Electronic Journal of Probability

Vol. 13 (2008), Paper no. 27, pages 852–879.

Journal URL
http://www.math.washington.edu/~ejpecp/

Malliavin Calculus in Lévy spaces and Applications to Finance

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Abstract

The main goal of this paper is to generalize the results of Fournié et al. [8] for markets generated by Lévy processes. For this reason we extend the theory of Malliavin calculus to provide the tools that are necessary for the calculation of the sensitivities, such as differentiability results for the solution of a stochastic differential equation .

Key words: Lévy processes, Malliavin Calculus, Monte Carlo methods, Greeks.

AMS 2000 Subject Classification: Primary 60H07, 60G5.

Submitted to EJP on June 5, 2007, final version accepted April 7, 2008.

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1 Introduction

In recent years there has been an increasing interest in Malliavin Calculus and its applications to finance. Such applications were first presented in the seminal paper of Fournié et al. [8]. In this paper the authors are able to calculate the Greeks using well known results of Malliavin Calculus on Wiener spaces, such as the chain rule and the integration by parts formula. Their method produces better convergence results from other established methods, especially for discontinuous payoff functions.
There have been a number of papers trying to produce similar results for markets generated by pure jump and jump-diffusion processes. For instance El-Khatib and Privault [6] have considered a market generated by Poisson processes. In Forster et al. [7] the authors work in a space generated by independent Wiener and Poisson processes; by conditioning on the jump part, they are able to calculate the Greeks using classical Malliavin calculus. Davis and Johansson [4] produce the Greeks for simple jump-diffusion processes which satisfy a separability condition. Each of the previous approaches has its advantages in specific cases. However, they can only treat subgroups of Lévy processes.
This paper produces a global treatment for markets generated by Lévy processes and achieves a similar formulation of the sensitivities as in Fournié et al. [8]. We rely on Malliavin calculus for discontinuous processes and expand the theory to fulfill our needs. Malliavin calculus for discontinuous processes has been widely studied as an individual subject, see for instance Bichteler et al. [3] for an overview of early works, Di Nunno et al. [5], Løkka [12] and Nualart and Vives [14] for pure jump Lévy processes, Solé et al. [16] for general Lévy processes and Yablonski [17] for processes with independent increments. It has also been studied in the sphere of finance, see for instance Benth et al. [2] and Léon et al. [11]. In our case we focus on square integrable Lévy processes.
The starting point of our approach is the fact that Lévy processes can be decomposed into a Wiener process and a Poisson random measure part. Hence we are able to use the results of Itô [9] on the chaos expansion property. In this way every square integrable random variable in our space can be represented as an infinite sum of integrals with respect to the Wiener process and the Poisson random measure. Having the chaos expansion we are able to introduce operators for the Wiener processes and Poisson random measure. With an application to finance in mind, the Wiener operator should preserve the chain rule property. Such a Wiener operator was introduced in Yablonski [17] for the more general class of processes with independent increments, using the classical Malliavin definition. In our case we adopt the definition of directional derivative first introduced in Nualart and Vives [14] for pure jump processes and then used in Léon et al. [11] and Solé et al.[16]. The chain rule formulation that is achieved for simple Lévy processes in Léon et al. [11], and for more general processes in Solé et al. [16], is only applicable to separable random variables. As Davis and Johansson [4] have shown, this form of chain rule restricts the scope of applications, for instance it excludes stochastic volatility models that allow jumps in the volatility. We are able to bypass the separability condition, by generalizing the chain rule in this setting. Following this, we define the directional Skorohod integrals, conduct a study of their properties and give a proof of the integration by parts formula. We conclude our theoretical part with the main result of the paper, the study of differentiability for the solution of a Lévy stochastic differential equation.
With the help of these tools we produce formulas for the sensitivities that have the same simplicity and easy implementation as the ones in Fournié et al [8].

The paper is organized as follows. In Section 2 we summarize results of Malliavin calculus, define the two directional derivatives, in the Wiener and Poison random measure direction, prove their equivalence to the classical Malliavin derivative and the difference operator in Løkka [12] respectively, and prove the general chain rule. In Section 3 we define the adjoint of the directional derivatives, the Skorohod integrals, and prove an integration by parts formula. In Section 4 we prove the differentiability of a solution of a Lévy stochastic differential equation and get an explicit form for the Wiener directional derivative. Section 5 deals with the calculation of the sensitivities using these results. The paper concludes in Section 6, with the implementation of the results and some numerical experiments.

2 Malliavin calculus for square integrable Lévy Processes

Let $Z = \{Z_t, t \in [0,T]\}$ be a centered square integrable Lévy process on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{0\le t\le T}$ is the augmented filtration generated by Z. Then the process can be represented as:

$$Z_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z(\mu - \pi)(dz, ds)$$

where $\{W_t\}_{t\in[0,T]}$ is the standard Wiener process and $\mu(\cdot,\cdot)$ is a Poisson random measure independent of the Wiener process defined by

$$\mu(A,t) = \sum_{s\le t} \mathbf{1}_A(\Delta Z_s)$$

where $A \in \mathcal{B}(\mathbb{R}_0)$. The compensator of the Lévy measure is denoted by $\pi(dz,dt) = \lambda(dt)\nu(dz)$ and the jump measure of the Lévy process by $\nu(\cdot)$, for more details see [1]. Since Z is square integrable the Lévy measure satisfies $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. Finally σ is a positive constant, λ the Lebesgue measure and $\mathbb{R}_0 = \mathbb{R}\setminus\{0\}$. In the following $\tilde{\mu}(ds,dz) = \mu(ds,dz) - \pi(ds,dz)$ will represent the compensated random measure.
In order to simplify the presentation, we introduce the following unifying notation for the Wiener process and the random Poisson measure

$$\begin{aligned} U_0 = [0,T] \quad &\text{and} \quad U_1 = [0,T]\times\mathbb{R} \\ dQ_0(\cdot) = dW. \quad &\text{and} \quad Q_1 = \tilde{\mu}(\cdot,\cdot) \\ d\langle Q_0\rangle = d\lambda \quad &\text{and} \quad d\langle Q_1\rangle = d\lambda\times d\nu \end{aligned}$$

$$u_k^l = \begin{cases} t_k & ,l=0 \\ (t_k,x), & l=1. \end{cases}$$

also we define an expanded simplex of the form:

$$G_{j_1,\dots,j_n} = \left\{ (u_1^{j_1},\dots,u_n^{j_n}) \in \prod_{i=1}^n U_{j_i} : 0 < t_1 < \dots < t_n < T \right\},$$

for $j_1,\dots,j_n = 0,1$.
Finally, $J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n})$ will denote the (n-fold) iterated integral of the form

$$J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n}) = \int_{G_{j_1,\dots,j_n}} g_{j_1,\dots,j_n}(u_1^{j_1},\dots,u_n^{j_n}) Q_{j_1}(du_1^{j_1})\dots Q_{j_n}(du_n^{j_n}) \tag{1}$$

where $g_{j_1,\dots,j_n}$ is a deterministic function in $L^2(G_{j_1,\dots,j_n}) = L^2(G_{j_1,\dots,j_n}, \bigotimes_{i=1}^n d\langle Q_{j_i}\rangle)$.

2.1 Chaos expansion

The theorem that follows is the chaos expansion for processes in the Lévy space $L^2(\Omega)$. It states that every random variable F in this space can be uniquely represented as an infinite sum of integrals of the form (1). This can be considered as a reformulation of the results in [9], or an expansion of the results in [12].

Theorem 1. *For every random variable $F \in L^2(\mathcal{F}_T, \mathbb{P})$, there exists a unique sequence $\{g_{j_1,\dots,j_n}\}_{n=0}^{\infty}$, $j_1,\dots,j_n = 0,1$, where $g_{j_1,\dots,j_n} \in L^2(G_{j_1,\dots,j_n})$ such that*

$$F = \sum_{n=0}^{\infty} \sum_{j_1,\dots,j_n=0,1} J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n}) \tag{2}$$

and we have the isometry

$$\|F\|^2_{L^2(P)} = \sum_{n=0}^{\infty} \sum_{j_1,\dots,j_n=0,1} \|J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n})\|^2_{L^2(G_{j_1,\dots,j_n})}.$$

2.1.1 Directional derivative

Given the chaos expansion we are able to define directional derivatives in the Wiener process and the Poisson random measure. For this we need to introduce the following modification to the expanded simplex

$$\begin{aligned} G^k_{j_1,\dots,j_n}(t) &= \Big\{(u_1^{j_1},\dots,\hat{u}_k^{j_k},\dots,u_n^{j_n}) \in G_{j_1,\dots,j_{k-1},j_{k+1},\dots,j_n} : \\ &\quad 0 < t_1 < \cdots < t_{k-1} < t < t_{k+1} \cdots < t_n < T\Big\}, \end{aligned}$$

where $\hat{u}$ means that we omit the u element. Note that $G^k_{j_1,\dots,j_n}(t) \cap G^l_{j_1,\dots,j_n}(t) = \emptyset$ if $k \neq l$. The definition of the directional derivatives follows the one in [11].

Definition 1. *Let $g_{j_1,\dots,j_n} \in L^2(G_{j_1,\dots,j_n})$ and $l = 0,1$. Then*

$$D^{(l)}_{u^l} J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n}) = \sum_{i=1}^{n} \mathbf{1}_{\{j_i=l\}} J_{n-1}^{(j_1,\dots,\hat{j}_i,\dots,j_n)}\left(g_{j_1,\dots,j_n}(\dots,u^l,\dots)\mathbf{1}_{G^i_{j_1,\dots,j_n}(t)}\right)$$

is called the derivative of $J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n})$ in the l-th direction.

We can show that this definition is reduced to the Malliavin derivative if we take $j_i = 0, \forall i = 1,\dots,n$, and to the definition of [12] if $j_i = 1, \forall i = 1,\dots,n$.

