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Stochastic nonlinear Schrödinger equations driven by a fractional noise Well-posedness, large deviations and support

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Abstract

We consider stochastic nonlinear Schrödinger equations driven by an additive noise. The noise is fractional in time with Hurst parameter H in $(0, 1)$ and colored in space with a nuclear space correlation operator. We study local well-posedness. Under adequate assumptions on the initial data, the space correlations of the noise and for some saturated nonlinearities, we prove sample path large deviations and support results in a space of Hölder continuous in time until blow-up paths. We consider Kerr nonlinearities when $H > 1/2$.

Key words: Large deviations, stochastic partial differential equations, nonlinear Schrödinger equation, fractional Brownian motion.

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1 Introduction

Nonlinear Schrödinger (NLS) equations are generic models for the propagation of the enveloppe of a wave packet in weakly nonlinear and dispersive media, see (10). They appear for example in optics, hydrodynamics, biology, field theory, crystals or Bose-Einstein condensates. Random perturbations of additive or multiplicative types, usually using the Gaussian space-time white noise, are often considered physics. In optics for example, spontaneous emission of noise is due to amplification along the fiber line that compensates loss.
The stochastic NLS equations studied here are written in Itô form

$$i\mathrm{d}u - (\Delta u + f(u))\mathrm{d}t = \mathrm{d}W^H, \tag{1.1}$$

where u is a complex valued function of time and space being $\mathbb{R}^d$, the initial datum u_0 is a function of some Sobolev space H^v based on L^2. W^H is a fractional Wiener process of the form ΦW_c^H where Φ is a bounded operator and W_c^H is a cylindrical fractional Wiener process on L^2, *i.e.* such that for any orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of L^2 there exists independent fractional Brownian motions (fBm) $\left(\beta_j^H(t)\right)_{t\geq 0}$ such that $W_c^H(t) = \sum_{j\in\mathbb{N}} \beta_j^H(t) e_j$. The fBm, with Hurst parameter $H \in (0,1)$, is a centered Gaussian processes with stationary increments

$$\mathbb{E}\left(\left|\beta^H(t) - \beta^H(s)\right|^2\right) = |t-s|^{2H}, \quad t, s > 0.$$

Only when Φ is Hilbert-Schmidt $W^H(t)$ for $t \geq 0$ have distributions which are Radon measures in Hilbert spaces. We consider pathwise weak, in the sense used in the analysis of PDEs, solutions equivalent to mild solutions

$$u(t) = U(t)u_0 - i\int_0^t U(t-s)f(u(s))\,ds - i\int_0^t U(t-s)dW^H(s) \tag{1.2}$$

where $(U(t))_{t\in\mathbb{R}}$ is the Schrödinger linear group on some H^s generated by the skew-adjoint unbounded operator $\left(-i\Delta, \mathrm{H}^{s+2}\right)$. This semi-group approach, see (4), is convenient for stochastic NLS equations where we use properties of the group and allows to consider the infinite dimensional stochastic integration in terms of the well studied integration with respect to the fBm. The last term in (1.2) is the stochastic convolution. It cannot be defined for cylindrical processes through regularization properties of the semi-group since $U(t)$ is an isometry on H^s. Also, since we work in $\mathbb{R}^d$, the stochastic convolution with a cylindrical process would be space wise translation invariant which is incompatible with it having paths in H^s. Thus, though in mild form, we keep the Hilbert-Schmidt operator. In optics the time variable in the NLS equation corresponds to space and reciprocally, thus noises considered for well-posedness are colored in time. These factional in time noises do not seem to have been considered in physics in the context of NLS equations, they are natural extensions of the previous studied noise. Different scaling properties could then be considered. Also like the paths of the fBm, the solutions of the NLS equation display self-similarity in space, near blow-up for supercritical nonlinearities, see (10). The coloration in time (space in optics) might be relevant in applications.
Kerr nonlinearities, of the form $f(u) = \lambda|u|^{2\sigma}u$ where $\lambda = \pm 1$, are often considered for NLS equations. It is proven in (2), with a white in time Gaussian noise, that the Cauchy problem is locally well-posed in H^1 for every σ when $d = 1, 2$ and when $\sigma < \frac{2}{d-2}$ for $d \geq 3$. For such σ

and $\lambda=-1$, defocusing case, the Cauchy problem is globally well-posed. When $\lambda=1$, focusing case, solutions may blow-up in finite time when $\sigma \geq \frac{2}{d}$, critical and supercritical nonlinearities. In (3), theoretical results on the influence of a noise on the blow-up have been obtained. Large deviations and a support theorem is given in (7), while (8) considers multiplicative noises. The results are applied to the problem of error in soliton transmission in (6; 7) and to the exit times in (7; 8). In (9), the exit from a basin of attraction for weakly damped equations is studied.
We first consider local well-posedness. Then we prove, for locally Lipschitz nonlinearities, a sample path large deviation principle (LDP) and a support theorem in a space of exploding and $H'-$Hölder continuous on time intervals before blow-up paths $(0<H'<H)$. Hölder regularity of the stochastic convolution cannot be transferred easily since the group is an isometry and we impose additional regularity on u_0 and Φ H^s-valued for s large enough. We treat Kerr nonlinearities for $H>\frac{1}{2}$ but do not consider Hölder continuity. General Gaussian noises such as derived from Volterra processes could be considered but we focus on fractional noises for computational convenience.

