

(2) *Take $t\in(0,1]$ and $n,k\in\{0,1,\dots\}$ such that $n+k\ge 1$. On the set $\{N_t=n\}\cap\{N_1-N_t=k\}$, X_t can be regarded as a function $X_t(s_1,\dots,s_{n+k})$ defined on $\{0<s_1<\cdots<s_{n+k}<1\}$, where $s_j=S_j(\omega)$ are the jumps of $N(\omega)$ in $[0,1]$. This function is continuously differentiable, and for any $j=1,2,\dots,n+k$,*

$$\frac{\partial X_t}{\partial s_j} = B(t,X)\frac{\partial X_0}{\partial s_j}\mathbf{1}_{\{1,\dots,n+k\}}(j) + A(s_j,t,X)\mathbf{1}_{\{1,\dots,n\}}(j) .$$

Proof: Since $X_0=\psi(\varphi_1(X_0))$, Proposition 2.4 and the Implicit Function Theorem ensure that X_0 is continuously differentiable, and we have

$$\frac{\partial X_0}{\partial s_j} = \frac{\psi'(\varphi_1(X_0))\left.\frac{\partial\varphi_1(x)}{\partial s_j}\right|_{x=X_0}}{1-\psi'(\varphi_1(X_0))\left.\frac{d\varphi_1(x)}{dx}\right|_{x=X_0}}$$
$$= \frac{\psi'(X_1)A(s_j,1,X)}{1-\psi'(X_1)B(1,X)} .$$

On the other hand, for $X_t = \varphi_t(X_0)$,

$$\begin{aligned}\frac{\partial X_t}{\partial s_j} =& \frac{d\varphi_t(x)}{dx}\Big|_{x=X_0}\frac{\partial X_0}{\partial s_j} + \frac{\partial \varphi_t(x)}{\partial s_j}\Big|_{x=X_0} \\ =& B(t,X)\frac{\partial X_0}{\partial s_j}\mathbf{1}_{\{1,\dots,n+k\}}(j) + A(s_j,t,X)\mathbf{1}_{\{1,\dots,n\}}(j)\ . \quad \square\end{aligned}$$

The following theorem is the counterpart of Proposition 2.5 for the case of boundary conditions. The proof follows the same lines but using at the end the decomposition

$$\begin{aligned}L_{X_t}(x) = & P\{X_t \le x,\, N_1 = 0\} \\ & + \sum_{n=1}^{\infty} P\{X_t \le x / N_t = 0,\ N_1 - N_t = n\} e^{-1}\frac{(1-t)^n}{n!} \\ & + \sum_{n=1}^{\infty} P\{X_t \le x / N_t = n\} e^{-t}\frac{t^n}{n!}\ .\end{aligned}$$

In order to apply the same change of variables of Proposition 2.5 we remark that, on the set $\{0 < s_1 < \cdots < s_n < t < s_{n+1} < \cdots < s_{n+k} < 1\}$,

$$\frac{\partial X_t}{\partial s_j} = \frac{A(s_j,t,X)}{1-\psi'(X_1)B(1,X)} \neq 0\ , \quad j = 1,2,\dots,n\ .$$

Theorem 3.5 *Let $f, F\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ and $\psi\colon\mathbb{R}\to\mathbb{R}$ satisfy the hypotheses of Proposition 3.4. Assume in addition that $\partial_2 F > -1$, $\psi' < 0$, and that condition (2.7) holds. Let x^* be the unique solution to $x = \psi(\Phi(0,1;x))$, and X the solution to (3.1). Then, the distribution function of X_t, $t\in(0,1]$, is*

$$L_{X_t}(x) = e^{-1}L^D_{X_t}(x) + (1-e^{-1})L^C_{X_t}(x)\ ,$$

with

$$L^D_{X_t}(x) = \mathbf{1}_{[\Phi(0,t;x^*),\infty)}(x)$$

and

$$L^C_{X_t}(x) = \frac{e^{-t}}{1-e^{-1}}\Big[e^{-(1-t)}\int_{-\infty}^{x}\sum_{n=1}^{\infty}\frac{(1-t)^n}{n!}h_{0n}(r)\,dr + \int_{-\infty}^{x}\sum_{n=1}^{\infty}\frac{t^n}{n!}h_n(r)\,dr\Big]\ ,$$

where h_{0n} is the density of X_t conditioned to $N_t = 0, N_1 = n$, and h_n is the density of X_t conditioned to $N_t = n$. For $t = 0$, the formula is also valid taking $h_n \equiv 0$.

4 The reciprocal property

Let $(\Omega,\mathfrak{F},P)$ be a probability space and let $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{B}$ be sub-σ-fields of $\mathfrak{F}$ such that $P(A_1\cap A_2|\mathfrak{B}) = P(A_1|\mathfrak{B})P(A_2|\mathfrak{B})$ for any $A_1\in\mathfrak{A}_1$, $A_2\in\mathfrak{A}_2$. Then the σ-fields $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are said to be *conditionally independent with respect to* $\mathfrak{B}$.