From the above we can reach the following definition for the space of random variables differentiable in the l-th direction, which we denote by $\mathbb{D}^{(l)}$, and its respective derivative $D^{(l)}$:

Definition 2. *1. Let $\mathbb{D}^{(l)}$ be the space of the random variables in $L^2(\Omega)$ that are differentiable in the l-th direction, then*

$$\begin{aligned}\mathbb{D}^{(l)} &= \{F \in L^2(\Omega), F = E[F] + \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=0,1} J_n^{(j_1,\ldots,j_n)}(g_n) : \\ &\sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=0,1} \sum_{i=1}^{n} \mathbf{1}_{\{j_i=l\}} \int_{U_{j_i}} \|g_{j_1,\ldots,j_n}(\cdot, u^l, \cdot)\|^2_{L^2(G^i_{i_1,\ldots,i_n})} \\ &\times d\langle Q_l\rangle(u^l) < \infty\}.\end{aligned}$$

2. Let $F \in \mathbb{D}^{(l)}$. Then the derivative on the l-th direction is:

$$D^{(l)}_{u^l} F = \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=0,1} \sum_{i=1}^{n} \mathbf{1}_{\{j_i=l\}} J_{n-1}^{(j_1,\ldots,\hat{j}_i,\ldots,j_n)} \left(g_{j_1,\ldots,j_n}(\cdot, u^l, \cdot) \mathbf{1}_{G^i_{j_1,\ldots,j_n}}(t) \right).$$

From the definition of the domain of the $l-$directional derivative, all the elements of $L^2(\Omega)$ with finite chaos expansion are included in $\mathbb{D}^{(l)}$. Hence, we can conclude that $\mathbb{D}^{(l)}$ is dense in $L^2(\Omega)$.

2.1.2 Relation between the Classical and the Directional Derivatives

In order to study the relation between the classical Malliavin derivative, see [13], the difference operator in [12] and the directional derivatives, we need to work on the canonical space.
The canonical Brownian motion is defined on the probability space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$, where $\Omega_W = C_0([0,1])$; the space of continuous functions on $[0,1]$ equal to null at time zero; $\mathcal{F}_W$ is the Borel σ-algebra and $\mathbb{P}_W$ is the probability measure on $\mathcal{F}_W$ such that $B_t(\omega) := \omega(t)$ is a Brownian motion.
Respectively, the triplet $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ denotes the space on which the canonical Poisson random measure. We denote with Ω_N the space of integer valued measures ω' on $[0,1] \times \mathbb{R}_0$, such that $\omega'(t,u) \leq 1$ for any point $(t,u) \in [0,1] \times \mathbb{R}_0$, and $\omega'(A \times B) < \infty$ when $\pi(A \times B) = \lambda(A)\nu(B) < \infty$, where ν is the σ-finite measure on $\mathbb{R}_0$. The canonical random measure on Ω_N is defined as

$$\mu(\omega', A \times B) := \omega'(A \times B).$$

With $\mathbb{P}_N$ we denote the probability measure on $\mathcal{F}_N$ under which μ is a Poisson random measure with intensity π. Hence, $\mu(A \times B)$ is a Poisson variable with mean $\pi(A \times B)$, and the variables $\mu(A_i \times B_j)$ are independent when $A_i \times B_j$ are disjoint.
In our case we have a combination of the two above spaces. With
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,1]}, \mathbb{P})$ we will denote the joint probability space,where
$\Omega := \Omega_W \otimes \Omega_N$ equipped with the probability measure $\mathbb{P} := \mathbb{P}_W \otimes \mathbb{P}_N$ and $\mathcal{F}_t := \mathcal{F}^W_t \otimes \mathcal{F}^N_t$.
Then there exists an isometry

$$L^2(\Omega_W \times \Omega_N) \simeq L^2(\Omega_W; L^2(\Omega_N)),$$

where

$$L^2(\Omega_W; L^2(\Omega_N)) = \{F : \Omega_W \to L^2(\Omega_N) : \int_{\Omega_W} \|F(\omega)\|^2_{L^2(\Omega_N)} dP_W(\omega) < \infty\}.$$

Therefore we can consider every $F \in L^2(\Omega_W; L^2(\Omega_N))$ as a functional $F : \omega \to F(\omega, \omega')$. This implies that $L^2(\Omega_W; L^2(\Omega_N))$ is a Wiener space on which we can define the classical Malliavin derivative D. The derivative D is a closed operator from $L^2(\Omega_W; L^2(\Omega_N))$ into $L^2(\Omega_W \times [0,1]; L^2(\Omega_N))$. We denote with $\mathbb{D}_{1,2}$ the domain of the classical Malliavin derivative. If $F \in \mathbb{D}_{1,2}$, then

$$\begin{aligned} DF &\in L^2(\Omega_W; L^2(\Omega_N; L^2([0,1]))) \\ &\simeq L^2(\Omega_W \times \Omega_N \times [0,1]) \end{aligned}$$

In the same way the difference operator $\tilde{D}$ defined in [12] with domain $\tilde{\mathbb{D}}_{1,2}$ is closed from $L^2(\Omega_N; L^2(\Omega_W))$ into $L^2(\Omega_N \times [0,1]; L^2(\Omega_W))$. If $F \in \tilde{\mathbb{D}}_{1,2}$, then

$$\begin{aligned} \tilde{D}F &\in L^2(\Omega_N; L^2(\Omega_W; L^2([0,1]))) \\ &\simeq L^2(\Omega_N \times \Omega_W \times [0,1]) \end{aligned}$$

As a consequence we have the following proposition.

Proposition 1. *On the space $\mathbb{D}^{(0)}$ the directional derivative $D^{(0)}$ is equivalent to the classical Malliavin derivative D, i.e. $D = D^{(0)}$. Respectively on $\mathbb{D}^{(1)}$ the directional derivative $D^{(1)}$ is equivalent the difference operator $\tilde{D}$, i.e. $\tilde{D} = D^{(1)}$.*

Given the directional derivatives D and $\tilde{D}$ we reach the subsequent proposition.

Proposition 2. *1. Let $F = f(Z, Z') \in L^2(\Omega)$, where Z depends only on the Wiener part and $Z \in \mathbb{D}^{(0)}$, Z' depends only on the Poisson random measure and $f(x, y)$ is a continuously differentiable function with bounded partial derivatives in x, then*

$$D^{(0)}F = \frac{\partial}{\partial x} f(Z, Z') D^{(0)} Z$$

2. Let $F \in \mathbb{D}^{(1)}$ then

$$D^{(1)}_{(t,z)} F = F \circ \epsilon^{+}_{(t,z)} - F,$$

where ϵ^+ is a transformation on Ω given by

$$\epsilon^{-}_{(t,z)} \omega(A \times B) = \omega(A \times B \cap (t,z)^c),$$

$$\epsilon^{+}_{(t,z)} \omega(A \times B) = \epsilon^{-}_{(t,z)} \omega(A \times B) + \mathbf{1}_A(t) \mathbf{1}_B(z).$$

2.1.3 Chain rule

The last proposition is an extension of the results in [11], where the authors consider only simple Lévy processes, and similar to corollary 3.6 in [16]. However, this chain rule is applicable to random variables that can be separated to a continuous and a discontinuous part;separable random variables, for more details see [4]. In what follows we provide the proof of chain rule with no separability requirements.
The first step is to find a dense linear span of Doléans-Dade exponentials for our space. To achieve this, as in [12], we use the continuous function

$$\gamma(z) = \begin{cases} e^z - 1 & , z < 0 \\ 1 - e^{-z} & , z > 0 \end{cases},$$

which is totally bounded and has an inverse. Moreover $\gamma \in L^2(\nu)$, $e^{\lambda\gamma} - 1 \in L^2(\nu)$, $\forall \lambda \in \mathbb{R}$ and for $h \in C([0,T])$ we have $e^{h\gamma} - 1 \in L^2(\pi)$, $h\gamma \in L^2(\pi)$, $e^{h\gamma} \in L^1(\pi)$.

Lemma 1. *The linear span $\mathcal{S}$ of random variables $Y = \{Y_t, t \in [0,T]\}$ of the form*

$$\begin{aligned} Y_t &= \exp\left\{\int_0^t \sigma h(s) dW_s + \int_0^t \int_{\mathbb{R}_0} h(s)\gamma(z)\tilde{\mu}(dz, ds)\right. \\ &- \left.\int_0^t \frac{\sigma^2 h(s)^2}{2} ds - \int_0^t \int_{\mathbb{R}_0} (e^{h(s)\gamma(z)} - 1 - h(s)\gamma(z))\pi(dz, ds)\right\} \end{aligned} \tag{3}$$

where $h \in C([0,T])$, is dense in $L^2(\Omega, \mathcal{F}, \mathbb{P})$

Proof. The proof follows the same steps of [12]. □

The proof of the chain rule requires the next technical lemma.

Lemma 2. *Let $F \in \mathbb{D}^{(0)}$ and $\{F_k\}_{k=1}^{\infty}$ be a sequence such that $F_k \in \mathbb{D}^{(0)}$ and $F_k \to F$ in $L^2(\mathbb{P})$. Then there exists a subsequence $\{F_{k_m}\}_{k_m=1}^{\infty}$ and a constant $0 < C < \infty$ such that $\|D^{(0)} F_{k_m}\|_{L^2([0,T]\times\Omega)} < C$, and*

$$D^{(0)} F = \lim_{m\to\infty} D^{(0)} F_{k_m}$$

in $L^2([0,T] \times \Omega)$.

