Electronic Journal of Probability

Vol. 14 (2009), Paper no. 32, pages 895–911.

Journal URL
http://www.math.washington.edu/~ejpecp/

Uniqueness of the stationary distribution and stabilizability in Zhang's sandpile model

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Abstract

We show that Zhang's sandpile model $(N, [a, b])$ on N sites and with uniform additions on $[a, b]$ has a unique stationary measure for all $0 \leq a < b \leq 1$. This generalizes earlier results of [6] where this was shown in some special cases.
We define the infinite volume Zhang's sandpile model in dimension $d \geq 1$, in which topplings occur according to a Markov toppling process, and we study the stabilizability of initial configurations chosen according to some measure μ. We show that for a stationary ergodic measure μ with density ρ, for all $\rho < \frac{1}{2}$, μ is stabilizable; for all $\rho \geq 1$, μ is not stabilizable; for $\frac{1}{2} \leq \rho < 1$, when ρ is near to $\frac{1}{2}$ or 1, both possibilities can occur.

Key words: Sandpile, stationary distribution, coupling, critical density, stabilizability.

AMS 2000 Subject Classification: Primary 60J27, 60F05, 60B10, 82B20.

Submitted to EJP on June 20, 2008, final version accepted April 6, 2009.

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1 Introduction

Zhang's sandpile model [14] is a variant of the more common abelian sandpile model [3], which was introduced in [1] as a toy model to study self-organized criticality. We define the model more precisely in the next section, but we start here with an informal discussion.

Zhang's model differs from the abelian sandpile model on a finite grid Λ in the following respects: The configuration space is $[0,1)^\Lambda$, rather than $\{0,1,\ldots,2d-1\}^\Lambda$. The model evolves, like the abelian sandpile model, in discrete time through additions and subsequent stabilization through topplings of unstable sites. However, in Zhang's model, an addition consists of a continuous amount, uniformly distributed on $[a,b] \subseteq [0,1]$, rather than one 'grain'. Furthermore, in a Zhang toppling of an unstable site, the entire height of this site is distributed equally among the neighbors, whereas in the abelian sandpile model one grain moves to each neighbor irrespective of the height of the toppling site.

Since the result of a toppling depends on the height of the toppling site, Zhang's model is not abelian. This means that 'stabilization through topplings' is not immediately well-defined. However, as pointed out in [6], when we work on the line, and as long as there are no two neighbouring unstable sites, topplings are abelian. When the initial configuration is stable, we will only encounter realizations with no two neighbouring unstable sites, and we have - a fortiori - that the model is abelian.

In [6], the following main results, in dimension 1, were obtained. Uniqueness of the stationary measure was proved for a number of special cases: (1) $a \geq \frac{1}{2}$; (2) $N = 1$, and (3) $[a,b] = [0,1]$. For the model on one site with $a = 0$, an explicit expression for the stationary height distribution was obtained. Furthermore, the existence of so called 'quasi-units' was proved for $a \geq 1/2$, that is, in the limit of the number of sites to infinity, the one-dimensional marginal of the stationary distribution concentrates on a single value $\frac{a+b}{2}$.

In the first part of the present paper, we prove, in dimension 1, uniqueness of the stationary measure for the general model, via a coupling which is much more complicated than the one used in [6] for the special case $a \geq 1/2$.

In Section 4, we study an infinite-volume version of Zhang's model in *any* dimension. A similar infinite-volume version of the abelian sandpile model has been studied in [10; 8; 7] and we will in fact also use some of the ideas in these papers.

For the infinite-volume Zhang model in dimension d, we start from a random initial configuration in $[0,\infty)^{\mathbb{Z}^d}$, and evolve it in time by Zhang topplings of unstable sites. We are interested in whether or not there exists a limiting stable configuration. Since Zhang topplings are not abelian, for a given configuration $\eta \in [0,\infty)^{\mathbb{Z}^d}$, for some sequence of topplings it may converge to a stable configuration but for others, it may not. Moreover we do not expect the final configuration - if there is any - to be unique. Therefore, we choose a random order of topplings as follows. To every site we attach an independent rate 1 Poisson clock, and when the clock rings, we topple this site if it is unstable; if it is stable we do nothing. For obvious reasons we call this the *Markov toppling process*.

We show that if we choose the initial configuration according to a stationary ergodic measure μ with density ρ, then for all $\rho < \frac{1}{2}$, μ is stabilizable, that is, the configuration converges to a final stable configuration. For all $\rho \geq 1$, μ is not stabilizable. For $\frac{1}{2} \leq \rho < 1$, when ρ is near to $\frac{1}{2}$ or 1, both possibilities can occur.

2 Model definition and notation

In this section, we discuss Zhang's sandpile on N sites, labelled $1, 2, \ldots, N$. We denoted by $\mathscr{X}_N = [0, \infty)^N$ the set of all possible configurations in Zhang's sandpile model. We will use symbols η and ξ to denote a configuration. We denote the value of a configuration η at site x by η_x, and refer to this value as the *height*, *mass* or *energy* at site x. We introduce a labelling of sites according to their height, as follows.

Definition 2.1. *For $\eta \in \mathscr{X}_N$, we call site x*

$$\begin{array}{lll} \textit{empty} & \textit{if} & \eta_x = 0, \\ \textit{anomalous} & \textit{if} & \eta_x \in (0, \frac{1}{2}), \\ \textit{full} & \textit{if} & \eta_x \in [\frac{1}{2}, 1), \\ \textit{unstable} & \textit{if} & \eta_x \geq 1. \end{array}$$

A site x is stable for η if $0 \leq \eta_x < 1$, and hence all the empty, anomalous and full sites are stable. A configuration η is called stable if all sites are stable, otherwise η is unstable. $\Omega_N = [0, 1)^N$ denotes the set of all stable configurations.

By $T_x(\eta)$ we denote the (Zhang) toppling operator for site x, acting on η and which is defined as follows.

Definition 2.2. *For all $\eta \in \mathscr{X}_N$ such that $\eta_x \geq 1$, we define*

$$T_x(\eta)_y = \begin{cases} 0 & \textit{if } x = y, \\ \eta_y + \frac{1}{2}\eta_x & \textit{if } |y - x| = 1, \\ \eta_y & \textit{otherwise}. \end{cases}$$

For all η such that $\eta_x < 1, T_x(\eta) = \eta$, for all x.

In other words, the toppling operator only changes η if site x is unstable; in that case, it divides its energy in equal portions among its neighbors. We say in that case that site x *topples*. If a boundary site topples, then half of its energy disappears from the system. Every configuration in $\mathscr{X}_N$ can stabilize, that is, reach a final configuration in Ω_N, through finitely many topplings of unstable sites, since energy is dissipated at the boundary.

