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First-passage competition with different speeds: positive density for both species is impossible

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Abstract

Consider two epidemics whose expansions on $\mathbb{Z}^d$ are governed by two families of passage times that are distinct and stochastically comparable. We prove that when the weak infection survives, the space occupied by the strong one is almost impossible to detect. Particularly, in dimension two, we prove that one species finally occupies a set with full density, while the other one only occupies a set of null density. Furthermore, we observe the same fluctuations with respect to the asymptotic shape as for the weak infection evolving alone. By the way, we extend the Häggström-Pemantle non-coexistence result "except perhaps for a denumerable set" to families of stochastically comparable passage times indexed by a continuous parameter.

Key words: First-passage percolation, competition, coexistence, random growth, moderate deviations.

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1 Introduction

Consider two species trying to colonize the graph $\mathbb{Z}^d$. As in first-passage percolation, the expansion of each species is governed by independent identically distributed random passage times attached to the bonds of the graph. Each vertex of the graph can be infected only once, by the first species that reaches it. Is it possible that both species simultaneously succeed in invading an infinite subset of the net, in other words does coexistence occur? That is the kind of question which was asked in the middle of the 90's by Häggström and Pemantle in two seminal papers [10; 11], where they gave the first results towards the following conjectures:

- If the two species travel at the same speed, coexistence is possible.
- If one of them travels faster than the other one, coexistence is impossible.

The problem of coexistence for two similar species has been solved by Häggström and Pemantle [10] in dimension two for exponential passage times, then by the authors of the present paper in any dimension for general passage times, under assumptions that are close to optimality [5]. Independently, Hoffman [12] gave a different proof involving tools that he will use again to get an extension to a larger number of species – see Hoffman [13].

On the contrary, the state of the art about the second conjecture – the non-coexistence problem – did not much change since its statement. More precisely, if one species travels according passage times following the exponential law with intensity 1, while the other one travels according passage times following the exponential law with intensity $\lambda \neq 1$, it is believed that coexistence is not possible. However, it is only known that coexistence is not possible "except perhaps for a denumerable set of values of λ", as it was proved by Häggström and Pemantle [11]. To sum up, if one denotes by Coex the set of intensities for the second particle that allow coexistence, we know that Coex $\supset \{1\}$ and Coex is denumerable, but we would like to have Coex $= \{1\}$. It follows that we are currently in the following perplexing situation: we know that for almost every value of λ, coexistence does not happen, but we are unable to exhibit any value of λ such that coexistence actually does not occur.

Therefore, the aim of this paper is to prove a weakened version of non-coexistence for epidemics with distinct speeds. Let us first introduce our framework: we consider two epidemics whose expansions are governed by two families of independent and identically distributed passage times whose laws are distinct and stochastically comparable. Obviously, this includes the case of exponential times. We say that *strong coexistence* occurs when each species finally occupies a set with positive natural density.

In dimension two, we prove that strong coexistence does not occur. More precisely, we show that almost surely, at infinite time, one species fills a set with full natural density, whereas the other one only fills a set with null natural density. In higher dimension, connectivity problems prevent us to obtain such a complete result. However, we show that, roughly speaking, a medium resolution satellite only sees one type of particles.

By the way, we also prove that the Häggström-Pemantle non-coexistence result "except perhaps for a denumerable set" can be extended to families of stochastically comparable passage times indexed by a continuous parameter. Note that the Häggström-Pemantle method [11] to prove denumerability of the set Coex has already been transposed to other models related to first-passage percolation: at first by Deijfen, Häggström, and Bagley for a model with spherical symmetry [2], then by the authors of the present paper for some percolating model [6].

Before giving more rigorous statements of our results, let us introduce general notations and give a formal description of the competition model.

General notations

We denote by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of non negative integers.

We endow the set $\mathbb{Z}^d$ with the set of edges $\mathbb{E}^d$ between sites of $\mathbb{Z}^d$ that are at distance 1 for the Euclidean distance: the obtained graph is denoted by $\mathbb{L}^d$. Two sites x and y that are linked by an edge are said to be *neighbors* and this relation is denoted: $x \sim y$. If A is a subset of $\mathbb{Z}^d$, we define the border of A:

$$\partial A = \{z \in A: \quad \exists y \in A^c \quad y \sim z\}.$$

A path in $\mathbb{Z}^d$ is a sequence $x_0, x_1, \dots, x_l$ of points in $\mathbb{Z}^d$ such that two successive points are neighbors. The integer l is called the length of the path.

The critical parameter for Bernoulli percolation (oriented percolation) on $\mathbb{Z}^d$ is denoted by $p_c = p_c(d)$ (respectively, $\overrightarrow{p_c} = \overrightarrow{p_c}(d)$).

Let us now recall the concept of stochastic domination: we say that a probability measure μ dominates a probability measure ν, which is denoted by $\nu \prec \mu$, if

$$\int f \; d\nu \le \int f \; d\mu$$

holds as soon as f is a non decreasing function.

The complementary event of A will mostly be denoted A^c. But sometimes, to improve readability, we prefer to use $\complement A$.

Assumptions on passage times

Let ν_{p_1} and ν_{p_2} be two probability measures on $[0, +\infty)$. We will always assume that

$$
\begin{array}{ll}
(H1) & \nu_{p_1} \succ \nu_{p_2} \text{ and } \nu_{p_1} \neq \nu_{p_2}. \\
(H2) & \forall k, l \in \mathbb{N} \quad \nu_{p_1}^{*k} \otimes \nu_{p_2}^{*l}(\{(x,x) : \, x \in [0,+\infty)\}) = 0. \\
(H3) & \text{for } i \in \{1,2\}, \; \nu_{p_i}(0) < p_c. \\
(H4) & \text{for } i \in \{1,2\}, \; \nu_{p_i}(\inf \operatorname{supp} \nu_{p_i}) < \overrightarrow{p_c}(d). \\
(H5) & \text{for } i \in \{1,2\}, \; \exists \gamma > 0 \text{ such that } \displaystyle\int_0^{+\infty} \exp(\gamma x) \, d\nu_{p_i}(x) < +\infty.
\end{array}
$$

In Assumption $(H3)$, $\inf \operatorname{supp} \nu$ denotes the infimum of the support of the measure ν. Note that Assumptions $(H2)$, $(H3)$, and $(H4)$ are clearly fulfilled when ν_{p_1} and ν_{p_2} are continuous with respect to Lebesgue's measure.

Assumption $(H2)$ exactly says that a sum of k independent random variables with common distribution ν_{p_1} and a sum of l independent random variables with common distribution ν_{p_2}, independent of the first family, have probability 0 to be equal: this will ensure that, during the competition process, no vertex of $\mathbb{Z}^d$ can be reached exactly at the same time by the two epidemics. Assumptions $(H1)$, $(H3)$ and $(H4)$ are the ones used by van den Berg and Kesten in [15] to prove that, in first-passage percolation, the time constant for ν_{p_2} is strictly smaller than the one for ν_{p_1}. Finally, assumption $(H5)$ gives access to large deviations and moderate deviations for fluctuations relatively to the asymptotic shape in first-passage percolation.

Construction of the competition model

The first infection (second infection) will use independent identically distributed passage times with common law ν_{p_1} (respectively, ν_{p_2}) and start from the source s_1 (respectively, s_2, distinct from s_1). As $\nu_{p_1} \succ \nu_{p_2}$, species 1 will be slower (or weaker) than species 2.

First, we couple the two measures ν_{p_1} and ν_{p_2} in agreement with the stochastic comparison relation $(H1)$: there exists a probability measure m on $[0,+\infty) \times [0,+\infty)$ such that $m(\{(x,y) \in [0,+\infty)^2; x \geq y\}) = 1$ and the marginals of m are ν_{p_1} and ν_{p_2}.

Now, we consider, on the set $\Omega = ([0,+\infty) \times [0,+\infty))^{\mathbb{E}^d}$, the probability measure $\mathbb{P} = m^{\otimes \mathbb{E}^d}$. For a realization $\omega = (\omega_e^1, \omega_e^2)_{e \in \mathbb{E}^d}$, the number ω_e^i represents the time needed by species i to cross edge e. Note that under $\mathbb{P}$, for each $i \in \{1,2\}$, the variables $(\omega_e^i)_{e \in \mathbb{E}^d}$ are independent and identically distributed with common law ν_{p_i}. Moreover, we almost surely have

$$\forall e \in \mathbb{E}^d \quad \omega_e^1 \geq \omega_e^2.$$

It remains to construct the two infections in a realization $\omega \in \Omega$. Let $E = ([0,+\infty] \times [0,+\infty])^{\mathbb{Z}^d}$. We recursively define a E-valued sequence $(X_n)_{n \geq 0}$ and a non-negative sequence $(T_n)_{n \geq 0}$. The sequence $(T_n)_{n \geq 0}$ contains the successive times of infections, while a point $\varepsilon = (\varepsilon^1(z), \varepsilon^2(z))_{z \in \mathbb{Z}^d} \in E$ codes, for each site z, its times of infection $\varepsilon^1(z)$ (its time of infection $\varepsilon^2(z)$) by the first infection (respectively, second infection). We start the process with two distinct sources s_1 and s_2 in $\mathbb{Z}^d$, we set $T_0 = 0$ and

$$X_0 = (X_0^1(z), X_0^2(z))_{z \in \mathbb{Z}^d} \text{ with } X_0^i(z) = 0 \text{ if } z = s_i, \text{ and } X_0^i(z) = +\infty \text{ otherwise.}$$

This means that at time 0, no point of $\mathbb{Z}^d$ has yet been infected but the two initial sources s_1 and s_2. Then, for $n \geq 0$, define the next time of infection:

$$T_{n+1} = \inf\{X_n^i(y) + \omega_{\{y,z\}}^i : \{y,z\} \in \mathbb{E}^d, \ i \in \{1,2\}, \ X_n^{3-i}(z) = +\infty\}.$$

Note that Assumption $(H2)$ ensures that if this infimum is reached for several triplets (i_1, y_1, z_1), $\dots$, (i_l, y_l, z_l), then $i_1 = \cdots = i_l = i$. In this case, the next infections are of type i from each y_j to each z_j. The set of infected points of type $3 - i$ has not changed:

$$\forall x \in \mathbb{Z}^d \quad X_{n+1}^{3-i}(x) = X_n^{3-i}(x),$$

while the points z_j has been infected by species i at time $X_n^i(y_j) + \omega_{\{y_j,z_j\}}^i$:

$$\begin{aligned} & \forall x \in \mathbb{Z}^d \backslash \{z_1, \dots, z_l\} \quad X_{n+1}^i(x) = X_n^i(x) \\ \text{and} \quad & \forall j \in \{1, \dots, l\} \quad X_{n+1}^i(z_j) = X_n^i(y_j) + \omega_{\{y_j,z_j\}}^i. \end{aligned}$$

Note that $X_n^i(y)$ and $X_n^{3-i}(y)$ can not be simultaneously finite, because each site is infected by at most one type of infection. Moreover, once $\min(X_n^1(x), X_n^2(x))$ is finite, its value – the time of infection of x – does not change any more with n.

Note however that this coupling is nothing else than a useful tool for our proofs: this does not constrain the evolution of the process. Particularly, the very definition of the evolution process tells us that the $(\omega_i^e)_{e\in\mathbb{E}^d, i\in\{1,2\}}$ could be independent as well without the law of the evolution process being changed.

For $i \in \{1,2\}$, we define

$$\forall t \in [T_n, T_{n+1}) \quad \eta^i(t) = \{z \in \mathbb{Z}^d : X_n^i(z) < +\infty\}.$$

The set $\eta^i(t)$ contains the points that are infected by type i at time t. We also define $\eta(t) = \eta^1(t) \cup \eta^2(t)$. Then, we introduce the sets of points that are finally infected by each epidemic:

$$\forall i \in \{1,2\} \quad \eta^i(\infty) = \underset{t\geq 0}{\cup} \eta^i(t).$$

The set $\mathcal{G}^i = \left\{|\eta^i(\infty)| = +\infty\right\}$ for $i = 1,2$, corresponds to the unbounded growth of species i, and coexistence is thus the event $\text{Coex} = \mathcal{G}^1 \cap \mathcal{G}^2$.

Coupling with first-passage percolation

The evolution of the two infections can be compared with a first-passage percolation.

Assume that ν is a probability measure on $[0,+\infty)$, such that

$$\nu(0) < p_c \text{ and } \exists \gamma > 0 \text{ such that } \int_0^{+\infty} \exp(\gamma x)\, d\nu(x) < +\infty.$$

Consider the probability measure $\mathbb{P} = \nu^{\otimes \mathbb{E}^d}$ on $\Omega = [0,+\infty)^{\mathbb{E}^d}$. Under $\mathbb{P}$, the coordinates $(\omega_e)_{e\in\mathbb{E}^d}$ are independent identically distributed random variables with common law ν. Then for each $x \in \mathbb{Z}^d$ and $t \geq 0$, we define the set of points reached from x in a time less than t:

$$B^x(t) = \left\{ y \in \mathbb{Z}^d : \text{ there exists a path } \gamma \text{ from } x \text{ to } y, \text{ with } \sum_{e\in\gamma} \omega_e \leq t \right\}.$$

The classical shape theorem gives the existence of a norm $\|.\|_\nu$ on $\mathbb{R}^d$ such that $B^0(t)/t$ almost surely converges to the unit ball $\mathcal{B}$ for $\|.\|_\nu$.

Note that the competition model contains two simple first-passage percolation models: for each $i \in \{1,2\}$, $x \in \mathbb{Z}^d$ and $t \geq 0$, the set

$$B_{p_i}^x(t) = \left\{ y \in \mathbb{Z}^d : \text{ there exists a path } \gamma \text{ from } x \text{ to } y, \text{ with } \sum_{e\in\gamma} \omega_e^i \leq t \right\}$$

is the random ball of radius t of first-passage percolation starting from x with passage time law ν_{p_i}. For simplicity, the related norm will be denoted by $\|.\|_{p_i}$, and its associated discrete balls $\mathcal{B}_{p_i}^x(t) = \{y \in \mathbb{Z}^d : \|y - x\|_{p_i} \leq t\}$.

There are some coupling relations between the competition model and first-passage percolation:

Lemma 1.1. $\forall t > 0 \quad \eta^1(t) \subset B_{p_1}^{s_1}(t)$ *and* $\eta^2(t) \subset B_{p_2}^{s_2}(t)$ *and* $B_{p_1}^{s_1}(t) \subset \eta(t)$.

We postpone the proof of these (not so) obvious properties to the next section.

Statement of results

We denote by $\|.\|_2$ the Euclidean norm on $\mathbb{R}^d$, by $\langle .,. \rangle$ the corresponding scalar product and by $\mathscr{S}$ the corresponding unit sphere: $\mathscr{S} = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$. Let $y, z \in \mathbb{R}^d$, $\overrightarrow{x} \in \mathscr{S}$, and $R, h > 0$. We define:

$$\begin{aligned}
d(y, \mathbb{R}\overrightarrow{x}) &= \|y - \langle y, \overrightarrow{x} \rangle \overrightarrow{x}\|_2 \text{ (the Euclidean distance from } y \text{ to the line } \mathbb{R}\overrightarrow{x}),\\
\mathrm{Cyl}_z(\overrightarrow{x}, R, h) &= \{y \in \mathbb{Z}^d : d(y - z, \mathbb{R}\overrightarrow{x}) \le R \text{ and } 0 \le \langle y - z, \overrightarrow{x} \rangle \le h\},\\
\mathrm{Cyl}_+(\overrightarrow{x}, R) &= \mathrm{Cyl}_0(\overrightarrow{x}, R, +\infty),\\
\mathrm{Cyl}(\overrightarrow{x}, R) &= \{y \in \mathbb{Z}^d : d(y, \mathbb{R}\overrightarrow{x}) \le R\}.
\end{aligned}$$

If A, B, C are subsets of $\mathbb{Z}^d$, we say that A disconnects B from infinity in C if every infinite path in C starting from some point in B must meet A. Obviously, this property always occurs when $B \cap C$ is empty.

We can then define the following events:

$$\begin{aligned}
\mathrm{Shadow}(\overrightarrow{x}, t, R) &= \left\{ \partial\eta(t) \cap \eta^2(t) \text{ disconnects } \eta^1(t) \text{ from infinity in } \mathrm{Cyl}_+(\overrightarrow{x}, R) \right\},\\
\mathrm{Shade}(t, R) &= \bigcup_{x \in \mathscr{S}} \mathrm{Shadow}(\overrightarrow{x}, t, R).
\end{aligned}$$

The event $\mathrm{Shadow}(\overrightarrow{x}, t, R)$ means that each infinite path starting from $\eta^1(t)$ and contained in the cylinder $\mathrm{Cyl}_+(\overrightarrow{x}, R)$ necessarily meets $\partial\eta(t) \cap \eta^2(t)$. Loosely speaking, on the event $\mathrm{Shadow}(\overrightarrow{x}, t, R)$, the strong infection casts a shadow of radius R on the weak infection.

Our main result says that if the strong infection occupies a too large portion of the frontier, *i.e.* if $\mathrm{Shade}(t, Rt^{1/2+\eta})$ occurs, then the survival probability of the slow species 1 is very small:

Theorem 1.2. *We suppose that assumptions $(H1)-(H5)$ hold. Consider $M > 0$ and $\eta \in (0, 1/2)$. There exist two strictly positive constants A, B such that*

$$\forall t \ge 0 \quad \mathbb{P}\left(\mathscr{G}^1 \cap \mathrm{Shade}(t, Mt^{1/2+\eta})\right) \le A\exp(-Bt^\eta).$$

Corollary 1.3. *Define R_t as the supremum of the r for which* $\mathrm{Shade}(t, r)$ *occurs. Let $\eta > 0$. Then, on the event $\mathscr{G}^1$, we almost surely have*

$$\lim_{t \to +\infty} \frac{R_t}{t^{1/2+\eta}} = 0.$$

Remember that, by Lemma 1.1 and the asymptotic shape result for first-passage percolation, the diameter of $\eta(t)$ is of order t. Thus, to obtain the absence of strong coexistence, we would also need a control on the number of such stains of species 2 on the surface of $\eta(t)$: in dimension larger or equal to three, the set $\partial\eta(t) \cap \eta^2(t)$ is not necessarily connected. On the contrary, in dimension two, we will see that the set $\partial\eta(t) \cap \eta^2(t)$ is connected, which enables us to prove the absence of strong coexistence. Consider any norm $\|.\|$ on $\mathbb{R}^2$, its discrete balls $\mathscr{B}(t) = \{y \in \mathbb{Z}^2 : \|y\| \le t\}$, and denote by $\mathscr{S}_{p_1}(t)$ the sphere with radius t for the $\|.\|_{p_1}$-norm:

Theorem 1.4. *We suppose that assumptions $(H1)-(H5)$ hold.*
For the two dimensional lattice, we have
1. For every $\beta \in (0, 1/2)$, there exists a constant $C > 0$ such that, almost surely on the event $\mathscr{G}^1$

$$\forall t > 0 \quad \frac{|\eta^2(\infty) \cap \mathscr{B}(t)|}{|\mathscr{B}(t)|} \le \frac{C}{t^{1/2-\beta}}.$$

2. For every $\beta > 0$, almost surely on the event $\mathscr{G}^1$

$$\overline{\lim_{t\to+\infty}} \frac{Diam\left(\eta^2(\infty)+[-1/2,1/2]^2\right)\cap \mathscr{S}_{p_1}(t))}{t^{1/2+\beta}} = 0.$$

3. Strong coexistence almost surely does not happen.

The next corollary of Theorem 1.2 makes Lemma 5.2 in Häggström-Pemantle [11] more precise: when coexistence occurs, the two species globally grow with the speed of the slow species, as if the slow species were alone. It also corresponds to a weak version of moderate deviations for first-passage percolation (see the results by Kesten and Alexander, recalled as Proposition 2.2 in the next section).

