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# Multifractal Analysis of a Class of Additive Processes with Correlated Non-Stationary Increments

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**Abstract**

We consider a family of stochastic processes built from infinite sums of independent positive random functions on $\mathbb{R}_+$. Each of these functions increases linearly between two consecutive negative jumps, with the jump points following a Poisson point process on $\mathbb{R}_+$. The motivation for studying these processes stems from the fact that they constitute simplified models for TCP traffic on the Internet. Such processes bear some analogy with Lévy processes, but they are more complex in the sense that their increments are neither stationary nor independent. Nevertheless, we show that their multifractal behavior is very much the same as that of certain Lévy processes. More precisely, we compute the Hausdorff multifractal spectrum of our processes, and find that it shares the shape of the spectrum of a typical Lévy process. This result yields a theoretical basis to the empirical discovery of the multifractal nature of TCP traffic.

**Keywords.** Multifractal processes, Hölder singularities, Hausdorff dimension, spectrum of singularities, Lévy processes, Internet Traffic Control Protocol.

**AMS Classification.** 28A80, 60G17, 60G30, 60J30.

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# 1 Background and Motivations

We study in this work a family of stochastic processes built from infinite sums of independent positive random functions on $\mathbb{R}_+$. Each of these functions increases linearly between two consecutive negative jumps, with the jump points following a Poisson point process on $\mathbb{R}_+$. The interest of this class of processes is twofold. The first is theoretical: It will be seen that the infinite sums of independent random positive functions that we study, though they have non-stationary and correlated increments, have connections with Lévy processes. The multifractal nature of Lévy processes has been demonstrated in [18]. A natural question is to enquire how the multifractal features of Lévy processes are modified when correlation and non-stationarity of the increments are present. It turns out that, at least in the frame we consider here, neither correlations nor non-stationarity modify the shape of the multifractal spectrum. More precisely, we compute the Hausdorff multifractal spectrum of our processes, and we show that it is the same as that of a typical Lévy process. Though the strategy developed in [17, 18] to study the multifractal nature of functions with a dense countable set of jump points applies partly here, our more complex setting requires different and/or refined arguments at key points of the study. In particular, we will need a refined version of Shepp's covering theorem for certain coverings of $\mathbb{R}_+$ by Poisson intervals.

The second interest stems from applications: The motivation for studying the processes considered here is that they constitute simplified but realistic models for TCP traffic on the Internet. Recent empirical studies, beginning with [23, 29], have shown that traffic on the Internet generated by the Traffic Control Protocol (TCP) is, under wide conditions, multifractal. This property has important consequences in practice. For instance, one may show that the queuing behavior of a multifractal traffic is significantly worse that the one of a non-fractal traffic (see [13] for details). It is therefore desirable to understand which features of TCP are responsible for multifractality, in order to try and reduce their negative impact on, e.g., the queuing behavior.

"Explaining" the multifractality of traffic traces from basic features of the Internet is a difficult task. Models investigated so far have been based on the paradigm of multiplicative cascades ([13],[24]). Indeed, with few exceptions (most notably [1, 15, 17, 18, 19]), multifractal analysis has mainly been applied to *multiplicative processes*. An obvious reason is that a multiplicative structure often leads naturally to multifractal properties ([25, 26, 8]).

However, there exists a number of real-world processes for which there is convincing experimental evidence of multifractality, but which do not display a naturally associated multiplicative structure. Among these, a major example is Internet traffic: Multiplicative models for TCP are not really convincing because there is no physical evidence that genuine traffic actually behaves as a cascading or multiplicative process. As a matter of fact, TCP traffic is rather an *additive* process, where the contributions of individual sources of traffic are merged in a controlled way.

The analysis developed below shows that *merely adding sources managed by TCP does lead to a multifractal behavior.* This result provides a theoretical confirmation to the empirical finding that TCP traffic is multifractal. Furthermore, it sheds light on the possible causes of this multifractality: Indeed, it indicates that it may be explained from the very nature of the protocol, with no need to invoke a hypothetical multiplicative structure. More precisely, multifractality in TCP already arises from the interplay between the additive increase multiplicative decrease (AIMD) mechanism and the variable synchronization of the sources. Finally, our computations allow to trace back, in a quantitative way, the main multifractal features of traces to specific mechanisms of TCP. This may have practical consequence in traffic engineering.

# 2 A simplified model of TCP traffic

The exact details of TCP are too intricate to allow for a tractable mathematical analysis. We consider a simplified model that captures the main ingredients of the congestion avoidance and flow control mechanisms of TCP. The reader interested with the precise features of TCP may consult [7, 24, 31]. Our model goes as follows (more details on the model may be found in [7]):

1. Each "source" of traffic $S_i$ sends "packets" of data at a time-varying rate. At time $t$, it sends $Z_i(t)$ packets.
2. Between two consecutive time instants $t$ and $t+1$, two things may happen: The source $i$ may experience a "loss", i.e. the flow control mechanisms of TCP detects that a packet sent by the source did not reach its destination. In this case, TCP tries to avoid congestion by forcing the source to halve the number of packets sent at time $t+1$ (multiplicative decrease mechanism). In other words, $Z_i(t+1) = Z_i(t)/2$. If there is no loss, the source is allowed to increase $Z_i(t)$ by 1, i.e. $Z_i(t+1) = Z_i(t)+1$ (additive increase mechanism).
3. The durations $(\tau_k^{(i)})_{k\geq 1}$ between time instants $t_k$ and $t_{k+1}$ where a given source $i$ experiences a loss are modeled by a sequence of independent exponential random variables with parameter $\lambda_i$.
4. The total traffic $Z$ is the sum of an infinite number of independent sources with varying rates $\lambda_i$, where $(\lambda_i)_{i\geq 1}$ is a non-decreasing sequence of positive numbers.

As compared to the true mechanisms of TCP, our model contains a number of simplifications (see [7]). However, except for one, these simplifications are not essential, at least as far as multifractality is concerned: Of all our assumptions, only the one of independence in (4) is clearly an oversimplification. Indeed, it is obvious that almost all losses are a consequence of congestion, which is caused by the fact that several sources are in competition. This gives rise to a strong correlation in the behavior of the sources. Unfortunately, introducing correlations leads to a significantly more complex analysis. One should remark nevertheless that the competition between sources is implicitly taken into account through the fact that sources indexed by large integers are subject to more frequent losses. We hope to investigate the effect of correlations on the multifractal behavior in a future work. Note also that most other approaches dealing with the fractal analysis of TCP make similar assumptions of independence. This is in particular the case in the models [4, 16, 22] discussed below.

Our model takes into account the main features of TCP, while allowing at the same time a thorough mathematical analysis: We show in the sequel that $Z$ is multifractal, and we compute its Hausdorff multifractal spectrum. Both the multifractality of $Z$ and the shape of its spectrum corroborates empirical findings ([23, 29]).

It is interesting to compare our approach with previous works dealing with the mathematical modeling of Internet traffic in relation with its (multi-) fractal behavior. A large number of studies ([16, 22, 27]) have given empirical evidence that many types of Internet traffic are "fractal", in the sense that they display self-similarity and/or long range dependence. Most theoretical models that have been developed so far have focused on explaining such behaviors. In that view, a popular class of models is based on the use of "ON/OFF" sources. An ON/OFF source is a source of traffic that is either idle, or sends data at a constant rate. Adequate assumptions on the distribution of the ON and/or OFF periods allow to obtain fractal properties. More precisely, the model in [22] considers independent and identically distributed ON/OFF sources, where the length of the ON and OFF periods are independent random variables. In addition, the distribution of the ON or/and of the OFF periods is assumed to have a regularly varying tail with exponent $\beta \in (1, 2)$. Then, when the number of sources tends to infinity, and if one rescales time slowly enough, the resulting traffic, properly normalized, tends to a fractional Brownian motion, with exponent $3/2 - \beta/2$. In [28], it is shown that the same model leads to a $\beta-$stable Lévy motion when the time rescaling is "fast". Finally, in a recent work, Gaigalas and Kaj ([14]) investigated the intermediate regime where time is rescaled proportionally to the number of sources. They found that the limit behavior in this case is neither a stable motion nor a fractional Brownian motion.

Another, elegant, model, which does not require a double re-normalization, is presented in [16]. It also uses a superposition of independent ON-OFF sources, but this time with a sequence of ratios for Poisson-idle and Poisson-active periods assumed to decay as a polynomial. Again, the resulting process display fractal features[1].

A major feature of the above models is that the sources, in their ON mode, send data at a constant rate. This is obviously a simplification, since one does not take into account the strong and rapid variations induced by the flow control mechanisms of TCP. This seems to be of no consequence for studying long range dependence or self-similarity: These properties are obtained through the slow decay of the probability of observing large busy or idle periods. These slow decays may in turn be traced back to certain large scale features, such as, *e.g.*, the distribution of the files sizes in the Internet ([11]). More generally, it is usually accepted that long memory is a property of the network.

In contrast, the use of ON/OFF sources does not allow a meaningful investigation of the multifractal properties of traffic: Contrarily to long range dependence, multifractality

[1]Note that the model that we consider does not require any kind of re-normalization.

is a short-time behavior. An ON/OFF modeling is clearly inadequate in this frame since it washes out all the (intra-source) high frequency content. At small time scales, the role of the protocol, *i.e.* TCP, becomes predominant ([4]). Incorporating some sort of modeling of TCP is thus necessary if one wants to perform a sensible high-frequency analysis: The local, rapid variations due to TCP, are determinant from the multifractal point of view.

In that view, it is interesting to note that the limiting behavior of the ON/OFF model which is usually considered is the one leading to fractional Brownian motion. It is therefore *not* multifractal. In contrast, the other limiting case gives rise to a stable motion, which *is* multifractal. A possible cause might be that, in this regime, the inter-source high frequency content (i.e. the rapid variations in the total traffic resulting from de-synchronized sources) is large enough to produce multifractality. However, it is not clear which actual mechanisms in the Internet would favor this particular regime. It would also be interesting to investigate whether the critical case studied in [14] is also multifractal.

Another approach that allows to "explain" the multifractal features of TCP is based on the use of "fluid models" ([4]): Rather than representing TCP at the packet level, one uses fluid equations to describe the joint evolution of throughput for sessions sharing a given router. The interest of this approach is that it represents the traffic as simple products of random matrices, while allowing to capture the AIMD mechanism of TCP. In particular, [4] shows through numerical simulations that this model does lead to a multifractal behavior. In other words, the fluid model indicates that the multifractality is already a consequence of the AIMD mechanism. This numerical result corroborates our theoretical findings. A network extension of the fluid model is studied in [5]. It also points to multifractality of the traces, with additional intriguing fractal features.

Note also that in a series of paper ([2, 3, 10]), F. Baccelli and collaborators have performed a fine analysis of TCP at the packet level. They have in particular shown that TCP is *Max-Plus linear*. A desirable extension of our work would be to study the multifractal properties of these more precise models.

# 3 A class of additive processes with non-stationary and correlated increments

We now describe our model in a formal way. Let $(\lambda_i)_{i\geq 1}$ be a non-decreasing sequence of positive numbers.

For every $i \geq 1$, let $(\tau_k^{(i)})_{k\geq 1}$ be a sequence of independent exponential random variables with parameter $\lambda_i$. Define $\tau_0^{(i)} = 0$. Set

$$T_k^{(i)} = \sum_{j=0}^{k} \tau_j^{(i)}.$$

The $\sigma$-algebras $\sigma(\tau_k^{(i)},\ k \geq 1)$ are assumed to be mutually independent.

We consider an infinite sequence of sources $(S_i)_{i\geq 1}$. The "traffic" $(Z_i(t))_{t\geq 0}$ generated by the source $S_i$, $i \geq 1$, is modeled by the following stochastic process

$$Z_i(t) = \begin{cases} Z_i(0) + t & \text{if } 0 \leq t < \tau_1^{(i)} \\ \dfrac{Z_i(T_{k-1}^{(i)}) + \tau_k^{(i)}}{\mu} + t - T_k^{(i)} & \text{if } T_k^{(i)} \leq t < T_{k+1}^{(i)} \text{ with } k \geq 1, \end{cases}$$

where $(Z_i(0))_{i\geq 1}$ is a sequence of non-negative random variables such that the series $\sum_{i\geq 1} Z_i(0)$ converge, and $\mu$ is a fixed real number larger than one (typically equal to 2 in the case of TCP).

The resulting "global traffic" is the stochastic process

$$Z(t) = \sum_{i\geq 1} Z_i(t) \quad (t \in \mathbb{R}_+).$$

Our first task is to give conditions under which $Z$ is almost surely everywhere finite.

**Proposition 1** *If $\sum_{i\geq 1} 1/\lambda_i < \infty$ then, with probability one, the stochastic process $Z$ is finite everywhere. If $\sum_{i\geq 1} 1/\lambda_i = \infty$ then, with probability one, $Z(t) = \infty$ almost everywhere with respect to the Lebesgue measure.*

The proof of Proposition 1 is postponed to Section 5.

We are interested in the multifractal nature of the sample paths of $Z$. In order to analyze this matter, it will be useful to decompose each elementary process $Z_i$ in the following way on $[T_k^{(i)}, T_{k+1}^{(i)})$:

$$Z_i = X_i + R_i$$

with

$$\begin{cases} X_i(t) = t - T_k^{(i)} \\ R_i(t) = \dfrac{Z_i(0)}{\mu^k} + \dfrac{1}{\mu^{k+1}} \displaystyle\sum_{j=1}^{k} \mu^j \tau_j^{(i)}. \end{cases}$$

Then, under the assumptions of Proposition 1, $Z$ is the sum of the two non-negative processes $X = \sum_{i\geq 1} X_i$ and $R = \sum_{i\geq 1} R_i$.

It will be shown that the processes $Z$ and $X$ share the multifractal spectrum of a Lévy process without Brownian part and whose characteristic measure is $\Pi = \sum_{i\geq 1} \lambda_i \delta_{-1/\lambda_i}$ (see [18] for the multifractal nature of Lévy processes).

A heuristic explanation of this fact is that the process $X$ "resembles" the Lévy process $L$ defined almost surely as $\lim_{N\to\infty} \sum_{i=1}^{N} L_i$ where $L_i(t) = t - 1/\lambda_i$ on $[T_k^{(i)}, T_{k+1}^{(i)})$ (see [9]): In particular, both $X$ and $L$ jump at each point $T_k^{(i)}$ $(i, k \geq 1)$; The jump sizes are mutually independent random variables for both processes; And, finally, at each $T_k^{(i)}$, the jump size of $L$ is the expectation of the jump size of $X$. A major difference is that the increments of $X$ are both correlated and not stationary. The same is true for the increments of $Z$. Moreover, the sizes of the jumps of $Z$ cease to be independent. This has important consequences in performing the multifractal analysis of $Z$: Even though the approach used by S. Jaffard in studying the multifractal nature of some functions with a countable dense set of jump points ([17], [18]) proves useful here, it is necessary to involve different and refined tools for the study of $X$ and $Z$. This will be discussed in more detail in Remark 1.

In the present work, the multifractal nature of $Z$ is investigated through the computation of its *spectrum of singularities* or *Hausdorff multifractal spectrum*. This spectrum gives a *geometrical information* on the singularity structure of $Z$. Another approach to multifractal analysis is based on a *statistical description* of the distribution of the singularities. It leads to the computation of the so-called *large deviation spectrum*. The large deviation spectrum and related quantities pertaining to the statistical analysis of $Z$ (as, e.g., its *Legendre multifractal spectrum*) are studied in the companion paper [6]. These quantities are the one usually considered in applications (see for instance [29, 23, 24, 13]).