Definition 4.1 We say that $X = \{X_t,\ t \in [0,1]\}$ is a *reciprocal process* if for every $0 \le a < b \le 1$, the σ-fields generated by $\{X_t,\ t \in [a,b]\}$ and $\{X_t,\ t \in (a,b)^c\}$ are conditionally independent with respect to the σ-field generated by $\{X_a, X_b\}$.

One can show that if X is a Markov process then X is reciprocal, and that the converse is not true. For a proof of this fact, we refer the reader to Alabert and Marmolejo [2] (Proposition 4.2), where we also established the next lemma (Lemma 4.6 of [2]).

Lemma 4.2 *If $\xi = \{\xi_t,\ t \in [0,1]\}$ has independent increments and g is a Borel function, then $X := \{g(\xi_1) + \xi_t,\ t \in [0,1]\}$ is a reciprocal process.*

In our previous work [2], we obtained several sufficient conditions on f for the solution to enjoy the reciprocal property when the Poisson noise appears additively, namely in

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t f(r, X_r)\, dr + \int_0^t dN_r \ , \\ X_0 = \psi(X_1) \ , \quad t \in [0,1] \ . \end{cases}$$

The main classes of functions f leading to this property are those of the form $f(t,x) = f_1(t) + f_2(t)x$ and those which are 1-periodic in the second variable, $f(t,x) = f(t, x+1)$. But we showed with examples that there are many more; we also obtained conditions on f ensuring that the solution will not be reciprocal. In contrast, for equations driven by the Wiener process, conditions which are at the same time necessary and sufficient have been obtained in a wide variety of settings, even with multiplicative noise.

With a multiplicative Poisson noise, the techniques currently known do not seem to allow a general analysis. We will restrict ourselves to linear equations. The main result contained in the next theorem is that if both coefficients are truly linear in the second variable (i.e. $f(t,x) = f_2(t)x$, and $F(t,x) = F_2(t)x$) then the solution is reciprocal. We have not been able to obtain necessary conditions, even when considering only the class of linear equations. Thus, we have to leave open the study of the general linear case, which for white noise driven equations (with boundary conditions also linear) was studied thoroughly in the seminal paper of Ocone and Pardoux [13].

Theorem 4.3 *Let $f_1, f_2, F_1, F_2 \colon [0,1] \to \mathbb{R}$ be continuous functions with $F_2(t) \ge -1$ for all $t \in [0,1]$, and $\psi \colon \mathbb{R} \to \mathbb{R}$ a continuous and non-increasing function. Let $X = \{X_t,\ t \in [0,1]\}$ be the solution of*

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t [f_1(r) + f_2(r) X_{r^-}]\, dr + \int_0^t [F_1(r) + F_2(r) X_{r^-}]\, dN_r \ , \\ X_0 = \psi(X_1), \quad t \in [0,1] \ . \end{cases}$$

In each of the following cases, X is a reciprocal process:

(1) *ψ is constant.*

(2) *$\psi(0) = 0$, $f_1 \equiv F_1 \equiv 0$.*

(3) $F_2 \equiv 0$.

(4) $F_2 > -1,\ f_1 = F_1 \equiv 0$.

(5) $F_2 \equiv -1,\ f_1 = F_1 \equiv 0$.

Proof: (1) reduces to the case of initial condition, while in (2) the solution is identically zero. Thus in both situations we obtain a Markov process.

(3) In this case the solution is

$$X_t = A(t)\Big[X_0 + \int_0^t \frac{f_1(r)}{A(r)}\,dr + \int_0^t \frac{F_1(r)}{A(r)}\,dN_r\Big]\ ,$$

where $A(t) = \exp\{\int_0^t f_2(r)\,dr\}$ as before, and X_0 solves

$$X_0 = \psi\Big(A(1)\Big[X_0 + \int_0^1 \frac{f_1(r)}{A(r)}\,dr + \int_0^1 \frac{F_1(r)}{A(r)}\,dN_r\Big]\Big)\ .$$

Defining $Y_t := \frac{X_t}{A(t)}$, we can write $Y_t = \xi_t + g(\xi_1)$, where g is a Borel function and

$$\xi_t = \int_0^t \frac{f_1(r)}{A(r)}\,dr + \int_0^t \frac{F_1(r)}{A(r)}\,dN_r\ .$$

Since ξ has independents increments, Lemma 4.2 implies that Y, and therefore X, is a reciprocal process.

