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THE LONGTIME BEHAVIOR OF BRANCHING RANDOM WALK IN A CATALYTIC MEDIUM

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Abstract We consider catalytic branching random walk (the *reactant*) where the state space is a countable Abelean group. The branching is critical binary and the local branching rate is given by a catalytic medium. Here the medium is itself an autonomous (ordinary) branching random walk (the *catalyst*) – maybe with a different motion law.

For persistent catalyst (transient motion) the reactant shows the usual dichotomy of persistence versus extinction depending on transience or recurrence of its motion.

If the catalyst goes to local extinction it turns out that the longtime behaviour of the reactant ranges (depending on its motion) from local extinction to free random walk with either deterministic or random global intensity of particles.

Keywords Branching random walk in random medium, reactant–catalyst systems, interacting particle systems, random media.

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The Longtime Behavior of Branching Random Walk, in a Catalytic Medium

A. Greven[1], A. Klenke[2], A. Wakolbinger[3]

March 3, 1998

Abstract

Consider a countable collection (ξ_t) of particles located on a countable group, performing a critical branching random walk where the branching rate of a particle is given by a random medium fluctuating both in space and time. Here we study the case where the time–space random medium (η_t) (called catalyst) is also a critical branching random walk evolving autonomously while the local branching rate of the reactant process (ξ_t) is proportional to the number of catalytic particles present at a site. The process (η_t) and the process (ξ_t) typically have different underlying motions.

Our main interest is to analyze the behavior of the bivariate system (η_t, ξ_t) as $t \to \infty$ and to exhibit features of $(\xi_t)_{t\geq 0}$ which are different from those of classical branching random walk. Some of these features have already been noticed for super processes.

First, we show first that if the symmetrized motion of the catalytic particles is transient then (ξ_t) behaves similarly as a classical branching random walk: depending on whether the symmetrized motion of the reactant is transient or recurrent we have an equilibrium with the original intensity, or we have local extinction.

Next, we consider the case where the symmetrized motion of the catalyst is recurrent. It is well known that in this case the catalyst goes locally to extinction; however, we discover new features of the reactant depending on the mobility both of catalyst and reactant. Now three regimes are possible. In one regime the reactant (ξ_t) behaves like a system of independent random walks in the limit $t \to \infty$, that is it converges to a Poisson system. In a second regime the reactant (ξ_t) approaches also a nontrivial stable state which is now a *mixture* of Poisson systems. Thirdly the reactant (ξ_t) can become locally extinct even in this case of a catalyst going to local extinction. We give examples for all these regimes and set the framework to develop a complete theory.

Keywords: Branching random walk in random medium, reactant–catalyst systems, interacting particle Systems, random media.

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Contents

1 Motivation and main results

(a) Motivation and background

A branching random walk is a process where particles move independently through space and branch according to some branching law independent of the motion. The space is usually given by $\mathbb{Z}^d$ or some other countable group as e.g. the hierarchical group introduced in population genetics by Sawyer and Felsenstein (compare [DG2]). Such spatially structured branching processes have been studied to quite some extent and it has been shown that their longtime behavior exhibits a rather interesting property: a dichotomy between local persistence and local extinction (depending on the underlying migration mechanism), and in the latter case various different regimes of clustering occur. For an overview see Wakolbinger (95) and Klenke (96).

The two crucial assumptions used in the analysis of the classical model are the *independence* of the evolution of the different families descending from distinct ancestors and the fact that the rate at which branching occurs is *constant* both *in space and time*. New phenomena occur if one drops this assumption of independence of subpopulations (Greven (91), Dawson and Hochberg (91)) or the assumption of a constant (in time and space) branching rate. Various models have been studied where the assumption of the homogeneity of the branching rate in *space* has been dropped and many new phenomena in the longtime behavior have been found, as for example phase transitions with the drift of the motion as crucial parameter (Greven and den Hollander (92)), while in other cases the behavior remains unchanged (Dawson and Fleischmann (83) and (85)).

Furthermore many papers study models without spatial structure in the case of a branching rate fluctuating in time (independent of the process) using the approach of a random dynamical system and Lyapunov exponents (Tanny [T]).

In this paper we are interested in spatial models for the case where the branching rate *fluctuates both in time and space*. Some pioneering work has been done here on the level of super processes on $\mathbb{R}^d$. Here, different from the particle model we are going to present, only $d \leq 3$ makes sense non–trivially since the super processes live on low dimensional sets. Processes of the above mentioned type (super process in a catalytic medium) have been constructed by Dawson and Fleischmann in [DF5] and [DF3]. There also the longtime behavior in $d = 1$ has been investigated while $d = 3$ is dealt with in [DF4]. The $d = 2$ case is studied in detail in Fleischmann and Klenke (98). Already on the level of super processes one sees new features in the longtime behaviors. For a detailed review of the work on spatial branching processes in random media see Klenke (99).

We will consider *particle systems* of the following type. First we have the *catalytic random medium*, which is a critical branching random walk (η_t) on some Abelian group ($\mathbb{Z}^d$ or the hierarchical group for example) and evolves autonomously. Then we have a *reactant process*, which is a branching random walk (ξ_t) whose branching rate at a site and at a time is proportional to the number of catalytic particles at that site at that time. This way we obtain the catalyst–reactant system $(\eta_t, \xi_t)_{t \geq 0}$.

The main question we want to resolve is: To what extent does the large time behavior of the reactant process differ from that of a classical branching random walk? For frozen media, as mentioned above the behavior of a system in random media can be drastically different from that of a classical system depending on finer properties of the underlying migration mechanism ([GH, 92]). Is this also the case for media which fluctuate in time as well?

Another aspect is that our model, though dealing with a "one way interaction" only, might help to give insight also into more complicated situations where the reactant influences also the evolution of the catalyst in some way. For material in this direction see the work on multitype contact processes as for example in Durrett and Schinazi ([DS]).

The essential parameters of the problem of the longtime behavior of the catalyst–reactant system are the *two random walk kernels* describing the motions of the catalyst and the reactant respectively. We will see that similarly as in the theory of one–particle evolutions in random media two situations can occur. On one hand there is a regime where the behavior is similar to the classical case, in a sense only "constants change", on the other hand there are also regimes where we obtain a completely different behavior. In fact, for a catalyst which goes to local extinction we have *three different possibilities* for the behavior of the reactant. Namely (i) the reactant eventually behaves like a system of countably many independent random walks (pure Poisson system), or (ii) like a mixture of Poisson systems, where the mixing law reflects qualitatively the randomness of both the catalyst and the reactant, or (iii) the reactant becomes locally extinct. Of particular interest is the case (ii) on the borderline between the regimes of persistent catalyst and (strongly) locally vanishing catalyst, where we find the *mixed* Poisson system in the longtime limit.

For the reader with a background in super processes we should remark that catalytic super Brownian motion shows an analogous behavior in the corresponding regimes (see [DF5, DF6, FK]): In $d = 3$ the joint distribution of catalyst and reactant has a non–trivial equilibrium. The cases $d = 1$ and $d = 2$ correspond to (i) and (ii) respectively. There are no analogues for the cases (iii) and persistent catalyst with reactant going to extinction.

It is not hard to believe that in $d = 1$ and $d = 3$ the behavior of catalytic super Brownian motion carries over to the random walk setting – at least if the random walks are symmetric Bernoulli. However we can show our results in a much greater generality. In $d = 2$ the situation is a lot more delicate. Note that the large deviation behavior of random walks differs from that of Brownian motion, resulting in the compact support property for super Brownian motion but not for super random walk. This property, however, is a crucial ingredient in the proof of the statement for two–dimensional catalytic super Brownian motion. This shows that these diffusion limits cannot be taken too naïvely.

Concluding we can say that in many – but not in all – cases there is a diffusion limit that is a good approximation of the particle system.

(b) The model

We continue with a precise description of the model. We need the following ingredients for this purpose.

Ingredients:

- a countable Abelian group G. The important examples are $G = \mathbb{Z}^d$ and $G = \Omega_N$, where Ω_N is the hierarchical group (with N symbols)

– two irreducible random walk kernels $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ on $G \times G$

– a translation invariant probability measure μ on $(\mathbb{N}_0 \times \mathbb{N}_0)^G$ which is ergodic and has finite intensity $\theta = (\theta_\eta, \theta_\xi) \in [0,\infty)^2$, i.e. under μ the expected size of a component indexed by $g \in G$ is $(\theta_\eta, \theta_\xi)$. The set of those measures is denoted in the sequel

$$\mathcal{E}_{\theta_\eta,\theta_\xi} \tag{1.1}$$

and elements of this set will be used as initial law of the process under consideration. The analogous set of measures on $(\mathbb{N}_0)^G$ with intensity $\theta \in [0,\infty)$ will be denoted by

$$\mathcal{E}_\theta. \tag{1.2}$$

Now we define a Markov process $(\eta_t, \xi_t)_{t\geq 0}$ on a suitable subset $E \subseteq (\mathbb{N}_0 \times \mathbb{N}_0)^G$ by the following rules

(i) $\mathcal{L}[(\eta_0, \xi_0)] = \mu$.

(ii) The process evolves (the initial state being independent of the evolution) according to the following mechanism

  – $(\eta_t)_{t\geq 0}$ evolves autonomously (i.e. is not influenced by $(\xi_t)_{t\geq 0}$). The evolution of η_t is a branching random walk. The branching mechanism is critical binary branching and the migration evolves via a continuous time random walk on G with transition rates given by the kernel $a(\cdot,\cdot)$.

  – Conditioned on $(\eta_t)_{t\geq 0}$ the process $(\xi_t)_{t\geq 0}$ is a branching random walk. The mechanism of branching for a particle in g at time t is critical binary branching at rate $\eta_t(g)$. The migration kernel of (ξ_t) is $b(\cdot,\cdot)$.

Definition We call η_t the *catalyst* and ξ_t the *reactant*.

Remark We will often view the configurations η or ξ as *measures* rather than functions on the countable set G, that is we define

$$\tilde{\eta} \in \mathcal{M}(G) : \tilde{\eta}(A) = \sum_{g\in A} \eta(g), \quad \text{for } A \subseteq G$$

but we will drop the $\sim$ and simply write η for notational convenience.

In the sequel we formally construct the process just described, this may be skipped at a first reading.

The process $(\eta_t, \xi_t)_{t\geq 0}$ can be constructed in two ways which we both present shortly. One way is to construct a Markov process on the configuration space by introducing the semigroup of the process via generators, the second way uses an explicit representation via single ancestor processes.

The construction of the process via semigroups forces us to give a rigorous description of the proper state space of the process, following the device of Liggett and Spitzer. For our model we can define

$$\text{(1.3)} \qquad ||(\eta,\xi)|| = \sum_g \gamma_1(g)\eta(g) + \gamma_2(g)\xi(g)$$

where γ_1 and γ_2 are weight functions which satisfy for some $M \in (1,\infty)\sum_g \gamma_1(g)a(g,g') \leq M\gamma_1(g')$, $\sum_g \gamma_2(g)b(g,g') \leq M\gamma_2(g')$ and which can be found as follows

$$\gamma_1(g) = \sum_n \sum_{g'} a^{(n)}(g,g')\beta(g')M^{-n}$$

and

$$\gamma_2(g) = \sum_n \sum_{g'} b^{(n)}(g,g')\ \beta(g')M^{-n}.$$

Here $a^{(n)}$ and $b^{(n)}$ denote the n-fold compositions of the transition kernels a respectively b and $\beta(\cdot)$ is a strictly positive summable function on G.

The state space $E \subseteq (\mathbb{N}_0 \times \mathbb{N}_0)^G$ is then given by

$$\text{(1.4)} \qquad E = \{(\eta,\xi) :\ ||\eta,\xi|| < \infty\}.$$

On E the semigroup is then constructed using approximations by processes with initially finitely many particles (compare Liggett (85)). The generator is described as follows:
Let $f(\eta,\xi)$ be a function depending only on finitely many components. Define for $\zeta \in \mathbb{N}_0^G$ and $k \in \mathbb{Z}$:

$$\text{(1.5)} \qquad \zeta_{g,g'} = \zeta - 1_g + 1_{g'},\ \ \zeta_g^k = \zeta + k1_g.$$

Let $(p_k)_{k\in\mathbb{N}_0}$, $(q_k)_{k\in\mathbb{N}_0}$ be probability distributions on $\mathbb{N}_0$ with finite mean. These are the offspring distribution laws of η and ξ. We carry out the construction in this generality but later we will always assume that the branching is critical binary: $p_0 = p_2 = q_0 = q_2 = \frac{1}{2}$.

Now the generator of the system (η_t,ξ_t) acts as follows $(c_i \in \mathbb{R}^+)$:

$$\begin{aligned}\text{(1.6)} \qquad (\mathcal{G}f)(\eta,\xi) &= c_1 \sum_{g,g'} [f(\eta_{g,g'},\xi) - f(\eta,\xi)]a(g,g')\eta(g) \\ &+ c_2 \sum_g \left[\sum_{k=0}^{\infty} p_k f(\eta_g^{k-1},\xi) - f(\eta,\xi)\right]\eta(g) \\ &+ c_3 \sum_{g,g'} [f(\eta,\xi_{g,g'}) - f(\eta,\xi)]b(g,g')\xi(g) \\ &+ c_4 \sum_g \left[\sum_{k=0}^{\infty} q_k f(\eta,\xi_g^{k-1}) - f(\eta,\xi)\right]\cdot\eta(g)\xi(g).\end{aligned}$$

Later we will need the continuous time transition kernels a_t and b_t defined by

$$(1.7) \qquad a_t = e^{-c_1 t} \sum_{n=0}^{\infty} \frac{(c_1 t)^n}{n!} a^{(n)}, \qquad b_t = e^{-c_3 t} \sum_{n=0}^{\infty} \frac{(c_3 t)^n}{n!} b^{(n)}. \qquad t \geq 0.$$

In the sequel we will in most cases tacitly assume $c_1 = c_2 = c_3 = c_4 = 1$ to avoid cumbersome notation.

We come now to the explicit construction, which gives as well the whole historical process. We focus on the construction of the reactant.

The *catalyst* $(\eta_t)_{t\geq 0}$ is defined to be the (rate 1) branching random walk on G with branching law (p_k) and random walk transition probabilities $a(\cdot,\cdot)$. It is well known how to construct this process from i.i.d. exponentially distributed branching times and collections of i.i.d. random walks, we do not repeat this here.

The construction of the *reactant* $(\xi_t)_{t\geq 0}$ is more complicated. Let $T = T(\omega)$ be a (random) Galton–Watson tree with branching law (q_k). By $\emptyset \in T$ we denote the root of T and by $\overleftarrow{v}$ we denote the ancestor of $v \in T$. Define the marked tree

$$(1.8) \qquad \overline{T} = ((v, \tau_v, (X_{v,t})_{t\geq 0}),\ v \in T),$$

where $\tau_v - \tau_{\overleftarrow{v}}$, $v \in T$, are for given T i.i.d. exponentially (mean one) distributed random variables and $(X_{v,t})_{t\geq 0}$, $v \in T$, are independent random walks on G according to $b(\cdot,\cdot)$ with $X_{v,0} = 0 \in G$ a.s. For notational convenience we introduce a formal point $\overleftarrow{\emptyset}$ and let $\tau_{\overleftarrow{\emptyset}} = 0$.

Choose $g \in G$ and let $Y_{\overleftarrow{\emptyset}} = g$ and $\sigma_{\overleftarrow{\emptyset}} = 0$. We define inductively the effective branching times σ_v, $v \in T$, and the paths $(Y_{v,t})_{t\in[\sigma_{\overleftarrow{v}},\sigma_v[}$, $v \in T$, by

$$(1.9) \qquad \sigma_v = \begin{cases} \sigma_{\overleftarrow{v}} + \inf\{t > 0 : \int_0^t \eta_{s+\sigma_{\overleftarrow{v}}}(X_{v,s} + Y_{\overleftarrow{v},\sigma_{\overleftarrow{v}}})\, ds \geq \tau_v - \tau_{\overleftarrow{v}}\}, & \sigma_{\overleftarrow{v}} < \infty \\ \infty, & \sigma_{\overleftarrow{v}} = \infty \end{cases}$$

$$(1.10) \qquad Y_{v,t} = Y_{\overleftarrow{v},\sigma_{\overleftarrow{v}}} + X_{v,t-\sigma_{\overleftarrow{v}}} \quad \text{for } t \in \begin{cases} [\sigma_{\overleftarrow{v}}, \sigma_v], & \sigma_v < \infty \\ [\sigma_{\overleftarrow{v}}, \sigma_v[, & \sigma_{\overleftarrow{v}} < \sigma_v = \infty \end{cases}.$$

We define the reactant $(\xi^g_t)_{t\geq 0}$ with initial state $\xi^g_0 = \delta_g$ (the Dirac measure on g) a.s. by

$$(1.11) \qquad \xi^g_t = \sum_{\substack{v\in T \\ t\in[\sigma_{\overleftarrow{v}},\sigma_v[}} \delta_{Y_{v,t}}, \quad t \geq 0.$$

For starting the reactant in the possibly random state $\xi_0 \in \mathcal{N}(G)$ (the integer valued measures on G) we take for given catalyst independent copies $(\xi^{g,i}_t)_{t\geq 0}$, $g \in G, i \in \mathbb{N}$ (with $\xi^{g,i}_0 = \delta_g$) and define

$$(1.12) \qquad \xi_t = \sum_{g\in G} \sum_{i=1}^{\xi_0(\{g\})} \xi^{g,i}_t.$$

It can be shown that we obtain in this way a Markov process $(\eta_t, \xi_t)_{t\geq 0}$ with state space E. Note that for all translation invariant initial laws with finite intensity $(\theta_\eta, \theta_\xi)$ the initial state lies a.s. in E and so do all the configurations evolving in time from this initial state. We are not going into further detail.

Though (1.3) defines a topology on E we will henceforth use the coarser product topology on E. In terms of $\mathcal{N}(G)$ this is the vague topology. In particular, weak convergence of probability measures on E will be equivalent to convergence of finite dimensional distributions.

(c) Results

In this section we will present the results we have on the behavior of (η_t, ξ_t) as $t \to \infty$. It is clear that besides the mobility of the reactant particles the other essential determinant for the behavior of the reactant ξ_t as $t \to \infty$ are the properties of the catalyst process. Therefore we will start by introducing in (i) a classification of the catalyst process according to its longtime behavior in three regimes. We continue in (ii) with the combined system (η_t, ξ_t) as $t \to \infty$ which are stated

(i) Longtime properties of the catalyst

In order to understand the catalyst process we need some basic facts from the theory of branching random walks. Recall that a classical branching random walk (which is critical) approaches either an equilibrium or it becomes locally extinct depending on whether the symmetrized underlying motion is transient or recurrent. In the latter case (the case of local extinction) various different regimes can be distinguished (concentrated clustering, diffusive clustering). See Klenke (96), Dawson and Greven (96).
The behavior of the reactant process depends both on the kernel b of its own motion and on the kernel a of the motion of the catalyst. To work out first the role of the catalyst motion we introduce a classification for the longtime behavior of the catalyst process.

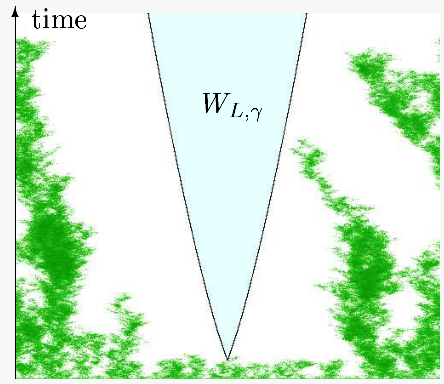

Figure 1 BRW on the torus $\mathbb{Z}/(500)$ and the empty parabola $W_{L,\gamma}$ (explained in Section 4)

In the following terminology for the process $(\eta_t)_{t\geq 0}$ of the catalyst, we always refer to a process whose initial law $\mathcal{L}[\eta_0]$ is translation invariant and ergodic with finite positive intensity.