**The spectrum of singularities.** We need the notion of pointwise regularity of a real valued function on a non-trivial subinterval $I$ of $\mathbb{R}$. If $f$ is such a function, $t_0 \in \mathrm{Int}(I)$ and $s \in \mathbb{R}_+$, then $f$ belongs to $C^s(t_0)$ if there exists $C > 0$ and a polynomial $P_{t_0}$ of degree at most $[s]$ such that in a neighborhood of $t_0$,

$$|f(t) - P_{t_0}(t)| \le C|t - t_0|^s.$$

The Hölder exponent of $f$ at $t_0$, denoted $h_f(t_0)$, is defined as

$$h_f(t_0) = \sup\{s : f \in C^s(t_0)\}.$$

The *spectrum of singularities* or *Hausdorff multifractal spectrum* of $f$ describes, for every $h \ge 0$, the "size" of the set $S_h$ of points in $\mathrm{Int}(I)$ where $f$ has Hölder exponent $h$. More precisely, let $\dim E$ denote the Hausdorff dimension of the set $E$ (we adopt the convention $\dim \emptyset = -\infty$). Then the spectrum of singularities of $f$ is the function: $h \mapsto \dim\{t : h_f(t) = h\}$.

The spectrum of singularities of the sample paths of $Z$ (here $I = \mathbb{R}_+$) is governed by the following index

$$\beta = \inf\{\gamma \ge 1;\ \sum_{i\ge 1} \frac{1}{\lambda_i^{\gamma-1}} < \infty\},$$

which is also the Blumenthal-Getoor [12] index of the Lévy process $L$ ($\beta \in [1, 2]$ under the assumptions of Proposition 1). Our main result is:

**Theorem 1** *Assume $\sum_{i\ge 1} 1/\lambda_i < \infty$. With probability one, $X$ and $Z$ are well defined and they share the following spectrum of singularities:*

$$\dim S_h = d_\beta(h) := \begin{cases} \beta h & \text{if } h \in [0, 1/\beta]; \\ -\infty & \text{otherwise.} \end{cases}$$

**Remark 1.** The spectra of $X$ and $Z$ are the same as that of the Lévy process $L$ defined above. The condition $\sum_{i\geq 1} 1/\lambda_i < \infty$ is also necessary and sufficient to define $L$, but [18] assumes slightly more than $\sum_{i\geq 1} 1/\lambda_i < \infty$ to derive the multifractal spectrum of $L$ when $\beta = 2$. More precisely, the additional assumption in [18] is $(\mathbf{C})$ : $\sum_{j\geq 1} 2^{-j}\sqrt{C_j \log(1+C_j)} < \infty$, where $C_j = \sum_{2^j \leq \lambda_i < 2^{j+1}} \lambda_i$. This restriction is due to the use of a certain Lemma by Stute in finding the lower bound estimate of the Hölder exponents. In fact this lemma gives an upper bound on the number of jump points of $\sum_{2^j \leq \lambda_i < 2^{j+1}} L_i(\cdot)$ in any dyadic interval. In [18], Stute's result is combined with a concentration inequality and the fact that the jump size is of the order of $2^{-j}$ at jump points of $\sum_{2^j \leq \lambda_i < 2^{j+1}} L_i(\cdot)$. Under $(\mathbf{C})$, this approach also yields a lower bound estimate for the Hölder exponents of $X$ (not for those of $Z$) if, on the one hand, one uses the same truncations of the $X_i$'s as those used in this paper, and on the other hand one interprets the $X_i$s as the difference between a drift and a pure jump process. Nevertheless, there remain problems with the lower bound estimates of the Hausdorff dimensions of the level sets $S_h$, as well as with the computation of the maximal Hölder exponent of $X$. This is due to the fact that the jump size $\delta$ at jump points of $\sum_{2^j \leq \lambda_i < 2^{j+1}} X_i(\cdot)$ ceases to be of the same order as $2^{-j}$ (more precisely, $\log\delta$ is not of the order of -$j$). In particular, in [18] the maximal Hölder exponent of $L$ is found using Shepp's Theorem on the covering of the real line by Poisson intervals centered at the jump points of $L$. Here, we need a refinement of Shepp's result for "economic" coverings by Poisson intervals centered at jump points of the $\sum_{2^j \leq \lambda_i < 2^{j+1}} X_i(\cdot)$s selected to satisfy that the jump size at each of those points is of the same order as $2^{-j}$ (Theorem 3).

Our lower bound estimate of the Hölder exponents of $X$ and $Z$ is not based on Stute's lemma. Rather, we rely on a classical concentration inequality (Bennett inequality, Lemma 3($ii$)). As a consequence, we avoid the restriction $(\mathbf{C})$ in the study of $X$ in Theorem 1 when $\beta = 2$.

Theorem 1 possesses the following natural extension: Let $(\mu_i)_{i\geq 1} \in (1,\infty)^{\mathbb{N}^*}$. For every $i \geq 1$ define

$$\widetilde{Z}_i(t) = \begin{cases} Z_i(0) + t & \text{if } 0 \leq t < \tau_1^{(i)} \\ \dfrac{\widetilde{Z}_i(T_{k-1}^{(i)}) + \tau_k^{(i)}}{\mu_i} + t - T_k^{(i)} & \text{if } T_k^{(i)} \leq t < T_{k+1}^{(i)} \text{ with } k \geq 1, \end{cases}$$

and

$$\widetilde{Z}(t) = \sum_{i\geq 1} \widetilde{Z}_i(t).$$

**Theorem 2** *Assume $(\mu_i)_{i\geq 1}$ is bounded, $|\log(\mu_i - 1)| = o(\log(\lambda_i))$ and $\sum_{i\geq 1} 1/(\mu_i - 1)\lambda_i < \infty$. With probability one, the process $\widetilde{Z}$ is well defined and its spectrum of singularities is $d_\beta$.*

In other words, the multifractal nature of the sum is not affected if $\mu$ is replaced by $\mu_i$ in $Z_i$ and if the sequence $(\mu_i)$ remains bounded and does not tend "too fast" to 1. Theorem 2 includes many potential or actual variants of TCP. For instance, one could imagine treating in different ways sources with different intensity $\lambda_i$: As long as the reduction factors are bounded and do not approach 1 too fast, the multifractal spectrum remains unchanged. This suggests that reducing the multifractality of TCP might require more drastic changes.

# 4 Proof of Theorem 1.

The proof of Theorem 1 is decomposed in several steps. In Section 4.1, we set some definitions useful in the sequel. Section 4.2 (resp. 4.3) gives lower (resp. upper) bounds for the Hölder exponents. Finally, Sections 4.4 and 4.5 computes the Hausdorff dimensions of the level sets $S_h$. Ancillary results needed for Sections 4.2 and 4.4-4.5 are grouped in Sections 5 and 6.

## 4.1 Definitions and notations

Due to the last assertion of Lemma 1 (Section 5) and the definition of the $R_i$s, the component involving $Z_i(0)$ is too small to play a role in computing the Hölder exponents of $Z$ on $(0,\infty)$. Consequently, we assume without loss of generality that $Z_i(0) = 0$ almost surely for all $i \geq 1$.

It is enough to establish that for every integer $T > 0$, the restrictions of $X$ and $Z$ to $(0, T)$ have almost surely the spectrum of singularities given in Theorem 1.

Therefore, in the sequel we fix $T \in \mathbb{N}^*$ and study $X$ and $Z$ on $(0, T)$.

Moreover, we may and will assume that $\inf_{i\geq 1} \lambda_i \geq 2$ without loss of generality, since we work under the assumption $\sum_{i\geq 1} 1/\lambda_i < \infty$.

We need some new definitions.

For every $i \geq 1$ and $t > 0$, define $T_t^{(i)} = \max\{T_k^{(i)};\ T_k^{(i)} \leq t\}$ and $k_t^{(i)}$ the integer $k$ such that $T_t^{(i)} = T_k^{(i)}$.

The following sets will prove to be useful.

For every $j \geq 1$ and $\delta > 0$ define

$$G_j = \{i \geq 1;\ 2^j \leq \lambda_i < 2^{j+1}\} \quad \text{and} \quad E_{j,\delta} = \bigcup_{i\in G_j} \ \bigcup_{k\geq 1:\ T_k^{(i)} \leq T} [T_k^{(i)} - 2^{-\delta j}, T_k^{(i)} + 2^{-\delta j}].$$

Then for every $\delta > 0$ define

$$E_\delta = \limsup_{j\to\infty} E_{j,\delta}.$$

For every $j \geq 1$ define

$$\beta_j = 1 + \frac{\log_2 \# G_j}{j},$$

where $\#G_j$ denotes the cardinal of the set $G_j$, with the convention $\log(0) = -\infty$. It follows from the definition of $\beta$ that

$$\beta = \limsup_{j \to \infty} \beta_j.$$

For $j \geq 1$ define

$$\gamma_j = \frac{6(j+1)}{2^j}.$$

For $j \geq 1$ and $t' > t > 0$ define

$$\begin{cases} X_{G_j}(t,t') = \sum_{i \in G_j} X_i(t') \mathbf{1}_{\{X_i(t') \leq \gamma_j\}} - X_i(t) \mathbf{1}_{\{X_i(t) \leq \gamma_j\}} \\ R_{G_j}(t,t') = \sum_{i \in G_j} \left( \widetilde{R}_i(t') - \widetilde{R}_i(t) \right) - \mathbb{E}\left( \widetilde{R}_i(t') - \widetilde{R}_i(t) \right), \end{cases}$$

where

$$\widetilde{R}_i(t) = \frac{1}{\mu^{k+1}} \sum_{j=1}^{k} \mu^j \tau_j^{(i)} \mathbf{1}_{\{\tau_j^{(i)} \leq \gamma_j\}} \text{ if } T_t^{(i)} = T_k^{(i)}$$

(Lemma 8$(ii)$ in Section 5 shows that if $t' > t > \gamma_j$ then $X_{G_j}(t,t')$ is a centered random variable).

Set

$$\begin{cases} b_{X_{G_j}}(t,t') = \left( \mathbb{E}(X_{G_j}(t,t')^2) \right)^{1/2} \\ b_{R_{G_j}}(t,t') = \left( \mathbb{E}(R_{G_j}(t,t')^2) \right)^{1/2}. \end{cases}$$

For every $\varepsilon > 0$ and $m \geq 1$ define

$$m(\beta, \varepsilon) = \frac{2m}{(\beta + \varepsilon)(3 - \beta + \varepsilon)}.$$

Notice that $\sup_{m \geq 1} \dfrac{m(\beta,\varepsilon)}{m} < 1$.

For every $J \geq 0$, denote by $D_J$ the set of dyadic points of the $J^{\text{th}}$ generation contained in $[0,T]$.

## 4.2 A lower bound for $h_Y(t_0)$, $Y \in \{X, R, Z\}$.

This section is devoted to the proof of the following proposition. It involves intermediate results stated and proved in Section 5.

**Proposition 2** *Assume the hypothesis of Theorem 1. Fix $\delta > \beta$. With probability one, for every $t_0 \in (0,T)$ and $Y \in \{X, R, Z\}$, if $t_0$ is not a jump point of $Y$ then*

$$t_0 \notin E_\delta \Rightarrow \; h_Y(t_0) \geq 1/\delta. \tag{1}$$

**Proof.** Due to the equality $Z = X + R$, and the fact the $X$, $R$, and $Z$ have the same jump points, we only have to deal with $Y \in \{X, R\}$.

Fix $\varepsilon > 0$ small enough so that: $(i)$ $3-\beta-2\varepsilon > 0$; $(ii)$ $\frac{3-\beta-2\varepsilon}{(\beta+\varepsilon)(3-\beta+\varepsilon)} > 1/\delta$ (in particular $1/(\beta+\varepsilon) > 1/\delta$); $(iii)$ $\beta+\varepsilon < 2$ if $\beta < 2$; $(iv)$ $1/2-\varepsilon > 1/\delta$ if $\beta = 2$.

Fix $\eta \in (0,T)$ and then $\Omega' = \Omega'(\eta)$ a subset of $\Omega$ of probability 1, such that for every $\omega \in \Omega'$, there exists $m_0(\omega) \geq 1$ such that for every $m \geq m_0(\omega)$, the conclusions of Corollary 3, Lemma 4 and Lemma 6$(ii)$ hold, as well as that of Lemma 1 (with $K = 6$) for $i \in G_j$ when $j \geq m/\delta$ and $G_j \neq \emptyset$, and also that of Lemma 5 and 7 if $j \geq (m+r_m)/\beta_j$. Fix such an $m_0(\omega)$ for every $\omega \in \Omega'$.

Now, fix $\omega \in \Omega'$, and then $t_0 \in (\eta, T)$ such that $t_0 \notin E_\delta(\omega)$ and $t_0$ is not a jump point of $Y(\omega)$. Since $t_0 \notin E_\delta(\omega)$, we can choose $j_0 \geq m_0(\omega)/\delta$ such that for every $j \geq j_0$, $t_0 \notin E_{j,\delta}$. The Hölder exponent of $Y$ at $t_0$ is the same as that of $\sum_{j\geq j_0}\sum_{i\in G_j} Y_i$. We also choose $j_0$ so that $\beta_j < \beta+\varepsilon < \delta$ and $(j+1)\sqrt{j} \leq 2^{\varepsilon j}$ for $j \geq j_0$. To conclude, we need the following three upper bounds $(a)$, $(b)$, $(c)$:

$(a)$ For every $m \geq \delta j_0$, $t \in (\eta, T)$ such that $2^{-m} \leq |t-t_0| \leq 2^{-m+1}$ and $j_0 \leq j \leq [m/\delta]-1$, $\sum_{i\in G_j} Y_i$ has no jump between $t$ and $t_0$. Consequently,

$$\begin{aligned}\left|\sum_{j_0\leq j\leq [m/\delta]-1}\sum_{i\in G_j} Y_i(t) - Y_i(t_0)\right| &= |t-t_0| \sum_{j_0\leq j\leq [m/\delta]-1} \#G_j \\ &= |t-t_0| \sum_{j_0\leq j\leq [m/\delta]-1} 2^{(\beta_j-1)j} \\ &\leq |t-t_0| \frac{2^{(\delta-1)m/\delta}}{2^{\delta-1}-1} \leq \frac{2}{2^{\delta-1}-1}|t-t_0|^{1/\delta}.\end{aligned}$$

$(b)$ By Lemma 6$(ii)$, for some constant $C = C(\omega)$, for every $m \geq \delta j_0$ and $t \in (\eta, T)$ such that $2^{-m} \leq |t-t_0| \leq 2^{-m+1}$, one has

$$\left|\sum_{[m/\delta]\leq j\leq (m+r_m)/\beta_j}\sum_{i\in G_j} Y_i(t) - Y_i(t_0)\right| \leq C|t-t_0|^{1/\delta}|\log(|t-t_0|)|^9.$$