(4) Here the solution is given by

$$X_t = X_0 \exp\Big\{\int_0^t f_2(r)\,dr + \int_0^t \log(1+F_2(r))\,dN_r\Big\}\ ,$$

where X_0 satisfies

$$X_0 = \psi\Big(X_0 \exp\Big\{\int_0^1 f_2(r)\,dr + \int_0^1 \log(1+F_2(r))\,dN_r\Big\}\Big)\ .$$

When $\psi(0) = 0$, we are in case (2). If $\psi(0) > 0$, then $X_0 > 0$, and setting $Y_t := \log(X_t)$ we obtain $Y_t = g(\xi_1) + \xi_t$, where g is a measurable function and $\xi_t = \int_0^t f_2(r)\,dr + \int_0^t \log[1+F_2(r)]\,dN_r$. We reach the conclusion as in case (3). If $\psi(0) < 0$, we can proceed analogously.

(5) In this situation, we obtain

$$X_t(\omega) = \begin{cases} x^* A(t)\ , & \text{if } S_1(\omega) > 1 \\ \psi(0)A(t)\mathbf{1}_{[0,S_1(\omega))}(t)\ , & \text{if } S_1(\omega) \le 1\ , \end{cases}$$

where x^* solves $x = \psi(xA(1))$. This process can be thought as the solution to the initial value problem

$$X_t = \eta + \int_0^t f_2(r)X_{r-}\,dr - \int_0^t X_{r-}\,dN_r\ ,$$

where η is the random variable

$$\eta(\omega) = \begin{cases} x^* \ , & \text{if } S_1(\omega) > 1 \\ \psi(0) \ , & \text{if } S_1(\omega) \le 1 \ . \end{cases}$$

Although η anticipates the Poisson process, X not only has the reciprocal property, but it is in fact a Markov process. Indeed, it is immediate to check that $Y_t := X_t/A(t)$ is a Markov chain taking at most three values. □

5 Backward and Skorohod equations

In this Section we consider the backward and Skorohod versions of our boundary value problems. There are very simple cases where the backward equation, even in the initial condition situation, does not possess a solution. For example, for $k \in \mathbb{R}$, the equation

$$\varphi_t = 1 + \int_0^t k\varphi_s \, dN_s \ ,$$

leads to $\varphi_{S_1} = 1 + k\varphi_{S_1}$ at $t = S_1$, which is absurd for $k = 1$. In general, for the existence of a solution of

$$\varphi_t(x) = x + \int_0^t f(r, \varphi_r(x))\, dr + \int_0^t F(r, \varphi_r(x))\, dN_r \ , \quad t \in [0,1] \ , \tag{5.1}$$

it is necessary that the mapping $A_r(y) := y - F(r, y)$ be invertible for each r.

Assume now that $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses (H_1) and (H_2) of Section 2, and that either

$$\forall t \in [0,1], \ \exists \alpha(t) < 1 : \ x > y \Rightarrow F(t,x) - F(t,y) \le \alpha(t)(x - y) \ , \tag{5.2}$$

or

$$\forall t \in [0,1], \ \exists \alpha(t) > 1 : \ x > y \Rightarrow F(t,x) - F(t,y) \ge \alpha(t)(x - y) \ . \tag{5.3}$$

Then there exists a unique process $\varphi(x) = \{\varphi_t(x), t \in [0,1]\}$ that satisfies the backward equation (5.1). This follows from Theorem 5.1 of León, Solé and Vives [8], where it is shown that φ is a solution to (5.1) if and only if φ is a solution to the forward equation

$$\varphi_t(x) = x + \int_0^t f(r, \varphi_{r^-}(x))\, dr + \int_0^t F(r, A_r^{-1}(\varphi_{r^-}(x)))\, dN_r \ .$$

The existence of A_r^{-1} is assured by (5.2) or (5.3).

We consider now the backward equation with boundary condition

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t f(r, X_r)\, dr + \int_0^t F(r, X_r)\, dN_r \ , \\ X_0 = \psi(X_1) \ , \quad t \in [0,1] \ . \end{cases} \tag{5.4}$$

Theorem 5.1 *Assume that f satisfies hypotheses (H_1) and (H_2) of Section 2 and that*

$$\beta(t)(x-y) \le F(t,x) - F(t,y) \le \alpha(t)(x-y)\ , \quad t \in [0,1]\ , \quad x > y\ , \tag{5.5}$$

for some functions α and β such that $\alpha - 1 \le \beta \le \alpha < 1$. Assume moreover that ψ satisfies hypothesis (H_3) of Theorem 3.1. Then (5.4) admits a unique solution $X = \{X_t, t \in [0,1]\}$, which is a càdlàg process.