Definition 1.1 We say that $(\eta_t)_{t\geq 0}$ is *site–recurrent*, iff

$$(1.13) \qquad \mathbf{P}[\sup\{t : \eta_t(g) > 0\} = +\infty] = 1 \quad \forall g \in G$$

and *site–transient* iff

$$(1.14) \qquad \mathbf{P}[\sup\{t:\ \eta_t(g) > 0\} < \infty] = 1 \quad \forall g \in G. \qquad \diamond$$

In the case of site–recurrent media we distinguish two different cases:

Definition 1.2 We say $(\eta_t)_{t \geq 0}$ is *strictly site–recurrent* iff the process is site–recurrent and in addition

$$(1.15) \qquad \liminf_{t \to \infty} \mathbf{P}[\eta_t(g) > 0] > 0$$

and the process is called *weakly site–recurrent*, iff it is site–recurrent and satisfies

$$(1.16) \qquad \limsup_{t \to \infty} \mathbf{P}[\eta_t(g) > 0] = 0. \qquad \diamond$$

With the help of the standard ergodic theory for branching random walks and the (shift)-ergodicity of the initial law one can show that we have a complete classification:

The first question is which regime occurs and the answer depends on the kernel $a(\cdot,\cdot)$. From the standard theory of branching random walks we know the following fact:

$$(1.17) \qquad (\eta_t)_{t \geq 0} \text{ is strictly site–recurrent } \iff \hat{a} \text{ is transient}$$

where $\hat{a}(\xi,\xi') = \frac{1}{2}(a(\xi,\xi') + a(\xi',\xi))$ denotes the kernel which describes the jump distribution of the distance between two independent random walks with jump kernel a.

Next one looks at the two remaining cases where one would like to settle the question as to when a catalyst process is site–transient or weakly site–recurrent, which as we already know from (1.17) can only occur for $\hat{a}$ being recurrent. On the other hand is $\{\sup\{t:\ \eta_t(g) > 0\} < \infty\}$ a translation invariant event (for our starting distribution). Hence by the ergodicity of the initial distribution the probability of that event is 0 or 1. Therefore we have in particular

Proposition 1.1 *Every catalyst process is either site–transient, strictly site–recurrent or weakly site–recurrent.* $\diamond$

The next question is which of the two cases occurs in the case $\hat{a}$ recurrent? This question is rather subtle and therefore a complete theory of this phenomenon would distract from our main point, hence we provide only some typical examples.

Proposition 1.2 *(a) Consider the case $G = \mathbb{Z}$ and let $a(\cdot,\cdot)$ be the kernel of symmetric simple random walk. Then the corresponding catalyst process $(\eta_t)_{t \geq 0}$ is site–transient (see Figure 1).*

(b) For $G = \mathbb{Z}^2$ and a symmetric transition kernel $a(\cdot,\cdot)$ with finite variance the catalyst process is weakly site–recurrent.

(c) If $a(\cdot,\cdot)$ is irreducible has a non–zero drift and finite variance then both in $G = \mathbb{Z}^1$ or $\mathbb{Z}^2$ the process is weakly site–recurrent. $\diamond$

The three different possible regimes for the catalyst process give rise to different features of the large time behavior of the reactant process.

(ii) The four regimes of longtime behavior of the catalyst–reactant system

According to the previous paragraph (i) we have three basic regimes for the behavior of the catalyst process. In the remainder we study the longtime behavior of ξ_t and of (η_t, ξ_t) in these three regimes which further splits these regimes since now for the reactant four possibilities occur

(a) convergence to an infinitely divisible system which is non–Poissonian and has a *local dependence structure,*

(b) local *extinction.*

(c) convergence to a *Poisson system,*

(d) convergence to a *mixed* Poisson system,

In Theorem 1 we will give a complete answer for the case of strictly site–recurrent catalysts, with two types of longtime behavior for the catalyst–reactant system namely (a) and (b). In particular we will find that the behavior in this case is as in the classical homogeneous case, where (a) or (b) occurs depending on whether the symmetrized reactant motion is transient or recurrent. In Theorem 2 and 3 we will describe besides local extinction the two other possible regimes of longtime behavior (Poisson systems and mixed Poisson systems) for classes of models on $\mathbb{Z}^1$ respectively $\mathbb{Z}^2$, where the catalyst is site–transient respectively weakly site–recurrent.

We begin our analysis with the simplest case, namely the one where the catalyst process $(\eta_t)_{t \geq 0}$ is *strictly site–recurrent.*

Here is our first theorem, which shows that for strictly site–recurrent catalytic media the reactant evolves after large times on a rough qualitative level in the same way as a classical branching random walk with the migration kernel a and branching rate given by $\mathbf{E}[\eta_t(g)]$. Recall that we call (ξ_t) or (η_t) persistent (stable) if the intensity of weak limit points as $t \to \infty$ of $\mathcal{L}[\eta_t]$ respectively $\mathcal{L}[\xi_t]$ are equal to the initial intensity.

Theorem 1 *Assume that $(\eta_t)_{t \geq 0}$ is strictly site–recurrent and the initial law μ of $(\eta_t, \xi_t)_{t \geq 0}$ is an element of $\mathcal{E}_{\theta_\eta, \theta_\xi}$ (see (1.1)). Then we have the following dichotomy in $\hat{b}$:*

(a) If $\hat{b}$ is transient then the catalyst–reactant system is persistent (stable) and furthermore:

$$(1.18) \qquad \mathcal{L}[(\xi_t, \eta_t)] \underset{t \to \infty}{\Longrightarrow} \nu_{\theta_\eta, \theta_\xi},$$

where $\nu_{\theta_\eta, \theta_\xi}$ is an invariant measure of the process (η_t, ξ_t), which satisfies:

$$(1.19) \qquad \nu_{\theta_\eta, \theta_\xi} \in \mathcal{E}_{\theta_\eta, \theta_\xi}$$

$$(1.20) \qquad \mathbf{E}[\xi_\infty(g) | \eta_\infty] = \theta_\xi \quad \textit{a.s. for all } g \in G, \ \textit{where } \mathcal{L}[(\xi_\infty, \eta_\infty)] = \nu_{\theta_\eta, \theta_\xi}$$

(b) If $\hat{b}$ is recurrent then the reactant clusters, i.e. with $\underline{0}$ being the configuration with all sites 0*:*

$$\mathcal{L}[(\eta_t, \xi_t)] \underset{t\to\infty}{\Longrightarrow} \nu_{\theta_\eta} \otimes \delta_{\underline{0}}, \tag{1.21}$$

where ν_{θ_η} is the unique extremal equilibrium

$\diamond$

Remark For given $(\eta_t)_{t\in\mathbb{R}}$ the law of the population consisting of the complete family of a reactant particle picked at random from the population at time t at a point $g \in G$ from the equilibrium process $(\eta_t, \xi_t)_{t\in\mathbb{R}}$ can be given explicitly. We give an explicit representation of this canonical Palm distribution of the reactant given the catalyst in Section 2(b), Proposition 2.2.

The other extreme case for the catalyst process will be our next topic, namely we consider *site–transient* catalyst processes. Here locally the catalyst disappears eventually. Therefore the reactant process should survive, at least if reactant particles do not travel too much.

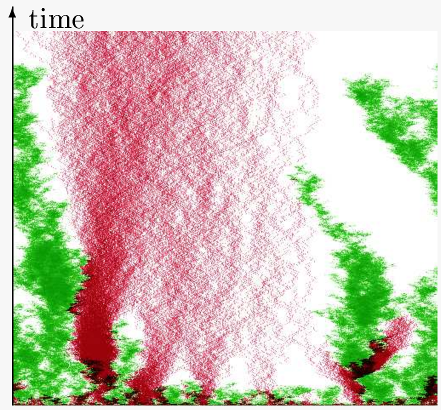

Figure 2 CBRW on the torus $\mathbb{Z}/(500)$. a and b are symmetric simple. The catalyst (green) is the same realization as

However by moving the reactant particles enough w
spots populated with catalysts since the latter have l

To illustrate this phenomenon we consider two cases on $G = \mathbb{Z}^1$, where the catalyst motion relative to the catalyst has zero respectively nonzero drift. In the case of nonzero drift the reactant becomes locally extinct. For technical reasons we can show this in Theorem 2(b) only in the case where $a(x,y) = \delta(x,y)$ is the kernel of the random "walk" that stands still. However the statement should also be true for a symmetric simple random walk, as suggested by the computer simulation of Figure 3.

In the case of zero drift the reactant eventually does not meet catalyst particles anymore and for large time t turns simply into a system of infinitely many inde-

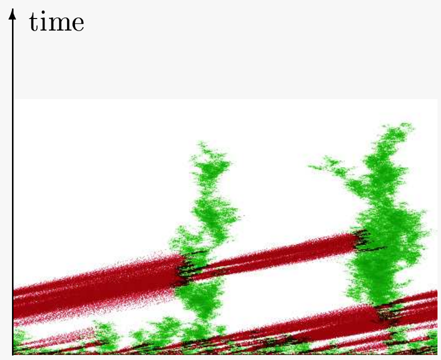

Figure 3 CBRW on the torus $\mathbb{Z}/(500)$. a is symmetric simple. The reactant has a drift to the right and dies out. See Theorem 2 (b).

pendent random walks on G. The latter system has as equilibrium states so called *Poisson–systems* with intensity θ, which we denote by $\mathcal{H}_\theta$. Formally for distinct points $g_i \in G$, $i = 1, \dots, j$, we define:

$$(1.22) \qquad \mathcal{H}_\theta(\{\eta \in \mathbb{N}_0^G : \eta(g_1) = k_1, \dots, \eta(g_j) = k_j\}) = \prod_{i=1}^{j} \left(e^{-\theta} \frac{\theta^{k_i}}{k_i!} \right).$$

Theorem 2 *Choose $G = \mathbb{Z}^1$.*

(a) *Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be transition kernels on $\mathbb{Z}^1$ with the property $\sum a(0,x)x = \sum b(0,x)x = 0$ and $\sum a(0,x)|x|^\alpha < \infty$ and $\sum b(0,x)|x|^\beta < \infty$ for some $\alpha > 2$ and $\beta > 1$. Then for $\mu \in \mathcal{E}_{\theta_\eta,\theta_\xi}$:*

$$(1.23) \qquad \mathcal{L}[(\eta_t, \xi_t)] \underset{t\to\infty}{\Longrightarrow} \delta_{\underline{0}} \otimes \mathcal{H}_{\theta_\xi}.$$

(b) *Let $a(\cdot,\cdot)$ be equal to $\delta(x,y)$ and $b(\cdot,\cdot)$ simple random walk with nonzero drift. Then:*

$$(1.24) \qquad \mathcal{L}[(\eta_t, \xi_t)] \underset{t\to\infty}{\Longrightarrow} \delta_{\underline{0}} \otimes \delta_{\underline{0}}. \qquad \diamond$$

A key ingredient for the proof of Theorem 2 (b) is a continuous time version of a result by Harry Kesten (1995) on the range of branching random walk. Although this statement is a little bit technical we state the proposition here since it might be of some interest on its own. The proof can be found in the appendix.

Proposition 1.3 *Let $(\phi_t)_{t\geq 0}$ be continuous time critical binary branching random walk on $\mathbb{Z}$, starting with one particle at $0 \in \mathbb{Z}$ at time $t = 0$. Assume that the migration kernel a satisfies $\sum_{x\in\mathbb{Z}} a(0,x)x = 0$ and for some $\alpha \geq 7$*

$$(1.25) \qquad \sum_{x\in\mathbb{Z}} a(0,x)|x|^{\alpha+\varepsilon} < \infty, \quad \text{for some } \varepsilon > 0.$$

Then

$$(1.26) \qquad \sup_{T\geq 1} \sup_{z\geq 2} Tz^3 \mathbf{P}\Big[\sup_{0\leq t\leq T} \max(\operatorname{supp} \phi_t) \geq zT^{1/2} \Big] < \infty.$$

Moreover if we assume that (1.25) holds only for $\alpha \geq 4$ then we still get that for every $\varepsilon > 0$

$$(1.27) \qquad \sup_{T\geq 1} \sup_{z\geq 2} \frac{\mathbf{P}[\sup_{0\leq t\leq T} \max(\operatorname{supp} \phi_t) \geq zT^{1/2}]}{z^{-\alpha/2}\, T^{-1} + z^{(1-\alpha)/2}\, T^{(2-\alpha)/4}} < \infty. \qquad \diamond$$

Now we are at the point to discuss the most interesting situation, namely the case of a catalyst which is *weakly site–recurrent*. This can be combined with different types of reactant motions, which can lead to various types of longtime behavior. We can distinguish first the cases of a transient and recurrent symmetrized reactant motion.

In the transient regime we will find no new phenomenon, but again we see pure Poisson states as limiting states. Precisely, start the system (η_t, ξ_t) in a Poisson configuration with intensity $(\theta_\eta, \theta_\xi)$ then:

$$(1.28) \qquad \mathcal{L}[(\eta_t, \xi_t)] \underset{t\to\infty}{\Longrightarrow} \delta_{\underline{0}} \otimes \mathcal{H}_{\theta_\xi}.$$

The proof of an even somewhat stronger result (Proposition 2.4) is given in Section 2(c).

We will focus here on the case of recurrent symmetrized reactant motion. Here a very interesting and new type (different from the homogeneous case) of longtime behavior occurs. We find *mixed* Poisson systems as limiting states. This occurs when catalyst and reactant have the same mobility. An analogous statement for super processes in $d = 2$ has recently been proved by Fleischmann and Klenke (98).

In this regime the point is the following. The observer sitting at site 0 sees catalyst clusters of increasing height passing by at random time points, which exhibit larger and larger distances. The question is: Do these catalyst clusters contain enough particles and are they met frequently enough by the reactant to cause local extinction? Or are too few catalyst clusters arriving so that eventually a Poisson system develops? Or do we see something intermediate?

Interestingly enough a situation occurs with sufficiently regular motion kernel, where at time T no catalyst is present in a spatial volume of order T and time window $\gamma T \quad (1 > \gamma > 0)$ back. Thus the spatial motion of the reactant can prevent extinction but is not strong enough to average out completely the randomness inherent in the time space catalyst process. Hence a mixed Poisson law of the reactant appears. We describe this phenomenon now in the following situation precisely.

Let $G = \mathbb{Z}^2$ and recall that we have critical binary branching and as underlying motions symmetric simple random walk. The goal will be to show that in $d = 2$ the reactant process is persistent and that in fact the limit law is a mixed Poisson random field $\mathbf{E}[\mathcal{H}_\zeta]$. I.e., it is a Poisson random field with *random intensity*, which has the special form

$$(1.29) \qquad \zeta\ell,$$

with ℓ counting measure on $\mathbb{Z}^2$ and where ζ is a random variable with values in $\mathbb{R}^+$. (Note that a system of pure random random walks on $\mathbb{Z}^d$ would have a Poisson random field as extremal invariant state. Hence ζ is all what is left in the longtime limit from the randomness created by the branching of the reactant.) What can we say about ζ?

This random variable should reflect the catalytic medium as experienced through time by a typical reactant particle, that is the moments of ζ should be given via a space–time average of the catalyst, with averaging weights reflecting the properties of the random walk kernel of the reactant. In order to make this precise the basic idea is to use an "invariance principle" and look at the diffusion limit, which is a special measure valued diffusion on $\mathbb{R}^d$, namely a super process in a random medium. We are going to explain this object now.

Let $(\varrho_t)_{t\geq 0}$ be super Brownian motion on $\mathbb{R}^2$ with initial state $\theta_\eta \cdot \ell$ (recall that this is the small mass fast branching continuum space diffusion limit of a branching random walk). Here ℓ is the Lebesgue measure on $\mathbb{R}^2$.

It is possible to construct super Brownian motion X^ϱ in the catalytic medium ϱ (Dawson and Fleischmann [DF5]). Here the branching rate of the reactant process X^ϱ is given by the additive functional of the collision local time of Brownian motion with (the catalyst) ϱ which is itself super Brownian motion. (For a treatment of collision local times see Barlow et al (92).) We will show in Proposition 1.4 that in fact this process X^ϱ occurs as the space–time rescaled limit of our catalytic branching random walk.

The process X^ϱ has the property that for initial condition for catalyst respectively reactant given by

$$(1.30)\qquad \varrho_0 = \theta_\eta \cdot \ell \quad \text{and} \quad X_0^\varrho = \theta_\xi \cdot \ell \ \text{ a.s.,}$$

for every $t > 0$, the reactant $X_t^\varrho \in \mathcal{M}(\mathbb{R}^2)$ ($:=$ the set of σ-finite Borel measures on $\mathbb{R}^2$) has a density (Fleischmann and Klenke (98)),Theorem 1, henceforth referred to as [FK]).

Let ζ denote the density of X_1^ϱ at the origin that is

$$(1.31)\qquad \zeta = \frac{X_1^\varrho(dx)}{dx}\Big|_{x=0}.$$

This definition makes sense for almost all ϱ. The $\mathbb{R}^+$-valued random variable ζ can be represented as the limit

$$(1.32)\qquad \zeta = \lim_{\delta\to 0}(p_\delta^B * X_{1-\delta}^\varrho)(0),$$

where p_t^B denotes the heat kernel. This relation makes precise the idea that ζ arises via a space–time average of the catalyst based on a "small" time window, a property which will be important below.

Furthermore the random density ζ has for given catalyst ϱ full expectation and finite variance:

$$(1.33)\qquad \mathbf{E}[\zeta|\varrho] = \theta_\xi \quad \text{a.s.}$$

$$(1.34)\qquad \mathbf{Var}[\zeta|\varrho] = 2\theta_\xi \int_0^1 dt \int_{\mathbb{R}^2} \varrho_t(dx) p_{1-t}^B(0,x)^2 \ < \infty \quad \text{a.s.}$$

This implies in particular (by a well known time - space - mass scaling property of X^ϱ) that X^ϱ is persistent and that X_t^ϱ converges to $\zeta\ell$ ([FK], Corollary 2):

$$\mathcal{L}[\mathcal{L}[X_t^\varrho|\varrho]] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}[\mathcal{L}[\zeta\ell|\varrho]].$$

In the case of branching random walk we can now prove:

Theorem 3 *Choose $G = \mathbb{Z}^2$. Assume that $\sum_{x\in\mathbb{Z}^2} a_1(0,x)x = \sum_{x\in\mathbb{Z}^2} b_1(0,x)x = 0$, that the second moments exists and that under $a_1(0,x)$ (respectively $b_1(0,x)$) the coordinates of $x = (x_1,x_2)$ are uncorrelated with variance 1. Further assume that*

$$(1.35)\qquad \sum_{x\in\mathbb{Z}^2} a(0,x)|x|^\alpha < \infty \qquad \textit{for some } \alpha > 6.$$

Then the reactant component of the process (η_t, ξ_t) converges as $t \to \infty$ to a mixed Poisson system:

$$(1.36) \qquad \mathcal{L}[\mathcal{L}[\xi_t|\eta]] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}[\mathbf{E}[\mathcal{H}_\zeta|\varrho]].$$

Here $(\varrho_t,\ X_t^\varrho)$ is the superprocess model defined in (1.30) and ζ is the density of X_1^ϱ (see (1.31)). $\diamond$

Remark The assumptions are fulfilled, e.g., if a and b are the kernels of symmetric simple random walk and if $c_1 = c_3 = 2$ (recall (1.7)).

The strategy of the proof follows the above described heuristics. We show

- The properly rescaled catalyst and reactant converge to the super–catalyst and super–reactant (Proposition 1.4).
- The catalyst η has time–space gaps on a macroscopic scale in which the fluctuations from the branching of the reactant can be sufficiently homogenized by the migration (Proposition 1.5).