$(c)$ For every $m \geq \delta j_0$, $t \in (\eta, T)$ such that $2^{-m} \leq |t-t_0| \leq 2^{-m+1}$ and $j \geq (m+r_m)/\beta_j$, fix $(d_t^{(j)}, d_{t_0}^{(j)}) \in D^2_{[2(\beta+\varepsilon)j]+1} \cap (\eta, T)$ such that $2^{-m} \leq |d_t^{(j)} - d_{t_0}^{(j)}| \leq 2^{-m+1}$ and $\max(|t-d_t^{(j)}|, |t_0-d_{t_0}^{(j)}|) \leq 2^{-[2(\beta+\varepsilon)j]-1}$. By Lemma 7, $\sum_{i\in G_j} Y_i$ has at most one jump point between $s$ and $d_s^{(j)}$ for $s \in \{t_0, t\}$. Consequently, by definition of the $X_i$ and $R_i$ and Lemma 5

$$\begin{aligned}
&\left|\sum_{i\in G_j} Y_i(t)-Y_i(t_0)-\sum_{i\in G_j} Y_i(d_t^{(j)})-Y_i(d_{t_0}^{(j)})\right| \\
\leq\ & 2(\#G_j)\max(|t-d_t^{(j)}|,|t_0-d_{t_0}^{(j)}|)+2Cj2^{-j} \\
\leq\ & 2^{(\beta_j-1)j}2^{-[2(\beta+\varepsilon)j]}+2Cj2^{-j}
\end{aligned}$$

and since $\beta_j<\beta+\varepsilon$

$$\begin{aligned}
&\sum_{j,\ (m+r_m)/\beta_j\leq j}\left|\sum_{i\in G_j} Y_i(t)-Y_i(t_0)-\sum_{i\in G_j} Y_i(d_t^{(j)})-Y_i(d_{t_0}^{(j)})\right| \\
=\ & O\left(\sum_{j\geq m/(\beta+\varepsilon)} 2^{-(2\beta-\beta_j+2\varepsilon+1)j}+j2^{-j}\right)=O\Big(2^{-\left(1+\frac{1}{\beta+\varepsilon}\right)m}\Big)+O(m2^{-\frac{m}{\beta+\varepsilon}}) \\
=\ & O(2^{-m/\delta})
\end{aligned}$$

by property $(ii)$ for $\varepsilon$. Moreover, due to Lemma 1, we have

$$\sum_{i\in G_j} X_i(d_t^{(j)})-X_i(d_{t_0}^{(j)})=X_{G_j}(d_t^{(j)},d_{t_0}^{(j)})$$

and

$$\sum_{i\in G_j} R_i(d_t^{(j)})-R_i(d_{t_0}^{(j)})=R_{G_j}(d_t^{(j)},d_{t_0}^{(j)})+\sum_{i\in G_j}\mathbb{E}\left(\widetilde{R}_i(d_t^{(j)})-\widetilde{R}_i(d_{t_0}^{(j)})\right).$$

Due to Corollary 3 and Lemma 4, this implies that

$$\begin{aligned}
&\sum_{j,\ m+r_m\leq j\beta_j}\left|\sum_{i\in G_j} Y_i(d_t^{(j)})-Y_i(d_{t_0}^{(j)})\right|\leq C2^{-m/\delta}+ \\
&C\sum_{j,\ (m+r_m)/\beta_j\leq j\leq m(\beta,\varepsilon)} m^2 2^{(\beta_j/2-1)j}|d_t^{(j)}-d_{t_0}^{(j)}|^{1/2}+C\sum_{j\geq m(\beta,\varepsilon)}(j+1)\sqrt{jm}2^{(\beta_j-3)j/2}.
\end{aligned}$$

On the one hand, since $\beta_j<\beta+\varepsilon$ and $|d_t^{(j)}-d_{t_0}^{(j)}|\leq 2^{-m+1}$, if $\beta<2$, property $(iii)$ for $\varepsilon$ yields

$$\begin{aligned}
&\sum_{j,\ (m+r_m)/\beta_j\leq j\leq m(\beta,\varepsilon)} m^2 2^{(\beta_j/2-1)j}|d_t^{(j)}-d_{t_0}^{(j)}|^{1/2}\leq 2m^2 2^{-m/2}\sum_{j\geq m/(\beta+\varepsilon)} 2^{\left((\beta+\varepsilon)/2-1\right)j} \\
=\ & \frac{2}{1-2^{(\beta+\varepsilon)/2-1}}m^2 2^{-m/2}2^{\left((\beta+\varepsilon)/2-1\right)\left(m/(\beta+\varepsilon)\right)}=\frac{2}{1-2^{(\beta+\varepsilon)/2-1}}m^2 2^{-m/(\beta+\varepsilon)} \\
=\ & O(m^2 2^{-m/\delta}),
\end{aligned}$$

and if $\beta = 2$, since $\beta_j < 2+\varepsilon$, property $(iv)$ for $\varepsilon$ yields

$$\begin{aligned}
&\sum_{j,\ (m+r_m)/\beta_j \le j \le m(\beta,\varepsilon)} m^2 2^{(\beta_j/2-1)j} |d_t^{(j)} - d_{t_0}^{(j)}|^{1/2} \\
\le\ & 2m^2 2^{-m/2} \sum_{j=1}^{m} 2^{\varepsilon j} = \frac{2^{1+\varepsilon}}{2^\varepsilon - 1} m^2 2^{-m(\frac{1}{2}-\varepsilon)} = O(m^2 2^{-m/\delta}).
\end{aligned}$$

On the other hand, since $\beta_j < \beta + \varepsilon$ and $(j+1)\sqrt{j} \le 2^{\varepsilon j}$, property $(ii)$ for $\varepsilon$ yields

$$\begin{aligned}
\sum_{j \ge m(\beta,\varepsilon)} (j+1)\sqrt{jm} 2^{(\beta_j-3)j/2} &\le \sqrt{m} \sum_{j \ge m(\beta,\varepsilon)} 2^{(\beta+2\varepsilon-3)j/2} \\
&= \frac{1}{1-2^{(\beta+2\varepsilon-3)/2}} \sqrt{m} 2^{-\frac{3-\beta-2\varepsilon}{(\beta+\varepsilon)(3-\beta+\varepsilon)} m} \\
&= O(\sqrt{m} 2^{-m/\delta}).
\end{aligned}$$

Finally, we get

$$\left| \sum_{j,\ (m+r_m)/\beta_j \le j} \sum_{i \in G_j} Y_i(t) - Y_i(t_0) \right| = O\Big( |t-t_0|^{1/\delta} |\log^2(|t-t_0|)| \Big).$$

From $(a)$, $(b)$ and $(c)$, we deduce that the Hölder exponent of $\sum_{j \ge j_0} \sum_{i \in G_j} Y_i$ and $Y$ at $t_0$ is at least $1/\delta$. So, for every $\omega \in \Omega'(\eta)$, if $t_0 \in (\eta, T)$ is not a jump point of $Y$, (1) holds. One concludes by considering $\Omega' = \cap_{n \ge 1} \Omega'(\frac{1}{n})$.

**Remark 2.** Under the condition $\sum_{i \ge 1} 1/\lambda_i < \infty$, the above computations imply the following property for $Y \in \{X, R\}$, *even without the knowledge of the finiteness of* $\sum_{i \ge 1} Y_i$: With probability one, for every $\eta \in (0,T)$ there exists $\alpha > 0$ such that if $t, t' \in (\eta, T)$ and $|t'-t| \le \alpha$ then $\lim_{J \to \infty} \sum_{j=1}^J \sum_{i \in G_j} Y_i(t') - \sum_{j=1}^J \sum_{i \in G_j} Y_i(t)$ exists. This is a key point in the proof of Proposition 1.

## 4.3 Upper bounds for $h_Y(t_0)$, $Y \in \{X, Z\}$

Let $\varphi : \mathbb{R}_+^* \to \mathbb{R}_+^*$. For $j \ge 1$ and $\delta \ge 0$ define

$$\widetilde{E}_{j,\delta,\varphi} = \bigcup_{i \in G_j} \bigcup_{k \ge 1: T_k^{(i)} \le T,\ \tau_k^{(i)} \ge \varphi(2^{-j})} [T_k^{(i)} - 2^{-\delta j}, T_k^{(i)} + 2^{-\delta j}]$$

and

$$\widetilde{E}_{\delta,\varphi} = \limsup_{j \to \infty} \widetilde{E}_{j,\delta,\varphi}.$$

**Proposition 3** *Suppose* $\lim_{j \to \infty} \frac{\log \varphi(2^{-j})}{\log 2^{-j}} = 1$. *Fix* $\delta > 0$. *With probability one, for every* $t_0 \in \widetilde{E}_{\delta,\varphi}$, *one has* $h_Y(t_0) \le 1/\delta$ *for* $Y \in \{X, Z\}$.

**Proof.** If $Y \in \{X, Z\}$ and $t > 0$ is a jump point of $Y$, $|\Delta_Y(t)|$ stands for the size of the corresponding jump.

Fix $t_0 \in \widetilde{E}_{\delta,\varphi}$. Fix a sequence $(r_{j_n})_{n\geq 1}$ of points such that for every $n \geq 1$ there exist $i \in G_{j_n}$ and $1 \leq k \leq N_T^{(i)}$ such that $r_{j_n} = T_k^{(i)}$ and $\tau_k^{(i)} \geq \varphi(2^{-j_n})$, and $t_0 \in [r_{j_n} - 2^{-\delta j_n}, r_{j_n} + 2^{-\delta j_n}]$.

By construction we have

$$|\Delta_X(r_{j_n})| = \tau_k^{(i)} \geq \varphi(2^{-j_n}).$$

Moreover,

$$|\Delta_Z(r_{j_n})| = (1 - 1/\mu) Z_i(r_{j_n}^-) \geq (1 - 1/\mu)\tau_k^{(i)} \geq (1 - 1/\mu)\varphi(2^{-j_n}).$$

Since $|r_{j_n} - t_0| \leq 2^{-\delta j_n}$, our assumption on $\varphi$ imply for $Y \in \{X, Z\}$

$$\liminf_{n\to\infty} \frac{\log |\Delta_Y(r_{j_n})|}{\log |r_{j_n} - t_0|} \leq 1/\delta.$$

The conclusion follows from Lemma 1 in [17] (also Lemma 4 in [18]).

In the next two subsections, the sets $S_h$ are the level sets of the Hölder exponents of $Y \in \{X, Z\}$.

## 4.4 $\dim S_h$ for $h \in [0, 1/\beta]$.

**Upper bound for** $\dim S_h$

**Proposition 4** *With probability one,* $\dim S_h \leq \beta h$ *for all* $0 \leq h \leq 1/\beta$.

**Proof.** The set of rational numbers being countable, the conclusion of Proposition 2 holds almost surely simultaneously for all rational $\delta > \beta$. Consequently, due to this proposition, with probability one, if $t_0 \in S_h$ then $t_0 \in \bigcap_{\delta\in\mathbb{Q},\ \delta<1/h} E_\delta$. Moreover, it is shown in [18] that $\dim E_\delta \leq \beta/\delta$.

**Lower bound for** $\dim S_h$

Let $\varphi$ be as in previous section. For $h \geq 0$, define

$$\widetilde{S}_{h,\varphi} = \bigcap_{\delta\in\mathbb{Q},\ \delta<1/h} \widetilde{E}_{\delta,\varphi} \setminus \bigcup_{\delta\in\mathbb{Q},\ \delta>1/h} E_\delta$$

$(1/0 := +\infty)$.

**Proposition 5** *Suppose* $\lim_{j\to\infty} \frac{\log\varphi(2^{-j})}{\log 2^{-j}} = 1$. *With probability one,* $\widetilde{S}_{h,\varphi} \subset S_h$ *for every* $0 \leq h \leq 1/\beta$.

**Proof.** We saw that the conclusion of Proposition 2 holds almost surely simultaneously for all rational $\delta > \beta$. This implies that with probability one, for every $h \in [0, 1/\beta]$, if $t_0 \in \widetilde{S}_{h,\varphi}$, then, due to Proposition 2, $h_Y(t_0) \geq 1/\delta$ for all rational $\delta$ such that $h > 1/\delta$, so $h_Y(t_0) \geq h$. Moreover, due to Proposition 3, if $t_0 \in \widetilde{S}_{h,\varphi}$ then $h_Y(t_0) \leq 1/\delta$ for all $\delta$ such that $h < 1/\delta$, so $h_Y(t_0) \leq h$.

Let $(j_n)_{n\geq 1}$ be an increasing sequence of integers such that $G_{j_n} \neq \emptyset$ and $\lim_{n\to\infty} \beta_{j_n} = \beta$. Then for $b \geq 1$ define

$$H_b = \limsup_{n\to\infty} \bigcup_{i\in G_{j_n}} \bigcup_{k\geq 1:\ T_k^{(i)} \leq T,\ \tau_k^{(i)} \geq 2^{-j_n}} [T_k^{(i)} - 2^{-b\beta_{j_n} j_n}, T_k^{(i)} + 2^{-b\beta_{j_n} j_n}].$$

For every $d \geq 0$, let $\mathcal{H}^d$ be the Hausdorff measure defined with the gauge function $x \geq 0 \mapsto (\log(x))^2 x^d$.

**Proposition 6** *With probability one*
*(i) $H_1$ is of full Lebesgue measure in $[0, T]$;*
*(ii) for all $b > 1$, $\mathcal{H}^{1/b}(H_b) > 0$.*

The proof is postponed to Section 6 (assertion $(i)$ is the only to be proved; the other one is a consequence of Theorem 2 in [18] or [19]).

**Corollary 1 (Lower bound for** $\dim \widetilde{S}_{h,id_{\mathbb{R}_+^*}}$ **)** *Suppose $\varphi(t) = t$, $t > 0$. With probability one, $\dim \widetilde{S}_{h,\varphi} \geq \beta h$ for every $h \in (0, 1/\beta]$.*

**Proof.** It is straightforward that, with probability one, for every $h \in (0, 1/\beta]$, $H_{1/(\beta h)} \subset \bigcap_{\delta\in\mathbb{Q},\ \delta<1/h} \widetilde{E}_{\delta,\varphi}$. Consequently, due to Proposition 6, $\mathcal{H}^{\beta h}(\bigcap_{\delta\in\mathbb{Q},\ \delta<1/h} \widetilde{E}_{\delta,\varphi}) > 0$. Moreover, by Proposition 4, $\mathcal{H}^{\beta h}(\bigcup_{\delta\in\mathbb{Q},\ \delta>1/h} E_\delta) = 0$. It follows that $\mathcal{H}^{\beta h}(\widetilde{S}_{h,\varphi}) > 0$.

Since $S_0 \neq \emptyset$ (it contains at least the jump points), it follows from Propositions 4 and 5, as well as Corollary 1 that with probability one, $\dim S_h = \beta h$ for all $h \in [0, 1/\beta]$.

## 4.5 $S_h = \emptyset$ for $h > 1/\beta$.

**Theorem 3** *["Economic covering result"] There exists $\varphi$ such that $\lim_{j\to\infty} \frac{\log \varphi(2^{-j})}{\log 2^{-j}} = 1$ and with probability one, $(0, T) \subset \widetilde{E}_{\delta,\varphi}$ for all $\delta < \beta$.*

We use the terminology "economic covering" with respect to the analogous property satisfied by the largest sets $E_\delta$: $(\mathcal{R})$ with probability one $(0, T) \subset E_\delta$ for all $\delta < \beta$. The property $(\mathcal{R})$ is used in [18] to prove for Lévy processes the result corresponding to Corollary 2 below. Moreover, $(\mathcal{R})$ is a consequence of Shepp's theorem for the covering of the real line by Poisson intervals.

The proof of Theorem 3 is postponed to Section 6.

**Corollary 2** *With probability one, $S_h = \emptyset$ for all $h > 1/\beta$.*

**Proof.** Combine Theorem 3 with Proposition 3.

## 5 Proofs of basic lemmas and propositions

Recall that we assumed without loss of generality that $\lambda_i \geq 2$ for all $i \geq 1$. For $t > 0$ and $i \geq 1$ define

$$N_t^{(i)} = \#\{T_k^{(i)};\ k \geq 1\} \cap [0,t].$$

$N_t^{(i)}$ is a Poisson random variable with intensity $\lambda_i t$.