2 Notations and preliminaries

We denote by L^2 the Hilbert space of complex Lebesgue square integrable functions with the inner product $(u,v)_{\mathrm{L}^2}=\mathfrak{Re}\int_{\mathbb{R}} u(x)\overline{v}(x)dx$. The Sobolev spaces H^r for $r\geq 0$ are the Hilbert spaces of functions f in L^2 such that their Fourier transform $\hat{f}$ satisfy $\|f\|_{\mathrm{H}^r}^2=\int_{\mathbb{R}^d}\left(1+|\xi|^2\right)^r|\hat{f}(\xi)|^2d\xi<\infty$. If I is an interval, $(E,\|\cdot\|_E)$ a Banach space and $r\in[1,\infty]$, then $\mathrm{L}^r(I;E)$ is the space of strongly Lebesgue measurable functions f such that $t\to\|f(t)\|_E$ is in $\mathrm{L}^r(I)$. The integral is of Bochner type. The space of bounded operators from B to C, two Banach spaces, is denoted by $\mathcal{L}_c(B,C)$. That of Hilbert-Schmidt operators Φ from E to F, two Hilbert spaces, is denoted by $\mathcal{L}_2(E,F)$. It is a Hilbert space when endowed with the norm $\|\Phi\|_{\mathcal{L}_2(E,F)}^2=\sum_{j\in\mathbb{N}}\|\Phi e_j\|_F^2$ where $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of E. We denote by $\mathcal{L}_2^{0,r}$ the space $\mathcal{L}_2(\mathrm{L}^2,\mathrm{H}^r)$. When A and B are two Banach spaces, $A\cap B$ with the norm defined as the maximum of the norms in A and in B, is a Banach space. A pair (r,p) of positive numbers is an admissible pair if p satisfies $2\leq p<\frac{2d}{d-2}$ when $d>2$ ($2\leq p<+\infty$ when $d=2$ and $2\leq p\leq+\infty$ when $d=1$) and r is such that $\frac{2}{r}=d\left(\frac{1}{2}-\frac{1}{p}\right)$.
When E is a Banach space we denote by $\mathrm{C}^{H'}([0,T];E)$ the space of $H'-$Hölder E-valued continuous functions on $[0,T]$ embedded with the norm

$$\|f\|_{H',T}=\sup_{t\in[0,T]}\|f(t)\|_E+\sup_{t,s\in[0,T],t\neq s}\frac{\|f(t)-f(s)\|_E}{|t-s|^{H'}}.$$

The space $\mathrm{C}^{H',0}([0,T];E)$ is the separable subset of the above such that

$$\lim_{|t-s|\to 0}\frac{\|f(t)-f(s)\|_E}{|t-s|^{H'}}=0.$$

We denote by $x\wedge y$ the minimum of x and y. A good rate function I is a function such that for every c positive, $\{x:\ I(x)\leq c\}$ is compact.

We use the approach to the stochastic calculus with respect to the fBm developed in (1) for general Volterra processes and defined as a Skohorod integral. These processes are of the form

$X(t)=\int_0^t K(t,s)d\beta(s)$ where K is a triangular ($K(t,s)=0$ if $s>t$) and locally square integrable kernel. We denote for $h\in \mathrm{L}^2(0,T)$ and $t\in[0,T]$ $Kh(t)=\int_0^T K(t,s)h(s)ds$ and by $\mathcal{E}$ the set of step functions with inner product defined through the covariance

$$\left\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]}\right\rangle_{\mathcal{H}} = \left(K(t,\cdot)\mathbb{1}_{[0,t]}, K(s,\cdot)\mathbb{1}_{[0,s]}\right)_{\mathrm{L}^2(0,T)}.$$

The operator defined initially on $\mathcal{E}$ with values in $\mathrm{L}^2(0,T)$ by

$$\left(K_T^*\varphi\right)(s) = \varphi(s)K(T,s) + \int_s^T (\varphi(t)-\varphi(s))K(dt,s) \tag{2.1}$$

is an isometry and can be extended to the closure $\mathcal{H}$ of $\mathcal{E}$ for the norm of the inner product . Also, for φ in $\mathcal{E}$ and h in $\mathrm{L}^2(0,T)$ the following duality holds

$$\int_0^T \left(K_T^*\varphi\right)(t)h(t)dt = \int_0^T \varphi(t)(Kh)(dt) \tag{2.2}$$

and extends integration of step function with respect to $Kh(dt)$ to integrands in $\mathcal{H}$. Also the Skohorod integral with respect to X could be written in Itô form

$$\delta^X(\varphi) = \int_0^T \left(K_T^*\varphi\right)(t)d\beta(t), \quad \varphi\in\mathcal{H}.$$

We now focus on the fBm which kernel, given $c_H=\left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{\frac{1}{2}}$, is

$$K(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2}-H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1-\left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du, \tag{2.3}$$

In a Hilbert space F, the operator K_T^*, the space $\mathcal{H}$ with functions taking values in F and the stochastic integration of F-valued integrands with respect to the fBm are defined similarly. It is such that a scalar product with an element of F is the stochastic integral of the scalar product of the integrand. We may indeed check that K_T^* commutes with the scalar product with an element of F. The stochastic integral with respect to a fractional Wiener processes in a Hilbert space F when the integrand is a bounded operator Λ from a Hilbert space E to F is defined for t positive as, see for example (11),

$$\int_0^t \Lambda(s)dW^H(s) = \sum_{j\in\mathbb{N}}\int_0^t \Lambda(s)\Phi e_j d\beta_j^H(s) = \sum_{j\in\mathbb{N}}\int_0^t \left(K_t^*\Lambda(\cdot)\Phi e_j\right)(s)d\beta_j(s),$$

when $(\Lambda(t))_{t\in[0,T]}$ is such that

$$\sum_{j\in\mathbb{N}}\int_0^T \|\left(K_T^*\Lambda(\cdot)\Phi e_j\right)(t)\|_F^2 dt < \infty$$

where $(e_j)_{j\in\mathbb{N}}$ is a complete orthonormal system of E, here assumed to be L^2. We may also check from (2.2) that $(U(t))_{t\in\mathbb{R}}$ commutes with K_T^*. We use several times the following property, that we may check using (2.1) and (2.3), that

$$\left(K_T^*\mathbb{1}_{[0,t]}\varphi\right)(s) = \left(K_t^*\varphi\right)(s)\mathbb{1}_{[0,t]}(s), \quad 0<t<T. \tag{2.4}$$

