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Brownian excursions, stochastic integrals, and representation of Wiener functionals

Jean Picard

Laboratoire de Mathématiques
CNRS (UMR 6620)
Université Blaise Pascal
63177 Aubière, France

Jean.Picard@math.univ-bpclermont.fr

Abstract

A stochastic calculus similar to Malliavin's calculus is worked out for Brownian excursions. The analogue of the Malliavin derivative in this calculus is not a differential operator, but its adjoint is (like the Skorohod integral) an extension of the Itô integral. As an application, we obtain an expression for the integrand in the stochastic integral representation of square integrable Wiener functionals; this expression is an alternative to the classical Clark-Ocone formula. Moreover, this calculus enables to construct stochastic integrals of predictable or anticipating processes (forward, backward and symmetric integrals are considered).

Key words: Brownian excursions, Malliavin calculus, Stochastic integrals, Stochastic integral representation, Anticipating calculus.

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1 Introduction

Consider the Wiener space $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$ of continuous paths $\omega : \mathbb{R}_+ \to \mathbb{R}^d$, with the Wiener measure $\mathbb{P}$, the standard Wiener process $W_t(\omega) = \omega(t)$, and the right continuous filtration $(\mathcal{F}_t)$ generated by W. The two main questions which are addressed in this article are:

- the representation of a variable F defined on Ω as a stochastic Itô integral with respect to W;
- the construction of stochastic integrals of predictable or anticipating processes with respect to W.

These two questions have been dealt with for a long time by means of basic stochastic calculus and of Malliavin's calculus; here, we want to describe an alternative approach based on Brownian excursions.

Consider first the representation problem. Let F be a square integrable random variable defined on Ω. It is well known that the martingale $\mathbb{E}[F|\mathcal{F}_t]$ can be written as

$$\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[F] + \int_0^t (Z_s, dW_s) \tag{1.1}$$

where Z_t is a $\mathbb{R}^d$-valued predictable process such that $\mathbb{E}\int |Z_t|^2 dt$ is finite; moreover $Z_t(\omega)$ is unique to a $\mathbb{P}(d\omega)\,dt$ negligible set. This formula has important theoretical consequences (this is the predictable representation property for the Wiener process), and it is also useful in some applications (mathematical finance is a recent example). Consequently, finding an expression of Z in terms of F is a natural question.

For instance, in the particular case $F = f(W_1)$, let

$$P_t f(x) = \mathbb{E}\big[f(x + W_t)\big]$$

denote the heat semigroup. Then the martingale $\mathbb{E}[F|\mathcal{F}_t]$, $0 \le t \le 1$, can be written as $P_{1-t} f(W_t)$, and an easy application of the Itô formula shows that for $t \le 1$,

$$Z_t = \nabla\big(P_{1-t} f\big)(W_t).$$

A classical extension of this particular case is based on the Malliavin calculus. If F is in a suitable functional space, one can consider its Malliavin derivative $D_t F$, and Z can be written by means of the Clark-Ocone formula ([7], Proposition 1.3.5 of [6])

$$Z_t = \mathbb{E}\big[D_t F \mid \mathcal{F}_t\big] \tag{1.2}$$

which expresses Z as the predictable projection of DF. However, in this formula, we have to assume that F is differentiable in the sense of Malliavin and this is a restriction (consider for instance variables related to the first exit time of an interval, or to some local time). In order to bypass this difficulty, we can write a similar formula for distributions on the Wiener space (in this case the derivative D_tF is considered in distribution sense, see [12]), but such a formula is not always very tractable; some other extensions have also been worked out in view of mathematical finance applications.

Our aim is to obtain another formula for Z which does not use any differentiation and which is valid in many cases. Notice that (1.2) follows from the fundamental formula of Malliavin's calculus (Definition 1.3.1 of [6])

$$\mathbb{E}\big[F\,\delta(\rho)\big] = \mathbb{E}\int \big(D_tF, \rho_t\big)dt.$$

This formula expresses the duality between the Malliavin derivative D and a divergence operator δ (the Skorohod integral); then (1.2) is a consequence of the fact that $\delta(\rho)$ coincides with the Itô integral for predictable processes ρ. Actually, any operator D satisfying the same property will provide a representation formula similar to (1.2). Here, we are going to replace D by an operator $\mathcal{D}$ which modifies the excursions of the Brownian path. It is not a differential operator; nevertheless, by working out a stochastic analysis of these excursions, we will prove a formula

$$\mathbb{E}\big[F\,\Phi(\rho)\big] = \mathbb{E}\int \big(\mathcal{D}_tF, \rho_t\big)dt \tag{1.3}$$

expressing the duality between $\mathcal{D}$ and an operator Φ which is again an extension of the Itô integration. Thus we will deduce that

$$Z_t = \mathbb{E}\big[\mathcal{D}_tF \bigm| \mathcal{F}_t\big]. \tag{1.4}$$

More explicitly, it will appear that the predictable projection in the right-hand side is an integral with respect to a measure which is strongly related to the Itô measure for Brownian excursions.

The operator $\Phi(\rho)$ coincides with Itô integrals for predictable processes ρ, so its study is related to our second question, namely the construction of stochastic integrals by means of the Brownian excursions. We will construct integrals with respect to a fixed component of the Wiener process, so we will write the Wiener process as $(W_t, V_t) \in \mathbb{R}\times\mathbb{R}^{d-1}$ and look for integrals with respect to the first component W. If ω is a one-dimensional continuous path with finite variation, the Lebesgue-Stieltjes integral with respect to ω can be obtained by counting the points of increase and decrease of ω (this will be explained in Section 3). If now ω is a Brownian path, it has no point of increase or decrease, but one can consider the excursions of ω above or below any level, and the beginning (respectively the end) of an excursion can be viewed as a point of increase or decrease on the right (respectively the left). This procedure provides generalised Lebesgue-Stieltjes measures; notice that we have not one measure, but four measures because we have the choice of considering beginnings or ends of excursions below or above any level. Predictable processes are not directly integrable with respect to these measures, but an approximation procedure leads to a construction of Itô, Stratonovich and backward integrals (depending on the choice of the measure). This approximation is equivalent to a regularisation of the Wiener path by means of a path with finite variation; in this sense, this procedure is similar to the approach of [11], though the approximation and the results are different. If now we integrate anticipating processes, we obtain forward, backward and symmetric integrals, and we can study the adjoint Φ of $\mathcal{D}$.

Technically, the study of these measures relies on classical formulas for excursions and more generally exit systems, see [2, 1]; it is related to a calculus which consists in appending and removing excursions to the Brownian path, and which is similar to appending and removing jumps to Poisson processes as in [8]; this calculus has features which are similar to the Malliavin calculus, but it is not a differential calculus.

Let us outline the contents of this article. The representation formula (1.4) of a variable F is stated in Section 2, and an elementary proof is given for simple functionals $F = f(W_{t_1}, \ldots, W_{t_n})$. The general case relies on the duality formula (1.3) between $\mathcal{D}$ and Φ, the proof of which is postponed; some examples are discussed.

In Section 3, the family of generalised Lebesgue-Stieltjes measures associated to a one-dimensional continuous path ω are introduced, and their stochastic analysis is worked out.

In Section 4, we deduce a construction of Itô, Stratonovich and backward integrals of predictable processes, and we complete the proof of the representation formula (1.4).

In Section 5, we construct the associated anticipating integrals, and explore the duality (1.3) between $\mathcal{D}$ and Φ.

We finish this introduction with a warning related to the general theory of processes. We will often consider predictable processes. Since we use the Wiener filtration, predictable and optional can be viewed as synonyms, but most of the results stated for predictable processes will not be true for processes which are only progressively measurable; this is usual in excursion theory and is due to the fact that the set of beginnings of excursions is progressively measurable but not predictable.

2 Representation of Wiener functionals

In this section, we first state in §2.1 the result, give some hints about the proof in §2.2 (it will be completed in Section 4), and describe some applications in §2.3.

2.1 The result

We first consider the one-dimensional case $d = 1$ (the extension to the multidimensional case is given at the end of the subsection). In order to introduce our result, we need some notation. If ω and θ are two paths indexed by $\mathbb{R}_+$, and if $t \geq 0$, we can consider the path

$\omega|t|\theta$ defined by

$$(\omega|t|\theta)(s) = \begin{cases} \omega(s) & \text{if } s < t, \\ \omega(t) + \theta(s-t) & \text{if } s \geq t. \end{cases}$$

If θ is a path, we can consider the hitting time

$$\zeta(\theta) = \inf\Big\{ s > 0;\ \theta(s) = 0 \Big\},$$

and define the path $\theta^\star$ by

$$\theta^\star(s) = \begin{cases} -\theta(s) & \text{if } s < \zeta(\theta), \\ \theta(s) & \text{if } s \geq \zeta(\theta). \end{cases} \tag{2.1.1}$$

The path $\theta^\star$ is the reflection of θ up to ζ.

Let $\mathcal{I}$ be the Itô measure of Brownian excursions (see Chapter XII of [10]); this is a σ-finite measure on Ω supported by

$$\Theta = \Big\{ \theta \in \Omega; \zeta(\theta) > 0; \theta(0) = 0; \theta(t) = 0 \text{ for } t \geq \zeta(\theta) \Big\}.$$

It can be characterised by the following property; under the restriction of $\mathcal{I}$ to $\{\zeta > t\}$ (which is a finite measure), conditionally on $\mathcal{F}_t$, the process $(\theta(t+s); s \geq 0)$ is a Wiener process stopped at 0, and its entrance measure is

$$\mathcal{I}\big\{\zeta(\theta) > t; \theta(t) \in dx\big\} = \frac{|x|}{\sqrt{2\pi t^3}} e^{-x^2/(2t)} dx. \tag{2.1.2}$$

Excursions are either positive or negative, and this leads to a decomposition $\Theta = \Theta^\uparrow \cup \Theta^\downarrow$ and $\mathcal{I} = \mathcal{I}^\uparrow + \mathcal{I}^\downarrow$. Notice that $\theta \mapsto \theta^\star$ transforms $\mathcal{I}^\uparrow$ and $\mathcal{I}^\downarrow$ into each other.

Under $\mathcal{I}$, the canonical process is stopped at ζ; it is useful to also consider the unstopped process, that is an Itô excursion followed by a Wiener path. More precisely we let $\mathcal{J}$ be the push forward of the measure $\mathcal{I}(d\theta)\,\mathbb{P}(d\omega)$ by the concatenation map

$$(\theta, \omega) \mapsto \theta|\omega.$$

It is a measure on the set of paths $\theta \in \Omega$ such that $\theta(0) = 0$ and $\zeta(\theta) > 0$. It has a characterisation similar to $\mathcal{I}$; on $\{\zeta > t\}$, conditionally on $\mathcal{F}_t$, the process $(\theta(t+s); s \geq 0)$ is a Wiener process, and

$$\mathcal{J}\big\{\zeta(\theta) > t; \theta(t) \in dx\big\} = \frac{|x|}{\sqrt{2\pi t^3}} e^{-x^2/(2t)} dx. \tag{2.1.3}$$

Notice that under $\mathcal{J}$, the map $\theta \mapsto \theta^\star$ consists of reflecting the first excursion without modifying the remaining of the path. Like $\mathcal{I}$, the measure $\mathcal{J}$ has a decomposition $\mathcal{J} = \mathcal{J}^\uparrow + \mathcal{J}^\downarrow$ involving the sign of the first excursion, and $\theta \mapsto \theta^\star$ transforms $\mathcal{J}^\uparrow$ and $\mathcal{J}^\downarrow$ into each other. We are now ready to state the main result.

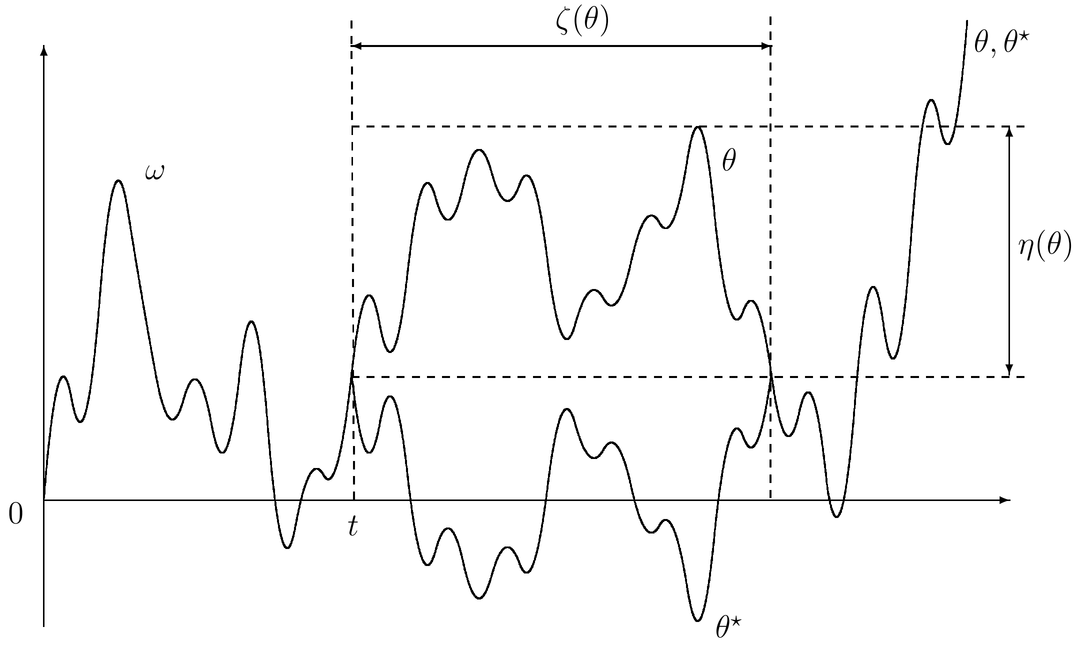

Figure 1: An example of paths $\omega|t|\theta$ and $\omega|t|\theta^\star$

Theorem 2.1.4. *On the one-dimensional Wiener space ($d = 1$), let F be a square integrable variable and define*

$$J_t F(\omega) = \int \big(F(\omega|t|\theta) - F(\omega|t|\theta^\star)\big)\mathcal{J}^\uparrow(d\theta) \tag{2.1.5}$$

on

$$\mathcal{A} = \Big\{(t,\omega);\ \int \big|F(\omega|t|\theta) - F(\omega|t|\theta^\star)\big|\mathcal{J}^\uparrow(d\theta) < \infty\Big\}.$$

Then the stochastic integral representation (1.1) *of* $\mathbb{E}[F|\mathcal{F}_t]$ *satisfies the relation*

$$Z_t(\omega) = J_t F(\omega) \tag{2.1.6}$$

almost everywhere on $\mathcal{A}$.

Remark 2.1.7. Variables F are often only defined almost surely. On the other hand, for t fixed, the law of $\omega|t|\theta$ under $\mathbb{P} \otimes \mathcal{J}^\uparrow$ is singular with respect to $\mathbb{P}$, so it seems that $J_t F$ is not well defined; however, it will appear that this law is absolutely continuous after a time integration, so $J_t F$ is actually well defined for almost any (t,ω) of $\mathcal{A}$.

Remark 2.1.8. A noteworthy and useful description of Brownian excursions is the Williams decomposition (Theorem XII.4.5 of [10]). Let

$$\eta(\theta) = \sup\Big\{|\theta(t)|;\ 0 \le t \le \zeta(\theta)\Big\}$$

be the height of the excursion. Then

$$\mathcal{I}^{\uparrow}(\eta \in da) = da \,/\, (2a^2), \tag{2.1.9}$$

and the law $\mathcal{I}^{\uparrow}$ conditioned on $\{\eta = a\}$ is described as follows; if $T_a(\theta)$ is the first hitting time of a, then $(\theta(t); 0 \le t \le T_a(\theta))$ and $(a - \theta(T_a(\theta) + t); 0 \le t \le \zeta(\theta) - T_a(\theta))$ are two independent Bessel processes of dimension 3, up to their first hitting time of a. One deduces a description of $\mathcal{J}^{\uparrow}(d\theta)$ by adding a Wiener path after $\zeta(\theta)$. This decomposition can be useful for the numerical simulation of Brownian excursions, and therefore for the computation of Z_t by means of a Monte-Carlo method.

Before entering the proof of the theorem, let us give its multidimensional extension. Similarly to $J_t F$ in (2.1.5), we define

$$\mathcal{J}_k^{\uparrow}(d\theta_1, \dots, d\theta_d) = \mathbb{P}(d\theta_1) \dots \mathbb{P}(d\theta_{k-1}) \mathcal{J}^{\uparrow}(d\theta_k) \mathbb{P}(d\theta_{k+1}) \dots \mathbb{P}(d\theta_d),$$

$$\theta^{\star}_{(k)} = (\theta_1, \dots, \theta_{k-1}, (\theta_k)^{\star}, \theta_{k+1}, \dots, \theta_d), \tag{2.1.10}$$

and

$$J_t^k F(\omega) = \int \big(F(\omega|t|\theta) - F(\omega|t|\theta^{\star}_{(k)}) \big) \mathcal{J}_k^{\uparrow}(d\theta) \tag{2.1.11}$$

on

$$\mathcal{A}^k = \Big\{ (t, \omega); \int \big| F(\omega|t|\theta) - F(\omega|t|\theta^{\star}_{(k)}) \big| \mathcal{J}_k^{\uparrow}(d\theta) < \infty \Big\}.$$

Theorem 2.1.12. *If F is a square integrable variable, then the k^{th} component of Z_t satisfies $Z_t^k = J_t^k F$ almost everywhere on $\mathcal{A}^k$.*

2.2 About the proof

We first prove Theorems 2.1.4 and 2.1.12 for a particular class of variables; the proof is indeed much simpler in this case, and is an application of the reflection principle for the Brownian motion. We then give some hints about the general case (the proof will be completed in Section 4).

Lemma 2.2.1. *Suppose that*

$$F = f(W_{t_1}, \dots, W_{t_n})$$

is a bounded simple functional. Then, with the notations of Theorem 2.1.4 (one-dimensional case), the set $\mathcal{A}$ has full measure and $Z_t(\omega) = J_t F(\omega)$ almost everywhere. Similarly, with the notations of Theorem 2.1.12 (multidimensional case), the set $\mathcal{A}^k$ has full measure and $Z_t^k(\omega) = J_t^k F(\omega)$ almost everywhere.