Here are the formal statements:

Proposition 1.4 (Diffusion limit) *Let X^ϱ be super Brownian motion in $\mathbb{R}^2$ in the catalytic medium ϱ and let (a_t) and (b_t) fulfill only the first and second moment assumption of Theorem 3. Furthermore let the initial conditions of η, ξ, ϱ, and X^ϱ be as in Theorem 3.*
(a) *Then the catalyst converges to super–Brownian motion:*

$$(1.37) \qquad \mathcal{L}\left[\left(T^{-1}\eta_{sT}(T^{1/2}\ \cdot)\right)_{s\geq 0}\right] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}\left[(\varrho_s)_{s\geq 0}\right].$$

(b) *Furthermore for any fixed $s > 0$,*

$$(1.38) \qquad \mathcal{L}[\mathcal{L}[T^{-1}\xi_{sT}(T^{1/2}\cdot)|\eta]] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}[\mathcal{L}[X_s^\varrho|\varrho]]. \qquad \diamond$$

For the next Proposition we need the higher moment assumption (1.35) of Theorem 3.

Proposition 1.5 (Empty time–space cylinder) *Let $(\eta_t)_{t\geq 0}$ be the catalyst process on $\mathbb{Z}^2$ where (a_t) fulfills the assumptions of Theorem 3. For $\varepsilon > 0$ there exist $\delta > 0$ and $r > 0$ such that (with $B(R)$ denoting the box with radius R)*

$$(1.39) \qquad \limsup_{T\to\infty} \mathbf{P}_{\mathcal{H}_1}\left[\sup_{(1-\delta)T\leq s\leq T} \eta_s(B(rT^{1/2})) > 0\right] < \varepsilon. \qquad \diamond$$

Since we talk about weak convergence and conditional distributions of measures on $\mathbb{R}^2$ in (1.38) we should mention that $\mathcal{M}(\mathbb{R}^2)$, equipped with the vague topology is a Polish space and so the standard theory applies to $\mathcal{M}(\mathbb{R}^2)$. See Kallenberg (83). Note that for $\mathbb{N}_0^G = \mathcal{N}(G) \subset \mathcal{M}(G)$ the vague topology coincides with the product topology that we have used in the previous statements.

If we are only interested in showing *persistence* of the reactant we can get this from Proposition 1.5 alone; Proposition 1.4 is only needed for the more detailed information on the limit law.

Corollary 1.6 *Consider the situation of Theorem 3, then (1.39) implies that the reactant (ξ_t) is persistent. In fact, for any weak limit point $\mathcal{L}[\mathcal{L}[\xi_\infty|\eta]]$ of $\mathcal{L}[\mathcal{L}[\xi_t|\eta]]$ we have*

$$\mathbf{P}[\mathbf{E}[\xi_\infty(0)|\eta] = \theta_\xi] = 1. \qquad \diamond \tag{1.40}$$

Outline The rest of the paper is organized as follows. In Section 2 we collect a number of tools for branching processes in random medium, among which is the appropriate version of Kallenberg's backward tree and Kallenberg's persistence–extinction criterion. Then we proceed in three sections each proving one of the three theorems in order of their appearance. In the appendix we show Proposition 1.3

2 Preliminaries

In this section we provide first of all in Subsection 2(a) some tools but then in Subsections 2(b) - 2(d) we explain and adapt the idea of Palm distributions, canonical measures and the Kallenberg backward tree criterion for persistence/extinction.

(a) Laplace functionals, moments and Poisson convergence

Throughout the proofs we will frequently make use of explicit calculations based on Laplace transforms, moments and convergence to Poisson systems due to migration. In this paragraph we collect the relevant facts.

Let $(Y_t)_{t\geq 0}$ be the random walk with transition probabilities given by $b_t(\cdot,\cdot)$ (recall (1.7)) and let $\overline{\kappa}$ be a (possibly random) additive functional of (Y_t) which is absolutely continuous w.r.t. Lebesgue measure. Let $Q(z) = \sum_{k=1}^\infty q_k z^k$ be the p.g.f. of (q_k). Recall that $\mathcal{N}(G)$ denotes the (nonnegative) integer valued measures on G.

Lemma 2.1 (Laplace functional)
There exists a unique multiplicative strong Markov process $(\zeta_t)_{t\geq 0}$ with values in $\mathcal{N}(G)$ starting in δ_g at time r whose Laplace functional (for $\varphi : G \to [0,\infty)$)

$$w_{r,t}^{\overline{\kappa}}(g) = \mathbf{E}_{r,\delta_g}^{\overline{\kappa}}[\exp\{-\langle \zeta_t, \varphi\rangle\}], \tag{2.1}$$

satisfies

$$w_{r,t}^{\overline{\kappa}}(g) = \mathbf{E}_{r,g}^{\overline{\kappa}}\left[e^{-\varphi(Y_t)}e^{-\overline{\kappa}((r,t))} + \int_r^t e^{-\overline{\kappa}((r,s))}Q(w_{s,t}^{\overline{\kappa}}(Y_s))\overline{\kappa}(ds)\right]. \tag{2.2}$$

Here $\mathbf{E}^{\overline{\kappa}}_{r,\delta_g}$ *(respectively* $\mathbf{E}^{\overline{\kappa}}_{r,g}$*) refers to the process* $(\zeta_t)_{t\geq r}$ *(respectively* $(Y_t)_{t\geq r}$*) started with one particle at* $g \in G$. $\diamond$

Proof See Dawson (93), Lemma 4.4.2 and the references given there. $\square$

Definition 2.1 *We refer to* $(\zeta_t)_{t\geq 0}$ *as the* $(\overline{\kappa}, Q, b)$ branching process. $\diamond$

Lemma 2.2 *Let* $(\eta_t)_{t\geq 0}$ *be a branching random walk. Let* $(Y_t)_{t\geq 0}$ *be the random walk with transition rate* b. *Choose* $\overline{\kappa}(dt) = \eta_t(Y_t)dt$ *as the additive functional. Consider the* $(\bar{\kappa}, Q, b)$*-branching process* (ζ_t). *Then* $(\eta_t, \zeta_t)_{t\geq 0}$ *is the catalyst–reactant process (CRP) introduced in (1.12).* $\diamond$

Proof It suffices to show that the CRP $(\eta_t, \xi_t)_{t\geq 0}$ solves (2.1) and (2.2). Define the conditional Laplace transforms $w^\eta_{r,t} = \mathbf{E}_{r,\delta_g}[\exp(-\langle \xi_t, \varphi\rangle)|\eta]$ and let $\mathcal{B}$ the generator of b_t. By construction $w^\eta_{r,t}(g)$ solves the backward equation

$$(2.3)\qquad -\frac{d}{dr} w^\eta_{r,t}(g) = \eta_r(g) Q(w^\eta_{r,t}(g)) - \eta_r(g) w^\eta_{r,t}(g) + \mathcal{B} w^\eta_{r,t}(g).$$

This equation has a unique solution with Feynman–Kac representation

$$(2.4)\qquad \begin{aligned} w^\eta_{r,t}(g) \;=\; \mathbf{E}^\eta_{r,g}\Big[& e^{-\varphi(Y_t)} \exp\{-\int_r^t \eta_s(Y_s)ds\} \\ & + \int_r^t \eta_s(Y_s) Q(w^\eta_{s,t}(Y_s)ds) \exp\{-\int_r^s \eta(Y_u)du\}\Big]. \end{aligned} \quad \square$$

Next we use this representation of the Laplace transform of a branching random walk in random medium to compare the probabilities of local extinction of two such processes associated with two media described by the two additive functionals $\bar{\kappa}_1$ and $\bar{\kappa}_2$, which satisfy:

$$(2.5)\qquad \bar{\kappa}_1 \geq \bar{\kappa}_2.$$

We denote the two resulting processes by $(\zeta^1_t)_{t\geq 0}$ and $(\zeta^2_t)_{t\geq 0}$ and the corresponding Laplace functionals by $w^i_{r,t}$, $i = 1, 2$. It is assumed that $\mathcal{L}[\zeta^1_0] = \mathcal{L}[\zeta^2_0]$. Then the following comparison result holds:

Lemma 2.3 (comparison) *Assume that (2.5) holds. Then the following is true.*

$$(2.6)\qquad w^1_{r,t} \geq w^2_{r,t}, \qquad t \geq r.$$

$$(2.7)\qquad \mathbf{P}[\zeta^1_t(A) = 0] \geq \mathbf{P}[\zeta^2_t(A) = 0], \quad A \subseteq G,\ t \geq 0. \qquad \diamond$$

Proof Define

$$(2.8)\qquad v_{r,t}(g) = w^1_{r,t}(g) - w^2_{r,t}(g), \quad g \in G,\ t \geq r.$$

Then $\{v_{r,t}(g), g \in G\}$ satisfy the following system of differential equations (recall $\mathcal{B}$ is the generator of b_t):

$$(2.9)\qquad -\frac{d}{dr} v_{r,t}(g) = -k_{r,t}(g) v_{r,t}(g) + (\mathcal{B} v_{r,t})(g) + h_{r,t}(g), \quad \forall g \in G$$

where (recall Q is the generating function of the offspring distribution (q_k)):

$$(2.10)\qquad \begin{aligned} k_{r,t}(g) &= \frac{\bar{\kappa}_2(dr)}{dr}\left(1 - \frac{Q(w^1_{r,t}(g)) - Q(w^2_{r,t}(g))}{v_{r,t}(g)}\right) \geq 0 \\ h_{r,t}(g) &= \frac{(\bar{\kappa}_1 - \bar{\kappa}_2)(dr)}{dr}\Big(Q(w^2_{r,t}(g) - w^1_{r,t}(g))\Big) \geq 0. \end{aligned}$$

Since the system of differential equations (2.9) has a unique solution, by the standard theory for differential equations in Banach spaces, here $L_\infty(G)$, the solution has the following Feynman–Kac representation (Y_t is the random walk generated by $\mathcal{B}$)

$$(2.11)\qquad v_{r,t}(g) = \mathbf{E}_{r,x}\left[\int_r^t h_{s,t}(Y_s) \exp\left\{-\int_r^s k_{u,t}(Y_u) du\right\} ds\right] \geq 0.$$

Manifestly we have $v_{r,t}(g) \geq 0$ proving (2.6). For the relation (2.7) recall that if we choose in the Laplace functional $\varphi = \lambda 1_A$ then we obtain the extinction probabilities in the limit $\lambda \to \infty$. This completes the proof. □

Now we will give formulas for the first and second moments of ξ_t conditioned on $(\eta_t)_{t\geq 0}$ in the case of critical binary branching of ξ_t, i.e. for $q_0 = q_2 = \frac{1}{2}$.

Lemma 2.4 (Moments) *For $f : G \to [0, \infty)$ bounded we have*

$$\begin{aligned} (2.12)\qquad \mathbf{E}_{0,\delta_g}[\langle \xi_t, f\rangle | \eta] &= (b_t f)(g) \qquad a.s. \\ (2.13)\qquad \mathbf{Var}_{0,\delta_g}[\langle \xi_t, f\rangle | \eta] &= b_t(f^2)(g) - (b_t f)^2(g) + \int ds \sum_{g\in G} b_s(g, g')\eta_s(\{g'\})(b_{t-s} f)^2(g'). \end{aligned}$$ ◇

Proof (2.12) follows from the criticality of the branching mechanism of ξ_t. (2.13) can be obtained by solving the backward equation for $\mathbf{E}_{r,\delta_g}[\langle \xi_t, f\rangle^2 | \eta]$. We omit the details. □

We will need the asymptotics of the probability of survival of the Galton Watson process $(Z_t)_{t\geq 0}$ with offspring distribution (q_k) and with $Z_0 = 1$ a.s.

Lemma 2.5 *If* (q_k) *has a finite variance* σ^2*, then*

$$\lim_{t\to\infty} t\mathbf{P}[Z_t > 0] = \frac{2}{\sigma^2}. \tag{2.14}$$

Furthermore in this case the rescaled process converges to an exponential mean $\sigma^2/2$ *random variable*

$$\mathcal{L}[Z_t/t \mid Z_t > 0] \underset{t\to\infty}{\Longrightarrow} \exp\left(\frac{\sigma^2}{2}\right). \tag{2.15}$$

In the special case $q_0 = q_2 = \frac{1}{2}$ *we have the equality*

$$\mathbf{P}[Z_t > 0] = \frac{2}{2+t},\ t \geq 0. \qquad \diamond \tag{2.16}$$

Proof See Athreya and Ney (72). $\square$

We come to a lemma that states convergence towards a mixed Poisson field of certain random populations.

Let $G = \mathbb{Z}^d$ or $G = \mathbb{R}^d$ and ℓ be the normed Haar measure on G (i.e., counting measure on $\mathbb{Z}^d$ or Lebesgue measure on $\mathbb{R}^d$). Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of random populations of particles on G (i.e., $\mathbb{N}_0$-valued random measures on G) and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of stochastic kernels on G. We write $\Phi_n \circ K_n$ for the random population obtained from Φ_n by independent displacements of the particles according to K_n. Further we write $\Phi_n K_n$ for the intensity measure of $\Phi_n \circ K_n$ given Φ_n, i.e., with $\mathcal{B}(G)$ denoting Borel sets

$$(\Phi_n K_n)(A) = \int \Phi_n(dg')K_n(g', A), \qquad A \in \mathcal{B}(G). \tag{2.17}$$

The followig proposition is well–known; we include a proof for the reader's convenience

Proposition 2.1 (Poisson convergence) *Assume that there is a nonnegative random variable* ζ *such that*

$$\mathcal{L}[\Phi_n K_n] \underset{n\to\infty}{\Longrightarrow} \mathcal{L}[\zeta\, \ell]. \tag{2.18}$$

Further assume that for compact $C \subset G$

$$\alpha_n(C) := \sup\{K_n(g, C),\ g \in G\} \underset{n\to\infty}{\longrightarrow} 0. \tag{2.19}$$

Then $\Phi_n \circ K_n$ *converges in distribution towards a mixed Poisson field with random intensity* ζ*:*

$$\mathcal{L}[\Phi_n \circ K_n] \underset{n\to\infty}{\Longrightarrow} \mathbf{E}[\mathcal{H}_\zeta]. \qquad \diamond \tag{2.20}$$

Proof Recall that $\mathcal{N}(G)$ carries the vague topology. Hence we have to show that for $\varphi \in C_c^+(G)$

$$\text{(2.21)} \qquad \mathbf{E}[\exp(-\langle \Phi_n \circ K_n, \varphi \rangle)] \underset{n\to\infty}{\longrightarrow} \mathbf{E}[\exp(-\zeta \langle \ell, 1-e^{-\varphi}\rangle)].$$

Note that if $\alpha_n(\mathrm{supp}(\varphi)) \leq 1/2$, hence for n large enough:

$$\begin{aligned} \text{(2.22)} \qquad & \left| \log\left[1 - \int K_n(g,dg')(1-e^{-\varphi(g')})\right] + \int K_n(g,dg')(1-e^{-\varphi(g')}) \right| \\ & \leq \left(\int K_n(g,dg')(1-e^{-\varphi(g')}) \right)^2 \\ & \leq \alpha_n(\mathrm{supp}(\varphi)) \int K_n(g,dg')(1-e^{-\varphi(g')}). \end{aligned}$$

Thus using the independence of the displacements and (2.22)

$$\begin{aligned} \text{(2.23)} \qquad \mathbf{E}\left[\exp(-\langle \Phi_n \circ K_n, \varphi\rangle)\right] &= \mathbf{E}\left[\exp\left(\left(\int \Phi_n(dg) \log\left(1 - \textstyle\int K_n(g,dg')[1-e^{-\varphi(g')}]\right)\right)\right)\right] \\ &= \mathbf{E}[\exp(-\langle \Phi_n K_n, 1-e^{-\varphi}\rangle) \cdot \exp(\beta_n)], \end{aligned}$$

where $|\beta_n| \leq \langle \Phi_n K_n, 1-e^{-\varphi}\rangle \cdot \alpha_n(\mathrm{supp}(\varphi))$. Then by assumption (2.18) and (2.19), $\beta_n \to 0$ stochastically as $n \to \infty$. We conclude (with (2.18)) that

$$\begin{aligned} \text{(2.24)} \qquad \lim_{n\to\infty} \mathbf{E}[\exp(-\langle \Phi_n \circ K_n, \varphi\rangle)] &= \lim_{n\to\infty} \mathbf{E}[\exp(-\langle \Phi_n K_n, 1-e^{-\varphi}\rangle)] \\ &= \mathbf{E}[\exp(-\zeta\langle \ell, 1-e^{-\varphi}\rangle)]. \end{aligned}$$

The r.h.s. above is for fixed ζ the Laplace transform of a Poisson system. This completes the proof. □

(b) The canonical Palm distributions of the reactant process

In this paragraph we present the tools arising from the infinite divisibility of $\mathcal{L}[\xi_t|\eta]$, namely *canonical measures* and *Palm distribution.*

Assume that the process $(\eta_t, \xi_t)_{t\geq 0}$ has as initial distribution (where ν_θ denotes the unique extremal intensity θ-equilibrium of $(\eta_t)_{t\geq 0}$):

$$\text{(2.25)} \qquad \mu = \nu_{\theta_\eta} \otimes \mathcal{H}_{\theta_\xi} \in \mathcal{E}_{\theta_\eta,\theta_\xi}.$$

This means that the catalyst starts in an ergodic measure with intensity θ_η which is an equilibrium and the reactant process is a Poisson system independent of the catalyst. In particular the law $\mu(\cdot|\eta)$ is *infinitely divisible* and has a canonical measure. Also the law $\mathcal{L}[\xi_t|(\eta_t)_{s\leq t}]$ is infinitely divisible and hence has a canonical measure. In other words for every realization of $(\eta_t)_{t\geq 0}$ there exists a σ-finite measure Q_t^η on $(\mathbb{N}_0)^G$ such that

$$\text{(2.26)} \qquad \mathbf{E}\left[e^{-\langle \xi_t, \varphi\rangle} | (\eta_s)_{s\leq t}\right] = \exp\left(-\int_{\mathbb{N}_0^G} (1-e^{-\langle \chi,\varphi\rangle}) Q_t^\eta(d\chi)\right),$$

where φ is a function $G \to \mathbb{R}^+$ which is 0 except at finitely many points. The measure Q_t^η describes the intensity of configurations of particles descending from the same ancestor. We view ξ_t below as a random measure on G rather than a configuration in $(\mathbb{N}_0)^G$.

Next recall the concept of Palm distributions of measures on the configuration space. Let $R \in \mathcal{M}(\mathcal{M}(G))$ and assume that R has locally finite intensity I, i.e.

$$(2.27) \qquad I(g) := \int \chi(\{g\}) R(d\chi) < \infty, \; g \in G.$$

Define the Campbell measure on $G \times \mathcal{M}(G)$:

$$(2.28) \qquad \bar{R}(A \times B) = \int \chi(A) \; 1_B(\chi) R(d\chi).$$

Then define:

$$(2.29) \qquad R_g(B) = \frac{\bar{R}(dg \times B)}{I(dg)}.$$

The family $\{R_g, \; g \in G\}$ is called the *Palm distribution* associated with R.

In other words for every g with $I(g) > 0$ the Palm distribution R_g arises through a re–weighting of the form of a local *size biasing* of R:

$$(2.30) \qquad R_g(B) = \frac{\int \chi(\{g\}) 1_B(\chi) R(d\chi)}{I(g)}, \quad B \subseteq \mathcal{M}(G).$$

Consider the situation when R is the canonical measure of an infinitely divisible law $\mathbf{P}$ on $\mathcal{M}(G)$. In this case (with a slight abuse of language) we make the following definition:

Definition 2.2 *The family* $(R_g, \; g \in G)$ *is called the family of* canonical Palm distributions *of* $\mathbf{P}$. $\diamond$

The best way to think of R_g is in terms of a two step sampling. Consider a Poisson system of configurations (ζ_i) with intensity measure R. Write $\zeta = \sum_i \zeta_i$ for its superposition. We first make a size biased sampling of ζ, that is we choose $d\zeta$ with probability $(\zeta(g)/I(g))\mathbf{P}[d\zeta]$. Given ζ we choose the configuration ζ_i with probability $\zeta_i(g)/\zeta(g)$. The law of the sampled configuration is R_g.