Proof. We follow the same steps as in Lemma 6 in [12]. Since F_k converges to F

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} \sum_{j_1,\ldots,j_n=0,1} \|g^k_{j_1,\ldots,j_n} - g_{j_1,\ldots,j_n}\|^2_{L^2(G_{j_1,\ldots,j_n})} = 0. \tag{4}$$

Since $F_k, F \in \mathbb{D}^{(0)}$ from the definition of the directional derivative we have

$$\begin{aligned} & E[\int_0^T (D_t^{(0)} F_k - D_t^{(0)} F)^2 dt] \\ = & \int_0^T \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=0,1} \sum_{i=1}^{n} \mathbf{1}_{\{j_i=0\}} \int_{U_{j_n}} \|g^k_{j_1,\ldots,j_n}(\cdot, t, \cdot) - g_{j_1,\ldots,j_n}(\cdot, t, \cdot)\|^2_{L^2(G^i_{j_1,\ldots,j_{n-1}})} \\ & \times d\langle Q_{j_n}\rangle dt \\ = & \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=0,1} \sum_{i=1}^{n} \mathbf{1}_{\{j_i=0\}} \int_{U_{j_n}} \|g^k_{j_1,\ldots,j_n}(\cdot) - g_{j_1,\ldots,j_n}(\cdot)\|^2_{L^2(G_{j_1,\ldots,j_{n-1}})} \\ & \times d\langle Q_{j_n}\rangle \end{aligned}$$

$$= \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=0,1} \sum_{i=1}^{n} \mathbf{1}_{\{j_i=0\}} \|g^k_{j_1,\ldots,j_n} - g_{j_1,\ldots,j_n}\|^2_{L^2(G_{j_1,\ldots,j_n})} < \infty.$$

From (4) we can choose a subsequence such that $\|g^{k_{m+1}}_{j_1,\dots,j_n} - g_{j_1,\dots,j_n}\|^2_{L^2(G_{j_1,\dots,j_n})} \leq \|g^{k_m}_{j_1,\dots,j_n} - g_{j_1,\dots,j_n}\|^2_{L^2(G_{j_1,\dots,j_n})}$ for all n. Hence

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=0,1} \sum_{i=1}^{n} \mathbf{1}_{\{j_i=0\}} \|g^{k_m}_{j_1,\dots,j_n} - g_{j_1,\dots,j_n}\|^2_{L^2(G_{j_1,\dots,j_n})} \\ \leq\ &\sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=0,1} \sum_{i=1}^{n} \mathbf{1}_{\{j_i=0\}} \|g^{k_1}_{j_1,\dots,j_n} - g_{j_1,\dots,j_n}\|^2_{L^2(G_{j_1,\dots,j_n})} < \infty \end{aligned}$$

However, $\lim_{m\to\infty} \|g^{k_m}_{j_1,\dots,j_n} - g_{j_1,\dots,j_n}\|^2_{L^2(G_{j_1,\dots,j_n})} = 0$. From the dominate convergence theorem we have

$$\begin{aligned} &\lim_{m\to\infty} \sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=0,1} \sum_{i=1}^{n} \mathbf{1}_{\{j_i=0\}} \|g^{k_m}_{j_1,\dots,j_n} - g_{j_1,\dots,j_n}\|^2_{L^2(G_{j_1,\dots,j_n})} = \\ &\sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=0,1} \sum_{i=1}^{n} \mathbf{1}_{\{j_i=0\}} \lim_{m\to\infty} \|g^{k_m}_{j_1,\dots,j_n} - g_{j_1,\dots,j_n}\|^2_{L^2(G_{j_1,\dots,j_n})} = 0 \end{aligned}$$

□

Using the fact that $D^{(0)}$ is a densely defined and closed operator, and that the elements of the linear span $\mathcal{S}$ are separable processes, we prove in the following theorem the chain rule for all processes in $\mathbb{D}^{(0)}$.

Theorem 2. ***(Chain Rule)*** *Let $F \in \mathbb{D}^{(0)}$ and f be a continuously differentiable function with bounded derivative. Then $f(F) \in \mathbb{D}^{(0)}$ and the following chain rule holds:*

$$D^{(0)} f(F) = f'(F) D^{(0)} F. \tag{5}$$

Proof. Let $F \in \mathbb{D}^{(0)}$. F can be approximated in $L^2(\Omega)$ by a sequence $\{F_n\}_{n=0}^{\infty}$, where $F_n \in \mathcal{S}$ for all $n \in \mathbb{N}$. Every term of F_n, as a linear combination of Lévy exponentials, is in $\mathbb{D}^{(0)}$. Then from Lemma 2 there exists a subsequence $\{F_{n_k}\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} D_t^{(0)} F_{n_k} = D_t^{(0)} F$ in $L^2([0,T]\times\Omega)$.
However, the elements of the sequence $\{F_{n_k}\}_{k=0}^{\infty}$ are separable processes. We can then apply the chain rule in Proposition 2 to the process $f(F_{n_k})$ and we have

$$D_t^{(0)} f(F_{n_k}) = f'(F_{n_k}) D_t^{(0)} F_{n_k}.$$

Since f is continuously differentiable with bounded derivative $\lim_{k\to\infty} f(F_{n_k}) = f(F)$ in $L^2(\Omega)$, and from the dominate convergence theorem we can conclude that $\lim_{k\to\infty} f'(F_{n_k}) = f'(F)$ in $L^2(\Omega)$. Hence
$\lim_{k\to\infty} f'(F_{n_k}) D_t^{(0)} F_{n_k} = f'(F) D_t^{(0)} F$, in $L^2([0,T]\times\Omega)$ for all $t \in [0,T]$.
Finally due to the closability of the operator $D_t^{(0)}$, $\lim_{k\to\infty} D_t^{(0)} f(F_{n_k}) = D_t^{(0)} f(F)$ in $L^2([0,T]\times\Omega)$ for all $t \in [0,T]$. The proof is concluded. □

Remark. The theory developed in this chapter also holds in the case that our space is generated by an d-dimensional Wiener process and k-dimensional random Poisson measures. However, we will have to introduce new notation for the directional derivatives in order to simplify things. For the multidimensional case, $D_t^{(0)}F$ will denote a row vector, where the element of the i-th row is the directional derivative for the Wiener process W^i , for all $i = 1, \ldots, d$. Similarly we define the row vector $D_{(t,z)}^{(1)}F$. Furthermore D^iF , $i = 1, \ldots, d$, will be scalars denoting the directional derivative of the i-th Wiener process W^i for $i = 1, \ldots, d$, and the derivative in the direction of the i-th random Poisson measure $\tilde{\mu}^i$ for $i = d+1, \ldots, d+k$.

3 Skorohod Integral

The next step after the definition of the directional derivatives is to define their adjoint, which are the Skorohod integrals in the Wiener and Poisson random measure directions.
The first two result of the section are the calculation of the Skorohod integral and the study of its relation to the Itô and Stieltjes-Lebesgue integrals. These are extensions of the results in [4] and [10] from simple Poisson processes to square integrable Lévy processes. The proof are performed in parallel ways as in [4] (or in more detail in [10]), therefore they are omitted. The main result however is an integration by parts formula. Although the separability result is yet again an extension of [4], having attained a chain rule for $D^{(0)}$ that does not require a condition, we are able to provide a simpler and more elegant proof. Finally the section closes with a technical result.

Definition 3. ***The Skorohod integral***
Let $\delta^{(l)}$ be the adjoint operator of the directional derivative $D^{(l)}$,$l = 0, 1$. The operator $\delta^{(l)}$ maps $L^2(\Omega \times U_l)$ to $L^2(\Omega)$. The set of processes $h \in L^2(\Omega \times U_l)$ such that:

$$\left| E\left[\int_{U_l} (D_u^{(l)}F)h_t d\langle Q_l\rangle\right] \right| \leq c\|F\|, \tag{6}$$

for all $F \in \mathbb{D}^{(l)}$, is the domain of $\delta^{(l)}$, denoted by $Dom\delta^{(l)}$.
For every $h \in Dom\delta^{(l)}$ we can define the Skorohod integral in the l-th direction $\delta^{(l)}(h)$ for which

$$E\left[\int_{U_l} (D_u^{(l)}F)h_t d\langle Q_l\rangle\right] = E[F\delta^{(l)}(h)] \tag{7}$$

for any $F \in \mathbb{D}^{(l)}$.

The following proposition provides the form of the Skorohod integral.

Proposition 3. *For $h(u) \in L^2(U^l)$ and $F \in L^2(\Omega)$ with chaos expansion*

$$F = E(F) + \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=0,1} J_n^{(j_1,\ldots,j_n)}(g_n)$$

Then the l-th directional Skorohod integral is

$$\begin{aligned}\delta^{(l)}(Fh) &= \int_{U_l} E[F]h(u_1)Q_l(du_1) \\ &+ \sum_{n=1}^{\infty}\sum_{j_1,\dots,j_n=0,1}\sum_{k=1}^{n}\int_{U_{j_n}}\cdots\int_{U_{j_{k+1}}}\int_{U_l}\int_{U_{j_k}}\int_{U_{j_1}} g_n(u_1^{j_1},\dots,u_n^{j_n}))h(u)\mathbf{1}_{G_{j_1,\dots,j_n}} \\ &\times \mathbf{1}_{\{t_k<t<t_{k+1}\}}Q_{j_1}(du_1^{j_1})\cdots Q_{j_k}(du_k^{j_k})Q_l(du)Q_{j_{k+1}}(du_{k+1}^{j_{k+1}})\cdots Q_{j_n}(du_n^{j_n})\end{aligned}$$

if the infinite sum converges in $L^2(\Omega)$.

Having the exact form of the Skorohod integral we can study its properties. For instance the Skorohod integral can be reduced to an Itô or Stieltjes-Lebesgue integral in the case of predicable processes.

Proposition 4. *Let h_t be a predictable process such that $E[\int_{U^l} h_t^2 d\langle Q_l\rangle] < \infty$. Then $h \in Dom$ $\delta^{(l)}$ for $l = 0, 1$ and*

$$\delta^{(l)}(h_t) = \int_{U^l} h_t Q_l(du^l).$$

We are now able to prove one of the main results, the integration by parts formula.

Proposition 5. *(**Integration by parts formula**) Let $Fh \in L^2(\Omega \times [0,T])$, where $F \in \mathbb{D}^{(0)}$, h_t is predictable square integrable process. Then $Fh \in Dom\delta^{(0)}$ and*

$$\delta^{(0)}(Fh) = F\int_0^T h_t dW_t - \int_0^T (D^{(0)}F)h_t dt,$$

if and only if the second part of the equation is in $L^2(\Omega)$.

Proof. From Theorem 2 we have

$$\begin{aligned}E\left[\int_0^T D_t^{(0)}G\cdot F\cdot udt\right] &= E\left[\int_0^T \left\{D_t^{(0)}(F\cdot G)\cdot u - G\cdot D_t^{(0)}F\cdot u\right\}dt\right] \\ &= E[F\cdot G\delta^{(0)}(u) - G\cdot\int_0^T D_t^{(0)}F\cdot udt]. \qquad (8)\end{aligned}$$

If $F\cdot G\delta^{(0)}(u) - G\cdot\int_0^T D_t^{(0)}F\cdot udt \in L^2(\Omega)$, then $F\cdot u \in Dom\delta^{(0)}$. Hence, from the definition of the Skorohod integral we have

$$E[\int_0^T D_t^{(0)}G\cdot F\cdot udt] = E[G\cdot\delta^{(0)}(F\cdot u)]. \qquad (9)$$

Combining (8), (9) and Proposition 4 the proof is concluded. □

Note that when F is an m-dimensional vector process and h a $m\times m$ matrix process the integration by part formula can be written as follows:

$$\delta^{(0)}(Fh) = F^*\int_0^T h_t dW_t - \int_0^T Tr\left(D^{(0)}Fh_t\right)dt.$$

The last proposition of this chapter will provide a relationship between the Itô and the Stieltjes-Lebesgue integrals and the directional derivatives.