Proof in the one-dimensional case. Fix t, and notice that for $t_j < t < t_{j+1}$, one has $F(\omega|t|\theta) = F(\omega|t|\theta^{\star})$ on $\{\theta; \zeta(\theta) \le t_{j+1} - t\}$; the complement of this event has finite measure for $\mathcal{J}^{\uparrow}$. Thus $\mathcal{A}$ has full measure. Moreover, the martingale has the form

$$\mathbb{E}[F|\mathcal{F}_t] = \int F(\omega|t|\theta) \mathbb{P}(d\theta) = g(W_{t_1}, \dots, W_{t_j}, W_t)$$

for the function

$$g(w_1,\dots,w_j,x)=\mathbb{E}\big[f(w_1,\dots,w_j,x+W_{t_{j+1}}-W_t,\dots,x+W_{t_n}-W_t)\big]$$

which is smooth with respect to the last variable. If g' denotes the derivative with respect to this last variable, then

$$\begin{aligned}
Z_t &= g'(W_{t_1},\dots,W_{t_j},W_t) \qquad (2.2.2)\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\big(g(W_{t_1},\dots,W_{t_j},W_t+\varepsilon)-g(W_{t_1},\dots,W_{t_j},W_t-\varepsilon)\big)\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\mathbb{E}\Big[f(W_{t_1},\dots,W_{t_j},W_{t_{j+1}}+\varepsilon,\dots,W_{t_n}+\varepsilon)\\
&\qquad -f(W_{t_1},\dots,W_{t_j},W_{t_{j+1}}-\varepsilon,\dots,W_{t_n}-\varepsilon)\;\Big|\;\mathcal{F}_t\Big]\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int\big(F(\omega|t|\theta+\varepsilon)-F(\omega|t|\theta-\varepsilon)\big)\mathbb{P}(d\theta). \qquad (2.2.3)
\end{aligned}$$

For the last line, notice that the paths $\omega|t|\theta\pm\varepsilon$ are not continuous at time t and are therefore not in Ω, but $F(\omega|t|\theta\pm\varepsilon)$ can still be defined for simple functionals F. The integral involves the paths $\theta\pm\varepsilon$ separately, so we can use another coupling of these two processes. If we choose the coupling based on the reflection principle, we get, with the notation (2.1.1),

$$Z_t=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int\big(F(\omega|t|\theta+\varepsilon)-F(\omega|t|(\theta+\varepsilon)^\star)\big)\mathbb{P}(d\theta). \qquad (2.2.4)$$

Define $\delta=t_{j+1}-t$ and let

$$\zeta^\varepsilon(\theta)=\inf\Big\{t\geq 0;\ \theta(t)=-\varepsilon\Big\}$$

be the coupling time of $\theta+\varepsilon$ and its reflected path; the difference in the integral of (2.2.4) is 0 on $\{\zeta^\varepsilon(\theta)\leq\delta\}$, so

$$Z_t=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int\big(F(\omega|t|\theta+\varepsilon)-F(\omega|t|(\theta+\varepsilon)^\star)\big)1_{\{\zeta^\varepsilon(\theta)>\delta\}}\mathbb{P}(d\theta).$$

In order to compute this expression, we need the law of $(\theta(\delta+s)+\varepsilon; s\geq 0)$ under $1_{\{\zeta^\varepsilon(\theta)>\delta\}}\mathbb{P}(d\theta)/(2\varepsilon)$. This process has Brownian increments, and its initial measure is obtained from the reflection principle; more precisely

$$\frac{1}{2\varepsilon}\mathbb{P}\Big\{\theta;\zeta^\varepsilon(\theta)>\delta,\theta(\delta)+\varepsilon\in dx\Big\}=\frac{1_{\mathbb{R}_+}(x)}{2\varepsilon\sqrt{2\pi\delta}}\Big(e^{-(x-\varepsilon)^2/(2\delta)}-e^{-(x+\varepsilon)^2/(2\delta)}\Big)dx.$$

By studying the limit as $\varepsilon\downarrow 0$, it appears that this process converges in law (for the convergence on all bounded Borel functions) to the process with Brownian increments and initial measure

$$\frac{x}{\sqrt{2\pi\delta^3}}e^{-x^2/(2\delta)}1_{\mathbb{R}_+}(x)dx.$$

But (recall (2.1.3)) this is also the law of $(\theta(\delta+s); s \geq 0)$ under the measure $1_{\{\zeta(\theta)>\delta\}}\mathcal{J}^{\uparrow}(d\theta)$, so

$$Z_t = \int \big(F(\omega|t|\theta) - F(\omega|t|\theta^{\star})\big) 1_{\{\zeta(\theta)>\delta\}}\mathcal{J}^{\uparrow}(d\theta).$$

Since the indicator can be removed, we deduce that $Z_t = J_t F$. □

Proof in the multidimensional case. Proving that $\mathcal{A}^k$ has full measure is similar. On the other hand, Z_t^k is given by a formula similar to (2.2.2), but involving the derivative of $g(w_1, \ldots, w_j, x)$ with respect to the k^{th} component of x. Then (2.2.3) becomes

$$Z_t^k(\omega) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int \big(F(\omega|t|\theta + \varepsilon e_k) - F(\omega|t|\theta - \varepsilon e_k)\big)\mathbb{P}(d\theta)$$

where $\theta \pm \varepsilon e_k$ is the path θ, the k^{th} component of which is translated by $\pm\varepsilon$. One can replace $\theta - \varepsilon e_k$ by the reflected path $(\theta + \varepsilon e_k)^{\star}_{(k)}$, and the coupling time is now

$$\zeta^{\varepsilon}_{(k)} = \inf\Big\{t \geq 0;\ \theta_k(t) = -\varepsilon\Big\}.$$

Then one checks that the law of $(\theta(\delta+s)+\varepsilon e_k; s \geq 0)$ under the measure $1_{\{\zeta^{\varepsilon}_{(k)}>\delta\}}\mathbb{P}(d\theta)/(2\varepsilon)$ converges as $\varepsilon \downarrow 0$ to the law of $(\theta(\delta+s); s \geq 0)$ under $1_{\{\zeta(\theta_k)>\delta\}}\mathcal{J}_k^{\uparrow}(d\theta)$, and one concludes like previously. □

Let us now consider the case of a general square integrable variable F. We can approximate it in L^2 by simple functionals F_n, so Z_t is the limit of $J_t F_n$. However, it is not evident in the general case to prove the convergence of $J_t F_n$ to $J_t F$, so we work out another method. The main tool is the following duality formula which will be proved in Section 4 (see after Theorem 4.1.11).

Lemma 2.2.5. *Let ρ_t be a real-valued bounded predictable process. Suppose $1 \leq k \leq d$ and let F be a bounded $\mathcal{F}_1$-measurable variable such that*

$$\mathbb{E}\int_0^1 \int \big|\rho_t\big| \, \big|F(\omega|t|\theta) - F(\omega|t|\theta^{\star}_{(k)})\big| \mathcal{J}_k^{\uparrow}(d\theta) dt < \infty, \tag{2.2.6}$$

so that $\rho_t\, J_t^k F$ is well defined almost everywhere. Then

$$\mathbb{E}\int_0^1 \rho_t\, J_t^k F\, dt = \mathbb{E}\Big[F \int_0^1 \rho_t dW_t^k\Big]. \tag{2.2.7}$$

Taking for granted this lemma, let us explain how the representation formula is deduced.

Proof of Theorems 2.1.4 and 2.1.12. We prove Theorem 2.1.4 (the multidimensional case of Theorem 2.1.12 is similar). It is sufficient to consider a $\mathcal{F}_1$ measurable variable F and to prove (2.1.6) for $0 \le t \le 1$. Let us first assume that F is bounded and put

$$Z'_t = \int \big|F(\omega|t|\theta) - F(\omega|t|\theta^\star)\big| \mathcal{J}^\uparrow(d\theta).$$

This is a predictable process. For any bounded predictable process ρ, we can apply Lemma 2.2.5 to $\rho 1_{\{Z' \le \mu\}}$ for $\mu > 0$, and deduce

$$\mathbb{E}\int_0^1 \rho_t Z_t 1_{\{Z'_t \le \mu\}} dt = \mathbb{E}\Big[F \int_0^1 \rho_t 1_{\{Z'_t \le \mu\}} dW_t\Big] = \mathbb{E}\int_0^1 \rho_t J_t F 1_{\{Z'_t \le \mu\}} dt.$$

Thus $Z_t = J_t F$ holds on $\{Z'_t \le \mu\}$ and therefore on $\mathcal{A}$ by letting μ tend to infinity. If F is not bounded but only square integrable, we approach it by

$$F_M = \max(\min(F, M), -M).$$

Let Z_t^M be the integrand involved in (1.1) for F_M. Then $J_t F_M$ converges to $J_t F$ on $\mathcal{A}$, and Z_t^M converges to Z_t in L^2, so we can take the limit in the formula for F_M and conclude. □

2.3 Examples

In the examples, we check that $\mathcal{A}$ has a negligible complement; this means

$$\int \big|F(\omega|t|\theta) - F(\omega|t|\theta^\star)\big| \mathcal{J}^\uparrow(d\theta) < \infty \tag{2.3.1}$$

for almost any (t, ω), or a similar relation with $\mathcal{J}_k^\uparrow$ in the multidimensional case. Then the representation formula (1.1) holds with $Z_t = J_t F$ almost everywhere. If θ is a path under $\mathcal{J}$, then $\zeta(\theta)$ and $\eta(\theta)$ will denote the length and height of the first excursion. Recall that $\mathcal{J}^\uparrow$ is the push forward of $\mathcal{I}^\uparrow \otimes \mathbb{P}$ by the concatenation map $(\theta, \omega) \mapsto \theta|\omega$; the law of $\eta(\theta)$ under $\mathcal{I}^\uparrow$ or $\mathcal{J}^\uparrow$ has already been given in (2.1.9), and the law of $\zeta(\theta)$ is deduced from (2.1.2) and is given by

$$\mathcal{I}^\uparrow(\zeta \in dt) = dt \,\big/\, \big(2\sqrt{2\pi t^3}\big) \tag{2.3.2}$$

on $(0, \infty)$. In particular, $\zeta^\gamma \wedge 1$ is integrable for $\gamma > 1/2$. The first examples are relevant to the one-dimensional case $d = 1$.

Example 2.3.3. Let $F = f(T_x)$, $f(L_1^x)$ or $f(g_1^x)$ be a bounded function of the first hitting time

$$T_x = \inf\Big\{t \ge 0;\ W_t = x\Big\}$$

of x, of the local time

$$L_t^x = \big|W_t - x\big| - |x| - \int_0^t \operatorname{sgn}(W_s - x) dW_s \tag{2.3.4}$$

at time $t = 1$ and level x, or of the last hitting time

$$g_1^x = \sup\{t \in [0,1];\ W_t = x\} \vee 0.$$

Then $F(\omega|t|\theta)$ and $F(\omega|t|\theta^\star)$ are different only when the height $\eta(\theta)$ of the excursion exceeds $|x - \omega(t)|$, so, from (2.1.9),

$$\mathcal{J}^\uparrow\Big\{\theta; F(\omega|t|\theta) \neq F(\omega|t|\theta^\star)\Big\} \leq \frac{1}{2|x-\omega(t)|} < \infty$$

almost everywhere; since we have assumed that F is bounded, we deduce (2.3.1).

Example 2.3.5. Let $F = T_{W_1}$ be the first time t at which W_t hits W_1. Then

$$\Big\{F(\omega|t|\theta) \neq F(\omega|t|\theta^\star)\Big\}$$
$$\subset \Big\{\eta(\theta) \geq \big(\sup_{[0,t]} \omega - \omega(t)\big) \wedge \big(\omega(t) - \inf_{[0,t]} \omega\big)\Big\} \cup \Big\{\zeta(\theta) \geq 1-t\Big\}$$

because on the complement of the right hand side, one has

$$W_1(\omega|t|\theta) = W_1(\omega|t|\theta^\star) \qquad \text{and} \qquad T_x(\omega|t|\theta) = T_x(\omega|t|\theta^\star)$$

for any x. This set has again finite measure for $\mathcal{J}^\uparrow$, so we can conclude as in Example 2.3.3. The same argument can be applied for instance to the first hitting time of $W_1/2$, or to the time of $[0,1]$ at which W is minimal.

Example 2.3.6. Consider $F = L_1^0$; this is similar to Example 2.3.3 but F is here unbounded. Put $\delta = 1-t$. Then

$$F(\omega|t|\theta) = L_t^0 + \ell_\delta^{-\omega(t)}$$

where ℓ denotes the local time of θ (it can be defined like L from Tanaka's formula (2.3.4)), and

$$F(\omega|t|\theta^\star) = L_t^0 + \ell_{\zeta(\theta)\wedge\delta}^{\omega(t)} + \ell_\delta^{-\omega(t)} - \ell_{\zeta(\theta)\wedge\delta}^{-\omega(t)},$$

so

$$\big|F(\omega|t|\theta) - F(\omega|t|\theta^\star)\big| = \ell_{\zeta(\theta)\wedge\delta}^{|\omega(t)|}$$

(at most one of two local times $\ell_{\zeta(\theta)\wedge\delta}^{\pm\omega(t)}$ is non zero). The right-hand side is zero except on $\{\eta(\theta) \geq |\omega(t)|\}$. Conditionally on this event, after hitting $|\omega(t)|$, the process θ is a Brownian motion, so

$$\frac{1}{\mathcal{J}^\uparrow\{\theta; \eta(\theta) \geq |\omega(t)|\}} \int \ell_{\zeta(\theta)\wedge\delta}^{|\omega(t)|}(\theta)\, 1_{\{\eta(\theta)\geq|\omega(t)|\}} \mathcal{J}^\uparrow(d\theta) \leq C\sqrt{\delta}$$

from Tanaka's formula. Thus

$$\int \big|F(\omega|t|\theta) - F(\omega|t|\theta^\star)\big| \mathcal{J}^\uparrow(d\theta) \leq C\sqrt{\delta}\, \mathcal{J}^\uparrow\{\theta; \eta(\theta) \geq |\omega(t)|\} = \frac{C\sqrt{\delta}}{2|\omega(t)|} < \infty$$

almost everywhere, so (2.3.1) again holds.

Notice that in this example the martingale $\mathbb{E}[F|\mathcal{F}_t]$ takes the form $L_t^0 + f_t(W_t)$ and $Z_t = f_t'(W_t)$. The next example is devoted to a non Markovian example.

Example 2.3.7. Consider the local time $F = L_1^{W_1}$ at level W_1. Then (2.3.1) again holds.

Proof. As in previous example, put $\delta = 1 - t$ and let ℓ denote the local time of θ. The difference $F(\omega|t|\theta) - F(\omega|t|\theta^\star)$ again involves ℓ but at two random levels; moreover, if $\zeta(\theta) > \delta$, one has $W_1(\omega|t|\theta) \neq W_1(\omega|t|\theta^\star)$ and it also involves the local time L of ω at two random levels. Thus we obtain the estimate

$$\left|F(\omega|t|\theta) - F(\omega|t|\theta^\star)\right| \leq 2\sup_x \ell^x_{\zeta\wedge\delta} + 2\sup_x L^x_t\, 1_{\{\zeta(\theta)>\delta\}}.$$

We also notice that the difference is zero if $|\theta(\delta)| > \eta(\theta)$, so

$$\begin{aligned}\left|F(\omega|t|\theta) - F(\omega|t|\theta^\star)\right| \leq & 2\sup_x \ell^x_{\zeta\wedge\delta}\, 1_{\{|\theta(\delta)|\leq\eta(\theta)\}} 1_{\{\zeta(\theta)\leq\delta/2\}} \\ & + 2\sup_x \ell^x_{\zeta\wedge\delta}\, 1_{\{\zeta(\theta)>\delta/2\}} + 2\sup_x L^x_t\, 1_{\{\zeta(\theta)>\delta\}}.\end{aligned}$$

An easy Gaussian estimate shows that

$$\mathcal{J}^\uparrow\Big[|\theta(\delta)| \leq \eta(\theta) \;\Big|\; \theta(s); 0 \leq s \leq \zeta(\theta)\Big] \leq \frac{2\eta(\theta)}{\sqrt{2\pi(\delta - \zeta(\theta))}} \leq \frac{2\eta(\theta)}{\sqrt{\pi\delta}}$$

on $\{\zeta(\theta) \leq \delta/2\}$. Thus (with a constant C depending on δ),

$$\begin{aligned}&\int \left|F(\omega|t|\theta) - F(\omega|t|\theta^\star)\right|\mathcal{J}^\uparrow(d\theta) \\ &\leq C\int \big(\sup_x \ell^x_{\zeta\wedge\delta}(\theta)\big)\big(\eta(\theta)1_{\{\zeta(\theta)\leq\delta/2\}} + 1_{\{\zeta(\theta)>\delta/2\}}\big)\mathcal{I}^\uparrow(d\theta) + C\sup_x L^x_t \\ &\leq C\left(\int \big(\sup_x \ell^x_{\zeta\wedge\delta}(\theta)\big)^2\mathcal{I}^\uparrow(d\theta)\right)^{1/2} \\ &\qquad \left(\int \big(\eta(\theta)1_{\{\zeta(\theta)\leq\delta/2\}} + 1_{\{\zeta(\theta)>\delta/2\}}\big)^2\mathcal{I}^\uparrow(d\theta)\right)^{1/2} + C\sup_x L^x_t.\end{aligned}$$

For the second term of the product, we know that $\mathcal{I}^\uparrow\{\zeta > \delta/2\}$ is finite, and we have

$$\begin{aligned}\int \eta(\theta)^2 1_{\{\zeta(\theta)\leq\delta/2\}}\mathcal{I}^\uparrow(d\theta) &\leq \int \Big(\sup_{[0,\zeta\wedge\delta/2]} \theta^2\Big)\mathcal{I}^\uparrow(d\theta) \\ &\leq C\int \big(\zeta(\theta)\wedge\delta/2\big)\mathcal{I}^\uparrow(d\theta) < \infty\end{aligned}$$

by applying Doob's inequality. For the first term of the product,

$$\int \big(\sup_x \ell^x_{\zeta\wedge\delta}(\theta)\big)^2\mathcal{I}^\uparrow(d\theta) \leq C\int \big(\zeta(\theta)\wedge\delta\big)\mathcal{I}^\uparrow(d\theta) < \infty$$

by applying maximal inequalities on local times (Theorem XI.2.4 of [10]). This completes the proof of (2.3.1). $\square$

Example 2.3.8. In the multidimensional setting, consider the first exit time

$$\tau = \inf\Big\{t \geq 0;\ W_t \notin D\Big\}$$

of some domain $D \ni 0$, and let $F = f(W_\tau)$ for f a Borel bounded function (we do not need any smoothness assumption on f nor on the boundary of D). Then $F(\omega|t|\theta) = F(\omega|t|\theta^\star_{(k)})$ as soon as the height $\eta(\theta)$ of the excursion is smaller than the distance between $\omega(t)$ and the complement of D, so (2.3.1) can be proved on $\{t < \tau\}$ as in Example 2.3.3. In this case, $\mathbb{E}[F|\mathcal{F}_t] = h(W_t)$ is given on $\{t < \tau\}$ by the solution h of the Dirichlet problem with boundary condition f, and $Z_t = \nabla h(W_t)$, so we obtain a stochastic representation $Z_t = J_t F$ for the gradient of h. Notice that this representation can be implemented by means of a simple Monte-Carlo method, whereas the classical Clark-Ocone formula is not directly applicable in this case; another classical stochastic interpretation of harmonic functions in Malliavin's calculus is to view them in duality with densities of some excessive measures (see for instance [9]), but again such an interpretation is not easy to use from a practical point of view. We can also consider $F = f(\tau, W_\tau)$.

Example 2.3.9. We now check that our result can also be applied to solutions of stochastic differential equations with Lipschitz coefficients. Let $F = f(X_1)$, where X_t is the solution of

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0,$$

with b and σ Lipschitz and bounded (we write the proof for real processes but the multidimensional case is similar). Then (2.3.1) holds true almost everywhere as soon as f satisfies

$$\big|f(y) - f(x)\big| \leq C\,|y - x|^\alpha$$

for some rate $\alpha > 1/2$.