Theorem 1.5. *Let $\beta > 0$ and $\eta \in (0,1/2)$. There exist two strictly positive constants A,B such that for every $t \geq 0$:*

$$\mathbb{P}\left(\mathscr{G}^1 \backslash \left\{\mathscr{B}_{p_1}\left(t-\beta t^{1/2+\eta}\right) \subset \eta(t) \subset \mathscr{B}_{p_1}\left(t+\beta t^{1/2+\eta}\right)\right\}\right) \leq A\exp(-Bt^{\eta}).$$

The estimates we obtained in this paper finally allow us to recover the "except perhaps for a denumerable set" non-coexistence result by Häggström and Pemantle, and to extend it to more general families of passage times indexed by a continuous parameter:

Theorem 1.6. *Let $(\nu_p)_{p\in I}$ be a family of probability measures indexed by a subset of $\mathbb{R}$. We assume that*

1. *for each $p \in I$, $\nu_p(0) < p_c$*
2. *for each $p \in I$, $\nu_p(\inf \operatorname{supp} \nu_p) < \overrightarrow{p_c}$*
3. *for each $p \in I$, there exists $\gamma > 0$ such that $\int_{[0,\infty)} \exp(\gamma x)\, d\nu_p(x) < \infty$,*
4. *for each $p,q \in I$, $p < q \Rightarrow \nu_p \succ \nu_q$ and $\nu_p \neq \nu_q$,*
5. *for each $p,q \in I$, $\forall k,l \in \mathbb{N}$, $\nu_p^{*k} \otimes \nu_q^{*l}(\{(x,x) : x \in [0,+\infty)\}) = 0$.*

*Denote by $\mathbb{P}_{p,q}$ the law of the competition process where species 1 (*resp. *2) uses passage times with law ν_p (*resp. *ν_q). Then, for each fixed $q \in I$, the set*

$$\{p \in I : \ p \leq q \text{ and } \mathbb{P}_{p,q}(\text{Coex}) > 0\}$$

is a subset of the points of discontinuity of the non-decreasing map $p \mapsto \mathbb{P}_{p,q}(\mathscr{G}^1)$, and is therefore at most denumerable.

Organization of the paper

In Section 2, we give a list of useful results in first-passage percolation. Most of them are classical and are recalled without proof. We also give there the proof of Lemma 1.1. Section 3 is mainly devoted to the proof of Theorem 1.3, which is the technical core of the paper. Section 4 establishes Theorem 1.5. In Section 5, we improve for the two dimensional lattice the results of Section 3 into the more friendly Theorem 1.4. The last section extends the Häggström-Pemantle Theorem to the present context as announced in Theorem 1.6.

2 Preliminary results

2.1 First-passage percolation results

Let us recall some classical results about first-passage percolation. We assume here that the passage times are independent identically distributed with common law ν satisfying

- $\nu(0) < p_c$;
- for some $\gamma > 0$, $\int_{[0,+\infty)} \exp(\gamma x)\, d\nu(x) < +\infty$.

Denote by $\|.\|_\nu$ the norm given by the shape theorem, and by $\mathscr{B}^x(t)$ the discrete ball relatively to $\|.\|_\nu$ with center x and radius t. The first two results give large deviations and moderate deviations for fluctuations with respect to the asymptotic shape, and the third one gives the strict monotonicity result for the asymptotic shape with respect to the distribution of the passage times:

Proposition 2.1 (Grimmett-Kesten [9]). *For any $\varepsilon > 0$, there exist two strictly positive constants A, B such that*

$$\forall t > 0 \quad \mathbb{P}\left(\mathscr{B}^0((1-\varepsilon)t) \subset B^0(t) \subset \mathscr{B}^0((1+\varepsilon)t)\right) \geq 1 - A\exp(-Bt).$$

Proposition 2.2 (Kesten [14], Alexander [1]). *For any $\beta > 0$, for any $\eta \in (0, 1/2)$, there exist two strictly positive constants A, B such that*

$$\forall t > 0 \quad \mathbb{P}\left(\mathscr{B}^0(t - \beta t^{1/2+\eta}) \subset B^0(t) \subset \mathscr{B}^0(t + \beta t^{1/2+\eta})\right) \geq 1 - A\exp(-Bt^\eta).$$

Proposition 2.3 (van den Berg-Kesten [15]). *Let ν_{p_1} and ν_{p_2} be two probability measures on $[0, +\infty)$ satisfying $(H1)$, $(H3)$, $(H4)$, $(H5)$.*

There exists a constant $C_{p_1,p_2} \in (0, 1)$ such that

$$\forall x \in \mathbb{R}^d \quad \|x\|_{p_2} \leq C_{p_1,p_2} \|x\|_{p_1}.$$

Note that in [15], the proof of this result is only written for the time constants. Nevertheless, it applies in any direction and computations can be followed in order to preserve a uniform control, whatever direction one considers. See for instance Garet and Marchand [6] for a detailed proof in an analogous situation. In the same way, the large deviation result of Proposition 2.1 is only stated in [9] for the time constant, but the result can be extended uniformly in any direction, as it is done in Garet and Marchand [7] for chemical distance in supercritical Bernoulli percolation. As far as Proposition 2.2 is concerned, it is a by-product of the proof of Theorem 3.1 in Alexander [1]. We include here a short proof for convenience.

Proof of Proposition 2.2. The outer bound for $B^0(t)$ follows from Kesten [14], Equation (1.19): there exist positive constants A_1, B_1 such that for all $t > 0$, we have

$$\mathbb{P}(B^0(t) \not\subset \mathscr{B}^0(t + \beta t^{1/2+\eta}) \leq A_1 \exp(-B_1 t^\eta).$$

Turning to the inner bound, we follow the lines of Alexander's proof: for $x, y \in \mathbb{Z}^d$, let us define the travel time between x and y by $T(x,y) = \inf\{t \geq 0 : \ y \in B^x(t)\}$. Then we have:

$$
\begin{aligned}
& \mathbb{P}(B^0(t) \not\supset \mathscr{B}^0(t - \beta t^{1/2+\eta})) \\
\leq \ & \mathbb{P}(B^0(t) \not\supset \mathscr{B}^0(t/2)) + \sum_{t/2 \leq \|x\|_\nu \leq t - \beta t^{1/2+\eta}} \mathbb{P}(T(0,x) \geq t) \\
\leq \ & A\exp(-Bt) + \sum_{t/2 \leq \|x\|_\nu \leq t - \beta t^{1/2+\eta}} \mathbb{P}(T(0,x) \geq t),
\end{aligned}
$$

where A and B are determined by Proposition 2.1 with $\varepsilon = 1/2$. By Alexander [1], Theorem 3.2, there exist positive constants C_4', M such that

$$\|x\|_\nu \geq M \Longrightarrow \mathbb{E}\ T(0,x) \leq \|x\|_\nu + C_4' \|x\|_\nu^{1/2} \log \|x\|_\nu.$$

Assume $t/2 \geq M$ and let x with $t/2 \leq \|x\|_\nu \leq t - \beta t^{1/2+\eta}$.
Let $C_M = \inf\{2^{1/2+\eta}\beta - C_4' y^{-\eta} \log y : \ y \geq M\}$. We assume that M is so large that $C_M > 0$. We have $\mathbb{E}\ T(0,x) \leq t - C_M \|x\|_\nu^{1/2+\eta} \leq t - C_M' \|x\|_2^{1/2+\eta}$, where C_M' is a positive constant, and then

$$\mathbb{P}(T(0,x) \geq t) \leq \mathbb{P}(T(0,x) - \mathbb{E}\ T(0,x) \geq C_M' \|x\|_2^{1/2+\eta}).$$

By Kesten's result [14], Equation (2.49) (see also Equation (3.7) in Alexander [1]), there exist positive constants C_5, C_6 such that

$$\mathbb{P}(T(0,x) - \mathbb{E}\ T(0,x) \geq C_M' \|x\|^{1/2+\eta}) \leq C_5 \exp(-C_6 \|x\|_2^\eta),$$

provided that M is large enough. Finally, it gives that

$$\sum_{t/2 \leq \|x\|_\nu \leq t - \beta t^{1/2+\eta}} \mathbb{P}(T(0,x) \geq t) \leq |\mathscr{B}^0(t)| C_5 \exp(-C_6 (t/2)^\eta) \leq C_5' \exp(-C_6' t^\eta),$$

where C_5', C_6' are positive constants. $\square$

Lemma 2.5 will ensure that the minimal time to cross the cylinder $\mathrm{Cyl}_z(\overrightarrow{x}, h, r)$ from bottom to top, using only edges in the cylinder, can not be much larger than the expected value $h\|\overrightarrow{x}\|_\nu$. We need an intermediary lemma.

For $x, y \in \mathbb{Z}^d$, denote by $I_{x,y}$ the length of the shortest path from x to y which is inside $\mathscr{B}_x(1.25\|x - y\|_\nu) \cap \mathscr{B}_y(1.25\|x-y\|_\nu)$. Of course $I_{x,y}$ as the same law that $I_{0,x-y}$, and we simply write $I_x = I_{0,x}$.

Lemma 2.4. *Let ε, a in $(0,1)$ and $\|.\|$ be any norm on $\mathbb{R}^d$. There exists M_0 such that for each $M \geq M_0$, there exist $\rho \in (0,1)$ and $t > 0$ such that*

$$\|x\| \in [aM, M/a] \Longrightarrow \mathbb{E}\ \exp(t(I_x - (1+\varepsilon)\|x\|_\nu)) \leq \rho.$$

Proof. Note that by norm equivalence, we can restrict ourselves to $\|.\|_1$. Let Y be a random variable with law ν and let $\gamma > 0$ be such that $\mathbb{E}\ e^{2\gamma Y} < +\infty$.

First, the large deviations result, Proposition 2.1, easily implies the following almost sure convergence:

$$\lim_{\|x\|_1\to+\infty} \frac{I_x}{\|x\|_v}=1.$$

By considering a deterministic path from 0 to x with length $\|x\|_1$, we see that I_x is dominated by a sum of $\|x\|_1$ independent copies of Y denoted by $Y_1,\dots,Y_{\|x\|_1}$. Thus, $I_x/\|x\|_1$ is dominated by

$$\frac{1}{\|x\|_1}\sum_{k=1}^{\|x\|_1} Y_i.$$

By the law of large numbers, this family is equi-integrable. Hence, $(T_x/\|x\|_1)_{x\in\mathbb{Z}^d\setminus\{0\}}$ and finally $(I_x/\|x\|_v)_{x\in\mathbb{Z}^d\setminus\{0\}}$ are also equi-integrable families. It follows that

$$\lim_{\|x\|_1\to+\infty} \frac{\mathbb{E}\ I_x}{\|x\|_v}=1. \tag{1}$$

Note now that for every $y\in\mathbb{R}$ and $t\in(0,\gamma]$,

$$e^{ty}\le 1+ty+\frac{t^2}{2}y^2e^{t|y|}\le 1+ty+\frac{t^2}{\gamma^2}e^{2\gamma|y|}.$$

Define $\tilde{I}_x=I_x-(1+\varepsilon)\|x\|_v$ and suppose that $t\in(0,\gamma]$. Then, as $|\tilde{I}_x|\le I_x+2\|x\|_v$, the previous inequality implies that

$$e^{t\tilde{I}_x}\le 1+t\tilde{I}_x+\frac{t^2}{\gamma^2}e^{4\gamma\|x\|_v}e^{2\gamma I_x}.$$

As $\|x\|_v\le\|x\|_1\|e_1\|_v$ and $I_x\le Y_1+\dots+Y_{\|x\|_1}$, if we set $R=e^{4\gamma\|e_1\|_v}\mathbb{E}\ e^{2\gamma Y}$, we obtain that

$$\mathbb{E}\ e^{t\tilde{I}_x}\le 1+t\left(\mathbb{E}\ \tilde{I}_x+\frac{t}{\gamma^2}R^{\|x\|_1}\right).$$

Considering Equation (1), let M_0 be such that $\|x\|\ge aM_0$ implies $\frac{\mathbb{E}\ I_x}{\|x\|_v}\le 1+\varepsilon/3$.
For x such that $\|x\|_1\ge aM_0$, we have $\mathbb{E}\ \tilde{I}_x\le-\frac{2}{3}\varepsilon\|x\|_v$, so

$$\mathbb{E}\ e^{t\tilde{I}_x}\le 1+t\left(-\frac{2}{3}\varepsilon\|x\|_v+\frac{t}{\gamma^2}R^{\|x\|_1}\right).$$

Therefore, we take $t=\min\left\{\gamma,\gamma^2\|e_1\|_v\frac{\varepsilon}{3}R^{-M/a}\right\}>0$ and $\rho=1-\frac{1}{3}\varepsilon\|e_1\|_v t<1$. □

Lemma 2.5. *For $z\in\mathbb{R}^d$, $\overrightarrow{x}\in\mathscr{S}$, and $h,r>0$ large enough, we can define the point s_0 (the point s_f) to be the integer point in $\mathrm{Cyl}_z(\overrightarrow{x},r,h)$ which is closest to z (respectively, $z+h\overrightarrow{x}$). We define the crossing time $t[\mathrm{Cyl}_z(\overrightarrow{x},h,r)]$ of the cylinder $\mathrm{Cyl}_z(\overrightarrow{x},r,h)$ as the minimal time needed to cross it from s_0 to s_f, using only edges in the cylinder.*

Then for any $\varepsilon>0$, and any function $f:\mathbb{R}_+\to\mathbb{R}_+$ with $\lim_{+\infty}f=+\infty$, there exist two strictly positive constants A and B such that

$$\begin{aligned}&\forall z\in\mathbb{R}^d\quad\forall\overrightarrow{x}\in\mathscr{S}\quad\forall h>0\\ &\mathbb{P}\left(t[\mathrm{Cyl}_z(\overrightarrow{x},h,f(h))]\ge\|\overrightarrow{x}\|_v(1+\varepsilon)h\right)\le A\exp(-Bh).\end{aligned}$$

Note that this gives the existence of a nearly optimal path from z to $z + h\overrightarrow{x}$ that remains at a distance less than $f(h)$ of the straight line. This result can be interesting on its own as we often lack of information on the position of the real optimal paths.

Proof. Let $\varepsilon \in (0,1)$ and consider the integer $M_0 \in \mathbb{N}$ given by Lemma 2.4. As $\|.\|_\nu$ is a norm, there exist two strictly positive constants c and C such that

$$\forall \overrightarrow{x} \in \mathscr{S} \quad c \leq \|\overrightarrow{x}\|_\nu \leq C. \tag{2}$$

Let $M_1 \geq M_0$ be an integer large enough to have

$$(1+2\varepsilon) \geq (1+\varepsilon)\left(1 + \frac{4\|e_1\|_\nu}{cM_1}\right). \tag{3}$$

Consider $h > M_1$ and set $N = 1 + \mathrm{Int}(h/M_1)$, where $\mathrm{Int}(x)$ denotes the integer part of x, and, for each $i \in \{0,\dots,N\}$ denote by x_i the integer point in the cylinder which is the closest to $z + \frac{ih\overrightarrow{x}}{N}$. Note that

$$\forall h \geq M_1 \quad \frac{M_1}{2} \leq \left(1 - \frac{1}{N}\right) M_1 \leq \frac{h}{N} \leq M_1. \tag{4}$$

and that for each $i, j \in \{0,\dots,N-1\}$,

$$\left| \|x_i - x_j\|_\nu - |j-i| \frac{h\|\overrightarrow{x}\|_\nu}{N} \right| \leq 2\|e_1\|_\nu. \tag{5}$$

1. Applying (5), (2) and (4), we obtain that for each $i \in \{0,\dots,N-1\}$,

$$\begin{aligned} \frac{h\|\overrightarrow{x}\|_\nu}{N} - 2\|e_1\|_\nu \leq \quad & \|x_i - x_{i+1}\|_\nu \quad \leq \frac{h\|\overrightarrow{x}\|_\nu}{N} + 2\|e_1\|_\nu \\ c\frac{M_1}{2} - 2\|e_1\|_\nu \leq \quad & \|x_i - x_{i+1}\|_\nu \quad \leq CM_1 + 2\|e_1\|_\nu. \end{aligned}$$

So we can find $a > 0$ such that, by increasing M_1 if necessary,

$$\forall i \in \{0,\dots,N-1\} \quad aM_1 \leq \|x_i - x_{i+1}\|_\nu \leq M_1/a. \tag{6}$$

2. Let $h_1 \geq 0$ be such that

$$\forall h \geq h_1 \quad f(h) \geq 2.5C(M_1+1)\|e_1\|_\nu + 1.$$

If we take now h larger than h_1, and if $y \in \mathscr{B}^{x_i}(1.25\|x_i - x_{i+1}\|_\nu)$ for some $i \in \{0,\dots,N-1\}$, then, with (2) and (5),

$$\begin{aligned} & d(y - z, \mathbb{R}\overrightarrow{x}) \\ = \quad & d(y, z + \mathbb{R}\overrightarrow{x}) \leq \left\| y - z - \frac{ih}{N}\overrightarrow{x} \right\|_2 \leq \|y - x_i\|_2 + \left\| x_i - z - \frac{ih}{N}\overrightarrow{x} \right\|_2 \\ \leq \quad & 1.25C\|x_i - x_{i+1}\|_\nu + 1 \leq 1.25C\left(\frac{h\|\overrightarrow{x}\|_\nu}{N} + 2\|e_1\|_\nu\right) + 1. \end{aligned}$$

But $h/N \leq M_1$ and $\|\overrightarrow{x}\|_v \leq \|\overrightarrow{x}\|_1 \|e_1\|_v \leq 2\|e_1\|_v$, thus

$$d(y-z, \mathbb{R}\overrightarrow{x}) \leq f(h).$$

On the other hand,

$$\begin{aligned} \langle y-z, \overrightarrow{x}\rangle &= \langle y-x_i, \overrightarrow{x}\rangle + \langle x_i - \left(z + \frac{ih}{N}\overrightarrow{x}\right), \overrightarrow{x}\rangle + \langle \frac{ih}{N}\overrightarrow{x}, \overrightarrow{x}\rangle \\ \text{i.e. } \left|\langle y-z, \overrightarrow{x}\rangle - \frac{ih}{N}\right| &\leq \left\|y - x_i\right\|_2 + 1 \leq 2.5C(M_1+1)\|e_1\|_v + 1. \end{aligned}$$

We choose then $i_0 \in \mathbb{N}$ such that:

$$i_0 \geq \frac{2}{M_1}(2.5C(M_1+1)\|e_1\|_v + 1).$$

Then, if h_2 is such that $1 + \mathrm{Int}(h_2/M_1) \geq 3i_0$, we obtain:

$$\forall h \geq h_2 \quad \forall i \in \{i_0, \dots, N-1-i_0\} \quad \mathscr{B}^{x_i}(1,25\|x_i - x_{i+1}\|_v) \subset \mathrm{Cyl}_z(\overrightarrow{x}, R, h). \tag{7}$$

3. There exists a deterministic path inside the cylinder from x_0 to x_{i_0} (from x_{N-i_0} to x_N) which uses less than $i_0\frac{h\|\overrightarrow{x}\|_1}{N} + 2$ edges: we denote by L_{start} (respectively, L_{end}) the random length of this path. By Equation (4), we have

$$\forall h \geq M_1 \quad i_0\frac{h\|\overrightarrow{x}\|_1}{N} + 2 \leq i_0\frac{2h}{N} + 2 \leq 2(i_0+1)M_1.$$

If $h > h_3 = \frac{3(i_0+1)M_1 \mathbb{E}\, Y}{\varepsilon\|\overrightarrow{x}\|_v}$, Chernoff's theorem gives the existence of two strictly positive constants A_1, B_1 such that

$$\forall h > 0 \quad \mathbb{P}\left(L_{start} > \varepsilon h\|\overrightarrow{x}\|_v\right) + \mathbb{P}\left(L_{end} > \varepsilon h\|\overrightarrow{x}\|_v\right) \leq A_1 e^{-B_1 h}. \tag{8}$$

4. So, provided that $h \geq h_2$, we have by (8), inside the cylinder, a path from x_0 to x_N with length

$$L_{start} + \sum_{i=i_0}^{N-i_0-1} I_{x_i, x_{i+1}} + L_{end}.$$

By Equation (5), if h is large enough, for each $i, j \in \{0, \dots, N-1\}$,

$$\begin{aligned} &\mathscr{B}^{x_i}(1.25\|x_i - x_{i+1}\|_v) \cap \mathscr{B}^{x_{i+1}}(1.25\|x_i - x_{i+1}\|_v) \\ &\cap \mathscr{B}^{x_j}(1.25\|x_j - x_{j+1}\|_v) \cap \mathscr{B}^{x_{j+1}}(1.25\|x_j - x_{j+1}\|_v) = \varnothing \end{aligned}$$

as soon as $|j - i| \geq 2$. We thus introduce, for $j \in \{0, 1\}$, the sums:

$$S_j = \sum_{\substack{I \leq i \leq N-I-1 \\ i=j \mod 2}} I_{x_i, x_{i+1}}.$$

Note that, with (5), (2) and (3) for each $j \in \{0,1\}$,

$$\begin{aligned}
\sum_{\substack{i_0 \le i \le N-i_0-1 \\ i=j \mod 2}} \|x_{i+1}-x_i\|_v &\le \frac{N-2i_0}{2}\left(\frac{h\|\overrightarrow{x}\|_v}{N}+2\|e_1\|_v\right) \\
&\le \frac{h\|\overrightarrow{x}\|_v}{2}\left(1+\frac{2N\|e_1\|_v}{h\|\overrightarrow{x}\|_v}\right) \\
&\le \frac{h\|\overrightarrow{x}\|_v}{2}\left(1+\frac{4\|e_1\|_v}{cM_1}\right) \le \left(\frac{1+2\varepsilon}{1+\varepsilon}\right)\frac{h\|\overrightarrow{x}\|_v}{2}.
\end{aligned}$$

Then, by independence of the terms in each S_j,

$$\begin{aligned}
&\mathbb{P}\left(S_j \ge \frac{h\|\overrightarrow{x}\|_v}{2}(1+2\varepsilon)\right) \\
\le\ &\mathbb{P}\left(S_j \ge (1+\varepsilon)\sum_{\substack{i_0 \le i \le N-i_0-1 \\ i=j \mod 2}} \|x_{i+1}-x_i\|_v\right) \\
\le\ &\mathbb{E}\ \exp\left(t\sum_{\substack{i_0 \le i \le N-i_0-1 \\ i=j \mod 2}} \left(I_{x_i,x_{i+1}}-(1+\varepsilon)\|x_{i+1}-x_i\|_v\right)\right) \\
\le\ &\prod_{\substack{i_0 \le i \le N-i_0-1 \\ i=j \mod 2}} \mathbb{E}\ \exp(t(I_{x_i,x_{i+1}}-(1+\varepsilon)\|x_{i+1}-x_i\|_v))
\end{aligned}$$

By (6), for each i, we have $\|x_i - x_{i+1}\|_v \in [aM_1, M_1/a]$, so we can apply Lemma 2.4: there exists some $\rho < 1$, such that for every h large enough,

$$\forall j \in \{0,1\} \quad \mathbb{P}\left(S_j \ge \frac{h\|\overrightarrow{x}\|_v}{2}(1+2\varepsilon)\right) \le \rho^{N/2} \le \rho^{h/(2M_1)}.$$

Together with (8), this proves the estimate of the lemma. □

2.2 Comparisons with first-passage percolation

We now prove Lemma 1.1, using an algorithmic construction analogous to the one used to define the competition process in the introduction.