**Lemma 1** *Assume $\sum_{i\geq 1} 1/\lambda_i < \infty$. For every $K \geq 2$, with probability one, there exists $i_0 \geq 1$ such that for every $i \geq i_0$,*

$$\begin{cases} N_T^{(i)} \leq M_i = T\lambda_i + 4\sqrt{T\lambda_i \log(T\lambda_i)} \\ \forall\ 1 \leq k \leq N_T^{(i)} + 1,\ \tau_k^{(i)} \leq K\log(\lambda_i)/\lambda_i. \end{cases}$$

*In particular, for every $i \geq i_0$ and $t \in (K\log(\lambda_i)/\lambda_i, T]$, one has $k_t^{(i)} \geq K^{-1}\lambda_i t/\log\lambda_i$.*

**Proof.** From Lemma 1 of [18], for $i \geq 1$ large enough $\mathbb{P}(N_T^{(i)} \geq M_i) \leq 1/(T\lambda_i)^7$. Moreover, for every $i \geq 1$

$$\begin{aligned} \mathbb{P}(\exists\ 1 \leq k \leq M_i + 1;\ \tau_k^{(i)} > \frac{K\log(\lambda_i)}{\lambda_i}) &= 1 - (\mathbb{P}(\tau_1^{(i)} \leq K\log(\lambda_i)/\lambda_i))^{M_i+1} \\ &= 1 - (1 - 1/\lambda_i^K)^{M_i+1} \\ &= T/\lambda_i^{K-1} + o(1/\lambda_i^{K-1}). \end{aligned}$$

Consequently

$$\sum_{i\geq 1} \mathbb{P}\Big(\{N_T^{(i)} \geq M_i\} \cup \Big\{\exists\ 1 \leq k \leq M_i + 1;\ \tau_k^{(i)} > \frac{K\log(\lambda_i)}{\lambda_i}\Big\}\Big) < \infty.$$

The first assertion of the lemma is a consequence of the Borel-Cantelli Lemma. The other one is a consequence of the first assertion and the definition of $k_t^{(i)}$.

**Proof of Proposition 1.** Suppose $\sum_{i\geq 1} 1/\lambda_i < \infty$. Assume we have shown the following property $(\mathcal{P})$: with probability one, the processes $X$ and $R$ are finite at every point $t$ of a dense countable subset of $\mathbb{R}_+$.

Then, since $X$ and $R$ are respectively the infinite sums of the nonnegative processes $X_i$ and $R_i$, the property obtained in Remark 2 after the proof of Proposition 2 and $(\mathcal{P})$ together show that almost surely $X$ and $R$ are finite everywhere, as well as their sum $Z$.

To see that $(\mathcal{P})$ holds, it is enough to show that for every $t > 0$ and $Y \in \{X, R\}$, $\sum_{i\geq 1} Y_i(t) < \infty$ almost surely. Fix $t > 0$. Due to Lemma 1, $k_t^{(i)}$ goes so fast to infinity that one can assume that $Z_i(0) = 0$ for all $i \geq 1$. Then, the computations done in the proofs of Lemma 8$(i)$ and Lemma 10$(i)$ show that $\sum_{i\geq 1} \mathbb{E}\big(X_i(t) + R_i(t)\big) < \infty$, hence the conclusion.

Suppose now that $\sum_{i\geq 1} 1/\lambda_i = \infty$. Then we can use Lemma 8$(ii)$ to show that for every $t > 0$, if $\gamma \in (0,t)$ then $\sum_{i\geq 1} \mathbb{E}\big(X_i(t)\mathbf{1}_{\{X_i(t)\leq\gamma\}}\big) = \sum_{i\geq 1} \frac{1}{\lambda_i}(1 - e^{-\lambda_i\gamma}) = \infty$.

Since the $X_i(t)$s are independent random variables, Kolmogorov's three series theorem (see [32] p. 106) shows that $\sum_{i\geq 1} X_i(t) = \infty$ almost surely. Moreover, the function $f : (t,\omega) \in \mathbb{R}_+^* \times \Omega \mapsto \mathbf{1}_{\{\infty\}}\Big(\sum_{i\geq 1} X_i(t)(\omega)\Big)$ is measurable. Consequently, the Fubini theorem applied for every $n \geq 1$ with the restriction of $f$ to $[0,n]\times\Omega$ and the products $\ell_n \otimes \mathbb{P}$, where $\ell_n$ denotes the restriction of the Lebesgue measure to $[0,n]$, implies that with probability one, $X(t) = \infty$ almost everywhere. The same holds for $Z$ since $Z \geq X$.

**Lemma 2** *Fix $\varepsilon > 0$, $\eta \in (0,T)$ and $Y \in \{X, R\}$.*

*(i) There exist $A, B > 0$ and $m_0 \geq 1$ such that for all $m \geq m_0$, if $m/3 \leq j \leq m(\beta,\varepsilon)$ is such that $G_j \neq \emptyset$ and $\eta < t, t' \leq T$ are such that $2^{-m} \leq t'-t \leq 2^{-m+1}$, then*

$$A \leq \frac{b_{Y_{G_j}}(t,t')}{2^{(\beta_j/2-1)j}|t'-t|^{1/2}} \leq Bj.$$

*For $m \geq 1$ define $r_m = 2\log_2\left(144 \max\left(1, \frac{1}{\mu-1}\right)\frac{B}{A^2}m^2(m+1)\right)$.*

*(ii) There exists $m_0 \geq 1$ such that for all $m \geq m_0$, if $(m+r_m)/\beta_j \leq j \leq m(\beta,\varepsilon)$, $\eta < t < t' \leq T$ and $2^{-m} \leq t'-t \leq 2^{-m+1}$ then*

$$\mathbb{P}\Big(|Y_{G_j}(t,t')| \geq 6Bm^2 2^{(\beta_j/2-1)j}|t'-t|^{1/2}\Big) \leq 2\exp(-9m^2).$$

*(iii) There exists $m_0 \geq 1$ such that for all $m \geq m_0$, if $j \geq m(\beta,\varepsilon)$, $\eta < t < t' \leq T$ and $t'-t \leq 2^{-m+1}$ then*

$$\mathbb{P}\Big(|Y_{G_j}(t,t')| \geq 48\max\left(1, \frac{1}{\mu-1}\right)(j+1)\sqrt{jm}2^{(\beta_j-3)j/2}\Big) \leq 2\exp(-8jm).$$

The proof of Lemma 2 uses the following well-known inequalities, which are essentially, *e.g.*, Lemma 1.5 and Bennett inequality (6.10) in [21].

**Lemma 3** *Let $(V_i)_{1\leq i\leq n}$ be a finite sequence of independent random variables with mean 0. Assume that there exists $\gamma > 0$ such that $|V_i| \leq \gamma$ almost surely for all $i$.*

*(i) For all $s > 0$,*

$$\mathbb{P}\Big(\Big|\sum_{i=1}^n V_i\Big| > s\gamma\sqrt{n}\Big) \leq 2e^{-s^2/2}.$$

*(ii) Define $b^2 = \mathbb{E}(\sum_{i=1}^n V_i^2)$. For all $0 < s \leq \frac{b^2}{2\gamma}$,*

$$\mathbb{P}\Big(\Big|\sum_{i=1}^n V_i\Big| > s\Big) \leq 2e^{-s^2/4b^2}.$$

**Proof of Lemma 2.** $(i)$ The case $Y = X$: One verifies that by our choice for $\gamma_j$, as $m \to \infty$, for $m/3 \le j \le m(\beta,\varepsilon)$ such that $G_j \ne \emptyset$ and $\eta < t < t' \le T$ such that $2^{-m} \le t' - t \le 2^{-m+1}$, if $i \in G_j$, one has $\lambda_i \gamma_j \to \infty$, $\gamma_j^2 e^{-\lambda_i \gamma_j} = o\Big(\frac{t'-t}{\lambda_i}\Big)$ and $\lambda_i(t'-t) \to 0$. It follows from Lemma 9 applied with $\gamma = \gamma_j$ ($\gamma_j < \eta$ for $j$ large enough) that as $m \to \infty$

$$\mathbb{E}\Big(\big(X_i(t')\mathbf{1}_{\{X_i(t') \le \gamma_j\}} - X_i(t)\mathbf{1}_{\{X_i(t) \le \gamma_j\}}\big)^2\Big) \sim (t'-t) \sum_{i \in G_j} \frac{2}{\lambda_i}.$$

Since by definition $(\#G_j)2^{-j} \le \sum_{i \in G_j} \frac{2}{\lambda_i} \le 2(\#G_j)2^{-j}$ this yields that for $m$ large enough, for $m/3 \le j \le m(\beta,\varepsilon)$ such that $G_j \ne \emptyset$ and $\eta < t < t' \le T$ such that $2^{-m} \le t' - t \le 2^{-m+1}$

$$\frac{1}{2} \le \frac{b_{Y_{G_j}}(t,t')}{2^{(\beta_j/2-1)j}|t'-t|^{1/2}} \le 2.$$

The case $Y = R$: It follows from a combination of Lemma 10$(i)$ and $(ii)$ that there exists $K > 0$ such that for $m$ large enough, for $m/3 \le j \le m(\beta,\varepsilon)$ and $\eta < t < t' \le T$ such that $2^{-m} \le t' - t \le 2^{-m+1}$, if $i \in G_j$

$$\Big(\mathbb{E}\big(\widetilde{R}_i(t') - \widetilde{R}_i(t)\big)\Big)^2 \le K \left(\gamma_j^2 e^{-\lambda_i \gamma_j} + \frac{\exp\big(-2(1-\mu^{-1})\lambda_i t\big)}{\lambda_i^2}\right).$$

By our choice $\gamma_j = 6(j+1)2^{-j}$, on the one hand $\gamma_j^2 e^{-\lambda_i \gamma_j} = O(2^{-6j})$ and $2^{-4j} = O\big(\frac{t'-t}{\lambda_i}\big)$ because $j \ge m/3$.

On the other hand, $\frac{\exp\big(-2(1-\mu^{-1})\lambda_i t\big)}{\lambda_i^2} \le 2^{-2j} \exp\big(-2(1-\mu^{-1})2^j \eta\big) = o(2^{-\alpha j})$ for all $\alpha > 0$. This implies that

$$\sum_{i \in G_j} \Big(\mathbb{E}\big(\widetilde{R}_i(t') - \widetilde{R}_i(t)\big)\Big)^2 = o(2^{(\beta_j-2)j}|t'-t|)$$

as $m \to \infty$, $m/3 \le j \le m(\beta,\varepsilon)$, $\eta < t < t' \le T$ and $2^{-m} \le t' - t \le 2^{-m+1}$.

Before applying Lemma 10$(iii)$, notice that for $i \in G_j$, $\gamma_j^2 e^{-\lambda_i \gamma_j/2}\lambda_i = O(j^2 2^{-j} e^{-3j}) = o(2^{-4j})$. Also before applying Lemma 10$(iv)$, use Lemma 1 (with $K = 6$) to get

$$\mathbb{P}(\{N_T^{(i)} \ge M_i\} \cup \{\exists\, 1 \le k \le M_i;\ \tau_j^{(i)} > \gamma_j\}) = O(2^{-5j});$$

next notice that if $i \in G_j$ and $\lambda_i(t'-t) \to 0$ then $\gamma_j^2(1 - e^{-\lambda_i(t'-t)}) = O(j^2(t'-t)2^{-j})$.

Then, it follows from Lemma 10$(iii)(iv)$ applied with $\gamma = \gamma_j$ that as $m \to \infty$, $m/3 \le j \le m(\beta,\varepsilon)$, $\eta < t < t' \le T$ and $2^{-m} \le t' - t \le 2^{-m+1}$,

$$\sum_{i \in G_j} \mathbb{E}\Big(\big(\widetilde{R}_i(t') - \widetilde{R}_i(t)\big)^2\Big) = O\big(j^2 2^{(\beta_j-2)j}(t'-t)\big).$$

To find the lower bound for $\sum_{i\in G_j}\mathbb{E}\Big(\big(\widetilde{R}_i(t')-\widetilde{R}_i(t)\big)^2\Big)$ when $G_j\neq\emptyset$, first write

$$\begin{aligned}
&\Big|\mathbb{E}\Big(\big(\widetilde{R}_i(t')-\widetilde{R}_i(t)\big)^2\Big)-\mathbb{E}\Big(\big(R_i(t')-R_i(t)\big)^2\Big)\Big|\\
=\ &\Big|\mathbb{E}\Big[\Big(R_i(t')-\widetilde{R}_i(t')-\big(R_i(t)-\widetilde{R}_i(t)\big)\Big)\Big(R_i(t')-R_i(t)+\widetilde{R}_i(t')-\widetilde{R}_i(t)\Big)\Big]\Big|
\end{aligned}$$

Then, the Cauchy-Schwarz inequality together with Lemma 10$(iii)(iv)$ and the above estimates show that

$$\Big|\mathbb{E}\Big(\big(\widetilde{R}_i(t')-\widetilde{R}_i(t)\big)^2\Big)-\mathbb{E}\Big(\big(R_i(t')-R_i(t)\big)^2\Big)\Big|=O\Big(j^2 2^{-j}e^{-3j/2}\sqrt{t'-t}\Big)=o\Big(\frac{t'-t}{\lambda_i}\Big)$$

as $m\to\infty$, $m/3\le j\le m(\beta,\varepsilon)$, $\eta<t<t'\le T$ and $2^{-m}\le t'-t\le 2^{-m+1}$. Then one uses Lemma 10$(v)$.

$(ii)$ Fix $m_0$ as in $(i)$. For $Y\in\{X,R\}$, $m\ge m_0$ and $(m+r_m)/\beta_j\le j\le m(\beta,\varepsilon)$, if $G_j\neq\emptyset$ and $\gamma_j<t<t'\le T$, $2^{-m}\le t'-t\le 2^{-m+1}$, the sum $Y_{G_j}(t,t')$ is made of centered random variables bounded by $\gamma=2\gamma_j$ if $Y=X$ and by $\gamma=2\gamma_j/(\mu-1)$ if $Y=R$. Moreover, by the left inequality in $(i)$ together with the properties $j\le m(\beta,\varepsilon)\le m$ and $\beta_j j\ge m+r_m$, one has

$$\frac{b^2_{Y_{G_j}}(t,t')}{2\gamma}\ge s:=6Bm^2 2^{(\beta_j/2-1)j}|t'-t|^{1/2}.$$

Consequently, we can apply Lemma 3$(ii)$ to get

$$\mathbb{P}\Big(|X_{G_j}(t,t')|\ge 6Bm^2 2^{(\beta_j/2-1)j}|t'-t|^{1/2}\Big)\le 2\exp\big(-s^2/4b^2_{Y_{G_j}}(t,t')\big).$$

Now using the right inequality in $(i)$ and again the fact that $j\le m$ yields $s^2/4b^2_{Y_{G_j}}(t,t')\ge 9m^2$, hence the conclusion.

$(iii)$ Follows from Lemma 3$(i)$ applied with $\gamma$ as in the proof of $(ii)$, $s=4\sqrt{jm}$ and $n=\#G_j=2^{(\beta_j-1)j}$ when $G_j\neq\emptyset$.