Proof. For ω fixed, we have to compare the solutions of the equation starting at $x = X_t(\omega)$ and driven respectively by the semimartingales θ and $\theta^\star$. More precisely, put $\delta = 1 - t$, let $\xi_s = \xi_s(\theta)$ be the solution of

$$d\xi_s = b(\xi_s)ds + \sigma(\xi_s)d\theta_s, \quad \xi_0 = X_t(\omega),$$

and similarly for $\xi^\star_s = \xi_s(\theta^\star)$; we have

$$\begin{aligned}\int \big|F(\omega|t|\theta) - F(\omega|t|\theta^\star)\big|\mathcal{J}^\uparrow(d\theta) &\leq C \int \big|\xi_\delta - \xi^\star_\delta\big|^\alpha d\mathcal{J}^\uparrow \\ &\leq C \int \big|\xi_{\zeta\wedge\delta} - \xi^\star_{\zeta\wedge\delta}\big|^\alpha d\mathcal{I}^\uparrow,\end{aligned}$$

where the last line is obtained by conditioning on the excursion $(\theta(s); 0 \leq s \leq \zeta(\theta))$ and by applying classical estimates on stochastic differential equations with Lipschitz coefficients. We want to prove that this term is finite. We have

$$\begin{aligned}\xi_{\zeta\wedge\delta} =& \xi_0 + b(\xi_0)\big(\zeta \wedge \delta\big) + \sigma(\xi_0)\theta_{\zeta\wedge\delta} \\ &+ \int_0^{\zeta\wedge\delta} (b(\xi_s) - b(\xi_0))ds + \int_0^{\zeta\wedge\delta} \big(\sigma(\xi_s) - \sigma(\xi_0)\big)d\theta_s\end{aligned}$$

and similarly for $\xi^\star$, so

$$\begin{aligned}\xi_{\zeta\wedge\delta}-\xi^\star_{\zeta\wedge\delta}=&2\sigma(\xi_0)\theta_{\zeta\wedge\delta}\\&+\int_0^{\zeta\wedge\delta}(b(\xi_s)-b(\xi^\star_s))ds+\int_0^{\zeta\wedge\delta}\big(\sigma(\xi_s)-\sigma(\xi^\star_s)\big)d\theta_s.\end{aligned}$$

For the first term, we have from (2.1.2)

$$\int\theta^\alpha_{\zeta\wedge\delta}\mathcal{I}^\uparrow(d\theta)=\int_{\{\zeta(\theta)>\delta\}}\theta^\alpha_\delta\mathcal{I}^\uparrow(d\theta)=\frac{1}{\sqrt{2\pi\delta^3}}\int_0^\infty x^{1+\alpha}e^{-x^2/(2\delta)}dx<\infty,$$

and for the others, it is sufficient (from Burkholder's inequalities) to prove that

$$I:=\int\Big(\int_0^{\zeta\wedge\delta}(\xi_s-\xi_0)^2ds\Big)^{\alpha/2}\mathcal{I}^\uparrow(d\theta)<\infty.$$

We have

$$\begin{aligned}\int_0^{\zeta\wedge\delta}(\xi_s-\xi_0)^2ds&\le\Big(\int_0^{\zeta\wedge\delta}s^pds\Big)^{1/p}\Big(\int_0^{\zeta\wedge\delta}\frac{|\xi_s-\xi_0|^{2q}}{s^q}ds\Big)^{1/q}\\&=C_q\big(\zeta\wedge\delta\big)^{2-1/q}\Big(\int_0^{\zeta\wedge\delta}\frac{|\xi_s-\xi_0|^{2q}}{s^q}ds\Big)^{1/q}\end{aligned}$$

for $1/p+1/q=1$. Thus

$$\begin{aligned}I&\le C_q^{\alpha/2}\int\big(\zeta\wedge\delta\big)^{\alpha(2q-1)/(2q)}\Big(\int_0^{\zeta\wedge\delta}\frac{|\xi_s-\xi_0|^{2q}}{s^q}ds\Big)^{\alpha/(2q)}\mathcal{I}^\uparrow(d\theta)\\&\le C_q^{\alpha/2}\Big(\int\big(\zeta\wedge\delta\big)^{\alpha(2q-1)/(2q-\alpha)}\mathcal{I}^\uparrow(d\theta)\Big)^{1-\alpha/(2q)}\\&\quad\Big(\iint_0^{\zeta\wedge\delta}\frac{|\xi_s-\xi_0|^{2q}}{s^q}ds\mathcal{I}^\uparrow(d\theta)\Big)^{\alpha/(2q)}\end{aligned}$$

for $q>\alpha/2\vee 1$. For the first term, we notice that the exponent of $\zeta\wedge\delta$ is greater than $1/2$ if q is chosen large enough (because $\alpha>1/2$), so this term is finite from (2.3.2). On the other hand, from classical estimates on Itô processes and by applying again (2.3.2),

$$\begin{aligned}\int(\xi_{s\wedge\zeta}-\xi_0)^{2q}\mathcal{I}^\uparrow(d\theta)&\le C'_q\int(s\wedge\zeta)^q\mathcal{I}^\uparrow(d\theta)\\&=C'_q\Big(\int_0^s\frac{t^{q-3/2}}{2\sqrt{2\pi}}dt+s^q\int_s^\infty\frac{t^{-3/2}}{2\sqrt{2\pi}}dt\Big)\\&=C''_q\,s^{q-1/2},\end{aligned}$$

so

$$\iint_0^{\zeta\wedge\delta}\frac{|\xi_s-\xi_0|^{2q}}{s^q}ds\mathcal{I}^\uparrow(d\theta)\le C''_q\int_0^\delta s^{-1/2}ds<\infty$$

We can conclude that I is finite. $\square$

3 Generalised Stieltjes measures

In this section, considering a real-valued function, we construct in §3.1 a family of measures which generalise the classical Lebesgue-Stieltjes measure, and we show in §3.2 and §3.3 how the calculus for Brownian excursions can be applied to them. This will be the main tool for the construction of stochastic integrals in next section, and in particular for the proof of Lemma 2.2.5. Throughout this section, except in §3.4, we suppose that we are in the one-dimensional case ($d = 1$).

3.1 A family of measures

If ω is a real-valued continuous increasing path, then

$$\int f(t) d\omega(t) = \int f(\omega^{-1}(x)) dx.$$

More generally, if ω is continuous and piecewise monotone, let $S_\omega(t)$ be $+1$ (respectively -1) on intervals where ω is increasing (respectively decreasing). Then the Lebesgue-Stieltjes integral with respect to ω can be written as

$$\int f(t) d\omega(t) = \int \sum_{t \in \omega^{-1}(x)} S_\omega(t) f(t) dx.$$

If now ω is a general continuous path (such as a typical Brownian path), this formula has no clear sense since the sum can be uncountable and the path may have no increase or decrease time (so S_ω is not well defined). Instead of considering all the times t in $\omega^{-1}(x)$, we will only consider times t such that ω is decreasing or increasing on the left or on the right of t. In other words, we will be concerned with beginnings or ends of excursions above or below any level x, and S_ω will be linked to the sign of these excursions. An end of excursion is obtained as a first hitting time

$$d_t^x = \inf\Big\{ s \geq t; \omega(s) = x \Big\}$$

(in particular $T_x = d_0^x$), and a beginning of excursion is obtained as a last hitting time

$$g_t^x = \sup\Big\{ s \leq t; \omega(s) = x \Big\}. \tag{3.1.1}$$

More precisely, consider

$$\zeta_t(\omega) = \inf\Big\{ s > 0; \omega(t+s) = \omega(t) \Big\}$$

which is almost surely finite for a Brownian path. We say that an excursion begins at time t if $\zeta_t(\omega) > 0$. In this case, we define the excursion $\Upsilon_t \omega \in \Theta$ beginning at t by

$$\Upsilon_t \omega : s \mapsto (\Upsilon_t \omega)(s) = \begin{cases} \omega(t+s) - \omega(s) & \text{if } s < \zeta_t(\omega), \\ 0 & \text{if } s \geq \zeta_t(\omega). \end{cases} \tag{3.1.2}$$

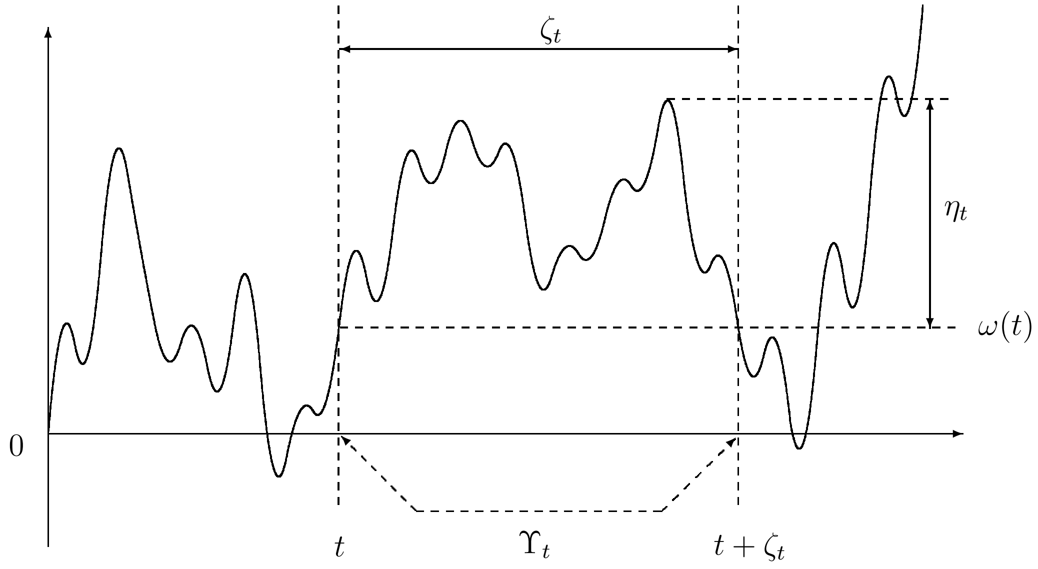

Figure 2: An excursion Υ_t

The length and the height of this excursion are ζ_t and $\eta_t = \eta(\Upsilon_t)$ (see Figure 2). For other values of t, we let $\Upsilon_t = 0$ be the null excursion. We obtain a path with values in $\Theta \cup \{0\}$; we will write $\Upsilon_t > 0$ or $\Upsilon_t < 0$ depending on the sign of the excursion. Then we let $\omega_x^{\nearrow}(dt)$ be the counting measure for the set of beginnings of excursions of the path ω above level x, that is

$$\omega_x^{\nearrow}(A) = \sum_{t:\omega(t)=x,\Upsilon_t(\omega)>0} 1_A(t)$$

for $A \subset \mathbb{R}_+$, and we define

$$\omega^{\nearrow}(A) = \int \omega_x^{\nearrow}(A)dx. \tag{3.1.3}$$

Lemma 3.1.4. *Let ω be a fixed real-valued continuous path. Then $\omega^{\nearrow}$ is a σ-finite measure on the set of beginnings of positive excursions of ω. Moreover, if $\Delta = (t_j)$ is a subdivision of $[0,1]$ and if $\omega_\Delta^{\nearrow}$ is the restriction of $\omega^{\nearrow}$ to*

$$B_\Delta(\omega) = \bigcup_j \Big\{ t;\ \Upsilon_t(\omega) > 0,\ t \le t_j \le t + \zeta_t(\omega) \Big\},$$

then

$$\omega_\Delta^{\nearrow}([r,s]) = \inf_{[s,t_{j+1}]} \omega - \inf_{[r,t_{j+1}]} \omega \tag{3.1.5}$$

for $t_j \le r \le s \le t_{j+1}$. The measure $\omega^{\nearrow}$ can be obtained on $[0,1]$ as the increasing limit of measures $\omega_{\Delta_n}^{\nearrow}$ for any refining sequence of subdivisions such that $|\Delta_n| = \max(t_{j+1}^n - t_j^n) \downarrow 0$. If ω has finite variation, then $\omega^{\nearrow}(dt)$ is the positive part $\omega^+(dt)$ of the Lebesgue-Stieltjes measure associated to ω.

Proof. We first verify the description (3.1.5) of the measure $\omega_\Delta^\nearrow$. The set $B_\Delta \cap [t_j, t_{j+1}]$ is the set of beginnings of positive excursions beginning in $[t_j, t_{j+1}]$ and straddling time t_{j+1}, so that (with notation (3.1.1))

$$B_\Delta \cap [t_j, t_{j+1}] = \Big\{ g_{t_{j+1}}^x;\ \inf_{[t_j, t_{j+1}]} \omega \le x \le \omega(t_{j+1}) \Big\}.$$

Thus, for $t_j \le r \le s \le t_{j+1}$,

$$\omega_\Delta^\nearrow([r,s]) = \int 1_{[r,s]}(g_{t_{j+1}}^x) dx = \inf_{[s,t_{j+1}]} \omega - \inf_{[r,t_{j+1}]} \omega$$

and (3.1.5) is proved. It is clear that $\omega^\nearrow$ is the limit of $\omega_{\Delta_n}^\nearrow$, and since $\omega_{\Delta_n}^\nearrow$ is finite, this implies that $\omega^\nearrow$ is σ-finite. Let us now consider the case of a path ω with finite variation on $[0,1]$. We deduce from (3.1.5) that

$$\omega_\Delta^\nearrow([r,s]) \le \omega(s) - \inf_{[r,s]} \omega \le \omega^+([r,s])$$

for $t_j \le r \le s \le t_{j+1}$, so at the limit, $\omega^\nearrow(dt) \le \omega^+(dt)$. On the other hand,

$$\omega_\Delta^\nearrow([t_j, t_{j+1}]) = \sup_{t_j \le s \le t_{j+1}} \big(\omega(t_{j+1}) - \omega(s)\big) \ge \big(\omega(t_{j+1}) - \omega(t_j)\big)^+,$$

so if Δ_n is the sequence of dyadic subdivisions (t_j^n) of $[0,1]$, we have

$$\omega_{\Delta_n}^\nearrow([0,1]) \ge \sum_j \big(\omega(t_{j+1}^n) - \omega(t_j^n)\big)^+.$$

At the limit, we obtain that $\omega^\nearrow([0,1]) \ge \omega^+([0,1])$; since $\omega^\nearrow \le \omega^+$, the proof of $\omega^\nearrow = \omega^+$ on $[0,1]$ is complete. $\square$

Remark 3.1.6. The integral with respect to $\omega_\Delta^\nearrow(dt)$ can be written as

$$\int_0^1 f(t) \omega_\Delta^\nearrow(dt) = \sum_j \int_{t_j}^{t_{j+1}} f(t) d\Big(\inf_{[s,t_{j+1}]} \omega\Big) = \sum_j \int_{\inf_{[t_j,t_{j+1}]} \omega}^{\omega(t_{j+1})} f(g_{t_{j+1}}^x) dx. \qquad (3.1.7)$$

In the definition (3.1.3), the measure $\omega^\nearrow(dt)$ is decomposed according to the level $x = \omega(t)$; we now see that it can also be decomposed according to the height $a = \eta_t$ of the excursion starting at time t; this will be useful in Section 4 (see the proof of Theorem 4.4.13).

Lemma 3.1.8. *The measure* $\omega^\nearrow$ *of* (3.1.3) *can be written as*

$$\omega^\nearrow(A) = \int \sum_{t : \Upsilon_t(\omega) > 0, \eta_t = a} 1_A(t)\, da. \qquad (3.1.9)$$

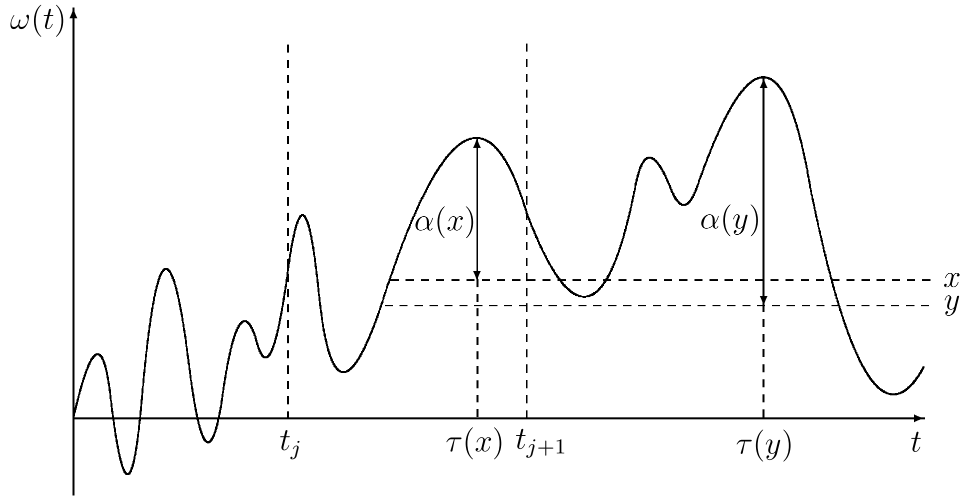

Figure 3: The function α (Proof of Lemma 3.1.8)

Proof. Following the notation of Lemma 3.1.4, it is sufficient to study the measure $\omega_\Delta^{\nearrow}$ on $[t_j, t_{j+1}]$. Consider the restriction of ω:

$$\omega_j : B_\Delta \cap [t_j, t_{j+1}] \to \mathbb{R} \qquad t \mapsto \omega(t).$$

This map is injective, its image is the interval $I = [\inf_{[t_j,t_{j+1}]} \omega, \omega(t_{j+1})]$, and the pull back $\omega_j^{-1}(dx)$ of the Lebesgue measure on I is the measure $\omega_\Delta^{\nearrow}$. On the other hand, for each level $x \in I$, let us consider the height $\alpha(x) = \eta_{g^x_{t_{j+1}}}$ of the excursion starting at $g^x_{t_{j+1}}$ (see Figure 3). Then

$$\alpha(x) = \omega(\tau(x)) - x$$

where $\tau(x)$ is the time at which this excursion is maximal. Actually, $x \mapsto \tau(x)$ is piecewise constant; it changes its value when the end $d^x_{t_{j+1}}$ of the excursion is a local minimum of ω, and the excursion starting at this minimum has a height greater than $\alpha(x)$; in this case one has $\alpha(x-) > \alpha(x)$. Thus $x \mapsto \alpha(x)$ is decreasing, and is the sum of an affine function with slope -1 and of jumps. We deduce that the pull back $\alpha^{-1}(da)$ of the Lebesgue measure on $\alpha(I)$ is the Lebesgue measure on I; by composition, the pull back $(\alpha \circ \omega)^{-1}(da)$ of the Lebesgue measure on $\alpha(I)$ is $\omega_\Delta^{\nearrow}$. This proves (3.1.9) when A is included in $B_\Delta \cap [t_j, t_{j+1}]$; thus it holds for $A \subset B_\Delta$, and therefore for any A by letting $|\Delta| \downarrow 0$. $\square$

Similarly to $\omega_x^{\nearrow}$ and $\omega^{\nearrow}$, we can consider the measures

$$\omega_x^{\searrow}(A) = \sum_{t:\omega(t)=x,\Upsilon_t(\omega)<0} 1_A(t), \qquad \omega^{\searrow}(A) = \int \omega_x^{\searrow}(A)dx$$

associated to beginnings of negative excursions. In the finite variation case, $\omega^{\searrow}$ is the negative part of the Lebesgue-Stieltjes measure of ω.

Let us now consider ends (instead of beginnings) of excursions; the excursion $\Upsilon_t^\dagger$ ending at time t will be defined later; now let us simply denote $\Upsilon_t^\dagger > 0$ or $\Upsilon_t^\dagger < 0$ depending on the sign of this excursion (the sign is well defined even if the excursion is incomplete). Then we can define

$$\omega_x^{\nwarrow}(A) = \sum_{t:\omega(t)=x, \Upsilon_t^\dagger(\omega)>0} 1_A(t), \qquad \omega^{\nwarrow}(A) = \int \omega_x^{\nwarrow}(A)dx,$$

$$\omega_x^{\swarrow}(A) = \sum_{t:\omega(t)=x, \Upsilon_t^\dagger(\omega)<0} 1_A(t), \qquad \omega^{\swarrow}(A) = \int \omega_x^{\swarrow}(A)dx.$$

Thus we have four elementary measures associated to ω. Then we put

$$\begin{aligned} \omega^{\rightarrow}(dt) = \omega^{\nearrow}(dt) - \omega^{\searrow}(dt), \quad \omega^{\leftarrow}(dt) = \omega^{\swarrow}(dt) - \omega^{\nwarrow}(dt), \\ \omega^{\uparrow}(dt) = \omega^{\nearrow}(dt) - \omega^{\nwarrow}(dt), \quad \omega^{\downarrow}(dt) = \omega^{\swarrow}(dt) - \omega^{\searrow}(dt). \end{aligned} \tag{3.1.10}$$

They are not finite signed measures, except if ω has finite variation; in that case, all of them coincide with the Lebesgue-Stieltjes measure (Lemma 3.1.4). In the other cases, they are only differences of σ-finite measures with disjoint supports. Relations between them can be obtained by means of the correction measures

$$\omega^{\diagup}(dt) = \omega^{\swarrow}(dt) - \omega^{\nearrow}(dt), \quad \omega^{\diagdown}(dt) = \omega^{\searrow}(dt) - \omega^{\nwarrow}(dt) \tag{3.1.11}$$

which are 0 in the finite variation case. More precisely, we have

$$\omega^{\leftarrow} = \omega^{\rightarrow} + \omega^{\diagup} + \omega^{\diagdown} = \omega^{\uparrow} + \omega^{\diagup} = \omega^{\downarrow} + \omega^{\diagdown}.$$

All these definitions are summarised in the following table, where GLSM stands for generalised Lebesgue-Stieltjes measure.