We define the $(N, [a, b])$ model as a discrete time Markov process with state space Ω_N, as follows. The process starts at time 0 from configuration $\eta(0) = \eta$. For every $t = 1, 2, \ldots$, the configuration $\eta(t)$ is obtained from $\eta(t-1)$ as follows: a random amount of energy $U(t)$, uniformly distributed on $[a, b]$, is added to a uniformly chosen random site $X(t) \in \{1, \ldots, N\}$, hence $\mathbb{P}(X(t) = j) = 1/N$ for all $j = 1, \ldots, N$. The random variables $U(t)$ and $X(t)$ are independent of each other and of the past of the process. We stabilize the resulting configuration through topplings (if it is already stable, then we do not change it), to obtain the new configuration $\eta(t)$. By $\mathbb{E}^\eta$ and $\mathbb{P}^\eta$, we denote expectation resp. probability with respect to this process.

3 Uniqueness of the stationary distribution

In Zhang's sandpile model, it is not obvious that the stationary distribution is unique, since the state space is uncountable. For the three cases: (1) $N = 1$, (2) $a \geq \frac{1}{2}$ and $N \geq 2$, (3) $a = 0, b = 1$ and

$N \geq 2$, it is shown in [6] that the model has a unique stationary distribution, and in addition, in case (2) and (3), for every initial distribution ν, the measure at time t, denoted by $\nu_t^{a,b,N}$, converges in total variation to the stationary distribution. In the case $N = 1$, there are values of a and b where we only have time-average total variation convergence, see Theorem 4.1 of [6].

In all these cases (except when $N = 1$) the proof consisted of constructing a coupling of two copies of Zhang's model with arbitrary initial configurations, in such a way that after some (random) time, the two coupled processes are identical. Each coupling was very specific for the case considered. In the proof for the case $a \geq \frac{1}{2}$ and $N \geq 2$, explicit use is being made of the fact that an addition to a full site always causes a toppling. The proof given for the case $a = 0$ and $b = 1$ and $N > 1$ can be generalized to other values of b, but $a = 0$ is necessary, since in the coupling we use that additions can be arbitrarily small. In the special case $N = 1$, the model is a renewal process, and the proof relies on that.

To prove the following result, we will again construct a coupling of two copies of Zhang's model with arbitrary initial configurations, in such a way that after some (random) time, the two coupled processes are identical. Such a coupling will be called 'successful', as in [13]. Here is the main result of this section; note that only the case $a = b$ is not included.

Theorem 3.1. *For every $0 \leq a < b \leq 1$, and $N \geq 2$, Zhang's sandpile model $(N, [a, b])$ has a unique stationary distribution which we denote by $\mu^{a,b,N}$. Moreover, for every initial distribution on $[0, 1)^N$, the distribution of the process at time t converges exponentially fast in total variation to $\mu^{a,b,N}$, as $t \to \infty$.*

We introduce some notation. Denote $\eta, \xi \in \Omega_N$ as the initial configurations, and $\eta(t), \xi(t)$ as two independent copies of the processes, starting from η and ξ respectively. The independent additions at time t for the two processes starting from η, ξ are $U^\eta(t)$ and $U^\xi(t)$, addition sites are $X^\eta(t)$,$X^\xi(t)$ respectively. Often, we will use 'hat'-versions of the various quantities to denote a coupling between two processes. So, for instance, $\hat{\eta}(t), \hat{\xi}(t)$ denote coupled processes (to be made precise below) with initial configurations η and ξ respectively. By $\hat{X}^\eta(t)$ and $\hat{X}^\xi(t)$ we denote the addition sites at time t in the coupling, and by $\hat{U}^\eta(t)$ and $\hat{U}^\xi(t)$ the addition amounts at time t.

In this section, we will encounter configurations that are such that they are empty at some site x, $1 \leq x \leq N$, and full at all the other sites. We denote the set of such configurations $\mathscr{E}_x$. By $\mathscr{E}_b$, we denote the set of configurations that have only one empty boundary site, and are full at all other sites, that is, $\mathscr{E}_b = \mathscr{E}_1 \cup \mathscr{E}_N$.

The coupling that we will construct is rather technical, but the ideas behind the main steps are not so difficult. In the first step, we make sure that the two copies of the process simultaneously reach a situation in which the N-th site is empty, and all other sites are full. In step 2, we make sure that the heights of the two copies at each vertex are within some small ϵ of each other. This can be achieved by carefully selecting the additions. Finally, in step 3, we show that once the heights at all sites are close to each other, then we can make the two copies of the process equal to each other by very carefully couple the amounts of mass that we add each time.

In order to give the proof of Theorem 3.1, we need the following three preliminary results, the proof of which will be given in Sections 3.1, 3.2 and 3.3 respectively.

Lemma 3.2. *For all η, ξ in Ω_N, $\eta(t)$ and $\xi(t)$ are a.s. simultaneously in $\mathscr{E}_b$ infinitely often.*

Lemma 3.3. *Let η and ξ be two configurations in $\mathcal{E}_N$ and let, for all $\epsilon > 0$,*

$$t_\epsilon = 2\lceil \frac{2}{a+b} \rceil \cdot \lceil \log_{(1-2^{-\lceil \frac{3N}{2} \rceil})}(\frac{2\epsilon}{N}) \rceil.$$

Consider couplings $(\hat{\eta}(t), \hat{\xi}(t))$ of the process starting at η and ξ respectively. Let, in such a coupling, T be the first time t with the property that

$$\max_{1 \le x \le N} \mid \hat{\eta}_x(t) - \hat{\xi}_x(t) \mid < \epsilon \tag{3.1}$$

and

$$\hat{\eta}(t) \in \mathcal{E}_N, \hat{\xi}(t) \in \mathcal{E}_N. \tag{3.2}$$

There exists a coupling such that the event $T \le t_\epsilon$ has probability at least $(2N)^{-t_\epsilon}$, uniformly in η and ξ.

Lemma 3.4. *Let*

$$\epsilon_{a,b,N} = \frac{b-a}{6 + 16\Pi_{l=1}^{N-1}(1 + 2^{N-2-l})}.$$

Consider couplings $(\hat{\eta}(t), \hat{\xi}(t))$ of the process starting at η and ξ respectively, with the property that

$$\max_{1 \le x \le N} \mid \eta_x - \xi_x \mid < \epsilon_{a,b,N}. \tag{3.3}$$

Let T' be the first time t with the property that $\hat{\eta}(t) = \hat{\xi}(t)$. Then there exists a coupling such that the event $T' < (N-1)\lceil \frac{1}{a+b} \rceil$ has probability bounded below by a positive constant that depends only on a, b and N.

We now present the coupling that constitutes the proof of Theorem 3.1, making use of the results stated in Lemma 3.2, Lemma 3.3 and Lemma 3.4.