Proof: By the relation between the forward and the backward equations with initial condition given above, the solution to (5.4) coincides with the solution to the forward equation with boundary condition

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t f(r, X_{r-})\, dr + \int_0^t F(r, A_r^{-1}(X_{r-}))\, dN_r\ , \\ X_0 = \psi(X_1)\ , \quad t \in [0,1]\ , \end{cases} \tag{5.6}$$

provided it exists. By Theorem 3.1, it is enough to show there exists a constant $k_2 \ge -1$ such that $\tilde F(t,x) := F(t, A_t^{-1}(x))$ satisfies $\tilde F(t,x) - \tilde F(t,y) \ge k_2(x-y)$, $\forall t \in [0,1]$, $x > y$. From (5.5),

$$0 < (1-\alpha(t))(x-y) \le A_t(x) - A_t(y) \le (1-\beta(t))(x-y)\ ,$$

hence

$$\frac{x-y}{1-\beta(t)} \le A_t^{-1}(x) - A_t^{-1}(y) \le \frac{x-y}{1-\alpha(t)}\ .$$

We find

$$\tilde F(t,x) - \tilde F(t,y) \ge \begin{cases} \dfrac{\beta(t)}{1-\beta(t)}(x-y)\ , & \text{if } \beta(t) \ge 0\ , \\ \dfrac{\beta(t)}{1-\alpha(t)}(x-y)\ , & \text{if } \beta(t) < 0\ , \end{cases}$$

and the conclusion follows. □

The study of the properties of backward equations can thus be reduced to the case of the forward equations when the above condition (5.5) on F holds. In particular, when $F(t,x) = F_1(t) + F_2(t)x$, we obtain

$$\tilde F(t,x) = \frac{F_1(t)}{1-F_2(t)} + \frac{F_2(t)}{1-F_2(t)}\, x\ ,$$

and condition (5.5) reduces to $F_2 < 1$.

Example 5.2 *(Linear backward equation).* Now consider the problem

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t [f_1(r) + f_2(r)X_r]\, dr + \int_0^t [F_1(r) + F_2(r)X_r]\, dN_r\ , \\ X_0 = \psi(X_1)\ , \quad t \in [0,1]\ , \end{cases}$$

where $f_1, f_2, F_1, F_2\colon [0,1] \to \mathbb{R}$ are continuous functions with $F_2 < 1$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ is a continuous and non-increasing function. By Theorem 5.1, this problem has unique solution, given by (see Example 3.3)

$$X_t = \eta_t \Big[X_0 + \int_0^t \frac{f_1(r)}{\eta_r}\, dr + \int_0^t \frac{F_1(r)}{(1-F_2(r))\eta_r}\, dN_r \Big] \ ,$$

where

$$\eta_t = \exp\Big\{ \int_0^t f_2(r)\, dr - \int_0^t \log(1-F_2(r))\, dN_r \Big\} \ ,$$

and X_0 solves

$$x = \psi\Big(\eta_1 \Big[x + \int_0^1 \frac{f_1(r)}{\eta_r}\, dr + \int_0^1 \frac{F_1(r)}{(1-F_2(r))\eta_r}\, dN_r \Big] \Big) \ . \quad \square$$

We turn now to the Skorohod equation with boundary condition

$$\left\{ \begin{array}{l} X_t = X_0 + \displaystyle\int_0^t f(r, X_r)\, dr + \int_0^t F(r, X_r)\, \delta \tilde{N}_r \ , \\ X_0 = \psi(X_1) \ , \quad t \in [0,1] \ . \end{array} \right. \tag{5.7}$$

We place ourselves in the canonical Poisson space $(\Omega, \mathfrak{F}, P)$ (see e.g. [8], [12] or [11] for a more detailed introduction to the analysis in this space). The elements of Ω are sequences $\omega = (s_1, \ldots, s_n)$, $n \geq 1$, with $s_j \in [0,1]$, together with a special point a. The canonical Poisson process is defined in $(\Omega, \mathfrak{F}, P)$ as the measure-valued process

$$N(\omega) = \left\{ \begin{array}{ll} 0 \ , & \text{if } \omega = a \ , \\ \sum_{i=1}^n \delta_{s_i} \ , & \text{if } \omega = (s_1, \ldots, s_n) \ , \end{array} \right.$$

where δ_{s_i} means the Dirac measure on s_i. Any square integrable random variable H in this space can be decomposed in Poisson-Itô chaos $H = \sum_{n=0}^\infty I_n(h_n)$, where $I_n(h_n)$ is the n-th multiple Poisson-Itô integral of a symmetric kernel $h_n \in L^2([0,1]^n)$ with respect to the compensated Poisson process $\tilde{N}$. For $u \in L^2(\Omega \times [0,1])$ with decomposition $u_t = \sum_{n=0}^\infty I_n(u_n^t)$ for almost all $t \in [0,1]$, Nualart and Vives [12] define its Skorohod integral as $\delta(u) := \int_0^1 u_s\, \delta \tilde{N}_s := \sum_{n=0}^\infty I_{n+1}(\tilde{u}_n)$, where $\tilde{u}_n$ is the symmetrization of u_n with respect to its $n+1$ variables, provided $u \in \operatorname{Dom} \delta$, that means, if $\sum_{n=0}^\infty (n+1)!\, \|\tilde{u}_n\|^2_{L^2([0,1]^{n+1})} < \infty$.

