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Points of positive density for smooth functionals

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Abstract. In this paper we show that the set of points where the density of a Wiener functional is strictly positive is an open connected set, assuming some regularity conditions.
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1 Introduction

The stochastic calculus of variations has been applied to derive properties of the support of a given Wiener functional. In [3] Fang proved that the support of a smooth Wiener functional is a connected set using the techniques of the quasi-sure analysis and the Ornstein-Uhlenbeck process. A simple proof of the connectivity of the support for random vectors whose components belong to $\mathbf{D}^{1,p}$ with $p > 1$ is given in [6, Proposition 4.1.1] using the Wiener chaos expansion.

An interesting question is to study the properties of the set where the density $p(x)$ of an m-dimensional Wiener functional F is positive. In the one-dimensional case we know that the density is always strictly positive in the interior of the support (which is a closed interval) if the random variable belongs to $\mathbf{D}^{1,p}$ with $p > 2$ and it possesses a locally Lipschitz density ([5]). In dimension bigger than one this result is not true. In [4] the authors present a simple example of a two-dimensional nondegenerate smooth Wiener functional whose density vanishes in the interior of the support. As a consequence, the set $\Gamma = \{x : p(x) > 0\}$ is, in general, strictly included in the interior of the support of the law of the functional.

In [4], using the approach introduced by Fang in [3] to handle the connectivity of the support, Hirsch and Song proved that the open set Γ is connected. The aim of this paper is to prove that the open set Γ is connected using the ideas introduced in the proof of the one-dimensional case, and assuming weak regularity assumptions on the Wiener functional.

2 Preliminaries

We will first introduce the basic notations and present some preliminary results that will be needed later.

Suppose that H is a real separable Hilbert space whose norm and inner product are denoted by $\|\cdot\|_H$ and $\langle\cdot\rangle_H$, respectively. We associate with H a Gaussian and centered family of random variables $W = \{W(h), h \in H\}$ such that

$$E(W(h)W(g)) = \langle h, g\rangle_H,$$

for all $h, g \in H$.

Let $\mathcal{S}$ denote the class of smooth random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)), \tag{2.1}$$

where f belongs to $C_p^\infty(\mathbf{R}^n)$ (i.e., f and all of its partial derivatives have polynomial growth order). If F has the form (2.1) we define its derivative DF as the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n))h_i. \tag{2.2}$$

For any real number $p \geq 1$ and any positive integer k will denote by $\mathbf{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p = E(|F|^p) + \sum_{j=1}^{k} E(\|D^j F\|_{H^{\otimes j}}^p),$$

where D^j denotes the jth iteration of the operator D. Set $\mathbf{D}^\infty = \cap_{p\geq 1} \cap_{k\geq 1} \mathbf{D}^{k,p}$. For any separable real Hilbert space V the spaces $\mathbf{D}^{k,p}(V)$ of V-valued functionals are introduced in a similar way.

We will denote by δ the adjoint of the operator D which is continuous from $\mathbf{D}^{k,p}(H)$ into $\mathbf{D}^{k-1,p}(H)$ for all $p > 1$, $k \geq 1$.

The following results are proved in [6, Lemma 1.4.2, Lemma 2.4.2].

Lemma 2.1 *Suppose that a set $A \in \mathcal{F}$ verifies $\mathbf{1}_A \in \mathbf{D}^{1,1}$. Then $P(A)$ is zero or one.*

Lemma 2.2 *Let $\{F_n, n \geq 1\} \in \mathbf{D}^{1,p}$, $p > 1$ be a sequence of random variables converging to F in L^p. Suppose that $\sup_n \|DF_n\|_{L^p(\Omega;H)} < \infty$. Then $F \in \mathbf{D}^{1,p}$, and there exists a subsequence $\{F_{n(i)}, i \geq 1\}$ which converges to F in the weak topology of $L^p(\Omega; H)$.*

The spaces $\mathbf{D}^{1,p}$ are stable under the composition with Lipschitz functions. More precisely, we have the following result ([5, Proposition 1.2.3]):