$\omega^{\nearrow}$		beginnings of positive excursions
$\omega^{\searrow}$		beginnings of negative excursions
$\omega^{\nwarrow}$		ends of positive excursions
$\omega^{\swarrow}$		ends of negative excursions
$\omega^{\uparrow}$	$\omega^{\nearrow} - \omega^{\nwarrow}$	GLSM based on positive excursions
$\omega^{\downarrow}$	$\omega^{\swarrow} - \omega^{\searrow}$	GLSM based on negative excursions
$\omega^{\rightarrow}$	$\omega^{\nearrow} - \omega^{\searrow}$	GLSM based on beginnings of excursions
$\omega^{\leftarrow}$	$\omega^{\swarrow} - \omega^{\nwarrow}$	GLSM based on ends of excursions
$\omega^{\diagup}$	$\omega^{\swarrow} - \omega^{\nearrow}$	a correction measure
$\omega^{\diagdown}$	$\omega^{\searrow} - \omega^{\nwarrow}$	a correction measure

Remark 3.1.12. It is known that positive excursions of a path can be viewed as a tree (see for instance [4, 3]). Then $\omega^{\nearrow}$ and $\omega^{\nwarrow}$ actually correspond to the Lebesgue measure on this tree. This point of view will be explored in another article.

If we apply this construction to the Wiener paths, we obtain the measures $W_x^{\nearrow}(dt)$, ... and $W^{\nearrow}(dt)$, ... Of course, a result for one of them can be transferred to the three others by time reversal and/or change of sign. Notice however that if we want to study predictable processes, there is a big difference between the measures $W^{\nwarrow}(dt)$ and $W^{\swarrow}(dt)$ which are predictable, and the two others which are supported by a set (beginnings of excursions) intersecting no graph of stopping time but only graphs of honest times, see [2]. Notice also that all of these measures are supported by dense Lebesgue negligible subsets of $\mathbb{R}_+$.

3.2 Calculus for excursions

We need to define some transformations of Ω consisting in adding and removing excursions. The transformation $\mathcal{E}_{t,\theta}^+$ consists in inserting the excursion $\theta \in \Theta$ at time t, so that

$$(\mathcal{E}_{t,\theta}^+\omega)(s) = \begin{cases} \omega(s) & \text{if } s < t, \\ \omega(t) + \theta(s-t) & \text{if } t \le s < t + \zeta(\theta), \\ \omega(s - \zeta(\theta)) & \text{if } s \ge t + \zeta(\theta). \end{cases} \tag{3.2.1}$$

On the other hand, $\mathcal{E}_t^-$ removes the excursion beginning at time t if an excursion indeed begins at this time, otherwise it does nothing; thus

$$(\mathcal{E}_t^-\omega)(s) = \begin{cases} \omega(s) & \text{if } s < t, \\ \omega(s + \zeta_t(\omega)) & \text{if } s \ge t. \end{cases} \tag{3.2.2}$$

Notice that if ϕ_t is a predictable process, then

$$\phi_t \circ \mathcal{E}_{t,\theta}^+ = \phi_t \circ \mathcal{E}_t^- = \phi_t. \tag{3.2.3}$$

This property is evident for $\phi_t = 1_{\{r<t\le s\}}F$ and F a $\mathcal{F}_r$-measurable variable, and it is extended to other predictable process with a monotone class theorem. Let us now study the effect of these transformations on non predictable processes.

Theorem 3.2.4. *If ϕ_t is a nonnegative process, one has*

$$\mathbb{E}\int \phi_t W^{\nearrow}(dt) = \mathbb{E}\iint \phi_t \circ \mathcal{E}_{t,\theta}^+ \mathcal{I}^{\uparrow}(d\theta)dt. \tag{3.2.5}$$

In order to prove this theorem, we first state the following basic lemma about excursions at a given level.

Lemma 3.2.6. *Fix some real x and let L_t^x be the local time at level x given by the Tanaka formula* (2.3.4)*. Let ϕ_t be a nonnegative measurable process. Then*

$$\mathbb{E}\sum_{t:W_t=x,\Upsilon_t\neq 0} \phi_t = \mathbb{E}\iint \phi_t \circ \mathcal{E}_{t,\theta}^+ \mathcal{I}(d\theta)dL_t^x. \tag{3.2.7}$$

In particular,

$$\mathbb{E}\int \phi_t W_x^{\nearrow}(dt) = \mathbb{E}\iint \phi_t \circ \mathcal{E}_{t,\theta}^{+}\mathcal{I}^{\uparrow}(d\theta)dL_t^x. \tag{3.2.8}$$

Proof. Let Υ_t' be the path $\Upsilon_t'(s) = W_{t+s} - W_t$, $s \geq 0$ and consider $\phi_t = \psi_t F(\Upsilon_t')$, where ψ_t is predictable. Then it is known (see Chapter XX of [2] or Chapter III of [1]) that

$$\mathbb{E}\sum_{t:W_t=x,\Upsilon_t\neq 0} \phi_t = \Big(\int F d\mathcal{J}\Big)\Big(\mathbb{E}\int \psi_t dL_t^x\Big). \tag{3.2.9}$$

On the other hand, if we look at the right-hand side of (3.2.7), we have

$$\phi_t \circ \mathcal{E}_{t,\theta}^{+} = \psi_t F(\theta|\Upsilon_t').$$

Under $\mathbb{P}(d\omega)\mathcal{I}(d\theta)$, the concatenated path $\theta|\Upsilon_t'(\omega)$ is independent of $\mathcal{F}_t$ and has law $\mathcal{J}$, so

$$\mathbb{E}\Big[\int \phi_t \circ \mathcal{E}_{t,\theta}^{+}\mathcal{I}(d\theta)\;\Big|\;\mathcal{F}_t\Big] = \psi_t \int F\, d\mathcal{J},$$

and

$$\mathbb{E}\iint \phi_t \circ \mathcal{E}_{t,\theta}^{+}\mathcal{I}(d\theta)dL_t^x = \Big(\int F d\mathcal{J}\Big)\Big(\mathbb{E}\int \psi_t dL_t^x\Big). \tag{3.2.10}$$

By joining (3.2.9) and (3.2.10), we obtain (3.2.7) for this class of processes ϕ. The general case follows from a monotone class argument, and (3.2.8) is obtained by replacing ϕ_t by $\phi_t 1_{\{\Upsilon_t>0\}}$. $\square$

Remark 3.2.11. Another proof can be worked out by using the Poisson calculus rather than the excursion calculus. More precisely, if we change the time with the inverse of $t \mapsto L_t^x$, then it is well known that the excursion process Υ_t becomes a Poisson point process with intensity measure $\mathcal{I}$. Applying the basic formula of [8, Corollaire 1] for Poisson measures also yields (3.2.7).

Proof of Theorem 3.2.4. We integrate (3.2.8) with respect to x. In the left-hand side we obtain the measure $W^{\nearrow}(dt)$, and in the right-hand side, integrating dL_t^x with respect to x gives the Lebesgue measure dt. $\square$

Of course, a similar formula holds for $W^{\searrow}(dt)$ and $\mathcal{I}^{\downarrow}(d\theta)$.

3.3 Iterated formulas

We now notice that Theorem 3.2.4 can be iterated to study integrals of higher order.

Theorem 3.3.1. *Let ϕ_{t_1,t_2} be a nonnegative process indexed by $\mathbb{R}_+^2$. Then*

$$\begin{aligned}\mathbb{E}\iint_{\{t_1<t_2\}} &\phi_{t_1,t_2} W^{\nearrow}(dt_1)W^{\nearrow}(dt_2)\\ &= \mathbb{E}\iiiint_{\{t_1<t_2\}} \phi_{t_1,t_2}\circ\mathcal{E}^+_{t_2,\theta_2}\circ\mathcal{E}^+_{t_1,\theta_1}\mathcal{I}^{\uparrow}(d\theta_1)\mathcal{I}^{\uparrow}(d\theta_2)dt_1\,dt_2.\end{aligned} \tag{3.3.2}$$

A similar result holds if $(W^{\nearrow}(dt_i),\mathcal{I}^{\uparrow}(d\theta_i))$ is replaced by $(W^{\searrow}(dt_i),\mathcal{I}^{\downarrow}(d\theta_i))$ for $i=1$ and/or 2; under an integrability condition, one can also replace it by $(W^{\rightarrow}(dt_i),\mathcal{I}^{\updownarrow}(d\theta_i))$, with $\mathcal{I}^{\updownarrow}=\mathcal{I}^{\uparrow}-\mathcal{I}^{\downarrow}$.

Proof. We iterate (3.2.5) by applying it firstly with respect to t_2, secondly with respect to t_1. Since we are on $\{t_1<t_2\}$, we notice that

$$W^{\nearrow}(dt_1)\circ\mathcal{E}^+_{t_2,\theta_2}=W^{\nearrow}(dt_1)$$

and obtain (3.3.2). The other cases are similar. □

In the right-hand side of (3.3.2), we first append an excursion at time t_1, then another one at time t_2. This may result in two different figures. If $t_2>t_1+\zeta(\theta_1)$, then we have appended two disjoint excursions. On the other hand, if the reverse inequality holds, we insert θ_2 inside θ_1 and obtain an augmented path $\mathcal{E}^+_{t_2-t_1,\theta_2}\theta_1$ containing θ_2 as an excursion; this augmented path always begins with an excursion; however it is not an excursion if θ_1 and θ_2 have opposite signs and $\eta(\theta_2)\geq\left|\theta_1(t_2-t_1)\right|$. We denote by $\widetilde{\theta}$ the first excursion of this augmented path (see Figure 4), so that

$$\widetilde{\theta}=\widetilde{\theta}(t_2-t_1,\theta_1,\theta_2):=\begin{cases}\Upsilon_0(\mathcal{E}^+_{t_2-t_1,\theta_2}\theta_1) & \text{if } t_2\leq t_1+\zeta(\theta_1),\\ \theta_1 & \text{otherwise.}\end{cases} \tag{3.3.3}$$

We are going to see that $\widetilde{\theta}$ plays an important role in our estimations.

Theorem 3.3.1 can be applied to processes of type $\phi_{t_1,t_2}=\phi_{t_1}\phi_{t_2}$, in order to deduce the L^2 norm or the variance of $\int\phi_t W^{\nearrow}(dt)$ or $\int\phi_t W^{\rightarrow}(dt)$. Let us explain this calculation in the simple case $\phi_t=1_B(t,\Upsilon_t)$ for a $B\subset\mathbb{R}_+\times\Theta$ of finite measure for $dt\,\mathcal{I}(d\theta)$. Relation (3.2.5) is written as

$$\mathbb{E}\int 1_B(t,\Upsilon_t)W^{\nearrow}(dt)=\iint 1_B(t,\theta)\mathcal{I}^{\uparrow}(d\theta)dt,$$

so

$$\begin{aligned}\Big(\mathbb{E}\int &1_B(t,\Upsilon_t)W^{\nearrow}(dt)\Big)^2\\ &=2\iiiint_{\{t_1<t_2\}}1_B(t_1,\theta_1)1_B(t_2,\theta_2)\mathcal{I}^{\uparrow}(d\theta_1)\mathcal{I}^{\uparrow}(d\theta_2)dt_1dt_2.\end{aligned} \tag{3.3.4}$$

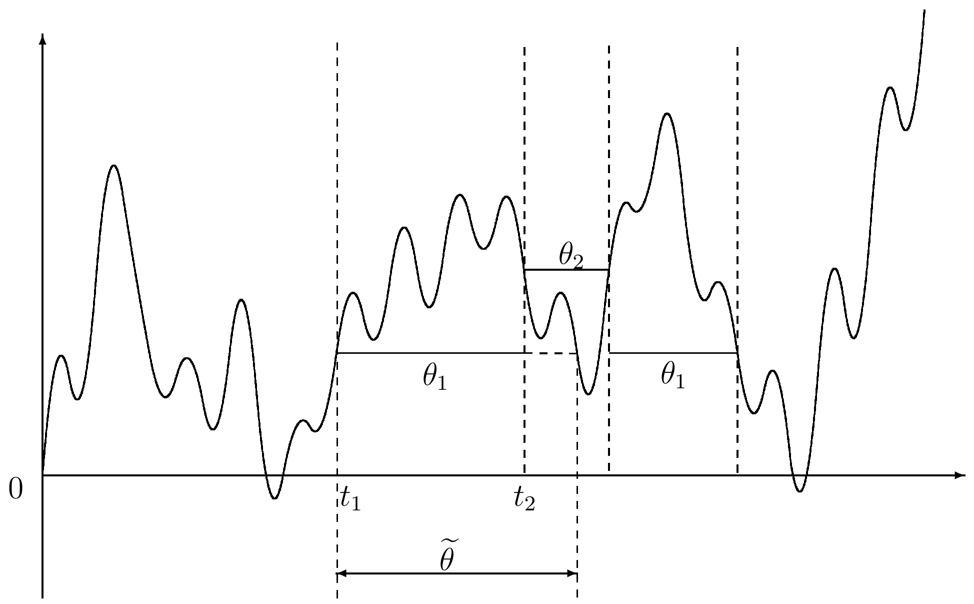

Figure 4: A path $\mathcal{E}^+_{t_2,\theta_2}\mathcal{E}^+_{t_1,\theta_1}\omega$ with $\widetilde{\theta}$

On the other hand, for $t_1 < t_2$,

$$1_B(t_2, \Upsilon_{t_2}) \circ \mathcal{E}^+_{t_2,\theta_2} \circ \mathcal{E}^+_{t_1,\theta_1} = 1_B(t_2, \theta_2), \quad 1_B(t_1, \Upsilon_{t_1}) \circ \mathcal{E}^+_{t_2,\theta_2} \circ \mathcal{E}^+_{t_1,\theta_1} = 1_B(t_1, \widetilde{\theta})$$

with the notation (3.3.3), so (3.3.2) is written as

$$\begin{aligned}&\mathbb{E}\Big(\int 1_B(t, \Upsilon_t) W^{\nearrow}(dt)\Big)^2 \\ &\quad = 2 \iiiint_{\{t_1<t_2\}} 1_B(t_1, \widetilde{\theta}) 1_B(t_2, \theta_2) \mathcal{I}^{\uparrow}(d\theta_1) \mathcal{I}^{\uparrow}(d\theta_2) dt_1 dt_2\end{aligned} \tag{3.3.5}$$

By comparing (3.3.4) and (3.3.5), we get the variance

$$\begin{aligned}&\mathrm{var}\Big(\int 1_B(t, \Upsilon_t) W^{\nearrow}(dt)\Big) \\ &\quad = 2 \iiiint_{\{t_1<t_2\}} (1_B(t_1, \widetilde{\theta}) - 1_B(t_1, \theta_1)) 1_B(t_2, \theta_2) \mathcal{I}^{\uparrow}(d\theta_1) \mathcal{I}^{\uparrow}(d\theta_2) dt_1 dt_2.\end{aligned} \tag{3.3.6}$$

Notice that in this integral we only have to consider times $t_2 \leq t_1 + \zeta(\theta_1)$ (otherwise $\widetilde{\theta} = \theta_1$). We can handle similarly negative excursions, and by considering opposite signs, we get the covariance

$$\begin{aligned}&\mathrm{cov}\Big(\int 1_B(t, \Upsilon_t) W^{\nearrow}(dt), \int 1_B(t, \Upsilon_t) W^{\searrow}(dt)\Big) \\ &\quad = \iiiint_{\{t_1<t_2\}} (1_B(t_1, \widetilde{\theta}) - 1_B(t_1, \theta_1)) 1_B(t_2, \theta_2) \\ &\qquad\qquad \big(\mathcal{I}^{\uparrow}(d\theta_1)\mathcal{I}^{\downarrow}(d\theta_2) + \mathcal{I}^{\downarrow}(d\theta_1)\mathcal{I}^{\uparrow}(d\theta_2)\big) dt_1 dt_2.\end{aligned} \tag{3.3.7}$$

We also deduce

$$\begin{aligned}&\operatorname{var}\Big(\int 1_B(t,\Upsilon_t)W^{\rightarrow}(dt)\Big)\\&\quad= 2\iiiint_{\{t_1<t_2\}}(1_B(t_1,\widetilde{\theta})-1_B(t_1,\theta_1))1_B(t_2,\theta_2)\mathcal{I}^{\updownarrow}(d\theta_1)\mathcal{I}^{\updownarrow}(d\theta_2)dt_1dt_2.\end{aligned}$$

Example 3.3.8. Let us study the excursions of height $\eta\geq a$ by considering $B=[0,1]\times\{\theta;\ \eta(\theta)\geq a\}$. If we look at positive excursions, we have

$$\mathbb{E}\int_0^1 1_{\{\eta_t\geq a\}}W^{\nearrow}(dt)=\mathcal{I}^{\uparrow}\{\eta\geq a\}=\frac{1}{2a}$$

from (2.1.9). For the variance, we use (3.3.6); we notice that

$$1_B(t_1,\widetilde{\theta})-1_B(t_1,\theta_1)\geq 0 \tag{3.3.9}$$

if θ_1 and θ_2 have the same sign, and it is non zero when θ_1 have an height less than a, but θ_2 has height greater than a and becomes a sub-excursion of θ_1. Thus

$$\begin{aligned}\operatorname{var}\int_0^1 1_{\{\eta_t\geq a\}}W^{\nearrow}(dt)&=2\iint_{[0,1]^2}\iint_{(\Theta^{\uparrow})^2}1_{\{\eta(\theta_1)<a\}}1_{\{\eta(\theta_2)\geq a\}}1_{\{t_1<t_2\leq t_1+\zeta(\theta_1)\}}\\&\qquad\qquad\mathcal{I}^{\uparrow}(d\theta_1)\mathcal{I}^{\uparrow}(d\theta_2)dt_1dt_2\\&=\frac{1}{a}\int_0^1\int_{\{\eta(\theta)<a\}}(\zeta(\theta)\wedge(1-t))\mathcal{I}^{\uparrow}(d\theta)dt.\end{aligned}\tag{3.3.10}$$

If X_t is a Bessel process (of dimension 3), then X_t^2-3t is a martingale, so the expected time of the first hitting time of a is $a^2/3$; consequently, the Williams decomposition recalled in Remark 2.1.8 shows that the expected length of an excursion of height a is $2a^2/3$. We deduce that the conditional expectation under $\mathcal{I}^{\uparrow}$ of $\zeta(\theta)$ given $\eta(\theta)$ is equal to $2\eta(\theta)^2/3$, and that

$$\frac{1}{a}\int_{\{\eta(\theta)<a\}}\zeta(\theta)\mathcal{I}^{\uparrow}(d\theta)=\frac{2}{3a}\int_{\{\eta(\theta)<a\}}\eta(\theta)^2\mathcal{I}^{\uparrow}(d\theta)=\frac{1}{3}. \tag{3.3.11}$$

We also check with the Williams decomposition that the conditional expectation of $\zeta(\theta)^2$ given $\eta(\theta)$ is finite and proportional to $\eta(\theta)^4$, so the difference between (3.3.10) and (3.3.11) is bounded by

$$\begin{aligned}\frac{1}{a}\int_0^1\int_{\{\eta(\theta)<a\}}\zeta(\theta)1_{\{\zeta(\theta)>1-t\}}\mathcal{I}^{\uparrow}(d\theta)dt&\leq\frac{1}{a}\int_{\{\eta(\theta)<a\}}\zeta(\theta)^2\mathcal{I}^{\uparrow}(d\theta)\\&=\frac{C}{a}\int_{\{\eta(\theta)<a\}}\eta(\theta)^4\mathcal{I}^{\uparrow}(d\theta)\\&=O(a^2).\end{aligned}$$

Thus the variance (3.3.10) of our variable is bounded by 1/3, and

$$\lim_{a\to 0} \operatorname{var} \int_0^1 1_{\{\eta_t \ge a\}} W^{\nearrow}(dt) = 1/3. \tag{3.3.12}$$

Similarly,

$$\lim_{a\to 0} \operatorname{var} \int_0^1 1_{\{\eta_t \ge a\}} W^{\searrow}(dt) = 1/3. \tag{3.3.13}$$