**Corollary 3** *Fix $\varepsilon>0$ and $\eta\in(0,T)$.*

*There exists $C>0$ such that with probability one, there exists $m_0\ge 1$ such that for all $m\ge m_0$ and $Y\in\{X,R\}$*
*$(i)$ For all $(m+r_m)/\beta_j\le j\le m(\beta,\varepsilon)$, for all $(d,d')\in D^2_{[2(\beta+\varepsilon)]j+1}\cap(\eta,T)^2$ such that $2^{-m}\le|d'-d|\le 2^{-m+1}$,*

$$|Y_{G_j}(d,d')|\le Cm^2 2^{(\beta_j/2-1)j}|d'-d|^{1/2}.$$

*$(ii)$ For all $j\ge m(\beta,\varepsilon)$, for all $(d,d')\in D^2_{[2(\beta+\varepsilon)]j+1}\cap(\eta,T)^2$ such that $2^{-m}\le|d'-d|\le 2^{-m+1}$,*

$$|Y_{G_j}(d,d')|\le C(j+1)\sqrt{jm}2^{(\beta_j-3)j/2}.$$

**Proof.** For every $m \geq 1$ large enough, if $j$ is such that $(m + r_m)/\beta_j \leq j \leq m(\beta, \varepsilon)$, one has $J = [2(\beta + \varepsilon)j] + 1 \in [2m, (2(\beta + \varepsilon)m)] + 1)]$. Moreover, the number of pairs $(d, d') \in D_J^2$ such that $2^{-m} \leq |d' - d| \leq 2^{-m+1}$ is bounded by $2^{2J-m+2}$. Consequently, it follows from Lemma 2$(ii)$ that there exists a constant $C > 0$ such that for $m$ large enough,

$$\mathbb{P}\left(\exists \begin{cases} (m + r_m)/\beta_j \leq j \leq m(\beta, \varepsilon), \\ \begin{cases} (d, d') \in D_{[2(\beta+\varepsilon)j]+1}, \\ 2^{-m} \leq |d' - d| \leq 2^{-m+1} \end{cases} \end{cases}, |Y_{G_j}(d, d')| > Cm^2 2^{(\beta_j/2-1)j} |d' - d|^{1/2}\right)$$
$$\leq \left(\sum_{J=2m}^{(2(\beta+\varepsilon)m]+1} 2^{2J-m+2}\right) 2\exp(-9m^2) \leq \exp(-8m^2).$$

Also, one has

$$\mathbb{P}\left(\exists \begin{cases} j \geq m(\beta, \varepsilon), \\ \begin{cases} (d, d') \in D_{[2(\beta+\varepsilon)j]+1}, \\ 2^{-m} \leq |d' - d| \leq 2^{-m+1} \end{cases} \end{cases}, |Y_{G_j}(d, d')| > C(j+1)\sqrt{jm} 2^{(\beta_j - 3)j/2}\right)$$
$$\leq \sum_{j \geq m(\beta,\varepsilon)} 2^{2([2(\beta+\varepsilon)j]+1)-m+2} 2\exp(-8jm) \leq \frac{\exp(-7m(\beta, \varepsilon)m)}{1 - \exp(-7m(\beta, \varepsilon)m)}.$$

Since $\sum_{m \geq 1} \exp(-8m^2) + \exp(-7m(\beta, \varepsilon)m)$ converge, the conclusion follows from the Borel-Cantelli lemma.

**Lemma 4** *Fix $\varepsilon > 0$ and $\eta \in (0, T)$. There exists $C > 0$ such that for every $m$ large enough,*

$$\sup_{\substack{\eta < t, t' \leq T \\ |t'-t| \leq 2^{-m+1}}} \sum_{j \geq m/3} \sum_{i \in G_j} \left| \mathbb{E}\left(\widetilde{R}_i(t') - \widetilde{R}_i(t)\right) \right| \leq C\, 2^{-m/\delta}.$$

**Proof.** We saw in the proof of Lemma 2$(i)$ that for every $j \geq 1$, $i \in G_j$, and $0 < \eta < t, t' \leq T$,

$$\left| \mathbb{E}\left(\widetilde{R}_i(t') - \widetilde{R}_i(t)\right) \right| = O(2^{-3j}) + O\big(2^{-j} \exp\big(-(1 - \mu^{-1})2^j \eta\big)\big).$$

Consequently

$$\sup_{\substack{\gamma < t, t' \leq T \\ |t'-t| \leq 2^{-m+1}}} \sum_{j \geq m/3} \sum_{i \in G_j} \left| \mathbb{E}\left(\widetilde{R}_i(t') - \widetilde{R}_i(t)\right) \right| = O(2^{-m}) = O(2^{-m/\delta}).$$

**Lemma 5** *Fix $\eta \in (0, T)$. With probability one, there exists $C > 0$ such that for $Y \in \{X, R\}$ and $j$ large enough, the jumps sizes of $\sum_{i \in G_j} Y_i$ in $(\eta, T)$ are bounded by $Cj2^{-j}$.*

**Proof.** Recall that $Z_i(0)$ is assumed to be 0 for all $i \geq 1$. Due to Lemma 1 and the respective definitions of $X_i$ and $R_i$ , with probability one, there exists $K > 0$ such that for $i$ large enough both $\sup_{t\in(\eta,T)} X_i(t)$ and $\sup_{t\in(\eta,T)} R_i(t)$ are less than $K \log(\lambda_i)/\lambda_i$. This is enough to conclude.

**Lemma 6** *(i) There exists $m_0 \geq 1$ such that for all $m \geq m_0$, for $Y \in \{X, R\}$, the probability of the event $E_m = \{$there exists $[m/\delta] \leq j \leq (m + r_m))/\beta_j$ such that $\sum_{i\in G_j} Y_i$ has on any of the $T2^{m-1}$ dyadic subintervals of length $2^{-m+1}$ of $[0, T]$ more than $m^8$ jumps$\}$ is bounded by $2^{-m}$.*

*(ii) Fix $\eta \in (0, T)$. With probability one, there exists $m_0 \geq 1$ and a constant $C_\eta > 0$ such that for $m \geq m_0$, for all $t, t' \in (\eta, T]$ such that $2^{-m} \leq |t' - t| \leq 2^{-m+1}$,*

$$\left| \sum_{[m/\delta]\leq j\leq(m+r_m)/\beta_j} \sum_{i\in G_j} Y_i(t') - Y_i(t) \right| \leq C_\eta |t' - t|^{1/\delta} \big| \log(|t' - t|)\big|^9.$$

**Proof.** $(i)$ Since the $Y_i$ are independent, the number of jump points of $\sum_{i\in G_j} Y_i$ in a dyadic interval of the $(m-1)^{\text{th}}$ generation is a Poisson variable of parameter $\Lambda_j = 2^{-m+1}\sum_{i\in G_j} \lambda_i \leq 2^{-m+1}(\#G_j)2^{j+1}$. Moreover, if $j \leq (m + r_m))/\beta_j$, $\Lambda_j \leq 4.2^{-m+\beta_j j} \leq 4.2^{r_m} = O(m^6) \leq m^7$ for $m$ large enough. Consequently, for $m$ large enough, the probability that $\sum_{i\in G_j} Y_i$ has on any of the $2^{m-1}T$ dyadic subintervals of length $2^{-m+1}$ of $[0, T]$ more than $m^8$ jumps is bounded by

$$2^{m-1}T \exp(-\Lambda_j) \sum_{k\geq m^8} \frac{m^{7k}}{k!} \leq 2^{m-1}T \sum_{k\geq m^8} \left(\frac{e}{m}\right)^k = O(2^{-m^8})$$

(we used Stirling's formula). It follows that $\mathbb{P}(E_m) = O\big((m + r_m)2^{-m^8}\big)$. This yields the first part of the lemma.

$(ii)$ Due to the first part of the lemma and the Borel-Cantelli lemma, with probability one, there exists $m_0 \geq 1$ such that if $m \geq m_0$ then $E_m$ does not happen, hence for all $t, t' \in (\eta, T]$ such that $2^{-m} \leq t' - t \leq 2^{-m+1}$ and all for all $j$ such that $[m/\delta] \leq j \leq (m + r_m)/\beta_j$, $\sum_{i\in G_j} Y_i$ has at most $2m^8$ jump points between $t$ and $t'$. By Lemma 5, $m_0$ can be also chosen so that these jumps are $O(j2^{-j})$, so $O(m)2^{-j}$ here. Consequently,

$$\begin{aligned}
\left| \sum_{[m/\delta]\leq j\leq(m+r_m)/\beta_j} \sum_{i\in G_j} Y_i(t') - Y_i(t) \right| &\leq \sum_{[m/\delta]\leq j\leq(m+r_m)/\beta_j} 2m^8 O(m)2^{-j} \\
&\quad + \sum_{[m/\delta]\leq j\leq(m+r_m)/\beta_j} |t' - t|2^{(\beta_j-1)j} \\
&\leq O(m^9)2^{-m/\delta} + |t' - t|2^{m+r_m} \sum_{m/\delta\leq j\leq(m+r_m)/\beta_j} 2^{-j} \\
&= O(m^9)2^{-m/\delta} + O(m^6)2^{-m/\delta},
\end{aligned}$$

that is the desired result.

**Lemma 7** *Fix $\varepsilon > 0$. With probability one, for $Y \in \{X, R\}$ and $j$ large enough, the distance between two consecutive jump points of $\sum_{i\in G_j} Y_i$ in $[0,T]$ is at least $2^{-2(\beta+\varepsilon)j}$.*

**Proof.** The jump points of $\sum_{i\in G_j} Y_i$ are also the jump points of a Poisson process of intensity $\Lambda_j = \sum_{i\in G_j} \lambda_i$. Consequently, there exists a sequence $(\theta_k^{(j)})_{k\geq 1}$ of independent exponential random variables with parameter $\Lambda_j$ such that almost surely the jump points of $\sum_{i\in G_j} Y_i$ in $[0,T]$ are described by the points of the increasing finite sequence of points of the form $\sum_{k=1}^m \theta_k^{(j)}$, $m \geq 1$, that belong to $[0,T]$. The problem is reduced to estimate the minimum distance between two consecutive of these points, i.e the minimum of $\{\theta_m^{(j)} : \sum_{k=1}^m \theta_k^{(j)} \in [0,T]\}$.

Define $\widetilde{N}_T^{(j)} = \#\{\theta_m^{(j)} : \sum_{k=1}^m \theta_k^{(j)} \in [0,T]\}$ and $\widetilde{M}_j = T\Lambda_j + 4\sqrt{T\Lambda_j \log(T\Lambda_j)}$. By Lemma 1 of [18], one has $\mathbb{P}(\widetilde{N}_T^{(j)} \geq \widetilde{M}_j) \leq 1/(T\Lambda_j)^7$. Moreover,

$$\begin{aligned}\mathbb{P}\left(\exists\, 1 \leq k \leq \widetilde{M}_j : \ \theta_k^{(j)} < 2^{-2(\beta+\varepsilon)j}\right) &= 1 - \left(\mathbb{P}(\theta_1^{(j)} \geq 2^{-2(\beta+\varepsilon)j})\right)^{\widetilde{M}_j} \\ &= 1 - \exp\big(-\widetilde{M}_j \Lambda_j 2^{-2(\beta+\varepsilon)j}\big).\end{aligned}$$

By construction, $\widetilde{M}_j \Lambda_j 2^{-2(\beta+\varepsilon)j} = O(2^{-\frac{\varepsilon}{2}j})$. It follows that

$$\sum_{j\geq 1} \mathbb{P}\left(\{\widetilde{N}_T^{(j)} \geq \widetilde{M}_j\} \cup \{\exists\, 1 \leq k \leq \widetilde{M}_j : \ \theta_k^{(j)} < 2^{-2(\beta+\varepsilon)j}\}\right) < \infty.$$

One concludes with the Borel-Cantelli Lemma.

Recall that for $k \geq 1$ the probability distribution of $T_k^{(i)}$ is the Gamma$(\lambda_i, k)$ distribution, that is

$$\mathbb{P}(T_k^{(i)} \in du) = \mathbf{1}_{[0,\infty)}(u) \frac{\lambda_i^k}{(k-1)!} u^{k-1} e^{-\lambda_i u} du. \tag{2}$$

It is then quite easy, and left to the reader, to verify the properties collected in the following lemma.

**Lemma 8** *Fix $i \geq 1$ and $\gamma > 0$. For every $t' > t > 0$,*
*(i)* $\mathbb{E}\big(X_i(t)\big) = \mathbb{E}(t - T_t^{(i)}) = \dfrac{1}{\lambda_i}\big(1 - e^{-\lambda_i t}\big)$.
*(ii) The stochastic process $\big(X_i(t)\mathbf{1}_{\{X_i(t)\leq\gamma\}}\big)_{t>\gamma}$ is stationary. More precisely, for $t > \gamma$, one has*

$$\begin{cases}\mathbb{P}(X_i(t)\mathbf{1}_{\{X_i(t)\leq\gamma\}} = 0) = e^{-\lambda_i t}, \\ \mathbb{P}\big(X_i(t)\mathbf{1}_{\{X_i(t)\leq\gamma\}} \in [u, u+du]\big) = \lambda_i e^{-\lambda_i u}\, du \quad (u \in (0,\gamma]).\end{cases}$$

*(iii) If $t' > t > \gamma \geq t' - t$, one has* $\mathbb{P}\big(\{T_t^{(i)} \geq t'-\gamma\} \cap \{T_t^{(i)} = T_{t'}^{(i)}\} \cap \{T_t^{(i)} \in [u, u+du]\}\big) = \mathbb{P}\big(\{T_t^{(i)} \in [u, u+du]\} \cap \{\tau^{(i)}_{k_t^{(i)}+1} \geq t' - u\}\big) = \lambda_i e^{-\lambda_i (t'-u)}\, du \quad (u \in [t'-\gamma, t])$.

*(iv) If $t > t' > \gamma \geq t' - t$, one has* $\mathbb{P}\big(\{T_t^{(i)} \geq t-\gamma\} \cap \{T_t^{(i)} < T_{t'}^{(i)}\} \cap \{T_t^{(i)} \in [u, u+du]\} \cap \{\tau^{(i)}_{k_t^{(i)}+1} \in [v, v+dv]\}\big) = \lambda_i^2 e^{-\lambda_i v}\, du dv \quad (u \in [t-\gamma, t],\ v \in (t-u, t'-u])$.

**Lemma 9** *One has*

$$\frac{\mathbb{E}\Big(\big(X_i(t')\mathbf{1}_{\{X_i(t')\le\gamma\}} - X_i(t)\mathbf{1}_{\{X_i(t)\le\gamma\}}\big)X_i(t)\big)^2\Big)}{2(t'-t)/\lambda_i} \longrightarrow 1 \quad \textit{as } (\mathcal{P}) \text{ holds},$$

*where*

$$(\mathcal{P}) = \begin{cases} (t'-t) \le \gamma < t < t' \\ \lambda_i\gamma \to \infty,\ \gamma^2 e^{-\lambda_i\gamma} = o\Big(\frac{t'-t}{\lambda_i}\Big) \\ \lambda_i(t'-t) \to 0. \end{cases}$$

**Proof.** Fix $i \ge 1$. Fix $0 < \gamma < t < t'$ so that $t' - t \le \gamma$. Denote $X_i(u)\mathbf{1}_{\{X_i(u)\le\gamma\}}$ by $Y_i(u)$ for $u \in \{t, t'\}$. By using Lemma 8($ii$) we get

$$\begin{aligned} \mathbb{E}\big(Y_i(t') - Y_i(t)\big)^2 &= 2\int_0^\gamma u^2\lambda_i e^{-\lambda_i u}\,du - 2\mathbb{E}\big(Y_i(t')Y_i(t)\big) \\ &= \frac{2}{\lambda_i^2}\big(2 - (2 + 2\lambda_i\gamma + \lambda_i^2\gamma^2)e^{-\lambda_i\gamma}\big) - 2\mathbb{E}\big(Y_i(t')Y_i(t)\big) \\ &= \frac{4}{\lambda_i^2} - 2\mathbb{E}\big(Y_i(t')Y_i(t)\big) + o\Big(\frac{t'-t}{\lambda_i}\Big) \text{ as } (\mathcal{P}) \text{ holds.} \end{aligned}$$

Now, on the one hand, using Lemma 8($iii$) we have

$$\begin{aligned} &\mathbb{E}\big(Y_i(t')Y_i(t)\mathbf{1}_{\{T_t^{(i)}=T_{t'}^{(i)}\}}\big) = \mathbb{E}\big([Y_i(t) + t' - t]Y_i(t)\mathbf{1}_{\{T_t^{(i)}=T_{t'}^{(i)}\}}\mathbf{1}_{\{T_t^{(i)}\ge t'-\gamma\}}\big) \\ &= \mathbb{E}\big([t - T_t^{(i)} + t' - t][t - T_t^{(i)}]\mathbf{1}_{\{T_t^{(i)}=T_{t'}^{(i)}\}}\mathbf{1}_{\{T_t^{(i)}\ge t'-\gamma\}}\big) \\ &= \int_{t'-\gamma}^t \lambda_i(t - u + t' - t)(t-u)e^{-\lambda_i(t'-u)}\,du \\ &= \frac{2}{\lambda_i^2}e^{-\lambda_i(t'-t)} + \frac{t'-t}{\lambda_i}e^{-\lambda_i(t'-t)} \\ &\qquad -\frac{e^{-\lambda_i\gamma}}{\lambda_i^2}\big(2 + 2\lambda_i(\gamma - t' + t) + \lambda_i^2(\gamma - t' + t)^2\big) - \frac{t'-t}{\lambda_i}e^{-\lambda_i\gamma}\big(1 + \gamma - t' + t\big) \\ &= \frac{2}{\lambda_i^2} - \frac{t'-t}{\lambda_i} + o\Big(\frac{t'-t}{\lambda_i}\Big) \text{ as } (\mathcal{P}) \text{ holds.} \end{aligned}$$

On the other hand we have

$$\mathbb{E}\big(Y_i(t')Y_i(t)\mathbf{1}_{\{T_t^{(i)}<T_{t'}^{(i)}\}}\big) = \mathbb{E}\big(Y_i(t')Y_i(t)\mathbf{1}_{\{t-\gamma\le T_t^{(i)}\le t\}}\mathbf{1}_{\{t-T_t^{(i)}<\tau^{(i)}_{k_t^{(i)}+1}\le t'-T_t^{(i)}\}}\big).$$

Due to the lack of memory of the exponential law, conditionally on $\{t - T_t^{(i)} < \tau^{(i)}_{k_t^{(i)}+1} \le t' - T_t^{(i)}\}$, the random variable $Y_i(t')$ has the probability distribution of $t' - T_t^{(i)} - \tau^{(i)}_{k_t^{(i)}+1}$.