Proposition 6. *Let h_t be a predictable square integrable process. Then*

- *if $h \in \mathbb{D}^{(0)}$ then*

$$D_t^{(0)} \int_0^T h_s dW_s = h_t + \int_t^T D_s^{(0)} h_s dW_s$$

$$D_t^{(0)} \int_0^T \int_{\mathbb{R}_0} h_s \tilde{\mu}(dz, ds) = \int_t^T \int_{\mathbb{R}_0} D_s^{(0)} h_s \tilde{\mu}(dz, ds)$$

- *if $h \in \mathbb{D}^{(1)}$ then*

$$D_{(t,z)}^{(1)} \int_0^T h_s dW_s = \int_t^T D_{(s,z)}^{(1)} h_s dW_s$$

$$D_{(t,z)}^{(1)} \int_0^T \int_{\mathbb{R}_0} h_s \tilde{\mu}(dz, ds) = h_t + \int_t^T \int_{\mathbb{R}_0} D_{(s,z)}^{(1)} h_s \tilde{\mu}(dz, ds)$$

Proof. This result can be easily deduced from the definition of the directional derivative. □

4 Differentiability of Stochastic Differential Equations

The aim of this section is to prove that under specific conditions the solution of a stochastic differential equation belongs to the domains of the directional derivatives. Having in mind the applications in finance, we will also provide a specific expression for the Wiener directional derivative of the solution.

Let $\{X_t\}_{t\in[0,T]}$ be an m-dimensional process in our probability space, satisfying the following stochastic differential equation:

$$\begin{aligned} dX_t &= b(t, X_{t^-})dt + \sigma(t, X_{t^-})dW_t + \int_{\mathbb{R}_0} \gamma(t, z, X_{t^-})\tilde{\mu}(dz, dt) \\ X_0 &= x \end{aligned} \tag{10}$$

where $x \in \mathbb{R}^m$, $\{W_t\}_{t\in[0,T]}$ is a d-dimensional Wiener process, $\tilde{\mu}$ is a compensated Poisson random measure. The coefficients $b : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^d$ and $\gamma : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}$, are continuously differentiable with bounded derivatives. The coefficients also satisfy the following linear growth condition:

$$\|b(t, x)\|^2 + \|\sigma(t, x)\|^2 + \int_{\mathbb{R}_0} \|\gamma(t, z, x)\|^2 \nu(dz) \leq C(1 + \|x\|^2),$$

for each $t \in [0, T], x \in \mathbb{R}^m$ where C is a positive constant. Furthermore there exists $\rho : \mathbb{R} \to \mathbb{R}$ with $\int_{\mathbb{R}_0} \rho(z)^2 \nu(dz) < \infty$, and a positive constant D such that

$$\|\gamma(t, z, x) - \gamma(t, z, y)\| \leq D|\rho(z)|\|x - y\|, \tag{11}$$

for all $x, y \in \mathbb{R}^m$ and $z \in \mathbb{R}_0$.
Under these conditions there exists a solution for (10) which is also unique[1]. For what follows we denote with σ_i the i-th column vector of σ and adopt the *Einstein convention* of leaving the summations implicit.
In the next theorem we prove that the solution $\{X_t\}_{t\in[0,T]}$ is differentiable in both directions of the Malliavin derivative. Moreover we reach the stochastic differentiable equations satisfied by the derivatives.

Theorem 3. *Let $\{X_t\}_{t\in[0,T]}$ be the solution of* (10). *Then*

1. *$X_t \in \mathbb{D}^{(0)}, \forall t \in [0,T]$ and the derivative $D_s^{(i)} X_t$ satisfies the following linear equation:*

$$\begin{aligned} D_s^i X_t &= \int_s^t \frac{\partial}{\partial x_k} b(r, X_{r^-}) D_s^i X_{r^-}^k dr \\ &+ \sigma_i(s, X_{s^-}) + \int_s^t \frac{\partial}{\partial x_k} \sigma_\alpha(r, X_{r^-}) D_s^i X_{r^-}^k dW_r^\alpha \\ &+ \int_s^t \int_{\mathbb{R}_0} \frac{\partial}{\partial x_k} \gamma(r, z, X_{r^-}) D_s^i X_{r^-}^k \tilde{\mu}(dz, dr) \end{aligned} \tag{12}$$

for $s \le t$ a.e. and

$$D_s^i X_t = 0, \quad a.e. \quad otherwise.$$

2. *$X_t \in \mathbb{D}^{(1)}, \forall t \in [0,T]$ and the derivative $D_{(s,z)}^{(1)} X_t$ satisfies the following linear equation*

$$\begin{aligned} D_{(s,z)}^{(1)} X_t &= \int_s^t D_{(s,z)}^{(1)} b(r, X_{r^-}) dr \\ &+ \int_s^t D_{(s,z)}^{(1)} \sigma(r, X_{r^-}) dW_r \\ &+ \gamma(s, z, X_{s^-}) + \int_s^t \int_{\mathbb{R}_0} D_{(s,z)}^{(1)} \gamma(r, z, X_{r^-}) \tilde{\mu}(dz, dr) \end{aligned} \tag{13}$$

for $s \le t$ a.e., with

$$D_{(s,z)}^{(1)} X_t = 0, \quad a.e. \quad otherwise.$$

Proof. 1. Using Picard's approximation scheme we introduce the following process

$$\begin{aligned} X_t^0 &= x_0 \\ X_t^{n+1} &= x_0 + \int_0^t b(s, X_{s^-}^n) ds + \int_0^t \sigma_j(s, X_{s^-}^n) dW_s^j \\ &+ \int_0^t \int_{\mathbb{R}_0} \gamma(s, z, X_{s^-}^n) \tilde{\mu}(dz, ds) \end{aligned} \tag{14}$$

[1] For existence and uniqueness see Theorem 6.2.3, Assumption 6.5.1 and discussion on page 312 in [1].

if $n \geq 0$.
We prove by induction that the following hypothesis holds true for all $n \geq 0$.
(H) $X_t^n \in \mathbb{D}^{(0)}$ for all $t \in [0,T]$,

$$\xi_n(t) := \sup_{0 \leq r \leq t} E[\sup_{r \leq s \leq t} |D_r^{(0)} X_s^n|^2] < \infty,$$

the left limit

$$D_r^{(0)} X_{s^-}^n = \lim_{t \uparrow s} D_r^{(0)} X_t^n$$

exists for all $s \geq r$, $D^{(0)} X_s^n$ is a predictable process and

$$\xi_{n+1}(t) \leq c_1 + c_2 \int_0^t \xi_n(s) ds$$

for some constants c_1, c_2.
It is straightforward that (H) is satisfied for $n = 0$. Let us assume that (H) is satisfied for $n \geq 0$. Then from Theorem 2, $b(s, X_{s^-}^n), \sigma(s, X_{s^-}^n)$ and $\gamma(s, z, X_{s^-}^n)$ are in $\mathbb{D}^{(0)}$. Furthermore, we have that

$$\begin{aligned}
D_r^{(0)} b^j(s, X_{s^-}^n) &= \frac{\partial b^j(s, X_{s^-}^n)}{\partial x_k} D_r^{(0)} X_{s^-}^{n,k} \mathbf{1}_{\{r \leq s\}} \\
D_r^{(0)} \sigma_\alpha^j(s, X_{s^-}^n) &= \frac{\partial \sigma_\alpha^j(s, X_{s^-}^n)}{\partial x_k} D_r^{(0)} X_{s^-}^{n,k} \mathbf{1}_{\{r \leq s\}} \\
D_r^{(0)} \gamma^j(s, z, X_{s^-}^n) &= \frac{\partial \gamma^j(s, z, X_{s^-}^n)}{\partial x_k} D_r^{(0)} X_{s^-}^{n,k} \mathbf{1}_{\{r \leq s\}},
\end{aligned}$$

Since the coefficients have continuously bounded first derivatives in the x direction and condition (11), there exists a constant K such that

$$|D_r^{(0)} b^j(s, X_{s^-}^n)| \leq K |D_r^{(0)} X_{s^-}^n| \tag{15}$$

$$|D_r^{(0)} \sigma_\alpha^j(s, X_{s^-}^n)| \leq K |D_r^{(0)} X_{s^-}^n| \tag{16}$$

$$|D_r^{(0)} \gamma^j(s, z, X_{s^-}^n)| \leq K |\rho(z)| |D_r^{(0)} X_{s^-}^n| \tag{17}$$

However, $\int_0^t \sigma_\alpha(s, X_{s^-}^n) dW_s^\alpha, \int_0^t \int_{\mathbb{R}_0} \gamma(s, z, X_{s^-}^n) \tilde{\mu}(dz, ds) \in \mathbb{D}^{(0)}$ from Proposition 6. Thus,

$$\begin{aligned}
D_r^i \int_0^t \sigma_\alpha^j(s, X_{s^-}^n) dW_s^\alpha &= \sigma_i^j(r, X_{r^-}^n) + \int_r^t D^i \sigma_\alpha^j(s, X_{s^-}^n) dW_r^\alpha \\
D_r^i \int_0^t \int_{\mathbb{R}_0} \gamma^j(s, z, X_{s^-}^n) \tilde{\mu}(dz, ds) &= \int_r^t \int_{\mathbb{R}_0} D_r^i \gamma^j(s, z, X_{s^-}^n) \tilde{\mu}(dz, ds).
\end{aligned}$$

Also $\int_0^t b(s, X_{s^-}^n) ds \in \mathbb{D}^{(0)}$, hence

$$D_r^i \int_0^t b^j(s, X_{s^-}^n) ds = \int_r^t D_r^i b^j(s, X_{s^-}^n) ds. \tag{18}$$

From the above we can conclude that $X_t^{n+1} \in \mathbb{D}^{(0)}$ for all $t \in [0,T]$. Furthermore

$$\begin{aligned} E\left[\sup_{r\le u\le t}|D_r^i X_s^{n+1}|^2\right] &\le 4\Big\{E\left[\sup_{r\le u\le t}|\sigma_i(r,X_{r^-}^n)|^2\right] \\ &+ E\left[\sup_{r\le u\le t}\Big|\int_r^u D_r^i b(s,X_{s^-})ds\Big|^2\right] \\ &+ E\left[\sup_{r\le u\le t}\Big|\int_r^u D_r^i \sigma_\alpha(s,X_{s^-})dW_s^\alpha\Big|^2\right] \\ &+ E\left[\sup_{r\le u\le t}\Big|\int_r^u\int_{\mathbb{R}_0} D_r^i \gamma(s,z,X_{s^-})\tilde{\mu}(dz,ds)\Big|^2\right]\Big\} \qquad (19) \end{aligned}$$