For a process u with integrable paths, define the random variable

$$\phi(u)(\omega) := \left\{ \begin{array}{ll} -\int_0^1 u_t(a)\, dt \ , & \text{if } \omega = a \ , \\ u_{s_1}(a) - \int_0^1 u_t(s_1)\, dt \ , & \text{if } \omega = (s_1) \ , \\ \sum_{j=1}^n u_{s_j}(\hat{\omega}_j) - \int_0^1 u_t(\omega)\, dt \ , & \text{if } \omega = (s_1, \ldots, s_n),\ n > 1 \ , \end{array} \right.$$

where $\hat{\omega}_j$ means $(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)$. One can also consider, for any random variable H and for almost all $t \in [0,1]$, the random variable

$$(\Psi_t H)(\omega) = \left\{ \begin{array}{ll} H(t) - H(a) \ , & \text{if } \omega = a \\ H(s_1, \ldots, s_n, t) - H(\omega) \ , & \text{if } \omega = (s_1, \ldots, s_n) \ . \end{array} \right.$$

The following Lemma is shown in Nualart and Vives [12].

Lemma 5.3 *With the notations introduced above, we have*

(a) If $u \in L^2(\Omega \times [0,1])$, then $\phi(u) \in L^2(\Omega)$ if and only if $u \in Dom\ \delta$, and in that case $\delta(u) = \phi(u)$.

(b) If $H = \sum_{n=0}^{\infty} I_n(h_n) \in L^2(\Omega)$, then $\Psi H \in L^2(\Omega \times [0,1])$ if and only if $\sum_{n=0}^{\infty} n\, n!\, \|h_n\|^2_{L^2([0,1]^n)} < \infty$, and in that case $\Psi_t H = \sum_{n=0}^{\infty} (n+1) I_n(h_{n+1}(t,\cdot))$.

Two concepts of solution for initial value Skorohod equations were introduced in [7] by León, Ruiz de Chávez and Tudor, which they called "strong solution" and "ϕ-solution". In the latter, the process $F(r, X_r)$ is only required to have square integrable paths, and its integral is interpreted as $\phi(F(r, X_r))$. If $F(r, X_r)$ belongs to Dom δ, Lemma 5.3 (a) ensures that both concepts coincide. We only need here a version of the first notion, which we will call simply "solution". We supplement the definition in [7] with the requirement of càdlàg paths, for the boundary condition to be meaningful.

Definition 5.4 A measurable process X is a *solution* of (5.7), if

(1) $f(\cdot, X.) \in L^1([0,1])$ with probability 1.

(2) $\mathbf{1}_{[0,t]}(\cdot) F(\cdot, X.) \in \text{Dom}\ \delta$ for almost all $t \in [0,1]$.

(3) The first equality in (5.7) is satisfied with probability 1, for almost all $t \in [0,1]$.

(4) With probability 1, X is càdlàg and $X_0 = \psi(X_1)$.

Theorem 5.5 *Let $f, F \colon [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfy hypotheses (H_1) and (H_2) of Section 2 with constants K_1, M_1 and K_2, M_2, respectively. Assume moreover that $\psi \colon \mathbb{R} \to \mathbb{R}$ satisfies*

(H_3'') ψ is a continuous and bounded function that verifies one of the following Lipschitz-type conditions:

(i) $x > y \Rightarrow \psi(x) - \psi(y) \le \eta \cdot (x - y)$, for some real constant $\eta < e^{-\tilde{K}}$,

(ii) $x > y \Rightarrow \psi(x) - \psi(y) \ge \eta \cdot (x - y)$, for some real constant $\eta > e^{\tilde{K}}$,

where $\tilde{K} = K_1 + K_2$.

Then (5.7) admits a unique solution.

Proof: Under our hypotheses, we can apply Theorems 3.7 and 3.13 of [7] to the equation

$$X_t = \zeta + \int_0^t f(r, X_r)\, dr + \int_0^t F(r, X_r)\, \delta \tilde{N}_r \ , \tag{5.8}$$

where ζ is a given bounded random variable, and the solution has a càdlàg version given by

$$X_t = \sum_{n=0}^{\infty} X_t^n(\omega) \mathbf{1}_{[0,1]^n}(\omega) \ ,$$

where $[0,1]^0 = \{a\}$, and X^n are the respective unique solutions of

$$X_t^0(a) = \zeta(a) + \int_0^t (f-F)(r, X_r^0(a))\, dr \ , \tag{5.9}$$

$$X_t^1(s_1) = \zeta(s_1) + \int_0^t (f-F)(r, X_r^1(s_1))\, dr + \mathbf{1}_{[0,t]}(s_1) F(s_1, X_{s_1}^0(a)) \ , \tag{5.10}$$

and for $n \geq 2$,

$$X_t^n(\omega) = \zeta(\omega) + \int_0^t (f-F)(r, X_r^n(\omega))\, dr + \sum_{j=1}^n \mathbf{1}_{[0,t]}(s_j) F(s_j, X_{s_j}^{n-1}(\tilde{\omega}_j)) \ , \tag{5.11}$$

with $\omega = (s_1, \ldots, s_n)$ and $\hat{\omega}_j = (s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)$.