Proposition 2.1 *Let $\phi : \mathbf{R}^m \to \mathbf{R}$ be a function such that*

$$|\phi(x) - \phi(y)| \leq \mathrm{K}|x - y|$$

for any $x, y \in \mathbf{R}^m$. Suppose that $G = (G^1, G^2, \dots, G^m)$ is a random vector whose components belong to the space $\mathbf{D}^{1,p}$, $p > 1$. Then $\phi(G)$ belongs to $\mathbf{D}^{1,p}$, and there exists a random vector $S = (S_1, S_2, \dots, S_m)$ bounded by K such that

$$D(\phi(G)) = \sum_{i=1}^{m} S_i DG^i.$$

3 Connectivity of the set of positive density

Let us introduce the following condition of a random vector F:

(H): $F = (F^1, F^2, \ldots, F^m)$ possesses a $\mathcal{C}^2$ density p with respect to the Lebesgue measure such that

$$M := \int_{\mathbf{R}^m} \sup_{|z-x|\leq 1} |\nabla^2 p(z)| dx < +\infty.$$

The main result of this section is the following theorem:

Theorem 3.1 *Let $F = (F^1, F^2, \ldots, F^m) \in (\mathbf{D}^{1,r})^m$, $r > 2$ be a random vector satisfying hypothesis (H). Set $\Gamma = \{x \in \mathbf{R}^m : p(x) > 0\}$. Then Γ is a pathwise connected open set of $\mathbf{R}^m$.*

Proof: It is sufficient to prove that Γ is connected because an open connected set of $\mathbf{R}^m$ is pathwise connected. Let A be a connected component of Γ.

For each $\varepsilon > 0$, let $f_\varepsilon : \mathbf{R}^m \to \mathbf{R}^+$ be the function defined by :

$$f_\varepsilon(x) = \frac{d(x, A^c)}{\varepsilon} \wedge 1.$$

That is,

$$f_\varepsilon(x) = \begin{cases} \frac{d(x,A^c)}{\varepsilon} & \text{if } \ 0 < d(x, A^c) < \varepsilon \\ 1 & \text{if } \ d(x, A^c) \geq \varepsilon \\ 0 & \text{if } \ x \in A^c \end{cases}$$

This implies clearly that f_ε is a Lipschitzian function with Lipschitz constant $\frac{1}{\varepsilon}$.

Set $\Phi_\varepsilon = f_\varepsilon(F)$. Using Proposition 2.1 with $\phi = f_\varepsilon$, $G = F$ and $p = r$, it is clear that the functional Φ_ε belongs to $\mathbf{D}^{1,r}$ and its derivative is given by the formula :

$$D\Phi_\varepsilon = \sum_{i=1}^m S_i DF^i$$

where the S_i verify $\sqrt{\sum_{i=1}^m S_i^2} \leq \frac{1}{\varepsilon}$. These random variables cancel almost surely outside the set $\{0 < d(F, A^c) < \varepsilon\}$ because $D\Phi_\varepsilon(F) = 0$ a.s. on the two sets : $\{F \in A^c\}$ and $\{d(F, A^c) \geq \varepsilon\}$, due to the local property of the derivative operator.

Clearly Φ_ε converges a.s. and in L^p for each $p \geq 1$ to $\mathbf{1}_A(F)$ as ε goes to zero. Hence, if we prove that

$$\sup_\varepsilon E(||D\Phi_\varepsilon||_H^p) < +\infty \tag{3.3}$$

for some $p > 1$, Lemma 2.2 will imply that $\mathbf{1}_A(F)$ belongs to $\mathbf{D}^{1,p}$. But, according to Lemma 2.1 $\mathbf{1}_A(F) \in \mathbf{D}^{1,p}$ is equivalent to $P(F \in A^c) = 0$ or 1. The definition of A implies that $P(F \in A) > 0$. Hence, $P(F \in A) = 1$, and the proof will be complete.