Proof of Theorem 3.1. Take two probability distributions ν_1, ν_2 on Ω_N, and choose η and ξ according to ν_1, ν_2 respectively. We present a successful coupling $\{\hat{\eta}(t), \hat{\xi}(t)\}$, with $\hat{\eta}(0) = \eta$ and $\hat{\xi}(0) = \xi$. If we assume that both ν_1 and ν_2 are stationary, then the existence of the coupling shows that $\nu_1 = \nu_2 = \nu$. If we take $\nu_1 = \nu$ and ν_2 arbitrary, then the existence of the coupling shows that any initial distribution ν_2 converges in total variation to ν.

The coupling consists of three steps, and is described as follows.

- *step 1.* We evolve the two processes independently until they encounter a configuration in $\mathcal{E}_b$ simultaneously. From Lemma 3.2 we know this happens a.s. By symmetry, we can assume that both configurations are in $\mathcal{E}_N$ as soon as both processes have reached a configuration in $\mathcal{E}_b$. From that moment on, we proceed to
- *step 2.* We use the coupling as described in the proof of Lemma 3.3. That amounts to choosing $\hat{X}^\xi(t) = \hat{X}^\eta(t) = X^\eta(t)$, and $\hat{U}^\xi(t) = \hat{U}^\eta(t) = U^\eta(t)$. As the proof of Lemma 3.3 shows, if $U^\eta(t)$ and $X^\eta(t)$ satisfy certain requirements for at most t_ϵ time steps, then we have that (3.1) and (3.2) occur, with $\epsilon = \epsilon_{a,b,N}$. If during this step, at any time step either $U^\eta(t)$ or $X^\eta(t)$ does not satisfy the requirements, then we return to step 1. But once we have (3.1) and (3.2) (which, by Lemma 3.3, has positive probability), then we proceed to

- *step 3.* Here, we use the coupling as described in the proof of Lemma 3.4. Again, we choose $\hat{X}^{\xi}(t) = \hat{X}^{\eta}(t) = X^{\eta}(t)$ and $\hat{U}^{\eta}(t) = U^{\eta}(t)$, but the dependence of $\hat{U}^{\xi}(t)$ on $U^{\eta}(t)$ is more complicated; the details can be found in the proof of Lemma 3.4. As the proof of Lemma 3.4 shows, if $U^{\eta}(t)$ and $X^{\eta}(t)$ satisfy certain requirements for at most $(N-1)\lceil\frac{1}{a+b}\rceil$ time steps, then we have that $\hat{\eta}(t) = \hat{\xi}(t)$ occurs, and from that moment on the two processes evolve identically. By Lemma 3.4, this event has positive probability. If during this step, at any time step, either $U^{\eta}(t)$ or $X^{\eta}(t)$ does not satisfy the requirements, we return to step 1.

In the coupling, we keep returning to step 1 until step 2 and subsequently step 3 are successfully completed, after which we have that $\hat{\eta}(t) = \hat{\xi}(t)$. Since each step is successfully completed with uniform positive probability, we a.s. need only finitely many attempts. Therefore we achieve $\hat{\eta}(t) = \hat{\xi}(t)$ in finite time, so that the coupling is successful. □

Now, we will proceed to give the proof of Lemma 3.2, Lemma 3.3 and Lemma 3.4.

3.1 Proof of Lemma 3.2

In this section, we show that starting two independent processes from any two configurations η and ξ, the two processes will a.s. be in $\mathscr{E}_b$ simultaneously infinitely often. The proof will be realized in two steps.

Lemma 3.5. *Let η be a configuration in Ω_N. The process starting from η visits $\mathscr{E}_N$ within $(N+1)\lceil\frac{1}{a+b}\rceil$ time steps, with probability at least $(\frac{1}{2N})^{(N+1)\lceil\frac{1}{a+b}\rceil}$.*

Proof. We prove this by giving an explicit event realizing this, that has the mentioned probability. In this step, we always make *heavy* additions to site N, that is, additions with value at least $(a+b)/2$.

First, starting from configuration η, we make heavy additions to site 1 until site 1 becomes unstable. Then an avalanche occurs and a new configuration with at least one empty site is reached. The leftmost empty site denoted by r_1. If $r_1 = N$ we are done. If $r_1 \neq N$, then it is easy to check that site r_1+1 is full. The total number of additions needed for this step is at most $2\lceil\frac{1}{a+b}\rceil$.

Then, if $r_1 \neq N$, we continue by making heavy additions to site r_1+1 until site r_1+1 becomes unstable. Then an avalanche starts from site r_1+1. During this avalanche, sites 1 to r_1-1 are not affected, site r_1 becomes full and we again reach a new configuration with at least one empty site, the leftmost of which is denoted by r_2. If $r_2 = N$ we are done. If not, note that $r_2 \geq r_1+1$ and that all sites $1,\ldots,r_2-1$ and r_2+1 are full. At most $\lceil\frac{1}{a+b}\rceil$ heavy additions are needed for this step.

If $r_2 \neq N$, we repeat this last procedure. After each avalanche, the leftmost empty site moves at least one site to the right, and hence, after the first step we need at most $N-1$ further steps.

Hence, the total number of heavy additions needed for the above steps is bounded above by $(N+1)\lceil\frac{1}{a+b}\rceil$. Every time step, with probability $\frac{1}{N}$, a fixed site is chosen and with probability $\frac{1}{2}$, an addition is a heavy addition. Therefore, the probability of this event is at least $(2N)^{-(N+1)\lceil\frac{1}{a+b}\rceil}$. □

Lemma 3.6. *Let $\xi(0) \in \mathscr{E}_b$, then $\xi(1) \in \mathscr{E}_b$ with probability at least $\frac{1}{N}$.*

Proof. Again, we give an explicit possibility with probability $\frac{1}{N}$. Starting in $\xi \in \mathscr{E}_b$, we make one addition to the site next to the empty boundary site. If this site does not topple, then of course $\xi(1)$ is still in $\mathscr{E}_b$. But if it does topple, then every full site will topple once, after which all sites will be full except for the opposite (previously full) boundary site. In other words, then $\xi(1)$ is also in $\mathscr{E}_b$. The probability that the addition site is the site next to the empty boundary site, is $\frac{1}{N}$. Then $\xi(1) \in \mathscr{E}_b$ with probability at least $\frac{1}{N}$. □

Proof of Lemma 3.2. From Lemma 3.5, it follows that the process starting from ξ is in $\mathscr{E}_b$ infinitely often. Let t_b^ξ be the first time that the process is in $\mathscr{E}_b$, and define

$$T_b^\eta = \min\{t : t \geq t_b^\xi, \eta(t) \in \mathscr{E}_b\}.$$

By the same lemma, the probability that $0 \leq T_b^\eta - t_b^\xi \leq (N+1)\lceil \frac{1}{a+b} \rceil$ is at least $(2N)^{-(N+1)\lceil \frac{1}{a+b} \rceil}$.