Denoting by $X^0(a,x)$ the solution of (5.9) starting at $\zeta(a) = x \in \mathbb{R}$, from Lema 2.1, for any $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$,

$$(x_2 - x_1)e^{-\tilde{K}t} \leq X_t^0(a, x_2) - X_t^0(a, x_1) \leq (x_2 - x_1)e^{\tilde{K}t} \ .$$

These inequalities and (H_3'') imply that there exists a unique point x^* such that

$$x^* = \psi(X_1^0(a, x^*)) \ ,$$

which we define as $X_0(a)$. Then, $X^0(a, X_0(a))$ satisfies (5.9) with $\zeta(a) = X_0(a)$ and the boundary condition of (5.7).

Once we know the path X^0, and given $\omega = s_1$, denoting by $X^1(s_1, x)$ the solution of (5.10) starting at $\zeta(s_1) = x \in \mathbb{R}$, one gets for any $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$,

$$(x_2 - x_1)e^{-\tilde{K}t} \leq X_t^1(s_1, x_2) - X_t^1(s_1, x_1) \leq (x_2 - x_1)e^{\tilde{K}t} \ .$$

Together with (H_3''), this implies that there exists a unique point x^* such that

$$x^* = \psi(X_1^1(s_1, x^*)) \ ,$$

which we define as $X_0(s_1)$. Then, $X^1(s_1, X_0(s_1))$ satisfies (5.10) with $\zeta(s_1) = X_0(s_1)$ and the boundary condition of (5.7).

In general, once we know the set of paths X^{n-1}, and given $\omega = (s_1, \ldots, s_n)$, using equation (5.11) one shows analogously that there exists a unique point x^* such that

$$x^* = \psi(X_1^n(\omega, x^*)) \ ,$$

which we define as $X_0(\omega)$. We have then that $X^n(\omega, X_0(\omega))$ satisfies (5.11) with $\zeta(\omega) = X_0(\omega)$ and the boundary condition of (5.7).

Since ψ is bounded, X_0 is a bounded random variable. The process thus constructed clearly satisfies (5.8) together with the boundary condition, and the theorem is proved. □

Remark 5.6 Skorohod equations can be converted to forward ones in special situations: When $F(t,x) \equiv F(t)$ or when $\psi \equiv x_0 \in \mathbb{R}$, the solution of (5.7) coincides with the solution of

$$\begin{cases} X_t = X_0 + \int_0^t (f - F)(r, X_{r-})\, dr + \int_0^t F(r, X_{r-})\, dN_r \ , \\ X_0 = \psi(X_1) \ , \quad t \in [0,1] \ . \end{cases} \qquad \square$$

The results of Sections 3 and 4 are automatically translated to Skorohod equations in the situations of the previous remark. In other cases, the inductive construction of the solution X, in which the value of $X_t(\omega)$ (with $\omega \in [0,1]^n$) depends on the values $X_t(\omega)$ (with $\omega \in [0,1]^{n-1}$), does not allow the equivalence.

In the last five years there have been some interest in finding the chaos decomposition of solutions to several type of equations in Poisson space (see e.g. [9], [8]). For instance, for

$$X_t = x + \int_0^t f_2(r) X_r\, dr + \int_0^t F_2(r) X_r\, \delta \tilde{N}_r \ , \quad x \in \mathbb{R} \ ,$$

one can find, using Lemma 3.10 of [9], the decomposition

$$X_t = x \exp\Big\{ \int_0^t f_2(r)\, dr \Big\} \sum_{n=0}^{\infty} I_n[(\mathbf{1}_{[0,t]}(\cdot) F_2(\cdot))^{\otimes n}]/n! \ .$$

We will give the chaos decomposition of the solution of two very specific linear equations with boundary conditions. First we discuss the resolution of Skorohod linear equations.

Example 5.7 *(Linear Skorohod equation).* Consider the problem

$$\begin{cases} X_t = X_0 + \int_0^t [f_1(r) + f_2(r) X_r]\, dr + \int_0^t [F_1(r) + F_2(r) X_r] \delta \tilde{N}_r \ , \\ X_0 = \psi(X_1) \ , \quad t \in [0,1] \ , \end{cases}$$

where $f_1, f_2, F_1, F_2 \colon [0,1] \to \mathbb{R}$ are continuous functions, and $\psi \colon \mathbb{R} \to \mathbb{R}$ satisfies (H_3'') of the Theorem 5.5. To describe X, let $Y_t(\omega) = \sum_{n=0}^{\infty} Y_t^n(\omega) \mathbf{1}_{[0,1]^n}(\omega)$, the solution to

$$Y_t = \int_0^t [f_1(r) + f_2(r) Y_r]\, dr + \int_0^t [F_1(r) + F_2(r) Y_r]\, \delta \tilde{N}_r \ .$$