From Cauchy-Schwartz and Burkholder-Davis-Gundy[2] inequality, (19) takes the following form

$$\begin{aligned} E\left[\sup_{r\le u\le t}|D_r^i X_s^{n+1}|^2\right] &\le c\Big\{E\left[\sup_{r\le u\le t}|\sigma_i(r,X_{r^-}^n)|^2\right] \\ &+ TE\left[\int_r^t \big|D_r^i b(s,X_{s^-})\big|^2 ds\right] \\ &+ E\left[\Big|\int_r^t D_r^i \sigma_\alpha(s,X_{s^-})dW_s^\alpha\Big|^2\right] \\ &+ E\left[\Big|\int_r^t\int_{\mathbb{R}_0} D_r^i \gamma(s,z,X_{s^-})\tilde{\mu}(dz,ds)\Big|^2\right]\Big\} \end{aligned}$$

$$\begin{aligned} &= c\Big\{E\left[\sup_{r\le u\le t}|\sigma_i(r,X_{r^-}^n)|^2\right] \\ &+ TE\left[\int_r^t \big|D_r^i b(s,X_{s^-})\big|^2 ds\right] \\ &+ E\left[\int_r^t |D_r^i \sigma_\alpha(s,X_{s^-})|^2 ds\right] \\ &+ E\left[\int_r^t\int_{\mathbb{R}_0} |D_r^i \gamma(s,z,X_{s^-})|^2\nu(dz)ds\right]\Big\} \qquad . \end{aligned}$$

From (15), (16) and (17) we have

$$\begin{aligned} E[\sup_{r\le s\le t}|D_r^i X_s^{n+1}|^2] &\le c\Big\{\beta + K^2\left(T+1+\int_{\mathbb{R}_0}\rho^2(z)\nu(dz)\right) \\ &\quad \times \int_r^t E(|D_r^i X_{s^-}^n|^2)ds\Big\}, \qquad (20) \end{aligned}$$

where $\beta = \sup_{n,i} E\left[\sup_{r\le s\le t}|\sigma_i(s,X_{s^-}^n)|^2\right]$.
Thus, hypothesis (H) holds for $n+1$. From Applebaum [1], Theorem 6.2.3, we have that

$$E\left(\sup_{s\le T}|X_s^n - X_s|^2\right) \to 0$$

[2] see [15] Theorem 48 p.193

as n goes to infinity.
By induction to the inequality (20), see for more details appendix A, we can conclude that the derivatives of X_s^n are bounded in $L^2(\Omega \times [0,T])$ uniformly in n. Hence $X_t \in \mathbb{D}^{(0)}$. Applying the chain rule to (12) we conclude our proof.

2. Following the same steps we can prove the second claim of the theorem.

□

With the previous theorem we have proven that the solution of (10) is in $\mathbb{D}^{(0)}$, and reached the stochastic differential equation that $D_s^{(0)} X_t$ satisfies. However, the Wiener directional derivative can take a more explicit form. As in the classical Malliavin calculus we are able to associate the solution of (12) with the process $Y_t = \nabla X_t$; first variation of X_t. Y satisfies the following stochastic differential equation[3]:

$$\begin{aligned} dY_t &= b^{'}(t, X_{t^-})Y_{t^-}dt + \sigma_i^{'}(t, X_{t^-})Y_{t^-}dW_t^i \\ &+ \int_{\mathbb{R}_0} \gamma^{'}(t, z, X_{t^-})Y_{t^-}\tilde{\mu}(dz, dt) \\ Y_0 &= I, \end{aligned} \tag{21}$$

where prime denotes the derivative and I the identity matrix. Hence, we reach the following proposition which provides us with a simpler expression for $D_s^{(0)} X_t$.

Proposition 7. *Let $\{X_t\}_{t\in[0,T]}$ be the solution of* (10)*. Then the derivative $D_t^{(0)}$ in the Wiener direction satisfies the following equation:*

$$D_r^{(0)} X_t = Y_t Y_{r^-}^{-1} \sigma(r, X_{r^-}), \quad \forall r \le t, \tag{22}$$

where $Y_t = \nabla X_t$ is the first variation of X_t.

Proof. The elements of the matrix Y satisfy the following equation:

$$\begin{aligned} Y_t^{ij} &= \delta_j^i + \int_0^t \frac{\partial}{\partial x_k} b^i(s, X_{s^-}) Y_s^{kj} ds \\ &+ \int_0^t \frac{\partial}{\partial x_\alpha} \sigma_k^i(s, X_{s^-}) Y_s^{kj} dW_s^\alpha \\ &+ \int_0^t \int_{\mathbb{R}_0} \frac{\partial}{\partial x_k} \gamma^i(s, z, X_{s^-}) Y_s^{kj} \tilde{\mu}(dz, ds), \end{aligned}$$

where δ is the Dirichlet delta.

[3]For the existence and uniqueness of the solution see [15] section V.7 Theorem 39.

Let $\{Z_t\}_{t\in[0,T]}$ be a $d\times d$ matrix valued process that satisfies the following equation

$$
\begin{aligned}
Z_t^{ij} &= \delta_j^i + \int_0^t \left(-\frac{\partial}{\partial x_j} b^k(t, X_{s^-}) + \frac{\partial}{\partial x_l}\sigma_\alpha^k(t, X_{s^-})\frac{\partial}{\partial x_j}\sigma_l^\alpha(t, X_{s^-})\right) Z_{s^-}^{ik} ds \\
&+ \int_0^t \int_{\mathbb{R}_0} \frac{\left(\frac{\partial}{\partial x_k}\gamma^i(t,z,X_{s^-})\right)^2}{1+\frac{\partial}{\partial x_k}\gamma^i(t,z,X_{s^-})} Z_{s^-}^{ik}\nu(dz)ds \\
&- \int_0^t \frac{\partial}{\partial x_l}\sigma_j^k(t, X_{s^-}) Z_{s^-}^{ik} dW_s^l \\
&- \int_0^t \int_{\mathbb{R}_0} \frac{\frac{\partial}{\partial x_k}\gamma^i(t,z,X_{s^-})}{1+\frac{\partial}{\partial x_k}\gamma^i(t,z,X_{s^-})} Z_{s^-}^{ik}\tilde{\mu}(dz,ds)
\end{aligned}
$$

By applying integration by parts formula we can prove that $Y_tZ_t = Z_tY_t = I$,

$$Z_t^{ij} Y_t^{jl} = \delta_l^i$$

Hence, $Z_t = Y_t^{-1}$. Furthermore it is easy to show applying again Itô's formula, that $Y_t^{il}Z_{r^-}^{lk}\sigma_j^k(r, X_{r^-})$ verifies (12) for all $r < t$. Hence the proof is concluded. □

5 Sensitivities

Using the Malliavin calculus developed in the previous sections we are able to calculate the sensitivities, i.e. the Greek letters. The Greeks are calculated for an m-dimensional process $\{X_t\}_{t\in[0,T]}$ that satisfies equation (10).
We denote the price of the contingent claim as

$$u(x) = E[\phi(X_{t_1}, \ldots, X_{t_n})], \tag{23}$$

where $\phi(X_{t_1}, \ldots, X_{t_n})$ is the payoff function, which is square integrable, evaluated at times $t_1, \ldots, t_n$ and discounted from maturity T.
In what follows we assume the following ellipticity condition for the diffusion matrix σ.

Assumption 1. *The diffusion matrix σ is elliptic. That implies that there exists $k > 0$ such that*

$$y^*\sigma^*(t,x)\sigma(t,x)y \ge k|y|^2, \quad \forall y, x \in \mathbb{R}^d.$$

5.1 Variation in the Drift Coefficient

Let us assume the following perturbed process

$$
\begin{aligned}
dX_t^\epsilon &= \left(b(t, X_{t^-}^\epsilon) + \epsilon\xi(t, X_{t^-}^\epsilon)\right) dt + \sigma(t, X_{t^-}^\epsilon)dW_t \\
&+ \int_{\mathbb{R}_0} \gamma(t,z,X_{t^-}^\epsilon)\tilde{\mu}(dz,dt), \\
X_0 &= x
\end{aligned}
$$

where ϵ is a scalar and ξ is a bounded function. Then we reach the following proposition.

Proposition 8. *Let σ be a uniformly elliptic matrix. We denote $u^\epsilon(x)$ the following payoff*

$$u^\epsilon(x) = E[\phi(X_T^\epsilon)].$$

Then

$$\frac{\partial u^\epsilon(x)}{\partial \epsilon}\bigg|_{\epsilon=0} = E[\phi(X_T)\int_0^T (\sigma^{-1}(t, X_{t^-})\xi(t, X_{t^-}))^* dW_t].$$

Proof. The proof is based on an application of Girsanov's theorem. For

$$M_T = \exp\left(-\epsilon\int_0^T (\sigma^{-1}(t, X_{t^-})\xi(X_{t^-}))^* dW_t) - \frac{\epsilon^2}{2}\int_0^T \|\sigma^{-1}(t, X_{t^-})\gamma(X_{t^-})\|^2 dt\right)$$

we have

$$E[\phi(X_T^\epsilon)] = E[M_T\phi(X_T)]$$

Hence

$$\begin{aligned}
\frac{\partial u(x)}{\partial \epsilon}\bigg|_{\epsilon=0} &= \lim_{\epsilon\to 0} E\Big[\frac{\phi(X_T^\epsilon) - \phi(X_T)}{\epsilon}\Big] \\
&= \lim_{\epsilon\to 0} E\Big[\phi(X_T)\frac{M_T^\epsilon - 1}{\epsilon}\Big] \\
&= E\Big[\phi(X_T)\int_0^T (\sigma^{-1}(t, X_{t^-})\xi(t, X_{t^-}))^* dW_t\Big]
\end{aligned}$$

□

5.2 Variation in the Initial Condition

In order to calculate the variation in the initial condition we will define the set Γ, as follows

$$\Gamma = \{\zeta \in L^2([0,T)) : \int_0^{t_i} \zeta(t)dt = 1, \forall i = 1, \ldots, n\}$$

where t_i are as in (23).