It follows from Lemma 8$(i)$ that $\mathbb{E}\big(Y_i(t')|t - T_t^{(i)} < \tau^{(i)}_{k_t^{(i)}+1} \le t' - T_t^{(i)}\big) = \frac{1}{\lambda_i}\big(1 - e^{-\lambda_i\left(t'-T_t^{(i)}-\tau^{(i)}_{k_t^{(i)}+1}\right)}\big)$. Consequently, using Lemma 8$(iv)$ we can get

$$\begin{aligned}
&\mathbb{E}\big(Y_i(t')Y_i(t)\mathbf{1}_{\{T_t^{(i)}<T_{t'}^{(i)}\}}\big)\\
= \;& \mathbb{E}\Big(\frac{1}{\lambda_i}\big(1 - e^{-\lambda_i\left(t'-T_t^{(i)}-\tau^{(i)}_{k_t^{(i)}+1}\right)}\big)(t - T_t^{(i)})\mathbf{1}_{\{t-\gamma\le T_t^{(i)}\le t\}}\mathbf{1}_{\{t-T_t^{(i)}<\tau^{(i)}_{k_t^{(i)}+1}\le t'-T_t^{(i)}\}}\Big)\\
= \;& \int_{\mathbb{R}_+^2} \mathbf{1}_{\{[t-\gamma,t]\}}(u)\mathbf{1}_{[t-u,t'-u]}(v)\lambda_i^2(t-u)\frac{1}{\lambda_i}\big(1 - e^{-\lambda_i(t'-u-v)}\big)e^{-\lambda_i v}\,dudv\\
= \;& \left(\frac{1-e^{-\lambda_i(t'-t)}}{\lambda_i^2} - \Big(\frac{t'-t}{\lambda_i} + \frac{(t'-t)^2}{2}\Big)e^{-\lambda_i(t'-t)}\right)\big(1-(1+\lambda_i\gamma)e^{-\lambda_i\gamma}\big)\\
= \;& o\Big(\frac{t'-t}{\lambda_i}\Big)
\end{aligned}$$

as $0 < t' - t \le \gamma < t < t'$, $\lambda_i\gamma \to \infty$ and $\lambda_i(t'-t) \to 0$. Adding the three above estimates yields the conclusion.

Fix $\gamma > 0$. Consider the stochastic process $\widetilde{R}_i^{(\gamma)} = \widetilde{R}_i$ defined on $[T_k^{(i)} \le t < T_{k+1}^{(i)})$ for every $k \ge 0$ and $i \ge 1$ by

$$\widetilde{R}_i(t) = \frac{1}{\mu^{k+1}}\sum_{j=1}^{k}\mu^j\tau_j^{(i)}\mathbf{1}_{\{\tau_j^{(i)}\le\gamma\}}.$$

**Lemma 10** *Fix $T \ge t' > t > 0$.*
$(i)$ $|\mathbb{E}(R_i(t')) - \mathbb{E}(R_i(t))| \le 4\dfrac{\mu}{(\mu-1)\lambda_i}\exp\big(-(1-\mu^{-1})\lambda_i t\big)$.

$(ii)$ *There exists a constant $K > 0$ such that if $\lambda_i\gamma \ge 1$ then*

$$0 \le \mathbb{E}\big(R_i(t) - \widetilde{R}_i(t)\big) \le K\gamma e^{-\lambda_i\gamma/2}.$$

$(iii)$ *There exists a constant $K > 0$ such that if $\lambda_i\gamma \ge 1$ and $t \ge \gamma$ then*

$$\mathbb{E}\big([R_i(t) - \widetilde{R}_i(t)]^2\big) \le K\gamma^2 e^{-\lambda_i\gamma/2}(\lambda_i T).$$

$(iv)$

$$\mathbb{E}\big([R_i(t') - R_i(t)]^2\big) \le \gamma^2(1 - e^{-\lambda_i(t'-t)}) + 4T^2\,\mathbb{P}(\{N_T^{(i)} \ge M_i\} \cup \{\exists\, 1 \le j \le M_i;\ \tau_j^{(i)} > \gamma\}).$$

$(v)$ $\mathbb{E}\Big(\mathbf{1}_{\{k_{t'}^{(i)}=k_t^{(i)}+1\}}[R_i(t') - R_i(t)]^2\Big) \ge \dfrac{2}{\mu^2}\Big(1-\dfrac{1}{\mu}\Big)^2\dfrac{t'-t}{\lambda_i} + o\Big(\dfrac{t'-t}{\lambda_i}\Big)$ *as $\lambda_i(t'-t) \to 0$ and $\lambda_i t \to \infty$.*

**Proof.** $(i)$ For every $k \geq 1$, the random variables $\tau_j^{(i)}$ have, conditionally on $\{k_t^{(i)} = k\}$, the same probability distribution. This yields

$$\begin{aligned}
\mathbb{E}(R_i(t)) &= \mathbb{E}\Big(\mathbf{1}_{\{k_t^{(i)}\geq 1\}}\frac{1}{\mu^{k_t^{(i)}+1}}\sum_{j=1}^{k_t^{(i)}}\mu^j\tau_j^{(i)}\Big) = \mathbb{E}\Big(\mathbf{1}_{\{k_t^{(i)}\geq 1\}}\frac{1}{k_t^{(i)}}\frac{1}{\mu^{k_t^{(i)}+1}}\sum_{j=1}^{k_t^{(i)}}\mu^j\big(\sum_{j=1}^{k_t^{(i)}}\tau_j^{(i)}\big)\Big)\\
&= (\mu-1)^{-1}\mathbb{E}\Big(\mathbf{1}_{\{k_t^{(i)}\geq 1\}}\frac{1}{k_t^{(i)}}(1-\frac{1}{\mu^{k_t^{(i)}}})T^{(i)}_{k_t^{(i)}}\Big) = (\mu-1)^{-1}\sum_{k=1}^{\infty}(1-\frac{1}{\mu^k})e^{-\lambda_i t}\int_0^t\frac{\lambda_i^k u^k}{k!}\,du\\
&= (\mu-1)^{-1}e^{-\lambda_i t}\Big(\int_0^t(e^{\lambda_i u}-e^{(\lambda_i/\mu)u})\,du\Big)\\
&= \frac{1}{(\mu-1)\lambda_i}\Big[1+(\mu-1)e^{-\lambda_i t}-\mu e^{-\lambda_i(1-1/\mu)t}\Big].
\end{aligned}$$

$(ii)$ and $(iii)$. On the one hand

$$\begin{aligned}
0 \leq \mathbb{E}\big(R_i(t)-\widetilde{R}_i(t)\big) &= \mathbb{E}\Big(\mathbf{1}_{\{k_t^{(i)}\geq 1\}}\frac{1}{\mu^{k_t^{(i)}+1}}\sum_{j=1}^{k_t^{(i)}}\mu^j\tau_j^{(i)}\mathbf{1}_{\{\tau_j^{(i)}>\gamma\}}\Big)\\
&= \mathbb{E}\Big(\mathbf{1}_{\{k_t^{(i)}\geq 1\}}\frac{1}{\mu^{k_t^{(i)}+1}}\big(\sum_{j=1}^{k_t^{(i)}}\mu^j\big)\tau_1^{(i)}\mathbf{1}_{\{\tau_1^{(i)}>\gamma\}}\Big) \leq \mathbb{E}\Big(\mathbf{1}_{\{k_t^{(i)}\geq 1\}}\tau_1^{(i)}\mathbf{1}_{\{\tau_1^{(i)}>\gamma\}}\Big)\\
&\leq \Big(\mathbb{E}\big((\tau_1^{(i)})^2\mathbf{1}_{\{\tau_1^{(i)}>\gamma\}}\big)\Big)^{1/2}\Big(\mathbb{P}(\{k_t^{(i)}\geq 1\})\Big)^{1/2} \leq \Big(\mathbb{E}\big((\tau_1^{(i)})^2\mathbf{1}_{\{\tau_1^{(i)}>\gamma\}}\big)\Big)^{1/2}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\mathbb{E}\big([R_i(t)-\widetilde{R}_i(t)]^2\big) = \mathbb{E}\Big(\mathbf{1}_{\{k_t^{(i)}\geq 1\}}\frac{1}{\mu^{2k_t^{(i)}+2}}\big(\sum_{j=1}^{k_t^{(i)}}\mu^j\tau_j^{(i)}\mathbf{1}_{\{\tau_j^{(i)}>\gamma\}}\big)^2\Big)\\
&\leq \mathbb{E}\Big(\mathbf{1}_{\{k_t^{(i)}\geq 1\}}\frac{k_t^{(i)}}{\mu^{2k_t^{(i)}+2}}\sum_{j=1}^{k_t^{(i)}}\mu^{2j}(\tau_j^{(i)})^2\mathbf{1}_{\{\tau_j^{(i)}>\gamma\}}\Big) = \mathbb{E}\Big(\mathbf{1}_{\{k_t^{(i)}\geq 1\}}\frac{k_t^{(i)}}{\mu^{2k_t^{(i)}+2}}\big(\sum_{j=1}^{k_t^{(i)}}\mu^{2j}\big)(\tau_1^{(i)})^2\mathbf{1}_{\{\tau_1^{(i)}>\gamma\}}\Big)\\
&\leq \mathbb{E}\Big(k_t^{(i)}\mathbf{1}_{\{k_t^{(i)}\geq 1\}}(\tau_1^{(i)})^2\mathbf{1}_{\{\tau_1^{(i)}>\gamma\}}\Big) \leq \Big(\mathbb{E}\big((\tau_1^{(i)})^4\mathbf{1}_{\{\tau_1^{(i)}>\gamma\}}\big)\Big)^{1/2}\Big(\sum_{k=1}^{\infty}k^2\mathbb{P}(T_t^{(i)}=T_k^{(i)})\Big)^{1/2}.
\end{aligned}$$

Then, simple computations show that for every $n \geq 1$, there exists $K > 0$ such that for all $i \geq 1$ and $t > 0$ such that $\lambda_i t \geq 1$ one has $\mathbb{E}\big((\tau_1^{(i)})^n\mathbf{1}_{\{\tau_1^{(i)}>\gamma\}}\big) \leq K\gamma^n e^{-\lambda_i\gamma}$ and $\big(\sum_{k=1}^{\infty}k^2\mathbb{P}(T_t = T_k^{(i)})\big)^{1/2} \leq K\lambda_i t$. This yields the conclusions.

$(iv)$ In the following computation, we use the fact that, conditionally on $\{\tau_j^{(i)} \leq \gamma\}$, for all $1 \leq j \leq k_{t'}^{(i)}$, both $R_i(t)$ and $R_i(t')$ are in $[0,\gamma]$, and so is $|R_i(t')-R_i(t)|$. We also use the obvious upper bound $R_i(t) \leq t$.

$$\begin{aligned}
&\mathbb{E}\big([R_i(t') - R_i(t)]^2\big) = \mathbb{E}\big(\mathbf{1}_{\{T_{t'} > T_t\}}[R_i(t') - R_i(t)]^2\big) \\
\leq\ & \gamma^2 \mathbb{P}(T_{t'} > T_t) + \mathbb{E}\big(\mathbf{1}_{\{\exists\ 1 \leq j \leq k_{t'}^{(i)};\ \tau_j^{(i)} > \gamma\}}[R_i(t') - R_i(t)]^2\big) \\
\leq\ & \gamma^2 \mathbb{P}(T_{t'} > T_t) + \mathbb{P}(\exists\ 1 \leq j \leq k_{t'}^{(i)};\ \tau_j^{(i)} > \gamma)(t + t')^2 \\
\leq\ & \gamma^2 (1 - e^{-\lambda_i(t'-t)}) + \mathbb{P}(\{N_T^{(i)} \geq M_i\} \cup \{\exists\ 1 \leq j \leq M_i;\ \tau_j^{(i)} > \gamma\})(t + t')^2.
\end{aligned}$$

$(v)$

$$\begin{aligned}
&\mathbb{E}\big(\mathbf{1}_{\{k_{t'}^{(i)} = k_t^{(i)}+1\}}[R_i(t') - R_i(t)]^2\big) \\
=\ & \mathbb{E}\left(\mathbf{1}_{\{k_{t'}^{(i)} = k_t^{(i)}+1\}}\left[\Big[\sum_{j=1}^{k_t^{(i)}} \mu^j\Big(\frac{1}{\mu^{k_t^{(i)}+2}} - \frac{1}{\mu^{k_t^{(i)}+1}}\Big)\tau_j^{(i)}\Big] + \frac{\tau^{(i)}_{k_t^{(i)}+1}}{\mu}\right]^2\right) \\
\geq\ & A_1 + A_2 + A_3
\end{aligned}$$

with

$$\begin{cases}
A_1 = \left(\frac{1}{\mu} - \frac{1}{\mu^2}\right)^2 \mathbb{E}\Big(\sum_{j=1}^{k_t^{(i)}} \mathbf{1}_{\{k_{t'}^{(i)} = k_t^{(i)}+1\}} \frac{\mu^{2j}}{\mu^{2k_t^{(i)}}}(\tau_j^{(i)})^2\Big) \\
A_2 = 2\mathbb{E}\left(\mathbf{1}_{\{k_{t'}^{(i)} = k_t^{(i)}+1\}}\Big[\sum_{j=1}^{k_t^{(i)}} \mu^j\Big(\frac{1}{\mu^{k_t^{(i)}+2}} - \frac{1}{\mu^{k_t^{(i)}+1}}\Big)\tau_j^{(i)}\Big]\frac{\tau^{(i)}_{k_t^{(i)}+1}}{\mu}\right) \\
A_3 = \mathbb{E}\left(\mathbf{1}_{\{k_{t'}^{(i)} = k_t^{(i)}+1\}} \frac{\left(\tau^{(i)}_{k_t^{(i)}+1}\right)^2}{\mu^2}\right).
\end{cases}$$

To find a lower bound of $A_1$ (resp. $A_3$) we use assertion $(ii)$ (resp. $(i)$) of the following lemma, whose elementary proof is left to the reader.