We now develop the tools that allows us later to analyze the infinitely divisible distributions $\mathcal{L}[\xi_t|\eta]$ in terms of Palm distribution and canonical Palm distributions.

We begin by constructing the canonical Palm distribution of $(\xi_t)_{t \geq 0}$ for given catalytic medium $(\eta_t)_{t \geq 0}$ where we use the initial law given in (2.25) for $(\eta_t, \xi_t)_{t \geq 0}$. We identify the canonical Palm distribution by giving an explicit representation of a realization, the so called *Kallenberg backward tree*. We need some ingredients.

For this purpose return to the branching random walk scenario of part (a) of this section for the additive functional κ given by the catalyst $(\eta_t)_{t\geq 0}$. Recall (2.26).

Let $(\bar{Y}_t^g)_{t\geq 0}$ be a random walk on G with transition kernel $\bar{b}$ $\quad(\bar{b}(g,g') := b(g',g))$ starting in g at time 0. Furthermore let

$$(2.31)\qquad \{(\xi_t^{g,s})_{t\geq s},\ g\in G,\ s\in \mathbb{R}^+\}$$

be a collection of independent branching random walks in catalytic medium with transition kernel b, starting in point g at time s with one particle. Next we consider a Poisson random measure $\nu^{g,T}$ on $G\times[0,T]$ with intensity measure

$$(2.32)\qquad \delta_{\bar{Y}^g_{T-s}}(dg')\eta_s(g')ds.$$

The ingredients introduced above allow to define the branching population

$$(2.33)\qquad \widehat{\xi}_T = \delta_g + \int\limits_{G\times[0,T]} \nu^{g,T}(dg',ds)\xi_T^{g',s}$$

which is embedded into a *backward tree.*

"Ego" is sampled at position g, which is the starting point of the ancestral line $\bar{Y}^g$ going back from time T into the past. The Poisson random measure ν marks the space–time points along the side–trees $\xi^{g',s}$.

As we will see in Proposition 2.2 (a) equation (2.33) provides a representation of the canonical Palm distribution. For a time discrete setting, this goes back to Kallenberg [K1]. For the time–continuous, time–homogeneous case a proof is given in [GRW]. We will outline the proof here both in order to be self–contained and to cope with the time–inhomogeneous situation we are interested in.

Later on, we will need that (2.33) provides a stochastic minorant for the Palm distribution of a system started off in a general initial distribution with homogeneous intensity. This assertion which we state in Proposition 2.3 follows rather directly from from Proposition 2.2 (a) together with the backward formula 1.9.1 in [LMW]; we will give a direct proof embedded into a general elementary argument.

Proposition 2.2 (a) (Backward tree representation, canonical Palm distribution)
A representation of the Palm distribution of the canonical measure Q_T^η corresponding to the point $g\in G$ is given for every given catalyst η by:

$$(2.34)\qquad (Q_T^\eta)_g = \mathcal{L}\left[\hat{\xi}_T|(\eta_t)_{t\in[0,T]}\right]. \qquad\qquad \diamond$$

Remark Recall that this realization describes for given catalyst the law of the cluster of a randomly picked reactant particle at time T at site g.

We can also describe the Palm distribution of the law of the reactant process starting with one particle in a fixed position. Consider the random walk bridge

$$(2.35)\qquad (Y_s^{(h,t),(g,T)})_{t\leq s\leq T},$$

which is obtained by conditioning Y to start at time t in h and to arrive at time T in g. Then:

Proposition 2.2 (b) (Backward tree representation, fixed ancestor) *The Palm distributions of the reactant process $\xi_T^{h,t}$ in the point g is given by the formula*

$$(2.36)\qquad (\mathcal{L}[\xi_T^{h,t}]|\eta)_g = \mathcal{L}\left[\delta_g + \int\limits_{G\times[t,T]} \nu^{(h,t),(g,T)}(dg',ds)\xi_T^{g',s}\Big|\eta\right],$$

where $\nu^{(h,t),(g,T)}(dg',ds)$ is a Poisson measure on $G\times[t,T]$ with random intensity measure

$$(2.37)\qquad \delta_{Y_s^{(h,t),(g,T)}}(dg')\eta_s(g')ds. \qquad \diamond$$

Proof of Proposition 2.2 The proof follows the argument in [GRW] in Section 2 of that paper. The only additional complication faced in the present situation is the *time–inhomogeneity* of the reactant process for given catalyst process. We give the complete argument here.

It is convenient (and simplifies notation) to think of a fixed realization η; let us then write

$$(2.38)\qquad \kappa(g',s) := \eta_s(g'), \quad g'\in G,\ s\geq 0.$$

We start by proving (2.36). For $\varphi\in C_c^+(G), g'\in G$ and $0\leq s\leq T$, define

$$(2.39)\qquad w_{s,T}^\eta(g') := \mathbf{E}\left[\exp\left(-\langle\xi_T^{g',s},\varphi\rangle\right)|\eta\right],$$

$$(2.40)\qquad v_{s,T}^\eta(g') := \mathbf{E}\left[\xi_T^{g',s}(g)\exp\left(-\langle\xi_T^{g',s},\varphi\rangle\right)|\eta\right].$$

Let $\mathcal{B}$ be the generator of the random walk Y. The function v is the solution of the following equation:

$$(2.41)\qquad \begin{aligned} \frac{\partial v_{s,T}^\eta(g')}{\partial s} &= -\mathcal{B}v_{s,T}^\eta(g') + \kappa(g',s)(1-w_{s,T}^\eta(g'))v_{s,T}^\eta(g'), \quad s\leq T,\\ v_{T,T}^\eta(g') &= 1_{\{g\}}(g')\, e^{-\varphi(g')}. \end{aligned}$$

Thus, by the Feynman–Kac formula we have

$$(2.42)\qquad \begin{aligned} v_{t,T}^\eta(h) &= b_{T-t}(h,g)e^{-\varphi(g)}\mathbf{E}\Big[\exp\Big(-\int_t^T \kappa(Y_s^{(h,t),(g,T)},s)(1-w_{s,T}^\eta(Y_s^{(h,0),(g,T)})\,ds\Big)|\eta\Big]\\ &= \mathbf{E}[\xi_T^{h,t}(g)]\mathbf{E}\Big[\exp\Big(-\Big\langle\delta_g+\int_{G\times[0,T]}\nu^{(h,t),(g,T)}(dg',ds)\xi_T^{g',s},\varphi\Big\rangle\Big)|\eta\Big]. \end{aligned}$$

This proves (2.36).

Now we turn to the proof of (2.34). Using (2.40) and (2.42) we have (recall (2.26) and (2.32)):

$$(2.43)\qquad \begin{aligned} \int Q_T^\eta(d\chi)\chi(\{g\})\exp(-\langle\chi,\varphi\rangle) &= \theta_\xi\sum_{h\in G}\mathbf{E}\Big[\xi_T^{h,0}(g)\exp\big(-\langle\xi_T^{h,0},\varphi\rangle\big)|\eta\Big]\\ &= \theta_\xi\sum_{h\in G}v_{0,T}^\eta(h) = \theta_\xi\mathbf{E}\Big[\exp\Big(-\Big\langle\delta_g+\int_{G\times[0,T]}\nu^{g,T}(dg',ds)\,\xi_T^{g',s},\varphi\Big\rangle\Big)|\eta\Big], \end{aligned}$$

where the last equality results from the fact that

$$(2.44) \qquad \mathcal{L}[(\bar{Y}^g_{T-s})_{0\le s\le T}] = \sum_{h\in G} b_T(h,g)\mathcal{L}\left[(Y_s^{(h,0),(g,T)})_{0\le s\le T}\right].$$

The proof of (2.34) is completed by observing that $\int Q^\eta_T(d\chi)\chi(\{g\}) = \theta_\xi$. $\square$

Finally we use the representations above to give a stochastic minorant for the *size–biased distributions* $(\mathcal{L}[\xi_T|\eta])g$ if we start the reactant system in a quite *general initial distribution.* Namely

Proposition 2.3 (Comparison for Palm distribution)
Take as initial distribution for the catalyst–reactant system any law satisfying:

$$(2.45) \qquad \mathbf{E}[\xi_0(g)] = \theta_\xi \in (0,\infty), \quad g\in G.$$

Then (recall (2.32) for ν) in the sense of stochastic ordering

$$(2.46) \qquad (\mathcal{L}[\xi_T|\eta])_g \ge \mathcal{L}\left[\int_{G\times[0,T]} \nu^{g,T}(dg',ds)\xi_T^{g',s}\Big|\eta\right]. \qquad \diamond$$

Proof We will derive this from Proposition 2.2 (b) and the following elementary fact:
Let I be a countable set and $\{X_i,\ i\in I\}$ a family of $\mathcal{M}(G)$-valued random variables with means

$$(2.47) \qquad e_i = \mathbf{E}[X_i(g)] \in [0,\infty), \qquad \sum_{i\in I} e_i \in (0,\infty).$$

Define

$$(2.48) \qquad X = \sum_{i\in I} X_i,$$

and denote (for those i with $e_i>0$) by X_i^g, X^g realizations of the size–biased (in g) distribution of X_i respectively X. This is $\mathcal{L}[X^g] = (\mathcal{L}[X])_g$ and $\mathcal{L}[X_i^g] = (\mathcal{L}[X_i])_g$. Note that we do not assume independence of either of the families $\{X_i,\ i\in I\}$ and $\{X_i^g,\ i\in I\}$. Furthermore define a random variable R which is independent of the family $\{X_i^g,\ i\in I\}$ by setting

$$(2.49) \qquad \mathbf{P}[R=i] = e_i/\mathbf{E}[X(g)].$$

Then

$$(2.50) \qquad \mathcal{L}[X^g] \ge \mathcal{L}[X_R^g].$$

This is verified by the following explicit calculation. For a test function $\varphi : G \to [0,\infty)$ and $K \geq 0$:

$$
\begin{aligned}
(2.51) \quad \mathbf{P}[\langle X^g, \varphi\rangle \geq K] &= \frac{1}{\mathbf{E}[X(g)]}\mathbf{E}[X(g); \langle X, \varphi\rangle \geq K] = \frac{1}{\mathbf{E}[X(g)]}\sum_{i\in I}\mathbf{E}[X_i(g); \langle X, \varphi\rangle \geq K] \\
&\geq \frac{1}{\mathbf{E}[X(g)]}\sum_{i\in I}\mathbf{E}[X_i(g);\ \langle X_i, \varphi\rangle \geq K] \\
&= \frac{1}{\mathbf{E}[X(g)]}\sum_{i\in I} e_i \mathbf{P}[\langle X_i^g, \varphi\rangle \geq K] \\
&= \mathbf{P}[\langle X_R^g, \varphi\rangle \geq K].
\end{aligned}
$$

We now apply formula (2.50) to the following situation. Let $I = G \times N$. To define the X_i let $\bar{X}_{(h,n)}$ denote independent versions of the reactant process started with one particle in h at time 0 and evaluated at time T. Define

$$(2.52) \qquad X_{(h,n)} = \bar{X}_{(h,n)} 1_{\{\xi_0(h)\geq n\}}.$$

and note that

$$(2.53) \qquad \mathcal{L}[X^g_{(h,n)}] = (\mathcal{L}[\xi_T^{h,0}|\eta])_g.$$

Writing $R = (R_1, R_2)$, i.e. $R_1 = h$ if $R = (h,n)$, we have $\mathbf{P}[R_1 = h] = b_T(h,g)$ and (recall (2.31))

$$(2.54) \qquad \mathcal{L}[X^g_R] = \mathbf{E}\left[\left(\mathcal{L}\left[\xi_T^{(R_1,0)}|R_1,\eta\right]\right)_g |\eta\right].$$

This in turn equals the r.h.s. of (2.46) by Proposition 2.2 (b). On the other hand we have $\mathcal{L}[X] = \mathcal{L}[\xi_T|\eta]$. Hence (2.54) equals also the l.h.s. of (2.46) and we have proved Proposition 2.3. $\square$

(c) A first application of the backward trees

We will now demonstrate the use of the backward tree representation in an easy example, which already gives some interesting information about our model and prove the statement (1.28) in the introduction.

Proposition 2.4 (recurrent catalyst motion, transient reactant motion) *Consider the model (η_t, ξ_t) introduced in Section 1 where the kernel $\hat{a}$ is recurrent and the kernel $\hat{b}$ is transient. Pick $\mathcal{L}[(\eta_0, \xi_0)] = \mu_\eta \otimes \mathcal{H}_{\theta_\xi}$, where $\mu_\eta \in \mathcal{E}_{\theta_\eta}$ (recall (1.2)). Then*

$$(2.55) \qquad \mathcal{L}[(\eta_t, \xi_t)] \underset{t\to\infty}{\Longrightarrow} \delta_{\underline{0}} \otimes \mathcal{H}_{\theta_\xi}. \qquad \diamond$$

Proof The proof proceeds in two steps, in the first step we use the Kallenberg technique to reduce the assertion to a statement on canonical Palm distributions which we prove in Step 2.

Step 1 Let $(R_t^\eta)_g$ denote the Palm distribution of the canonical measure Q_t^η of ξ_t for given catalyst $(\eta_t)_{t\geq 0}$. We will prove in Step 2 that for every $B \subseteq G$ with $|B| < \infty$, there exists a t_0 such that for every $t \geq t_0$:

$$\text{(2.56)} \qquad \mathbf{P}[(R_t^\eta)_g(\{\chi : (\chi - \delta_g)(B) > 0\})] < \varepsilon, \quad \forall g \in B.$$

The above relation implies that

$$\text{(2.57)} \qquad (R_t^\eta)_g\Big|_B \underset{t\to\infty}{\Longrightarrow} \delta_{(\delta_g)} \quad \text{stochastically.}$$

However note that the r.h.s. is the canonical Palm distribution of $\mathcal{H}_{\theta_\xi}$. Since the intensity of ξ_t for given η is θ_ξ for all t we find that the $(R_t^\eta)_g$ converge as $t \to \infty$ to $\delta_{(\delta_g)}$ and the intensities of ξ_t converge to the intensity of $\mathcal{H}_{\theta_\xi}$. It follows from [K1], Lemma 10.8, that $\mathcal{L}[\xi_t|\eta] \underset{t\to\infty}{\Longrightarrow} \mathcal{H}_{\theta_\xi}$ stochastically. Hence

$$\text{(2.58)} \qquad \mathcal{L}[\xi_t] \underset{t\to\infty}{\Longrightarrow} \mathcal{H}_{\theta_\xi}.$$

Since $\hat{a}$ is recurrent we know that

$$\text{(2.59)} \qquad \mathcal{L}[\eta_t] \underset{t\to\infty}{\Longrightarrow} \delta_{\underline{0}}.$$

The combination of the last two relations gives the assertion.

Step 2 We now prove (2.56). Choose T large enough such that

$$\text{(2.60)} \qquad \theta_\eta \cdot \int_T^\infty \hat{b}_s(g, B)ds < \frac{\varepsilon}{2}.$$

Next choose $C \supseteq B$, with $|C| < \infty$ such that for the random walk (Y_t) associated with $\bar{b}$ (recall that $\bar{b}(g, g') = b(g', g)$):

$$\text{(2.61)} \qquad \mathbf{P}[A_1] > 1 - \varepsilon/4, \quad A_1 := \{\bar{Y}_s^g \in C, \quad \forall s \in [0, T]\}.$$

Furthermore we can choose T so large that in addition:

$$\text{(2.62)} \qquad \mathbf{P}[A_2] > 1 - \varepsilon/4, \quad A_2 := \{\eta_{t-T}(C) = 0, \quad \forall s \in [0, T]\}.$$

Using the Kallenberg representation of Proposition 2.2 (a) and Chebyshev's inequality we get

$$\begin{aligned}
\text{(2.63)} \qquad & \mathbf{P}[(R_t^\eta)_g(\{\chi : (\chi - \delta_g)(B) > 0\}) > 0;\ A_1 \cap A_2] \\
& \leq \mathbf{E}[1_{A_1} \int_0^t b_s(\bar{Y}_s^g, B)\eta_{t-s}(\bar{Y}_s^g)ds;\ A_2] \\
& \leq \mathbf{E}\Big[\int_T^t b_s(\bar{Y}_s^g, B)\eta_{t-s}(\bar{Y}_s^g)ds\Big] \leq \frac{\varepsilon}{2}.
\end{aligned}$$

Since $\mathbf{P}[A_1^c \cup A_2^c] < \varepsilon/2$ by construction we obtain the assertion (2.56). □

(d) Kallenberg criterion

For spatial critical branching processes there exists a well known criterion for persistence versus local extinction, the so called Kallenberg criterion based on the explicit representation of the canonical Palm distribution (compare Section 2(b)). That is, the question whether a spatial branching process goes to local extinction or is persistent can be answered by deciding whether the pedigree of a sampled individual is locally clumping as $t \to \infty$ or not.

This idea is due to Kallenberg [K1], who treated time discrete homogeneous models. Later this was extended to various classes of time continuous models, see [LW] and the references given there.

Here we generalize the Kallenberg criterion to the case of *branching in random medium.* Namely we establish the Kallenberg criterion for branching random walk in the time inhomogeneous medium. Due to the discrete space we can give a fairly short (and self–contained) proof avoiding the technicalities of continuum models.

The random medium we use is $\kappa(g,-t), t \geq 0, g \in G$. That is if we start the branching random walk at time $-t$ then the additive functional of the random walk $(Y_s)_{-t \leq s \leq 0}$ is given by

$$\overline{\kappa}((-r,-s)) = \int_{-r}^{-s} \kappa(Y_u, u) du$$

(see Lemma 2.1).

Let ξ_s^{-t} be the population at time s of a branching random walk in the (fixed) catalytic medium κ started at time $-t$ in the random state of a spatially homogeneous Poisson field with mean 1. We assume that the motion is given by the random walk Y with transition kernel $b_s(\cdot,\cdot)$. We fix a $g \in G$ and denote by $(\bar{Y}_s^g)_{s \geq 0}$ a random walk with kernel $\bar{b}$ started at time 0 in g.

Proposition 2.5 (Kallenberg criterion)

(a) *There is a random population* $\xi_0^{-\infty}$ *such that* $\mathcal{L}[\xi_0^{-t}] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}[\xi_0^{-\infty}]$.

(b) *Fix* $g \in G$. *Let* $\nu^{g,0}(dg', ds)$ *be a Poisson measure on* $G \times (-\infty, 0]$ *with random intensity* $\kappa(\bar{Y}_{-s}^g, s)\, ds$. *Let* $\{(\xi_t^{g,s})_{t \geq s}, g \in G, s \leq 0\}$ *be an independent family of branching random walks moving with kernel* b *in the medium* κ, *starting at time* s *in* δ_g.

The following conditions are equivalent:

(2.64) (i) $\displaystyle\int_{G\times(-\infty,0]} \nu^{g,0}(dg', ds)\xi_0^{g',s}(g)$ *is stochastically bounded as* $t \to \infty$.