Repeatedly applying Lemma 3.6 gives that the event that $\xi(t_b^\xi + 1) \in \mathscr{E}_b, \xi(t_b^\xi + 2) \in \mathscr{E}_b, \dots, \xi(T_b^\eta) \in \mathscr{E}_b$, occurs with probability bounded below by $(\frac{1}{N})^{T_b^\eta - t_b^\xi}$.

We have showed that when $\xi(t) \in \mathscr{E}_b$, within at most $(N+1)\lceil \frac{1}{a+b} \rceil$ time steps, the two processes are in $\mathscr{E}_b$ simultaneously with probability at least $(\frac{1}{2N^2})^{(N+1)\lceil \frac{1}{a+b} \rceil}$. Combining this with the fact that the process starting from ξ is in $\mathscr{E}_b$ infinitely often, we conclude the two processes are in $\mathscr{E}_b$ simultaneously infinitely often. □

3.2 Proof of Lemma 3.3

In this part, we couple two processes starting from $\eta, \xi \in \mathscr{E}_N$. The coupling consists of choosing the addition amounts and sites equal at each time step. For this coupling, we present an event that has probability $(2N)^{-t_\epsilon}$, with $t_\epsilon = 2\lceil \frac{2}{a+b} \rceil \cdot \lceil \log_{(1-2^{-\lceil \frac{3N}{2} \rceil})}(\frac{2\epsilon}{N}) \rceil$, and which is such that if it occurs, then (3.1) and (3.2) are satisfied.

The event we need is that for t_ϵ time steps,

1. All additions are heavy.
2. The additions occur to site N until site N becomes unstable, then to site 1 until site 1 becomes unstable, then to site N again, etcetera.

The probability for an addition to be heavy is $\frac{1}{2}$ and the probability for the addition to occur to a fixed site is $\frac{1}{N}$. Therefore, the probability of this event is $(2N)^{-t_\epsilon}$.

Now we show that if this event occurs, then (3.1) and (3.2) are satisfied. Let $\hat{U}(t)$ be the addition amount at time t. Define a series of stopping times $\{\tau_k\}_{k \geq 0}$ by

$$\tau_0 = 0, \tau_k := \min\{t > \tau_{k-1} : \sum_{t=\tau_{k-1}+1}^{\tau_k} \hat{U}(t) \geq 1\}, \text{ for } k \geq 1, \tag{3.4}$$

and denote

$$S_k = \sum_{t=\tau_{k-1}+1}^{\tau_k} \hat{U}(t). \tag{3.5}$$

The times τ_k ($k > 0$) are such that in both configurations, *only* at these times an avalanche occurs. Indeed, for the first avalanche this is clear because we only added to site N, which was empty before we started adding. But whenever an avalanche starts at a boundary site, and all other sites are full, then every site topples exactly once and after the avalanche, the opposite boundary site is empty. Thus the argument applies to all avalanches.

Since we make only heavy additions,

$$\tau_k - \tau_{k-1} \leq \lceil \frac{2}{a+b} \rceil, \text{ for all } k. \tag{3.6}$$

After the k-th avalanche, the height $\hat{\eta}_y(\tau_k)$ is a linear combination of $S_1, ..., S_k$ and $\eta_1, ..., \eta_{N-1}$, which we write as

$$\hat{\eta}_y(\tau_k) \quad = \quad \sum_{l=1}^{k} A_{ly}(k) S_l + \sum_{m=1}^{N} B_{my}(k)\eta_m, \text{ for } 1 \leq y \leq N, \tag{3.7}$$

and a similar expression for $\hat{\xi}_y(\tau_k)$. From Proposition 3.7 of [6], we have that

$$B_{my}(k) \leq (1 - 2^{-\lceil \frac{3N}{2} \rceil}) \max_x B_{mx}(k-1).$$

By induction, we find

$$B_{my}(k) \leq (1 - 2^{-\lceil \frac{3N}{2} \rceil})^k$$

and hence

$$\begin{aligned} \max_{1 \leq y \leq N} \mid \hat{\eta}_y(\tau_k) - \hat{\xi}_y(\tau_k) \mid &\leq N(1 - 2^{-\lceil \frac{3N}{2} \rceil})^k \max_{1 \leq x \leq N} \mid \eta_x - \xi_x \mid \\ &\leq \frac{N}{2}(1 - 2^{-\lceil \frac{3N}{2} \rceil})^k, \end{aligned}$$

where we use the fact that $\eta, \xi \in \mathscr{E}_N$ implies $\max_{1 \leq x \leq N} \mid \eta_x - \xi_x \mid \leq \frac{1}{2}$.

For each $\epsilon > 0$, choose $k_\epsilon = 2\lceil \log_{(1-2^{-\lceil \frac{3N}{2} \rceil})}(\frac{2\epsilon}{N})\rceil$. Then $\frac{N}{2}(1 - 2^{-\lceil \frac{3N}{2} \rceil})^{k_\epsilon} \leq \epsilon$, so that

$$\max_{1 \leq x \leq N} \mid \hat{\eta}_x(\tau_{k_\epsilon}) - \hat{\xi}_x(\tau_{k_\epsilon}) \mid < \epsilon,$$

and moreover, an even number of avalanches occurred, which means that at time τ_{k_ϵ}, both processes are in $\mathscr{E}_N$. By (3.6), $\tau_{k_\epsilon} \leq t_\epsilon = k_\epsilon \lceil \frac{2}{a+b} \rceil$. Thus, τ_{k_ϵ} is a random time T as in the statement of Lemma 3.3. $\square$

3.3 Proof of Lemma 3.4

As in the proof of Lemma 3.3, we will describe the coupling, along with an event that has probability bounded below by a constant that only depends on a, b and N, and is such that if it occurs, then within $(N-1)\lceil\frac{1}{a+b}\rceil$ time steps, $\hat{\eta}(t)=\hat{\xi}(t)$. First we explain the idea behind the coupling and this event, after that we will work out the mathematical details.

The idea is that in both processes we add the same amount, only to site 1, until an avalanche is about to occur. We then add slightly different amounts while still ensuring that an avalanche occurs in both processes. After the first avalanche, all sites contain a linear combination of the energy before the last addition, plus a nonzero amount of the last addition. We choose the difference D_1 between the additions that cause the first avalanche, such that site N will have the same energy in both processes after the first avalanche, where $|D_1|$ is bounded above by a value that only depends on a, b and N. Sites $N-1$ will become empty, and the differences between the two new configurations on all other sites will be larger than those before this avalanche, but can be controlled.