On the other hand, the covariance is given by (3.3.7) and

$$1_B(t_1, \widetilde{\theta}) - 1_B(t_1, \theta_1) \le 0 \tag{3.3.14}$$

if θ_1 and θ_2 have opposite signs, and is non zero when excursions θ_1 and θ_2 have height greater than a, but the height of $\widetilde{\theta}$ is less than a; this happens when $t_2 - t_1$ is less than the first hitting time $T_a(\theta_1)$ of a (or $-a$) by θ_1. The contributions of $\mathcal{I}^{\uparrow}(d\theta_1)\mathcal{I}^{\downarrow}(d\theta_2)$ and $\mathcal{I}^{\downarrow}(d\theta_1)\mathcal{I}^{\uparrow}(d\theta_2)$ are equal, and

$$\begin{aligned}
&\operatorname{cov}\Big(\int_0^1 1_{\{\eta_t \ge a\}} W^{\nearrow}(dt), \int_0^1 1_{\{\eta_t \ge a\}} W^{\searrow}(dt)\Big) \\
&= -2 \iint_{[0,1]^2} \iint 1_{\{\eta(\theta_1)\ge a\}} 1_{\{\eta(\theta_2)\ge a\}} 1_{\{t_1 < t_2 \le t_1 + T_a(\theta_1)\}} \mathcal{I}^{\uparrow}(d\theta_1)\mathcal{I}^{\downarrow}(d\theta_2) dt_1 dt_2 \\
&= -\frac{1}{a} \int_0^1 \int_{\{\eta(\theta)\ge a\}} (T_a(\theta) \wedge (1-t)) \mathcal{I}^{\uparrow}(d\theta) dt.
\end{aligned} \tag{3.3.15}$$

Conditionally on $\{\eta(\theta) \ge a\}$, $T_a(\theta)$ is the hitting time of a by a Bessel process, so it has expectation $a^2/3$ and

$$-\frac{1}{a} \int_{\{\eta(\theta)\ge a\}} T_a(\theta) \mathcal{I}^{\uparrow}(d\theta) = -\frac{a}{3} \mathcal{I}^{\uparrow}\{\eta \ge a\} = -\frac{1}{6}. \tag{3.3.16}$$

The difference between (3.3.15) and (3.3.16) is bounded by

$$\begin{aligned}
\frac{1}{a} \int_0^1 \int_{\{\eta(\theta)\ge a\}} T_a(\theta) 1_{\{T_a(\theta) > 1-t\}} \mathcal{I}^{\uparrow}(\theta) dt &\le \frac{1}{a} \int_{\{\eta(\theta)\ge a\}} T_a(\theta)^2 \mathcal{I}^{\uparrow}(d\theta) \\
&= C\, a^3 \mathcal{I}^{\uparrow}\{\eta \ge a\} = O(a^2),
\end{aligned}$$

so

$$\lim_{a\to 0} \operatorname{cov}\Big(\int_0^1 1_{\{\eta_t \ge a\}} W^{\nearrow}(dt), \int_0^1 1_{\{\eta_t \ge a\}} W^{\searrow}(dt)\Big) = -1/6. \tag{3.3.17}$$

We deduce from (3.3.12), (3.3.13) and (3.3.17) that

$$\lim_{a\to 0} \operatorname{var} \int_0^1 1_{\{\eta_t \ge a\}} W^{\rightarrow}(dt) = 1. \tag{3.3.18}$$

Example 3.3.19. Excursions of length greater than ε are studied with $B = [0,1]\times\{\theta;\zeta(\theta) \geq \varepsilon\}$. The variance of $\int_0^1 1_{\{\zeta_t\geq\varepsilon\}} W^{\nearrow}(dt)$ is obtained from combined excursions where θ_1 has length less than ε. One has $\mathcal{I}^{\uparrow}\{\zeta \geq \varepsilon\} = 1/\sqrt{2\pi\varepsilon}$ from (2.3.2), so

$$\begin{aligned}\operatorname{var}\int_0^1 1_{\{\zeta_t\geq\varepsilon\}} W^{\nearrow}(dt) &= 2\iint_{[0,1]^2}\iint 1_{\{\zeta(\theta_1)<\varepsilon\}} 1_{\{\zeta(\theta_2)\geq\varepsilon\}} 1_{\{t_1<t_2\leq t_1+\zeta(\theta_1)\}} \\ &\qquad\qquad \mathcal{I}^{\uparrow}(d\theta_1)\mathcal{I}^{\uparrow}(d\theta_2)dt_1 dt_2 \\ &= \frac{2}{\sqrt{2\pi\varepsilon}}\int_0^1\int_{\{\zeta(\theta)<\varepsilon\}} (\zeta(\theta)\wedge(1-t))\mathcal{I}^{\uparrow}(d\theta)dt.\end{aligned}$$

By proceeding as in previous example, we can prove that this variance is dominated by its limit

$$\lim_{\varepsilon\to 0}\operatorname{var}\int_0^1 1_{\{\zeta_t\geq\varepsilon\}} W^{\nearrow}(dt) = \lim_{\varepsilon\to 0}\frac{2}{\sqrt{2\pi\varepsilon}}\frac{1}{2\sqrt{2\pi}}\int_0^{\varepsilon}\frac{dy}{\sqrt{y}} = \frac{1}{\pi},$$

and similarly for $W^{\searrow}$. One can also write the formula involving mixed excursions for the covariance, but calculations are more complicated. Subsequently, we will only use the boundedness of the covariance, which follows from the boundedness of the variances.

3.4 The multidimensional case

In this short subsection, we describe how results of §3.2 and §3.3 are extended to the multidimensional case $\Omega = C(\mathbb{R}_+,\mathbb{R}^d)$. Excursions of each component have to be considered separately, so let us suppose that we want to study the first component; then we write the Wiener process as $(W_t, V_t) \in \mathbb{R}\times\mathbb{R}^{d-1}$. We let Θ_1 be the space of d-dimensional continuous paths such that the first component θ_1 is an excursion with length $\zeta(\theta)$, and the other components are paths stopped at time $\zeta(\theta)$. On this space, we consider the measures $\mathcal{I}_1^{\uparrow}$ and $\mathcal{I}_1^{\downarrow}$ under which the first component θ_1 is a Brownian positive or negative excursion, and the other components are independent Wiener paths up to time $\zeta(\theta)$; the main difference with respect to the one-dimensional case is that now $\theta(\zeta(\theta)) \neq 0$. Then $\mathcal{E}^+_{t,\theta}$ again consists in inserting θ at t; the difference with respect to (3.2.1) is that

$$\mathcal{E}^+_{t,\theta}\omega(s) = \omega(s-\zeta(\theta)) + \theta(\zeta(\theta))$$

for $s > t+\zeta(\theta)$. Similarly, $\mathcal{E}^-_t$ is defined like (3.2.2), but

$$\mathcal{E}^-_t\omega(s) = \omega(s+\zeta_t(\omega)) - \omega(t+\zeta_t(\omega)) + \omega(t)$$

for $s > t$. The process Υ_t is defined as in (3.1.2) (its first component is an excursion). Then Theorems 3.2.4 and 3.3.1 hold with the measures $W^{\nearrow}(dt)$ and $\mathcal{I}_1^{\uparrow}(d\theta)$ (one first consider processes ϕ_t which are the product of a functional of W and of a functional of V, then the general case follows from a standard monotone class theorem).

4 Stochastic integration

In this section, we use the generalised Lebesgue-Stieltjes measures (3.1.10) in order to construct stochastic integrals of predictable processes. As in §3.4, we write the Wiener process as $(W_t, V_t) \in \mathbb{R}\times\mathbb{R}^{d-1}$, and we want to construct integrals with respect to the first component W. More precisely, we check that the measure $W^{\rightarrow}(dt)$ leads to Itô integrals, that the measures $W^{\uparrow}(dt)$ and $W^{\downarrow}(dt)$ lead to Stratonovich integrals, and that $W^{\leftarrow}(dt)$ leads to so-called backward integrals. Each type of integral is studied by means of an approximation procedure; more precisely, the approximation consists in restricting the generalised Lebesgue-Stieltjes measures to subsets of the time interval in order to obtain finite signed measures. As an application, the construction of Itô integrals is closely related to the duality formula of Lemma 2.2.5, so we are able to complete the proof of the representation theorem for Wiener functionals (Theorems 2.1.4 and 2.1.12). Results of this section are summarised in §4.5.

Notice however that this is a nonlinear approach to integration; we can separately construct an integral with respect to each component of the Wiener process, but if now we consider the integral with respect to the vector-valued process as the sum of these integrals, then the result depends on the choice of the frame on $\mathbb{R}^d$. For instance, the measures $W_1^{\rightarrow} + W_2^{\rightarrow}$ and $(W_1 + W_2)^{\rightarrow}$ are not equal (a beginning of excursion of W_1 is not a beginning of excursion of $W_1 + W_2$), though they would be equal if W_1 and W_2 were of finite variation (since they would coincide with Lebesgue-Stieltjes measures).

It is sometimes easier for notational convenience to consider processes indexed by $\mathbb{R}$. In this case, $(W_t, V_t; t \geq 0)$ and $(W_{-t}V_{-t}; t \geq 0)$ are independent standard Wiener processes. However, we always construct integrals on the time interval $[0, 1]$; this means that we integrate a process ρ_t, $0 \leq t \leq 1$, which is extended by $\rho \equiv 0$ out of this interval.

4.1 Itô integrals

It can be seen from results of previous section that predictable processes cannot be directly integrated with respect to $W^{\rightarrow}(dt)$ (consider for instance a non zero constant process). However (with the notations of §3.4), if B is a subset of $[0,1]\times\Theta_1$ which has finite measure for $dt\,\mathcal{I}_1(d\theta)$, we have

$$\mathbb{E}\int 1_B(t,\Upsilon_t)W^{\nearrow}(dt) = \int_B \mathcal{I}_1^{\uparrow}(d\theta)dt < \infty$$

and similarly for $W^{\searrow}(dt)$, so we can consider

$$\Psi_B^{\rightarrow}(\rho) = \int \rho_t 1_B(t,\Upsilon_t)W^{\rightarrow}(dt) = \int_0^t \rho_s dW_s^{\rightarrow}(B) \tag{4.1.1}$$

for any bounded process ρ, where

$$W_t^{\rightarrow}(B) = \Psi_B^{\rightarrow}(1_{[0,t]}) = \int_0^t 1_B(s,\Upsilon_s)W^{\rightarrow}(ds) \tag{4.1.2}$$

is an anticipating process with finite variation. Since we want positive and negative excursions to compensate, we assume that B is symmetric, that is

$$(t,\theta)\in B \Leftrightarrow (t,\theta^\star)\in B$$

with $\theta^\star(t)=(-\theta_1(t),\theta_2(t),\ldots,\theta_d(t))$ (it was denoted by $\theta^\star_{(1)}$ in (2.1.10) but we prefer a lighter notation for this section). Then we can define $\Psi^\rightarrow(\rho)$ as the limit in some sense of $\Psi^\rightarrow_B(\rho)$ when B grows to $[0,1]\times\Theta_1$, if this limit exists. In this subsection, we suppose that ρ is predictable; we check that the convergence holds for a weak topology and that the limit $\Psi^\rightarrow(\rho)$ coincides with the Itô integral. We also check that the convergence may be stronger under some assumptions.

4.1a The general case

We now introduce the analogue of the Malliavin derivative in our calculus; this is the difference operator

$$\mathcal{D}^1_t F=\int\big(F\circ\mathcal{E}^+_{t,\theta}-F\circ\mathcal{E}^+_{t,\theta^\star}\big)\mathcal{I}^\uparrow_1(d\theta), \tag{4.1.3}$$

which is defined for (t,ω) such that the integrand of the right-hand side is integrable. Notice in particular that the operator J^1_t defined in (2.1.11) can be written as the predictable projection

$$J^1_t F=\mathbb{E}\big[\mathcal{D}^1_t F \;\big|\; \mathcal{F}_t\big].$$

We have checked in Lemma 2.2.1 that for simple bounded functionals F, $J^1_t F$ is the process Z^1_t involved in the integral representation formula (1.1), so we deduce that if F is simple, bounded, and ρ_t is predictable, bounded, then

$$\mathbb{E}\Big[F\int_0^1\rho_t dW_t\Big]=\mathbb{E}\int_0^1\rho_t J^1_t F\,dt=\mathbb{E}\int_0^1\rho_t\mathcal{D}^1_t F\,dt. \tag{4.1.4}$$

We can now state the general convergence result of $\Psi^\rightarrow_B(\rho)$ to the Itô integral of ρ.

Theorem 4.1.5. *Let $F=f(W_{t_1},V_{t_1},\ldots,W_{t_N},V_{t_N})$ be a bounded simple variable, and let $(\rho_t;0\le t\le 1)$ be a bounded predictable process. Then for any sequence B_n of symmetric sets with finite measure increasing to $[0,1]\times\Theta_1$, one has*

$$\lim\mathbb{E}\big[F\,\Psi^\rightarrow_{B_n}(\rho)\big]=\mathbb{E}\Big[F\int_0^1\rho_t dW_t\Big]. \tag{4.1.6}$$

This theorem states that $\Psi^\rightarrow_B(\rho)$ converges to the Itô integral of ρ for the weak topology induced by bounded simple variables. The basic result used in the proof is the following lemma.

Lemma 4.1.7. *Let ρ_t be a bounded predictable process, and let F be a bounded variable such that*

$$\mathbb{E}\int_0^1 \int |\rho_t| \left|F \circ \mathcal{E}^+_{t,\theta} - F \circ \mathcal{E}^+_{t,\theta^\star}\right| \mathcal{I}^\uparrow_1(d\theta)dt < \infty. \tag{4.1.8}$$

Then, for any (B_n) as in Theorem 4.1.5,

$$\lim \mathbb{E}\big[F\, \Psi^{\rightarrow}_{B_n}(\rho)\big] = \mathbb{E}\int_0^1 \rho_t \mathcal{D}^1_t F\, dt. \tag{4.1.9}$$

Proof. First notice as in (3.2.3) that since ρ_t is predictable, one has $\rho_t \circ \mathcal{E}^+_{t,\theta} = \rho_t$. Then Theorem 3.2.4 (generalised in §3.4) implies that

$$\mathbb{E}\Big[F\int_0^1 \rho_t 1_{B_n}(t,\Upsilon_t) W^{\nearrow}(dt)\Big] = \mathbb{E}\iint_0^1 1_{B_n}(t,\theta)\rho_t\big(F\circ\mathcal{E}^+_{t,\theta}\big)\mathcal{I}^\uparrow_1(d\theta)dt$$

and similarly for the $W^{\searrow}$-integral, so, from the symmetry of B_n,

$$\mathbb{E}\big[F\,\Psi^{\rightarrow}_{B_n}(\rho)\big] = \mathbb{E}\iint_0^1 1_{B_n}(t,\theta)\rho_t\big(F\circ\mathcal{E}^+_{t,\theta} - F\circ\mathcal{E}^+_{t,\theta^\star}\big)\mathcal{I}^\uparrow_1(d\theta)dt.$$

Assumption (4.1.8) enables to take the limit in the right-hand side and to deduce the lemma. □

Proof of Theorem 4.1.5. If (t_j) are the times associated to the simple variable F, one has

$$\left|F\circ\mathcal{E}^+_{t,\theta} - F\circ\mathcal{E}^+_{t,\theta^\star}\right| \le C\, 1_{\{\zeta(\theta)\ge t_{j+1}-t\}}$$

for $t_j \le t < t_{j+1}$. On the other hand, from (2.3.2),

$$\int_{t_j}^{t_{j+1}}\int 1_{\{\zeta(\theta)\ge t_{j+1}-t\}}\mathcal{I}^\uparrow(d\theta)dt = \int_{t_j}^{t_{j+1}} \frac{dt}{\sqrt{2\pi(t_{j+1}-t)}} < \infty$$

so (4.1.8) holds. We can apply Lemma 4.1.7, and the conjunction of (4.1.4) and (4.1.9) enables to conclude. □

4.1b The weak L^2 convergence

The convergence of Theorem 4.1.5 holds for a large class of sequences (B_n). By choosing particular sequences, we can have a better (but still weak) convergence, namely for the weak topology of L^2. In the same time, we are able to complete the proof of Lemma 2.2.5, and therefore of the integral representation formula (Theorems 2.1.4 and 2.1.12).

Lemma 4.1.10. *There exist sequences (B_n) such that the convergence result* (4.1.6) *of Theorem 4.1.5 holds for any $F \in L^2$ and any bounded predictable process ρ_t.*

Bounded sequences of L^2 are relatively compact for the weak L^2 topology, so it is easy to see that the statement of Lemma 4.1.10 holds true as soon as $\Psi^{\rightarrow}_{B_n}(\rho)$ is bounded in L^2 for any bounded predictable ρ; these sequences (B_n) are subsequently called good approximating sequences. We give below three examples of good approximating sequences (B_n), so this will complete the proof of the lemma.

An important consequence of Lemma 4.1.10 is the following duality formula.

Theorem 4.1.11. *The duality formula*

$$\mathbb{E}\int_0^1 \mathcal{D}_t^1 F\,\rho_t dt = \mathbb{E}\Big[F\int_0^1 \rho_t dW_t\Big] \tag{4.1.12}$$

holds for any bounded F and any bounded predictable process ρ satisfying (4.1.8)*.*

Proof. The left-hand side of (4.1.12) is expressed in (4.1.9) as a limit, and the fact that this limit is the also the right-hand side of (4.1.12) is exactly Lemma 4.1.10 if we choose a good sequence (B_n). □

Proof of Lemma 2.2.5 (for $k = 1$). Since a path under $\mathcal{J}_1^{\uparrow}$ is made (for its first component) with a positive excursion followed by an independent Wiener path, we notice that the assumptions (2.2.6) and (4.1.8) are equivalent, as well as the conclusions (2.2.7) and (4.1.12), so Lemma 2.2.5 follows from Theorem 4.1.11. □

We now look for examples of good approximating sequences (B_n). It is sufficient to put conditions on the first component of θ, so we suppose $d = 1$ in these examples; the generalisation to the multidimensional case is straightforward.

Example 4.1.13. Consider excursions with a height greater than some $a > 0$; more precisely, let $B = B_0(a)$ be defined by

$$B_0(a) = \Big\{(t,\theta) \in [0,1]\times\Theta;\ \eta(\theta) \geq a\Big\}.$$

We have already studied in Example 3.3.8 the case $\rho \equiv 1$, and check that $\Psi^{\rightarrow}_B(1)$ is centred with bounded variance. We now have to consider other bounded predictable processes ρ; the approximation $\Psi^{\rightarrow}_B(\rho)$ is again centred and we have to compute the variance. First notice that if ϕ_t is a bounded symmetric process in the sense that

$$\phi_t \circ \mathcal{E}^+_{t,\theta} = \phi_t \circ \mathcal{E}^+_{t,\theta^\star},$$

then, from (3.2.5),

$$\mathbb{E}\int \phi_t 1_B(t,\Upsilon_t) W^{\rightarrow}(dt) = 0.$$

Consequently,

$$\mathbb{E}\iint_{\{t_1<t_2\}} \rho_{t_1}\rho_{t_2} 1_B(t_1,\Upsilon_{t_1}\circ\mathcal{E}^-_{t_2}) 1_B(t_2,\Upsilon_{t_2}) W^{\rightarrow}(dt_1) W^{\rightarrow}(dt_2) = 0$$

because this is the integral with respect to $W^{\rightarrow}(dt_2)$ of a symmetric process. Thus

$$\begin{aligned}\mathbb{E}\Psi_B^{\rightarrow}(\rho)^2 &= 2\,\mathbb{E}\iint_{\{t_1<t_2\}} \rho_{t_1}\rho_{t_2}\big(1_B(t_1,\Upsilon_{t_1}) - 1_B(t_1,\Upsilon_{t_1}\circ\mathcal{E}_{t_2}^-)\big)1_B(t_2,\Upsilon_{t_2}) \\ &\qquad W^{\rightarrow}(dt_1)W^{\rightarrow}(dt_2) \qquad (4.1.14)\\ &\le C\,\mathbb{E}\iint_{\{t_1<t_2\}} \big|1_B(t_1,\Upsilon_{t_1}) - 1_B(t_1,\Upsilon_{t_1}\circ\mathcal{E}_{t_2}^-)\big|1_B(t_2,\Upsilon_{t_2}) \\ &\qquad \big|W^{\rightarrow}(dt_1)\big|\big|W^{\rightarrow}(dt_2)\big| \\ &= C\iiiint_{\{t_1<t_2\}} \big|1_B(t_1,\theta_1) - 1_B(t_1,\widetilde{\theta})\big|1_B(t_2,\theta_2)\mathcal{I}(d\theta_1)\mathcal{I}(d\theta_2)dt_1dt_2\end{aligned}$$

by applying Theorem 3.3.1, and with the notation $\widetilde{\theta}$ given in (3.3.3). By applying the decomposition $\mathcal{I} = \mathcal{I}^{\uparrow} + \mathcal{I}^{\downarrow}$, the sign of $1_B(t_1,\widetilde{\theta}) - 1_B(t_1,\theta_1)$ given in (3.3.9) and (3.3.14), and formulas (3.3.6) and (3.3.7), we check that the quadruple integral is half of the variance of $\Psi_B^{\rightarrow}(1)$. It is dominated by the variances of $\Psi_B^{\nearrow}(1)$ and $\Psi_B^{\searrow}(1)$ which have been proved to be bounded in Example 3.3.8. Thus $\Psi_B^{\rightarrow}(\rho)$ is bounded in L^2, and it is sufficient to choose a sequence $B_n = B_0(a_n)$ corresponding to a sequence $a_n \downarrow 0$ to obtain a good approximating sequence.