**Lemma 11** $(i)$ $\mathbb{P}\Big(2 \leq k_{t'}^{(i)} = k_t^{(i)} + 1,\ T^{(i)}_{k_t^{(i)}} \in [u, u + du],\ \tau^{(i)}_{k_t^{(i)}+1} \in [v, v + dv]\Big) = \lambda_i^2 e^{-\lambda_i(t'-u)}\, dudv \quad (u \in [0, t],\ v \in [t - u, t' - u])$.

$(ii)$ $\mathbb{P}\Big(3 \leq k_{t'}^{(i)} = k_t^{(i)}+1,\ T^{(i)}_{k_t^{(i)}-1} \in [u, u+du],\ \tau^{(i)}_{k_t^{(i)}} \in [v, v+dv],\ \tau^{(i)}_{k_t^{(i)}+1} \in [w, w+dw]\Big) = \lambda_i^3 e^{-\lambda_i(t'-u)}\, dudvdw \quad (u \in [0, t],\ v \in [0, t - u],\ w \in [t - (u + v), t' - (u + v)])$.

Also, in finding a lower bound for $A_1$ and $A_2$ we use the fact that conditionally on $\{k_{t'}^{(i)} = k_t^{(i)} + 1\}$, the $\tau_j^{(i)}$, $1 \leq j \leq k_t^{(i)}$, have the same probability distribution.

$$
\begin{aligned}
A_1 &= \Big(\frac{1}{\mu}-\frac{1}{\mu^2}\Big)^2\mathbb{E}\Big(\sum_{j=1}^{k_t^{(i)}}\mathbf{1}_{\{k_{t'}^{(i)}=k_t^{(i)}+1\}}\frac{\mu^{2j}}{\mu^{2k_t^{(i)}}}(\tau^{(i)}_{k_t^{(i)}})^2\Big)\\
&\geq \Big(\frac{1}{\mu}-\frac{1}{\mu^2}\Big)^2\int_0^t\int_0^{t-u}\int_{t-(u+v)}^{t'-(u+v)}\lambda_i^3v^2e^{-\lambda_i(t'-u)}\,dudvdw\\
&= \Big(\frac{1}{\mu}-\frac{1}{\mu^2}\Big)^2\frac{t'-t}{3\lambda_i}e^{-\lambda_i(t'-t)}\int_0^{\lambda_i t}u^3e^{-u}du\\
&= 2\Big(\frac{1}{\mu}-\frac{1}{\mu^2}\Big)^2\frac{t'-t}{\lambda_i}+o\Big(\frac{t'-t}{\lambda_i}\Big)
\end{aligned}
$$

as $\lambda_i(t'-t)\to 0$ and $\lambda_i t\to\infty$.

$$
\begin{aligned}
&A_2\\
&= -2\mathbb{E}\left(\mathbf{1}_{\{2\leq k_{t'}^{(i)}=k_t^{(i)}+1\}}\frac{1}{k_t^{(i)}}\Big[\sum_{j=1}^{k_t^{(i)}}\frac{\mu^j}{\mu^{k_t^{(i)}+2}}\Big]\Big(1-\frac{1}{\mu}\Big)T^{(i)}_{k_t^{(i)}}\tau^{(i)}_{k_t^{(i)}+1}\right)\\
&= -2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{\mu^2}(1-\frac{1}{\mu^{k+1}})\int_0^t u\frac{\lambda_i^ku^{k-1}}{(k-1)!}e^{-\lambda_i u}\int_{t-u}^{t'-u}v\lambda_ie^{-\lambda_i v}e^{-\lambda_i(t'-u-v)}dvdu\\
&\geq -2e^{-\lambda_i t'}\sum_{k=1}^{\infty}\frac{1}{\mu^2}\int_0^t\frac{\lambda_i^ku^k}{k!}\int_{t-u}^{t'-u}\lambda_i v\,dvdu\\
&\geq -e^{-\lambda_i t'}\frac{(t'-t)}{\mu^2}\int_0^t\lambda_ie^{\lambda_i u}(t'+t-2u)\,du\\
&= -e^{-\lambda_i t'}\frac{(t'-t)}{\mu^2}\left[(t'+t)(e^{\lambda_i t}-1)-\frac{2}{\lambda_i}[(\lambda_i t-1)e^{\lambda_i t}+1]\right]\\
&\geq -2\frac{t'-t}{\mu^2\lambda_i}+o\Big(\frac{t'-t}{\lambda_i}\Big)
\end{aligned}
$$

as $\lambda_i(t'-t)\to 0$ and $\lambda_i t\to\infty$.

$$
\begin{aligned}
A_3 &= \frac{1}{\mu^2}\int_0^t\int_{t-u}^{t'-u}\lambda_i^2v^2e^{-\lambda_i(t'-u)}\,dudv=\frac{1}{\mu^2}\int_0^t\lambda_i^2e^{-\lambda_i(t'-u)}\frac{(t'-u)^3-(t-u)^3}{3}\,du\\
&= \frac{t'-t}{3\mu^2}\int_0^t\lambda_i^2e^{-\lambda_i(t'-u)}((t'-u)^2+(t'-u)(t-u)+(t-u)^2)\,du\\
&= \frac{t'-t}{3\mu^2}\int_{t'-t}^{t'}\lambda_i^2e^{-\lambda_i v}(3v^2+3v(t-t')+(t'-t)^2)\,dv\\
&\geq \frac{t'-t}{\mu^2\lambda_i}\int_{\lambda_i(t'-t)}^{\lambda_i t'}v^2e^{-v}\,dv-\frac{(t'-t)^2}{\mu^2}\int_{\lambda_i(t'-t)}^{\lambda_i t'}ve^{-v}\,dv\\
&= 2\frac{t'-t}{\mu^2\lambda_i}+o\Big(\frac{t'-t}{\lambda_i}\Big)
\end{aligned}
$$

as $\lambda_i(t'-t) \to 0$ and $\lambda_i t \to \infty$. The conclusion follows by adding the respective lower bounds of $A_1$, $A_2$ and $A_3$.

# 6 Proofs of covering results

**Proof of Proposition 6.**$(i)$ Suppose we have shown that for every $t \in (0,T]$, $\mathbb{P}(t \in H_1) = 1$. Then the Fubini theorem applied to the measurable function $f(\omega,t) = \mathbf{1}_{\{H_1(\omega)\}}(t)$ and the measure $\mathbb{P} \otimes \ell_T$, where $\ell_T$ is the restriction of the Lebesgue measure to $(0,T]$, yields the conclusion.

Fix $t \in (0,T]$. To see that $\mathbb{P}(t \in H_1) = 1$, first notice that the event $\{t \in H\}$ is also the limsup of the events

$$A_{n,i} = \Big\{ t \in \bigcup_{\substack{1 \le k \le N_T^{(i)} \\ \tau_k^{(i)} \ge 2^{-j_n}}} [T_k^{(i)} - 2^{-\beta_{j_n} j_n}, T_k^{(i)} + 2^{-\beta_{j_n} j_n}] \Big\} \quad (n \ge 1,\ i \in G_{j_n}),$$

as $n \to \infty$. Since by construction these events are independent, by the Borel-Cantelli Lemma it is enough to show that

$$\sum_{n \ge 1} \sum_{i \in G_{j_n}} \mathbb{P}(A_{n,i}) = +\infty.$$

For every $n$ such that $2^{-\beta_{j_n} j_n} + 2^{-j_n} \le t$ we have

$$\begin{aligned}
& \mathbb{P}(A_{n,i}) \\
\ge\ & \mathbb{P}\big(k_t^{(i)} \ge 2,\ T_t^{(i)} \le t < T_t^{(i)} + 2^{-\beta_{j_n} j_n},\ \tau_{k_t}^{(i)} \ge 2^{-j_n}\big) \\
=\ & \mathbb{P}\left( \begin{cases} k_t^{(i)} \ge 2,\ T^{(i)}_{k_t^{(i)}-1} \le t - 2^{-j_n}, \\ \max\big(2^{-j_n}, t - T^{(i)}_{k_t^{(i)}-1} - 2^{-\beta_{j_n} j_n}\big) \le \tau^{(i)}_{k_t^{(i)}} \le t - T^{(i)}_{k_t^{(i)}-1}, \end{cases} \right) \\
\ge\ & P_{n,i}
\end{aligned}$$

where

$$P_{n,i} = \mathbb{P}\left(k_t^{(i)} \ge 2,\ T^{(i)}_{k_t^{(i)}-1} \le t - 2^{-j_n} - 2^{-\beta_{j_n} j_n},\ t - T^{(i)}_{k_t^{(i)}-1} - 2^{-\beta_{j_n} j_n} \le \tau^{(i)}_{k_t^{(i)}} \le t - T^{(i)}_{k_t^{(i)}-1}\right).$$

To compute $P_{n,i}$ we use (2) to get that $\mathbb{P}\big(k_t^{(i)} \ge 2,\ T^{(i)}_{k_t^{(i)}-1} \in [u, u+du], \tau^{(i)}_{k_t^{(i)}} \in [v, v+dv]\big) = \mathbf{1}_{\{u+v \le t\}} \mathbf{1}_{\{0 \le u\}} \lambda_i^2 e^{-\lambda_i(t-u)} du dv \quad (u \in [0, t - 2^{-j_n} - 2^{-\beta_{j_n} j_n}],\ v \in [t-u-2^{-\beta_{j_n} j_n}, t-u])$.

We obtain

$$P_{n,i} = \lambda_i 2^{-\beta_{j_n} j_n} \exp\big(-\lambda_i(2^{-j_n} + 2^{-\beta_{j_n} j_n})\big) - \lambda_i 2^{-\beta_{j_n} j_n} \exp(-\lambda_i t).$$

Notice that since $t$ is a fixed positive number and $\# G_{j_n} = 2^{(\beta_{j_n}-1)j_n}$, one has

$$\sum_{n \ge 1} \sum_{i \in G_{j_n}} \lambda_i 2^{-\beta_{j_n} j_n} \exp(-\lambda_i t) < \infty.$$

So we have to prove that $\sum_{n\geq 1}\sum_{i\in G_{j_n}} P'_{n,i} = +\infty$ where $P'_{n,i} = \lambda_i 2^{-\beta_{j_n} j_n} \exp\big(-\lambda_i(2^{-j_n} + 2^{-\beta_{j_n} j_n})\big)$. Since by construction $\beta_{j_n} \geq 1$, we have $-\lambda_i(2^{-j_n} + 2^{-\beta_{j_n} j_n}) \geq -4$ for $i \in G_{j_n}$. It follows that $\sum_{i\in G_{j_n}} P'_{n,i} \geq e^{-4}\sum_{i\in G_{j_n}} \lambda_i 2^{-\beta_{j_n} j_n} \geq e^{-4}$ since $\lambda_i 2^{-\beta_{j_n} j_n} \geq 2^{-(\beta_{j_n}-1)j_n} = (\#G_{j_n})^{-1}$. Consequently, $\sum_{n\geq 1}\sum_{i\in G_{j_n}} P'_{n,i} \geq \sum_{n\geq 1} e^{-4}$.

$(ii)$ Consequence of $(i)$ and Theorem 2 of [18].

**Proof of Theorem 3.** Since for every choice of $\varphi$ the family $\widetilde{E}_{\delta,\varphi}$, $0 \leq \delta < \beta$, is non-increasing, it is enough to find a function $\varphi$ such that for every $\eta \in (0,T)$ and $\delta \in (0,\beta)$, with probability one, $(\eta, T) \subset \widetilde{E}_{\delta,\varphi}$.

We distinguish the cases $\beta = 1$ and $\beta > 1$.

**Case $\beta = 1$:** fix $0 < \delta < 1$. In fact, the following stronger result holds: with probability one, there exists $i_0 \geq 1$ such that for all $i \geq i_0$

$$(\eta, T) \subset \bigcup_{1\leq k\leq N_T^{(i)},\ \tau_k^{(i)}\geq 2\lambda_i^{-1}} [T_k^{(i)} - \lambda_i^{-\delta}, T_k^{(i)} + \lambda_i^{-\delta}].$$

To see this, denote by $\widetilde{M}_i$ the smallest integer larger than $\lambda_i^{1-\delta}/(2\log(\lambda_i))$, and notice, using Lemma 1 and its proof, that

$$\begin{aligned}
&\mathbb{P}(\exists\ 1 \leq k \leq N_T^{(i)} :\ \forall\ 0 \leq l \leq \widetilde{M}_i,\ \tau_{k+l}^{(i)} < 2\lambda_i^{-1}) \\
\leq\ & \mathbb{P}(N_T^{(i)} > M_i) + M_i\mathbb{P}(\forall\ 0 \leq l \leq \widetilde{M}_i,\ \tau_{1+l}^{(i)} < 2\lambda_i^{-1}) \\
\leq\ & (T\lambda_i)^{-7} + M_i(1-e^{-2})^{\widetilde{M}_i+1},
\end{aligned}$$

where $M_i = T\lambda_i + 4\sqrt{T\lambda_i\log(T\lambda_i)}$. The right hand side of the previous inequality is summable. By the Borel-Cantelli Lemma, this implies that with probability one, for $i$ large enough, the number of consecutive points of the form $T_k^{(i)}$ in $(0,T)$ such that $\tau_k^{(i)} < 2\lambda_i^{-1}$ is at most $\widetilde{M}_i + 1$. Moreover, due to Lemma 1 (applied with $K = 2$), the distance between two $T_k^{(i)}$'s is at most $2\log(\lambda_i)/\lambda_i$ for $i$ large enough. Consequently, for $i$ large enough, two consecutive intervals of the form $[T_k^{(i)} - \lambda_i^{-\delta}, T_k^{(i)} + \lambda_i^{-\delta}]$ with $\tau_k^{(i)} \geq 2\lambda_i^{-1}$ are overlapping. This yields the conclusion.

**Case $\beta > 1$:** Let $(j_n)_{n\geq 1}$ be an increasing sequence of integers such that $\beta = \lim_{n\to\infty}\beta_{j_n}$ and $G_{j_n} \neq \emptyset$ for all $n \geq 1$. Then choose a sequence $(\delta_{j_n})_{n\geq 1}$ such that $\beta_{j_n} \geq \delta_{j_n}$, $\lim_{n\to\infty}\beta_{j_n} - \delta_{j_n} = 0$ and $\log(j_n) = o\big((\beta_{j_n} - \delta_{j_n})j_n\big)$. Also define $\psi(n) = 2^{(\beta_{j_n}-\delta_{j_n})j_n} j_n^2$ and choose the function $\varphi$ to be

$$\varphi(t) = \begin{cases} \dfrac{2^{-j_n}}{\psi(n)} & \text{if } t = 2^{-j_n} \\ t & \text{otherwise.} \end{cases}$$

By construction $\lim_{t\to 0^+}\frac{\log\varphi(t)}{\log(t)} = 1$.

We prove the following result, which is stronger than the desired one because $\lim_{n\to\infty} \delta_{j_n} = \beta$: with probability one, for all $n_0$ large enough,

$$(\eta, \infty) \subset \bigcup_{n \geq n_0} \widehat{E}_{n,\delta,\varphi},$$

where

$$\widehat{E}_{n,\delta,\varphi} = \bigcup_{i \in G_{j_n}} \bigcup_{\substack{k \geq 1 \\ \tau_k^{(i)} > \varphi(2^{-j_n})}} (T_k^{(i)}, T_k^{(i)} + 2^{-j_n \delta_{j_n}}).$$

The method is inspired from Kahane's approach for Shepp's Theorem in [20].