Proposition 9. *Assume that the diffusion matrix σ is uniformly elliptic. Then for all $\zeta(t) \in \Gamma$*

$$(\triangledown u(x))^* = E\Big[\phi(X_{t_1}, \ldots, X_{t_n})\int_0^T \zeta(t)(\sigma^{-1}(t, X_{t^-})Y_{t^-})^* dW_t\Big]$$

Proof. Let ϕ be a continuously differentiable function with bounded gradient. Then we can differentiate inside the expectation[4] and we have

$$\begin{aligned}
\nabla u(x) &= E[\sum_{i=1}^n \nabla_i\phi(X_{t_1}, \ldots, X_{t_n})\frac{\partial}{\partial x}X_{t_i}] \\
&= E[\sum_{i=1}^n \nabla_i\phi(X_{t_1}, \ldots, X_{t_n})Y_{t_i}], \qquad (24)
\end{aligned}$$

[4]For details see [4] and [8]

where $\nabla_i \phi(X_{t_1}, \ldots, X_{t_n})$ is the gradient of ϕ with respect to X_{t_i}, and $\frac{\partial}{\partial x} X_{t_i}$ is the $d \times d$ matrix of the first variation of the d-dimensional process X_{t_i}.
From (22) we have

$$Y_{t_i} = D_t^{(0)} X_{t_i} \sigma^{-1}(t, X_{t-}) Y_{t-}.$$

Hence for any $\zeta \in \Gamma$

$$Y_{t_i} = \int_0^T \zeta(t) D_t^{(0)} X_{t_i} \sigma^{-1}(t, X_{t-}) Y_{t-} dt$$

and inserting the above to (24) follows that

$$\nabla u(x) = E[\int_0^T \sum_{i=1}^n \nabla_i \phi(X_{t_1}, \ldots, X_{t_n}) \zeta(t) D_t^{(0)} X_{t_i} \sigma^{-1}(t, X_{t-}) Y_{t-} dt]$$

from Theorem 2 $\phi(X_{t_1}, \ldots, X_{t_n}) \in \mathbb{D}^{(0)}$, thus

$$\nabla u(x) = E[\int_0^T D_t^{(0)} \phi(X_{t_1}, \ldots, X_{t_n}) \zeta(t) \sigma^{-1}(t, X_{t-}) Y_{t-} dt]$$

From the definition of the Skorohod integral we reach

$$(\nabla u(x))^* = E[\phi(X_{t_1}, \ldots, X_{t_n}) \delta^{(0)} \big(\zeta(\cdot)(\sigma^{-1}(\cdot, X_\cdot) Y_\cdot)^* \big)]$$

However, $\zeta(t)(\sigma^{-1}(t, X_{t-}) Y_{t-})^*$ is a predictable process, thus the Skorohod integral coincides with the Wiener. Since the family of continuously differentiable functions is dense in L^2, the result hold for any $\phi \in L^2$, see [8] and [4] for more details. □

5.3 Variation in the diffusion coefficient

For this section we consider the following perturbed process

$$\begin{aligned} dX_t^\epsilon &= b(t, X_{t-}^\epsilon) dt + \Big(\sigma(t, X_{t-}^\epsilon) + \epsilon \xi(t, X_{t-}^\epsilon) \Big) dW_t \\ &+ \int_{\mathbb{R}_0} \gamma(t, z, X_{t-}^\epsilon) \tilde{\mu}(dz, dt), \\ X_0 &= x \end{aligned}$$

where ϵ is a scalar and ξ is a continuously differentiable function with bounded gradient. We also introduce the variation process in respect to ϵ, $Z_t^\epsilon = \frac{\partial}{\partial \epsilon} X_t^\epsilon$, which satisfies the following sde

$$\begin{aligned} dZ_t^\epsilon &= b'(t, X_{t-}^\epsilon) Z_{t-}^\epsilon dt + \Big(\sigma'(t, X_{t-}^\epsilon) + \epsilon \xi'(t, X_{t-}^\epsilon) \Big) Z_{t-}^\epsilon dW_t \\ &+ \xi(t, X_{t-}^\epsilon) dW_t \\ &+ \int_{\mathbb{R}_0} \gamma'(t, z, X_{t-}^\epsilon) Z_{t-}^\epsilon \tilde{\mu}(dz, dt), \\ Z_0^\epsilon &= 0 \end{aligned}$$

In this case, we introduce the set Γ_n, where

$$\Gamma_n = \{\zeta \in L^2([0,T)) : \int_{t_{i-1}}^{t_i} \zeta(t)dt = 1, \forall i = 1, \dots, n\}.$$

Proposition 10. *Assume that the diffusion matrix σ is uniformly elliptic, and that for $\beta_{t_i} = Y_{t_i}^{-1} Z_{t_i}$, $i = 1, \dots, n$ we have $\sigma^{-1}(t, X_{t-}) Y_t \beta_t \in Dom\delta^{(0)}$, for all $t \in [0,T]$. We denote $u^\epsilon(x)$ the following payoff*

$$u^\epsilon(x) = E[\phi(X_{t_1}^\epsilon, \dots, X_{t_n}^\epsilon)].$$

Then for all $\zeta(t) \in \Gamma$

$$\left.\frac{\partial u^\epsilon(x)}{\partial \epsilon}\right|_{\epsilon=0} = E[\phi(X_{t_1}, \dots, X_{t_n}) \delta^{(0)}(\sigma^{-1}(t, X_{t-}) Y_{t-} \tilde{\beta}_t)],$$

where

$$\tilde{\beta}_t = \sum_{i=1}^n \zeta(t)(\beta_{t_i} - \beta_{t_{i-1}}) \mathbf{1}_{\{t_i \leq t < t_i\}}$$

for $t_0 = 0$. Moreover, if $\beta \in \mathbb{D}^{(0)}$ then

$$\begin{aligned} \delta^0(\sigma^{-1}(t, X_{t-}) Y_{t-} \tilde{\beta}_t) &= \sum_{i=1}^n \left\{ \beta_{t_i}^* \int_{t_{i-1}}^{t_i} \zeta(t)(\sigma^{-1}(t, X_{t-}) Y_{t-})^* dW_t \right. \\ &- \int_{t_{i-1}}^{t_i} \zeta(t) Tr\left((D_t^{(0)} \beta_{t_i}) \sigma^{-1}(t, X_{t-}) Y_{t-} \right) dt \\ &- \left. \int_{t_{i-1}}^{t_i} \zeta(t)(\sigma^{-1}(t, X_{t-}) Y_{t-} \beta_{t_{i-1}})^* dW_t \right\}. \end{aligned}$$

Proof. Let ϕ be a continuously differentiable function with bounded gradient as in Proposition 9,we can differentiate inside the expectation. Hence

$$\begin{aligned} \frac{\partial u^\epsilon(x)}{\partial \epsilon} &= E[\sum_{i=1}^n \nabla_i \phi(X_{t_1}, \dots, X_{t_n}) \frac{\partial}{\partial x} X_{t_i}] \\ &= E[\sum_{i=1}^n \nabla_i \phi(X_{t_1}^\epsilon, \dots, X_{t_n}^\epsilon) Z_{t_i}^\epsilon], \end{aligned} \tag{25}$$

however $\beta_t = Y_t^{-1} Z_t$, $Z_t^\epsilon|_{\epsilon=0} = Z_t$. Then from (22) we have

$$\begin{aligned} & \int_0^T D_t^{(0)} X_{t_i} \sigma^{-1}(t, X_{t-}) Y_{t-} \tilde{\beta}_t dt \\ = & \int_0^{t_i} Y_{t_i} \tilde{\beta}_t dt \\ = & Y_{t_i} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \zeta(t)(\beta_{t_i} - \beta_{t_{i-1}}) \mathbf{1}_{\{t_i \leq t < t_i\}} \\ = & Y_{t_i} \beta_{t_i} = Z_{t_i}. \end{aligned}$$

Inserting the above to (25) we have the following

$$\frac{\partial u^\epsilon(x)}{\partial \epsilon} = E[\int_0^T \sum_{i=1}^n \nabla_i \phi(X^\epsilon_{t_1},\dots,X^\epsilon_{t_n}) D^{(0)}_t X_{t_i} \sigma^{-1}(t,X_{t-}) Y_t \tilde{\beta}_t dt],$$

since $\sigma^{-1}(t,X_{t-})Y_t\tilde{\beta}_t \in Dom\delta^{(0)}$, for all $t\in[0,T]$

$$\frac{\partial u^\epsilon(x)}{\partial \epsilon} = E[\phi(X^\epsilon_{t_1},\dots,X^\epsilon_{t_n})\delta^0(\sigma^{-1}(t,X_{t-})Y_{t-}\tilde{\beta}_t)],$$

the result follows. If $\beta\in\mathbb{D}^{(0)}$ using Proposition 5 we can calculate the Skorohod integral. □

5.4 Variation in the jump amplitude

For this section we consider the following perturbed process

$$\begin{aligned} dX^\epsilon_t &= b(t,X^\epsilon_{t-})dt + \sigma(t,X^\epsilon_{t-})dW_t \\ &+ \int_{\mathbb{R}_0} \Big(\gamma(t,z,X^\epsilon_{t-}) + \epsilon\xi(t,X^\epsilon_{t-})\Big)\tilde{\mu}(dz,dt), \\ X_0 &= x \end{aligned}$$

where ϵ is a scalar and ξ is a continuously differentiable function with bounded gradient. As in the previous section, we will also introduce the variation process in respect to ϵ, $Z^\epsilon_t = \frac{\partial}{\partial\epsilon}X^\epsilon_t$, which satisfies the following sde

$$\begin{aligned} dZ^\epsilon_t &= b'(t,X^\epsilon_{t-})Z^\epsilon_{t-}dt + \sigma'(t,X^\epsilon_{t-})Z^\epsilon_{t-}dW_t \\ &+ \int_{\mathbb{R}_0} \Big(\gamma'(t,z,X^\epsilon_{t-}) + \epsilon\xi'(t,X^\epsilon_{t-})\Big) Z^\epsilon_{t-}\tilde{\mu}(dz,dt) \\ &+ \int_{\mathbb{R}_0} \xi(t,X^\epsilon_{t-})\tilde{\mu}(dz,dt), \\ Z^\epsilon_0 &= 0 \end{aligned}$$

And we will use the set Γ_n as it is defined in the previous section.