Remark 4.1.15. A disadvantage of the approximation $B = B_0(a)$ for the integral on $[0,1]$ is that $\Psi_B^{\rightarrow}(\rho)$ is not $\mathcal{F}_1$-measurable, because knowing the height of an excursion beginning in $[0,1]$ may involve information on the Wiener path after time 1. To bypass this point, a possibility is to add to $B_0(a)$ all (t,θ) such that $\zeta(\theta) \ge 1-t$ and to consider

$$B_1(a) = \Big\{(t,\theta) \in [0,1]\times\Theta;\ \eta(\theta) \ge a \text{ or } \zeta(\theta) \ge 1-t\Big\}.$$

This means that we consider all excursions straddling time 1, independently of their height. The approximation $\Psi_B^{\rightarrow}(\rho)$ for $B = B_1(a)$ is now $\mathcal{F}_1$-measurable, and it converges like previous one, because the difference between this approximation and previous one is dominated by a. Subsequently, as mentioned in the beginning of the section, it will be useful to consider paths indexed by $\mathbb{R}$; in this case, we also add to $B_1(a)$ excursions beginning before time 0 and ending in $[0,1]$; this means that we put

$$B(a) = B_1(a) \cup \Big\{(t,\theta) \in (-\infty,0]\times\Theta;\ -t \le \zeta(\theta) \le 1-t\Big\}. \qquad (4.1.16)$$

Approximations of integrals based on $B(a)$ will subsequently be called height-based approximations.

Example 4.1.17. A similar procedure can be worked out by considering excursions of length $\zeta(\theta) \ge \varepsilon$, more precisely

$$\begin{aligned}B'(\varepsilon) = \Big\{(t,\theta) \in& [0,1]\times\Theta;\ \zeta(\theta) \ge \varepsilon\wedge(1-t)\Big\} \\ &\cup\Big\{(t,\theta) \in (-\infty,0]\times\Theta;\ -t \le \zeta(\theta) \le 1-t\Big\}.\end{aligned}$$

We are again reduced to the boundedness of the variances corresponding to $\rho \equiv 1$ (Example 3.3.19). These approximations will be called length-based approximations.

Example 4.1.18. Consider a subdivision $\Delta = (t_j)$ of $[0,1]$, and let $B = B(\Delta)$ be the set of (t,θ) such that the interval $[t, t+\zeta(\theta)]$ contains a point of the subdivision. With the notation $\omega_\Delta^{\nearrow}$ introduced in (3.1.5), a similar notation $\omega_\Delta^{\searrow}$ and $\omega_\Delta^{\rightarrow} = \omega_\Delta^{\nearrow} - \omega_\Delta^{\searrow}$, we have from (3.1.7)

$$\int_0^1 f(t)\omega_\Delta^{\rightarrow}(dt) = \sum_k \int_{\omega[t_j,t_{j+1}]} f(g^x_{t_{j+1}}) \operatorname{sgn}(\omega(t_{j+1}) - x)dx$$

where $\omega[r,s]$ is the range of ω on $[r,s]$. Thus

$$\Psi_B^{\rightarrow}(\rho) = \int_0^1 \rho_t W_\Delta^{\rightarrow}(dt) = \sum_k Y_{[t_j,t_{j+1}]}$$

where

$$Y_{[r,s]} = \int_r^s \rho_t 1_{\{\zeta_t > s-t\}} W^{\rightarrow}(dt) = \int_{W[r,s]} \rho_{g_s^x} \operatorname{sgn}(W_s - x)dx.$$

An application of the duality formula (3.2.5) shows that

$$\mathbb{E}\big[F\, Y_{[r,s]}\big] = \mathbb{E}\int_r^s \rho_t \mathcal{D}_t^{s-t} F\, dt \tag{4.1.19}$$

with

$$\mathcal{D}_t^\delta F = \int_{\{\zeta(\theta) > \delta\}} \big(F \circ \mathcal{E}^+_{t,\theta} - F \circ \mathcal{E}^+_{t,\theta^\star}\big)\mathcal{I}^\uparrow(d\theta).$$

We check that $Y_{[r,s]}$ is $\mathcal{F}_s$-measurable, and an application of (4.1.19) with a $\mathcal{F}_r$-measurable variable F enables to deduce

$$\mathbb{E}\big[Y_{[r,s]} \bigm| \mathcal{F}_r\big] = 0$$

because $\mathcal{D}_t^{s-t} F = 0$ for $r \le t \le s$. Thus the variables $Y_{[t_k,t_{k+1}]}$ are centred and orthogonal in L^2. Moreover

$$\big|Y_{[r,s]}\big| \le C \int_{W[r,s]} dx = C\big(\sup_{[r,s]} W - \inf_{[r,s]} W\big),$$

so the variance of $Y_{[t_j,t_{j+1}]}$ is of order $t_{j+1} - t_j$, and the sum $\sum Y_{[t_j,t_{j+1}]}$ is bounded in L^2. Thus Theorem 4.1.11 can be applied to $B_n = B(\Delta_n)$ for any sequence $\Delta_n = (t_j^n)$ of subdivisions of $[0,1]$ such that $\max_j(t^n_{j+1} - t^n_j) \to 0$.

4.1c The strong L^2 convergence

Notice that the approximations $\Psi_B^{\rightarrow}(\rho)$ generally do not converge strongly in L^2. For instance, for $\rho \equiv 1$, consider the approximation based on time discretization of Example 4.1.18. In this case we have

$$Y_{[r,s]} = \int_{W[r,s]} \operatorname{sgn}(W_s - x)dx = 2W_s - \inf_{[r,s]} W - \sup_{[r,s]} W.$$

From the invariance of the Wiener measure by time reversal, we check that

$$\mathbb{E}\Big[\Big(W_s + W_r - \inf_{[r,s]} W - \sup_{[r,s]} W\Big)(W_s - W_r)\Big] = 0.$$

Thus

$$\mathbb{E}\big[Y_{[r,s]}(W_s - W_r)\big] = \mathbb{E}\big[(W_s - W_r)^2\big] = s - r,$$

so $Y_{[r,s]}$ is the sum of $W_s - W_r$ and of a non trivial orthogonal variable and

$$\mathbb{E}\big(Y_{[r,s]}\big)^2 = C(s-r) \quad \text{with } C > 1.$$

Thus

$$\mathbb{E}\big(\Psi_B^{\rightarrow}(1)\big)^2 = \mathbb{E}\Big(\sum_j Y_{[t_j,t_{j+1}]}\Big)^2 = C$$

does not converge to $\mathbb{E}[W_1^2] = 1$, and the convergence $\lim \Psi_B^{\rightarrow}(1) = W_1$ does not hold strongly in L^2.

Nevertheless, we now prove the strong convergence for a particular approximation when ρ is continuous.

Theorem 4.1.20. *If we use height-based approximations $B = B(a)$ given by* (4.1.16) *and if ρ is a bounded continuous predictable process, then the convergence of $\Psi_B^{\rightarrow}(\rho)$ to the Itô integral of ρ holds for the strong topology of L^2.*

Proof. We already have the weak convergence, so it is sufficient to prove that

$$\lim \mathbb{E}\Big(\Psi_B^{\rightarrow}(\rho)\Big)^2 = \mathbb{E}\int_0^1 \rho_t^2 dt \tag{4.1.21}$$

for $B = B(a)$ and $a \downarrow 0$. This has been checked in (3.3.18) for $\rho \equiv 1$, and we have to write the formulas in the general case. The variance has already been estimated in Example 4.1.13; we again use (4.1.14), but we handle it differently, taking advantage of the continuity of ρ. We have

$$\mathbb{E}\Big(\Psi_B^{\rightarrow}(\rho)\Big)^2 = I_1 + I_2$$

with

$$I_1 = 2\,\mathbb{E}\iint_{\{t_1<t_2\}} \rho_{t_1}^2\big(1_B(t_1,\Upsilon_{t_1}) - 1_B(t_1,\Upsilon_{t_1}\circ\mathcal{E}_{t_2}^-)\big)1_B(t_2,\Upsilon_{t_2}) \\ W^{\rightarrow}(dt_1)W^{\rightarrow}(dt_2),$$

$$I_2 = 2\,\mathbb{E}\iint_{\{t_1<t_2\}} \rho_{t_1}\big(\rho_{t_2} - \rho_{t_1}\big)\big(1_B(t_1,\Upsilon_{t_1}) - 1_B(t_1,\Upsilon_{t_1}\circ\mathcal{E}_{t_2}^-)\big)1_B(t_2,\Upsilon_{t_2}) \\ W^{\rightarrow}(dt_1)W^{\rightarrow}(dt_2).$$

The first term is studied from Theorem 3.3.1 (notice that ρ_{t_1} is not affected by $\mathcal{E}^+_{t_1,\theta_1}$ and $\mathcal{E}^+_{t_2,\theta_2}$), and we get

$$I_1 = 2\mathbb{E}\iiiint_{\{t_1<t_2\}} \rho^2_{t_1}\big(1_B(t_1,\widetilde{\theta}) - 1_B(t_1,\theta_1)\big)1_B(t_2,\theta_2)\mathcal{I}_1^{\updownarrow}(d\theta_1)\mathcal{I}_1^{\updownarrow}(d\theta_2)dt_1dt_2.$$

By proceeding as in Example 3.3.8, we check

$$\lim I_1 = \mathbb{E}\int_0^1 \rho_t^2 dt.$$

For I_2, we have enumerated in Example 3.3.8 the cases where the difference between the indicators is non zero; this can happen only when t_2 belongs to the excursion beginning at t_1, and before this excursion hits the level $\pm a$, so we must have

$$t_2 \le \inf\big\{t > t_1;\ |W_t - W_{t_1}| \ge a\big\}$$

and

$$t_1 \in \big\{g^x_{t_2};\ W_{t_2} - a \le x \le W_{t_2} + a\big\}.$$

The first condition shows that

$$|\rho_{t_2} - \rho_{t_1}| \le R_a = \sup\Big\{|\rho_s - \rho_r|; \max_{[r,s]} W - \min_{[r,s]} W \le a\Big\},$$

and the second one shows that for t_2 fixed,

$$\int \big|1_B(\Upsilon_{t_1}) - 1_B(\Upsilon_{t_1}\circ\mathcal{E}^-_{t_2})\big|\,\big|W^{\rightarrow}(dt_1)\big| \le 2a,$$

so

$$I_2 \le C\,a\,\mathbb{E}\Big[|R_a|\int 1_B(\Upsilon_t)\big|W^{\rightarrow}(dt)\big|\Big].$$

This term is shown to converge to 0 from the Cauchy-Schwarz inequality because R_a converges almost surely to 0 and $\int 1_B(\Upsilon_t)|W^{\rightarrow}(dt)|$ is of order $1/a$ in L^2. Adding the asymptotic behaviours of I_1 and I_2 yields (4.1.21). □

Remark 4.1.22. In §4.4, in Remark 4.4.10, we will see that almost sure convergence holds in some cases.

Remark 4.1.23. We do not know whether the same result holds for length-based approximations of Example 4.1.17, but we doubt it. From the proof which is given here, the strong convergence for height-based approximations may seem to come from the miraculous formula (3.3.18). Actually, the particularity of height-based approximations with respect to others will appear more clearly in §4.3.

4.2 Backward integrals

In this subsection we consider the measure $W^{\leftarrow}(dt)$ based on ends of excursions. More precisely, we suppose that $(W_t, V_t; t \geq 0)$ and $(W_{-t}, V_{-t}; t \geq 0)$ are independent standard Wiener processes. We can define the length of the excursion of W ending at t by

$$\zeta_t^{\dagger} = \inf\Big\{ s > 0;\ W_{t-s} = W_t \Big\},$$

and the excursion ending at t by

$$\Upsilon_t^{\dagger} : s \mapsto W_{t-\zeta_t^{\dagger}+s} - W_{t-\zeta_t^{\dagger}} \quad \text{for } 0 \leq s \leq \zeta_t^{\dagger}.$$

Then, similarly to (4.1.1) and (4.1.2), we consider the operator

$$\Psi_B^{\leftarrow}(\rho) = \int 1_B(t - \zeta_t^{\dagger}, \Upsilon_t^{\dagger}) \rho_t W^{\leftarrow}(dt) = \int_0^1 \rho_t dW_t^{\leftarrow}(B) \tag{4.2.1}$$

with

$$W_t^{\leftarrow}(B) = \Psi_B^{\leftarrow}(1_{[0,t]}).$$

The process $W_t^{\leftarrow}(B)$ also has finite variation, but an important difference with respect to $W_t^{\rightarrow}(B)$ is that it is predictable. If we use for instance height-based approximations (4.1.16), then $\Psi_{B(a)}^{\leftarrow}(\rho)$ is measurable with respect to $(W_t, V_t; 0 \leq t \leq 1)$ (see the discussion of Remark 4.1.15). We are going to check that when ρ is a semimartingale, then $\Psi_B^{\leftarrow}(\rho)$ is an approximation of what is usually called the backward integral of ρ.

Theorem 4.2.2. *Suppose that $(\rho_t; 0 \leq t \leq 1)$ is a bounded Itô process*

$$\rho_t = \rho_0 + \int_0^t \alpha_s ds + \int_0^t b_s dW_s + \int_0^t (\beta_s, dV_s) \tag{4.2.3}$$

for bounded predictable processes α_t, b_t and β_t. Let $B = (B_n)$ be a sequence such that $W_t^{\leftarrow}(B_n)$ is bounded in L^2. Then $\Psi_B^{\leftarrow}(\rho)$ converges for the weak topology of L^2 to

$$\Psi^{\leftarrow}(\rho) = \int_0^1 \rho_t dW_t + \langle \rho, W \rangle_1 = \int_0^1 \rho_t dW_t + \int_0^1 b_t dt. \tag{4.2.4}$$

For height based approximations $B = B(a)$ of (4.1.16), the convergence holds for the strong topology of L^2.

Proof. Theorem 4.1.5 and a time reversal imply that

$$\lim \mathbb{E}\big[F\, W_t^{\leftarrow}(B_n)\big] = \mathbb{E}\big[F\, W_t\big] \tag{4.2.5}$$

for any bounded simple F; from our assumption about the L^2 boundedness of $W_t^{\leftarrow}(B_n)$, this convergence also holds for any $F \in L^2$. On the other hand, Itô's formula applied to (4.2.1) and (4.2.4) yields

$$\begin{aligned}\Psi_B^{\leftarrow}(\rho) =& \rho_1 W_1^{\leftarrow}(B) - \int_0^1 W_t^{\leftarrow}(B)\alpha_t dt - \int_0^1 W_t^{\leftarrow}(B) b_t dW_t \\ &- \int_0^1 W_t^{\leftarrow}(B)(\beta_t, dV_t)\end{aligned} \tag{4.2.6}$$

and

$$\Psi^{\leftarrow}(\rho) = \rho_1 W_1 - \int_0^1 W_t\alpha_t dt - \int_0^1 W_t b_t dW_t - \int_0^1 W_t(\beta_t, dV_t).$$

If now F is a square integrable variable, it can be written from (1.1) as a stochastic integral of a predictable square integrable process $Z_t = (Z_t^1, Z_t')$ with respect to (W_t, V_t), and we have

$$\begin{aligned}\mathbb{E}\big[F\,\Psi_B^{\leftarrow}(\rho)\big] =& \mathbb{E}\big[F\rho_1 W_1^{\leftarrow}(B)\big] - \int_0^1 \mathbb{E}\big[F W_t^{\leftarrow}(B)\alpha_t\big]dt \\ &- \int_0^1 \mathbb{E}\big[Z_t^1 W_t^{\leftarrow}(B) b_t\big]dt - \int_0^1 \mathbb{E}\big[W_t^{\leftarrow}(B)(Z_t', \beta_t)\big]dt\end{aligned} \tag{4.2.7}$$

and

$$\begin{aligned}\mathbb{E}\big[F\,\Psi^{\leftarrow}(\rho)\big] =& \mathbb{E}\big[F\rho_1 W_1\big] - \int_0^1 \mathbb{E}\big[F W_t\alpha_t\big]dt \\ &- \int_0^1 \mathbb{E}\big[Z_t^1 W_t b_t\big]dt - \int_0^1 \mathbb{E}\big[W_t(Z_t', \beta_t)\big]dt.\end{aligned} \tag{4.2.8}$$

From (4.2.5), each term of the right-hand side of (4.2.7) converges to the corresponding term of (4.2.8), so the first statement of the theorem is proved. For height based approximations, one proves as in Theorem 4.1.20 that $W_t^{\leftarrow}(B(a))$ converges strongly to W_t, so one can take the limit in (4.2.6) and obtain the strong convergence of $\Psi_{B(a)}^{\leftarrow}(\rho)$. □

Remark 4.2.9. As for Itô integrals, we will see in Remark 4.4.10 that the convergence can be almost sure.

Remark 4.2.10. We deduce of course from Theorems 4.1.11 and 4.2.2 that the difference $\Psi_B^{\leftarrow}(\rho) - \Psi_B^{\rightarrow}(\rho)$ is a weak approximation of the quadratic covariation $\langle \rho, W\rangle_1$. Next subsection is devoted to another approximation of this covariation.

4.3 Quadratic covariation

We now study the correction measure $W^{\diagup}(dt)$ given by (3.1.11) (one can of course study similarly $W^{\diagdown}(dt)$). We consider like previously paths indexed by $\mathbb{R}$ and put

$$\begin{aligned}\Psi_B^{\diagup}(\rho) &= \int \rho_t\Big(1_B(t-\zeta_t^\dagger,\Upsilon_t^\dagger)W^{\swarrow}(dt) - 1_B(t,\Upsilon_t)W^{\nearrow}(dt)\Big)\\ &= \int \rho_t dW_t^{\diagup}(B)\end{aligned}$$

with

$$W_t^{\diagup}(B) = \Psi_B^{\diagup}(1_{[0,t]}).$$

As for $\Psi_B^{\rightarrow}$ and $\Psi_B^{\leftarrow}$, we are interested in the asymptotic behaviour of this expression. However, we only consider height-based approximations $B = B(a)$ defined in (4.1.16). We suppose that ρ is a smooth Itô process, that is a process given by (4.2.3) for bounded α, b, and β, such that

$$b_t - b_r = O(\sqrt{t-r}) \tag{4.3.1}$$

in the spaces L^p, $1 \le p < \infty$.