3 The stochastic convolution

In this section we present a few properties of the stochastic convolution which corresponds to the Ornstein-Uhlenbeck case when there is no nonlinearity.
When we consider locally Lipshitz nonlinearities, also called saturated, precise assumptions are in Section 4, we treat singular kernels and state the results in spaces of Hölder continuous functions. We use the following Banach spaces

$$C_T^{H',H} = \mathrm{C}\left([0,T];\mathrm{H}^{1+2H}\right) \cap \mathrm{C}^{H'}\left([0,T];\mathrm{H}^1\right)$$

and

$$C_T^{H',H,0} = \mathrm{C}\left([0,T];\mathrm{H}^{1+2H}\right) \cap \mathrm{C}^{H',0}\left([0,T];\mathrm{H}^1\right).$$

The latter space is separable. For measurability issue, we define

$$C_\infty^{H,0} = \bigcap_{T>0,0<H'<H} C_T^{H',H,0}$$

equipped with the projective limit topology. It is a separable metrisable space. We make the following assumption

Assumption (N1)

$$\Phi \textit{ belongs to } \mathcal{L}_2\left(\mathrm{L}^2,\mathrm{H}^{1+2(H+\alpha)}\right) \textit{ with } \left(\frac{1}{2}-H\right)\mathbb{1}_{H<\frac{1}{2}} < \alpha < (1-H)\mathbb{1}_{H<\frac{1}{2}} + \mathbb{1}_{H\geq\frac{1}{2}}.$$

This assumption is used along with the fact that for γ in $[0,1)$ and t positive

$$\|U(t)-I\|_{\mathcal{L}_c(\mathrm{H}^{1+2\gamma},\mathrm{H}^1)} \leq 2^{1-\gamma}|t|^\gamma; \tag{3.1}$$

it could be proved using the Fourier transform.
When we consider Kerr nonlinearities when the space dimension is such that $d>2$ we impose

Assumption (N2)

$$\Phi \in \mathcal{L}_2^{0,2} \textit{ and } H > \frac{1}{2}.$$

In (2), the authors impose weaker assumptions on Φ, namely $\Phi \in \mathcal{L}_2^{0,1}$, and check the required integrabilty of the stochastic convolution. It is more intricate for a fractional noise. This integrability follows from the Strichartz inequalities under (N2), however this assumption is certainly too strong.

Under (N1), the following result on the stochastic convolution holds.

Lemma 3.1. *The stochastic convolution $Z: t \mapsto \int_0^t U(t-s)dW^H(s)$ is well defined. It has a modification in $C_\infty^{H,0}$ and defines a $C_\infty^{H,0}-$ random variable. Moreover, the direct images $\mu^{Z,T,H'}$ of its law μ^Z by the restriction on $C_T^{H',H,0}$ for T positive and $0<H'<H$ are centered Gaussian measures.*

Proof. The stochastic convolution is well defined since for t positive

$$\begin{aligned}&\textstyle\sum_{j\in\mathbb{N}}\int_0^t \|(K_t^* U(t-\cdot)\Phi e_j)(u)\|^2_{\mathrm{H}^{1+2H}}\,du\\ &=\textstyle\sum_{j\in\mathbb{N}}\int_0^t \left\|U(-u)\Phi e_j K(t,u)+\int_u^t (U(-r)-U(-u))\,\Phi e_j K(dr,u)\right\|^2_{\mathrm{H}^{1+2H}}du\\ &\le 2\|\Phi\|^2_{\mathcal{L}_2^{0,1+2H}}\textstyle\int_0^t K(t,u)^2du+2\sum_{j\in\mathbb{N}}\int_0^t\left\|\int_u^t (U(-r)-U(-u))\,\Phi e_j K(dr,u)\right\|^2_{\mathrm{H}^{1+2H}}du\\ &\le 2(T_1+T_2).\end{aligned}$$

Note that we used the continuous embedding of $\mathrm{H}^{1+2(H+\alpha)}$ into H^{1+2H}. The integral in T_1 is equal to $\mathbb{E}[(\beta^H(t))^2]=t^{2H}$. Using (3.1), we obtain

$$T_2\le 4^{1-\alpha}\|\Phi\|^2_{\mathcal{L}_2^{0,1+2(H+\alpha)}}c_H^2\left(\frac{1}{2}-H\right)^2\int_0^t\left(\int_u^t (r-u)^{H-\frac{3}{2}+\alpha}\left(\frac{r}{u}\right)^{H-\frac{1}{2}}dr\right)^2du$$

thus

$$T_2\le 4^{1-\alpha}\|\Phi\|^2_{\mathcal{L}_2^{0,1+2(H+\alpha)}}c_H^2\left(\frac{1}{2}-H\right)^2\int_0^t\left(\int_u^t (r-u)^{H-\frac{3}{2}+\alpha}dr\right)^2du,$$

the integral is well defined since $H-\frac{3}{2}+\alpha>-1$. We finally obtain

$$T_2\le\frac{4^{\frac{1}{2}-\alpha}\|\Phi\|^2_{\mathcal{L}_2^{0,1+2(H+\alpha)}}}{H+\alpha}\left(\frac{c_H\left(H-\frac{1}{2}\right)}{H-\frac{1}{2}+\alpha}\right)^2 t^{2H+2\alpha}.$$

Note that when $H>\frac{1}{2}$, the assumption on α is not necessary, indeed the kernel is null on the diagonal and its derivative is integrable. We could obtain directly

$$T_2\le\|\Phi\|^2_{\mathcal{L}_2^{0,1+2(H+\alpha)}}\int_0^t K(t,u)^2du=\|\Phi\|^2_{\mathcal{L}_2^{0,1+2(H+\alpha)}}t^{2H}.$$