Proposition 11. *Assume that the diffusion matrix σ is uniformly elliptic, and that for $\beta_{t_i} = Y^{-1}_{t_i}Z_{t_i}$, $i=1,\dots,n$ we have $\sigma^{-1}(t,X_{t-})Y_t\tilde{\beta}_t \in Dom\delta^{(0)}$, for all $t\in[0,T]$. We denote $u^\epsilon(x)$ the following payoff*

$$u^\epsilon(x) = E[\phi(X^\epsilon_{t_1},\dots,X^\epsilon_{t_n})].$$

Then for all $\zeta(t)\in\Gamma$

$$\left.\frac{\partial u^\epsilon(x)}{\partial \epsilon}\right|_{\epsilon=0} = E[\phi(X^\epsilon_{t_1},\dots,X^\epsilon_{t_n})\delta^{(0)}(\sigma^{-1}(t,X_{t-})Y_{t-}\tilde{\beta}_t)],$$

where

$$\tilde{\beta}_t = \sum_{i=1}^n \zeta(t)(\beta_{t_i} - \beta_{t_{i-1}})\mathbf{1}_{\{t_i\le t<t_i\}}$$

for $t_0 = 0$. *Moreover, if* $\beta \in \mathbb{D}^{(0)}$ *then*

$$\begin{aligned}
\delta^{(0)}(\sigma^{-1}(t, X_{t^-})Y_{t^-}\tilde{\beta}_t) &= \sum_{i=1}^{n} \left\{ \beta^*_{t_i} \int_{t_{i-1}}^{t_i} \zeta(t)(\sigma^{-1}(t, X_{t^-})Y_{t^-})^* dW_t \right. \\
&- \int_{t_{i-1}}^{t_i} \zeta(t) Tr\left((D_t^{(0)}\beta_{t_i})\sigma^{-1}(t, X_{t^-})Y_{t^-} \right) dt \\
&- \left. \int_{t_{i-1}}^{t_i} \zeta(t)(\sigma^{-1}(t, X_{t^-})Y_{t^-}\beta_{t_{i-1}})^* dW_t \right\}.
\end{aligned}$$

Proof. The same as in Proposition 10. □

6 Examples

In this section, we explicitly calculate the Greeks for a general stochastic volatility model with jumps both in the underlying and the volatility (**SVJJ**). This is followed by some numerical results on two specific cases of SVJJ, the Bates and the SVJ model.

Example 1. *Let* $X_t = (X_t^1, X_t^2)$ *be a two dimensional stochastic process satisfying the following stochastic differential equation.*

$$\begin{aligned}
dX_t^1 &= rX_{t^-}^1 dt + \sqrt{X_{t^-}^2} X_{t^-}^1 dW_t^1 \\
&+ X_{t^-}^1 \int_{\mathbb{R}_0} \gamma_1(t, z)\tilde{\mu}(dz, dt) \\
X_0^1 &= x_1 \\
dX_t^2 &= k(m - X_{t^-}^2)dt + \sigma\sqrt{X_{t^-}^2} dW_t^2 \\
&+ X_{t^-}^2 \int_{\mathbb{R}_0} \gamma_2(t, z)\tilde{\mu}(dz, dt) \\
X_0^2 &= x_2
\end{aligned}$$

with $E[dW_t^1 dW_t^2] = \rho dt$. *The sensitivities for the payoff*

$$u = E[e^{-rT}\phi(X_T^1)],$$

have the form

$$
\begin{aligned}
Rho &= E\Big[e^{-rT}\phi(X_T^1)\int_0^T \frac{dW_t^1}{\sqrt{X_{t^-}^2}} - \frac{\rho}{\sqrt{1-\rho^2}}\int_0^T \frac{dW_t^2}{\sqrt{X_{t^-}^2}}\Big] \\
&- E[Te^{-rT}\phi(X_T^1)\Big] \\
Delta &= E\Big[e^{-rT}\phi(X_T^1)\frac{1}{x_1}\left(\int_0^T \frac{dW_t^1}{\sqrt{X_{t^-}^2}} - \frac{\rho}{\sqrt{1-\rho^2}}\int_0^T \frac{dW_t^2}{\sqrt{X_{t^-}^2}}\right)\Big] \\
Gamma &= E\Big[e^{-rT}\phi(X_T^1)\frac{1}{x_1^2T^2}\left(\left(\int_0^T \frac{dW_t^1}{\sqrt{X_{t^-}^2}} - \frac{\rho}{\sqrt{1-\rho^2}}\int_0^T \frac{dW_t^2}{\sqrt{X_{t^-}^2}}\right)^2\right. \\
&- \left.\frac{1}{1-\rho^2}\int_0^T \frac{dt}{X_{t^-}^2}\right) - T\left(\int_0^T \frac{dW_t^1}{\sqrt{X_{t^-}^2}} - \frac{\rho}{\sqrt{1-\rho^2}}\int_0^T \frac{dW_t^2}{\sqrt{X_{t^-}^2}}\right)\Big] \\
Vega &= E\Big[e^{-rT}\phi(X_T^1)\frac{1}{T}\left(\left(W_T^1 - \int_0^T \sqrt{X_{t^-}^2}\,dt\right)\right. \\
&\times \left.\left(\int_0^T \frac{dW_t^1}{\sqrt{X_{t^-}^2}} - \frac{\rho}{\sqrt{1-\rho^2}}\int_0^T \frac{dW_t^2}{\sqrt{X_{t^-}^2}}\right) - \int_0^T \frac{dt}{\sqrt{X_{t^-}^2}}\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
Alpha &= E\Big[e^{-rT}\phi(X_T^1)\frac{1}{T}\left\{\int_0^T\int_{\mathbb{R}_0} z\tilde{\mu}(dz,dt) - \int_0^T\int_{\mathbb{R}_0} z\gamma_1(z,t)\pi(dz,dt)\right\} \\
\times &\left(\int_0^T \frac{dW_t^1}{\sqrt{X_{t^-}^2}} - \frac{\rho}{\sqrt{1-\rho^2}}\int_0^T \frac{dW_t^2}{\sqrt{X_{t^-}^2}}\right))\Big]
\end{aligned}
$$

For details on the calculations refer to the appendix.
In order to illustrate the effectiveness of Malliavin calculus in the calculation of the Greeks, we have performed a number of simulations estimating the Delta for digital options for different models. Our main aim is to compare the results of the Malliavin approach with the finite difference method. This simulations have been performed using standard Euler schemes.

Bates model

In Figures 1 we plot the delta for a digital option for an underlying that satisfies the Bates model. The Bates is an extension of the Heston model, where the underlying has jumps. The sde are given by

$$
\begin{aligned}
dX_t^1 &= rX_{t^-}^1dt + \sqrt{X_{t^-}^2}X_{t^-}^1dW_t^1 \\
&+ X_{t^-}^1\int_{\mathbb{R}_0}(e^z-1)\tilde{\mu}(dz,dt) \\
X_0^1 &= x_1 \\
dX_t^2 &= k(m - X_{t^-}^2)dt + \sigma\sqrt{X_{t^-}^2}dW_t^2 \\
X_0^2 &= x_2.
\end{aligned}
$$

The Lévy measure of μ is $\nu(z) = \lambda \frac{1}{\sqrt{2\pi}b} e^{\frac{(z-a)^2}{b^2}}$, thus the intensity of the Poisson process μ is λ and the jump sizes follow the normal distribution with parameters a, b.

Stochastic volatility model with jump (SVJ) model

In Figures 2 we plot the delta for a digital option for an underlying that has a stochastic volatility with jumps (SVJ). The SVJ is an extension of the Heston model, where the stochastic volatility includes jumps. The sde are given by

$$\begin{aligned}
dX_t^1 &= rX_{t^-}^1 dt + \sqrt{X_{t^-}^2} X_{t^-}^1 dW_t^1 \\
X_0^1 &= x_1 \\
dX_t^2 &= k(m - X_{t^-}^2)dt + \sigma \sqrt{X_{t^-}^2} dW_t^2 \\
&+ X_{t^-}^2 \int_{\mathbb{R}_0} z\tilde{\mu}(dz, dt) \\
X_0^2 &= x_2.
\end{aligned}$$

The Lévy measure of μ is $\nu(z) = \lambda e^{-a}$, thus the intensity of the Poisson process μ is λ and the jump sizes follow the exponential distribution with parameters a, b.

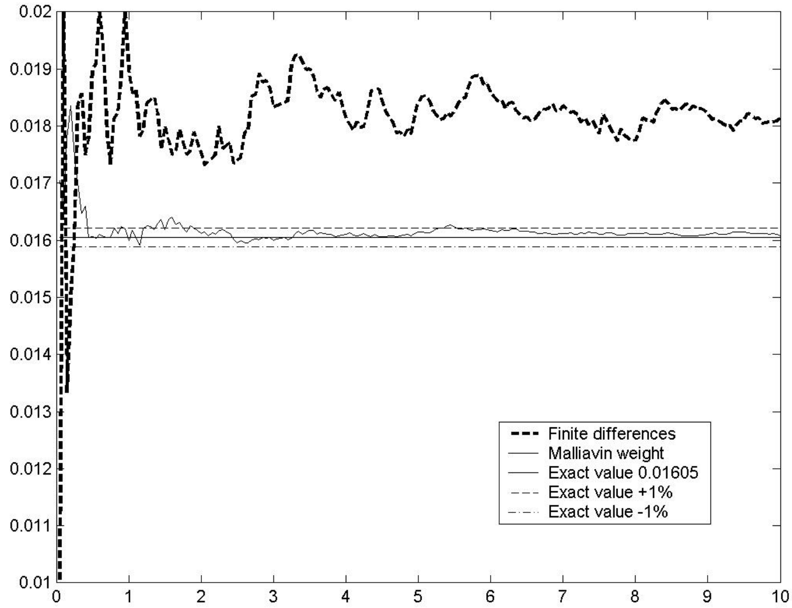

Figure 1: Delta for a digital option in the Bates model, with parameters $T = 1, X_0^1 = 100, K = 100, r = 0.05, \sigma = 0.04, X_0^2 = 0.04, k = 1, m = 0.05, \rho = -0.8, \lambda = 1, a = -0.1$ and $b = 0.001$