The particularity of height-based approximations is that positive and negative excursions of height greater than a are in bijection with each other, and actually, this explains the particular behaviour of these approximations which was already noticed in Theorems 4.1.20 and 4.2.2. Let us define

$$\sigma_t = \inf\big\{s \ge t;\ W_s \ge W_t + a\big\}. \tag{4.3.2}$$

If t is the beginning of a positive excursion of height greater than a, then σ_t is the end of a negative excursion of height greater than a, and $t \mapsto \sigma_t$ is a bijection between these two sets. This remark is important for the following result.

Theorem 4.3.3. *Suppose that ρ is a smooth Itô process* (4.2.3) *satisfying* (4.3.1)*. Then*

$$\lim \Psi_B^{\diagup}(\rho) = \frac{1}{2}\langle \rho, W\rangle_1 = \frac{1}{2}\int_0^1 b_t dt$$

in probability for height-based approximations. If $\beta \equiv 0$ (in particular in the one-dimensional case $d = 1$), then the convergence is almost sure.

We first prove the following lemma.

Lemma 4.3.4. *One has*

$$\lim_{a\to 0} a \int 1_B(t,\Upsilon_t) b_t W^{\nearrow}(dt) = \frac{1}{2}\int_0^1 b_s ds \tag{4.3.5}$$

almost surely.

Proof. Let us first consider the case $b \equiv 1$. The difference between the two sides is of order a in L^2 (recall the calculation of Example 3.3.8). We can deduce from the Borel-Cantelli lemma that the almost sure convergence holds for the sequence $a_n = 1/n$. Moreover, the monotonicity of $a \mapsto B(a)$ implies that

$$\begin{aligned}\frac{n}{n+1} a_n \int 1_{B(a_n)}(t, \Upsilon_t) W^{\nearrow}(dt) &\leq a \int 1_{B(a)}(t, \Upsilon_t) W^{\nearrow}(dt) \\ &\leq \frac{n+1}{n} a_{n+1} \int 1_{B(a_{n+1})}(t, \Upsilon_t) W^{\nearrow}(dt)\end{aligned}$$

for $a_{n+1} \leq a \leq a_n$. We deduce the whole almost sure convergence as $a \downarrow 0$. The case $b_t = 1_{[0,r]}(t)$ is studied similarly, so almost surely, the measure $a 1_B(t, \Upsilon_t) W^{\nearrow}(dt)$ converges weakly to $dt/2$, and the lemma is proved for any continuous b. But the continuity of b follows from (4.3.1) and the Kolmogorov lemma (Theorem I.1.8 of [10]). $\square$

Remark 4.3.6. For $b \equiv 1$, if we factorise $W^{\nearrow}(dt)$ into $W_x^{\nearrow}(dt)dx$ as in (3.1.3), a well known approximation of the local time by the number of upcrossings (Theorem VI.1.10 of [10]) states that

$$\lim_{a \to 0} a \int 1_B(t, \Upsilon_t) W_x^{\nearrow}(dt) = \frac{1}{2} L_t^x,$$

and the convergence (4.3.5) for $b \equiv 1$ is the integrated form of this result.

Another step in the proof of Theorem 4.3.3 consists in estimating double integrals

$$A_{rt} = \int_r^t (b_s - b_r) dW_s.$$

Lemma 4.3.7. *For any fixed* $0 < \gamma < 1$*, one has*

$$|b_t - b_r| \leq K(t-r)^{\gamma/2}, \qquad |A_{rt}| \leq K(t-r)^{\gamma}$$

for any $0 \leq r \leq t \leq 1$*, and for some finite variable* K*.*

Remark 4.3.8. If b were assumed to be a semimartingale, this type of result can be used to check that the process (W, b) with its Lévy area has finite p-variation for $p > 2$ (see [5]). The multiplicative property satisfied by rough paths is here written as

$$A_{rt} = A_{rs} + A_{st} + (b_s - b_r)(W_t - W_s) \tag{4.3.9}$$

for $r \leq s \leq t$.

Proof of Lemma 4.3.7. The first estimate is classical from the Kolmogorov lemma and relies on (4.3.1). In particular, from (4.3.9),

$$\left| A_{rt} - A_{rs} - A_{st} \right| \leq K_0 (t-r)^{\gamma} \tag{4.3.10}$$

for $r \leq s \leq t$ and for some finite K_0. On the other hand, the variable

$$A'_{rt} = \sup_{r \leq u \leq t} \Big| \int_r^u (b_s - b_r) dW_s \Big|$$

is of order $t - r$ in L^p (apply Doob's inequality), so

$$\mathbb{P}\big[A'_{rt} \geq (t-r)^\gamma\big] \leq C(t-r)^{p(1-\gamma)}.$$

We apply this inequality to $r = t_k^n = k2^{-n}$, $t = t_{k+1}^n$, and obtain

$$\sum_n \sum_{k=0}^{2^n-1} \mathbb{P}\big[A'_{t_k^n, t_{k+1}^n} \geq (t_{k+1}^n - t_k^n)^\gamma\big] \leq C \sum_n 2^n 2^{-pn(1-\gamma)} < \infty$$

if p is chosen large enough. Thus the Borel-Cantelli lemma shows that

$$A'_{t_k^n, t_{k+1}^n} \leq K_1 (t_{k+1}^n - t_k^n)^\gamma \tag{4.3.11}$$

for some finite K_1. Now, for $0 < r < t < 1$, let $n = n(r,t)$ be the maximal integer such that

$$t_{k-1}^n \leq r \leq t_k^n \leq t \leq t_{k+1}^n \tag{4.3.12}$$

for some k. Then

$$2^{-n-1} \leq t - r \leq 2^{-n+1} \tag{4.3.13}$$

because if it is smaller, then $n+1$ satisfies (4.3.12). The almost additive property (4.3.10) shows that

$$\begin{aligned} A_{rt} &= A_{rt_k^n} + A_{t_k^n t} + O((t-r)^\gamma) \\ &= A_{t_{k-1}^n t_k^n} - A_{t_{k-1}^n r} + A_{t_k^n t} + O(2^{-n\gamma}), \end{aligned}$$

so

$$|A_{rt}| \leq 2A'_{t_{k-1}^n t_k^n} + A'_{t_k^n t_{k+1}^n} + K_2 2^{-n\gamma} \leq K(t-r)^\gamma$$

from (4.3.11) and (4.3.13). □

Proof of Theorem 4.3.3 when $\beta \equiv 0$. By using the time σ_t defined in (4.3.2), the bijection $t \mapsto \sigma_t$ transforms $1_{\{\eta_t \geq a\}} W^{\nearrow}(dt)$ into $1_{\{\eta_t^\dagger \geq a\}} W^{\swarrow}(dt)$. Thus we can write

$$\Psi_B^{\not\,}(\rho) = \int 1_{\{0 \leq t \leq \sigma_t \leq 1\}} 1_{\{\eta_t \geq a\}} (\rho_{\sigma_t} - \rho_t) W^{\nearrow}(dt) + O(a)$$

because all the beginnings and ends of excursions of $\Psi_B^{\not\,}$ are taken into account in the integral of the right side, except a small part of the first and last hitting times. On the other hand, notice that Lemma 4.3.7 and (4.2.3) imply that

$$\rho_t - \rho_r = b_r (W_t - W_r) + O((t-r)^\gamma) \tag{4.3.14}$$

for $0 < \gamma < 1$. Thus

$$\Big|\Psi^{\swarrow}_B(\rho) - a\int 1_{\{0\le t\le\sigma_t\le 1\}}1_{\{\eta_t\ge a\}}b_t W^{\nearrow}(dt)\Big| \le K\int_0^1 ((\sigma_t - t)^\gamma \wedge 1)W^{\nearrow}(dt) + C\,a. \quad (4.3.15)$$

The integral in the left-hand side is close to the integral of the left-hand side of (4.3.5). For the right-hand side, notice that $(\sigma_t - t)^\gamma \wedge 1$ converges almost surely to 0 as $a \downarrow 0$, and is bounded by $\zeta_t^\gamma \wedge 1$. Moreover,

$$\mathbb{E}\int_0^1 (\zeta_t^\gamma \wedge 1)W^{\nearrow}(dt) = \int (\zeta(\theta)^\gamma \wedge 1)\mathcal{I}^{\uparrow}(d\theta) < \infty$$

for $\gamma > 1/2$, so

$$\int_0^1 (\zeta_t^\gamma \wedge 1)W^{\nearrow}(dt) < \infty \quad (4.3.16)$$

almost surely. Thus we deduce from the dominated convergence theorem that the right-hand side of (4.3.15) converges to 0. This completes the proof of the almost sure convergence of $\Psi^{\swarrow}_B(\rho)$. □

Proof of Theorem 4.3.3 in the general case. We have already considered the case $\beta \equiv 0$, so we can now suppose $\alpha \equiv b \equiv 0$. We use the filtration $\mathcal{F}'_t$ generated by $(W_s; 0 \le s \le 1)$ and $(V_s; 0 \le s \le t)$. Then $W^{\swarrow}_t(B)$ is $\mathcal{F}'_0$-measurable, the process V_t is a $\mathcal{F}'_t$-Wiener process, and

$$\Psi^{\swarrow}_B(\rho) = \int \rho_t dW^{\swarrow}_t(B) = \rho_1 W^{\swarrow}_1(B) - \int W^{\swarrow}_t(B)(\beta_t, dV_t)$$

where the stochastic integral is understood in the filtration $(\mathcal{F}'_t)$. Similarly to the previous case, we can prove that $W^{\swarrow}_t(B) = \Psi^{\swarrow}_B(1_{[0,t]})$ converges almost surely to 0, uniformly in t, so we deduce the convergence in probability of $\Psi^{\swarrow}_B(\rho)$ to 0. □

4.4 Stratonovich integrals

In this subsection, we consider the measures $W^{\uparrow}(dt)$ and $W^{\downarrow}(dt)$ on $\mathbb{R}$, and define

$$\begin{aligned}\Psi^{\uparrow}_B(\rho) &= \int \rho_t\Big(1_B(t,\Upsilon_t)W^{\nearrow}(dt) - 1_B(t-\zeta_t^{\dagger},\Upsilon_t^{\dagger})W^{\nwarrow}(dt)\Big) \\ &= \int_0^1 \rho_t dW^{\uparrow}_t(B)\end{aligned} \quad (4.4.1)$$

with

$$W^{\uparrow}_t(B) = \Psi^{\uparrow}_B(1_{[0,t]}), \quad (4.4.2)$$

and similarly

$$\Psi^{\downarrow}_B(\rho) = \int \rho_t\Big(1_B(t-\zeta_t^{\dagger},\Upsilon_t^{\dagger})W^{\swarrow}(dt) - 1_B(t,\Upsilon_t)W^{\searrow}(dt)\Big).$$

Actually, we focus our study on $\Psi_B^{\uparrow}$ since the other one is similar. Notice that

$$\Psi_B^{\uparrow}(\rho) = \Psi_B^{\leftarrow}(\rho) - \Psi_B^{\swarrow}(\rho), \tag{4.4.3}$$

so we deduce from Theorems 4.2.2 and 4.3.3 that for a smooth Itô process and for height-based approximations, it converges in probability to the Stratonovich integral. Our aim is to prove that the convergence in probability actually holds for any approximation and that it is almost sure when $\beta \equiv 0$ or for height-based approximations. Notice that if $\Psi_B^{\uparrow}(\rho)$ converges in probability as $B \uparrow \Theta_1^{\uparrow}$, the limit is necessarily the Stratonovich integral (since this is the limit on a subsequence).

The definition (4.4.1) of $\Psi_B^{\uparrow}$ involves the beginnings and ends of the same excursions, so we can write

$$\Psi_B^{\uparrow}(\rho) = \int 1_B(t, \Upsilon_t)\big(\rho_t - \rho_{t+\zeta_t}\big)W^{\nearrow}(dt). \tag{4.4.4}$$

Thus if

$$\int \big|\rho_t - \rho_{t+\zeta_t}\big| W^{\nearrow}(dt) < \infty \tag{4.4.5}$$

almost surely, then $\Psi_B^{\uparrow}(\rho)$ converges almost surely to

$$\int_0^1 \rho_t \circ dW_t = \int \big(\rho_t - \rho_{t+\zeta_t}\big)W^{\nearrow}(dt). \tag{4.4.6}$$

Theorem 4.4.7. *Let ρ be smooth Itô process satisfying* (4.2.3) *and* (4.3.1)*. If $\beta \equiv 0$, the integrability condition* (4.4.5) *holds true, so the Stratonovich integral is given by* (4.4.6)*, and $\Psi_B^{\uparrow}(\rho)$ converges almost surely to it.*

Proof. Since ρ is bounded, in order to prove (4.4.5), we can neglect the first and last incomplete excursions and only prove that

$$\int \big|\rho_{t+\zeta_t} - \rho_t\big| 1_{\{0 \le t < t+\zeta_t \le 1\}} W^{\nearrow}(dt) < \infty. \tag{4.4.8}$$

We deduce from (4.3.14) that

$$\big|\rho_{t+\zeta_t} - \rho_t\big| \le K(\zeta_t^{\gamma} \wedge 1),$$

so (4.3.16) implies (4.4.8). The other statements of the theorem follow from the above discussion. $\square$

Remark 4.4.9. The approximation (4.4.1) of the Stratonovich integral is written as the integral with respect to the anticipating process with finite variation $W_t^{\uparrow}(B)$. For instance, for the height-based approximations of (4.1.16), the path $W_t^{\uparrow}(B(a))$ is depicted in Figure 5, and one has $W_t - a \le W_t^{\uparrow}(B(a)) \le W_t$. For approximations associated to subdivisions $\Delta = (t_j)$ of $[0,1]$ (Example 4.1.18), one has

$$W_t^{\uparrow}(B(\Delta)) = \sum_j 1_{[t_j,t_{j+1})}(t)\Big(\inf_{[t_j,t]} W \vee \inf_{[t,t_{j+1}]} W\Big).$$

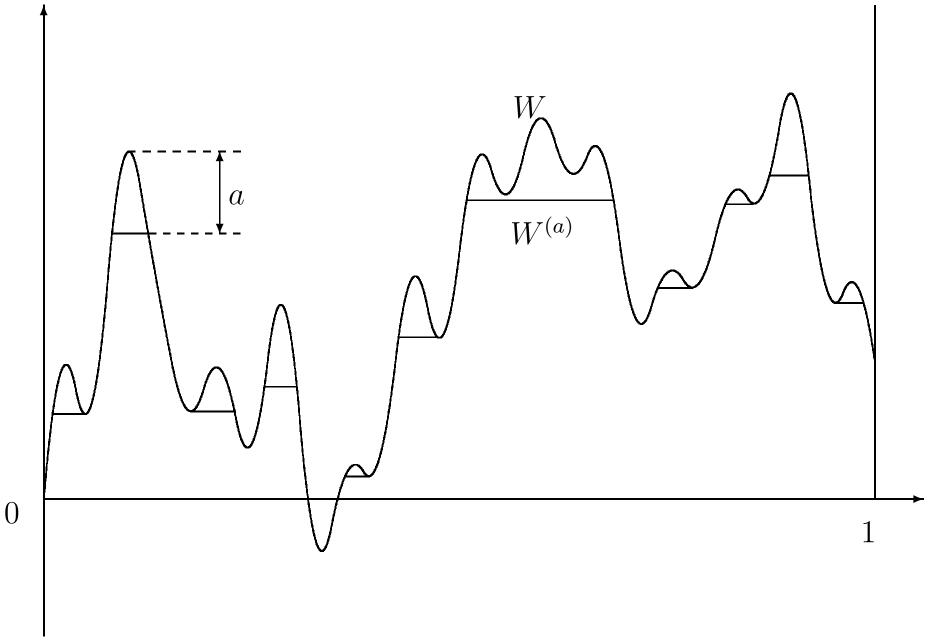

Figure 5: Paths W_t and $W_t^{(a)} := W_t^{\uparrow}(B(a))$

Remark 4.4.10. We can deduce from (4.4.3), Theorems 4.3.3 and 4.4.7 that for height-based approximations and when $\beta \equiv 0$, the approximations $\Psi_B^{\leftarrow}(\rho)$ converge almost surely to the backward integral. Similarly, by using

$$\Psi_B^{\rightarrow}(\rho) = \Psi_B^{\downarrow}(\rho) - \Psi_B^{\swarrow}(\rho),$$

we check that $\Psi_B^{\rightarrow}(\rho)$ converges almost surely to the Itô integral.

Theorem 4.4.11. *If ρ is a smooth Itô process satisfying* (4.2.3) *and* (4.3.1)*, the convergence of $\Psi_B^{\uparrow}(\rho)$ to the Stratonovich integral holds in probability.*

Proof. The case $\beta \equiv 0$ has already been studied in Theorem 4.4.7, so we now suppose $\alpha \equiv b \equiv 0$ (in particular, Itô, backward and Stratonovich integrals coincide). We use as in Theorem 4.3.3 the enlarged filtration $\mathcal{F}'_t$ generated by $(W_s, 0 \le s < \infty; V_s, 0 \le s \le t)$. The process ρ_t is a semimartingale for this filtration, and the process $W_t^{\uparrow}(B)$ given by (4.4.2) is $\mathcal{F}'_0$-measurable. We can write

$$\Psi_B^{\uparrow}(\rho) = \int_0^1 \rho_t dW_t^{\uparrow}(B) = \int_0^1 (W_1^{\uparrow}(B) - W_t^{\uparrow}(B))(\beta_t, dV_t). \tag{4.4.12}$$

We have proved in Theorem 4.4.7 that $W_t^{\uparrow}(B)$ converges almost surely to W_t. Moreover,

$$\inf_{[0,t]} W \le W_t^{\uparrow}(B) \le W_t - \inf_{[0,t]} W,$$

so

$$\left|W_t^{\uparrow}(B)\right| \leq 2 \sup_{[0,t]} |W|.$$

This estimation is sufficient to deduce the convergence in probability of $\Psi^{\uparrow}_B(\rho)$ in the form (4.4.12). ☐

We now show that in the case of height-based approximations, almost sure convergence to the Stratonovich integral holds even when $\beta \not\equiv 0$.

Theorem 4.4.13. *If ρ is a smooth Itô process satisfying* (4.2.3) *and* (4.3.1)*, then $\Psi^{\uparrow}_B(\rho)$ converges almost surely for height based approximations* (4.1.16) *to the Stratonovich integral.*

Proof. The result has already been proved for $\beta \equiv 0$ in Theorem 4.4.7, so we now suppose $\alpha \equiv b \equiv 0$, and consider the enlarged filtration $\mathcal{F}'_t$. We use Lemma 3.1.8 to write (4.4.4) in the form

$$\Psi^{\uparrow}_{B(a)}(\rho) = \int_a^{\infty} \sum_{t:\Upsilon_t>0,\eta_t=b} \left(\rho_t - \rho_{t+\zeta_t}\right) db + O(a)$$

where the '$O(a)$' involves the small excursions straddling time 0 or 1. Thus the theorem will proved if we check that

$$\int_{0+} \left| \sum_{t:\Upsilon_t>0,\eta_t=a} \left(\rho_t - \rho_{t+\zeta_t}\right) \right| da < \infty \tag{4.4.14}$$

almost surely. But

$$\sum_{t:\Upsilon_t>0,\eta_t=a} \left(\rho_t - \rho_{t+\zeta_t}\right) = -\int_0^1 \left(\sum_{t:\Upsilon_t>0,\eta_t=a} 1_{[t,t+\zeta_t]}(s) \right) (\beta_s, dV_s)$$

is bounded in L^1 as $a \to 0$ because the sum in the right-hand side is 0 or 1, so (4.4.14) holds true. ☐

4.5 Summary

We summarise in a table the results obtained in this section about convergence to the stochastic integrals and the quadratic covariation in various frameworks, when ρ is a smooth semimartingale satisfying (4.2.3) and (4.3.1), and either for any approximating sequence, or for the height-based approximations $B(a)$ of (4.1.16).