We now prove that for any positive T and $0<H'<H$, Z has a modification in $C_T^{H',H,0}$. We prove that it has a modification which is in $\mathrm{C}^{H''}\left([0,T],\mathrm{H}^1\right)$ for some H'' such that $H'<H''<H$. It will thus belong to $\mathrm{C}^{H',0}\left([0,T],\mathrm{H}^1\right)$. Note that, as we are dealing with a centered Gaussian process, upper bounds on higher moments could be deduced from an upper bound on the second order moment; see (4) for a proof in the infinite dimensional setting. It is therefore enough to show that there exists positive C and $\gamma>H''$ such that for every $(t,s)\in[0,T]^2$.

$$\mathbb{E}\left[\|Z(t)-Z(s)\|^2_{\mathrm{H}^1}\right]\le C|t-s|^{2\gamma},$$

and then conclude with the Kolmogorov criterion.
When $0<s<t$, we have

$$\begin{aligned}Z(t)-Z(s)=\;&U(s)\,(U(t-s)-I)\textstyle\sum_{j\in\mathbb{N}}\int_0^T\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(-\cdot)\Phi e_j\right)(w)d\beta_j(w)\\ &+U(s)\textstyle\sum_{j\in\mathbb{N}}\int_0^T\left((K_t^*U(-\cdot)\Phi e_j)(w)-(K_s^*U(-\cdot)\Phi e_j)(w)\right)d\beta_j(w)\\ =\;&\tilde{T}_1(t,s)+\tilde{T}_2(t,s).\end{aligned}$$

We have

$$\begin{aligned}&\mathbb{E}\left[\left\|\tilde{T}_1(t,s)\right\|^2_{\mathrm{H}^1}\right]\\ &\le\|U(t-s)-I\|^2_{\mathcal{L}_c\left(\mathrm{H}^{1+2(H+\alpha)},\mathrm{H}^1\right)}\textstyle\sum_{j\in\mathbb{N}}\int_0^T\left\|\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(-\cdot)\Phi e_j\right)^2(w)\right\|^2_{\mathrm{H}^{1+2(H+\alpha)}}dw\\ &\le C(T,H,\alpha)\|\Phi\|^2_{\mathcal{L}_2^{0,1+2(H+\alpha)}}|t-s|^{2(H+\alpha)},\end{aligned}$$

where $C(T,H,\alpha)$ is a constant, and

$$\begin{array}{l}\mathbb{E}\left[\left\|\tilde{T}_2(t,s)\right\|_{\mathrm{H}^1}^2\right]\\ =\sum_{j\in\mathbb{N}}\int_0^T \|U(-u)\Phi e_jK(t,u)+\int_u^t\left(U(-r)-U(-u)\right)\Phi e_jK(dr,u)\\ \qquad -U(-u)\Phi e_jK(s,u)-\int_u^s\left(U(-r)-U(-u)\right)\Phi e_jK(dr,u)\quad \|_{\mathrm{H}^1}^2du\\ \leq \sum_{j\in\mathbb{N}}\left(\tilde{T}_{21}^j+\tilde{T}_{22}^j+\tilde{T}_{23}^j\right),\end{array}$$

where, using the fact that the kernel is triangular,

$$\begin{array}{ll}\tilde{T}_{21}^j &= \int_0^s\left\|U(-u)\Phi e_j\left(K(t,u)-K(s,u)\right)+\int_s^t\left(U(-r)-U(-u)\right)\Phi e_jK(dr,u)\right\|_{\mathrm{H}^1}^2du,\\ \tilde{T}_{22}^j &= 2\int_s^t\|U(-u)\Phi e_jK(t,u)\|_{\mathrm{H}^1}^2du\\ \tilde{T}_{23}^j &= 2\int_s^t\left\|\int_u^t\left(U(-r)-U(-u)\right)\Phi e_jK(dr,u)\right\|_{\mathrm{H}^1}^2du.\end{array}$$

We have

$$\begin{array}{ll}\tilde{T}_{21}^j &= \int_0^s\left\|\int_s^tU(-r)\Phi e_jK(dr,u)\right\|_{\mathrm{H}^1}^2du\\ &= \|\Phi e_j\|_{\mathrm{H}^1}^2\int_0^s\left(\int_s^t|K(dr,u)|\right)^2du\\ &= \|\Phi e_j\|_{\mathrm{H}^1}^2\int_0^s\left(K(t,u)-K(s,u)\right)^2du\end{array}$$

thus

$$\begin{array}{ll}\tilde{T}_{21}^j &\leq \|\Phi e_j\|_{\mathrm{H}^1}^2\int_0^t\left(K(t,u)-K(s,u)\right)^2du\\ &\leq \|\Phi e_j\|_{\mathrm{H}^1}^2\,\mathbb{E}\left[\left(\beta^H(t)-\beta^H(s)\right)^2\right]\\ &\leq \|\Phi e_j\|_{\mathrm{H}^1}^2|t-s|^{2H},\end{array}$$

and

$$\begin{array}{ll}\tilde{T}_{22}^j &= 2\|\Phi e_j\|_{\mathrm{H}^1}^2\int_s^tK(t,u)^2du\\ &= 2\|\Phi e_j\|_{\mathrm{H}^1}^2\int_s^t\left(K(t,u)-K(s,u)\right)^2du\end{array}$$

thus

$$\begin{array}{ll}\tilde{T}_{22}^j &\leq 2\|\Phi e_j\|_{\mathrm{H}^1}^2\int_0^t\left(K(t,u)-K(s,u\right)^2du\\ &\leq 2\|\Phi e_j\|_{\mathrm{H}^1}^2|t-s|^{2H},\end{array}$$

finally the same computations as above shows that when $H-\frac{3}{2}+\alpha>-1$ (used for integrability issue when $H<\frac{1}{2}$), we have

$$\begin{array}{ll}\tilde{T}_{23}^j &\leq 4^{1-(H+\alpha)}\|\Phi\|_{\mathcal{L}_2^{0,1+2(H+\alpha)}}^2c_H^2\left(H-\frac{1}{2}\right)^2\int_s^t\left(\int_u^t(r-u)^{2H-\frac{3}{2}+\alpha}\left(\frac{r}{u}\right)^{H-\frac{1}{2}}dr\right)^2du\\ &\leq \frac{4^{\frac{1}{2}-(H+\alpha)}\|\Phi\|_{\mathcal{L}_2^{0,1+2(H+\alpha)}}^2}{2H+\alpha}\left(\frac{c_H\left(H-\frac{1}{2}\right)}{2H-\frac{1}{2}+\alpha}\right)^2(t-s)^{4H+2\alpha}.\end{array}$$