(ii) $$\int_0^\infty b_t(\bar{Y}_t^g, g)\kappa(\bar{Y}_t^g, -t)\, dt < \infty \quad a.s.$$

(iii) $\mathbf{E}[\xi_0^{-\infty}(g)] = 1$ *(i.e.* (ξ_t) *is persistent).*

(iv) $\mathbf{E}[\xi_0^{-\infty}(g)] > 0.$

(c) *Suppose we are in the situation given in (i)-(iv) above. Then the Palm distributions* $(\widetilde{\mathcal{L}[\xi_0^{-\infty}]})_g$ *of the canonical measure of* $\mathcal{L}[\xi_0^{-\infty}]$ *have the representation*

$$(2.65) \qquad (\widetilde{\mathcal{L}[\xi_0^{-\infty}]})_g = \mathcal{L}[\delta_g + \int_{G\times(-\infty,0]} \nu^{g,0}(dg',ds)\xi_0^{g',s}], \quad g \in G. \qquad \diamond$$

Proof We prove each of the statements (a) - (c). in a separate step.

(a) Since $\mathbf{E}[\xi_0^{-t}] = \ell$ for all $t \geq 0$ the family $\mathcal{L}[\xi_0^{-t}], t \geq 0\}$ is tight. Hence it suffices to show convergence of the Laplace transforms. For $f \in C_c^+(G)$ let

$$(2.66) \qquad u(t,g;f) = 1 - \mathbf{E}_{-t,\delta_g}[\exp(-\langle \xi_0, f\rangle)],$$

where $(\xi_s)_{-t\leq s\leq 0}$ is branching random walk in the medium κ and starting time and configuration are encoded as subscript of $\mathbf{E}$. If we can show that $\mathbf{E}[\exp(-\langle \xi_0^{-t}, f\rangle)]$ is decreasing in t, then the limit $\xi_0^{-\infty}$ exists. However, by the branching property and the independence of particles in the initial configuration

$$(2.67) \qquad \mathbf{E}[\exp(-\langle \xi_0^{-t}, f\rangle)] = \exp(-\langle \ell, u(t,\cdot,f)\rangle).$$

We know that u solves the partial differential equation

$$(2.68) \qquad \partial_t u = \mathcal{B}u - \kappa(g,-t)u^2,$$

where $\mathcal{B}$ is the generator of (Y_t). Differentiating the r.h.s. of (2.67) and inserting (2.68) yields a non–positive number and we are done.

(b) The equivalence of (i) with (iii) is immediate from the representation of the canonical Palm distribution in Part 3 of this proposition. We are now going to show the equivalence of (ii), (iii), and (iv). The proof has two parts. Part 1 shows that (ii) $\Longleftrightarrow$ (iii) and Part 2 shows that (iii) $\Longleftrightarrow$ (iv) of which only (iv)$\Rightarrow$(iii) is nontrivial. For transparency each part is broken in steps.

(ii) $\Longleftrightarrow$ (iii)

Step 1 We begin proving an identity between distribution, Palm distribution and canonical Palm distribution. Fix $g \in G$ and define $N_t = \xi_0^{-t}(g)$, $0 \leq t \leq \infty$. Let $\widehat{N_t}$ have the size biased distribution of N_t, i.e. (note that $\mathbf{E}[N_t] = 1$)

$$(2.69) \qquad \mathbf{P}[\widehat{N_t} = n] = n\mathbf{P}[N_t = n].$$

Further let $\widehat{\widetilde{N_t}}$ have the size biased distribution of the canonical measure of N_t. We assume that N_t and $\widehat{\widetilde{N_t}}$ are independent. Then

$$(2.70) \qquad \widehat{N_t} \stackrel{\mathcal{D}}{=} N_t + \widehat{\widetilde{N_t}}.$$

This is easily shown by a small calculation with Laplace transforms. Let Q_t be the canonical measure of $\mathcal{L}[N_t]$. Then by definition for $\lambda \geq 0$

$$(2.71)\qquad \mathbf{E}[e^{-\lambda N_t}] = \exp\left(-\sum_{m=0}^{\infty} Q_t(\{m\})(1-e^{-\lambda m})\right).$$

Differentiating both sides of (2.71) gives (note that $\mathbf{P}[N_t = 0] > 0$, hence Q_t is finite)

$$(2.72)\qquad \mathbf{E}[e^{-\lambda \widehat{N}_t}] = \mathbf{E}[e^{-\lambda N_t}] \sum_{m=0}^{\infty} Q_t(\{m\}) m e^{-\lambda m} \;=\; \mathbf{E}[e^{-\lambda N_t}]\, \mathbf{E}[e^{-\lambda \widehat{\widehat{N}}_t}].$$

Step 2 Here we prove (ii) $\Longrightarrow$ (iii). Now $\mathbf{E}[\xi_0^{-\infty}(g)] = 1$ is equivalent to uniform integrability of $\{N_t,\, t \geq 0\}$ which in turn is equivalent to the tightness of $\mathcal{L}\left[\widehat{N}_t\right]$ since $\mathbf{E}[N_t] = 1$. By (2.70) this is equivalent to the tightness of $\left\{\mathcal{L}\left[\widehat{\widehat{N}}_t\right],\, t \geq 0\right\}$.

In order to check tightness of $\left\{\mathcal{L}\left[\widehat{\widehat{N}}_t\right],\, t \geq 0\right\}$ we recall from (2.34), $\widehat{\widehat{N}}_t$ that

$$(2.73)\qquad \mathcal{L}[\widehat{\widehat{N}}_t] = \mathcal{L}[S_t],$$

where S_t is the following functional of the branching tree

$$(2.74)\qquad S_t = 1 + \int_{G \times (-\infty, 0]} \nu^{g,0}(dg', ds)\, \xi^{g',s}(g).$$

Hence we can define the monotone limit S:

$$(2.75)\qquad S = \lim_{t \to \infty} S_t$$

Note that a.s. finiteness of S is equivalent to tightness of $\{\mathcal{L}[S_t],\, t \geq 0\}$, and hence to persistence of (ξ_0^{-t}).

Then we get (ii)$\Rightarrow$(iii) by the equality

$$(2.76)\qquad \mathbf{E}[S|\bar{Y}^g] = \int_0^{\infty} b_t(\bar{Y}_t^g, g)\kappa(\bar{Y}_t^g, -t)\, dt.$$

Step 3 It remains to show that (iii) implies (ii). We do so by proving that $S < \infty$ implies (ii). Assume that $S < \infty$ a.s. Let $\mathcal{F}_\infty := \bigcap_{t \geq 0} \sigma(\bar{Y}_s^g,\, s \geq t)$ be the terminal σ-field of $(\bar{Y}_t^g)$. Let $\widehat{\xi_0^{g',-t}}(g)$ have the size–biased distribution of $\xi_0^{g',-t}(g)$. Then we have the backward tree representation

$$(2.77)\qquad \mathcal{L}[S_t | (\bar{Y}_s^g)_{s \geq t}] = \mathcal{L}\left[\widehat{\xi_0^{\bar{Y}_t^g,-t}}(g) | \bar{Y}_t^g\right].$$

This follows by using for the l.h.s. (2.74) and for the r.h.s. (2.36).

With this identity we can continue as follows: By the martingale convergence theorem we get:

$$(2.78) \quad \begin{aligned} \limsup_{n\to\infty}\limsup_{t\to\infty}\mathbf{P}\left[\widehat{\xi_0^{\bar Y_t^g,-t}}(g)\geq n|\bar Y^g\right] &= \limsup_{n\to\infty}\limsup_{t\to\infty}\mathbf{P}\left[\widehat{\xi_0^{\bar Y_t^g,-t}}(g)\geq n|\bar Y_t^g\right] \\ &= \limsup_{n\to\infty}\limsup_{t\to\infty}\mathbf{P}\left[S_t\geq n|(\bar Y_s^g)_{s\geq t}\right] \\ &\leq \limsup_{n\to\infty}\limsup_{t\to\infty}\mathbf{P}\left[S\geq n|(\bar Y_s^g)_{s\geq t}\right] \\ &= \limsup_{n\to\infty}\mathbf{P}[S\geq n|\mathcal{F}_\infty]=0 \end{aligned}$$

by the assumption $S<\infty$. Hence $\{\mathcal{L}[\widehat{\xi_0^{\bar Y_{-s}^g,s}}(g)|\bar Y^g],\ s\leq 0\}$ is tight a.s., and this implies the existence of a constant $\gamma=\gamma(\bar Y^g)>0$ such that for $s\leq 0$

$$(2.79) \quad \begin{aligned} \mathbf{P}\left[\xi_0^{\bar Y_{-s}^g,s}(g)>0|\bar Y^g\right] &\geq \gamma\mathbf{E}\left[\xi_0^{\bar Y_{-s}^g,s}(g)|\bar Y^g\right] \\ &= \gamma b_{-s}(\bar Y_{-s}^g,g). \end{aligned}$$

Next we note

$$(2.80) \quad S\geq\int_{G\times(-\infty,0]}\nu^{g,0}(dg',ds)1(\xi_0^{g',s}(g)>0).$$

Hence S can be bounded below by a Poisson random variable with mean

$$(2.81) \quad \gamma\int_0^\infty b_t(\bar Y_t^g,g)\kappa(\bar Y_t^g,-t)dt,$$

which is finite iff this expression is finite. In other words $S<\infty$ a.s. implies (i) and we are done.

(iii) $\Longrightarrow$ (ii).
Assume that $\mathbf{E}[N_\infty]>0$. Let ξ_{-s}^{-t} be the population at time $-s$ of a branching random walk in the catalytic medium κ started at time $-t$, $-t\leq -s$, in the initial law $\mathcal{H}_1$. By the same reasoning as in the proof of Part 1 we infer that there exists a random population $\xi_{-s}^{-\infty}$ such that $\mathcal{L}[\xi_{-s}^{-t}]\underset{t\to\infty}{\Longrightarrow}\mathcal{L}[\xi_{-s}^{-\infty}]$, and that $\mathcal{L}[\xi_{-s}^{-\infty}]$ is transported into $\mathcal{L}[\xi_0^{-\infty}]$ through the branching dynamics in the medium κ. The random population $\xi_0^{-\infty}$ can be represented as

$$(2.82) \quad \mathcal{L}[\xi_0^{-\infty}]=\int_{\mathcal{N}(G)}\mathbf{P}[\xi_{-t}^{-\infty}\in dm]\,\mathcal{L}\left[\sum_{h\in G}\sum_{i=1}^{m(\{h\})}\xi_0^{i,h,-t}\right],$$

where the family $\{\xi^{i,h,-t},\ i\in\mathbb{N},\ h\in G\}$ is independent. We thus obtain for all $c>0$:

$$(2.83) \quad \begin{aligned} &\mathbf{E}[\xi_0^{-\infty};\ \{\xi_0^{-\infty}>c\}]= \\ &= \int_{\mathcal{N}(G)}\mathbf{P}[\xi_{-t}^{-\infty}\in dm]\,\mathbf{E}\left[\sum_{h\in G}\sum_{i=1}^{m(\{h\})}\xi_0^{i,h,-t}(g);\ \sum_{h\in G}\sum_{i=1}^{m(\{h\})}\xi_0^{i,h,-t}(g)>c\right] \end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\mathcal{N}(G)} \mathbf{P}[\xi_{-t}^{-\infty} \in dm] \int_G m(dh)\, \mathbf{E}[\xi_0^{1,h,-t}(g);\ \xi_0^{1,h,-t}(g) > c] \\
&= \int_{\mathcal{N}(G)} \mathbf{P}[\xi_{-t}^{-\infty} \in dm] \int_G m(dh)\, b_t(h,g)\ \mathbf{P}[\widehat{\xi_0^{1,h,-t}}(g) > c] \\
&= \int_{\mathcal{N}(G)} \mathbf{P}[\xi_{-t}^{-\infty} \in dm] \int_G m(dh)\, b_t(h,g) \\
&\qquad \mathbf{P}\Big[1 + \int_{G\times[-t,0]} \nu^{(h,-t),(g,0)}(dg',ds)\, \xi_0^{g',s}(g) > c\Big],
\end{aligned}
$$

where we used (2.36) in the last equality and defined $\nu^{(h,-t),(g,0)}$ as the Poisson measure on $G \times (-\infty, 0]$ with intensity (recall (2.35) and compare with (2.37))

$$
(2.84) \qquad \delta_{Y_s^{(h,-t),(g,0)}}(dg')\kappa(g',s)ds.
$$

Since by assumption $\mathbf{E}[\xi_0^{-\infty}(h)] = \varepsilon > 0$ for all $h \in G$ we know that also $\mathbf{E}[\xi_{-t}^{-\infty}(h)] = \varepsilon > 0$ and hence the r.h.s. in (2.83) equals

$$
(2.85) \qquad \varepsilon \mathbf{P}\Big[1 + \int_{G\times[-t,0]} \nu^{g,0}(dg',ds)\xi_0^{g',s}(g) > c\Big] = \varepsilon \mathbf{P}[S_t > c].
$$

This shows that the random variable S (recall (2.75)) is a.s. finite. As stated in Step 2, this is equivalent to persistence.

(c) Denote the canonical measure of $\xi_0^{-\infty}$ by R_∞ and by R_t the canonical measure of the population of the process at time 0 if we start at time $-t$, in the initial distribution $\mathcal{H}_1$. Observe that $\int R_\infty(d\psi)\psi(g) = 1 = \ell(g)$. Furthermore $\psi(g)$ is uniformly integrable under the measures R_t. Since for all $f \in C_c^+(G)$ the finite measures $R_t(d\psi)(1 - e^{-\langle \psi, f\rangle})$ converge weakly as $t \to \infty$ to $R_\infty(d\psi)(1 - e^{-\langle \psi, f\rangle})$, we get

$$
(2.86) \qquad \int R_\infty(d\psi)\psi(g)(1 - e^{-\langle \psi, f\rangle}) = \lim_{t\to\infty} \int R_t(d\psi)\psi(g)(1 - e^{-\langle \psi, f\rangle}).
$$

With Proposition 2.2 (a) we continue this equation

$$
\begin{aligned}
&= \lim_{t\to\infty} \mathbf{E}\Big[1 - \exp\Big(-\Big\langle \delta_g + \int_{G\times[-t,0]} \nu^{g,0}(dg',ds)\xi_0^{g',s}, f\Big\rangle\Big)\Big] \\
&= \mathbf{E}\Big[1 - \exp\Big(-\Big\langle \delta_g + \int_{G\times(-\infty,0]} \nu^{g,0}(dg',ds)\xi_0^{g',s}, f\Big\rangle\Big)\Big].
\end{aligned}
$$

This completes the proof. $\square$

3 Persistent catalysts: Proof of Theorem 1

The proof of Theorem 1 will be based on the fact that the law of the reactant process given the catalyst process is (for suitable initial law) *infinitely divisible*. Hence it is associated with a canonical measure so that ξ_t can be decomposed into the clans of related particles which correspond to the point process generated on $\mathcal{N}(G)$ by the canonical measure. Then following Section 2(b) it is possible to give an explicit construction of the *canonical Palm distribution* (see Definition 2.2).

Having this Palm distribution for given catalyst process in an explicit representation at our disposal we will establish the local divergence or convergence of a realization of this measure depending on whether $\hat{b}$ is recurrent or transient. From these results we deduce the needed facts on $\mathcal{L}[\xi_t]$ itself using the Kallenberg criterion of Section 2(d). Since the behavior of $\mathcal{L}[\eta_t]$ as $t \to \infty$ is well known, we can use coupling techniques to make assertions about $\mathcal{L}[(\eta_t, \xi_t)]$ as $t \to \infty$.

In order to analyze the canonical Palm distribution a prominent role is played by certain functionals generated by *random walks in random scenery* and we therefore study the latter in a separate Subsection 3(a), before we prove Theorem 1. in Subsection 3(b).

(a) Preparation: Random walk in random scenery

In this section we prove some statements concerning objects controlling the law of the Kallenberg backward tree, which appeared in Section 2(b).

An important quantity in studying the Kallenberg backward tree is the expected number of relatives of the randomly chosen particle located at the particular site 0, conditioned on the random walk path of the randomly chosen particle. According to Section 2(b) (Proposition 2.2(a)) this conditional expectation is equal to:

$$\mathbb{E}^1 \left[\int_0^T \eta_t \left(\bar{Y}^1_{T-t} \right) \, 1 \, \left(\bar{Y}^1_{T-t} = \bar{Y}^2_{T-t} \right) dt \right], \tag{3.1}$$

where $(\bar{Y}^i_t)_{t \geq 0}$ $i = 1, 2$ are independent realizations of random walks starting in 0 at time 0 and which have transition kernel $\bar{b}$ ($\bar{b}(g, g') = b(g', g)$) and $\mathbb{E}^i$, $i = 1, 2$, denotes expectation with respect to the i-th walk.

We saw in Section 2(b), Proposition 2.5(b), that depending on whether this quantity in (3.1) remains stochastically bounded or not the laws $\mathcal{L}[\hat{\xi}_t | (\eta_s)_{s \geq 0}]$ form a tight family or not. We will therefore continue by studying the expression (3.1) and prove:

Proposition 3.1 (Catalytic occupation functional)
Let $(Z^i_t)_{t \geq 0}$ $i = 1, 2$ be two independent irreducible random walks on G, which start in the points g and g'. Furthermore consider $(\eta_t)_{t \geq 0}$ with $\mathcal{L}[\eta_0] = \nu_\theta \in \mathcal{E}_\theta$ with $\theta > 0$, where ν_θ is the unique extremal equilibrium of the branching random walk with intensity θ. Then

$$\int_0^\infty \eta_t(Z^1_t) \, 1 \, (Z^1_t = Z^2_t) dt = \begin{matrix} +\infty \\ < \infty \end{matrix} \quad (\eta_t)_{t \geq 0} \text{ - } a.s., \tag{3.2}$$

depending on whether $(Z^1_t - Z^2_t)_{t \geq 0}$ is recurrent or transient. $\diamond$

Proof For $(Z_t^1 - Z_t^2)_{t\geq 0}$ transient the statement is trivial. In fact, by the independence of η and the random walks even the expectation is finite:

$$
\begin{aligned}
(3.3)\qquad \mathbf{E}\left[\int_0^\infty \eta_t(Z_t^1)\, 1_{\{0\}}(Z_t^1 - Z_t^2)\, dt\right] &= \int_0^\infty \mathbf{E}[\eta_t(Z_t^1)]\, \mathbf{P}[Z_t^1 - Z_t^2 = 0]\, dt \\
&= \theta \cdot \int_0^\infty \mathbf{P}[Z_t^1 - Z_t^2 = 0]\, dt \ < \infty.
\end{aligned}
$$

Now consider the case $(Z_t^1 - Z_t^2)_{t\geq 0}$ recurrent. Here we give a proof using a standard variance calculation. Below follows a more systematic approach in a situation without variances.

Define the inverse collision time

$$T_t = \inf\left\{ s : \int_0^s dr\ 1_{Z_r^1 = Z_r^2} \geq t \right\}$$

and let $\chi_t = \eta_{T_t}(Z_{T_t}^1)$. Note that by recurrence $T_t < \infty$ a.s. for all t and that $Z_{T_t-}^1 = Z_{T_t-}^2$, $t \geq 0$. Hence

$$\int_0^{T_t} \eta_s(Z_s^1) 1_{Z_s^1 = Z_s^2}\ ds = \int_0^t \chi_s\ ds.$$

Notice that $\mathbf{E}[\int_0^t \chi_s\ ds | Z^1, Z^2] = t\theta$ a.s. and that

$$
\begin{aligned}
(3.4)\qquad \mathbf{Var}[\int_0^t \chi_s\ ds] &= \mathbf{E}[\mathbf{Var}[\int_0^t \chi_s\ ds|\ Z^1, Z^2]] \\
&= \mathbf{E}[\int_0^t dr \int_0^t ds\ \mathbf{Cov}[\eta_{T_r}(Z_{T_r}^1), \eta_{T_s}(Z_{T_s}^1) | Z^1, Z^2]].
\end{aligned}
$$

Note that $\sup_{g,h\in G} \mathbf{Cov}[\eta_t(g), \eta_s(h)]$ is finite, depends only on $|t-s|$ and vanishes as $|t-s| \to \infty$. Hence

$$\lim_{t\to\infty} t^{-2} \mathbf{Var}[\int_0^t \chi_s\ ds] = 0.$$

Thus there exists a sequence $t_n \uparrow \infty$ such that $t_n^{-1}\chi_{t_n} \to \theta$ a.s. In particular $\chi_t \to \infty$, $t \to \infty$, a.s. $\square$

Note that in the proof above we needed second moments only in the recurrent case, but even there they are not necessary as we will now develop in a more general set–up. Observe that with the irreducibility of the random walks we can assume w.l.o.g. that $g = g' = 0$ and we will do so in the rest of this subsection.