Proof of Lemma 1.1. The inclusion $\eta^1(t) \subset B_{p_1}^{s_1}(t)$ is obvious: by construction of the process, if $x \in \eta^1(t)$, there exists a path between s_1 and x included in $\eta^1(t)$, and whose travel time is thus less than t. The second inclusion $\eta^2(t) \subset B_{p_1}^{s_2}(t)$ is proved in the same way.

Let us now prove the third inclusion $B_{p_1}^{s_1}(t) \subset \eta(t)$. We are going to build the first-passage process by Dijkstra's algorithm, in a formalism analogous to the one used to define the competition process.

Recall that $\Omega = ([0,+\infty) \times [0,+\infty))^{\mathbb{E}^d}$ is endowed with the measure $\mathbb{P} = m^{\otimes \mathbb{E}^d}$. Consider a fixed configuration $\omega \in \Omega$.

Let $E' = [0,+\infty]^{\mathbb{Z}^d}$. We recursively define a E'-valued sequence $(X'_n)_{n\geq 0}$ and a non-negative sequence $(T'_n)_{n\geq 0}$. The sequence $(T'_n)n \geq 0$ contains the successive times of infections, while a point $\varepsilon = (\varepsilon(z))_{z\in\mathbb{Z}^d} \in E$ codes, for each site z, its times of infection $\varepsilon(z)$. We start the process with the single source s_1, and set $T'_0 = 0$ and

$$X'_0 = (X_0(z))_{z\in\mathbb{Z}^d} \text{ with } X'_0(s_1) = 0 \text{ and } X'_0(z) = +\infty \text{ if } z \neq s_1.$$

Then, for $n \geq 0$, define the next time of infection:

$$T'_{n+1} = \inf\{X'_n(y) + \omega^1_{\{y,z\}} : \quad \{y,z\} \in \mathbb{E}^d\}.$$

The infimum is reached for some couples (y_i, z_i), meaning that the z_i are being infected from the y_i:

$$\forall x \in \mathbb{Z}^d \backslash \{z_i\},\ X'_{n+1}(x) = X'_n(x) \text{ and } X'_{n+1}(z_i) = X'_n(y_i) + \omega^1_{\{y_i,z_i\}}.$$

We also note that $\eta'(t)$, the set of infected points at time t is given by

$$\forall n \in \mathbb{N} \quad \forall t \in [T_n, T_{n+1}) \quad \eta'(t) = \{z \in \mathbb{Z}^d : X'_n(z) < +\infty\}.$$

By Dijkstra's algorithm, $\eta'(t)$ is exactly the set $B^{s_1}_{p_1}(t)$.

We now proceed by induction to prove that for every $n \in \mathbb{N}$

$$(H_n) \quad \forall x \in \mathbb{Z}^d \quad X'_n(x) \geq \min(X^1_n(x), X^2_n(x)).$$

Clearly, (H_0) is true. Assume that (H_n) holds. We have the following alternative:

- If $X'_{n+1}(x) = +\infty$, it is obvious that $X'_{n+1}(x) \geq \min(X^1_{n+1}(x), X^2_{n+1}(x))$.
- If $X'_n(x) < +\infty$ then $X'_{n+1}(x) = X'_n(x)$ and, as $X'_n(x) \geq \min(X^1_n(x), X^2_n(x))$, the number $\min(X^1_n(x), X^2_n(x))$ is also finite, and thus $\min(X^1_n(x), X^2_n(x)) = \min(X^1_{n+1}(x), X^2_{n+1}(x))$ – recall that $X^1_n(x)$ and $X^2_n(x)$ can not be simultaneously finite. Consequently, $X'_{n+1}(x) \geq \min(X^1_{n+1}(x), X^2_{n+1}(x))$.
- If $X'_{n+1}(x) < +\infty$ and $X'_n(x) = +\infty$, the point x is being infected at time T'_{n+1} through the edge e from the point y, which is consequently such that $X'_n(y) < +\infty$. As $X'_n(y) \geq \min(X^1_n(y), X^2_n(y))$, and $\omega^1_e \geq \omega^2_e$,

$$X'_{n+1}(x) = X'_n(y) + \omega^1_e \geq \min(X^1_{n+1}(y), X^2_{n+1}(y)).$$

Note that

$$\begin{aligned} \eta(t) &= \{z \in \mathbb{Z}^d : \exists n \in \mathbb{N},\ \min(X^1_n(z), X^2_n(z)) \leq t\}, \\ \eta'(t) &= \{z \in \mathbb{Z}^d : \exists n \in \mathbb{N},\ X'_n(z) \leq t\}. \end{aligned}$$

It is then obvious that $\eta'(t) \subset \eta(t)$. $\square$

3 Coexistence can not be observed by a medium resolution satellite

The proof of Theorem 1.2 follows, at least in its main lines, the strategy initiated by Häggström and Pemantle: the aim is to prove that an event, suspected to be incompatible with the survival of the weak, allows the strong to grant themselves, with high probability, a family of shells that surround the weak, preventing thus coexistence. In the Häggström-Pemantle paper [11], the objective is to show that when coexistence occurs, it is unlikely that the strong can advance significantly beyond the weak. The event considered here is of a different nature: we must prove that the strong can not occupy too large a region on the frontier of the infected zone. Obviously, this requires finer controls. Moreover, the use of the shape theorem is not sufficient: moderate deviations for the fluctuations with respect to the asymptotic shape provide sharper estimates. Some more technical difficulties also follow from the loss of some nice properties of exponential laws. However, this last kind of difficulties has already been overcome by the authors of this paper in the previous article [6]. We refer the reader to this paper for some more comments.

To prove Theorem 1.2, we first prove an analogous result, Lemma 3.6, in a fixed given direction. Theorem 1.3 follows then from a Borel-Cantelli type of argument.

Definitions.

Let $\mathscr{S}_{p_2}$ be the unit sphere for the norm $\|.\|_{p_2}$. We define the *shells*: for each $A \subset \mathscr{S}_{p_2}$, and every $0 < r < r'$, we set

$$\text{Shell}(A, r, r') \quad = \quad \{x \in \mathbb{Z}^d : \, x/\|x\|_{p_2} \in A \text{ and } r \leq \|x\|_{p_2} \leq r'\}.$$

So roughly speaking, we should think of A as the set of possible directions for the points in the shell, while $[r, r']$ is the set of radii.

For $A \subset \mathscr{S}_{p_2}$ and $\varphi > 0$, define the following enlargement of A:

$$A \oplus \varphi \quad = \quad (A + \mathscr{B}^0_{p_2}(\varphi)) \cap \mathscr{S}_{p_2}.$$

Let us state first three geometric lemmas:

Lemma 3.1. *For any norm $|.|$ on $\mathbb{R}^d$, one has*

$$\forall x, y \in \mathbb{R}^d \backslash \{0\} \quad \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \leq \frac{2|x-y|}{\max\{|x|, |y|\}}.$$

Lemma 3.2. *For any norm $|.|$ on $\mathbb{R}^d$, there exist a constant $C > 0$ such that the unit sphere for $|.|$ can be covered with $C(1+\frac{1}{\varepsilon})^{d-1}$ balls of radius ε having their centers on the unit sphere.*

Proof. When $|.| = \|.\|_\infty$, it is easy to see that the sphere can be covered with $2d(1+\frac{2}{\varepsilon})^{d-1}$ balls of radius ε.

Now let A, B be two strictly positive constants such that

$$\forall x \in \mathbb{R}^d \quad A\|x\|_\infty \leq |x| \leq B\|x\|_\infty$$

and let $K = \frac{A}{2B}$. Using the remark above, we can find and integer $n \le 2d(1+\frac{2}{K\varepsilon})^{d-1}$ and a family of balls $(\mathscr{B}_\infty^{x_i}(K\varepsilon))_{1\le i\le n}$ that covers the unit sphere $\{x \in \mathbb{R}^d : \|x\|_\infty = 1\}$ and satisfies $\|x_i\|_\infty = 1$ for each $i \in \{1,\dots,n\}$. Now note $\Psi(x) = \frac{x}{|x|}$ and let y in the unit sphere for $|.|$: there exists $i \in \{1,\dots,n\}$ such that

$$\|x_i - \frac{y}{\|y\|_\infty}\|_\infty \le K\varepsilon.$$

Since $\Psi(\frac{y}{\|y\|_\infty}) = y$, Lemma 3.1 ensures that

$$|\Psi(x_i) - y| \le \frac{2|x_i - \frac{y}{\|y\|_\infty}|}{|x_i|} \le \frac{2BK\varepsilon}{A} = \varepsilon.$$

So, the unit sphere for $|.|$ can be covered with $2d(1+\frac{4B}{A}\frac{1}{\varepsilon})^{d-1}$ balls of radius ε having their centers on the unit sphere. $\square$

Lemma 3.3. *Let $R \ge 0$. If $\overrightarrow{u}, \overrightarrow{v} \in \mathscr{S}$ are such that $\|\overrightarrow{u} - \overrightarrow{v}\|_2 \le (\frac{h}{2R})^2$, then*

$$\mathrm{Cyl}(\overrightarrow{v}, h/2) \cap \mathscr{B}_2(R) \subset \mathrm{Cyl}(\overrightarrow{u}, h).$$

Moreover, if $\|\overrightarrow{u} - \overrightarrow{v}\|_2 \le 1/2$, then

$$\mathrm{Cyl}_+(\overrightarrow{v}, h/2) \cap \mathscr{B}_2(R) \cap \mathscr{B}_2(h)^c \subset \mathrm{Cyl}_+(\overrightarrow{u}, h).$$

Proof. We denote $\theta = \|\overrightarrow{u} - \overrightarrow{v}\|_2$. Then $d(y, \mathbb{R}\overrightarrow{u})^2 = \|y - \langle y, \overrightarrow{u}\rangle \overrightarrow{u}\|_2^2 = \|y\|_2^2 - \langle y, \overrightarrow{u}\rangle^2$ and

$$\begin{aligned} d(y,\mathbb{R}\overrightarrow{u})^2 - d(y,\mathbb{R}\overrightarrow{v})^2 &= \langle y, \overrightarrow{v}\rangle^2 - \langle y, \overrightarrow{u}\rangle^2 \\ &\le \langle y, \overrightarrow{v} - \overrightarrow{u}\rangle \langle y, \overrightarrow{u} + \overrightarrow{v}\rangle \le 2\theta \|y\|_2^2. \end{aligned}$$

Suppose first that $y \in \mathrm{Cyl}(\overrightarrow{v}, h/2) \cap \mathscr{B}_2(R)$: we have

$$d(y,\mathbb{R}\overrightarrow{u})^2 - d(y,\mathbb{R}\overrightarrow{v})^2 \le 2\theta R^2 \le h^2/2.$$

So $d(y,\mathbb{R}\overrightarrow{u})^2 = d(y,\mathbb{R}\overrightarrow{v})^2 + (d(y,\mathbb{R}\overrightarrow{u})^2 - d(y,\mathbb{R}\overrightarrow{v})^2) \le h^2/4 + h^2/2 \le h^2$, which means that $y \in \mathrm{Cyl}(\overrightarrow{u}, h)$.

Now suppose that $\theta \le 1/2$ and $\|y\|_2 \ge h$. We have $d(y, \mathbb{R}\overrightarrow{v}) = \|y - \langle y, v\rangle \overrightarrow{v}\|_2 \le h/2$ and $\langle y, \overrightarrow{v}\rangle \ge 0$, so $\langle y, \overrightarrow{v}\rangle = |\langle y, \overrightarrow{v}\rangle| \ge \|y\|_2 - h/2$. This implies

$$\langle y, \overrightarrow{u}\rangle = \langle y, \overrightarrow{v}\rangle - \langle y, \overrightarrow{v} - \overrightarrow{u}\rangle \ge \langle y, \overrightarrow{v}\rangle - \|y\|_2\theta \ge \|y\|_2(1-\theta) - \frac{h}{2} \ge \|y\|_2\left(\frac{1}{2} - \theta\right) \ge 0,$$

which ends the proof. $\square$

The next lemma ensures that if the event $\mathrm{Shadow}(\overrightarrow{x}, t, Rt^{1/2+\eta})$ occurs, then there is a large probability that the strong infection manages to colonize a small shell. Then, he gets a positional advantage. Let $K_1 > 0$ be such that

$$\forall x \in \mathbb{R}^d \quad \|x\|_2 \le K_1 \|x\|_{p_1}. \tag{9}$$

Lemma 3.4. *Let $R>0$, $\eta\in(0,1/2)$ and $\delta\in(0,R/K_1)$. Choose then $\delta'>0$ with $\delta<\delta'<\min\left\{\delta/C_{p_1,p_2},R/K_1\right\}$. Choose γ,γ' such that $1<\gamma<\gamma'$ and $\theta>0$. For any $\overrightarrow{x}\in\mathscr{S}$, any $t>0$, any γ,γ' such that $1<\gamma<\gamma'$ and any $\theta>0$, we define the following events, depending on $\overrightarrow{x},t,\gamma,\gamma'$ and θ:*

$$\begin{aligned}
E^1 &= \{\eta^1(t+\delta t^{1/2+\eta})\subset \mathscr{B}^0_{p_1}(t+\delta' t^{1/2+\eta})\},\\
E^2 &= \Big\{\eta^2(t+\delta t^{1/2+\eta})\\
&\quad \supset \mathit{Shell}\left(\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|_{p_2}}\oplus(\theta t^{-\frac{1}{2}+\eta}),\frac{\|\overrightarrow{x}\|_{p_2}}{\|\overrightarrow{x}\|_{p_1}}(t+\gamma\delta' t^{\frac{1}{2}+\eta}),\frac{\|\overrightarrow{x}\|_{p_2}}{\|\overrightarrow{x}\|_{p_1}}(t+\gamma'\delta' t^{\frac{1}{2}+\eta})\right)\Big\},\\
E &= E^1\cap E^2.
\end{aligned}$$

Then there exist $\gamma_0'>1$ and $\theta_0>0$ such that for any γ,γ',θ satisfying $1<\gamma<\gamma'<\gamma_0'$ and $0<\theta<\theta_0$, there exist two strictly positive constants A,B such that

$$\forall \overrightarrow{x}\in\mathscr{S}\quad \forall t>0\quad \mathbb{P}(\mathrm{Shadow}(\overrightarrow{x},t,Rt^{1/2+\eta})\backslash E)\le A\exp(-Bt^{\eta}).$$

Proof. Let $R>0$, $\eta\in(0,1/2)$ and $\delta\in(0,R/K_1)$. Choose then $\delta'>0$ with $\delta<\delta'<\min\left\{\delta/C_{p_1,p_2},R/K_1\right\}$.

Choice of constants. Choose now δ_1,ε and β positive such that

$$\delta'<\delta_1<\frac{\delta}{C_{p_1,p_2}}, \tag{10}$$

$$\delta'<\delta_1(1-\varepsilon), \tag{11}$$

$$\beta<\min\{\delta'-\delta,\delta_1\varepsilon\}. \tag{12}$$

Step 1. Control of the slow p_1-infection. As $\eta<1/2$, we have

$$(t+\delta t^{1/2+\eta})+\beta(t+\delta t^{1/2+\eta})^{1/2+\eta}=t+(\delta+\beta)t^{1/2+\eta}+o(t^{1/2+\eta}).$$

By condition (12), $\delta+\beta<\delta'$. Thus, using the moderate deviation result (Proposition 2.2), there exist two strictly positive constants A_1,B_1 such that $\forall t>0$

$$\mathbb{P}((E^1)^c)\le A_1\exp(-B_1t^{\eta}). \tag{13}$$

We can thus assume in the following that E^1 occurs.

Step 2. Control of the competition process. Define:

$$F_1=F_1(t)=\left\{\eta^1(t)\subset\mathscr{B}^0_{p_1}(t+\beta t^{1/2+\eta})\right\}\cap\left\{\mathscr{B}^0_{p_1}(t-\beta t^{1/2+\eta})\subset\eta^1(t)\cup\eta^2(t)\right\}.$$

By Proposition 2.2 and Lemma 1.1, there exist two strictly positive constants A_2,B_2 such that $\forall t>0$

$$\mathbb{P}(F_1^c)\le A_2\exp(-B_2t^{\eta}). \tag{14}$$

We can thus assume in the following that F_1 occurs.

Define an integer approximation of the line $\mathbb{R}\overrightarrow{x}$ as follows:

$$D_{\overrightarrow{x}} = \{y \in \mathbb{Z}^d \,:\, \exists z \in \mathbb{R}\overrightarrow{x},\ \|y - z\|_\infty \le 1/2\}.$$

Note that $D_{\overrightarrow{x}}$ is connected. Let now $s_0 \in \eta^2(t) \cap D_{\overrightarrow{x}}$ be a point which realizes the maximum $\max_{y \in \eta^2(t) \cap D_{\overrightarrow{x}}} \langle y, \overrightarrow{x}\rangle$. On the event Shadow$(\overrightarrow{x}, t, Rt^{1/2+\eta}) \cap F_1$, the point s_0 is well defined and satisfies

$$\|s_0\|_{p_1} \ge t - \beta t^{1/2+\eta}. \tag{15}$$

Step 3. The weak infection can not fill the hole. Define

$$\begin{aligned} F_2 = F_2(t) &= \left\{\forall x \in \mathscr{B}_{p_1}(t + \beta t^{1/2+\eta}) \quad B^x_{p_1}(\delta t^{1/2+\eta}) \subset \mathscr{B}^x_{p_1}(\delta' t^{1/2+\eta})\right\}, \\ r &= R - K_1\delta' > 0. \end{aligned}$$

As $\delta' > \delta$, by the large deviation result (Proposition 2.1), there exist two strictly positive constants A_3, B_3 such that $\forall t > 0$

$$\mathbb{P}(F_2^c) \le |\mathscr{B}_{p_1}(t + \beta t^{1/2+\eta})| A_3 \exp(-B_3 t^{1/2+\eta}) \le A_3' \exp(-B_3' t^{1/2+\eta}). \tag{16}$$

We can thus assume in the following that F_2 occurs. Note that as $\mathscr{B}^x_{p_1}(\delta') \subset \mathscr{B}^x_2(K_1\delta')$ by the very definition of K_1, on the event Shadow$(\overrightarrow{x}, t, Rt^{1/2+\eta}) \cap F_1 \cap F_2$, there will exist a corridor Cyl$(\overrightarrow{x}, rt^{1/2+\eta})$ where we are sure that the strong infection will not be bothered in its progression between time t and time $t + (\beta + \delta_1 C_{p_1,p_2})t^{1/2+\eta}$ by the weak infection – see Figure 1.