When $H>\frac{1}{2}$ the kernel is zero on the diagonal, its derivative has constant sign and it is integrable thus we can obtain without the assumption on α

$$\begin{array}{ll}\tilde{T}_{23}^j &\leq 4\int_s^t\|\Phi e_j\|_{\mathrm{H}^1}^2\left(\int_u^t|K(dr,u)|\right)^2du\\ &\leq 4\|\Phi e_j\|_{\mathrm{H}^1}^2\int_s^tK(t,u)^2du\\ &\leq 4\|\Phi e_j\|_{\mathrm{H}^1}^2\,\mathbb{E}\left[|\beta^H(t)-\beta^H(s)|^2\right]\\ &\leq 4\|\Phi e_j\|_{\mathrm{H}^1}^2|t-s|^{2H}.\end{array}$$

Thus Z admits a modification with $H''-$Hölder continuous sample paths with $H' < H'' < H$. We now explain why Z has a modification which is in $\mathrm{C}\left([0,T],\mathrm{H}^{1+2H}\right)$. Since the group is an isometry we have

$$\begin{aligned}\|Z(t)-Z(s)\|_{\mathrm{H}^{1+2H}} \leq\ & \left\|(U(t-s)-I)\sum_{j\in\mathbb{N}}\int_0^T\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(-\cdot)\Phi e_j\right)(w)d\beta_j(w)\right\|_{\mathrm{H}^{1+2H}}\\ & +\left\|\tilde{T}_2(t,s)\right\|_{\mathrm{H}^{1+2H}}.\end{aligned}$$

Since the group is strongly continuous and since, from the above,

$$\sum_{j\in\mathbb{N}}\int_0^T\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(-\cdot)\Phi e_j\right)(w)d\beta_j(w)$$

belongs to H^{1+2H}, the first term of the right hand side goes to zero as s converges to t. Also, we may write

$$\left\|\tilde{T}_2(t,s)\right\|_{\mathrm{H}^{1+2H}} \leq \|Y(t)-Y(s)\|_{\mathrm{H}^{1+2H}}$$

where $(Y(t))_{t\in[0,T]}$, defined for $t\in[0,T]$ by

$$Y(t)=\sum_{j\in\mathbb{N}}\int_0^T\left(K_t^*U(-\cdot)\Phi e_j\right)(w)d\beta_j(w),$$

is a Gaussian process. We again conclude, with the same bounds for $\tilde{T}_{21}^j$ and $\tilde{T}_{22}^j$ and an upper of the order of $(t-s)^{2H+2\alpha}$ for $\tilde{T}_{23}^j$ and using the Kolmogorov criterion, that $Y(t)$ admits a modification with continuous sample paths. Thus, for such a modification of Y, Z has continuous sample paths.
The fact that $\mu^{Z,T}$ are Gaussian measures follows from the fact that Z is defined as

$$\sum_{j\in\mathbb{N}}\int_0^t\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(t-\cdot)\Phi e_j\right)(s)d\beta_j(s).$$

The law is Gaussian since the law of the action of an element of the dual is a pointwise limit of Gaussian random variables; see for example (7).
It is a standard fact to prove that the process defines a $C_\infty^{H,0}$ random variable, see for example (7) for similar arguments. We use the fact that the process takes its values in a separable metrisable space. □

Remark 3.2. *The assumption on α seems too strong to have the desired Hölder exponent. It is required only for integrability in the upper bounds of T_2 and $\tilde{T}_{23}^j$. Also, the assumption that Φ is Hilbert-Schmidt in a Sobolev space of exponent at least $1+2H$ is only required in order that the convolution is a H^{1+2H} valued process. Indeed, there is a priori no reason that a Hölder continuous stochastic convolution gives rise to a Hölder continuous solution to the stochastic NLS equations. It is obtained by assuming extra space regularity of the solution.*

In the following we consider such a modification. The next lemma characterizes the Reproducing Kernel Hilbert Space (RKHS) of such Gaussian measures.

Lemma 3.3. *The covariance operator of Z on $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ is given by*

$$\begin{aligned}\mathcal{Q}h(t) = \textstyle\sum_{j\in\mathbb{N}}\int_0^T\int_0^{t\wedge u} & \left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(t-\cdot)\Phi e_j\right)(s) \\ & \left(\left(K_T^*\mathbb{1}_{[0,u]}(\cdot)U(u-\cdot)\Phi e_j\right)(s),h(u)\right)_{\mathrm{L}^2} dsdu,\end{aligned}$$

for $h\in\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$. When $H>\frac12$ we may write $\mathcal{Q}h(t)$ as

$$c_H^2\left(H-\frac12\right)^2\beta\left(2-2H,H-\frac12\right)\int_0^T\int_0^t\int_0^s|u-v|^{2H-2}U(t-v)\Phi\Phi^*U(u-s)h(s)dudvds.$$

For T positive and $0<H'<H$, the RKHS of $\mu^{Z,T,H'}$ is $\operatorname{im}\mathcal{Q}^{\frac12}$ *with the norm of the image structure. It is also* $\operatorname{im}\mathcal{L}$ *where for $h\in\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$*

$$\mathcal{L}h(t)=\sum_{j\in\mathbb{N}}\int_0^t\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(t-\cdot)\Phi e_j\right)(s)(h(s),e_j)_{\mathrm{L}^2}ds=\int_0^t U(t-s)\Phi Kh(ds).$$