When we keep adding to site 1, in the next avalanche only the sites $1,\ldots,N-2$ will topple. We choose the addition amounts such that after the second avalanche, sites $N-1$ will have the same energy. Since site N did not change in this avalanche, we now have equality on two sites. After the second avalanche, site $N-2$ is empty, and the configurations are still more different on all other sites, but the difference can again be controlled.

We keep adding to site 1 until, after a total of $N-1$ avalanches, the configurations are equal on all sites. After each avalanche, we have equality on one more site, and the difference increases on the nonequal sites. We deal with this increasing difference by controlling the maximal difference between the corresponding sites of the two starting configurations by the constant $\epsilon_{a,b,N}$, so that we can choose each addition of both sequences from a nonempty interval in $[a,b]$. The whole event takes place in finite time, and will therefore have positive probability.

Proof of Lemma 3.4. The coupling is as follows. We choose $\hat{X}^{\eta}(t)=X^{\eta}(t)$, and $\hat{U}^{\eta}(t)=U^{\eta}(t)$. We choose the addition sites $\hat{X}^{\eta}(t)$ and $\hat{X}^{\xi}(t)$ equal, and the addition amounts $\hat{U}^{\eta}(t)$ and $\hat{U}^{\xi}(t)$ either equal, or not equal but dependent. In the last case, $\hat{U}^{\xi}(t)$ is always of the form $a+(\hat{U}^{\eta}(t)+D-a)$ mod $(b-a)$, where D does not depend on $\hat{U}^{\eta}(t)$. The reader can check that, for any D, $\hat{U}^{\xi}(t)$ is then uniformly distributed on $[a,b]$.

The event we need is as follows. First, all additions are heavy. For the duration of $N-1$ avalanches, which is at most $(N-1)\lceil\frac{1}{a+b}\rceil$ time steps, the additions to η occur to site 1, and the amount is for every time step in a certain subinterval of $[a,b]$, to be specified next.

We denote $\frac{a+b}{2}=a''$ and recursively define

$$\epsilon_{k+1}=(1+2^{N-k-2})\epsilon_k,$$

with $\epsilon_1=\epsilon_{a,b,N}$. Between the $(k-1)$-st and k-th avalanche, the interval is $[a'',a''+2\epsilon_k]$, and at the time where the k-th avalanche occurs, the interval is a subinterval of $[a',b]=[a''+3\epsilon_k,b]$ of length $\frac{b-a'}{2}$; see below.

The probability of at most $(N-1)\lceil\frac{1}{a+b}\rceil$ additions occurring to site 1, is bounded from below by $N^{-\lceil\frac{1}{a+b}\rceil(N-1)}$. Since ϵ_k is increasing in k, the probability of all addition amounts occurring in the appropriate intervals, is bounded below by $(2\epsilon_1)^{(N-1)(\lceil\frac{1}{a+b}\rceil-1)}\cdot(\frac{b-a}{4}-\frac{3\epsilon_{(N-1)}}{2})^{N-1}$. The probability of the event is bounded below by the product of these two bounds.

Now we define the coupling such that if this event occurs, then after $N-1$ avalanches, we have that $\hat{\eta}(t)=\hat{\xi}(t)$. In the remainder, we suppose that the above event occurs.

We start with discussing the time steps until the first avalanche. Suppose, without loss of generality, that $\eta_1 \geq \xi_1$. We make equal additions in $[a'', a''+2\epsilon_1]$, until the first moment t such that $\eta_1(t) > 1-a''-2\epsilon_1$. We then know that $\xi_1(t) > 1-a''-3\epsilon_1$. If we now choose the last addition for both configurations in $[a', b]=[a''+3\epsilon_1, b]$, then both will topple.

Define

$$D_1 := \sum_{y=1}^{N-1} 2^{y-1}(\eta_y - \xi_y). \tag{3.8}$$

Let τ_1 be the time at which the first avalanche occurs. Then we choose for all $t<\tau_1$, $\hat{U}^{\eta}(t)=\hat{U}^{\xi}(t)$, and $\hat{U}^{\xi}(\tau_1)-\hat{U}^{\eta}(\tau_1)=D_1$. Since $|D_1|<2^{N-1}\epsilon_1 \leq \frac{b-a'}{4}$, when $\hat{U}^{\eta}(\tau_1) \in [\frac{3a'+b}{4}, \frac{a'+3b}{4}]$ (the middle half of $[a', b]$)

$$a' \leq \hat{U}^{\eta}(\tau_1) + D_1 < b. \tag{3.9}$$

So, if $\hat{U}^{\eta}(\tau_1)$ is uniformly distributed on $[\frac{3a'+b}{4}, \frac{a'+3b}{4}]$, then $\hat{U}^{\xi}(\tau_1)=\hat{U}^{\eta}(\tau_1)+D_1$ is uniformly distributed on $[\frac{3a'+b}{4}+D_1, \frac{a'+3b}{4}+D_1] \subset [a', b]$.

Let $R_1=\sum_{t=1}^{\tau_1-1}\hat{U}^{\eta}(t)$. Then at time τ_1, for $1 \leq x \leq N-2$ we have

$$\hat{\eta}_x(\tau_1) = \frac{1}{2^{x+1}}(\eta_1 + R_1 + \hat{U}^{\eta}(\tau_1)) + \frac{1}{2^x}\eta_2 + \cdots + \frac{1}{2}\eta_{x+1},$$

and

$$\hat{\eta}_{N-1}(\tau_1)=0; \hat{\eta}_N(\tau_1)=\hat{\eta}_{N-2}(\tau_1),$$

and a similar expression for $\hat{\xi}_x(\tau_1)$. It follows that

$$\hat{\eta}_N(\tau_1) - \hat{\xi}_N(\tau_1) = -2^{(1-N)}D_1 + \sum_{y=1}^{N-1} 2^{y-N}(\eta_y - \xi_y) = 0$$

which means that the two coupled processes at time τ_1 are equal at site N.