Taking into account Remark 5.6, Y is the solution to the forward equation

$$Y_t = \int_0^t [(f_1 - F_1)(r) + (f_2 - F_2)(r) Y_{r-}]\, dr + \int_0^t [F_1(r) + F_2(r) Y_{r-}]\, dN_r \ ,$$

which is given in Example 3.3. Then $X_t = Y_t + Z_t$, where Z_t satisfies

$$\begin{cases} Z_t = Z_0 + \int_0^t f_2(r) Z_r\, dr + \int_0^t F_2(r) Z_r \delta \tilde{N}_r \ , \\ Z_0 = \psi(Y_1 + Z_1) \ , \quad t \in [0,1] \ . \end{cases}$$

We know (see proof of Theorem 5.5) that $Z_t = \sum_{n=0}^{\infty} Z_t^n(\omega)\mathbf{1}_{[0,1]^n}(\omega)$, where, writing $\tilde{A}(t) := \exp\{\int_0^t (f_2 - F_2)(r)\,dr\}$,

$$\frac{Z_t^0(a)}{\tilde{A}(t)} = Z_0(a)$$

and $Z_0(a)$ is the solution to

$$x = \psi(\tilde{A}(1)x + Y_1(a)) \ .$$

For $\omega = (s_1)$,

$$\frac{Z_t^1(s_1)}{\tilde{A}(t)} = Z_0(s_1)\mathbf{1}_{[0,s_1)}(t) + [Z_0(s_1) + F_2(s_1)Z_0(a)]\mathbf{1}_{[s_1,1]}(t)$$

and $Z_0(s_1)$ solves

$$x = \psi\big(\tilde{A}(1)[x + F_2(s_1)Z_0(a)] + Y_1(s_1)\big) \ .$$

In general, for $\omega = (s_1, \ldots, s_n)$ with $0 < s_1 < \cdots < s_n < 1$ we have

$$\begin{aligned}\frac{Z_t^n(\omega)}{\tilde{A}(t)} &= Z_0(\omega)\mathbf{1}_{[0,s_1)}(t) + \ldots \\ &\quad \cdots + \Big[Z_0(\omega) + \sum_{k=1}^{i} \sum_{\substack{j_1,\ldots,j_k=1 \\ \text{distinct}}}^{i} F_2(s_{j_1})\ldots F_2(s_{j_k})Z_0(\tilde{\omega}_{(s_{j_1},\ldots,s_{j_k})})\Big]\mathbf{1}_{[s_i,s_{i+1})}(t) + \ldots \\ &\quad \cdots + \Big[Z_0(\omega) + \sum_{k=1}^{n} \sum_{\substack{j_1,\ldots,j_k=1 \\ \text{distinct}}}^{n} F_2(s_{j_1})\ldots F_2(s_{j_k})Z_0(\tilde{\omega}_{(s_{j_1},\ldots,s_{j_k})})\Big]\mathbf{1}_{[s_n,1]}(t) \ ,\end{aligned}$$

where

$$\hat{\omega}_{(s_{j_1},\ldots,s_{j_k})} = (\ldots, s_{j_1-1}, s_{j_1+1}, \ldots, s_{j_k-1}, s_{j_k+1}, \ldots) \ , \quad \hat{\omega}_{(s_1,\ldots,s_n)} = a \ ,$$

and $Z_0(\omega)$ is the solution to

$$x = \psi\Big(\tilde{A}(1)\Big[x + \sum_{k=1}^{n} \sum_{\substack{j_1,\ldots,j_k=1 \\ \text{distinct}}}^{n} F_2(s_{j_1})\ldots F_2(s_{j_k})Z_0(\tilde{\omega}_{(s_{j_1},\ldots,s_{j_k})})\Big] + Y_1(\omega)\Big) \ . \quad \square$$

Example 5.8 *(Chaos decompositions)*

(1) Consider the problem

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t [f_1(r) + f_2(r)X_r]\,dr + \int_0^t F_1(r)\,\delta\tilde{N}_r \ , \\ X_0 = aX_1 + b \ , \quad t \in [0,1] \ , \end{cases}$$

where f_1, f_2, F_1 are continuous functions, and $a, b \in \mathbb{R}$ with $a \neq \exp\{-\int_0^1 f_2(r)\,dr\}$. Its solution coincides with the solution of the forward equation

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t [(f_1 - F_1)(r) + f_2(r)X_{r^-}]\,dr + \int_0^t F_1(r)\,dN_r \ , \\ X_0 = aX_1 + b \ , \quad t \in [0,1] \ , \end{cases}$$

which is

$$X_t = A(t)\Big[X_0 + \int_0^t \frac{(f_1 - F_1)(r)}{A(r)}\,dr + \int_0^t \frac{F_1(r)}{A(r)}\,dN_r\Big] \ ,$$

where $A(t) = \exp\{\int_0^t f_2(r)\,dr\}$ and

$$X_0 = \frac{aA(1)}{1 - aA(1)}\Big[\int_0^1 \frac{(f_1 - F_1)(r)}{A(r)}\,dr + \int_0^1 \frac{F_1(r)}{A(r)}\,dN_r\Big] + \frac{b}{1 - aA(1)} \ .$$