Let us prove the uniform estimate (3.3) for the derivatives. We have :

$$||D\Phi_\varepsilon||_H \leq \frac{1}{\varepsilon}||DF||_H \mathbf{1}_{\{0<d(F,A^c)<\varepsilon\}}.$$

Hölder's inequality implies that for every $1 \leq p \leq r$:

$$E[||D\Phi_\varepsilon||_H^p] \leq \frac{1}{\varepsilon^p}[E(||DF||_H^r)]^{\frac{p}{r}}[P(0 < d(F,A^c) < \varepsilon)]^{\frac{r-p}{r}}$$

We can express

$$P\{0 < d(F,A^c) < \varepsilon\} = \int_{\{0<d(x,A^c)<\varepsilon\}} p(x)dx.$$

Let $x \in \mathbf{R}^m$ be a point such that $0 < d(x,A^c) < \varepsilon$. The set A^c being closed, we can find a point $\bar{x}$ in A^c such that $d(x,A^c) = d(x,\bar{x})$. The point $\bar{x}$ belongs to the boundary of A. This implies $p(\bar{x}) = 0$ which corresponds to a minimum of the function p, so $\nabla p(\bar{x}) = 0$. Using the Tayor expansion, we can write :

$$p(x) = p(x) - p(\bar{x}) = \int_0^1 (1-\theta)\sum_{i,j}(\nabla_i\nabla_j p)(\bar{x}+\theta(x-\bar{x}))(x_i-\bar{x}_i)(x_j-\bar{x}_j)d\theta.$$

This implies that for $0 < \varepsilon < 1$, one has the bound

$$p(x) \leq \frac{\varepsilon^2}{2}\sup_{|z-x|\leq 1}|\nabla^2 p(z)|.$$

Coming back to the estimate and using hypothesis (H), we have :

$$E[||D\Phi_\varepsilon||_H^p] \leq \frac{M}{2}\frac{1}{\varepsilon^p}[E(||DF||^r)]^{\frac{p}{r}}\varepsilon^{\frac{2(r-p)}{r}}.$$

It remains to note that given $r > 2$, there exists $p = \frac{2r}{r+2} > 1$ proving the uniform estimate. $\square$

4 Appendix

We give here a sufficient condition for (H). Let γ be the Malliavin covariance matrix of F:

$$\gamma^{ij} = \langle DF^i, DF^j \rangle_H.$$

Proposition 4.1 *Suppose that there exist real numbers s_1 and s_2 depending on m such that F satisfies:*

$$\begin{array}{ll} \text{(i)} & (\det \gamma)^{-1} \in L^{s_1} \\ \text{(ii)} & F \in \mathbf{D}^{m+3,s_2}. \end{array}$$

Then (H) is fulfilled.

Proof: We decompose the integral appearing in (H) into 2^m integrals on each of the 2^m quadrants of $\mathbf{R}^m$:

$$M = \sum_{n=1}^{2^m} \int_{Q^n} \sup_{|x-z|\leq 1} |\nabla^2 p(z)| dx = \sum_{n=1}^{2^m} M_n.$$

We take Q^n as a generic quadrant of $\mathbf{R}^m$ that we write as (using an eventual permutation of coordinates):

$$Q^n = \{x_1 \geq 0, \cdots, x_k \geq 0, x_{k+1} \leq 0, \cdots, x_m \leq 0\}.$$

Then, we use an adequate representation of $|\nabla^2 p(z)|$ well fitted to the quadrant:

$$\frac{\partial^2 p(z)}{\partial x_i \partial x_j} = (-1)^k E\left[\mathbf{1}_{F_1 > z_1, F_2 > z_2, \cdots, F_k > z_k, F_{k+1} < z_{k+1}, \cdots, F_m < z_m} H_{ij}\right],$$

where $H_{ij} = H_i \circ H_j \circ H_m \circ \cdots \circ H_1(1)$, with $H_k(G)$ being defined for any $k = 1, \ldots, m$ and any $G \in \cup_{r>1} \mathbf{D}^{1,r}$ by the formula

$$H_k(G) = \delta\left(\sum_{j=1}^{m} G(\gamma^{-1})^{kj} DF^j\right).$$

This implies that (we take the norm in $\mathbf{R}^m$ defined by $|x| = \sup_i |x_i|$):

$$\begin{aligned} M_n &= \int_{Q^n} \sup_{|x-z|\leq 1} \sup_{i,j=1,\ldots,m} \left| E(\mathbf{1}_{A_k(z)} H_{ij} \right| dx \\ &\leq \int_{Q^n} \sup_{|x-z|\leq 1} \sup_{i,j=1,\ldots,m} E\left(\mathbf{1}_{A_k(z)} |H_{ij}|\right) dx, \end{aligned}$$

where

$$A_k(z) = \{F_1 > z_1, F_2 > z_2, \cdots, F_k > z_k, F_{k+1} < z_{k+1}, \cdots, F_m < z_m\}.$$