After this first avalanche, the differences on sites $1, \ldots, N-3$ have been increased. Ignoring the fact that sites $N-2$ happen to be equal (to simplify the discussion), we calculate

$$\begin{aligned}
\max_{1\leq x\leq N} \mid \hat{\eta}_x(\tau_1) - \hat{\xi}_x(\tau_1) \mid &\leq \max_{1\leq x\leq N}\left\{\frac{1}{2^{x+1}} \mid D_1 \mid + \max_{1\leq x\leq N} \mid \eta_y - \xi_y \mid \sum_{l=1}^{x+1}\frac{1}{2^l}\right\} \\
&\leq \max_{1\leq x\leq N}\left\{\frac{2^{N-1}}{2^{x+1}} + (1-\frac{1}{2^x})\right\} \max_{1\leq x\leq N} \mid \eta_y - \xi_y \mid \\
&\leq (1+2^{N-3}) \max_{1\leq x\leq N} \mid \eta_y - \xi_y \mid \\
&\leq (1+2^{N-3})\epsilon_1 = \epsilon_2.
\end{aligned} \tag{3.10}$$

We wish to iterate the above procedure for the next $N-2$ avalanches. We number the avalanches $1, \ldots, N-1$, and define τ_k as the time at which the k-th avalanche occurs. As explained for the case

$k=1$, we choose all additions equal, except at times τ_k, where we choose $\hat{U}^{\xi}(\tau_k)-\hat{U}^{\eta}(\tau_k)=D_k$, with

$$D_k=\sum_{y=1}^{N-k}2^{y-1}[\hat{\eta}_y(\tau_{k-1})-\hat{\xi}_y(\tau_{k-1})]$$

and

$$|D_k|\leq 2^{N-k}\max_{1\leq x\leq N}|\hat{\eta}_x(\tau_{k-1})-\hat{\xi}_x(\tau_{k-1})|.$$

The maximal difference between corresponding sites in the two resulting configurations has the following bound:

$$\begin{aligned}
&\max_{1\leq x\leq N}|\hat{\eta}_x(\tau_k)-\hat{\xi}_x(\tau_k)|\\
&\leq \max_{1\leq x\leq N}\left\{\frac{1}{2^{x+1}}|D_k|+\sum_{l=1}^{x+1}\frac{1}{2^l}\max_{1\leq x\leq N}|\hat{\eta}_x(\tau_{k-1})-\hat{\xi}_x(\tau_{k-1})|\right\}\\
&\leq (1+2^{N-k-2})\max_{1\leq x\leq N}|\hat{\eta}_x(\tau_{k-1})-\hat{\xi}_x(\tau_{k-1})|\\
&\leq (1+2^{N-k-2})\epsilon_k=\epsilon_{k+1}.
\end{aligned}\tag{3.11}$$

Hence for all k, $|D_k|$ is bounded from above by $\epsilon_{k+1}2^{N-k}$, where ϵ_{k+1} only depends on ϵ_k and N. With induction, we find,

$$|D_k|\leq 2^{N-k}\Pi_{l=1}^{k-1}(1+2^{N-l-2})\epsilon_1:=d_k.$$

We choose $\epsilon_1=\epsilon_{a,b,N}$ such that $D_{N-1}\leq\frac{b-a'}{4}$. As the upper bound d_k is increasing in k, we get $D_k\leq\frac{b-a'}{4}$, for all $k=1,\dots,N-1$. □

It follows from our proof that the convergence to the stationary distribution goes in fact exponentially fast. Indeed, every step of the coupling is such that a certain good event occurs with a certain minimal probability within a bounded number of steps. Hence, there exists a probabiliy $q>0$ and a number $M>0$ such that with probability q, the coupling is succesfull within M steps, uniformly in the initial configuration. This implies exponential convergence.

4 Zhang's sandpile in infinite volume

4.1 Definitions and main results

In this section we work in general dimension d. We let $\mathscr{X}=[0,\infty)^{\mathbb{Z}^d}$ denote the set of infinite-volume height configurations in dimension d and $\Omega=[0,1)^{\mathbb{Z}^d}$ the set of all stable configurations. For $x\in\mathbb{Z}^d$, the (Zhang) toppling operator T_x is defined as

$$T_x(\eta)_y=\begin{cases}0 & \text{if } x=y,\\ \eta_y+\frac{1}{2d}\eta_x & \text{if } |y-x|=1,\\ \eta_y & \text{otherwise.}\end{cases}$$

The infinite-volume version of Zhang's sandpile model is quite different from its abelian sandpile counterpart. Indeed, in the infinite-volume abelian sandpile model, it is shown that if a configuration can reach a stable one via some order of topplings, it will reach a stable one by every order of topplings and the final configuration as well as the topplings numbers per site are always the same, see [8; 10; 7].

In Zhang's sandpile model in infinite volume, the situation is not nearly as nice. Not only does the final stable realization depend on the order of topplings, the very stabilizability itself also does. We illustrate this with some examples.

Consider the initial configuration (in $d = 1$)

$$\eta = (\dots, 0, 0, 1.4, 1.2, 0, 0, \dots).$$

We can reach a stable configuration in any order of the topplings, but the final configuration as well as the number of topplings per site depend on which unstable site we topple first. We can choose to start toppling at the left or right unstable site, or to topple the two sites in parallel (that is, at the same time); the different results are presented in Table 1.

start at site	toppling numbers	final configuration
left	$(\dots, 0, 0, 1, 1, 0, 0, \dots)$	$(\dots, 0, 0.7, 0.95, 0, 0.95, 0, \dots)$
right	$(\dots, 0, 1, 2, 3, 1, 0, \dots)$	$(\dots, 0.5, 0.5, 0.525, 0, 0.525, 0.55 \dots)$
parallel	$(\dots, 0, 0, 1, 1, 0, 0, \dots)$	$(\dots, 0, 0.7, 0.6, 0.7, 0.6, 0, \dots)$

Table 1: The three possible stabilizations of $(\dots, 0, 0, 1.4, 1.2, 0, 0, \dots)$

For a second example, let

$$\xi = (\dots, 0.9, 0.9, 0, 1.4, 1.2, 0, 0.9, 0.9, \dots).$$

This is a configuration that evolves to a stable configuration in some orders of topplings, but not by others. Indeed, if we start to topple the left unstable site first, we obtain the stable configuration

$$(\dots, 0.9, 0.9, 0.7, 0.95, 0, 0.95, 0.9, 0.9, \dots),$$

but if we topple the right unstable site first, after two topplings, we reach

$$\xi' = (\dots, 0.9, 0.9, 1, 0, 1, 0.6, 0.9, 0.9, \dots).$$

By arguing as in the proof of the forthcoming Theorem 4.3, one can see that this configuration cannot evolve to a stable configuration.

In view of these examples, we have to be more precise about the way we perform topplings. In the present paper, we will use the *Markov toppling process*: to each site we associate an independent rate 1 Poisson process. When the Poisson clock 'rings' at site x and x is unstable at that moment, we perform a Zhang-toppling at that site. If x is stable, we do nothing. By $\eta(t)$, we denote the random configuration at time t. We denote by $M(x, t, \eta)$ the (random) number of topplings at site x up to and including time t.

Definition 4.1. *A configuration $\eta \in \mathscr{X}$ is said to be* stabilizable *if for every $x \in \mathbb{Z}^d$,*

$$\lim_{t\to\infty} M(x, t, \eta) < \infty$$

a.s. In that case we denote the limit random variable by $M(x, \infty, \eta)$.