Part 1
Take a (time–space) random field, i.e.

$$(3.5)\qquad \{L(s,g);\ s \in \mathbb{R},\ g \in G\}$$

which satisfies (time–space homogeneity):

$$
\begin{aligned}
(3.6)\qquad &\mathcal{L}[\{L(s+s',\ g+g');\ s \in \mathbb{R},\ g \in G\}] \\
&= \mathcal{L}[\{L(s,g);\ s \in \mathbb{R},\ g \in G\}] \qquad \forall\ s' \in \mathbb{R},\ g' \in G
\end{aligned}
$$

$$(3.7) \qquad \{L(s,g),\ s \in \mathbb{R},\ g \in G\} \quad \text{is mixing.}$$

We will study for two independent random walks $(Z_t^i)_{t \geq 0},\ i = 1,2$, with the same initial point and (possibly different) irreducible transition kernels the object

$$(3.8) \qquad \int_0^\infty L(s, Z_s^1)\, 1\, (Z_s^1 = Z_s^2) ds.$$

The key result is a generalization of Proposition 3.1 to a general mixing field L instead of η (Lemma 3.1). In fact, Proposition 3.1 is a corollary of Lemma 3.1 once we have showed that η fulfills the assumptions of Lemma 3.1. This is the content of the subsequent Lemma 3.3.

Lemma 3.1 *Let $(Z_t^1)_{t \geq 0}$ and $Z_t^2)_{t \geq 0}$ be independent (possibly different) random walks on G. If $\mathbf{E}[L(s,g)] = \theta > 0$ and if $(Z_t^1 - Z_t^2)_{t \geq 0}$ is recurrent then*

$$(3.9) \qquad \int_0^t L(s, Z_s^1)\, 1\, (Z_s^1 = Z_s^2)\, ds \Big/ \int_0^t 1\, (Z_s^1 = Z_s^2)\, ds \underset{t \to \infty}{\longrightarrow} \theta \quad a.s. \qquad \diamond$$

Lemma 3.2 *Consider a branching random walk on G with transient symmetrized migration kernel. The equilibrium process associated with the extremal equilibrium measure ν_θ induces a field*

$$(3.10) \qquad \{\eta_t(g),\ (t,g) \in \mathbb{R} \times G\}$$

which is stationary and mixing. $\diamond$

With Lemma 3.2 we see that we can use Lemma 3.1 which then gives immediately the assertion of Proposition 3.1 in the recurrent case.

All which remains to conclude the proof of Proposition 3.1 is now to verify Lemma 3.1 and Lemma 3.2. This we do in the next two parts below.

Part 2

Proof of Lemma 3.1 The proof proceeds in two steps. In order to handle the quantities above we need to establish that the *field seen from the random walk* $(Z_t^1)_{t \geq 0}$ remains stationary and mixing. To that extent we need the following tool (Lemma 3.3) formulated and proved for a discrete random walk in Step 1, then Step 2 completes the proof of Lemma 3.1 based on the discrete time Lemma 3.3.

Step 1 Let $\widetilde{L} = \{(\widetilde{L}(n,g)),\ n \in \mathbb{Z},\ g \in G\}$ be a mixing random field and let $(T_n, X_n)_{n \in \mathbb{N}}$ be a random walk in $\mathbb{Z} \times G$, independent of $\tilde{L}$ and satisfying

$$(3.11) \qquad \mathbf{P}[T_{n+1} \geq T_n + 1] = 1.$$

We continue the random walk for indices $n \in \mathbb{Z}^-$ by means of the reversed random walk.

Lemma 3.3 *The random field $K = (K_n)_{n\in\mathbb{Z}}$ with*

(3.12) $\quad K_n = \widetilde{L}(T_n, X_n)$

is stationary and mixing. $\diamond$

The key to this property is the fact that by (3.11) this space–time random walk is transient.

Proof Clearly $(K_n)_{n\in\mathbb{Z}}$ is stationary. So it suffices to check the mixing property. We will show

(3.13) $\quad |\mathbf{P}[K \in A,\ \sigma_k K \in B] - \mathbf{P}[K \in A]\mathbf{P}[K \in B]| \underset{k\to\infty}{\longrightarrow} 0$

where σ_k denotes the shift by the element $k \in \mathbb{Z}$ and with cylinder sets $A, B \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}})$ depending only on the coordinates $0, \ldots, N$ ($N \in \mathbb{N}$). First note that we may assume $\mathbf{P}[T_0 = 0] = 1$.

By the mixing property of $\widetilde{L}$ for $A, B \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}\times G})$ and $\varepsilon > 0$ there exists $M > 0$ such that for $k \geq M$ and all $g \in G$

(3.14) $\quad \left|\mathbf{P}\left[\widetilde{L} \in A,\ \widetilde{L}(k+\cdot, g+\cdot) \in B\right] - \mathbf{P}\left[\widetilde{L} \in A\right]\mathbf{P}\left[\widetilde{L} \in B\right]\right| \leq \varepsilon.$

We have to introduce some notation. For $(t, g) = (t_n, g_n)_{n\in\mathbb{Z}} \in (\mathbb{Z} \times G)^{\mathbb{Z}}$ and $l, k \in \mathbb{Z}$, $l \leq k$, let

(3.15)
$$\begin{aligned} \vartheta_k(t,g) &= (t_{n+k}, g_{n+k})_{n\in\mathbb{Z}}, \\ \overline{\vartheta}_k(t,g) &= (t_{n+k} - t_k, g_{n+k} - g_k)_{n\in\mathbb{Z}}, \\ (t,g)_{l,k} &= (t_{n+l}, g_{n+l})_{n=0,\ldots,k-l}. \end{aligned}$$

Note that $(T, X)_{0,k}$ and $(\overline{\vartheta}_{k+1}(T, X))_{0,l}$ are by the random walk property independent for $k, l \geq 0$.

For $h \in (\mathbb{N}_0 \times G)^{\{0,\ldots,N\}}$ let

(3.16)
$$\begin{aligned} A_h &= \{a \in \mathbb{R}^{(\mathbb{N}_0\times G)} : a \circ h \in A\} \\ B_h &= \{b \in \mathbb{R}^{(\mathbb{N}_0\times G)} : b \circ h \in B\}. \end{aligned}$$

Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ nd $\mathcal{J} \subset (\mathbb{N}_0 \times G)^{\{0,\ldots,N\}}$ finite such that

(3.17) $\quad \mathbf{P}[(T, X)_{0,N} \in \mathcal{J}] \geq 1 - \varepsilon.$

Choose $M \geq N$ large enough such that (3.14) holds for any choice $A' = A_g$, $B' = B_h$ ($g, h \in \mathcal{J}$). For $k \geq M$ and $g, h \in \mathcal{J}$ let

(3.18) $\quad H_k(g, h) = \{j \in (\mathbb{N}_0 \times G)^{\mathbb{Z}} : j_{0,N} = g,\ (\overline{\vartheta}_k(j))_{0,N} = h\}$

and define H_k as the disjoint union

(3.19) $\quad H_k = \bigcup_{g,h\in\mathcal{J}} H_k(g, h).$

Note that

$$(3.20) \qquad \mathbf{P}[(T,X) \in H_k(g,h)] = \mathbf{P}[(T,X)_{0,N} = g] \cdot \mathbf{P}[(T,X)_{0,N} = h].$$

Denote by c_ε any quantity of absolute value $\le \varepsilon$. Then for $k \ge M$ (note that this implies $T_k \ge M$ a.s.)

$$\begin{aligned}
(3.21) \qquad & \mathbf{P}[K \in A,\, \sigma_k K \in B] \\
&= \mathbf{P}[K \in A,\, \sigma_k K \in B,\, (T,X) \in H_k] + c_\varepsilon \\
&= \sum_{g,h \in \mathcal{J}} \int_{H_k(g,h)} \mathbf{P}[(T,X) \in d(t,f)] \cdot \mathbf{P}\left[\widetilde{L} \in A_g,\, \widetilde{L}(t_k + \cdot\,, f_k + \cdot) \in B_h\right] + c_\varepsilon \\
&= \sum_{g,h \in \mathcal{J}} \int_{H_k(g,h)} \mathbf{P}[(T,X) \in d(t,f)] \cdot \mathbf{P}\left[\widetilde{L} \in A_g\right] \cdot \mathbf{P}\left[\widetilde{L} \in B_h\right] + 2c_\varepsilon \\
&= \left(\sum_{g \in \mathcal{J}} \mathbf{P}\left[(T,X)_{0,N} = g\right] \mathbf{P}[\widetilde{L} \in A_g]\right) \left(\sum_{h \in \mathcal{J}} \mathbf{P}[(T,X)_{0,N} = h] \mathbf{P}[\widetilde{L} \in B_h]\right) + \; 2c_\varepsilon \\
&= \mathbf{P}[K \in A]\mathbf{P}[K \in B] + 4c_\varepsilon .
\end{aligned}$$

This proves the assertion (3.13) and hence Lemma 3.3. □

Step 2 We are ready to complete the proof of Lemma 3.1. Since we can build the given continuous time random walks $(Z_t^i)_{t \ge 0}$ $i = 1, 2$ via a Poisson process and the jump chain, we can reduce the problem of showing (3.9) to a discrete time problem which then allows to apply Step 1 and the standard ergodic theorem to conclude the argument.

Let λ^i and $c^i(\cdot,\cdot)$ be the jump rate and jump distribution of $(\bar{Z}_t^i)$, $i = 1, 2$. We assume that λ^1 and λ^2 are minimal in the sense that either $c^1(0,0) = 0$ or $c^2(0,0) = 0$. (This ensures that $(\bar{Z}_t^1 - \bar{Z}_t^2)$ does not make jumps on the spot.) Let $(N_t)_{t \ge 0}$ a Poisson process with rate $\lambda^1 + \lambda^2$ and with $N_0 = 0$. Let

$$(3.22) \qquad \widetilde{N}_n = \inf\{t \ge 0 : N_t \ge n\},\; n \in \mathbb{N}_0$$

be the time of the n-th jump. Let $(\widetilde{Z}_n)_{n \in \mathbb{N}_0} = (\widetilde{Z}_n^1, \widetilde{Z}_n^2)_{n \in \mathbb{N}_0}$ be a random walk in $G \times G$ with transition probabilities

$$\begin{aligned}
(3.23) \qquad \mathbf{P}\left[\widetilde{Z}_{n+1} - \widetilde{Z}_n = (g,0)\right] &= \frac{\lambda^1}{\lambda^1 + \lambda^2} c^1(0,g) \\
\mathbf{P}\left[\bar{Z}_{n+1} - \widetilde{Z}_n = (0,g)\right] &= \frac{\lambda^2}{\lambda^1 + \lambda^2} c^2(0,g)
\end{aligned}$$

Thus

$$(3.24) \qquad \mathcal{L}\left[(\widetilde{Z}_{N_t})_{t \ge 0}\right] = \mathcal{L}\left[(Z_t^1, Z_t^2)_{t \ge 0}\right].$$

Let $T_0 = 0$ and define for $n \in \mathbb{N}$ inductively the times of the n-th entrance in the diagonal $D \subset G \times G$

$$T_n = \inf\{m > T_{n-1} + 1 : \widetilde{Z}^1_m = \widetilde{Z}^2_m\} \tag{3.25}$$

(note that by the minimality assumption excluding jumps on the spot: $\mathbf{P}\left[\widetilde{Z}^1_{T_m+1} \neq \widetilde{Z}^2_{T_m+1}\right] = 1\ \forall m$). We assume that all these random objects are independent of the field L.

Let

$$\widetilde{L}(n,g) = \int_{\widetilde{N}_n}^{\widetilde{N}_n+1} L(t,g)\,dt. \tag{3.26}$$

Then $(\widetilde{L}(n,g),\ n \in \mathbb{N}_0,\ g \in G)$ is also mixing and has mean $\frac{\theta}{\lambda^1+\lambda^2}$.

Let $M_t = \sup\{n \in \mathbb{N}_0 : T_n \leq N_t\}$. Then up to an asymptotically negligible nuisance term, the l.h.s. of (3.9) equals

$$\frac{\sum_{n=0}^{M_t} \widetilde{L}(T_n, \widetilde{Z}^1_{T_n})}{\sum_{n=0}^{M_t} (\widetilde{N}_{T_n+1} - \widetilde{N}_{T_n})}. \tag{3.27}$$

Note that by the recurrence assumption on $(Z^1_t - Z^2_t)$, we have $M_t \to \infty$ a.s.

Now by the law of large numbers for i.i.d. random variables we get:

$$\frac{1}{n}\sum_{k=0}^{n} (\widetilde{N}_{T_k+1} - \widetilde{N}_{T_k}) \underset{n\to\infty}{\longrightarrow} \frac{1}{\lambda^1+\lambda^2} \quad \text{a.s..} \tag{3.28}$$

Hence it suffices to show

$$\frac{1}{n}\sum_{k=0}^{n} \widetilde{L}(T_k, \widetilde{Z}^1_{T_k}) \underset{n\to\infty}{\longrightarrow} \frac{\theta}{\lambda^1+\lambda^2} \quad \text{a.s..} \tag{3.29}$$

However, this is a consequence of (3.12) in Lemma 3.3 together with the classical ergodic theorem for stationary processes. □

Part 3
Proof of Lemma 3.2
It is a classical result that the field η under the equilibrium measure is mixing (in space) see Theorem 1.7 in [F]. A proof can be based on the representation of the canonical measure of the equilibrium distribution (recall Section 2(b)). Namely the extremal equilibrium ν_θ of a branching random walk is a Poisson system in the space of clans, that is $\mathcal{N}(G)$, and its intensity measure is given by the canonical measure of the infinitely divisible law ν_θ. Denote this intensity measure on $\mathcal{N}(G)$ by Q, a realization of the Poisson–system of clans by $\widehat{\eta}_\infty$. Observe that with this notation for $g, g' \in G$

$$\begin{aligned} Q(\widehat{\eta}_\infty(g) > 0,\ \widehat{\eta}_\infty(g') > 0) &\leq \theta Q_g(\widehat{\eta}_\infty(g') > 0) \\ &\leq \theta \sum_{l=1}^{\infty} Q_g(\widehat{\eta}_\infty(g') = l) \\ &= \theta \bar{a}_0(g', g) \ + \ \theta \int_0^\infty \widehat{a}_{2t}(g, g')dt \underset{g\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.30}$$

This implies that $\{\eta_t(g),\ g \in G\}$ is mixing for all $t \in \mathbb{R}$.

Hence all we need is to show that we have the mixing property in time for the equilibrium process:

$$(3.31) \qquad \mathcal{L}[(\eta_t(g), \eta_{t+s}(g)] \underset{s\to\infty}{\longrightarrow} (\mathcal{L}[\eta_t(g)])^{*2}, \quad \text{for all } g \in G.$$

Now we have to refine the picture from the previous argument and we look at the complete branching tree. Here we introduce the cemetery Δ, that is every particle dying in the usual picture is now making a jump into Δ instead. Corresponding to the equilibrium process $(\eta_t)_{t\in\mathbb{R}}$ we can construct the process of all paths associated with a particle present at some time. This object decomposes now into a Poisson system of clans of related particles in the space of paths $\mathcal{N}(D((-\infty,0), G \cup \{\Delta\}))$, with canonical measure $\bar{Q}$. This system can be obtained by starting a branching dynamic at time $-s$ in ν_θ, weighting all paths of descent with 1 and then letting $s \to \infty$. Denote by φ an element of $\mathcal{N}(D(\mathbb{R}, G \cup \Delta))$ describing one clan. Let $T \in \mathbb{R}$ and $r > 0$. Similarly as in (3.30) we size–bias $\bar{Q}$ in the space–time point (g,T). This distribution $\bar{Q}_{(g,T)}$ has a backward tree representation as in Proposition 2.2 now involving the paths of the side–trees. Hence as in (3.30) we get (note that the first term corresponds to the backbone):

$$(3.32) \qquad \bar{Q}(\varphi_T(g) > 0,\ \varphi_{T-r}(g') > 0) = \theta \bar{a}_r(g',g) \ + \ \theta \int_0^\infty \sum_{g''\in G} \bar{a}_t(g',g'') a_{t+r}(g'',g) dt.$$

The right hand side of the inequality above tends to 0 if either $r \to \infty$ or $d(g,g') \to \infty$, since $\hat{a}$ (and hence a) is transient.

Therefore the event that at the time point t and $t+r$ we find in g a particle of the same clan has probability independent of t and tending to 0 as $r \to \infty$. This means that the equilibrium branching random walk is mixing with respect to time–shifts. $\square$

(b) Completion of the proof of Theorem 1

The proof of this theorem proceeds differently for the case of transient and recurrent kernels $\hat{b}$. In the recurrent case we exploit the Proposition 3.1 from the previous section together with the backward techniques from chapter 2. The transient case is treated in a somewhat different manner adding *coupling* to the tool kit. We therefore begin with the recurrent case.

The case $\hat{b}$ recurrent
Now we are in the case $\widehat{b}$ *recurrent* and we have to obtain the local extinction of the reactant for general initial laws in $\mathcal{E}_{\theta_\eta,\theta_\xi}$. Since we know that $\mathcal{L}[\eta_t] \underset{t\to\infty}{\Longrightarrow} \nu_{\theta_\eta}$ as $t \to \infty$ it suffices to prove here that $\mathcal{L}[\xi_t] \underset{t\to\infty}{\Longrightarrow} \delta_{\underline{0}}$ as $t \to \infty$. We will establish below that

$$(3.33) \qquad \mathcal{L}[\xi_t(0)|\eta] \underset{t\to\infty}{\Longrightarrow} \delta_0 \quad \text{stochastically.}$$

For that relation above it suffices to show that the size–biased distributions of $\mathcal{L}[\xi_t(0)|\eta]$ diverge stochastically as $t \to \infty$. Note that if $(\eta_t)_{t\geq 0}$ were the equilibrium process then we would obtain this divergence by combining Proposition 2.3 with Proposition 3.1 and Proposition 2.5, Part 2. Our strategy is therefore to make a comparison to that situation of a catalyst in equilibrium.

In order to set the scene for that comparison argument we think of the catalyst–reactant system started at time $-t$ in μ and evaluated at time 0. We denote that system by $(\eta_s^{-t},\ \xi_s^{-t})_{s\in\mathbb{R}}$, where we use the convention that for $s \leq -t$ we keep the system constant, i.e. equal to $(\eta_{-t}^{-t},\ \xi_{-t}^{-t})$. From the convergence of $\mathcal{L}[\eta_t]$ to ν_{θ_η} as $t \to \infty$ we conclude that

$$(3.34)\qquad \mathcal{L}\Big[(\eta_t^{-T})_{t\in\mathbb{R}}\Big] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}\Big[(\eta_t^{-\infty})_{t\in\mathbb{R}}\Big],$$

where $(\eta_t^{-\infty})_{t\in\mathbb{R}}$ denotes the equilibrium process with marginal ν_{θ_η}.