	Any B		$B = B(a)$	
	Any β	$\beta \equiv 0$	Any β	$\beta \equiv 0$
$\Psi^{\rightarrow}_B(\rho) \to$ Itô	(w) if B good		(P)	(a.s.)
$\Psi^{\leftarrow}_B(\rho) \to$ backward	(w) if B good		(P)	(a.s.)
$\Psi^{\nearrow}_B(\rho) \to$ covariation			(P)	(a.s.)
$\Psi^{\uparrow}_B(\rho) \to$ Stratonovich	(P)	(a.s.)	(a.s.)	

The weak L^2 convergence ((w) in the table) holds if $\Psi_B^{\rightarrow}(\rho)$ is bounded in L^2 (for the Itô integral) or when $\Psi_B^{\leftarrow}(1)$ is bounded in L^2 (for the backward integral).

5 Anticipating calculus

In this section, we explain how techniques of previous section can be applied to study $\Psi_B^{\rightarrow}(\rho)$, ... for some anticipating processes ρ and to construct anticipating integrals $\Psi^{\rightarrow}(\rho)$, ... We also obtain a duality between the transformation $\mathcal{D}$ of the Wiener path and an operator Φ; this duality is similar to the duality between the Malliavin derivative and the Skorohod integral (see for instance [6]).

5.1 Anticipating integrals

We again consider the space generated by the Wiener process (W_t, V_t), and we want to integrate with respect to W processes ρ_t, $0 \le t \le 1$, which are not predictable for the Wiener filtration. We say that the bounded process ρ_t is in the domain of $\Psi^{\rightarrow}$ with integral $\Psi^{\rightarrow}(\rho)$ if $\Psi_B^{\rightarrow}(\rho)$ defined in (4.1.1) converges in probability to $\Psi^{\rightarrow}(\rho)$ for height-based approximations $B = B(a)$ of (4.1.16), $a \downarrow 0$. We can give a similar definition for $\Psi^{\leftarrow}$, $\Psi^{\uparrow}$, $\Psi^{\downarrow}$. In view of the predictable case, we can say that $\Psi^{\rightarrow}(\rho)$ is a forward integral, that $\Psi^{\leftarrow}(\rho)$ is a backward integral, and that $\Psi^{\uparrow}(\rho)$ and $\Psi^{\downarrow}(\rho)$ are symmetric integrals; these two last integrals were equal in the predictable case and are also equal in the examples below.

Remark 5.1.1. One can consider other definitions, by requiring for instance the weak L^2 convergence instead of the convergence in probability.

Example 5.1.2. In the case $d\rho_t = \alpha_t dt$, it is not necessary to assume that α is adapted. Even in the anticipative case, one can write

$$\Psi_B^{\rightarrow}(\rho) = \int_0^1 \rho_t dW_t^{\rightarrow}(B) = \rho_1 W_1^{\rightarrow}(B) - \int_0^1 W_t^{\rightarrow}(B)\alpha_t dt$$

and similarly for the other integrals, so the four types of integrals exist and coincide with

$$\Psi(\rho) = \rho_1 W_1 - \int_0^1 W_t \alpha_t dt.$$

A similar result holds in the case $d\rho_t = \beta_t dV_t$, where β_t is non anticipating with respect to V but may be anticipating with respect to W; this means that we use (like previously) the filtration $\mathcal{F}'_t$ generated by $(W_s; s \ge 0)$ and $(V_s; 0 \le s \le t)$.

Example 5.1.3. Suppose that $\rho_t = \rho'_{t+\varepsilon}$, where ρ'_t is an Itô process (4.2.3). Then we divide the time interval into intervals of length ε. On each of these intervals, ρ is an integral with respect to the Wiener increments on the next interval, and these increments can be viewed as an independent process. Thus we are reduced to Example 5.1.2, and all the integrals coincide with

$$\Psi(\rho) = \rho_1 W_1 - \int_0^1 W_s d\rho_s$$

where the integral is computed for the filtration $\mathcal{F}_{t+\varepsilon}$.

Example 5.1.4. If

$$|\rho_t - \rho_s| \le K\,|t-s|^\gamma$$

for some finite variable K and some $1/2 < \gamma \le 1$, then $\rho_{t+\zeta_t} - \rho_t$ is integrable with respect to $W^{\nearrow}$ ((4.4.5) holds true), and $\Psi^{\uparrow}(\rho)$ can be defined by the right-hand side of (4.4.6). We can approach it by

$$\Psi^{\uparrow}_{B(\Delta)}(\rho) = \sum_k \Big(\int_{\inf_{[t_j,t_{j+1}]} W}^{W_{t_{j+1}}} \rho_{g^x_{t_{j+1}}}\, dx - \int_{\inf_{[t_j,t_{j+1}]} W}^{W_{t_j}} \rho_{d^x_{t_j}}\, dx \Big)$$

for a subdivision $\Delta = (t_j)$ of $[0,1]$. If we compare with Riemann sums, it appears that

$$\begin{aligned} &\Big| \Psi^{\uparrow}_{B(\Delta)}(\rho) - \sum_j \rho_{t_j}(W_{t_{j+1}} - W_{t_j}) \Big| \\ &\qquad \le \sum_j \sup_{t_j \le t \le t_{j+1}} |\rho_t - \rho_{t_j}| \Big(W_{t_j} + W_{t_{j+1}} - 2 \inf_{[t_j,t_{j+1}]} W \Big). \end{aligned}$$

Estimates on the modulus of continuity of W and ρ show that this expression converges to 0, so $\Psi^{\uparrow}(\rho)$ is the limit of Riemann sums and therefore coincides with the Young integral (described in §3.3.2 of [5]). On the other hand, the study of $\Psi^{\nearrow}_B(\rho)$ for $B = B(a)$ shows that it is dominated by

$$\int 1_{\{\eta_t \ge a\}} (\sigma_t - t)^\gamma W^{\nearrow}(dt).$$

Recall (Example 3.3.8) that

$$\int 1_{\{\eta(\theta)\ge a\}} T_a(\theta) \mathcal{I}^{\uparrow}_1(d\theta) = \frac{a^2}{3} \mathcal{I}^{\uparrow}_1 \big\{ \eta(\theta) \ge a \big\},$$

so

$$\int 1_{\{\eta(\theta)\ge a\}} T_a(\theta)^\gamma \mathcal{I}^{\uparrow}_1(d\theta) = O(a^{2\gamma})\, \mathcal{I}^{\uparrow}_1 \big\{ \eta(\theta) \ge a \big\} = O(a^{2\gamma-1}).$$

Thus

$$\mathbb{E} \int_0^1 1_{\{\eta_t \ge a\}} (\sigma_t - t)^\gamma W^{\nearrow}(dt) = O(a^{2\gamma-1})$$

converges to 0, so $\Psi^{\nearrow}_B(\rho)$ (and also $\Psi^{\searrow}_B(\rho)$) converges to 0, and the four types of integrals coincide.

Example 5.1.5. In the one-dimensional case, consider $\rho'_t = f(\rho_t, .)$, where $f(\rho, \omega)$ is a random function which is C_b^2 with respect to ρ, and ρ_t is a smooth Itô process (4.2.3). Then

$$\left|\rho'_{t+\zeta_t} - \rho'_t\right| \le C\left|\rho_{t+\zeta_t} - \rho_t\right|,$$

so ρ' is like ρ in the domain of $\Psi^\uparrow$, and $\Psi^\uparrow(\rho')$ is again given by (4.4.6). On the other hand, from Taylor's formula,

$$\begin{aligned}\rho'_{\sigma_t} - \rho'_t &= f'(\rho_t)\big(\rho_{\sigma_t} - \rho_t\big) + O\big(\left|\rho_{\sigma_t} - \rho_t\right|^2\big) \\ &= ab_t f'(\rho_t) + O\big((\sigma_t - t)^\gamma\big)\end{aligned}$$

for $0 < \gamma < 1$ (recall (4.3.14)). One can deduce at the limit (Lemma 4.3.4 is valid for anticipating processes) that

$$\Psi^{\nearrow}(\rho') = \Psi^{\searrow}(\rho') = \frac{1}{2}\int_0^t f'(\rho_s) b_s ds.$$

In particular, $\Psi^\uparrow(\rho')$ and $\Psi^\downarrow(\rho')$ again coincide.

5.2 A duality property

We now describe for the multidimensional Wiener process $(W_t^1, \dots, W_t^d)$ the relation between our integrals and the adjoint Φ of $\mathcal{D} = (\mathcal{D}^1, \dots, \mathcal{D}^d)$ in L^2, where each operator $\mathcal{D}^k$ is defined similarly to (4.1.3). If we compare this calculus with standard anticipating calculus, the operator Φ can be viewed as an analogue of the Skorohod integral, so our problem is similar to the description of some anticipating integrals (which are constructed as some limits) by means of the Skorohod integral, as in §3.1 of [6].

Let $\mathcal{S}$ be the class of simple functionals $F = f(W_{t_1}, \dots, W_{t_n})$ with f bounded and Lipschitz; in particular,

$$\int \left|F \circ \mathcal{E}^+_{t,\theta} - F \circ \mathcal{E}^+_{t,\theta^\star}\right| \mathcal{J}_k^\uparrow(d\theta) \le C \sum_j \int \theta_k(t_{j+1} - t)\mathcal{I}_k^\uparrow(d\theta) = C\,n$$

for any $1 \le k \le d$ and for $t_j < t < t_{j+1}$, so $\mathcal{D}_t F$ is bounded for these functionals. We define the domain of Φ to be the set of processes ρ_t in $L^2(\mathbb{R}_+ \times \Omega; \mathbb{R}^d)$ such that

$$\mathbb{E}\int (\rho_t, \mathcal{D}_t F)\, dt \le C\,\mathbb{E}\big[F^2\big]^{1/2}$$

for any $F \in \mathcal{S}$, and for some $C \ge 0$. Then $\Phi(\rho)$ is defined in L^2 by

$$\mathbb{E}\int (\rho_t, \mathcal{D}_t F)\, dt = \mathbb{E}\big[F\,\Phi(\rho)\big]$$

for any $F \in \mathcal{S}$.

In particular, $\Phi(\rho)$ is centred. Since $\mathcal{S}$ is dense in L^2, it is classical to verify that $\Phi(\rho)$ is uniquely defined by this relation. Moreover, Φ is a closed operator.

We use the notation

$$\rho^-_{k,t} = \rho_{k,t} \circ \mathcal{E}^-_{k,t}$$

where $\rho_{k,t}$ is the k^{th} component of ρ_t, and $\mathcal{E}^-_{k,t}$ removes the excursion of the k^{th} Wiener component starting at time t (see §3.4 for $k = 1$); this excursion is denoted by $\Upsilon_{k,t}$. In particular, if ρ is predictable, then $\rho^- = \rho$.

Theorem 5.2.1. *Let ρ be a bounded process on $[0, 1]$ which is extended by 0 out of this interval, and consider height-based approximations $B = B(a)$ of (4.1.16). Then ρ is in the domain of Φ if and only if*

$$\Psi^{\rightarrow}_{B(a)}(\rho^-) := \sum_{k=1}^{d} \int \rho^-_{k,t} 1_{B(a)}(t, \Upsilon_{k,t}) W^{\rightarrow}_k(dt)$$

converges for the weak topology of L^2 induced by $\mathcal{S}$, and $\Phi(\rho)$ is its limit.

Remark 5.2.2. Roughly speaking, by applying §5.1, the theorem says that $\Phi(\rho) = \Psi^{\rightarrow}(\rho^-)$.

Proof. We deduce from Theorem 3.2.4 that

$$\mathbb{E}\Big[F \int 1_B(t, \Upsilon_{k,t}) \rho^-_{k,t} W^{\nearrow}_k(dt)\Big] = \mathbb{E} \iint_B \big(F \circ \mathcal{E}^+_{t,\theta}\big) \rho_{k,t} \mathcal{I}^{\uparrow}_k(d\theta) dt.$$

If B is symmetric, we have a similar relation with $W^{\nearrow}_k(dt)$ and $\mathcal{E}^+_{t,\theta}$ replaced by $W^{\searrow}_k(dt)$ and $\mathcal{E}^+_{t,\theta^\star_{(k)}}$, so by taking the difference between these two relations, we obtain for $F \in \mathcal{S}$

$$\mathbb{E}\Big[F \int 1_B(t, \Upsilon_{k,t}) \rho^-_{k,t} W^{\rightarrow}_k(dt)\Big] = \mathbb{E} \iint_B \big(F \circ \mathcal{E}^+_{t,\theta} - F \circ \mathcal{E}^+_{t,\theta^\star_{(k)}}\big) \rho_{k,t} \mathcal{I}^{\uparrow}_k(d\theta) dt.$$

Thus

$$\lim_{a \downarrow 0} \mathbb{E}\big[F\, \Psi^{\rightarrow}_{B(a)}(\rho^-)\big] = \mathbb{E} \int (\mathcal{D}_t F, \rho_t)\, dt$$

and the theorem is proved. □

Example 5.2.3. If ρ is predictable, then $\rho^- = \rho$ and both $\Phi(\rho)$ and $\Psi^{\rightarrow}(\rho)$ are the Itô integral.

Example 5.2.4. In the one-dimensional case, consider $\rho_t = g(W_1)\, 1_{[0,1]}(t)$ for a bounded Lipschitz function g. It is easy to check that

$$\Psi^{\rightarrow}(\rho) = g(W_1)\, W_1.$$

We now prove that ρ is in the domain of Φ and that

$$\Phi(\rho) = g(W_1) W_1 + \int_{W[0,1]} \Gamma_x dx$$

with

$$\Gamma_x = \lim_{a\to 0} \sum_{t:W_t=x} 1_{B(a)}(t,\Upsilon_t)\,\mathrm{sgn}(\Upsilon_t)\big(g(W_1)-g(W_{1+\zeta_t})\big)$$

strongly in L^2.

Proof. We already know $\Psi^{\rightarrow}(\rho)$ so we have to study $\Psi^{\rightarrow}(\rho-\rho^-)$. We have

$$\rho_t^- = 1_{[0,1]}(t)g(W_{1+\zeta_t}),$$

so

$$\begin{aligned}\Psi_B^{\rightarrow}(\rho-\rho^-) &= \int_0^1 1_B(t,\Upsilon_t)\big(g(W_1)-g(W_{1+\zeta_t})\big)W^{\rightarrow}(dt)\\ &= \int \sum_{t:W_t=x,\zeta_t>0} 1_B(t,\Upsilon_t)\,\mathrm{sgn}(\Upsilon_t)\big(g(W_1)-g(W_{1+\zeta_t})dx.\end{aligned}$$

If we separate the last excursion from the others, we get

$$\begin{aligned}\Psi_B^{\rightarrow}(\rho-\rho^-) = &\int_{W[0,1]} 1_B(g_1^x,\Upsilon_{g_1^x})\,\mathrm{sgn}(\Upsilon_{g_1^x})\big(g(W_1)-g(W_{1+\zeta_{g_1^x}})\big)dx\\ &+\int \sum_{t:W_t=x,\zeta_t>0} 1_B(t,\Upsilon_t)1_{\{\zeta_t<1-t\}}\,\mathrm{sgn}(\Upsilon_t)\big(g(W_1)-g(W_{1+\zeta_t})\big)dx.\end{aligned}$$

The first term is easily studied as $B\uparrow\mathbb{R}_+\times\Theta$, and leads to the last excursion in Γ_x. Let us write the second term as $\int\Gamma_x^B dx$. Then Γ_x^B is a sum of orthogonal variables because the signs of the excursions are independent, and independent from the absolute excursions and the future $(W_t;t\ge 1)$. Thus

$$\begin{aligned}\mathbb{E}[(\Gamma_x^B)^2] &= \mathbb{E}\sum_{t:W_t=x,\zeta_t>0} 1_B(t,\Upsilon_t)1_{\{\zeta_t<1-t\}}\big(g(W_1)-g(W_{1+\zeta_t})\big)^2\\ &\le C\mathbb{E}\sum_{t:W_t=x,\zeta_t>0} 1_B(t,\Upsilon_t)1_{\{\zeta_t<1-t\}}\zeta_t\\ &\le C\mathbb{P}[T_x<1]\end{aligned}$$

where in the second line we apply a conditional expectation given $\mathcal{F}_1$, and in the third line we notice that the total length $\sum\zeta_t$ of excursions is bounded by 1, and that there is no term if the process does not hit x before time 1. Moreover, if we consider $B=B(a_n)$ for $a_n\downarrow 0$ and if Γ_x^n are the associated variables, then $(\Gamma_x^{n+1}-\Gamma_x^n;n\ge 0)$ are orthogonal (for the above reason of independent signs), so Γ_x^n converges strongly in L^2, and it is not difficult to deduce that $\int\Gamma_x^n dx$ also converges ($\|\Gamma_x^n-\Gamma_x^\infty\|_2$ converges to 0 for any x, and is dominated by $\mathbb{P}[T_x<1]$ which is integrable, so we apply the dominated convergence theorem). We deduce the desired result. ∎

Example 5.2.5. Returning to the multidimensional case, let $\mathcal{P}$ be the set of processes $\rho_t = (\rho_{k,t}; 1 \le k \le d)$ of the form

$$\rho_{k,t} = \sum_{j=0}^{n-1} g_{k,j}(W^k_{t_0}, \dots, W^k_{t_n}; W^1, \dots, W^{k-1}, W^{k+1}, \dots, W^d) 1_{[t_j, t_{j+1}]}$$

where $(t_0, \dots, t_n)$ is a subdivision of $[0,1]$ and $g_{k,j}$ is bounded and Lipschitz with respect to its $(n+1)$ first arguments. By proceeding as in previous example, we can prove that $\mathcal{P}$ is included in the domain of Φ. In particular, the domain of Φ is dense in the space of square integrable processes.

We can define the adjoint $\overline{\mathcal{D}}$ of Φ as follows. The domain of $\overline{\mathcal{D}}$ is the set of variables $F \in L^2$ such that

$$\mathbb{E}\big[F\,\Phi(\rho)\big] \le C\,\mathbb{E}\Big[\int |\rho_t|^2 dt\Big]^{1/2}$$

for any ρ in the domain of Φ, and for some $C \ge 0$. If F is in this domain, $\overline{\mathcal{D}}F$ is defined in L^2 by

$$\mathbb{E}\int (\overline{\mathcal{D}}_t F, \rho_t) dt = \mathbb{E}\big[F\,\Phi(\rho)\big] \tag{5.2.6}$$

for any ρ in the domain of Φ. Then $\overline{\mathcal{D}}$ is a closed operator. Let us compare $\mathcal{D}$ and $\overline{\mathcal{D}}$; it is clear that they coincide on $\mathcal{S}$.

Theorem 5.2.7. *The operator $\overline{\mathcal{D}}$ is the closure in L^2 of the restriction of $\mathcal{D}$ to $\mathcal{S}$.*

Proof. The domain of Φ is dense in L^2 (Example 5.2.5), so $\overline{\mathcal{D}}$ is closed, and it is sufficient to prove that $\mathcal{S}$ is dense in the domain of $\overline{\mathcal{D}}$ endowed with the norm

$$\|F\| = \mathbb{E}\Big[F^2 + \int |\overline{\mathcal{D}}_t F|^2 dt\Big]^{1/2}.$$

Let G be in the domain of $\overline{\mathcal{D}}$ and orthogonal to $\mathcal{S}$; we want to check that $G = 0$. For any $F \in \mathcal{S}$, we have

$$\mathbb{E}\Big[FG + \int (\mathcal{D}_t F, \overline{\mathcal{D}}_t G) dt\Big] = \mathbb{E}\Big[FG + \int (\overline{\mathcal{D}}_t F, \overline{\mathcal{D}}_t G) dt\Big] = 0,$$

so $\overline{\mathcal{D}}G$ is in the domain of Φ, and

$$G + \Phi(\overline{\mathcal{D}}G) = 0.$$

Thus

$$\begin{aligned} 0 &= \mathbb{E}\Big[G^2 + \Phi(\overline{\mathcal{D}}G)^2 + 2G\Phi(\overline{\mathcal{D}}G)\Big] \\ &= \mathbb{E}\Big[G^2 + \Phi(\overline{\mathcal{D}}G)^2 + 2\int |\overline{\mathcal{D}}_t G|^2 dt\Big] \end{aligned}$$

from (5.2.6), and $G = 0$. ∎

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