Denote the collection of all stabilizable configurations by $\mathscr{S}$. It is not hard to see that $\mathscr{S}$ is shift-invariant and measurable with respect to the usual Borel sigma field. Hence, if μ is an ergodic stationary probability measure on $\mathscr{X}$, $\mu(S)$ is either 0 or 1.

Definition 4.2. *A probability measure μ on $\mathscr{X}$ is called* stabilizable *if $\mu(\mathscr{S}) = 1$.*

The next theorem is the main result in this section. When the density of an ergodic translation-invariant measure μ is at least 1, μ is not stabilizable, and when it is smaller than $\frac{1}{2}$, μ is stabilizable. The situation when $\frac{1}{2} \le \rho < 1$ is not nearly as elegant. Clearly, when we take μ_ρ to be the point mass at the configuration with constant height ρ, then μ_ρ is stabilizable for all $\frac{1}{2} \le \rho < 1$. On the other hand, the following theorem shows that there are measures μ with density close to $\frac{1}{2}$ and close to 1 which are not stabilizable.

Theorem 4.3. *Let μ be an ergodic translation-invariant probability measure on $\mathscr{X}$ with $\mathbb{E}_\mu(\eta_0) = \rho < \infty$. Then*

1. *If $\rho \ge 1$, then μ is not stabilizable, that is, $\mu(\mathscr{S}) = 0$.*
2. *If $0 \le \rho < \frac{1}{2}$, then μ is stabilizable, that is, $\mu(\mathscr{S}) = 1$.*
3. *For all $1/2 \le \rho < d/(2d-1)$ and $(2d-1)/(2d) < \rho < 1$, there exists an ergodic measure μ_ρ with density ρ which is not stabilizable.*

Remark There is no obvious monotonicity in the density as far as stabilizability is concerned. Hence we cannot conclude from the previous theorem that for *all* $1/2 \le \rho < 1$ there exists an ergodic measure μ_ρ which is not stabilizable.

5 Proofs for the infinite-volume sandpile

For an initial measure μ, $\mathbb{E}_\mu$ and $\mathbb{P}_\mu$ denote the corresponding expectation and probability measures in the stabilization process. We first show that no mass is lost in the toppling process.

Proposition 5.1. *Let μ be an ergodic shift-invariant probability measure on $\mathscr{X}$ with*

$$\mathbb{E}_\mu(\eta_0) = \rho < \infty$$

which evolves according to the Markov toppling process. Then we have

1. *for $0 \le t < \infty$, $\mathbb{E}_\mu \eta_0(t) = \rho$,*
2. *if μ is stabilizable, then $\mathbb{E}_\mu \eta_0(\infty) = \rho$.*

Proof. We prove 1. via the well known mass transport principle. Let the initial configuration be denoted by η. Imagine that at time $t = 0$ we have a certain amount of mass at each site, and we colour all mass white, except the mass at a special site x which we colour black. Whenever a site topples, we further imagine that the black and white mass present at that site, are *both* equally distributed among the neighbours. So, for instance, when x topples for the first time, all its neighbours receive a fraction $1/(2d)$ of the original black mass at x, plus possibly some white mass. We denote by $B(y,t)$ the total amount of black mass at site y at time t. First, we argue that at any finite time t,

$$\sum_{y\in\mathbb{Z}^d} B(y,t) = \eta_x, \tag{5.1}$$

that is, no mass is lost at any finite time t. Indeed, had this not been the case, then we define t^* to be the infimum over those times t for which (5.1) is not true. Since mass must be subsequently passed from one site to the next, this implies that there is a path $(x = x_0, x_1, \ldots)$ of neighbouring sites to infinity, starting at x such that the sites x_i all topple before time t^*, in the order given. Moreover, since t^* is the infimum, the toppling times t_i of x_i satisfies $\lim_{i\to\infty} t_i = t^*$. Hence, for any $\epsilon > 0$, we can find i_0 so large that for all $i > i_0$, $t_i > t^* - \epsilon$. Call a site open of its Poisson clock 'rings' in the time interval $(t^* - \epsilon, t^*)$, and closed otherwise. This constitutes an independent percolation process, and if ϵ is sufficiently small, the open sites do not percolate. Hence a path as above cannot exist, and we have reached a contradiction. It follows that no mass is lost at any finite time t, and we can now proceed to the routine proof of 1. via mass-transport.

We denote by $X(x,y,t,\eta)$ the amount of mass at y at time t which started at x. From mass preservation, we have

$$\eta_x = \sum_{y\in\mathbb{Z}^d} X(x,y,t,\eta) \tag{5.2}$$

and

$$\eta_y(t) = \sum_{x\in\mathbb{Z}^d} X(x,y,t,\eta). \tag{5.3}$$

Since all terms are non-negative and by symmetry, this gives

$$\begin{aligned}\mathbb{E}_\mu \eta_0(0) &= \sum_{y\in\mathbb{Z}^d} \mathbb{E}_\mu X(0,y,t,\eta) \\ &= \sum_{y\in\mathbb{Z}^d} \mathbb{E}_\mu X(y,0,t,\eta) = \mathbb{E}_\mu \eta_0(t).\end{aligned}$$

To prove 2., we argue as follows. From 1. we have that for every $t < \infty$, $\mathbb{E}_\mu \eta_0(t) = \rho$. Using Fatou's lemma we obtain

$$\mathbb{E}_\mu(\eta_0(\infty)) = \mathbb{E}_\mu(\lim_{t\to\infty} \eta_0(t)) \le \liminf_{t\to\infty} \mathbb{E}_\mu(\eta_0(t)) = \rho, \tag{5.4}$$

and therefore it remains to show that $\mathbb{E}_\mu(\eta_0(\infty)) \ge \rho$. This can be shown in the same way as Lemma 2.10 in [7], using the obvious identity

$$\eta_x(t) = \eta_x - L(x,t,\eta) + \frac{1}{2d} \sum_{|y-x|=1} L(y,t,\eta) \tag{5.5}$$

instead of (3) in [7], where $L(x,t,\eta)$ (for $0 \le t \le \infty$) denotes the total amount of mass that is lost from site x via topplings, until and including time t. $\square$

Proposition 5.2. *Let $\eta(t)$ be obtained by the Markov toppling process starting from $\eta \in \mathscr{X}$. Let Λ be a finite subset of $\mathbb{Z}^d$, such that all sites in Λ toppled at least once before time t. Let β_Λ be the number of internal bonds of Λ. Then*

$$\sum_{x\in\Lambda} \eta_x(t) \geq \frac{1}{2d}\beta_\Lambda. \tag{5.6}$$

Proof. Let (x, y) be an internal bond of Λ. By assumption, both x and y topple before time t. Suppose that x is the last to topple among x and y. As a result of this toppling, at least mass $1/(2d)$ is transferred from x to y and this mass will stay at y until time t since y does not topple anymore before time t. In this way, we associate with each internal bond, an amount of mass of at least $1/(2d)$, which is present in Λ at time t. Hence the total amount of mass in Λ at time t is at least $1/(2d)$ times the number of internal bonds. ☐

We can now prove Theorem 4.3.