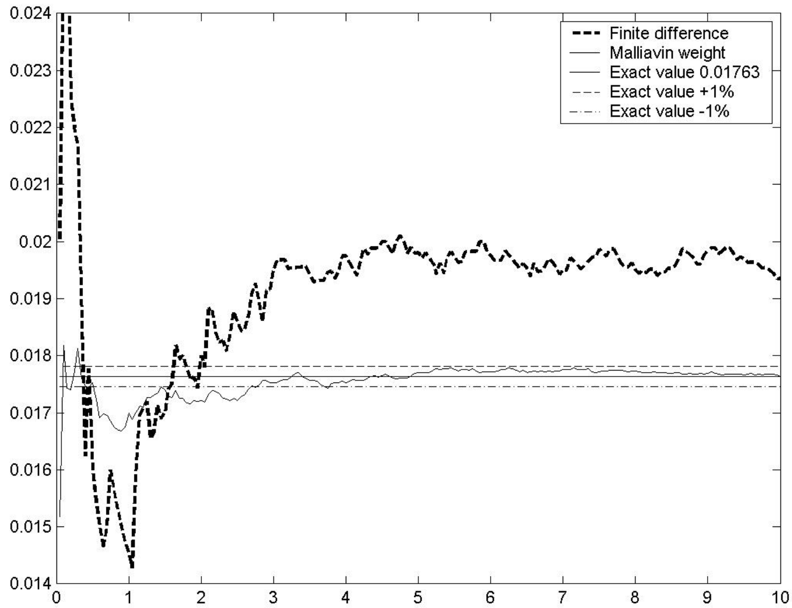

Figure 2: Delta for a digital option in the SVJ model, with parameters $T = 1, X_0^1 = 100, K = 100, r = 0.05, \sigma = 0.04, X_0^2 = 0.04, \kappa = 1, \eta = 0.05, \rho = -0.8, \lambda = 1$ and $a = 0.001$

It is obvious from the figures that the Malliavin weight performs better in the case of discontinues payoffs as we would expect.

A Bounded derivative of the Picard approximation scheme

From inequality (20) we have

$$\xi_{n+1}(T) \le c_1 + \int_r^T \xi_n(t)dt,$$

where $c_1 = c\beta$, $c_2 = cK^2\left(T + 1 + \int_{\mathbb{R}_0} \rho^2(z)\nu(dz)\right)$ and $\xi_n(t) = E[\sup_{r\le s\le t} |D_r^i X_s^n|^2]$. We will prove by induction that

$$\xi_n(T) \le c_1 \sum_{j=0}^{n} \frac{c_2^j (T-r)^j}{j!}. \tag{26}$$

For $n = 0$ it is obvious that (26) holds. Let (26) hold for n. Then

$$\begin{aligned}
\xi_{n+1} &\le c_1 + c_2 \int_r^T \xi_n(s)ds \\
&\le c_1 + c_1 c_2 \int_r^T \sum_{j=0}^{n} \frac{c_2^j (s-r)^j}{j!} ds \\
&\le c_1 + c_1 c_2 \sum_{j=1}^{n+1} \frac{c_2^j (T-r)^j}{j!} \\
&\le c_1 \sum_{j=0}^{n+1} \frac{c_2^j (T-r)^j}{j!}
\end{aligned}$$

Since $\lim_{n\to\infty} \sum_{j=0}^{n} \frac{c_2^j (T-r)^j}{j!} = e^{c_2(T-r)} < \infty$, we can conclude that $\xi_n(T) < \infty$.

B Calculation of the Greeks for the SVJJ model

The system of stochastic differential equations of the extended Heston model can be rewritten in a matrix form:

$$dX_t = b(t, X_{t^-})dt + \sigma(t, X_{t^-})dW_t + \int_{\mathbb{R}_0} \gamma(t, z, X_{t^-})\tilde{\mu}(dz, dt),$$

where $b^*(t, X_{t^-}) = (rX_{t^-}^1, k(m - X_{t^-}^2))^*$,

$$\sigma(t, X_{t^-}) = \begin{pmatrix} X_{t^-}^1 \sqrt{X_{t^-}^2} & 0 \\ \sigma\rho\sqrt{X_{t^-}^2} & \sigma\sqrt{1-\rho^2}\sqrt{X_{t^-}^2} \end{pmatrix}$$

and

$$\gamma(t,z,X_{t^-}) = \begin{pmatrix} X^1_{t^-}\gamma_1(t,z) \\ X^2_{t^-}\gamma_2(t,z) \end{pmatrix}$$

The inverse of σ is

$$\sigma(t,X_{t^-})^{-1} = \frac{1}{\sigma X^1_{t^-}X^2_{t^-}\sqrt{1-\rho^2}} \begin{pmatrix} \sigma\sqrt{1-\rho^2}\sqrt{X^2_{t^-}} & 0 \\ -\sigma\rho\sqrt{X^2_{t^-}} & X^1_{t^-}\sqrt{X^2_{t^-}} \end{pmatrix}$$

- **Rho** In the case of the sensitivity with respect to r we perturb the drift with $\xi(t,x) = (x_1,0)^*$. Hence,

$$\left(\sigma^{-1}(t,X_{t^-})\xi(t,X_{t^-})\right)^* = \left(\frac{1}{\sqrt{X^2_{t^-}}}, -\frac{\rho}{\sqrt{1-\rho^2}\sqrt{X^2_{t^-}}}\right)$$

Applying Lemma 8 we reach the result.

- **Delta** The first variation process has the following form

$$\begin{aligned} dY_t &= b'(t,X_{t^-})Y_{t^-}dt + \sigma_1^{'}(t,X_{t^-})Y_{t^-}dW^1_t + \sigma_2^{'}(t,X_{t^-})Y_{t^-}dW^2_t \\ &+ Y_{t^-}\int_{\mathbb{R}_0}\gamma'(t,z,X_{t^-})\tilde{\mu}(dz,dt), \\ Y_0 &= I \end{aligned} \tag{27}$$

where $b^{'}(t,X_{t^-}) = \begin{pmatrix} r & 0 \\ 0 & -k \end{pmatrix}$,

$$\sigma_1^{'}(t,X_{t^-}) = \begin{pmatrix} \sqrt{X^2_{t^-}} & \frac{X^1_{t^-}}{2\sqrt{X^2_{t^-}}} \\ 0 & \frac{\sigma\rho}{2\sqrt{X^2_{t^-}}} \end{pmatrix},$$

$$\sigma_2^{'}(t,X_{t^-}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma\sqrt{1-\rho^2}}{2\sqrt{X^2_{t^-}}} \end{pmatrix},$$

and

$$\gamma'(t,z,X_{t^-}) = \begin{pmatrix} \gamma_1(t,z) & 0 \\ 0 & \gamma_2(t,z) \end{pmatrix}$$

Since $Y^{1,1}_t = \frac{X^1_t}{x_1}$ and $Y^{2,1}_t = 0$ the matrix $(\sigma^{-1}(t,X_{t^-})Y_t)^*$ has the following form

$$\begin{pmatrix} \frac{1}{X^1_{t^-}\sqrt{X^2_{t^-}}}Y^{1,1}_{t^-} & \frac{-\rho}{\sqrt{1-\rho^2}X^1_{t^-}\sqrt{X^2_{t^-}}}Y^{1,1}_{t^-} \\ \frac{1}{X^1_{t^-}\sqrt{X^2_{t^-}}}Y^{2,1}_{t^-} & \frac{1}{\sqrt{1-\rho^2}\sqrt{X^2_{t^-}}}\left(\frac{-\rho Y^{1,2}_{t^-}}{X^1_{t^-}} + \frac{Y^{2,2}_{t^-}}{\sigma}\right) \end{pmatrix}$$

Applying Proposition 9 we reach the result.

- **Gamma** In the case of the second derivative with respect the initial condition x by applying again Lemma 9 to the *Delta* we reach our result.

- **Vega** In order to calculate $Vega$ we perturb X_t with $\xi(t,x) = \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix}$. From Itô's formula it is easy to verify that $Z_t^1 = X_t^1\Big(W_t^1 - \int_0^t \sqrt{X_s^2}ds\Big)$. Furthermore, since X_t^2 does not depend on ϵ we can deduce that $Z_t^2 = 0$. Then $\beta_t = Y_t^{-1}Z_t = \begin{pmatrix} x_1\Big(W_t^1 - \int_0^t \sqrt{X_t^2}ds\Big) \\ 0 \end{pmatrix}$.

  The Wiener directional derivative for $\sqrt{X_{s^-}^2}$ has the following form

$$\begin{aligned} D_t^{(0)}\sqrt{X_{s^-}^2} &= \frac{1}{\sqrt{X_{s^-}^2}} D_t^{(0)} X_{s^-}^2 \\ &= \frac{1}{\sqrt{X_{s^-}^2}} \begin{pmatrix} \rho\sqrt{X_{t^-}^2}\frac{Y_{s^-}^{2,2}}{Y_{t^-}^{2,2}}\mathbf{1}_{\{t<s\}} \\ \sqrt{1-\rho^2}\sqrt{X_{t^-}^2}\frac{Y_{s^-}^{2,2}}{Y_{t^-}^{2,2}}\mathbf{1}_{\{t<s\}} \end{pmatrix} \end{aligned}$$

$$D_t\beta_T = x_1\begin{pmatrix} 1 - \frac{\rho\sigma}{2}\int_t^T \frac{\sqrt{X_{t^-}^2}Y_{s^-}^{2,2}}{\sqrt{X_s^2}Y_{t^-}^{2,2}}dW_t^1 & -\frac{\sqrt{1-\rho^2}\sigma}{2}\int_t^T \frac{\sqrt{X_{t^-}^2}Y_{s^-}^{2,2}}{\sqrt{X_s^2}Y_{t^-}^{2,2}}dW_t^2 \\ 0 & 0 \end{pmatrix}$$

  then

$$Tr((D_t^{(0)}\beta_T)\sigma^{-1}(t,X_{t^-})Y_{t^-}) = \frac{1}{\sqrt{X_{t^-}^2}}$$

  Using Proposition 10 we reach the wanted expression.

- **Alpha** In order to calculate the sensitivity with respect to the jump amplitude we perturb X_t with $\xi(t,x) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$. From Itô's formula it is easy to verify that $Z_t^1 = X_t^1\Big(\int_0^t\int_{\mathbb{R}_0} z\tilde{\mu}(dz,ds) - \int_0^t\int_{\mathbb{R}_0} z\gamma(z,s)\pi(dz,ds)\Big)$. Following the same steps as in $Vega$ we reach the wanted result.

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