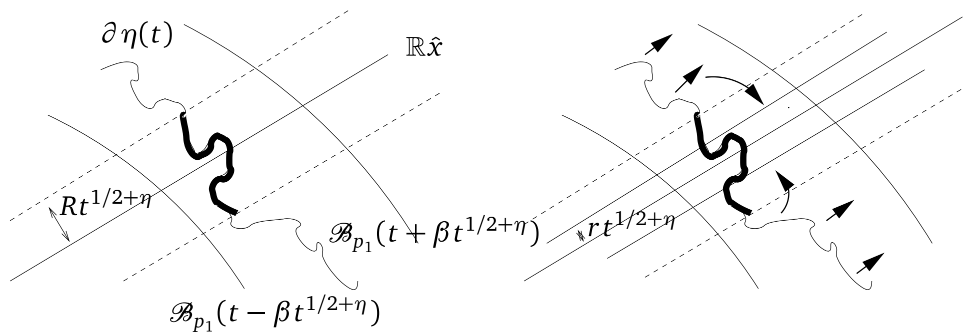

Figure 1: On the left hand side, Shadow$(x, t, Rt^{1/2+\eta})$: the strong are in bold. On the right hand side, the evolution of the weak: to enter Cyl$(x, rt^{1/2+\eta})$, it has to cross a gap of order $(R-r)t^{1/2+\eta}$.

Step 4. The strong manage to escape. Remember that, by the choice (11), $\delta' < \delta_1(1-\varepsilon)$. Define s_f as a point in $D_{\overrightarrow{x}} \cap \mathscr{B}_{p_1}(t + \delta_1(1-\varepsilon)t^{1/2+\eta})$ such that $(s_f + [-1/2, 1/2]^d) \cap \mathscr{B}_{p_1}(t + \delta_1(1-\varepsilon)t^{1/2+\eta})^c \neq \varnothing$. Then, by Estimate (15), we have

$$\begin{aligned} \|s_f - s_0\|_2 &\le \|s_f - \frac{\|s_f\|_{p_1}}{\|\overrightarrow{x}\|_{p_1}}\overrightarrow{x}\|_2 + \frac{\|s_f\|_{p_1}}{\|\overrightarrow{x}\|_{p_1}} - \frac{\|s_0\|_{p_1}}{\|\overrightarrow{x}\|_{p_1}} + \|\frac{\|s_0\|_{p_1}}{\|\overrightarrow{x}\|_{p_1}}\overrightarrow{x} - s_0\|_2 \\ &\le 2\sqrt{d} + \frac{\delta_1(1-\varepsilon) + \beta}{\|\overrightarrow{x}\|_{p_1}} t^{1/2+\eta}. \end{aligned}$$

By Condition (12) there exists $\varepsilon' > 0$ such that $\delta_1(1-\varepsilon)+\beta < \delta_1/(1+\varepsilon')$, and by definition of C_{p_1,p_2} in Proposition 2.3, for t large enough,

$$\|s_f - s_0\|_2 \leq \frac{\delta_1 C_{p_1,p_2}}{(1+\varepsilon')\|\overrightarrow{x}\|_{p_2}} t^{1/2+\eta}.$$

Define

$$F_3 = F_3(\overrightarrow{x}, t) = \{s_f \in \eta^2(t + \delta_1 C_{p_1,p_2} t^{1/2+\eta})\}.$$

Applying Lemma 2.5 with $h = \frac{\delta_1 C_{p_1,p_2}}{\|\overrightarrow{x}\|_{p_2}} t^{1/2+\eta}$, $\varepsilon = \varepsilon'$ and $f(h) = rt^{1/2+\eta}$ gives two strictly positive constants A_4, B_4 such that $\forall \overrightarrow{x} \in \mathscr{S} \quad \forall t > 0$

$$\mathbb{P}\left(\text{Shadow}(\overrightarrow{x}, t, Rt^{1/2+\eta}) \cap F_1 \cap F_2 \cap F_3^c\right) \leq A_4 \exp(-B_4 t^{1/2+\eta}). \tag{17}$$

Step 5. The strong colonizes a small shell. Let K_2 be a strictly positive constant such that $\forall x \in \mathbb{R}^d \quad \|x\|_{p_2} \leq K_2\|x\|_2$, and choose $\delta_2 > 0$ such that

$$\delta_2 < \min\left\{\delta - C_{p_1,p_2}\delta_1, \frac{r}{K_2}\right\}.$$

Let $\varepsilon' > 0$ be such that $(1+\varepsilon')K_2\delta_2 < r$. Define

$$F_4 = F_4(\overrightarrow{x}, t) = \{\mathscr{B}_{p_2}^{s_f}((1-\varepsilon')\delta_2 t^{1/2+\eta}) \subset \eta_2(t + \delta t^{1/2+\eta})\}.$$

Here, the choice we made for ε' ensures that, for t large enough

$$\mathscr{B}_{p_2}^{s_f}((1+\varepsilon')\delta_2 t^{1/2+\eta}) \subset (\text{Cyl}(\overrightarrow{x}, rt^{1/2+\eta}) \cap \mathscr{B}_{p_1}(t + \delta' t^{1/2+\eta})^c).$$

Thus, by the large deviation result (Proposition 2.1), there exist two strictly positive constants A_5, B_5 such that $\forall \overrightarrow{x} \in \mathscr{S} \quad \forall t > 0$,

$$\mathbb{P}(\text{Shadow}(\overrightarrow{x}, t, Rt^{1/2+\eta}) \cap F_1 \cap F_2 \cap F_3 \cap F_4^c) \leq A_5 \exp(-B_5 t^{1/2+\eta}). \tag{18}$$

Choose now $\theta_0 > 0$ and $\gamma_0' > 1$ such that

$$\begin{aligned} &\text{Shell}\left(\left\{\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|_{p_2}}\right\} \oplus \theta_0 t^{-1/2+\eta}, \frac{\|\overrightarrow{x}\|_{p_2}}{\|\overrightarrow{x}\|_{p_1}}(t + \delta' t^{1/2+\eta}), \frac{\|\overrightarrow{x}\|_{p_2}}{\|\overrightarrow{x}\|_{p_1}}(t + \gamma_0'\delta' t^{1/2+\eta})\right) \\ \subset \quad &\mathscr{B}_{p_2}^{s_f}((1-\varepsilon')\delta_2 t^{1/2+\eta}). \end{aligned}$$

Then $F_4 \backslash E^2 = \varnothing$, and collecting estimates (13), (14), (16), (17), and (18), we get the estimate of the lemma. □

The next lemma describes the typical progression of the strong infection from one shell to the next one.

Lemma 3.5. *Let $\varphi \in (0,2]$, $h \in (0,1/2)$ and $\alpha \in (1,2)$ be fixed parameters such that*

$$(1+h)\left(1+\frac{3\varphi}{h}\right) < \alpha. \tag{19}$$

For any S subset of $\mathscr{S}$ and for any $r,s>0$, we define the following event $E = E(S,r,s)$:
"Any point in the big Shell$(S \oplus \frac{\varphi r}{2(s+r)} \oplus \frac{\varphi(1+h)r}{2(s+(1+h)r)}, s+(1+h)r, s+(1+h)^2 r)$ is linked to a point in the small Shell$(S \oplus \frac{\varphi r}{2(s+r)}, s+r, s+(1+h)r)$ by an open path whose length is less than $\alpha h r$."
Then there exist two strictly positive constants A and B, only depending on φ, h, α, such that for any $r,s>0$ and any S of $\mathscr{S}$, we have

$$\mathbb{P}(E^c) \le A(s+r)^d \exp(-Br).$$

Moreover, we can assume that all the infection paths needed in E are completely included in the bigger Shell$(S \oplus \frac{(\varphi+2\alpha h)r}{s+r}, s+[1-3\varphi](1+h)r, \infty)$.

Proof. Let $\varphi \in (0,2]$, $h \in (0,1)$ and $\alpha \in (1,2)$ be fixed parameters satisfying Equation (19) and choose, in this order, $\alpha' > 1$, $\varepsilon > 0$ and $\rho > 0$ such that

$$(1+h)\varphi + (1+h)^2(1+\varphi) - (1+h-2\rho) \le \alpha' h < \alpha h, \tag{20}$$
$$4\rho\alpha' h < \varphi \text{ and } 4\rho\alpha' h < \varphi(1+h-2\rho), \tag{21}$$
$$h - 2\rho - \rho\alpha' h > 0, \tag{22}$$
$$(1+\varepsilon)^2(1+\rho)\alpha' \le \alpha. \tag{23}$$

Note that (20) is allowed by (19). Let S be any subset of $\mathscr{S}$.
Define $T = S \oplus \frac{\varphi r}{2(s+r)} \oplus \frac{\varphi(1+h)r}{2(s+(1+h)r)}$. Let $z \in T$:

$$\exists v \in S,\ u_1 \in \mathscr{B}^0_{p_2}\left(\frac{\varphi(1+h)r}{2(s+(1+h)r)}\right),\ u_2 \in \mathscr{B}^0_{p_2}\left(\frac{\varphi r}{2(s+r)}\right)$$

such that $z = v + u_1 + u_2$.

- As $v \in S$, we have $v \oplus \dfrac{\varphi r}{2(s+r)} \subset S \oplus \dfrac{\varphi r}{2(s+r)}$.
- Moreover, $\|z-v\|_{p_2} = \|u_1+u_2\|_{p_2} \le \|u_1\|_{p_2} + \|u_2\|_{p_2} \le \dfrac{\varphi(1+h)r}{(s+(1+h)r)}$.

Thus for any $z \in \text{Shell}(T, s+(1+h)r, s+(1+h)^2 r)$, we can choose $\overrightarrow{v}_z \in S$ such that

$$\left\|\frac{z}{\|z\|_{p_2}} - \overrightarrow{v}_z\right\|_{p_2} \le \frac{\varphi(1+h)r}{s+(1+h)r} \text{ and } \overrightarrow{v}_z \oplus \frac{\varphi r}{2(s+r)} \subset S \oplus \frac{\varphi r}{2(s+r)}.$$

We set $v_z = [s + (1+h-2\rho)r]\overrightarrow{v}_z$. Let us first estimate $\|z - v_z\|_{p_2}$: on the one hand,

$$
\begin{aligned}
\|z - v_z\|_{p_2} &\leq \|z - \|z\|_{p_2}\overrightarrow{v}_z\|_{p_2} + |\,\|z\|_{p_2} - s - (1+h-2\rho)r| \\
&\leq \|z\|_{p_2}\frac{\varphi(1+h)r}{s+(1+h)r} + \|z\|_{p_2} - s - (1+h-2\rho)r \\
&\leq \|z\|_{p_2}\left(1 + \frac{\varphi(1+h)r}{s+(1+h)r}\right) - s - (1+h-2\rho)r \qquad (24) \\
&\leq s\frac{\varphi(1+h)r}{s+(1+h)r} + [(1+h)^2\left(1 + \frac{\varphi(1+h)r}{s+(1+h)r}\right) - (1+h-2\rho)]r \\
&\leq [(1+h)\varphi + (1+h)^2(1+\varphi) - (1+h-2\rho)]r \qquad (25) \\
&\leq \alpha' hr \text{ thanks to (20),} \qquad (26)
\end{aligned}
$$

and, on the other hand,

$$\|z - v_z\|_{p_2} \geq \|z\|_{p_2} - \|v_z\|_{p_2} \geq 2\rho r. \qquad (27)$$

Geometrical fact. Let us see that, for every $z \in \mathrm{Shell}(T, s + (1+h)r, s + (1+h)^2 r)$,

$$\mathscr{B}_{p_2}^{v_z}(\rho\|z - v_z\|_{p_2}) \subset \mathscr{B}_{p_2}^{z}((1+\rho)\|z - v_z\|_{p_2}) \cap \mathrm{Shell}(S \oplus \frac{\varphi r}{2(s+r)}, s+r, s+(1+h)r). \qquad (28)$$

The triangle inequality ensures the first inclusion $\mathscr{B}_{p_2}^{v_z}(\rho\|z - v_z\|_{p_2}) \subset \mathscr{B}_{p_2}^{z}((1+\rho)\|z - v_z\|_{p_2})$. Let then $u \in \mathscr{B}_{p_2}^{v_z}(\rho\|z - v_z\|_{p_2})$, then, we apply Lemma 3.1 to $\|.\|_{p_2}$ and get:

$$
\begin{aligned}
\|\frac{u}{\|u\|_{p_2}} - \overrightarrow{v}_z\|_{p_2} &= \|\frac{u}{\|u\|_{p_2}} - \frac{v_z}{\|v_z\|_{p_2}}\|_{p_2} \\
&\leq \frac{2\rho\|z - v_z\|_{p_2}}{\|v_z\|_{p_2}} \\
&\leq \frac{2\rho\alpha' hr}{s + (1+h-2\rho)r} \text{ by Equation (26) and definition of } v_z \\
&\leq \frac{\varphi r}{2(s+r)} \text{ thanks to Equation (21)}
\end{aligned}
$$

and thus $\frac{u}{\|u\|_{p_2}} \in S \oplus \frac{\varphi r}{2(s+r)}$. For the norm of u, by definition of v_z and Equation (26), we have:

$$
\begin{aligned}
\|v_z\|_{p_2} - \rho\|z - v_z\|_{p_2} \leq \; &\|u\|_{p_2} \leq \|v_z\|_{p_2} + \rho\|z - v_z\|_{p_2} \\
s + (1+h-2\rho)r - \rho\alpha' hr \leq \; &\|u\|_{p_2} \leq s + (1+h-2\rho)r + \rho\alpha' hr \\
s + r \leq \; &\|u\|_{p_2} \leq s + (1+h)r,
\end{aligned}
$$

thanks to Equations (22) and (20). This proves the geometrical fact (28).

Probabilistic estimate. We can then estimate the probability of E^c. Note that, with (28), we have

$$
\begin{aligned}
E^c \subset \bigcup \quad & \left\{\mathscr{B}_{p_2}^{z}((1+\rho)\|z - v_z\|_{p_2}) \not\subset B_{p_2}^{z}((1+\rho)(1+\varepsilon)\|z - v_z\|_{p_2})\right\} \\
\cup \quad & \{B_{p_2}^{z}((1+\rho)(1+\varepsilon)\|z - v_z\|_{p_2}) \not\subset \mathscr{B}_{p_2}^{z}((1+\rho)(1+\varepsilon)^2\|z - v_z\|_{p_2})\},
\end{aligned}
$$

where the union is for $z \in \mathrm{Shell}(T, s+(1+h)r, s+(1+h)^2 r)$. By Proposition 2.1, there exist two strictly positive constants A_2 and B_2 such that for every S, for every $s, r > 0$,

$$\begin{aligned} \mathbb{P}(E^c) &\leq \sum_{z \in \mathrm{Shell}(T,s+(1+h)r,s+(1+h)^2 r)} A_2 \exp(-B_2(1+\rho)(1+\varepsilon)\|z - v_z\|_{p_2}) \\ &\leq |\mathrm{Shell}(T, s+(1+h)r, s+(1+h)^2 r)| \times A_2 \exp(-B_2(1+\rho)2\rho r) \end{aligned}$$

thanks to (27). Then, for every $z \in \mathrm{Shell}(T, (1+h)r, (1+h)^2 r)$, thanks to (26) and (23), one has $(1+\varepsilon)(1+\rho)\|z - v_z\|_{p_2} \leq \alpha h r$, which proves the exponential estimate of the lemma.

Control of the infection paths. It remains to estimate the minimal room needed to perform this infection, or in other words to control

$$\bigcup_{z \in \mathrm{Shell}(T,s+(1+h)r,s+(1+h)^2 r)} \mathscr{B}^z_{p_2}((1+\varepsilon)^2(1+\rho)\|z - v_z\|_{p_2}).$$

Let $z \in \mathrm{Shell}(T, (1+h)r, (1+h)^2 r)$ and $u \in \mathscr{B}^z_{p_2}((1+\varepsilon)^2(1+\rho)\|z - v_z\|_{p_2})$. We have:

$$\begin{aligned} & \|u\|_{p_2} \\ \geq\ & \|z\|_{p_2} - (1+\varepsilon)^2(1+\rho)\|z - v_z\|_{p_2} \\ \geq\ & (1+\varepsilon)^2(1+\rho)[s + (1+h-2\rho)r] \\ & - \left[(1+\varepsilon)^2(1+\rho)\left(1 + \frac{\varphi(1+h)r}{s+(1+h)r}\right) - 1\right]\|z\|_{p_2} \qquad \text{thanks to (24)} \\ \geq\ & (1+\varepsilon)^2(1+\rho)[s + (1+h-2\rho)r] \\ & - \left[(1+\varepsilon)^2(1+\rho)\left(1 + \frac{\varphi(1+h)r}{s+(1+h)r}\right) - 1\right](s + (1+h)^2 r) \\ \geq\ & s + r[(1+\varepsilon)^2(1+\rho)[(1+h-2\rho) - \varphi - (1+\varphi(1+h)r)(1+h)^2] + (1+h)^2]. \end{aligned}$$

This last term tends to $s + r(1+h-\varphi-(1+h)^3\varphi)$ when ε and ρ tend to 0. By decreasing if necessary ε and ρ, we obtain, as $h < 1/2$:

$$\|u\|_{p_2} \geq s + (1-3\varphi)(1+h)r.$$

Finally, by applying Lemma 3.1 and then Inequality (26), we have

$$\|\overrightarrow{u} - \overrightarrow{z}\|_{p_2} \leq \frac{2\|u - z\|_{p_2}}{\|z\|_{p_2}} \leq \frac{2(1+\varepsilon)^2(1+\rho)\alpha' h r}{s + (1+h)^2 r} \leq \frac{2\alpha h r}{s+r}.$$

Thus $u \in \mathrm{Shell}(T \oplus \frac{2\alpha h r}{s+r}, s + (1-3\varphi)(1+h)r, \infty)$, which ends the proof of the lemma. $\square$

Lemma 3.6. *Let $R > 0$ and $\eta \in (0, 1/2)$. There exist two strictly positive constants A, B such that*

$$\forall \overrightarrow{v} \in \mathscr{S} \quad \forall t > 0 \quad \mathbb{P}\left(\mathscr{G}^1 \cap \mathrm{Shadow}(\overrightarrow{v}, t, Rt^{1/2+\eta})\right) \leq A \exp(-Bt^\eta).$$

Proof. For convenience, we note $\hat{x} = v/\|v\|_{p_2} \in \mathscr{S}_{p_2}$. Therefore, $\overrightarrow{v} = \frac{\hat{x}}{\|\hat{x}\|_2} \in \mathscr{S}$. Let $\delta > 0$.

Idea of the proof. The idea is quite natural: start the progression thanks to the initialization Lemma 3.4. Then, apply recursively the progression Lemma 3.5 until the stronger infection surrounds the weaker one. The point is to ensure that this progression is not disturbed by the spread of the weaker infection.