Proof. We may first check with the same computations as those used in Lemma 3.1 that $\mathcal{L}$ is well defined and that for h in $\mathrm{L}^2(0,T;\mathrm{L}^2)$, $\mathcal{L}h$ belongs to $\mathrm{L}^2(0,T;\mathrm{L}^2)$. Take h and k in $\mathrm{L}^2(0,T;\mathrm{L}^2)$, we have

$$\begin{aligned}&\mathbb{E}\left[\int_0^T(Z(u),h(u))_{\mathrm{L}^2}du\int_0^T(Z(t),k(t))_{\mathrm{L}^2}dt\right]\\ &=\textstyle\sum_{j\in\mathbb{N}}\mathbb{E}\Big[\int_0^T\int_0^T\left(\int_0^T\left(K_T^*\mathbb{1}_{[0,u]}(\cdot)U(u-\cdot)\Phi e_j\right)(s)d\beta_j(s),h(u)\right)_{\mathrm{L}^2}\\ &\qquad\qquad\left(\int_0^T\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(t-\cdot)\Phi e_j\right)(v)d\beta_j(v),k(t)\right)_{\mathrm{L}^2}\Big]\\ &=\int_0^T(\mathcal{Q}h(t),k(t))_{\mathrm{L}^2}dt\end{aligned}$$

where $\mathcal{Q}$ is defined in the lemma. When $H>\frac12$ the inner product in $\mathcal{H}$ takes a simpler form, see for example (11), which gives the corresponding expression of the covariance operator. Checking that for $k\in\mathrm{L}^2(0,T;\mathrm{L}^2)$,

$$\mathcal{L}^*k(s)=\sum_{j\in\mathbb{N}}\int_s^T\left(\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(t-\cdot)\Phi e_j\right)(s),k(t)\right)_{\mathrm{L}^2}e_jdt,$$

we obtain that $\mathcal{Q}=\mathcal{L}\mathcal{L}^*$. We may thus deduce, see for example (7), that the RKHS of $\mu^{Z,T,H'}$ is also $\operatorname{im}\mathcal{L}$ with the norm of the image structure. □

When we impose (N2) we can prove as above that the stochastic convolution Z has a modification in $\mathrm{C}\left([0,\infty);\mathrm{H}^2\right)$ embedded with the projective limit topology letting the time interval go to infinity. Thus from the Sobolev embeddings, for T positive and $(r(p),p)$ an admissible pair, Z belongs to $X^{(T,p)}=\mathrm{C}\left([0,T];\mathrm{H}^1\right)\cap\mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right)$. This space is usually considered in the fixed point argument proving the local well-posedness for Kerr nonlinearities. We may check

Lemma 3.4. *Z defines a $\mathrm{C}\left([0,\infty);\mathrm{H}^2\right)$-random variable. The laws of the projections $\mu^{Z,T}$ on $\mathrm{C}\left([0,T];\mathrm{H}^2\right)$ for T positive are centered Gaussian measures of RKHS* $\operatorname{im}\mathcal{L}$.

We deduce the following result that we push forward to obtain results for the solution of the SPDE. The proof is classical, see for example (7).

Proposition 3.5. *The direct image measures for ϵ positive of $x \mapsto \sqrt{\epsilon}x$ on $C_\infty^{H,0}$, respectively* $\mathrm{C}\left([0,\infty);\mathrm{H}^2\right)$*, satisfy a LDP of speed ϵ and good rate function*

$$I^Z(f) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\ \mathcal{L}(h)=f} \left\{ \|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} \right\}.$$

Under (N1) the support of the measure μ^Z is given by

$$\mathrm{supp}\ \mu^Z = \overline{\mathrm{im}\ \mathcal{L}}^{C_\infty^{H,0}},$$

under (N2) the same result holds replacing $C_\infty^{H,0}$ by $\mathrm{C}\left([0,\infty);\mathrm{H}^2\right)$.

4 Local well-posedness of the Cauchy problem

We consider two cases. Either we assume (N1), an initial datum such that $u_0 \in \mathrm{H}^{1+2H}$ and

Assumption (NL)

$$\begin{array}{rl} (i) & f \text{ is Lipschitz on the bounded sets of } \mathrm{H}^{1+2H} \\ (ii) & f(0) = 0. \end{array}$$

or we assume (N2), $u_0 \in \mathrm{H}^1$ and f is a Kerr nonlinearity.
We first recall the following important fact. If $v^{u_0}(z)$ denotes the solution of

$$\left\{ \begin{array}{l} i\frac{\mathrm{d}v}{\mathrm{d}t} = \Delta v + f(v - iz) \\ u(0) = u_0. \end{array} \right. \tag{4.1}$$

where z belongs to $C_\infty^{H,0}$ (respectively $\mathrm{C}\left([0,\infty),\mathrm{H}^2\right)$) and $\mathcal{G}^{u_0}$ is the mapping

$$\mathcal{G}^{u_0}:\ z \mapsto v^{u_0}(z) - iz$$

then the solution u^{ϵ,u_0} of stochastic NLS equation is such that $u^{\epsilon,u_0} = \mathcal{G}^{u_0}(\sqrt{\epsilon}Z)$ where Z is the stochastic convolution.
We may check with a fixed point argument the following result, see (2).

Theorem 4.1. *Assume that u_0 is $\mathcal{F}_0$ measurable and belongs to* H^{1+2H} *(respectively* H^1*); then there exists a unique solution to* (1.2) *with continuous* H^{1+2H} *(respectively* H^1*) valued paths. The solution is defined on a random interval* $[0,\tau^*(u_0,\omega))$ *where* $\tau^*(u_0,\omega)$ *is either* ∞ *or a finite blow-up time.*

In the next sections we state sample paths LDPs and support theorems, the proofs are not specific to stochastic NLS equations and similar proofs could be found in (7). With the first set of assumptions we state a result in a space of Hölder continuous paths with any value of the Hurst parameter. In the last section we consider the case of Kerr nonlinearities when $H > \frac{1}{2}$.