Therefore the chaos decomposition of X_t is

$$\begin{aligned} X_t = {} & \frac{bA(t)}{1 - aA(1)} + \int_0^1 A(t)\Big(\mathbf{1}_{[0,t]}(r) + \frac{aA(1)}{1 - aA(1)}\Big)\frac{f_1(r)}{A(r)}\,dr \\ & + I_1\Big[A(t)\Big(\mathbf{1}_{[0,t]}(\cdot) + \frac{aA(1)}{1 - aA(1)}\Big)\frac{F_1(\cdot)}{A(\cdot)}\Big] \ . \end{aligned}$$

(2) Consider the problem

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t f_2(r)X_{r^-}\,dr - \int_0^t X_{r^-}\,dN_r \ , \\ X_0 = \psi(X_1) \ , \quad t \in [0,1] \ , \end{cases}$$

where $f_2\colon [0,1] \to \mathbb{R}$ is a continuous function, and $\psi\colon \mathbb{R} \to \mathbb{R}$ is a continuous and non-increasing function. The solution is

$$X_t(\omega) = \begin{cases} x^* A(t) \ , & \text{if } S_1(\omega) > 1 \\ \psi(0)A(t)\mathbf{1}_{[0,S_1(\omega))}(t) \ , & \text{if } S_1(\omega) \le 1 \ , \end{cases}$$

where x^* is the unique solution to $x = \psi(xA(1))$. We can write in the Poisson space

$$X_t = A(t)\Big[(x^* - \psi(0))\mathbf{1}_{\{a\}} + \psi(0)\mathbf{1}_{\{t<S_1\}}\Big] \ .$$

Using Lemma 5.3 (b) one obtains the following chaos decomposition:

$$X_0 = \psi(0) + (x^* - \psi(0))e^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}I_n(1) \ ,$$

$$X_1 = A(1)x^* e^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}I_n(1) \ ,$$

and for $t \in (0,1)$ we have

$$X_t = A(t)\Big\{(x^* - \psi(0))e^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}I_n(1) + \psi(0)e^{-t}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}I_n\big[(\mathbf{1}_{[0,t]}(\cdot))^{\otimes n}\big]\Big\} \ . \quad \square$$

References

[1] A. Alabert, M. Ferrante, and D. Nualart. Markov field property of stochastic differential equations. *Ann. Probab.*, 23:1262–1288, 1995.

[2] A. Alabert and M. A. Marmolejo. Differential equations with boundary conditions perturbed by a Poisson noise. *Stochastic Process. Appl.*, 91:255–276, 2001.

[3] S. Bernstein. Sur les liaisons entre les grandeurs aléatoires. In *Verh. Internat. Math.-Kongr., Zurich*, pages 288–309, 1932.

[4] R. Buckdahn and D. Nualart. Skorohod stochastic differential equations with boundary conditions. *Stochastics Stochastics Rep.*, 45:211–235, 1993.

[5] E. Carlen and E. Pardoux. Differential calculus and integration by parts on Poisson space. *In Stochastics, Algebra and Analysis in Classical and Quantum Dynamics*, pages 63–73, 1990.

[6] C. Donati-Martin. Quasi-linear elliptic stochastic partial differential equations. Markov property. *Stochastics Stochastics Rep.*, 41:219–240, 1992.

[7] J. A. León, J. Ruiz de Chávez, and C. Tudor. Strong solutions of anticipating stochastic differential equations on the Poisson space. *Bol. Soc. Mat. Mexicana*, 2(3):55–63, 1996.

[8] J. A. León, J. Solé, and J. Vives. A pathwise approach to backward and forward stochastic differential equations on the Poisson space. *Stochastic Anal. Appl. 19 (2001), no. 5, 821–839*, 1997.

[9] J. A. León and C. Tudor. Chaos decomposition of stochastic bilinear equations with drift in the first Poisson-Itô chaos. *Statist. Probab. Lett.*, 48:11–22, 2000.

[10] D. Nualart and E. Pardoux. Boundary value problems for stochastic differential equations. *Ann. Probab.*, 19:1118–1144, 1991.

[11] D. Nualart and J. Vives. Anticipative calculus for the Poisson process based on the Fock space. *Séminaire de probabilités XXIV 1988/89, Lect. Notes Math. 1426*, pages 154–165, 1990.

[12] D. Nualart and J. Vives. A duality formula on the Poisson space and some applications. *In: Proccedings of the Ascona Conference on Stochastic Analysis. Progress in Probability, Birkhaüser*, pages 205–213, 1995.

[13] D. Ocone and E. Pardoux. Linear stochastic differential equations with boundary conditions. *Probab. Theory Related Fields*, 82:489–526, 1989.