Step 0. Choice of constants. Choose $R > 0$, $\eta \in (0, 1/2)$ and $\delta \in (0, R/K_1)$. Choose then $\delta' > \delta$ such that $\delta' < \min\{\delta/C_{p_1,p_2}, R/K_1\}$. Lemma 3.4 gives then $\theta_0 > 0$ and $\gamma'_0 > 0$. Remember that $C_{p_1,p_2} < 1$ and choose, in this order, $\alpha, \gamma, \theta, h, \varphi, \varepsilon$:

$$1 < \alpha < \min\left\{\frac{1}{C_{p_1,p_2}}, 2\right\}, \tag{29}$$

$$1 < \gamma < \gamma'_0 \text{ and } 0 < \theta < \theta_0, \tag{30}$$

$$\begin{aligned} &0 < h < \frac{1}{2} \text{ and } 1 + h < \min\left\{\alpha, \frac{\gamma'_0}{\gamma}\right\} \\ &\quad \text{and } 8\alpha h C_{p_1,p_2} C_{p_2,p_1} < 1 - \max\left\{\frac{1}{\gamma}, C_{p_1,p_2}\alpha\right\}, \end{aligned} \tag{31}$$

$$\begin{aligned} &\varphi > 0 \text{ and } (1+h)\left(1 + \frac{3\varphi}{h}\right) < \alpha \text{ and } \varphi\gamma\delta' < 2\theta \\ &\quad \text{and } \varphi\left(4C_{p_1,p_2}C_{p_2,p_1}\left(\frac{1}{2h} + 1\right) + 3\right) + 8\alpha h C_{p_1,p_2}C_{p_2,p_1} \\ &\quad < 1 - \max\left\{\frac{1}{\gamma}, C_{p_1,p_2}\alpha\right\}, \end{aligned} \tag{32}$$

$$\begin{aligned} &\varepsilon > 0 \text{ and } \varphi\left(4C_{p_1,p_2}C_{p_2,p_1}(\frac{1}{2h} + 1) + 3\right) + 8\alpha h C_{p_1,p_2}C_{p_2,p_1} \\ &\quad + \varepsilon\alpha C_{p_1,p_2} < 1 - \max\left\{\frac{1}{\gamma}, C_{p_1,p_2}\alpha\right\}. \end{aligned} \tag{33}$$

Set $\gamma' = (1+h)\gamma < \gamma'_0$.

Step 1. Initialization of the spread. Let us introduce the following notations:

$$\begin{aligned} E_1^1(\hat{x}, t) &= \{\eta^1(t + \delta t^{1/2+\eta}) \subset \mathscr{B}^0_{p_1}(t + \delta' t^{1/2+\eta})\}, \\ E_1^2(\hat{x}, t) &= \Big\{\eta^2(t + \delta t^{1/2+\eta}) \\ &\qquad \supset \text{Shell}\left(\hat{x} \oplus \frac{\varphi r_0 \|\hat{x}\|_{p_1}}{2(t + r_0\|\hat{x}\|_{p_1})}, \frac{t + \gamma\delta' t^{1/2+\eta}}{\|\hat{x}\|_{p_1}}, \frac{t + \gamma'\delta' t^{1/2+\eta}}{\|\hat{x}\|_{p_1}}\right)\Big\}, \\ E_1(\hat{x}, t) &= E_1^1(\hat{x}, t) \cap E_1^2(\hat{x}, t). \end{aligned}$$

From Assumption $\varphi\gamma\delta' < 2\theta$ in (32), it follows that $\frac{\varphi r_0\|\hat{x}\|_{p_1}}{2(t+r_0\|\hat{x}\|_{p_1})} \le \theta t^{-1/2+\eta}$ for t large enough. Then, by Lemma 3.4, there exist two strictly positive constants A_1 and B_1 such that for every $\hat{x} \in \mathscr{S}_{p_2}$, for every $t > 0$, we have

$$\mathbb{P}\left(\text{Shadow}\left(\frac{\hat{x}}{\|\hat{x}\|_2}, t, Rt^{1/2+\eta}\right) \backslash E_1(\hat{x}, t)\right) \le A_1 \exp(-B_1 t^\eta). \tag{34}$$

Thus, if Shadow$(\frac{\hat{x}}{\|\hat{x}\|_2}, t, Rt^{1/2+\eta})$ occurs, then at the slightly larger time $t_1(\hat{x},t) = t + \delta t^{1/2+\eta}$, the first shell

$$S_1(\hat{x},t) = \text{Shell}\left(\hat{x} \oplus \frac{\varphi r_0 \|\hat{x}\|_{p_1}}{2(t + r_0\|\hat{x}\|_{p_1})}, \frac{t + \gamma\delta' t^{1/2+\eta}}{\|\hat{x}\|_{p_1}}, \frac{t + \gamma'\delta' t^{1/2+\eta}}{\|\hat{x}\|_{p_1}}\right)$$

is with high probability colonized by the p_2-infection. We want now to extend this colonization to larger and larger shells by applying recursively Lemma 3.5.

Notations. We still need to introduce a certain number of notations, inspired by Lemma 3.5.

$k = 1$
$r_0 = r_0(\hat{x},t) = \frac{1}{\|\hat{x}\|_{p_1}}\gamma\delta' t^{1/2+\eta}$ $r_1 = r_1(\hat{x},t) = (1+h)r_0$ $t_1 = t_1(\hat{x},t) = t + \delta t^{1/2+\eta}$ $A_1 = A_1(\hat{x},t) = \{\hat{x}\} \oplus \frac{\varphi r_0\|\hat{x}\|_{p_1}}{2(t + r_0\|\hat{x}\|_{p_1})}$ $S_1 = S_1(\hat{x},t) = \text{Shell}\left(A_1, \frac{t}{\|\hat{x}\|_{p_1}} + r_0, \frac{t}{\|\hat{x}\|_{p_1}} + r_1\right)$
$k \geq 2$
$r_k = r_k(\hat{x},t) = (1+h)r_{k-1} = (1+h)^k r_0$ and $r_k^{\min} = [1-3\varphi](1+h)r_{k-2}$ $t_k = t_k(\hat{x},t) = t_{k-1} + h\alpha r_{k-2} = t + \delta t^{1/2+\eta} + r_0\alpha[(1+h)^{k-1} - 1]$ $A_k = A_k(\hat{x},t) = A_{k-1} \oplus \frac{\varphi r_{k-1}\|\hat{x}\|_{p_1}}{2(t + r_{k-1}\|\hat{x}\|_{p_1})}$ $A_k^+ = A_k^+(\hat{x},t) = A_{k-2} \oplus \frac{(\varphi + 2\alpha h)r_{k-2}\|\hat{x}\|_{p_1}}{t + r_{k-2}\|\hat{x}\|_{p_1}}$ $S_k = S_k(\hat{x},t) = \text{Shell}\left(A_k, \frac{t}{\|\hat{x}\|_{p_1}} + r_{k-1}, \frac{t}{\|\hat{x}\|_{p_1}} + r_k\right)$ $S_k^+ = S_k^+(\hat{x},t) = \text{Shell}\left(A_k^+, \frac{t}{\|\hat{x}\|_{p_1}} + r_k^{\min}, \infty\right)$

Define also the following events, for $k \geq 2$ and $x \in \mathbb{Z}^d \backslash \{0\}$:

$$\begin{aligned}
E_k^1 &= E_k^1(\hat{x},t) = \{\eta^1(t_k(\hat{x},t)) \cap S_k^+(\hat{x},t) = \varnothing\},\\
E_k^2 &= E_k^2(\hat{x},t) = \{\eta^2(t_k(\hat{x},t)) \supset S_k(\hat{x},t)\},\\
E_k &= E_k(\hat{x},t) = E_k^1(\hat{x},t) \cap E_k^2(\hat{x},t).
\end{aligned}$$

The aim is the following: we want to apply Lemma 3.5 to prove that if $E_k^2(\hat{x},t)$ is realized, then with high probability $E_{k+1}^2(\hat{x},t)$ is also realized. But we need first to control the spread of the slow p_1-infection, and to see that it will not disturb the spread of the fast p_2-infection from $S_k(\hat{x},t)$ to $S_{k+1}(\hat{x},t)$.

Step 2. Rough control of the slow p_1-infection. Here, for convenience, the complementary event of A is denoted by $\complement(A)$. Let $\varepsilon > 0$. Define, for every $k \geq 2$

$$F_k^1(\hat{x},t) = \left\{\eta^1(t_k(\hat{x},t)) \subset \mathscr{B}_{p_1}^0(t_1 + (\delta' - \delta)t^{1/2+\eta} + (1+\varepsilon)(t_k - t_1))\right\}.$$

Let us prove that there exist two strictly positive constants A_2 and B_2 such that

$$\forall t>0 \quad \forall \hat{x} \in \mathscr{S}_{p_2} \quad \mathbb{P}\left(E_1^1(\hat{x},t) \cap \complement\left(\bigcap_{k\geq 2} F_k^1(\hat{x},t)\right)\right) \leq A_2 \exp(-B_2 t^{1/2+\eta}). \tag{35}$$

Note that

$$\{B_{p_1}^0(s) \subset \mathscr{B}_{p_1}^0(s')\} \cap \{B_{p_1}^0(s+t) \not\subset \mathscr{B}_{p_1}^0(s'+t')\} \subset \bigcup_{x \in \mathscr{B}_{p_1}^0(s')} \{B_{p_1}^x(t) \not\subset \mathscr{B}_{p_1}^x(t')\},$$

and thus that

$$\mathbb{P}\left(\{B_{p_1}^0(s) \subset \mathscr{B}_{p_1}^0(s')\} \cap \{B_{p_1}^0(s+t) \not\subset \mathscr{B}_{p_1}^0(s'+t')\}\right) \leq |\mathscr{B}_{p_1}^0(s')| \mathbb{P}\left(B_{p_1}^0(t) \not\subset \mathscr{B}_{p_1}^0(t')\right).$$

In our context, this gives

$$\begin{aligned}
&\mathbb{P}\left(E_1^1(\hat{x},t) \cap \complement\left(\bigcap_{k\geq 2} F_k^1(\hat{x},t)\right)\right) \\
\leq\ & \sum_{k\geq 2} \mathbb{P}\left(E_1^1(\hat{x},t) \cap \{B_{p_1}^0(t_k) \not\subset \mathscr{B}_{p_1}^0(t_1 + (\delta'-\delta)t^{1/2+\eta} + (1+\varepsilon)(t_k - t_1)\}\right) \\
\leq\ & \sum_{k\geq 2} |\mathscr{B}_{p_1}^0(t+\delta' t^{1/2+\eta})| \mathbb{P}\left(B_{p_1}^0(t_k - t_1) \not\subset \mathscr{B}_{p_1}((1+\varepsilon)(t_k - t_1))\}\right).
\end{aligned}$$

The large deviation result, Proposition 2.1, gives then two strictly positive constants A, B such that

$$\begin{aligned}
\mathbb{P}\left(E_1^1(\hat{x},t) \cap \complement\left(\bigcap_{k\geq 2} F_k^1(\hat{x},t)\right)\right) &\leq |\mathscr{B}_{p_1}^0(t+\delta' t^{\frac{1}{2}+\eta})| \sum_{k\geq 2} A \exp\left(-B(t_k - t_1)\right) \\
&\leq |\mathscr{B}_{p_1}^0(t+\delta' t^{\frac{1}{2}+\eta})| A \sum_{k\geq 2} \exp\left(-B\alpha r_0 (k-1)h\right) \\
&\leq A|\mathscr{B}_{p_1}^0(t+\delta' t^{\frac{1}{2}+\eta})| \frac{\exp\left(-B\alpha r_0 h\right)}{1-\exp\left(-B\alpha r_0 h\right)} \\
&\leq A_2 \exp(-B_2 t^{\frac{1}{2}+\eta}),
\end{aligned}$$

since $r_0 = \frac{1}{\|\hat{x}\|_{p_1}} \gamma \delta' t^{1/2+\eta}$.

Step 3. Estimates for angles. Let us see that for any $\hat{x} \in \mathscr{S}_{p_2}$, for any $\varphi, \psi \geq 0$,

$$(\hat{x} \oplus \varphi) \oplus \psi \subset \hat{x} \oplus (\varphi + \psi).$$

Let $z \in (\hat{x} \oplus \varphi) \oplus \psi$: there exist $y \in \hat{x} \oplus \varphi$ and $v \in \mathscr{B}_{p_2}(\psi)$ such that $z = y + v$. As $y \in \hat{x} \oplus \varphi$, there exists $w \in \mathscr{B}_{p_2}(\varphi)$ such that $y = \hat{x} + w$. Thus

$$\|z - \hat{x}\|_{p_2} \leq \|v\|_{p_2} + \|w\|_{p_2} \leq \varphi + \psi.$$

This implies

$$\forall k \geq 2 \quad A_k(\hat{x},t) \quad \subset \quad \hat{x} \oplus \left(\sum_{j=0}^{k-1} \frac{\varphi r_j \|\hat{x}\|_{p_1}}{2(t + r_j \|\hat{x}\|_{p_1})}\right) \text{ and } A_k^+(\hat{x},t) \subset \hat{x} \oplus \theta_k^+,$$

$$\text{with } \theta_k^+ = \sum_{j=0}^{k-2} \frac{\varphi r_j \|\hat{x}\|_{p_1}}{2(t + r_j \|\hat{x}\|_{p_1})} + \frac{(\varphi + 2\alpha h) r_{k-2} \|\hat{x}\|_{p_1}}{t + r_{k-2} \|\hat{x}\|_{p_1}}.$$

Then, for all $k \geq 2$, for t large enough,

$$\begin{aligned}
\frac{t}{\|\hat{x}\|_{p_1}}\theta_k^+ &= \frac{t}{\|\hat{x}\|_{p_1}}\left(\sum_{j=0}^{k-2}\frac{\varphi r_j\|\hat{x}\|_{p_1}}{2(t+r_j\|\hat{x}\|_{p_1})}+\frac{(\varphi+2\alpha h)r_{k-2}\|\hat{x}\|_{p_1}}{t+r_{k-2}\|\hat{x}\|_{p_1}}\right)\\
&\leq \frac{\varphi}{2}\sum_{j=0}^{k-2}r_j\|\hat{x}\|_{p_1}+(\varphi+2\alpha h)r_{k-2}\\
&\leq r_0\left((1+h)^{k-1}\left(\frac{\varphi}{2h}+\varphi+2\alpha h\right)\right). \qquad (36)
\end{aligned}$$

Step 4. The weak can not bother the strong.

Let us see that for all $k \geq 2$

$$\mathscr{B}^0_{p_1}(t+\delta' t^{1/2+\eta}+(1+\varepsilon)r_0\alpha[(1+h)^{k-1}-1])\cap S_k^+(\hat{x},t)=\emptyset. \qquad (37)$$

Let y such that $\|y\|_{p_1} = t+\delta' t^{1/2+\eta}+(1+\varepsilon)r_0\alpha[(1+h)^{k-1}-1])$ and $\|\hat{y}-\hat{x}\|_{p_2} \leq \theta_k^+$. As $y=\|y\|_{p_1}\hat{y}/\|\hat{y}\|_{p_1}$, for t large enough

$$\begin{aligned}
&\|y\|_{p_2}\\
\leq\ & \frac{t+\delta' t^{1/2+\eta}}{\|\hat{x}\|_{p_1}}+(t+\delta' t^{1/2+\eta})\left|\frac{1}{\|\hat{y}\|_{p_1}}-\frac{1}{\|\hat{x}\|_{p_1}}\right|+\frac{(1+\varepsilon)\alpha r_0[(1+h)^{k-1}-1]}{\|\hat{y}\|_{p_1}}\\
\leq\ & \frac{t+\delta' t^{1/2+\eta}}{\|\hat{x}\|_{p_1}}+\frac{(t+\delta' t^{1/2+\eta})C_{p_2,p_1}\|\hat{y}-\hat{x}\|_{p_2}}{\|\hat{x}\|_{p_1}\|\hat{y}\|_{p_1}}+\frac{(1+\varepsilon)\alpha r_0[(1+h)^{k-1}-1]}{\|\hat{y}\|_{p_1}}\\
\leq\ & \frac{t}{\|\hat{x}\|_{p_1}}+\frac{r_0}{\gamma}+\frac{2tC_{p_1,p_2}C_{p_2,p_1}\theta_k^+}{\|\hat{x}\|_{p_1}}+C_{p_1,p_2}(1+\varepsilon)\alpha r_0[(1+h)^{k-1}-1]\\
\leq\ & \frac{t}{\|\hat{x}\|_{p_1}}+\frac{r_0}{\gamma}+2C_{p_1,p_2}C_{p_2,p_1}r_0(1+h)^{k-1}\left(\frac{\varphi}{2h}+\varphi+2\alpha h\right)\\
&+C_{p_1,p_2}(1+\varepsilon)\alpha r_0[(1+h)^{k-1}-1],
\end{aligned}$$

where the last inequality follows from (36). Then

$$\begin{aligned}
&\frac{1}{r_0}\left(\|y\|_{p_2}-\frac{t}{\|x\|_{p_1}}-r_k^{\min}\right)\\
\leq\ & \frac{1}{\gamma}+2C_{p_1,p_2}C_{p_2,p_1}(1+h)^{k-1}\left(\frac{\varphi}{2h}+\varphi+2\alpha h\right)\\
&+C_{p_1,p_2}(1+\varepsilon)\alpha[(1+h)^{k-1}-1]-(1-3\varphi)(1+h)^{k-1}\\
\leq\ & (1+h)^{k-1}\left(2C_{p_1,p_2}C_{p_2,p_1}\left(\frac{\varphi}{2h}+\varphi+2\alpha h\right)+C_{p_1,p_2}(1+\varepsilon)\alpha-1+3\varphi\right)\\
&+\left(\frac{1}{\gamma}-C_{p_1,p_2}(1+\varepsilon)\alpha\right). \qquad (38)
\end{aligned}$$

We want to prove that this quantity is negative for every $k \geq 2$. Conditions (32) and (33) ensure that the coefficient in $(1+h)^{k-1}$ in (38) is negative. Thus, asymptotically in k, the right-hand side in (38) is negative. To ensure it is negative for every $k \geq 2$, we only need to see that it is true for $k = 2$, which is ensured by Conditions (32) and (33). This proves (37). Note that this also implies

$$\forall k \geq 2 \quad F_k^1 \subset E_k^1. \tag{39}$$

Equations (39) and (35) together give:

$$\forall t > 0 \quad \forall \hat{x} \in \mathscr{S}_{p_2} \quad \mathbb{P}\left(E_1^1(\hat{x}, t) \cap \left(\bigcup_{k\geq 2} E_k^1(\hat{x}, t)^c\right)\right) \leq A_2 \exp(-B_2 t^{1/2+\eta}). \tag{40}$$

Step 5. Control of the fast p_2-infection. Let A_3 and B_3 be the two strictly positive constants given by Lemma 3.5 for the choice for α, h, φ we made in (29), (31) and (32). Note that for every $k \geq 2$, we have

$$E_k^1 \cap E_{k-1}^2 \cap E(A_{k-1}, r_{k-2}, \frac{t}{\|\hat{x}\|_{p_1}}) \subset E_k^2$$

where the event $E(.,.,.)$ was defined in Lemma 3.5. Thus, the application of Lemma 3.5 implies that for any $\hat{x} \in \mathscr{S}_{p_2}$, any $t > 0$, for every $k \geq 2$,

$$\begin{aligned}
\mathbb{P}((E_k^2)^c \cap E_k^1 \cap E_{k-1}^2) &\leq \mathbb{P}(E(A_{k-1}, r_{k-2}, \frac{t}{\|\hat{x}\|_{p_1}})^c) \\
&\leq A_3(\frac{t}{\|\hat{x}\|_{p_1}} + r_{k-2})^d \exp(-B_3 r_{k-2}).
\end{aligned}$$

Thus, for each $t \geq 1$, each $\hat{x} \in \mathscr{S}_{p_2}$,

$$\begin{aligned}
\sum_{k\geq 2} \mathbb{P}\left((E_k^2)^c \cap (E_k^1 \cap E_{k-1}^2\right) &\leq A_3 \sum_{k\geq 2} \left(\frac{t}{\|\hat{x}\|_{p_1}} + r_{k-2}\right)^d \exp(-B_3 r_{k-2}) \\
&\leq A_3 \sum_{k\geq 0} (C_{p_1,p_2} t + 1)^d (r_k + 1)^d \exp(-B_3 r_k) \\
&\leq A_4 (C_{p_1,p_2} t + 1)^d \sum_{k\geq 0} \exp(-B_4 r_k) \\
&\leq A_4 (C_{p_1,p_2} t + 1)^d \sum_{k\geq 0} \exp(-B_4 r_0 (1 + kh)) \\
&\leq A_4 (C_{p_1,p_2} t + 1)^d \exp(-B_4 r_0)(1 - \exp(-B_4 r_0 h))^{-1} \\
&\leq A_5 \exp(-B_5 t^{1/2+\eta}).
\end{aligned} \tag{41}$$

where A_4, A_5 and B_4, B_5 are strictly positive constants.