5 The case of a nonlinearity satisfying (NL)

Since solutions may blow-up in finite time, we proceed as in (7) to define proper path spaces where we can state the LDP and support result. Here we consider a space where paths are $H'-$Hölder continuous ($0 < H' < H$) with values in H^1 on compact time intervals before blow-up. We add a point Δ to the space H^{1+2H} and embed the space with the topology such that its open sets are the open sets of H^{1+2H} and the complement in $\mathrm{H}^{1+2H} \cup \{\Delta\}$ of the closed bounded sets of H^{1+2H}. The set $\mathrm{C}([0,\infty); \mathrm{H}^{1+2H} \cup \{\Delta\})$ is now well defined. We denote the blow-up time of f in $\mathrm{C}([0,\infty); \mathrm{H}^{1+2H} \cup \{\Delta\})$ by $\mathcal{T}(f) = \inf\{t \in [0,\infty) : \ f(t) = \Delta\}$, with the convention that $\inf \emptyset = \infty$. We define, setting Δ as a cemetery,

$$\begin{aligned}\mathcal{E}^H\left(\mathrm{H}^1\right) = \ & \big\{ f \in \mathrm{C}([0,\infty); \mathrm{H}^{1+2H} \cup \{\Delta\}) : \ f(t_0) = \Delta \Rightarrow \forall t \geq t_0, \ f(t) = \Delta; \\ & \forall T < \mathcal{T}(f), \ \forall\, 0 < H' < H, \ f \in \mathrm{C}^{H'}\left([0,T]; \mathrm{H}^1\right) \Big\}.\end{aligned}$$

It is endowed with the topology defined by the neighborhood basis

$$V_{T,R,H'}(\varphi_1) = \left\{ \varphi \in \mathcal{E}^H\left(\mathrm{H}^1\right) : \ \mathcal{T}(\varphi) > T, \ \|\varphi_1 - \varphi\|_{C_T^{H',H}} \leq R \right\},$$

of φ_1 in $\mathcal{E}^H\left(\mathrm{H}^1\right)$ given $T < \mathcal{T}(\varphi_1)$ and R positive. The space is a Hausdorff topological space and thus we can apply the Varadhan contraction principle.

Lemma 5.1. *The mapping*

$$\begin{array}{ccc} C_\infty^{H,0} & \to & \mathcal{E}^H\left(\mathrm{H}^1\right) \\ z & \mapsto & \mathcal{G}^{u_0}(z) \end{array}$$

is continuous.

Proof. This could be done by revisiting the fixed point argument, this time in $C_{T^*}^{H'}$ for T^* small enough depending on the norm of the initial data and z in $C_T^{H'}$ for some fixed T and some $H' < H$ fixed. Though with different norms, the remaining of the argument allowing to prove the continuity of $v^{u_0}(z)$ with respect to z, detailed in (2), holds. In the computations we use (3.1) in order to treat the Hölder norms. □

Lemma 3.1 and 5.1 give that u^{1,u_0} and the mild solutions u^{ϵ,u_0} of

$$\left\{ \begin{array}{l} i\mathrm{d}u - (\Delta u + f(u))\mathrm{d}t = \sqrt{\epsilon}\mathrm{d}W^H, \\ u(0) = u_0 \ \in \mathrm{H}^{1+2H}. \end{array} \right. \tag{5.1}$$

define $\mathcal{E}^H\left(\mathrm{H}^1\right)$ random variables. We denote by $\mu^{u^{\epsilon,u_0}}$ the laws. We deduce from Lemma 3.3 and 5.1, the fact that $\left(\mathcal{G}^{u_0} \circ \mathcal{L}\right)(\cdot) = \mathbf{S}(u_0, \cdot)$, and the Varadhan contraction principle the following theorem.

Theorem 5.2. *The probability measures $\mu^{u^{\epsilon,u_0}}$ on $\mathcal{E}^H\left(\mathrm{H}^1\right)$ satisfy a LDP of speed ϵ and good rate function*

$$I^{u_0}(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2): \ \mathbf{S}(u_0,h)=w} \left\{ \|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} \right\},$$

where $\mathbf{S}(u_0,h)$ denotes the mild solution in $\mathcal{E}^H\left(\mathrm{H}^1\right)$ of the following control problem

$$\left\{ \begin{array}{l} i\frac{\partial u}{\partial t} - (\Delta u + f(u)) = \Phi \dot{K} h, \\ u(0) = u_0 \ \in \mathrm{H}^{1+2H}, \ h \in \mathrm{L}^2\left(0,\infty; \mathrm{L}^2\right); \end{array} \right. \tag{5.2}$$

it is called the skeleton. Only the integral, or the integral in the mild formulation, of the right hand side is defined; it is by means of the duality relation.

Remark 5.3. *We could also prove a uniform LDP as for example in (8).*

The following theorem characterizes the support of the solutions.

Theorem 5.4. *The support of the law $\mu^{u^{1,u_0}}$ on $\mathcal{E}^H\left(\mathrm{H}^1\right)$ is given by*

$$\operatorname{supp} \mu^{u^{1,u_0}} = \overline{\operatorname{im} \mathbf{S}}^{\mathcal{E}^H\left(\mathrm{H}^1\right)}.$$

6 The case of Kerr nonlinearities

In this section we consider Kerr nonlinearities when $d \geq 2$ and $\sigma < \frac{2}{d-2}$. This time, we do not state a result in a space of Hölder continuous functions with values in H^1. We would need that the convolution which involves the nonlinearity is Hölder continuous. Thus, in order to use (3.1), we would have to compute the Sobolev norm of the nonlinearity in some space $\mathrm{H}^{1+2\gamma}$ where γ is positive.