Let $n_\eta > 0$ be such that $\eta > \varphi(2^{-j_n}) + 2^{-j_n \delta_{j_n}}$ for $n \geq n_\eta$. Then fix $n_0 \geq n_\eta$. For $N \geq n_0$ set

$$\tau_N = \inf \Big\{ t \geq \eta : \ t \notin \widehat{E}_N := \bigcup_{n=n_0}^{N} \widehat{E}_{n,\delta,\varphi} \Big\}.$$

The sequence $\tau_N$ is non-decreasing, and the conclusion will follow if we show that a.s. $\tau_N \to \infty$ as $N \to \infty$, or equivalently $\mathbb{E}(\exp(-\tau_N)) \to 0$ as $N \to \infty$. We shall prove that there exists $C > 0$ such that for $N$ large enough

$$\mathbb{E}(e^{-\tau_N}) \leq C 2^{(j_N+1)\delta_{j_N}} \exp(-\big(2^{j_N(\beta_{j_N} - \delta_{j_N})}(1 - 2^{-\delta_{j_N}})\big). \tag{3}$$

By our choice for $\delta_{j_N}$, the right hand side in (3) tends to 0 as $N \to \infty$ and the conclusion follows.

To establish (3), we look at the integral

$$I_N = \mathbb{E} \int_{[\eta,\infty)} e^{-t} \mathbf{1}_{\{t \notin \widehat{E}_N\}} \, dt.$$

in two ways. Fix $N \geq n_0$. First, in order to obtain (5) below we write

$$I_N = \int_{[\eta,\infty)} e^{-t} \mathbb{P}(t \notin \widehat{E}_N) \, dt$$

with

$$\begin{aligned} \mathbb{P}(t \notin \widehat{E}_N) &= \prod_{n=n_0}^{N} \prod_{i \in G_{j_n}} \Big(1 - \mathbb{P}(\exists\ k \geq 1 : \ \tau_k^{(i)} > \varphi(2^{-j_n}), \ t - 2^{-j_n \delta_{j_n}} < T_k^{(i)} < t)\Big) \\ &\leq \prod_{n=n_0}^{N} \prod_{i \in G_{j_n}} \big(1 - P_i(t)\big) \end{aligned} \tag{4}$$

where

$$P_i(t) = \mathbb{P}(\exists\ k \geq 1 : \ \tau_k^{(i)} > \varphi(2^{-j_n}), \tau_{k+1}^{(i)} > 2^{-j_n \delta_{j_n}}, \ t - 2^{-j_n \delta_{j_n}} < T_k^{(i)} < t).$$

Let us compute $P_i(t)$ for $i \in G_{j_n}$. Due to its definition, $P_i(t)$ simplifies to be

$$P_i(t) = \mathbb{P}(\tau^{(i)}_{k^{(i)}_t} > \varphi(2^{-j_n}),\ \tau^{(i)}_{k^{(i)}_t+1} > 2^{-j_n\delta_{j_n}},\ t - 2^{-j_n\delta_{j_n}} < T^{(i)}_t < t).$$

On the one hand, since $t \geq \varphi(2^{-j_n}) + 2^{-j_n\delta_{j_n}}$ for $n \geq n_\eta$ and $t \geq \eta$ we have

$$\begin{aligned}
P_1 &:= \mathbb{P}(k^{(i)}_t = 1,\ \tau^{(i)}_1 > \varphi(2^{-j}),\ \tau^{(i)}_2 > 2^{-j_n\delta_{j_n}},\ t - 2^{-j_n\delta_{j_n}} < \tau^{(i)}_1 < t) \\
&= e^{-\lambda_i 2^{-j_n\delta_{j_n}}} \int_{t-2^{-j_n\delta_{j_n}}}^{t} \lambda_i e^{-\lambda_i u}\, du = e^{-\lambda_i t} - e^{-\lambda_i(t+2^{-j_n\delta_{j_n}})}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
P_2 &:= \mathbb{P}\big(k^{(i)}_t \geq 2,\ \tau^{(i)}_{k^{(i)}_t} > \varphi(2^{-j}),\ \tau^{(i)}_{k^{(i)}_t+1} > 2^{-j_n\delta_{j_n}},\ t - 2^{-j_n\delta_{j_n}} < T^{(i)}_t < t) \\
&= \mathbb{P}\left(\begin{cases} k^{(i)}_t \geq 2,\ T^{(i)}_{k^{(i)}_t-1} < t - \varphi(2^{-j_n}), \\ \max\big(\varphi(2^{-j_n}), t - T^{(i)}_{k^{(i)}_t-1} - 2^{-j_n\delta_{j_n}}\big) < \tau^{(i)}_{k^{(i)}_t} \leq t - T^{(i)}_{k^{(i)}_t-1}, \\ \tau^{(i)}_{k^{(i)}_t+1} > 2^{-j_n\delta_{j_n}}. \end{cases}\right)
\end{aligned}$$

To evaluate that probability, we use the fact $\mathbb{P}\big(k^{(i)}_t \geq 2,\ T^{(i)}_{k^{(i)}_t-1} \in [u, u+du],\ \tau^{(i)}_{k^{(i)}_t} \in [v, v+dv],\ \tau^{(i)}_{k^{(i)}_t+1} > 2^{-j_n\delta_{j_n}}\big) = e^{-\lambda_i 2^{-j_n\delta_{j_n}}} \lambda_i^2 e^{-\lambda_i v}\, du dv, \quad (0 \leq u \leq t - \varphi(2^{-j_n}),\ \max\big(\varphi(2^{-j_n}), t - u - 2^{-j_n\delta_{j_n}}\big) < v \leq t - u)$. A computation yields

$$P_2 = \lambda_i 2^{-j_n\delta_{j_n}} \exp\big(-\lambda_i(\varphi(2^{-j_n}) + 2^{-j_n\delta_{j_n}})\big) - \exp(-\lambda_i t) + \exp\big(-\lambda_i(t + 2^{-j_n\delta_{j_n}})\big).$$

Adding $P_1$ and $P_2$ yields $P_i(t) = \lambda_i 2^{-j_n\delta_{j_n}} \exp\big(-\lambda_i(\varphi(2^{-j_n}) + 2^{-j_n\delta_{j_n}})\big)$ which does not depend on $t$. Write $P_i$ for $P_i(t)$. Equation (4) now yields

$$I_N \leq e^{-\eta} \prod_{n=n_0}^{N} \prod_{i \in G_{j_n}} (1 - P_i). \tag{5}$$

The integral $I_N$ can also be written

$$\begin{aligned}
I_N &= \mathbb{E}\left(e^{-\tau_N} \int_{[0,\infty)} e^{-s} \mathbf{1}_{\{\tau_N + s \notin \widehat{E}_N\}}\, ds\right) \\
&= \mathbb{E}\left(e^{-\tau_N} \mathbb{E}\left(\int_{[0,\infty)} e^{-s} \mathbf{1}_{\{\tau_N + s \notin \widehat{E}_N\}}\, ds | \tau_N\right)\right) \\
&= \mathbb{E}\left(e^{-\tau_N} \int_{[0,\infty)} e^{-s} \mathbb{P}(\tau_N + s \notin \widehat{E}_N\, |\tau_N)\, ds\right).
\end{aligned}$$

For every $s \geq 0$ and $t \geq \eta$, one has

$$\begin{aligned}
& \mathbb{P}(\tau_N + s \notin \widehat{E}_N | \tau_N = t) \\
= \ & \mathbb{P}\Big( \bigcap_{n=n_0}^{N} \bigcap_{i \in G_{j_n}} \{ \ \nexists k : \ \tau_k^{(i)} > \varphi(2^{-j_n}), \ T_k^{(i)} \in (\max(t, t+s-2^{-j_n\delta_{j_n}}), t+s)\Big\} \Big| \tau_N = t \Big) \\
\geq \ & \mathbb{P}\Big( \bigcap_{n=n_0}^{N} \bigcap_{i \in G_{j_n}} \{ \ \nexists k : \ T_k^{(i)} \in (\max(t, t+s-2^{-j_n\delta_{j_n}}), t+s)\} \Big| \tau_N = t \Big) \\
= \ & \prod_{n=n_0}^{N} \prod_{i \in G_{j_n}} \widetilde{P}_i(s,t)
\end{aligned}$$

where, for $i \in G_{j_n}$, $\widetilde{P}_i(s,t)$ is the probability that there is no integer $k \geq 1$ such that $T_k^{(i)} \in (\max(t, t+s-2^{-j_n\delta_{j_n}}), t+s)$. The last equality is due to the lack of memory of the exponential law as well as the independence between sources. The probability $\widetilde{P}_i(s,t)$ is well known to be equal to $e^{-\lambda_i |I|}$ where $|I|$ denotes the length of the interval $I = (\max(t, t+s-2^{-j_n\delta_{j_n}}), t+s)$. Since that length does not depend on $t$, we denote $\widetilde{P}_i(s,t)$ by $\widetilde{P}_i(s)$, which is given by

$$\widetilde{P}_i(s) = \begin{cases} e^{-\lambda_i s} & \text{if } 0 \leq s \leq 2^{-j_n\delta_{j_n}} \\ e^{-\lambda_i 2^{-j_n\delta_{j_n}}} & \text{otherwise.} \end{cases}$$

It turns out that

$$I_N \geq \mathbb{E}(e^{-\tau_N}) \int_{\mathbb{R}_+} e^{-s} \prod_{n=n_0}^{N} \prod_{i \in G_{j_n}} \widetilde{P}_i(s) \, ds.$$

Now we can use (5) to get

$$\begin{aligned}
e^{-\eta} \ & \geq \ \mathbb{E}(e^{-\tau_N}) \int_{\mathbb{R}_+} e^{-s} \prod_{n=n_0}^{N} \prod_{i \in G_{j_n}} \frac{\widetilde{P}_i(s)}{1 - P_i} \, ds \\
& = \ \mathbb{E}(e^{-\tau_N}) \int_{\mathbb{R}_+} e^{-s} \prod_{n=n_0}^{N} \Pi_{1,n}(s) \Pi_{2,n}(s) \, ds, \qquad (6)
\end{aligned}$$

where

$$\begin{cases} \Pi_{1,n}(s) = \begin{cases} \prod_{i \in G_{j_n}} \frac{\exp(-\lambda_i s)}{1 - \lambda_i 2^{-j_n\delta_{j_n}} \exp\left(-\lambda_i(\varphi(2^{-j_n}) + 2^{-j_n\delta_{j_n}})\right)} & \text{if } s \leq 2^{-j_n\delta_{j_n}} \\ 1 & \text{otherwise,} \end{cases} \\ \Pi_{2,n}(s) = \begin{cases} 1 & \text{if } s \leq 2^{-j_n\delta_{j_n}} \\ \prod_{i \in G_{j_n}} \frac{\exp\left(-\lambda_i 2^{-j_n\delta_{j_n}}\right)}{1 - \lambda_i 2^{-j_n\delta_{j_n}} \exp\left(-\lambda_i(\varphi(2^{-j_n}) + 2^{-j_n\delta_{j_n}})\right)} & \text{otherwise.} \end{cases} \end{cases}$$

We claim that $\prod_{n=n_0}^{N}\Pi_{2,n}(s)$ is bounded by below independently of $s$ by a positive constant. To prove this fact first we notice that for every $n\geq n_0$ and $i\in G_{j_n}$ one has $\frac{\exp\left(-\lambda_i 2^{-j_n\delta_{j_n}}\right)}{1-\lambda_i 2^{-j_n\delta_{j_n}}\exp\left(-\lambda_i(\varphi(2^{-j_n})+2^{-j_n\delta_{j_n}})\right)}\leq 1$ since $\frac{e^{-x}}{1-xe^{-x}}\leq 1$ on $\mathbb{R}_+$. Consequently, for every $n\geq n_0$

$$\Pi_{2,n}(s)\geq\prod_{i\in G_{j_n}}\frac{\exp\big(-\lambda_i 2^{-j_n\delta_{j_n}}\big)}{1-\lambda_i 2^{-j_n\delta_{j_n}}\exp\big(-\lambda_i(\varphi(2^{-j_n})+2^{-j_n\delta_{j_n}})\big)}.$$

Then, we use the fact that for $x\in(0,1)$ one has $e^{-x}\geq 1-x$ and $\frac{1}{1-x}\geq(1+x)$. This yields

$$\begin{aligned}&\frac{\exp\big(-\lambda_i 2^{-j_n\delta_{j_n}}\big)}{1-\lambda_i 2^{-j_n\delta_{j_n}}\exp\big(-\lambda_i(\varphi(2^{-j_n})+2^{-j_n\delta_{j_n}})\big)}\\ \geq\;&\big(1-\lambda_i 2^{-j_n\delta_{j_n}}\big)\Big(1+\lambda_i 2^{-j_n\delta_{j_n}}\exp\big(-\lambda_i(\varphi(2^{-j_n})+2^{-j_n\delta_{j_n}})\big)\Big).\end{aligned}\tag{7}$$

Moreover, the definition of $\varphi$ and the fact that $\lim_{n\to\infty}\delta_{j_n}=\beta>1$ together imply that $\lambda_i(\varphi(2^{-j_n})+2^{-j_n\delta_{j_n}})\to 0$ as $n\to\infty$. A computation using this fact shows that the logarithm of the right hand side of (7) is equivalent to $-\lambda_i^2 2^{-j_n(1+\delta_{j_n})}/\psi(n)$. It follows that $\log P_{2,n(s)}$ is larger than or equal to a term equivalent to $u_n:=-\sum_{i\in G_{j_n}}\lambda_i^2 2^{-j_n(1+\delta_{j_n})}/\psi(n)$. Moreover $u_n\geq -4\#G_{j_n}2^{2j_n}2^{-j_n(1+\delta_{j_n})}/\psi(n)=-4j_n^{-2}$ by construction. Since $\sum_{n\geq n_0}j_n^{-2}<\infty$ we have the conclusion.

So there exists $A>0$ such that

$$e^{-\eta}\geq A\,\mathbb{E}(e^{-\tau_N})\int_{\mathbb{R}_+}e^{-s}\prod_{n=n_0}^{N}\Pi_{1,n}(s)\,ds.$$

A last computation shows that there exists $B>0$ such that

$$\log\Pi_{1,n}(s)\geq Bu_n+\sum_{i\in G_{j_n}}\lambda_i(2^{-j_n\delta_{j_n}}-s)\geq Bu_n+2^{j_n\beta_{j_n}}(2^{-j_n\delta_{j_n}}-s),\quad(0\leq s\leq 2^{-j_n\delta_{j_n}}).$$

So there exists $A'>0$ such that

$$\begin{aligned}e^{-\eta}&\geq A'\,\mathbb{E}(e^{-\tau_N})\int_{\mathbb{R}_+}e^{-s}\prod_{\substack{n_0\leq n\leq N\\ s\leq 2^{-j_n\delta_{j_n}}}}\exp\big(2^{j_n\beta_{j_n}}(2^{-j_n\delta_{j_n}}-s)\big)\,ds\\ &\geq A'\,\mathbb{E}(e^{-\tau_N})\int_0^{2^{-j_N\delta_{j_N}}}e^{-s}\exp\big(2^{j_N\beta_{j_N}}(2^{-j_N\delta_{j_N}}-s)\big)\,ds\\ &\geq A'\,\mathbb{E}(e^{-\tau_N})\int_0^{2^{-(j_N+1)\delta_{j_N}}}e^{-s}\exp\big(2^{j_N(\beta_{j_N}-\delta_{j_N})}(1-2^{-\delta_{j_N}})\big)\,ds\\ &\geq A'\,\mathbb{E}(e^{-\tau_N})2^{-(j_N+1)\delta_{j_N}}\exp\big(-2^{-(j_N+1)\delta_{j_N}}\big)\exp\big(2^{j_N(\beta_{j_N}-\delta_{j_N})}(1-2^{-\delta_{j_N}})\big)\end{aligned}$$

and (3) follows since $\exp\big(-2^{-(j_N+1)\delta_{j_N}}\big)\to 1$ as $N\to\infty$.

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