Conclusion. For k large enough, the set S_k disconnects 0 from infinity, and thus the event $\bigcap_{k\geq 1} E_k$ implies that the slow p_1-infection is surrounded by the fast p_2-infection and thus dies out. So, using

(34), (40) and (41), we obtain:

$$\begin{aligned}
& \mathbb{P}\left(\mathscr{G}^1 \cap \text{Shadow}\left(\frac{\hat{x}}{\|\hat{x}\|_2}, t, Rt^{1/2+\eta}\right)\right) \\
\leq\ & \mathbb{P}\left(\text{Shadow}\left(\frac{\hat{x}}{\|\hat{x}\|_2}, t, Rt^{1/2+\eta}\right) \cap \bigcup_{k\geq 1} E_k(\hat{x},t)^c\right) \\
\leq\ & \mathbb{P}\left(\text{Shadow}\left(\frac{\hat{x}}{\|\hat{x}\|_2}, t, Rt^{1/2+\eta}\right) \cap E_1(\hat{x},t)^c\right) + \mathbb{P}\left(E_1^1(\hat{x},t) \cap \bigcup_{k\geq 2} E_k^1(\hat{x},t)^c\right) \\
& + \sum_{k\geq 2} \mathbb{P}\left((E_k^2(\hat{x},t))^c \cap (E_k^1(\hat{x},t) \cap E_{k-1}^2(\hat{x},t)\right) \\
\leq\ & A\exp(-Bt^\eta),
\end{aligned}$$

which completes the proof. □

Proof of Theorem 1.2. Proposition 2.1 and Lemma 1.1 give the existence of strictly positive constants α, β, A_1, B_1 such that the event

$$F_t = \{\forall y \in \partial\eta(t) \quad \|y\|_2 \in (\alpha t, \beta t)\}$$

has a probability larger than $1 - A_1 \exp(-aB_1 t)$, so we only have to control the probability of the event

$$\mathscr{G}^1 \cap \text{Shade}(t, Mt^{1/2+\eta}) \cap F_t.$$

Assume that t is large enough to have $\alpha t > Mt^{1/2+\eta}$ and set

$$\theta = \min\left\{\frac{1}{2}, \left(\frac{Mt^{1/2+\eta}}{2\beta t}\right)^2\right\}.$$

By Lemma 3.2, there exists a subset T of the unit sphere $\mathscr{S}$ with $|T| \leq C(1+\frac{1}{\theta})^{d-1}$ such that $\mathscr{S} \subset \bigcup_{\overrightarrow{x}\in T} \mathscr{B}_2^{\overrightarrow{x}}(\theta)$.

Assume now that $\text{Shade}(t, Mt^{1/2+\eta}) \cap F_t$ occurs: there exists $\overrightarrow{u} \in \mathscr{S}_2$ such that $\text{Shadow}(\overrightarrow{u}, t, Mt^{1/2+\eta})$ happens. Let $\overrightarrow{x} \in S$ be such that $\|\overrightarrow{u} - \overrightarrow{x}\|_2 \leq \theta$. Let us prove that $\text{Shadow}(\overrightarrow{x}, t, \frac{M}{2}t^{1/2+\eta})$ happens.

Let γ be an infinite path in $\text{Cyl}_+(\overrightarrow{x}, \frac{M}{2}t^{1/2+\eta})$ starting at some point $y \in \eta(t)$. We must prove that γ meets $\partial\eta(t) \cap \eta^2(t)$. We can suppose without loss of generality that y is the last point of γ in $\eta(t)$ and thus $y \in \partial\eta(t)$. Note that the points in γ after y are in $\mathscr{B}_2(\alpha t)^c \subset \mathscr{B}_2(Mt^{1/2+\eta})^c$.

- Either $y \in \eta^2(t)$, and we are done.
- Or $y \in \eta^1(t)$. Let z be the point after y along γ where γ exits from $\mathscr{B}_2(\beta t)$: between y and z, the path is in $\text{Cyl}_+(\overrightarrow{x}, \frac{M}{2}t^{1/2+\eta}) \cap \mathscr{B}_2(\beta t) \cap \mathscr{B}_2(Mt^{1/2+\eta})^c \subset \text{Cyl}_+(\overrightarrow{u}, Mt^{1/2+\eta}) \cap \mathscr{B}_2(\beta t)$ thanks to Lemma 3.3. We build now a path γ' inside the $\text{Cyl}_+(\overrightarrow{u}, Mt^{1/2+\eta})$ by concatenating the portion of γ between y and z and any infinite path starting from z and staying in $\text{Cyl}_+(\overrightarrow{u}, Mt^{1/2+\eta}) \cap \mathscr{B}_2(\beta t)^c$: this path prevents the occurrence of $\text{Shadow}(\overrightarrow{u}, t, Mt^{1/2+\eta})$, as it starts from a point in $\eta^1(t)$ and do not visit any other point in $\eta(t)$. The assumption $y \in \eta^1(t)$ is thus contradicted.

So $y \in \eta^2(t)$, which means that Shadow$(\overrightarrow{x}, t, \frac{M}{2} t^{1/2+\eta})$ happens, and implies

$$\text{Shade}(t, Mt^{1/2+\eta}) \cap F_t \subset \bigcup_{\overrightarrow{x} \in T} \text{Shadow}\left(\overrightarrow{x}, t, \frac{M}{2} t^{1/2+\eta}\right).$$

Finally, Theorem 3.6 give two strictly positive constants A', B' such that

$$\begin{aligned}
\mathbb{P}(\mathscr{G}^1 \cap \text{Shade}(t, Mt^{1/2+\eta}) \cap F_t) &\leq |T| A' \exp(-B' t^\eta) \\
&\leq A' \left(1 + \frac{1}{\theta}\right)^{d-1} \exp(-Bt^\eta) \\
&\leq A' \left(1 + \frac{4\beta^2}{M^2} t^{1-2\eta}\right)^{d-1} \exp(-Bt^\eta),
\end{aligned}$$

which ends the proof. □

Proof of Corollary 1.3. We can assume that $\eta \in (0, 1)$. For every $n \geq 1$, we define $t_n = n^2$ and

$$A(n, M) = \bigcup_{t \in [t_n, t_{n+1}]} \text{Shade}(t, Mt^{1/2+\eta}).$$

For each $M > 0$, we have to prove that $\mathbb{P}\left(\mathscr{G}^1 \cap \overline{\lim_{n \to +\infty}} A(n, M)\right) = 0$. By the Borel-Cantelli lemma, it is sufficient to prove that

$$\sum_{n=1}^{+\infty} \mathbb{P}(\mathscr{G}^1 \cap A(n, M)) < +\infty.$$

Let then $M > 0$. Obviously,

$$\begin{aligned}
&\mathbb{P}(\mathscr{G}^1 \cap A(n, M)) \\
\leq\ &\mathbb{P}\left(\mathscr{G}^1 \cap \text{Shade}\left(t_{n+1}, \frac{M}{2} t_{n+1}^{1/2+\eta}\right)\right) + \mathbb{P}\left(\eta^1(t_{n+1}) \not\subset \mathscr{B}_{p_1}(2t_{n+1})\right) \\
&+ \mathbb{P}\left(\left(A(n, M) \backslash \text{Shade}\left(t_{n+1}, \frac{M}{2} t_{n+1}^{1/2+\eta}\right)\right) \cap \{\eta^1(t_{n+1}) \subset \mathscr{B}_{p_1}(2t_{n+1})\}\right).
\end{aligned}$$

Lemma 1.2 gives two strictly positive constants A_1, B_1 such that

$$\mathbb{P}\left(\mathscr{G}^1 \cap \text{Shade}\left(t_{n+1}, \frac{M}{2} t_{n+1}^{1/2+\eta}\right)\right) \leq A_1 \exp(-B_1 n^{2\eta}),$$

while Proposition 2.1 gives two strictly positive constants A_2, B_2 such that

$$\mathbb{P}\left(\eta^1(t_{n+1}) \not\subset \mathscr{B}_{p_1}(2t_{n+1})\right) \leq A_2 \exp(-B_2 n^2).$$

So it only remains to prove that

$$\sum_{n=1}^{+\infty} \mathbb{P}\left(\left(A(n, M) \backslash \text{Shade}\left(t_{n+1}, \frac{M}{2} t_{n+1}^{1/2+\eta}\right)\right) \cap \{\eta^1(t_{n+1}) \subset \mathscr{B}_{p_1}(2t_{n+1})\}\right) < +\infty.$$

Assume now that $(A(n,M)\setminus \text{Shade}(t_{n+1},\frac{M}{2}t_{n+1}^{1/2+\eta}))\cap\{\eta^1(t_{n+1})\subset\mathscr{B}_{p_1}(0,2t_{n+1})\}$ holds: there exists $\overrightarrow{x}$ and $t\in[t_n,t_{n+1})$ such that $\text{Shadow}(\overrightarrow{x},t,Mt^{1/2+\eta})$ holds but not $\text{Shadow}(\overrightarrow{x},t_{n+1},\frac{M}{2}t_{n+1}^{1/2+\eta})$. Since $\text{Shadow}(\overrightarrow{x},t_{n+1},\frac{M}{2}t_{n+1}^{1/2+\eta})$ is not fulfilled, there exists some infinite path γ in $\text{Cyl}_+(\overrightarrow{x},\frac{M}{2}t_{n+1}^{1/2+\eta})$ starting in some point $v\in\eta^1(t_{n+1})$ and that never meets $\eta^2(t_{n+1})$. By the definition of the process $(\eta^1(t))_{t\geq 0}$, there exist $u\in\eta^1(t_n)$ and a path γ' from u to v such that

$$\sum_{e\in\gamma'}\omega_e^1\leq t_{n+1}-t_n$$

and the path γ' does not meet any point in $\eta^2(\infty)$. The path γ' can not stay completely in $\text{Cyl}_+(\overrightarrow{x},Mt_n^{1/2+\eta})$, otherwise concatenating γ' and γ together would contradict $\text{Shadow}(\overrightarrow{x},t,Mt^{1/2+\eta})$. Let w be a point in $\gamma'\cap\text{Cyl}_+(\overrightarrow{x},Mt_n^{1/2+\eta})^c$: the portion γ'' of γ' between $w\in\eta^1(t_{n+1})\cap\text{Cyl}_+(\overrightarrow{x},Mt_n^{1/2+\eta})^c$ and $v\in\eta^1(t_{n+1})\cap\text{Cyl}_+(\overrightarrow{x},\frac{M}{2}t_{n+1}^{1/2+\eta})$ satisfies

$$\sum_{e\in\gamma''}\omega_e^1\leq t_{n+1}-t_n.$$

Provided that n is large enough, we can say that $\|w-v\|_{p_1}\geq 2(t_{n+1}-t_n)$ – note that $t_n^{1/2+\eta}\gg t_{n+1}-t_n$. Using the fact that $\eta^1(t_{n+1})\subset\mathscr{B}_{p_1}(2t_{n+1})$, we obtain

$$\begin{aligned}&(A(n,M)\setminus \text{Shade}(t_{n+1},Mt_{n+1}^{1/2+\eta}/2))\cap\{\eta^1(t_{n+1})\subset\mathscr{B}_{p_1}(0,2t_{n+1})\}\\ \subset\quad &\bigcup_{w\in\mathscr{B}_{p_1}(2t_{n+1})}\{B^w_{p_1}(t_{n+1}-t_n)\not\subset\mathscr{B}^w_{p_1}(2(t_{n+1}-t_n))\}.\end{aligned}$$

So it follows from Proposition 2.1 that there exist strictly positive constants A_3,B_3 such that

$$\begin{aligned}&\mathbb{P}((A(n,M)\setminus \text{Shade}(t_{n+1},Mt_{n+1}^{1/2+\eta}/2))\cap\{\eta^1(t_{n+1})\subset\mathscr{B}_{p_1}(0,2t_{n+1})\})\\ \leq\quad & |\mathscr{B}_{p_1}(2t_{n+1})|A_3\exp(-B_3(t_{n+1}-t_n))\\ \leq\quad & Kn^{2d}A_3\exp(-2B_3n),\end{aligned}$$

which is summable. This ends the proof of the theorem. □

Note that the order $t^{1/2+\eta}$ that appears in our results follows from moderate deviations for fluctuations with respect to the asymptotic shape given in Proposition 2.2. However, the conjectured order for these fluctuations is rather 1/3. The proofs would apply with an estimate of the type

$$\forall t>0\quad \mathbb{P}\left(\mathscr{B}^0(t-\beta t^{1/3+\eta})\subset B^0(t)\subset\mathscr{B}^0(t+\beta t^{1/3+\eta})\right)\geq 1-A\exp(-Bt^\eta).$$

which would lead to replace $t^{1/2+\eta}$ by $t^{1/3+\eta}$ in our results.

4 Moderate deviations for the global growth of the epidemics

This section is devoted to the proof of Theorem 1.5: when the weak survives, we recover the same fluctuations with respect to the asymptotic shape as in the case where the weak infection evolves alone.

In fact, the only point is to see that if the strong infection at time t admits points outside $\mathscr{B}_{p_1}\left(t+\beta t^{1/2+\eta}\right)$, then this positional advantage enables us to create an event of type Shade$(T, MT^{1/2+\eta})$ at the slightly larger time $T = t + \alpha t^{1/2+\eta}$, and the probability of such an event is controlled by Lemma 1.2.

Proof of Theorem 1.5. Let $\beta > 0$ and $\eta \in (0,1/2)$. We want to prove that there exist two strictly positive constants A, B such that

$$\forall t>0 \quad \mathbb{P}\left(\mathscr{G}^1 \cap \left\{\mathscr{B}_{p_1}\left(t-\beta t^{1/2+\eta}\right) \subset \eta(t) \subset \mathscr{B}_{p_1}\left(t+\beta t^{1/2+\eta}\right)\right\}^c\right) \le A\exp(-Bt^{\eta}).$$

We can first easily rule out the cases where $\eta^1(t) \not\subset \mathscr{B}_{p_1}\left(t+\beta t^{1/2+\eta}\right)$ and where $\mathscr{B}_{p_1}\left(t-\beta t^{1/2+\eta}\right) \not\subset \eta(t)$, by Proposition 2.2 and Lemma 1.1: there exist two strictly positive constants A_1, B_1 such that for every $t>0$, we have

$$\begin{aligned}
&\mathbb{P}\left(\mathscr{G}^1 \cap \left\{\mathscr{B}_{p_1}\left(t-\beta t^{1/2+\eta}\right) \subset \eta(t) \subset \mathscr{B}_{p_1}\left(t+\beta t^{1/2+\eta}\right)\right\}^c\right)\\
\le\ & \mathbb{P}\left(\mathscr{B}_{p_1}\left(t-\beta t^{1/2+\eta}\right) \not\subset \eta(t)\right) + \mathbb{P}\left(\eta^1(t) \not\subset \mathscr{B}_{p_1}\left(t+\beta t^{1/2+\eta}\right)\right)\\
& + \mathbb{P}\left(\mathscr{G}^1 \cap \left\{\eta^2(t) \not\subset \mathscr{B}_{p_1}\left(t+\beta t^{1/2+\eta}\right)\right\}\right)\\
\le\ & \mathbb{P}\left(\mathscr{G}^1 \cap \left\{\eta^2(t) \not\subset \mathscr{B}_{p_1}\left(t+\beta t^{1/2+\eta}\right)\right\}\right) + A_1\exp(-B_1 t^{\eta}).
\end{aligned}$$

Remember that C_{p_2,p_1} is a strictly positive constant such that

$$\forall x \in \mathbb{R}^d \quad \|x\|_{p_1} \le C_{p_2,p_1}\|x\|_{p_2}.$$

Choose now $\alpha > 0$ such that $(2C_{p_2,p_1}+3)\alpha < \beta$. Define now the following events:

$$\begin{aligned}
E &= \left\{\forall x \in \mathscr{B}_{p_1}(2t) \quad \mathscr{B}^x_{p_2}\left(\frac{\alpha}{2} t^{1/2+\eta}\right) \subset B^x_{p_2}\left(\alpha t^{1/2+\eta}\right) \subset \mathscr{B}^x_{p_2}\left(2\alpha t^{1/2+\eta}\right)\right\},\\
F &= \left\{\eta^2(t) \not\subset \mathscr{B}_{p_1}\left(t+\beta t^{1/2+\eta}\right)\right\} \cap \left\{\eta^1(t+\alpha t^{1/2+\eta}) \subset \mathscr{B}_{p_1}\left(t+3\alpha t^{1/2+\eta}\right)\right\}.
\end{aligned}$$

By Proposition 2.1 applied to the p_2-epidemic, there exist positive constants A_2, A'_2, B_2, B'_2 such that

$$\mathbb{P}(E^c) \le |\mathscr{B}_{p_1}(2t)| A'_2 \exp(-B'_2 t^{1/2+\eta}) \le A_2\exp(-B_2 t^{\eta}),$$

and, by Proposition 2.2 applied to the p_1-epidemic, there exist positive constants A_3, B_3 such that

$$\mathbb{P}\left(\left\{\eta^1(t+\alpha t^{1/2+\eta}) \not\subset \mathscr{B}_{p_1}\left(t+3\alpha t^{1/2+\eta}\right)\right\}\right) \quad \le \quad A_3\exp(-B_3 t^{\eta}).$$

It is thus sufficient to prove that

$$\mathbb{P}(\mathscr{G}_1 \cap E \cap F) \le A\exp(-Bt^{\eta}).$$

Assume now that $E \cap F$ happens: there exists x with $t+\beta t^{1/2+\eta} \le \|x\|_{p_1} \le 2t$ and $x \in \eta^2(t)$. Note first that $\mathscr{B}_{p_1}(t+3\alpha t^{1/2+\eta}) \cap \mathscr{B}^x_{p_2}(2\alpha t^{1/2+\eta}) = \varnothing$: if $z \in \mathscr{B}^x_{p_2}(2\alpha t^{1/2+\eta})$, then, by the choice we made for α,

$$\begin{aligned}
\|z\|_{p_1} &\ge \|x\|_{p_1} - \|z-x\|_{p_1} \ge t+\beta t^{1/2+\eta} - C_{p_2,p_1}\|z-x\|_{p_2}\\
&\ge t+\beta t^{1/2+\eta} - 2C_{p_2,p_1}\alpha t^{1/2+\eta} > t+3\alpha t^{1/2+\eta}.
\end{aligned}$$

Now, by the event F, we have $\eta^1(t+\alpha t^{1/2+\eta}) \subset \mathscr{B}_{p_1}(t+3\alpha t^{1/2+\eta})$ and, by the event E, we also have $B^x_{p_2}(\alpha t^{1/2+\eta}) \subset \mathscr{B}^x_{p_2}(2\alpha t^{1/2+\eta})$. Thus, as $x \in \eta^2(t)$ and $\mathscr{B}_{p_1}(t+3\alpha t^{1/2+\eta}) \cap \mathscr{B}^x_{p_2}(2\alpha t^{1/2+\eta}) = \varnothing$, we obtain that

$$\eta^2(t+\alpha t^{1/2+\eta}) \supset B^x_{p_2}(\alpha t^{1/2+\eta}) \supset \mathscr{B}^x_{p_2}\left(\frac{\alpha}{2} t^{1/2+\eta}\right).$$

Let $C > 0$ be such that $\forall x \in \mathbb{R}^d$, $\|x\|_2 \geq C\|x\|_{p_2}$. Define $M = \frac{C\alpha}{2(1+\beta)^{1/2+\eta}}$ and $T = t + \alpha t^{1/2+\eta}$. Then, for every $t \geq 1$,

$$\mathscr{B}^x_{p_2}\left(\frac{\alpha}{2} t^{1/2+\eta}\right) \supset \mathscr{B}^x_2(MT^{1/2+\eta}).$$

This, with the previous inclusion, implies that $\eta^2(T) \supset \mathscr{B}^x_2(MT^{1/2+\eta})$, then that $\eta^2(T)$ disconnects $\eta^1(T)$ from infinity in $\mathrm{Cyl}_+(x/\|x\|_2, MT^{1/2+\eta})$, then that $\mathrm{Shade}(T, MT^{1/2+\eta})$ occurs. Then

$$\mathbb{P}(\mathscr{G}_1 \cap E \cap F) \leq \mathbb{P}(\mathscr{G}_1 \cap \mathrm{Shade}(T, MT^{1/2+\eta}) \leq A\exp(-Bt^{\eta})$$

by Lemma 1.2. □

5 Density of the strong in the two dimensional case

This section is devoted to the proof of Theorem 1.4: we prove that in dimension two, when coexistence occurs, the strong epidemic finally occupies a subset of $\mathbb{Z}^2$ with null density. We first need some definitions:

Definition 5.1. *For any $t \geq 0$, denote by $C_{ext}(t)$ the infinite connected component of $\eta(t)^c$, and by $\partial^{\infty}\eta(t)$ the external boundary of $\eta(t)$:*

$$\partial_{ext}\eta(t) = \{z \in \eta(t) : \exists y \in C_{ext}(t),\ y \sim z\}.$$

What is specific to the two dimensional case is that the external boundary of the fast infection $\partial_{\text{ext}}\eta(t) \cap \eta^2(\infty)$ is $*$-connected – this will be proved in Lemmas 5.2 and 5.3. Loosely speaking, by Theorem 1.5, the external boundary $\partial_{\text{ext}}\eta(t)$ of the infection at time t is included in a very thin annulus with radius t and width $t^{1/2+\eta}$. Then, Theorem 1.3, combined with the $*$-connectivity of $\partial_{\text{ext}}\eta(t) \cap \eta^2(\infty)$, ensures that the shadow cast by the fast infection on the slow infection has a diameter smaller than $t^{1/2+\eta}$. However, it remains to control the points in $\eta^2(\infty)$ that are never in a position to create shadow, see Figure 2.