But on Q^n we can replace

$$\sup_{|x-z|\le 1} E\left(\mathbf{1}_{A_k(z)}|H_{ij}|\right)$$

by $E\left(\mathbf{1}_{A_k(x+1_k)}|H_{ij}|\right)$, where 1_k is the point with first k coordinates equal to -1 and the $m-k$ remaining coordinates equal to 1. So, we have to evaluate for all $i,j = 1,\ldots,m$

$$M_n^{ij} = \int_{Q^n} E\left(\mathbf{1}_{A_k(x+1_k)}|H_{ij}|\right) dx.$$

By Fubini's theorem we obtain

$$\begin{aligned}
M_n^{ij} &= E\left(|H_{ij}| \int_{Q^n} \mathbf{1}_{A_k(x+1_k)} dx\right) \\
&= E\Bigg(|H_{ij}| \int_0^{F_1+1} dx_1 \int_0^{F_2+1} dx_2 \cdots \int_0^{F_k+1} dx_k \\
&\quad \times \int_{F_{k+1}-1}^{0} dx_{k+1} \cdots \int_{F_m-1}^{0} dx_m\Bigg) \\
&= E\left(|H_{ij}|(F_1+1)\cdots(F_k+1)(1-F_{k+1})\cdots(1-F_m)\mathbf{1}_{B_k}\right),
\end{aligned}$$

where $B_k = \{F_1 > -1, F_2 > -1, \cdots, F_k > -1, F_{k+1} < 1, \cdots, F_m < 1\}$. As a consequence, we can estimate M_n^{ij} by

$$M_n^{ij} \le c_m E(|H_{ij}|(|F|^m + 1)),$$

and by Hölder's inequality we get

$$E(|H_{ij}||F|^m) \le \|H_{ij}\|_s \||F|^m\|_{s'},$$

where $\frac{1}{s} + \frac{1}{s'} = 1$. Let us first estimate $\|H_k(G)\|_q$ for any $k = 1,\ldots,m$ and for any random variable $G \in \cup_{r>1}\mathbf{D}^{1,2q}$ and some $q > 1$. We have by Meyer's inequality

$$\begin{aligned}
\|H_k(G)\|_q &\le c_q \left\| G \sum_{j=1}^m (\gamma^{-1})^{kj} DF^j \right\|_{\mathbf{D}^{1,q}(H)} \\
&\le c_q \|G\|_{\mathbf{D}^{1,2q}} \|T_k\|_{\mathbf{D}^{1,2q}(H)},
\end{aligned}$$

where

$$T_k = \sum_{j=1}^{m} (\gamma^{-1})^{kj} DF^j.$$

As a consequence,

$$\|H_1(1)\|_q \leq c_q \|T_1\|_{\mathbf{D}^{1,q}(H)},$$

$$\|H_2 \circ H_1(1)\|_q \leq c_q^2 \|T_2\|_{\mathbf{D}^{1,2q}(H)} \|T_1\|_{\mathbf{D}^{2,2q}(H)},$$

and by iteration,

$$\begin{aligned}
&\|H_i \circ H_j \circ H_m \circ \cdots \circ H_1(1)\|_q \\
&\leq c_q^{m+2} \|T_i\|_{\mathbf{D}^{1,2q}(H)} \|T_j\|_{\mathbf{D}^{2,2q}(H)} \|T_m\|_{\mathbf{D}^{3,4q}(H)} \cdots \|T_1\|_{\mathbf{D}^{m+2,2^{m+1}q}(H)}.
\end{aligned}$$

This estimate completes the proof of the proposition. □

We could specify the exponents s_1 and s_2 appearing in the statement of Proposition 4.1. To do this it suffices to estimate $\|T_k\|_{\mathbf{D}^{m+2,2^{m+1}q}(H)}$ by Sobolev norms of F and L^p norms of $(\det \gamma)^{-1}$.

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