To continue recall the notation in Proposition 2.2 and Proposition 2.5 and define for $0 \leq t \leq T \leq \infty$ and a fixed element $g \in G$:

$$(3.35)\qquad N_{-t}^{-T} = \int_{-t}^{0} \nu_{-T}^{g,0}(dg', ds)\xi_0^{g',s}(g),$$

where the Poisson point measure $\nu_{-T}^{g,0}$ is defined with respect to the medium $(\eta_t^{-T})_{t\in\mathbb{R}}$. As pointed out above we know already that:

$$(3.36)\qquad \mathbf{P}[N_{-t}^{-\infty} \to \infty \text{ as } t \to \infty] = 1.$$

Since by (3.34) we have

$$(3.37)\qquad \mathcal{L}[N_{-t}^{-T}] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}[N_{-t}^{-\infty}]$$

we can conclude that

$$(3.38)\qquad N_{-t}^{-t} \underset{t\to\infty}{\longrightarrow} \infty \text{ stochastically.}$$

From Proposition 2.3 together with the fact

$$(3.39)\qquad (\mathcal{L}[\widehat{\xi_T(g)}]) = \mathbf{E}[\mathcal{L}[\widehat{\xi_T(g)}|\eta]]$$

we get

$$(3.40)\qquad \mathcal{L}[N_{-t}^{-t}] \leq \mathcal{L}[\widehat{\xi_t(g)}].$$

Hence $\mathcal{L}[\widehat{\xi_t(g)}]$ diverges and therefore $\mathcal{L}[\xi_t(g)]$ converges to δ_0 and we have completed the proof of (3.33). $\square$

The case $\hat{b}$ transient

In the case $\hat{b}$ transient we start in Step 1 proving Theorem 1 based on Section 2 and 3(a) but only under the assumption that the initial law μ is of a special form implying in particular two things: (i) The catalyst is given by the equilibrium process for every initial configuration for the reactant and (ii) for every catalyst realization the law of the reactant is infinitely divisible. Note that the methods of infinite divisible laws and canonical measures apply only to $\mathcal{L}[\xi_t|(\eta_s)_{s\le t}]$ but *not* to $\mathcal{L}[(\eta_t,\xi_t)]$. This approach of first looking at special initial states allows us on the way also to identify the canonical Palm distribution of $\mathcal{L}[\xi_0|(\eta_t)_{t\le 0}]$ of the equilibrium process. We remove in Step 2 the assumption on the initial state using *coupling* techniques.

Step 1 Assume now that μ is of the form:

$$\mu = \nu_{\theta_\eta} \otimes \mathcal{H}_{\theta_\xi}. \tag{3.41}$$

Then the catalyst $(\eta_t)_{t\ge 0}$ is in an extremal *equilibrium* with intensity θ_η: $\mathcal{L}[\eta_t] = \mathcal{L}[\eta_0] \quad \forall t \ge 0$. Furthermore $\mathcal{L}[\xi_t|(\eta_s)_{s\ge 0}]$ is *infinitely divisible* and we can apply the Kallenberg criterion given in Section 2(d) in Proposition 2.5.

Our arguments are grouped into three pieces: (i) We show that starting in μ (recall (3.41)) the law $\mathcal{L}[(\eta_t,\xi_t)]$ converges for $t\to\infty$ towards a limit law $\nu_{\theta_\eta,\theta_\xi}$ which we specify. (ii) We prove that $\nu_{\theta_\eta,\theta_\xi} \in \mathcal{E}_{\theta_\eta,\theta_\xi}$. (iii) We establish that $\nu_{\theta_\eta,\theta_\xi}$ is an invariant measure for the catalyst–reactant dynamic.

(i) Consider the catalyst equilibrium process $(\eta_t)_{t\in\mathbb{R}}$ which is prescribed by $\mathcal{L}[\eta_0] = \nu_{\theta_\eta}$, where $\nu_{\theta_\eta} \in \mathcal{E}_{\theta_\eta}$ is the unique extremal invariant measure with intensity θ_η of the catalyst process. For every realization of the catalyst process we can construct according to Lemma 2.2 the reactant process $(\xi_s^{-t})_{s\ge -t}$ which starts at time $-t$ in $\mathcal{H}_{\theta_\xi}$. In this context again the Kallenberg criterion applies and from Part 1 (a) of Proposition 2.5 we can conclude that there exists $\xi_0^{-\infty}$ such that:

$$\mathcal{L}[\xi_0^{-t}|(\eta_s)_{s\in\mathbb{R}}] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}[\xi_0^{-\infty}|(\eta_s)_{s\in\mathbb{R}}], \quad (\eta_s)_{s\in\mathbb{R}} - \text{a.s.} \tag{3.42}$$

For a given medium $\widehat{\eta} = (\eta_s)_{s\in\mathbb{R}}$ we denote this limit by $K(\widehat{\eta},\cdot)$. From Part (b), (ii) and (iii), of Proposition 2.5 we know that

$$\int K(\widehat{\eta}, d\xi)\, \xi(g) = \theta_\xi, \quad \forall g \in G \quad \text{a.s.}, \tag{3.43}$$

provided that, with $(\bar{Y}_t^g)_{t\ge 0}$ denoting a random walk with transition kernel $\bar{b}$, the following holds:

$$\int_0^\infty b_s(\bar{Y}_s^g, g)\eta_{-s}(\bar{Y}_s^g)ds < \infty \qquad \{(\bar{Y}_u^g)_{u\ge 0},\ (\eta_u)_{u\in(-\infty,0]}\} - \text{a.s.} \tag{3.44}$$

This latter statement is however (compare the recurrent case) according to Proposition 3.1 equivalent to the transience of $\bar{b}$. Hence by assumption (3.43) holds.

We define now $\nu_{\theta_\eta,\theta_\xi}$ by setting for every bounded measurable function F on $\mathbb{N}_0^G \times \mathbb{N}_0^G$ and with the abbreviation $\widehat{\nu}_{\theta_\eta} = \mathcal{L}[\widehat{\eta}]$ for the law of the equilibrium process (on path space):

$$\begin{aligned} \int \nu_{\theta_\eta,\theta_\xi}(d\widetilde{\eta}, d\widetilde{\xi}) F(\widetilde{\eta},\widetilde{\xi}) &= \int \widehat{\nu}_{\theta_\eta}(d\widehat{\eta}) K(\widehat{\eta}, d\widetilde{\xi}) F(\widehat{\eta}_0, \widetilde{\xi}) \\ &= \int \nu_{\theta_\eta}(d\widetilde{\eta}) K_0(\widetilde{\eta}, d\widetilde{\xi}) F(\widetilde{\eta},\widetilde{\xi}), \end{aligned} \tag{3.45}$$

where K_0 is the average of $K(\widehat{\eta},\cdot)$ over $\mathcal{L}[\widehat{\eta}|\widehat{\eta}_0]$.

If we abbreviate by $K^{-t}(\widehat{\eta},\cdot) = \mathcal{L}[\xi_0^{-t}|(\eta_s)_{s\in\mathbb{R}}]$ then we want to show finally that for every test function $F \in C_b(\mathbb{N}_0^G \times \mathbb{N}_0^G)$:

$$(3.46) \qquad \int\int K^{-t}(\widehat{\eta}, d\widetilde{\xi})\widehat{\nu}_{\theta_\eta}(d\widehat{\eta})F(\widehat{\eta}_0,\widetilde{\xi}) \underset{t\to\infty}{\longrightarrow} \int\int K(\widehat{\eta}, d\widetilde{\xi})\widehat{\nu}_{\theta_\eta}(d\widehat{\eta})F(\widehat{\eta}_0,\widetilde{\xi}).$$

This however follows with Lebesgue dominated convergence from (3.42) since F is in particular continuous in $\tilde{\xi}$ and is bounded in both variables.

Altogether we have established that with $\mathcal{L}[(\eta_0,\xi_0)] = \mu$ as defined in (3.41):

$$(3.47) \qquad \mathcal{L}[(\eta_t,\xi_t)] \underset{t\to\infty}{\Longrightarrow} \nu_{\theta_\eta,\theta_\xi}.$$

(ii) The next point is to prove that $\nu_{\theta_\eta,\theta_\xi} \in \mathcal{E}_{\theta_\eta,\theta_\xi}$. Begin with

(3.48) $\quad \nu_{\theta_\eta,\theta_\xi}$ is shift invariant.

Recall that the equilibria ν_{θ_η} and $\widehat{\nu}_{\theta_\eta}$ of the catalyst are shift invariant as well as $\mathcal{H}_{\theta_\xi}$. Call a kernel K shift invariant if $K(\tau_g\cdot,\tau_g\cdot) = K(\cdot,\cdot)$ with σ_g denoting the shift of the configuration by g. Then by construction $\mathcal{L}[\xi_0^{-t}|(\eta_s)_{s\in\mathbb{R}}]$ is shift invariant and hence also its weak limit of (3.46) as $t\to\infty$. This implies then by the definition of $\nu_{\theta_\eta,\theta_\xi}$ the shift invariance of the latter.

Next we check that the means of both components have the right value. Since $\nu_{\theta_\eta,\theta_\xi}$ has a projection on the catalyst component which is ν_{θ_η} and furthermore due to (3.43) we know that

$$(3.49) \qquad \mathbf{E}_{\nu_{\theta_\eta,\theta_\xi}}[\eta(g)] = \theta_\eta, \quad \mathbf{E}_{\nu_{\theta_\eta,\theta_\xi}}[\xi(g)] = \theta_\xi.$$

The final point is now to show that $\nu_{\theta_\eta,\theta_\xi}$ is shift–ergodic.

We will prove the stronger property that the random field $(\eta_\infty(g),\xi_\infty(g))$, $g\in G$, with $\mathcal{L}[\eta_\infty,\xi_\infty] = \nu_{\theta_\eta,\theta_\xi}$, is *mixing.* The basic idea is that a spatial branching population in equilibrium can be decomposed into independent random family clusters of related particles (the realization of the point process on configuration space induced by the canonical measure) and each cluster has the property that the probability to find a member of the cluster in the point g *and* the point g' goes to 0 as $d(g,g')$ tends to infinity. (Recall the proof of Proposition 3.1 below (3.31)).

This implies that $\{\eta_\infty(g_i),\ i=1,\dots,n\}$ becomes asymptotically independent as $d(g_i,g_j)\to\infty$ for all i,j with $i\neq j$. A similar argument as below (3.8) can be carried out for ξ_∞ if we consider a fixed realization of the catalyst and start the reactant process at time $-t$ and observe it at time 0. We omit further details at this point.

It then remains to prove that also

$$(3.50) \qquad \{\eta_\infty(g_i),\ i=1,\dots,n\}, \quad \{\xi_\infty(g_i'),\ i=1,\dots,m\}$$

become asymptotically independent if the distance between the two sets $\{g_i;\ i=1,\dots,n\}$ and $\{g_i';\ i=1,2,\dots,m\}$ tends to ∞. For this it is enough to observe that the catalyst process in equilibrium is mixing in time. This allows us to proceed as follows:

(α) We first choose a space–time set of the form $A \times [t, T]$, $(t < T)$ containing $\{g'_i; i = 1, \dots, m\} \times \{T\}$ with $T - t$ and A large enough so that $\{\xi_T(g'_i);\ i = 1, \dots, m\}$ depends up to an ε only on $A \times [t, T]$, that is: the distribution of ξ_t on $\{g'_i;\ i = 1, \dots, m\}$ at time T and the corresponding finite dimensional distribution of the reactant $(\xi'_s)_{s \geq t}$ evolving from time t on with the catalyst $(\eta_s 1_A)_{s \geq t}$, is in variational distance ε-close. (This is easily seen from the backward tree representation.)

(β) Next we choose the $\{g_i;\ i = 1, \dots, n\}$ so far away, that the catalyst on $\{g_i;\ i = 1, ;n\} \times [t, T]$ is up to an ε, independent of the catalyst in $A \times [t, T]$. The latter is possible because of the time–space mixing property of the catalyst in equilibrium already used to prove Proposition 3.1.

Combining the observations (α) and (β) proves (3.50).

(iii) In order to verify that $\nu_{\theta_\eta, \theta_\xi}$ is invariant under the catalyst–reactant dynamic we observe first that the following weakened Feller property holds:

$$(3.51) \qquad \Big\{\mu_n \in \mathcal{E}_{\theta_\eta, \theta_\xi}, \mu_n \underset{n \to \infty}{\Longrightarrow} \mu,\ \mathbf{E}_\mu[\eta(0) + \xi(0)] = \theta_\eta + \theta_\xi\Big\}$$
$$\text{implies } \Big\{\mathcal{L}_{\mu_n}[(\eta_t, \xi_t)] \underset{n \to \infty}{\Longrightarrow} \mathcal{L}_\mu[(\eta_t, \xi_t)] \text{ for all } t > 0\Big\}.$$

This is easily deduced for example from the couplings we present in (3.55) and (3.65) below.

Denote by $(S(t))_{t \geq 0}$ the semigroup associated with $(\eta_t, \xi_t)_{t \geq 0}$. Then the relation (3.51) implies together with (3.47):

$$(3.52) \qquad (\nu_{\theta_\eta, \theta_\xi}) S(t) = \mathrm{w} - \lim_{s \to \infty} (\nu_{\theta_\eta} \otimes \mathcal{H}_{\theta_\xi}) S(t + s) = \nu_{\theta_\eta, \theta_\xi},$$

which shows that $\nu_{\theta_\eta, \theta_\xi}$ is invariant under $(S(t))_{t \geq 0}$.

Step 2 In this step we have to remove the assumption of the special form of our initial state in the convergence statement for $\mathcal{L}[(\eta_t, \xi_t)]$ in the case of *transient* $\widehat{b}$. The strategy is to compare the initial state $\nu_{\theta_\eta} \otimes \mathcal{H}_{\theta_\xi}$ with an arbitrary element $\mu \in \mathcal{E}_{\theta_\eta, \theta_\xi}$ by going through an intermediate situation. Namely let π_η denote the projection on the η-component and define

$$(3.53) \qquad \nu := \pi_\eta(\mu) \in \mathcal{E}_{\theta_\eta}.$$

Denote again with $(S(t))_{t \geq 0}$ the semigroup of the process $(\eta_t, \xi_t)_{t \geq 0}$. We will show

$$(3.54) \qquad \begin{array}{ll} \text{(i)} & (\nu_{\theta_\eta} \otimes \mathcal{H}_{\theta_\xi} - \nu \otimes \mathcal{H}_{\theta_\xi}) S(t) \underset{t \to \infty}{\Longrightarrow} 0 - \text{measure}, \\ \text{(ii)} & (\mu - \nu \otimes \mathcal{H}_{\theta_\xi}) S(t) \underset{t \to \infty}{\Longrightarrow} 0 - \text{measure}. \end{array}$$

Combining relations (3.54) part (i) and (ii) with (3.47) of Step 1 gives finally the convergence assertion of the theorem. Let us now prove relations (i) and (ii) of (3.54).

(i) It is well known (compare [G], Proof of Theorem 1 specialized in the notation of that paper to the case $p = 0$, or [L] for the treatment of similar models) that if ν and μ are two homogeneous, ergodic laws on $(\mathbb{N}_0)^G$ with the same intensity θ, then we can construct a bivariate process $(\eta_t^1, \eta_t^2)_{t \geq 0}$ such that $(\eta_t^i)_{t\geq 0}$, $i = 1, 2$, is a version of the catalyst process $(\eta_t)_{t\geq 0}$ with initial law ν respectively μ and the property (since $\hat{a}$ is transient)

$$\text{(3.55)} \qquad \mathbf{E}[|\eta_t^1(g) - \eta_t^2(g)|] = \mathbf{E}[|\eta_t^1(0) - \eta_t^2(0)|] \searrow 0 \text{ as } t \to \infty.$$

For this purpose one defines the mechanism in the bivariate process by letting migration and branching occur at rate $\eta_t^1(g) \wedge \eta_t^2(g)$ in both components and with rate $(\eta_t^1(g) - \eta_t^2(g))^{+,-}$ in component one respectively two. Then the result (3.55) is a special case of the model treated in [G]. See also proof of (3.54) Part (ii) below for more details of such arguments.

The coupling relation implies in particular that the processes agree in boxes of arbitrary size for large values of t with probability close to one. By choosing the box B around A large enough and using the preservation of the mean we can guarantee that for any given time interval no migration from the complement of B into A occurs. Hence for every $A \subset G$, $|A| < \infty$ and for every $T > 0$:

$$\text{(3.56)} \qquad \mathbf{P}\left[\eta_{t+s}^1(g) = \eta_{t+s}^2(g), \quad \forall g \in A,\ s \in [0, T]\right] \underset{t\to\infty}{\longrightarrow} 1.$$

From this fact we will deduce the desired convergence result based on the analysis of the infinitely divisible laws $\mathcal{L}[\xi_t^i | (\eta_s)_{s\in\mathbb{R}}]$, $i = 1, 2$, where the reactant processes are constructed on the same probability space using the *coupled* catalyst. Here are the details.

We start now the two coupled catalyst processes η^1 and η^2 from above at time $-t$ by using the realization of (η_s^1, η_s^2) and considering on the same probability space

$$\text{(3.57)} \qquad (\eta_s^{1,-t},\ \eta_s^{2,-t})_{s\geq -t} = (\eta_{t+s}^1,\ \eta_{t+s}^2)_{s\geq -t}.$$

For both these processes we construct on one probability space the two reactant processes $(\xi_s^{i,-t})_{s\geq 0}$ which start at time $-t$ in the *same* realization of $\mathcal{H}_{\theta_\xi}$ for every given realization of the catalyst. We abbreviate

$$\text{(3.58)} \qquad \Gamma_{\eta^i}^{-t} = \mathcal{L}[\xi_0^{i,-t} | (\eta_s^{i,-t})_{s\geq -t}], \quad i = 1, 2.$$

Let $(\widetilde{\Gamma}_{\eta^i}^{-t})_g$ denote the canonical Palm distributions of $\Gamma_{\eta^i}^{-t}$ in the point $g \in G$.

We will prove that (3.56) implies the relation:

$$\text{(3.59)} \qquad (\widetilde{\Gamma}_{\eta^1}^{-t})_g - (\widetilde{\Gamma}_{\eta^2}^{-t})_g \underset{t\to\infty}{\Longrightarrow} 0 - \text{measure, in} \quad (\eta^1, \eta^2) - \text{probability}.$$

Then by Lemma 10.8 in [K2]

$$\text{(3.60)} \qquad (\Gamma_{\eta^1}^{-t}) - (\Gamma_{\eta^2}^{-t}) \underset{t\to\infty}{\Longrightarrow} 0 - \text{measure, in} \quad (\eta^1, \eta^2) - \text{probability}.$$

From this relation we obtain as before in (3.46) that

$$\text{(3.61)} \qquad \mathcal{L}\left[(\eta_0^{1,-t},\ \xi_0^{1,-t})\right] - \mathcal{L}\left[(\eta_0^{2,-t},\ \xi_0^{2,-t})\right] \underset{t\to\infty}{\Longrightarrow} 0 - \text{measure}.$$

This concludes the proof once we have verified (3.59).

At this point in order to prove (3.59) we bring into play the coupling result (3.56) and the explicit representation of the canonical Palm distribution. This representation was described in Section 2(b). The key is the following. If we consider a reactant starting with a single particle, then we can achieve that with high probability it sees the same medium whether we work with $\eta^{1,-t}$ or $\eta^{2,-t}$ provided his descendents do not migrate too far. Here are the details.