The proof breaks down in higher dimension, as we can imagine a configuration where the fast infection occupies a tree whose branches simultaneously widen and ramify. For instance, we can assume that the radius at height t is of order $t^{1/2}$ and the number of branches at height t is of order $t^{d-3/2}$.

Let us now recall the graphical duality of the square lattice. Let $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$, $\mathbb{E}^2_* = \{\{a, b\} : a, b \in \mathbb{Z}^2_* \text{ and } \|a - b\|_2 = 1\}$ and $\mathbb{L}^2_* = (\mathbb{Z}^2_*, \mathbb{E}^2_*)$, which is isomorphic to $\mathbb{L}^2$. For each bond $e = \{a, b\}$ of $\mathbb{L}^2$ (*resp.* $\mathbb{L}^2_*$), let us denote by $s(e)$ the only subset $\{i, j\}$ of $\mathbb{Z}^2_*$ (*resp.* $\mathbb{Z}^2$) such that the quadrangle $aibj$ is a square in $\mathbb{R}^2$. The application s is clearly an involution.

For any finite set $A \subset \mathbb{Z}^2$, we denote by $\mathscr{P}\mathrm{eierls}(A)$ the set of Peierls contours associated to A, that is

$$\mathscr{P}\mathrm{eierls}(A) = \{e \in \mathbb{E}^2_* : \ 1\!\!1_A \text{ is not constant on } s(e)\}.$$

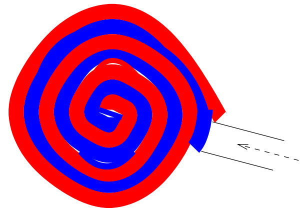

Figure 2: The shadow created by fast/blue on slow/red has a small diameter, while fast/blue fills a set with positive density.

If, on the plane $\mathbb{R}^2$, we draw the edges which are in $\mathscr{P}\text{eierls}(A)$, we obtain a family of curves – the so-called Peierls contours – which are exactly the boundary of the subset $A+[-1/2,1/2]^2$ of $\mathbb{R}^2$. If $A\subset\mathbb{Z}^2$ is a bounded connected subset of $\mathbb{L}^2$, there exists a unique set of bonds $\Gamma(A)\subset\mathscr{P}\text{eierls}(A)\subset\mathbb{E}^2_*$ which form a cycle surrounding A, in the sense that every infinite connected subset of bonds $D\subset\mathbb{E}^2$ satisfying $D\cap A\neq\varnothing$ also satisfies $D\cap s(\Gamma(A))\neq\varnothing$. If we draw $\Gamma(A)$ on the plane $\mathbb{R}^2$, we get the external boundary of $A+[-1/2,1/2]^2$, *i.e.* the boundary of the infinite connected component of $(A+[-1/2,1/2]^2)^c$.

Note also that if γ is a Jordan curve on $\mathbb{L}^2_*$, the set $\text{Int}(\gamma)$ ($\text{Ext}(\gamma)$) composed by the points in $s(\gamma)$ that are in the bounded (respectively, unbounded) connected component of $\mathbb{R}^2\backslash\gamma$ is $*$-connected.

We begin with two lemmas to prove the $*$-connectivity of the set $\eta^2(t)\cap\partial_{\text{ext}}\eta(t)$ in dimension 2.

Lemma 5.2. *Let A, B be two disjoint finite connected subsets of $\mathbb{Z}^2$ such that A, B, and $A\cup B$ are connected. We define*

$$\Delta_A=\{e\in\mathbb{E}^2_*:\ s(e)\cap A\neq\varnothing\}\quad\textit{and}\quad\Delta_B=\{e\in\mathbb{E}^2_*:\ s(e)\cap B\neq\varnothing\},$$
$$E^A_{A\cup B}=\Gamma(A\cup B)\cap\Delta_A\quad\textit{and}\quad E^B_{A\cup B}=\Gamma(A\cup B)\cap\Delta_B.$$

Then $E^A_{A\cup B}$ and $E^B_{A\cup B}$ are connected.

Proof. Since $\Gamma(A\cup B)=E^A_{A\cup B}\cup E^B_{A\cup B}$ is connected, we can assume without loss of generality that $E^A_{A\cup B}$ and $E^B_{A\cup B}$ are non-empty. The contour $\Gamma(A\cup B)$ is a cycle that we denote as a sequence of distinct consecutive edges $e_0,e_1,\dots,e_{f-1}$, where the "end" of e_{f-1} is the "start" of e_0. To these edges we associate a sequence $x_0,\dots,x_{f-1}$ of points in $\mathbb{Z}^2$ such that $(A\cup B)\cap s(e_i)=\{x_i\}$. We also denote by $m_0,\dots,m_{f-1}$ the middle points of the edges $e_0,e_1,\dots,e_{f-1}$.

Assume that $x_0\in A$ and suppose by contradiction that $E^A_{A\cup B}$ is not connected: there exist p,q with $1<p<q<f$ with $x_i\in A$ for $i\in\{0\}\cup\{p,\dots,q-1\}$ whereas $x_i\in B$ for $i\in\{1,\dots,p-1\}\cup\{q\}$.

Since A is connected, there exists a simple path in A from x_0 to x_{q-1} which corresponds to a path in $\mathbb{Z}^2_*$ from the start of e_0 to the end of e_{q-1}. The union of this path with the path $(e_0,e_1,\dots e_{q-1})$ makes a Jordan curve γ.

Obviously, $m_q\notin\gamma$. Since m_q is on the outer boundary of the connected set $(A\cup B)+[-1/2,1/2]^2$, there exists an (infinite) path in $(A\cup B)^c$ joining m_q to infinity. So, we can say that m_q is in the infinite component of γ^c. Since B is connected, there exists a path γ' in B from x_q to x_1. Let e be the first edge of γ' which crosses γ.

By construction, we know that each edge $e \in \mathbb{L}^2$ which crosses γ' from the unbounded component to a point in B must be one of the $s(e_i)$'s. But there is a contradiction because no $s(e_i)$ can have both ends in $A \cup B$. $\square$

Lemma 5.3. *In dimension 2, the set $\eta^2(t) \cap \partial_{ext}\eta(t)$ is $*$-connected.*

Proof. By the very definition of the evolution process, $\eta^1(t)$ and $\eta^2(t)$ are connected. If $\eta(t) = \eta^1(t) \cup \eta^2(t)$ is connected , it follows from Lemma 5.2 that $\eta^2(t) \cap \partial_{\text{ext}}\eta(t) = \text{Int}(\Gamma(\eta(t)) \cap \Delta_{\eta^2(t)})$ is connected. Otherwise, $\eta^2(t) \cap \partial_{\text{ext}}\eta(t) = \text{Int}(\eta^2(t))$, which is also connected. $\square$

We can now proceed to the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $\alpha > 0$, and $\eta > 0$ such that $1 + \varepsilon = (2+\alpha)(1/2+\eta) < 1+\alpha$. Let $\alpha' > \alpha$ be such that $2+\alpha > 1+\alpha'$ (this last condition is only necessary to ensure the good definition of the event A_n^2 introduced below). For $n > 0$, we define:

$$\begin{aligned}
A_n^1 &= \left\{\mathscr{B}_{p_1}\left(n^{2+\alpha} - 4n^{1+\varepsilon}\right) \subset \eta\left(n^{2+\alpha} - 2n^{1+\varepsilon}\right) \subset \mathscr{B}_{p_1}\left(n^{2+\alpha}\right)\right\}, \\
A_n^2 &= \text{Shade}\left(n^{2+\alpha} - 2n^{1+\varepsilon}, n^{1+\alpha'}\right)^c, \\
A_n^3 &= \bigcap_{x \in \mathscr{B}_{p_1}\left(n^{2+\alpha}\right) \setminus \mathscr{B}_{p_1}\left(n^{2+\alpha} - 5n^{1+\varepsilon}\right)} \left\{\mathscr{B}_{p_2}^x\left(4(2+\alpha)n^{1+\alpha}\right) \subset B_{p_2}^x\left(5(2+\alpha)n^{1+\alpha}\right)\right\}, \\
A_n &= \bigcap_{1 \le i \le 3} A_n^i.
\end{aligned}$$

<u>Step 1.</u> Let us see that

$$\mathbb{P}\left(\mathscr{G}^1 \cap \overline{\lim_{n \to +\infty}} A_n^c\right) = 0. \tag{42}$$

Indeed, if $\varphi(n) = n^{2+\alpha} - 2n^{1+\varepsilon}$, then for n large enough:

$$n^{2+\alpha} - 4n^{1+\varepsilon} \le \varphi(n) - \varphi(n)^{1/2+\eta} \le \varphi(n) + \varphi(n)^{1/2+\eta} \le n^{2+\alpha}.$$

By Lemma 1.5, there exist two strictly positive constants A_1 and B_1 such that

$$\forall n \ge 1 \quad \mathbb{P}\left(\mathscr{G}^1 \cap (A_n^1)^c\right) \le A_1 \exp(-B_1 n^{\eta(2+\alpha)}).$$

Then, as $\varphi(n)^{1/2+\eta} = o\left(n^{1+\alpha'}\right)$, by Lemma 1.2, there exist two strictly positive constants A_2 and B_2 such that

$$\forall n \ge 1 \quad \mathbb{P}\left(\mathscr{G}^1 \cap (A_n^2)^c\right) \le A_2 \exp(-B_2 n^{\eta(2+\alpha)}).$$

Finally, by the large deviations result (Proposition 2.1) for the p_2-infection, there exist two strictly positive constants A_3 and B_3 such that

$$\forall n \ge 1 \quad \mathbb{P}((A_n^3)^c) \le |(\mathscr{B}_{p_1}\left(n^{2+\alpha}\right))| A_3 \exp(-B_3 n^{1+\alpha}).$$

Collecting the previous estimates, as $\eta(2+\alpha) < 1+\alpha$, there exist two strictly positive constants A_4 and B_4 such that

$$\forall n \ge 1 \quad \mathbb{P}(\mathscr{G}^1 \cap A_n^c) \le \sum_{1 \le i \le 3} \mathbb{P}(\mathscr{G}^1 \cap (A_n^i)^c) \le A_4 \exp\left(-B_4 n^{\eta(2+\alpha)}\right),$$

which proves (42) by the Borel-Cantelli lemma.

Denote by

$$\begin{aligned}
\Gamma_n &= \Gamma\left(\eta\left(n^{2+\alpha}-2n^{1+\varepsilon}\right)\right),\\
F_n &= s(\Gamma_n)\cap\eta\left(n^{2+\alpha}-2n^{1+\varepsilon}\right),\\
F_n^i &= s(\Gamma_n)\cap\eta^i\left(n^{2+\alpha}-2n^{1+\varepsilon}\right)=\eta^i(\infty)\cap\partial_{\text{ext}}\eta\left(n^{2+\alpha}-2n^{1+\varepsilon}\right).
\end{aligned}$$

If n is large enough, $\eta\left(n^{2+\alpha}-2n^{1+\varepsilon}\right)$ is connected and Γ_n is thus a circuit. Note also that by Lemma 5.3, F_n^2 is $*$-connected.

Step 2. Assume that $\mathscr{G}_1\cap A_{n-1}\cap A_n$ occurs for some n large enough. Let us prove that there exists $y\in\mathscr{B}_{p_1}\left(n^{2+\alpha}\right)\setminus\mathscr{B}_{p_1}\left(n^{2+\alpha}-5n^{1+\varepsilon}\right)$ such that

$$F_n^2 \subset \mathscr{B}_2^y\left(5n^{1+\alpha'}\right)\cap\mathscr{B}_{p_1}\left(n^{2+\alpha}\right)\setminus\mathscr{B}_{p_1}\left(n^{2+\alpha}-5n^{1+\varepsilon}\right). \tag{43}$$

By A_n^1, $F_n\subset\mathscr{B}_{p_1}\left(n^{2+\alpha}\right)\setminus\mathscr{B}_{p_1}\left(n^{2+\alpha}-5n^{1+\varepsilon}\right)$. Assume then by contradiction that

$$(H)\quad \exists z_1,z_2\in F_n^2 \text{ such that } \|z_1-z_2\|_2\geq 5n^{1+\alpha'}.$$

As in dimension 2, F_n^2 is $*$-connected, and as $\mathscr{G}_1$ occurs, we can define γ_n as the portion of Γ_n between z_1 and z_2 such that:

$$s(\gamma_n)\cap\eta\left(n^{2+\alpha}-2n^{1+\varepsilon}\right)\subset\eta^2(\infty).$$

As the width of the annulus $\mathscr{B}_{p_1}\left(n^{2+\alpha}\right)\setminus\mathscr{B}_{p_1}\left(n^{2+\alpha}-5n^{1+\varepsilon}\right)$ is of order $5n^{1+\varepsilon}=o(n^{1+\alpha'})$, there exists $\overrightarrow{x}_0\in\mathscr{S}_2$ such that in $\text{Cyl}_+(\overrightarrow{x}_0,2n^{1+\alpha'})$, the set $s(\gamma_n)\cap\eta\left(n^{2+\alpha}-2n^{1+\varepsilon}\right)\subset F_n^2$ disconnects 0 from infinity. By A_n^2, the event $\text{Shade}(n^{2+\alpha}-2n^{1+\varepsilon},2n^{1+\alpha'})$ can not occur, thus in $\text{Cyl}_+(\overrightarrow{x}_0,2n^{1+\alpha'})$, the set F_n^2 does not disconnect $\eta^1(n^{2+\alpha}-2n^{1+\varepsilon})$ from infinity. Let $z\in\eta^1(n^{2+\alpha}-2n^{1+\varepsilon})$ be in the same connected component as infinity in $\text{Cyl}_+(\overrightarrow{x}_0,2n^{1+\alpha'})$ deprived of F_n^2. Note that – this is a key point – the infection path from s_1 to z has to enter $\text{Cyl}_+(\overrightarrow{x}_0,2n^{1+\alpha'})\cap\mathscr{B}_{p_1}\left(n^{2+\alpha}\right)\setminus\mathscr{B}_{p_1}\left(n^{2+\alpha}-5n^{1+\varepsilon}\right)$ by crossing the border of the cylinder.

If n is large enough, then $(n-1)^{2+\alpha}<n^{2+\alpha}-5n^{1+\varepsilon}$.

By A_{n-1}^1, the set $\eta\left((n-1)^{2+\alpha}-2(n-1)^{1+\varepsilon}\right)$ is contained in $\mathscr{B}_{p_1}\left((n-1)^{2+\alpha}\right)$, and thus, an infection path from s_1 to z has to visit some vertex $s\in\eta^1(\infty)$ satisfying $s\in\mathscr{B}_{p_1}\left(n^{2+\alpha}-5n^{1+\varepsilon}\right)\setminus\mathscr{B}_{p_1}\left((n-1)^{2+\alpha}\right)$, and thus such that

$$s\notin B_{p_1}^{s_1}((n-1)^{2+\alpha}-2(n-1)^{1+\varepsilon}).$$

This implies

$$z\in B_{p_1}^s(n^{2+\alpha}-2n^{1+\varepsilon}-(n-1)^{2+\alpha}+2(n-1)^{1+\varepsilon})\subset B_{p_1}^s(2(2+\alpha)n^{1+\alpha}).$$

Thus, z has to be at a random distance for the p_2-infection less than $2(2+\alpha)n^{1+\alpha}$ of a point in the border of the cylinder, and, by A_n^3 at a distance $\|.\|_{p_2}$ less than $5(2+\alpha)n^{1+\alpha}$ of a point in the border of the cylinder; and this is also the case for any point $z\in\eta^1(n^{2+\alpha}-2n^{1+\varepsilon})$ in the same connected component as infinity in $\mathrm{Cyl}_+(\overrightarrow{x}_0, 2n^{1+\alpha'})$ deprived of F_n^2. As $1+\alpha' > 1+\alpha$, this implies, for n large enough, that in $\mathrm{Cyl}_+(\overrightarrow{x}_0, n^{1+\alpha'})$, the set F_n^2 disconnects $\eta^1(n^{2+\alpha}-2n^{1+\varepsilon})$ from infinity, which contradicts A_n^2, and thus (H). This completes the proof of (43).

Step 3. Assume now that n is large enough and that $\mathscr{G}_1 \cap A_{n-1} \cap A_n \cap A_{n+1} \cap A_{n+2}$ occurs. Let $y \in \mathscr{B}_{p_1}\left(n^{2+\alpha}\right) \setminus \mathscr{B}_{p_1}\left(n^{2+\alpha}-5n^{1+\varepsilon}\right)$ be such that (43) is satisfied. Let us prove that

$$\eta^2(\infty) \cap \left(\mathscr{B}_{p_1}\left((n+1)^{2+\alpha}\right) \setminus \mathscr{B}_{p_1}\left(n^{2+\alpha}\right)\right) \subset \mathscr{B}_2^y(6n^{1+\alpha'}). \tag{44}$$

Consider

$$z \in \eta^2(\infty) \cap \left(\mathscr{B}_{p_1}\left((n+1)^{2+\alpha}\right) \setminus \mathscr{B}_{p_1}\left(n^{2+\alpha}\right)\right).$$

If n is large enough, $(n+1)^{2+\alpha} \le (n+2)^{2+\alpha} - 4(n+2)^{1+\varepsilon}$, and thus, by A_{n+2}^1, the infection time of z is less than $(n+2)^{2+\alpha}-2(n+2)^{1+\varepsilon}$. By A_n^1, as $z \notin \mathscr{B}_{p_1}\left(n^{2+\alpha}\right)$, its infection time is also strictly larger than $n^{2+\alpha}-2n^{1+\varepsilon}$. Thus the infection path from s_2 to z has to visit some point $s \in F_n^2$. By (43), the point s is not in $\mathscr{B}_{p_1}\left(n^{2+\alpha}-5n^{1+\varepsilon}\right)$; thus, as $(n-1)^{2+\alpha} < n^{2+\alpha}-5n^{1+\varepsilon}$ for n large enough, A_{n-1}^1 ensures that the infection time for s is larger than $(n-1)^{2+\alpha}-2(n-1)^{1+\varepsilon}$, which leads, for n large enough, to

$$z \in B_{p_2}^s((n+2)^{2+\alpha}+2(n+2)^{1+\varepsilon}-((n-1)^{2+\alpha}-2(n-1)^{1+\varepsilon})) \subset B_{p_2}^s(4(2+\alpha)n^{1+\alpha}).$$

As $s \in \mathscr{B}_{p_1}\left(n^{2+\alpha}\right) \setminus \mathscr{B}_{p_1}\left(n^{2+\alpha}-5n^{1+\varepsilon}\right)$, by A_n^3, we have $z \in \mathscr{B}_{p_2}^s(5(2+\alpha)n^{1+\alpha}))$, and thus

$$z \in \bigcup_{s \in \mathscr{B}_2^y(5n^{1+\alpha'})} \mathscr{B}_{p_2}^s(5(2+\alpha)n^{1+\alpha})),$$

which is included in $\mathscr{B}_2^y(6n^{1+\alpha'})$ for n large enough. This proves (44).