Proof of Theorem 4.3. We prove 1. first. Let μ be any ergodic shift-invariant measure with $\mathbb{E}_\mu(\eta_0) = \rho \geq 1$ and suppose μ is stabilizable. According to Proposition 5.1, we have

$$\mathbb{E}_\mu(\eta_0(\infty)) = \mathbb{E}_\mu(\eta_0) = \rho \geq 1, \tag{5.7}$$

which contradicts the assumption that $\eta(\infty)$ is stable.

For 2., let μ be any ergodic shift-invariant probability measure on $\mathscr{X}$ with $\mathbb{E}_\mu(\eta_0) = \rho < \frac{1}{2}$, and suppose that μ is not stabilizable. We will now show that this leads to a contradiction.

Define $C_n(t)$ to be the event that before time t, every site in the box $[-n, n]^d$ topples at least once. Since μ is not stabilizable we have that $\mathbb{P}_\mu(C_n(t)) \to 1$ as $t \to \infty$. Indeed, if a configuration is not stabilizable, all sites will topple infinitely many times as can be easily seen.

Choose $\epsilon > 0$ such that $1 - \epsilon > 2\rho$. For this ϵ, there exists a non-random time $T^\epsilon > 0$ such that for all $t > T^\epsilon$,

$$\mathbb{P}_\mu(C_n(t)) > 1 - \epsilon. \tag{5.8}$$

From Proposition 5.2 we have that at time $t \geq T^\epsilon > 0$, with probability at least $1 - \epsilon$, the following inequality holds:

$$\frac{\sum_{x\in[-n,n]^d} \eta_x(t)}{(2n+1)^d} \geq \frac{1}{2}\frac{(2n)^d}{(2n+1)^d}. \tag{5.9}$$

Therefore, we also have

$$\mathbb{E}_\mu\left(\frac{\sum_{x\in[-n,n]^d} \eta_x(t)}{(2n+1)^d}\right) \geq \frac{1}{2}(1-\epsilon)\frac{(2n)^d}{(2n+1)^d}.$$

Since $2\rho < 1$, we can choose n so large that

$$(1-\epsilon)\frac{(2n)^d}{(2n+1)^d} > 2\rho.$$

Using the shift-invariance of μ and the toppling process, for $t \geq T^\epsilon$, we find

$$\mathbb{E}_\mu\eta_0(t) = \mathbb{E}_\mu\left(\frac{\sum_{x\in[-n,n]^d} \eta_x(t)}{(2n+1)^d}\right) \geq \frac{1}{2}(1-\epsilon)\frac{(2n)^d}{(2n+1)^d} > \rho. \tag{5.10}$$

However, from Proposition 5.1, we have for any finite t that $\mathbb{E}_\mu \eta_0(t) = \mathbb{E}_\mu \eta_0(0) = \rho$.

Next we prove 3. We start with $\rho > (2d-1)/(2d)$. To understand the idea of the argument, it is useful to first assume that we have an unstable configuration η on a *bounded* domain Λ (with periodic boundary conditions) with the property that $\eta_x \geq 1 - 1/(2d)$, for all $x \in \Lambda$. On such a bounded domain, we can order the topplings according to the time at which they occur. Hence we can find a sequence of sites $x_1, x_2, \ldots$ (not necessarily all distinct) and a sequence of times $t_1 < t_2 < \cdots$ such that the i-th toppling takes place at site $x_i \in \Lambda$ at time t_i. At time t_1, x_1 topples, so all neighbours of x_1 receive at least $1/(2d)$ from x_1. This means that all neighbours of x_1 become unstable at time t_1, and therefore they will all topple at some moment in the future. As a result, x_1 itself will also again be unstable after all its neighbours have toppled, and hence x_1 will topple again in the future.

In an inductive fashion, assume that after the k-th toppling (at site x_k at time t_k), we have that it is certain that all sites that have toppled so far, will topple again in the future, that is, after time t_k. Now consider the next toppling, at site x_{k+1} at time t_{k+1}. If none of the neighbours of x_{k+1} have toppled before, then a similar argument as for x_1 tells us that x_{k+1} will topple again in the future. If one or more neighbours of x_{k+1} have toppled before, then the inductive hypothesis implies that they will topple again after time t_{k+1}. Hence, we conclude that *all* neighbours of x_{k+1} will topple again which implies, just as before, that x_{k+1} itself will topple again. We conclude that each sites which topples, will topple again in the future, and therefore the configuration can not be stabilized.

This argument used the fact that we work on a bounded domain, since only then is there a well-defined sequence of consecutive topplings. But with some extra work, we can make a similar argument work for the infinite-volume model as well, as follows.

Let $s_0 > 0$ be so small that the probability that the Poisson clock at the origin 'rings' before time s_0 is smaller than the critical probability for independent site percolation on the d-dimensional integer lattice. Call a site *open* if its Poisson clock rings before time s_0. By the choice of s_0, all components of connected open sites are finite. For each such component of open sites, we now order the topplings that took place between time 0 and time s_0. For each of these components, we can argue as in the first paragraphs of this proof, and we conclude that all sites that toppled before time s_0, will topple again at some time larger than s_0. We then repeat this procedure for the time interval $[s_0, 2s_0]$, $[2s_0, 3s_0], \ldots$, and conclude that at any time, a site that topples, will topple again in the future. This means that the configuration is not stabilizable. Hence, if we take a measure μ_ρ such that with μ_ρ-probability 1, all configurations have the properties we started out with, then μ_ρ is not stabilizable.

Next, we consider the case where $1/2 \leq \rho < d/(2d-1)$. Consider a measure μ_ρ whose realizations are a.s. 'checkerboard' patterns in the following way: any realization is a translation of the configuration in which all sites whose sum of the coordinates is even obtain mass 2ρ, and all sites whose sum of coordinates is odd obtain zero mass. Consider a site x with zero mass. Since all neighbours of x are unstable, these neighbours will all topple at some point. By our choice of ρ, x will become unstable precisely at the moment that the *last* neighbour topples - this follows from a simple computation. By an argument pretty much the same as in the first case, we now see that all sites that originally obtained mass 2ρ, have the property that after they have toppled, *all* their neighbours will topple again in the future, making the site unstable again. This will go on forever, and we conclude that the configuration is not stabilizable. Hence, μ_ρ is not stabilizable. $\square$

Remark The arguments in case of parallel topplings are simpler: the case $\rho < 1/2$ can be done as above, while for *all* $\rho \geq 1/2$, the checkerboard pattern is preserved at all times, preventing stabilization.

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