Remark 6.1. *When $H < \frac{1}{2}$, we could state a weaker result than in the previous section imposing that $u_0 \in \mathrm{H}^1$ and $\Phi \in \mathcal{L}_2^{0,2+\alpha}$. The fixed point could be conducted in $\mathrm{C}^{H'}\left([0,T];\mathrm{H}^{1-2H}\right) \cap \mathrm{C}\left([0,T];\mathrm{H}^1\right) \cap \mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right)$ where $(r(p),p)$ is an admissible pair and $0 < H' < H$ and uses the Strichartz inequalities. Indeed, from the Sobolev embeddings, the stochastic convolution has a modification in $\mathrm{C}\left([0,T];\mathrm{H}^2\right) \cap \mathrm{C}^{H'}\left([0,T];\mathrm{H}^{2-2H}\right)$ and thus belongs to the desired space.*

Let us return to the case $H > \frac{1}{2}$. Since under (N2) Z has a modification in $X^{(T,p)}$, we can use the continuity of the solution with respect to the convolution of the perturbation with $U(t)$ of (7) and repeat the argument. Thus, for initial data in H^1, we may state a LDP and support result in the space

$$\begin{aligned}\mathcal{E}_\infty = \Big\{ & f \in \mathrm{C}([0,\infty);\mathrm{H}^1 \cup \{\Delta\}) : \ f(t_0) = \Delta \Rightarrow \forall t \geq t_0, \ f(t) = \Delta; \\ & \forall T < \mathcal{T}(f), \ \forall \, p \in \left[2, \tfrac{2d}{d-2}\right), f \in \mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right)\Big\}.\end{aligned}$$

When $d = 2$ or $d = 1$ we write $p \in [2,\infty)$. The space is embedded with the topology defined by the neighborhood basis

$$W_{T,p,R}(\varphi_1) = \left\{\varphi \in \mathcal{E}_\infty : \mathcal{T}(\varphi) \geq T, \ \|\varphi_1 - \varphi\|_{X^{(T,p)}} \leq R\right\}, \quad \text{for } \varphi_1 \in \mathcal{E}_\infty.$$

Theorem 6.2. *$\mu^{u^{\epsilon,u_0}}$ satisfy a LDP on $\mathcal{E}_\infty$ of speed ϵ and good rate function*

$$I^{u_0}(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2): \ \mathbf{S}(u_0,h)=w} \left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\},$$

where $\mathbf{S}(u_0,h)$ is the mild solution of

$$\left\{\begin{array}{l} i\frac{\partial u}{\partial t} - (\Delta u + \lambda |u|^{2\sigma} u) = \Phi \dot{K} h, \\ u(0) = u_0 \ \in \mathrm{H}^1, \ h \in \mathrm{L}^2\left(0,\infty;\mathrm{L}^2\right); \end{array}\right. \tag{6.1}$$

Theorem 6.3. *The support of the law $\mu^{u^{1,u_0}}$ on $\mathcal{E}_\infty$ is given by*

$$\text{supp } \mu^{u^{1,u_0}} = \overline{\text{im } \mathbf{S}}^{\mathcal{E}_\infty}.$$

In the case where all the parameters are set to one, it is interesting to study the effect a fractional noise would have on the error in transmission by soliton in fibers compared to what has been obtained for white in time noises in (6; 7). We believe that the Hurst parameter of the fractional Wiener process has an important effect on the tails of the mass and arrival time of a the signal at the end of the line. Upper bounds have been obtained using energy inequalities for the controlled PDEs. When $H > 3/2$, upper bounds of the liminf as the amplitude of the noise goes to zero of the log probability of the tails of $-T^{2H}$ and $-T^{2+2H}$ respectively for the mass and arrival time, T is the length of the fiber line, are easily obtained. For smaller Hurst parameters it seems difficult to carry similar computations since the operator $\dot{\mathrm{K}}\mathrm{h}$ is not necessarily continuous. We have given sense to the stochastic convolution by convoluting with the group and assuming extra smoothness assumptions on Φ. This idea is also used when the noise is replaced by a control. However, the energy inequalities used do not rely on the mild form and integrals may be singular. Lower bounds were obtained restricting to controls giving rise to a well chosen soliton ansatz with fluctuating parameters and studying the associated problem of the calculus of variations. For $H \neq 1/2$ the operator $\dot{\mathrm{K}}\mathrm{h}$ has to be inverted as well.
The order in T of the upper bounds makes sense and we expect it could hold for smaller H as well. If similar lower bounds could be obtained it would indicate that the tails increase with H. For $H > 1/2$ tails would be larger than the white in time noise case and induced by the positive correlation of past and future increments of the fractional Wiener process or long range dependence. This would indicate that the process is more bold to explore the space further from the deterministic soliton solution. In contrast when $H < 1/2$ and the correlation of past and future increments of the driving process is negative, we expect the random solution to explore less the space away from the soliton giving rise to smaller tails of the processes impairing soliton transmission and ultimately exponentially reducing the annoying fluctuation of the arrival time called timing-jitter. Note that control elements are mostly suggested as a solution to reduce the undesirable timing-jitter.

Remark 6.4. *Similar difficulties as those above arise if we study the tails of the blow-up times. As mentioned in the introduction, it would be interesting to see if the self-similarity of the driving process interacts with the self-similarity in space of the NLS solution near blow-up. Concerning the exit problem, we used in (9) for a white in time noise the strong Markov property. This no longer holds for fractional noises. Also, treating multiplicative noises is much more involved than additive noise since, for example, the stochastic convolution defined through regular integral with respect to the Brownian motion is anticipating. It is for that reason that we do not investigate the global existence since the Itô formula applied to the Hamiltonian and mass to a certain power as in (2) gives rise to anticipating stochastic integrals. We expect that a rough paths approach could be applied to treat multiplicative noises.*

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