Step 4. Let us prove that there exists a constant C such that

$$\forall n \ge 1 \quad |\eta^2(\infty) \cap \mathscr{B}_{p_1}(n^{2+\alpha})| \le C n^{3+2\alpha'}. \tag{45}$$

By (42), on $\mathscr{G}^1$, there almost surely exists m such that A_n occurs for every $n \ge m$. Thus, almost surely, for every $n > m$, by (43) and (44) – and by increasing m if necessary –, there exists $y_n \in \mathscr{B}_{p_1}\left(n^{2+\alpha}\right) \setminus \mathscr{B}_{p_1}\left(n^{2+\alpha}-5n^{1+\varepsilon}\right)$ such that

$$\eta^2(\infty) \cap \left(\mathscr{B}_{p_1}\left((n+1)^{2+\alpha}\right) \setminus \mathscr{B}_{p_1}\left(n^{2+\alpha}\right)\right) \subset \mathscr{B}_2^{y_n}(6n^{1+\alpha'}).$$

Thus, there exist constants C_i such that for $n \ge m$,

$$\begin{aligned} |\eta^2(\infty) \cap \mathscr{B}_{p_1}(n^{2+\alpha})| &\le C_1|\mathscr{B}_{p_1}(m^{2+\alpha})| + \sum_{k=m}^{n-1} C_2|\mathscr{B}_2(6k^{1+\alpha'})| \\ &\le C_3 m^{4+2\alpha} + C_4 \sum_{k=m}^{n-1} k^{2+2\alpha'} \le C_5 n^{3+2\alpha'}. \end{aligned}$$

Step 5. Consider $\beta \in (0,1/2)$. We can choose $\alpha' > \alpha > 0$ such that $\frac{3+2\alpha'}{2+\alpha} = 3/2 + \beta$ and such that $1+\alpha' < 2+\alpha$.

For t large enough, choose n such that $n^{2+\alpha} \le t < (n+1)^{2+\alpha}$. Thus, by the previous step,

$$|\eta^2(\infty) \cap \mathscr{B}_{p_1}(t)| \le |\eta^2(\infty) \cap \mathscr{B}_{p_1}((n+1)^{2+\alpha})| \le C(n+1)^{3+2\alpha'} \sim Ct^{\frac{3+2\alpha'}{2+\alpha}} = Ct^{3/2+\beta}.$$

The norm equivalence implies the analogous result for any norm on $\mathbb{R}^2$, which proves the first point of Theorem 1.4.

Let us now prove the second point: for every $\beta > 0$,

$$\overline{\lim_{t \to +\infty}} \frac{\mathrm{Diam}\left(\left(\eta^2(\infty) + [-1/2, 1/2]^2\right) \cap \mathscr{S}_{p_1}(t)\right)}{t^{1/2+\beta}} = 0.$$

It is clearly sufficient to consider $\beta \in (0,1/2)$. We can choose $\alpha' > \alpha > 0$ such that $\frac{1+\alpha'}{2+\alpha} < 1/2 + \beta$. For t large enough, choose n such that $n^{2+\alpha} < t \le (n+1)^{2+\alpha}$. Then, by (44), there exists $y \in \mathbb{R}^2$ such that

$$\left(\eta^2(\infty) + [-1/2, 1/2]^2\right) \cap \mathscr{S}_{p_1}(t) \cap \mathbb{Z}^2 \subset \mathscr{B}_{p_1}((n+1)^{2+\alpha}) \backslash \mathscr{B}_{p_1}(n^{2+\alpha}) \subset \mathscr{B}_2^y(6n^{1+\alpha'}).$$

As $n \sim t^{\frac{1}{2+\alpha}}$, this ends the proof of the second point.

Turning to the proof of the last assertion, consider the following alternative:

- If the weak species (type 1) does not unboundedly grow, its natural density is zero, while the density of the strong is one.
- If the weak species grows unboundedly, the first point of the present theorem ensures that the strong species has null density, and therefore that the weak have full density.

□

6 Non-coexistence except perhaps for a denumerable set

In this section, we prove Theorem 1.6. Remember that Häggström and Pemantle proved non-coexistence for two epidemics progressing according exponential laws with parameter 1 and λ "except perhaps for a denumerable set" for λ, and Theorem 1.6 extends this result to families of laws depending on a continuous parameter.

The first step consists in coupling all possible competition models on the same probability space, respecting the stochastic order of the laws. This will give natural inclusions between sets of infected points for competition with distinct parameters, as stated in Lemma 6.1. Assume that $p < r$, that the slow infection (respectively the strong one) uses the law with parameter p (respectively, r) and that both infections manage to grow unboundedly. Let also choose $q \in (p, r)$. Then, we can expect that strengthening the slow infection by increasing its parameter from p to q makes it strong enough to win and surround the fast one. This is formalized in Lemma 6.3. The proof is based on Theorem 1.5. Finally, we show that for fixed $q \in I$, the set of $p < q$ such that $\mathbb{P}(\mathscr{G}^1_{p,q} \cap \mathscr{G}^2_{p,q}) > 0$ is a subset of the discontinuity set of an increasing function, and is thus at most denumerable. This will conclude the proof of Theorem 1.6.

Coupling

We first couple all passage times for varying parameters thanks to the generalized inverse of the repartition function. Thus, all the competition models can live on the same probability space. This generalizes the construction presented in the introduction.

On $\Omega = [0,1]^{\mathbb{E}^d}$, consider the probability measure $\mathbb{P} = \mathscr{U}[0,1]^{\otimes \mathbb{E}^d}$, where $\mathscr{U}[0,1]$ denotes the uniform law on the set $[0,1]$. For each $\omega \in \Omega$ and $p \in I$, define

$$t_e^p = \inf\{x \in \mathbb{R} : \nu_p((-\infty, x]) \geq \omega_e\}.$$

Under $\mathbb{P}$, the variables $(t_e^p)_{e \in \mathbb{E}^d}$ are independent identically distributed with common law ν_p. Moreover

$$\forall e \in \mathbb{E}^d \quad \forall (p,q) \in I^2 \quad p \leq q \Longrightarrow t_e^p \geq t_e^q.$$

We build now, for a given $(p_1, p_2) \in I^2$, the competition process in a realization $\omega \in \Omega$. Let $E = ([0,+\infty] \times [0,+\infty])^{\mathbb{Z}^d}$. We recursively define a E-valued sequence $(X_n^{p_1,p_2})_{n \geq 0}$ and a non-negative sequence $(T_n^{p_1,p_2})_{n \geq 0}$. The sequence $(T_n^{p_1,p_2})_{n \geq 0}$ contains the successive times of infections, while a point $\varepsilon = (\varepsilon^1(z), \varepsilon^2(z))_{z \in \mathbb{Z}^d} \in E$ codes, for each site z, its times of infection $\varepsilon^1(z)$ ($\varepsilon^2(z)$) by the first (respectively, the second) infection. We start the process with two distinct sources s_1 and s_2 in $\mathbb{Z}^d$, and set $T_0^{p_1,p_2} = 0$ and

$$X_0^{p_1,p_2} = (X_0^{1,p_1,p_2}(z), X_0^{2,p_1,p_2}(z))_{z \in \mathbb{Z}^d} \text{ with } \begin{cases} X_0^{i,p_1,p_2}(z) = 0 \text{ if } z = s_i, \\ X_0^{i,p_1,p_2}(z) = +\infty \text{ otherwise.} \end{cases}$$

This means that at time 0, no point of $\mathbb{Z}^d$ has been infected yet but the two initial sources s_1 and s_2. Then, for $n \geq 0$, define the next time of infection:

$$T_{n+1}^{p_1,p_2} = \inf\{X_n^{i,p_1,p_2}(y) + t_{\{y,z\}}^{p_i} : \quad \{y,z\} \in \mathbb{E}^d,\ i \in \{1,2\},\ X_n^{3-i,p_1,p_2}(z) = +\infty\}.$$

Note that the infimum in the definition of T_{n+1} is always taken on a finite set. Moreover, Assumption (5) ensures that if this infimum is reached by several triplets (i, y, z), all these triplets have the same first coordinate, which means that a point can be infected by the same species from distinct neighbors at the same time, but not by the two species simultaneously. For such a triplet, the next infection is of type i from (one of the) y to z. The set of infected points of type $3-i$ has not changed:

$$\forall x \in \mathbb{Z}^d \quad X_{n+1}^{3-i,p_1,p_2}(x) = X_n^{3-i,p_1,p_2}(x),$$

while the point z has been infected by species i at time $X_n^{i,p_1,p_2}(y) + t_{\{y,z\}}^{p_i}$:

$$\forall x \in \mathbb{Z}^d \backslash \{z\} \quad X_{n+1}^{i,p_1,p_2}(x) = X_n^{i,p_1,p_2}(x) \text{ and } X_{n+1}^{i,p_1,p_2}(z) = X_n^{i,p_1,p_2}(y) + t_{\{y,z\}}^{p_i}.$$

Note that $X_n^{i,p_1,p_2}(y)$ and $X_n^{3-i,p_1,p_2}(y)$ can not be simultaneously finite, which corresponds to the fact that each site is infected by at most one type of infection. Moreover, once $\min(X_n^{1,p_1,p_2}(x), X_n^{2,p_1,p_2}(x))$ is finite, its value – the time of infection of x – does not change any more.

As $\nu_{p_1} \succ \nu_{p_2}$, species 1 is slower than species 2.
We also define the sets $\eta^{p_1,p_2}(t), \eta^{1,p_1,p_2}(t), \eta^{2,p_1,p_2}(t)$ that are respectively infected points, infected points of type 1, infected points of type 2 at time t, by

$$\forall i \in \{1,2\} \quad \forall t \in [T_n^{p_1,p_2}, T_{n+1}^{p_1,p_2}) \quad \eta^{i,p_1,p_2}(t) = \{z \in \mathbb{Z}^d; X_n^{i,p_1,p_2}(z) < +\infty\}$$

and $\eta^{p_1,p_2}(t) = \eta^{1,p_1,p_2}(t) \cup \eta^{2,p_1,p_2}(t)$. We also define

$$\begin{aligned} \eta^{i,p_1,p_2}(\infty) &= \bigcup_{t\geq 0} \eta^{i,p_1,p_2}(t), \\ \mathcal{G}^{i,p_1,p_2} &= \left\{|\eta^{i,p_1,p_2}(\infty)| = +\infty\right\} \text{ for } i = 1,2. \end{aligned}$$

The set $\mathcal{G}^{i,p_1,p_2}$ corresponds to the survival of type i.

Lemma 6.1. *Let $t \geq 0$.*

- *$\eta^{1,p,q}(t)$ is non-decreasing in p and non-increasing in q,*
- *$\eta^{2,p,q}(t)$ is non-decreasing in q and non-increasing in p.*

Proof. We only prove the monotonicity with respect to q. Let then $q < r$. We prove by induction that for every $n \in \mathbb{N}$, we have

$$(H_n) \quad \forall x \in \mathbb{Z}^d \quad X_n^{1,p,q}(x) \leq X_n^{1,p,r}(x) \text{ and } X_n^{2,p,q}(x) \geq X_n^{2,p,r}(x).$$

Clearly, (H_0) is true. Assume that (H_n) holds.
1. Let us first prove that $X_{n+1}^{1,p,q}(x) \leq X_{n+1}^{1,p,r}(x)$ for each x in $\mathbb{Z}^d$. We have the following alternative:

- If $X_{n+1}^{1,p,r}(x) = +\infty$, it is obvious that $X_{n+1}^{1,p,q}(x) \leq X_{n+1}^{1,p,r}(x)$.
- If $X_n^{1,p,r}(x) < +\infty$ then, by (H_n), $X_n^{1,p,q}(x)$ is also finite, and thus their values do not change when we go from n to $n+1$, and the inequality is preserved.
- If $X_{n+1}^{1,p,r}(x) < +\infty$ and $X_n^{1,p,r}(x) = +\infty$, there exists $y \sim x$ which has infected x at the $(n+1)$-th step of the construction with type 1 species. In other words:

  $$X_{n+1}^{1,p,r}(x) = X_n^{1,p,r}(y) + t_{\{y,x\}}^p \text{ and } \forall z \sim x,\ X_n^{1,p,r}(y) + t_{\{y,x\}}^p < X_n^{2,p,r}(z) + t_{\{z,x\}}^r.$$

  By (H_n), since $t^r \leq t^q$, we have for each $z \sim x$:

  $$X_n^{1,p,q}(y) + t_{\{y,x\}}^p \leq X_n^{1,p,r}(y) + t_{\{y,x\}}^p < X_n^{2,p,r}(z) + t_{\{z,x\}}^r \leq X_n^{2,p,q}(z) + t_{\{z,x\}}^q.$$

  This says that, in the (p,q) competition, x is infected by the species 1 and that $X_{n+1}^{1,p,r}(x) = X_n^{1,p,r}(y) + t_{\{y,x\}}^p \geq X_{n+1}^{1,p,q}(x)$.

2. Let us now prove that $X_{n+1}^{2,p,q}(x) \geq X_{n+1}^{2,p,r}(x)$ for each $x \in \mathbb{Z}^d$. We have the following alternative:

- If $X_{n+1}^{2,p,q}(x) = +\infty$, it is obvious that $X_{n+1}^{2,p,q}(x) \geq X_{n+1}^{2,p,r}(x)$.

- If $X_n^{2,p,q}(x) < +\infty$ then, by (H_n), $X_n^{2,p,r}(x)$ is also finite, and thus their values do not change when we go from n to $n+1$, and the inequality is preserved.
- If $X_{n+1}^{2,p,q}(x) < +\infty$ and $X_n^{2,p,q}(x) = +\infty$, there exists $y \sim x$ which has infected x at the $(n+1)$-th step of the construction with type 2 species. In other words:
$$X_{n+1}^{2,p,q}(x) = X_n^{2,p,q}(y) + t_{\{y,x\}}^q \text{ and } \forall z \sim x,\ X_n^{2,p,q}(y) + t_{\{y,x\}}^q < X_n^{1,p,q}(z) + t_{\{z,x\}}^p.$$
By (H_n), since $t^r \le t^q$, we have for each $z \sim x$:
$$X_n^{2,p,r}(y) + t_{\{y,x\}}^r \le X_n^{2,p,q}(y) + t_{\{y,x\}}^q < X_n^{1,p,q}(z) + t_{\{z,x\}}^p \le X_n^{1,p,r}(z) + t_{\{z,x\}}^p.$$
This says that, in the (p,r) competition, x is infected by the species 2 and that $X_{n+1}^{2,p,r}(x) \le X_n^{2,p,r}(y) + t_{\{y,x\}}^r = X_{n+1}^{2,p,q}(x)$.

To conclude, note that $\eta^{i,p,q}(t) = \{z \in \mathbb{Z}^d : \exists n \in \mathbb{N},\ X^{i,p,q}(z) \le t\}$. It is then obvious that $\eta^{1,p,q}(t)$ is non-increasing in q, and $\eta^{2,p,q}(t)$ is non-decreasing in q. □

Lemma 6.2. *For $A \subset \mathbb{Z}^d$, we define*
$$|A|_p = \sup\{\|x\|_p : x \in A\} \quad \textit{and} \quad |A|_{*,p} = \inf\{\|x\|_p : x \in \mathbb{Z}^d \backslash A\}.$$
Then, $\mathbb{P}$-almost surely,
$$\frac{|B_p^a(t)|_p}{t} \to 1 \quad \textit{and} \quad \frac{|B_p^a(t)|_{*,p}}{t} \to 1.$$

Proof. These are direct consequences of the large deviation result, Proposition 2.1. □

Lemma 6.3. *If $p < q < r$, then $\mathbb{P}(\mathcal{G}^{1,p,r} \cap \mathcal{G}^{2,q,r}) = 0$.*

Proof. By Lemma 6.1, $\mathcal{G}^{2,p,r} \supset \mathcal{G}^{2,q,r}$, thus we have $\mathcal{G}^{1,p,r} \cap \mathcal{G}^{2,q,r} = (\mathcal{G}^{1,p,r} \cap \mathcal{G}^{2,p,r}) \cap \mathcal{G}^{2,q,r}$. So we can assume that $\mathcal{G}^{1,p,r} \cap \mathcal{G}^{2,p,r}$ occurs and prove that $\mathcal{G}^{2,q,r}$ can not happen. By Theorem 1.5, we have
$$\overline{\lim_{t\to+\infty}} \frac{|\eta^{2,p,r}(t)|_p}{t} \le 1, \text{ which implies, by Lemma 6.1, } \overline{\lim_{t\to+\infty}} \frac{|\eta^{2,q,r}(t)|_p}{t} \le 1.$$
Now, by Proposition 2.3, we have
$$\overline{\lim_{t\to+\infty}} \frac{|\eta^{2,q,r}(t)|_q}{t} \le C_{p,q} < 1.$$
Using the coupling Lemma 1.1 and Lemma 6.2 together, we get
$$\underline{\lim_{t\to+\infty}} \frac{|\eta^{1,q,r}(t) \cup \eta^{2,q,r}(t)|_{*,q}}{t} \ge 1.$$
Now, let t be large enough to ensure that
$$\frac{|\eta^{2,q,r}(t)|_q}{t} \le \frac{2C_{p,q}+1}{3} = \alpha \text{ and } \frac{|\eta^{1,q,r}(t) \cup \eta^{2,q,r}(t)|_{*,q}}{t} \ge \frac{C_{p,q}+2}{3} = \beta.$$
Then every point x such that $\alpha t < \|x\|_q < \beta t$ belongs to $\eta^{1,q,r}(t) \backslash \eta^{2,q,r}(t)$. This prevents the occurrence of the event $\mathcal{G}^{2,q,r}$. □

Proof of Theorem 1.6. Let $q \in I$ be fixed and consider the maps $s^i : p \mapsto \mathbb{P}(\mathscr{G}^{i,p,q})$. By Lemma 6.1, s^1 is non-decreasing, whereas s^2 is non-increasing. Suppose now that $p < q$. We prove that $\mathbb{P}(\mathscr{G}^{1,p,q} \cap \mathscr{G}^{2,p,q}) = 0$ if s^1 is left-continuous at p. By Lemma 6.3,

$$\begin{aligned}
\mathbb{P}(\mathscr{G}^{1,p,q} \cap \mathscr{G}^{2,p,q}) &= \mathbb{P}(\mathscr{G}^{1,p-1/n,q} \cap \mathscr{G}^{1,p,q} \cap \mathscr{G}^{2,p,q}) \\
&\quad + \mathbb{P}((\mathscr{G}^{1,p-1/n,q})^c \cap \mathscr{G}^{1,p,q} \cap \mathscr{G}^{2,p,q}) \\
&= \mathbb{P}((\mathscr{G}^{1,p-1/n,q})^c \cap \mathscr{G}^{1,p,q} \cap \mathscr{G}^{2,p,q}) \\
&\leq \mathbb{P}((\mathscr{G}^{1,p-1/n,q})^c \cap \mathscr{G}^{1,p,q}) \\
&\leq \mathbb{P}(\mathscr{G}^{1,p,q}) - \mathbb{P}(\mathscr{G}^{1,p-1/n,q}) \\
&\leq s^1(p) - s^1(p-1/n).
\end{aligned}$$

Thus, the set of $p < q$ such that $\mathbb{P}(\mathscr{G}^1_{p,q} \cap \mathscr{G}^2_{p,q}) > 0$ is a subset of the discontinuity set of the non-decreasing function s^1. Therefore, it is at most denumerable. Note that we could prove that $\mathbb{P}(\mathscr{G}^{1,p,q} \cap \mathscr{G}^{2,p,q}) = 0$ if s^2 is right-continuous at p in the very same way. ☐

7 Concluding remarks

Since we are coming to the end of our study, it is worth questioning the relevance of the notion of strong coexistence and its relationship with the Häggström-Pemantle conjecture.

Strong coexistence is obviously stronger, but how far away is it from the notion of coexistence? In the case where the two species have the same passage times law, a partial answer is given by a recent work by Gouéré [8]: some of his results imply that – under classical assumptions implying coexistence – one can find initial configurations that give rise to strong coexistence with positive probability. The restriction on the initial conditions can however be dropped by a modification argument as in Garet-Marchand [5] or in Deijfen-Häggström [3]. Actually, in the case where the two species have the same passage times law, simulations let think that, as soon as coexistence occurs, each species occupies a cone, and thus strong coexistence occurs.

The above remarks seem to show the relevance of the notion of strong coexistence, but we must now wonder how far we are from the Häggström-Pemantle conjecture. It could be interesting to note that Theorem 1.6 allows a reformulation of this conjecture: it is sufficient to prove that the map $p \mapsto \mathbb{P}_{p,q}(\mathscr{G}^1)$ has a unique point of discontinuity. Note also that this reformulation is quite close to some recent result by Deijfen and Häggström concerning competition with exponential speeds on general graphs (Theorem 4.1 in [4]). However, it is not evident that this remark can be exploited to prove the conjecture. In this perspective, it is more natural to look at Theorem 1.2 and Corollary 1.3: it seems highly unlikely that the strong could survive while it is constrained to occupy only a negligible portion of the aerial surface, but we did not succeed to prove it at this time.

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