For $t \geq T > 0$, consider branching random walks $(\xi_s^{i,-t,-T,g})_{s \geq -T}$, $i = 1$ or 2, with migration kernel b starting at time $-T$ with one particle in g and using the medium $\eta^{i,-t}$, $i = 1,2$ (recall the catalyst process $\eta^{i,-t}$ starts at time $-t$). As a consequence of the coupling result in (3.56) we conclude that for every $A \subseteq G$ with $|A| < \infty$ and $T > 0$ we can couple the reactant processes for $i = 1$ and $i = 2$ such that for every fixed $g' \in G$:

$$\text{(3.62)} \qquad \mathbf{P}\Big[\xi_s^{1,-t,-T,g}(g') = \xi_s^{2,-t,-T,g}(g') \text{ for all } g \in A,\ s \in [-T,0]\Big] \underset{t\to\infty}{\longrightarrow} 1.$$

Namely we simply have to choose $B \supseteq A$ such that $\mathbf{P}[Y_s^g \in B,\ \forall s \in [0,T]] \geq 1 - \varepsilon/2$ for all $g \in A$ and then $t \geq t_0(\varepsilon)$ such that $\mathbf{P}[\eta_s^{1,-t}(g) = \eta_s^{2,-t}(g),\ \forall g \in B,\ s \in [-T,0]] \geq 1 - \varepsilon/2$ (see (3.56)). Then conclude that the l.h.s. of (3.62) exceeds $1-\varepsilon$ for $t \geq t_0(\varepsilon)$, if we use in the construction of $\xi_s^{i,-t,-T,g}$, $i = 1,2$, the same branching events in B and the same collection of walks for the jointly generated particles.

Finally to handle particles performing large excursions, that is particles breaking off the Kallenberg tree in A but leave B between $[-T,0]$ or particles breaking of before time $-T$, consider the random variable which is defined as follows. Let $(\bar{Y}_t^{g'})$ be a random walk with transition rates $\bar{b}$ starting in g' and consider a Poisson point process on $[-t,0]$ with intensity $\eta_s^{i,-t}(\bar{Y}_{-s}^{g'})ds$ at the time point $s \in [-t,0)$. Denote the realizations by $s_1, s_2, \ldots$. Define the random variable:

$$\text{(3.63)} \qquad R_t^{i,T} = \sum_{k \in I_{t,T}} \xi_0^{i,-t,-s_k,g_k}(g'), \quad g_k = \bar{Y}_{-s_k}^{g'}$$
$$\text{with } I_{t,T} = \{k : s_k \leq -T \text{ or } g_k \notin B\}.$$

Then for every $t \geq T$ (probability over random walk and Poisson point process):

$$\text{(3.64)} \qquad \mathbf{P}\left[R_t^{i,T} > 0\right] \leq \theta_\eta \cdot \int_T^\infty \hat{b}_{2s}(g',g')ds + \theta_\eta \int_0^T \sum_{g \in G \setminus B} b_s(g',g)b_s(g,g')ds \underset{T\to\infty}{\longrightarrow} C_B,$$

where $C_B \downarrow 0$ as $B \uparrow \mathbb{Z}^d$. Return now to the representation of the canonical Palm distributions $(\tilde{\Gamma}_{\eta^i}^{-t})_g$, $i = 1,2$, in Proposition 2.2 and note that the contributions can be split in two such that either (3.62) or (3.64) applies. Now the combination of (3.62) and (3.64) proves (3.59).

(ii) Recall (3.53) and (3.54). Now we prove the second part of (3.54). Since both initial laws of the processes which we compare have the same projection on the η-component we can construct (η_0^1, ξ_0^1) and (η_0^2, ξ_0^2) in such a way that $\eta_0^1 = \eta_0^2$. Then we work with one given medium $(\eta_t)_{t \geq 0}$.

It is easy to see that for a given medium $(\eta_t)_{t\geq 0}$ we can construct a coupling between two processes ξ^1_t, ξ^2_t starting in different initial states but moving in the *same medium.*

- migration of one particle from g to g' in both components occurs at rates $b(g,g')$ $(\xi^1_t(g) \wedge \xi^2_t(g))$ and only in one of the components $i=1,2$ at rates $b(g,g')(\xi^1_t(g)-\xi^2_t(g))^{\pm}$.
- Branching of one particle in both components in g occurs at rate $\eta_t(g)(\xi^1_t(g) \wedge \xi^2_t(g))$ and at rate $\eta_t(g)(\xi^1_t(g)-\xi^2_t(g))^{\pm}$ only in one of the components $i=1,2$.

Then going over the proof of (3.55) (for some details see below) one sees that for $\hat{b}$ transient the coupling is successful, that is

$$(3.65) \qquad \mathbf{E}\left[|\xi^1_t(g)-\xi^2_t(g)|\right] = \mathbf{E}\left[|\xi^1_t(0)-\xi^2_t(0)|\right] \searrow 0 \quad \text{as} \quad t \to \infty.$$

In order to understand this note that the convergence to 0 in (3.65) is due to the *migration* and has nothing to do with the branching mechanism as long as the latter is critical for given medium. Next we give some information on the formal details.

The key point is that for the coupled dynamic $(\eta_t, \xi^1_t, \xi^2_t)$ we have with $\mathcal{G}$ denoting the generator of this evolution and with $h((\eta,\xi^1,\xi^2)) = |\xi^1(g)-\xi^2(g)|$:

$$(3.66) \qquad \begin{aligned}(\mathcal{G}h)((\eta,\xi^1,\xi^2)) \quad &= \quad \sum_{g'\in G} b(g',g)\left(|\xi^1(g')-\xi^2(g')| - |\xi^1(g)-\xi^2(g)|\right) \\ &\quad -2\sum_{g'\in G} b(g',g)|\xi^1(g')-\xi^2(g')|\Delta_{g,g'}(\xi^1,\xi^2),\end{aligned}$$

where

$$(3.67) \qquad \Delta_{g,g'}(\xi^1,\xi^2) = 1((\xi^1(g)-\xi^2(g))\cdot(\xi^1(g')-\xi^2(g'))<0).$$

Next note that

$$(3.68) \qquad \frac{d}{dt}\mathbf{E}[|\xi^1_t(g)-\xi^2_t(g)|] = \mathbf{E}[(\mathcal{G}h)(\eta_t,\xi^1_t,\xi^2_t)]$$

and that the r.h.s. can be evaluated explicitely using (3.66).

In the expression on the r.h.s. of (3.66) the medium does not enter explicitly. Nevertheless for given catalyst (ξ^1_t,ξ^2_t) is not spatially homogeneous. But after averaging over the medium again as in the homogeneous case the first term in (3.66) disappears, so that $\mathbf{E}|\xi^2_t(g)-\xi^2_t(g)|$ is non–increasing in t. In fact this expectation is strictly decreasing as long as the discrepancies between the two processes in points g and g' with $b(g',g)>0$ have different sign. This fact together with the irreducibility of b and the ergodicity of limiting states gives (3.65). This part of the argument carries over from the homogeneous case, we do not repeat all the details at this point and refer to [G].

Recall that $\mathcal{N}(G)$ carries the vague topology. Hence from (3.65) it is clear that the weak limits of $\mathcal{L}[(\eta_t,\xi^1_t)]$ and $\mathcal{L}[(\eta_t,\xi^2_t)]$ for $t\to\infty$ agree. We get therefore the convergence of $\mathcal{L}[(\eta_t,\xi^i_t))$ to the same limit as $t\to\infty$ if we start in μ respectively $\nu\otimes\mathcal{H}_{\theta_\xi}$, which completes the proof of (3.54) part (ii).

4 Catalysts locally dying out: Proof of Theorem 2

(a) The reactant is asymptotically Poisson: Proof of Theorem 2 (a)

Since under the assumptions of the theorem, the catalyst becomes locally extinct, it suffices to show that this implies indeed that the reactant converges to a Poisson system with intensity θ_ξ. The proof of the latter fact will be based on the following idea. The catalyst is site–transient and we can show that larger and larger areas are eventually vacant. As a consequence, since also for a typical reactant particle there is a positive probability to survive until it reaches such a catalyst free area and then remain in it forever it can in fact survive forever with positive probability. This we will prove in Step 1. Together with preservation of intensity for finite times this will give in Step 2 the assertion.

Step 1 (Surviving reactant particle)
For $T > 0$ and $x \in \mathbb{Z}$ define $(\xi_t^{x,T})_{t \geq T}$ to be the reactant process in the catalytic medium $(\eta_t)_{t \geq 0}$ starting with one particle at x at time $T \geq 0$. Define the probability σ_T that a single reactant particle started in 0 at time T will never branch by

$$\sigma_T = \mathbf{P}[\|\xi_t^{0,T}\| \equiv 1,\, t \geq T]. \tag{4.1}$$

Lemma 4.1 *For $\mathcal{L}[\eta_0]$ stationary with $\mathbf{E}[\eta_0(x)] = \theta_\eta < \infty$ the following holds*

$$\liminf_{T \to \infty} \sigma_T = 1. \qquad \diamond \tag{4.2}$$

Proof Recall that $b(\cdot,\cdot)$ has finite moments of order β. We begin by introducing the following two types of time dependent sets in the time–space diagram. These sets depend on two parameters L and γ, where $\gamma \in (\frac{1}{\beta} \vee (\frac{1}{2} + \frac{1}{\alpha}),\, 1)$ is fixed and $L > 0$ will be adapted later on to obtain with high probability two things: (1) catalyst free regions in the sense that all catalyst particles descending from an ancestor in that region have died out and (2) regions which the random walk (Y_t) does not leave. These two facts we will prove below in (i) and (ii) respectively.

(i) In order to define a catalyst free region we need three objects:

$$U_{L,\gamma}(x) = LK_0(|x|+1)^{1/\gamma}, \qquad x \in \mathbb{Z}, \tag{4.3}$$

where $K_0 = K_0(\gamma)$ is given by

$$K_0 = 2\theta_\eta \sum_{x \in \mathbb{Z}} (|x|+1)^{-1/\gamma}, \tag{4.4}$$

$$V_{L,\gamma}(t) = \{x \in \mathbb{Z} : U_{L,\gamma}(x) \leq t\} = \left\{x \in \mathbb{Z} \,:\, |x| \leq \left(\frac{t}{K_0 L}\right)^\gamma - 1\right\} \tag{4.5}$$

$$W_{L,\gamma}(t) = \left\{x \in \mathbb{Z} \,:\, |x| \leq \frac{1}{2}\left(\frac{t}{K_0 L}\right)^\gamma - 1\right\}. \tag{4.6}$$

See Figure 1 on page 12 for an illustration of the set

$$W_{L,\gamma} = \{(t,x) :\, x \in W_{L,\gamma}(t)\}. \tag{4.7}$$

Next we define the event $B_{L,\gamma}$ that at time t all catalyst particles started at time 0 in $V_{L,\gamma}(t)$ have died out already. Recall that by construction of the process we can decompose $(\eta_t)_{t\geq 0}$ into independent branching random walks $\{(\eta_t^{x,i})_{t\geq 0},\ x\in\mathbb{Z}, i\in\mathbb{N}\}$,

$$\eta_t = \sum_{i=1}^{\eta_0(x)} \eta_t^{x,i}. \tag{4.8}$$

Let

$$B_{L,\gamma} = \left\{ \|\eta^{x,i}_{U_{L,\gamma}(x)}\| = 0 \text{ for all } x\in\mathbb{Z},\, 1\leq i\leq \eta_0(x) \right\}. \tag{4.9}$$

Since for all $x\in\mathbb{Z}$ and $i\in\mathbb{N}$, $\mathbf{P}[\|\eta_t^{x,i}\|>0] = \frac{2}{2+t}$ (see Lemma 2.5), we have

$$\begin{aligned}\mathbf{P}[B_{L,\gamma}] &\geq 1-\sum_{x\in\mathbb{Z}} \mathbf{E}[\eta_0(x)]\frac{2}{2+U_{L,\gamma}(x)} \\ &= 1-\sum_{x\in\mathbb{Z}} \frac{2\theta_\eta}{2+U_{L,\gamma}(x)} \geq 1-\frac{1}{L}.\end{aligned} \tag{4.10}$$

Hence for $\varepsilon>0$ and $L>2/\varepsilon$

$$\mathbf{P}[B_{L,\gamma}] \geq 1-\frac{\varepsilon}{2}. \tag{4.11}$$

(ii) Now we define the event that a realization of the random walk with transition probabilities (b_t), denoted by (Y_t), does not leave $W_{L,\gamma}$ at large times:

$$A_{T,L,\gamma} = \{Y_{t-T}\in W_{L,\gamma}(t)\ \forall\, t\geq T\}, \quad T>0. \tag{4.12}$$

The next argument makes use of the assumption that $\gamma>1/\beta$. Recall that $\mathbf{E}[Y_1]=0$ and $\mathbf{E}[|Y_1|^\beta]<\infty$. By a sharpened version of the strong law of large numbers (see, e.g., Durrett (96), Ch. 1, Thm. 8.8) for $\varepsilon>0$ there is a $T_0\geq LK_0$ such that for $T\geq T_0$,

$$\mathbf{P}[A_{T,L,\gamma}] \geq 1-\frac{\varepsilon}{2}. \tag{4.13}$$

Now we are ready to return to σ_T (recall (4.1)). Clearly,

$$\sigma_T = \mathbf{E}\left[\exp\left(-\int_T^\infty \eta_t(\{Y_{t-T}\})\,dt\right)\right]. \tag{4.14}$$

Hence (abbreviating $A=A_{T,L,\gamma}$ and $B=B_{L,\gamma}$)

$$\begin{aligned}\sigma_T &\geq \mathbf{E}\left[1_{A\cap B}\exp\left(-\int_T^\infty \eta_t(\{Y_{t-T}\})\,dt\right)\right] \\ &= \mathbf{E}\left[\exp\left(-1_{A\cap B}\int_T^\infty \eta_t(\{Y_{t-T}\})\,dt\right)\right] - \mathbf{P}[(A\cap B)^{\mathrm{c}}] \\ &\geq \exp\left(-\int_T^\infty \mathbf{E}\left[1_{A\cap B}\,\eta_t(\{Y_{t-T}\})\right]dt\right) - \varepsilon.\end{aligned} \tag{4.15}$$

Now (recall that (X_t) is a random walk with transition kernels (a_t))

$$
\begin{aligned}
\mathbf{E}\left[1_{A\cap B}\,\eta_t(\{Y_{t-T}\})\right] &\le \sup\left\{\mathbf{E}\left[1_B\eta_t(\{x\})\right],\; x\in W_{L,\gamma}(t)\right\}\\
&= \sup\left\{\theta_\eta \sum_{y\in(V_{L,\gamma}(t))^c} a_t(y,x),\; x\in W_{L,\gamma}(t)\right\} \qquad (4.16)\\
&\le \theta_\eta \mathbf{P}\left[|X_t|\ge \frac{1}{2}\left(\frac{t}{K_0L}\right)^{\gamma}\right].
\end{aligned}
$$

Recall that we assumed that for some $\alpha > 2$

$$C := \mathbf{E}[|X_1|^\alpha] < \infty.$$

Hence using an inequality of Marcinkiewicz–Zygmund (see, e.g., Dharmadhikari, Fabian and Jogdeo (1968), cf. also Burkholder (1973), Theorem 3.2) we get

$$\mathbf{E}[|X_t|^\alpha] \le C\,C_\alpha\, t^{\alpha/2}, \qquad t\ge 1, \tag{4.17}$$

for some universal constant $C_\alpha < \infty$. Using Chebyshev's inequality we get

$$
\begin{aligned}
\mathbf{P}\left[|X_t|\ge \frac{1}{2}\left(\frac{t}{K_0L}\right)^{\gamma}\right] &\le C\,C_\alpha \frac{t^{\alpha/2}}{((t/K_0L)^\gamma/2)^\alpha} \qquad (4.18)\\
&= 2^\alpha (K_0L)^{\alpha\gamma} C\,C_\alpha t^{(\alpha/2)-\alpha\gamma}.
\end{aligned}
$$

Since we assumed $\gamma > \frac{1}{2}+\frac{1}{\alpha}$, we have $\frac{\alpha}{2}-\alpha\gamma < -1$. Plugging this estimate in (4.16) we get

$$\int_T^\infty dt\;\mathbf{E}\left[1_{A\cap B}\,\eta_t(\{Y_{t-T}\})\right] \le \theta_\eta \frac{2^{1+\alpha}(K_0L)^{\alpha\gamma}C\,C_\alpha}{2+\alpha-\alpha\gamma} T^{1-\alpha\gamma-\alpha/2} \longrightarrow 0, \quad T\to\infty. \tag{4.19}$$

Hence by (4.15)

$$\liminf_{T\to\infty} \sigma_T \ge 1-\varepsilon. \tag{4.20}$$

Since $\varepsilon > 0$ was arbitrary, (4.2) holds. $\square$

With a little more effort we can extend the proof of Lemma 4.1 and show that $W_{L,\gamma}$ becomes vacant for large L if $\alpha > 4$ (see Figure 1 on page 12). Technically we will not make use of that result but we state it here since we think that it is instructive.

Lemma 4.2 *Assume that $\alpha > 4$. Then*

$$\liminf_{L\to\infty}\; \mathbf{P}[\eta_t(W_{L,\gamma}(t)) = 0 \textit{ for all } t\ge 0] = 1.$$

Proof We give an elementary proof only for the case where a is the kernel of a Bernoulli random walk. The more general case can be handled using Kesten's result on the range of BRW (see Proposition 1.3).

Fix $\varepsilon > 0$. Let $L > 2/\varepsilon$ and define

$$x_t = \sup \left\{ x \in \mathbb{N}_0 \,:\, x \le \frac{1}{2}\left(\frac{t}{K_0 L}\right)^{\gamma} - 1 \right\}.$$

Recall that $B = B_{L,\gamma}$ from (4.10). Using (4.16) and (4.18) we can choose T_0 such that for $T \ge T_0$

$$\int_T^\infty \mathbf{E}[1_B \eta_t(\{-x_t - 1, -x_t, -x_t + 1, x_t - 1, x_t, x_t + 1\})]\, dt \le \varepsilon/2. \tag{4.21}$$

Hence for T large enough

$$\mathbf{P}\left[1_B \eta_t(\{-x_t, x_t\}) > 0 \text{ for some } t \ge T\right] \le \varepsilon. \tag{4.22}$$

Together with (4.11) we have

$$\mathbf{P}\left[\eta_t(\{-x_t, x_t\}) = 0 \text{ for all } t \ge T\right] \ge 1 - 2\varepsilon. \tag{4.23}$$

Given the event in (4.23) all particles in $W_{L,\gamma}(t)$ for some $t \ge T$ stem from ancestors in $W_{L,\gamma}(T)$ at time T. However the probability that any of these particles has an offspring alive at time t is bounded by

$$|W_{L,\gamma}(T)| \cdot \theta_\eta \cdot \frac{2}{2 + (t - T)} \to 0, \quad t \to \infty. \tag{4.24}$$

Hence for $T' > T$ large enough

$$\mathbf{P}[\eta_t(W_{L,\gamma}(t)) = 0 \text{ for all } t \ge T'] \ge 1 - 3\varepsilon. \tag{4.25}$$

Note that $W_{L,\gamma}(t)$ is decreasing in L. Hence we can assume that L is so large that $W_{L,\gamma}(T') = \emptyset$ making the condition $t \ge T'$ void:

$$\mathbf{P}[\eta_t(W_{L,\gamma}(t)) = 0 \text{ for all } t \ge 0] \ge 1 - 3\varepsilon. \tag{4.26}$$

This concludes the proof. □

Step 2 (Preservation of intensity)
With Lemma 4.1 at hand the statement of Theorem 2 seems rather obvious: For a fixed time $T > 0$ there is preservation of mass in the reactant

$$\mathbf{E}[\xi_T] = \theta_\xi \ell.$$

Lemma 4.1 says that after a large time T the particles of (ξ_t) essentially perform independent random walks and hence ξ_t converges in distribution to a Poisson field with intensity θ_ξ. Although this argument is quite convincing it might be necessary to give a formal proof.

We will define for fixed (large) time T two processes $(\xi_{T,t})_{t\geq T}$ and $(\xi'_{T,t})_{t\geq T}$. The initial state of both processes is assumed to coincide with ξ_T. For $t \geq T$ the particles of $(\xi_{T,t})$ evolve as independent random walks with transition kernel given by (b_t). For $t \geq T$ the motion of particles of $(\xi'_{T,t})$ coincides with that of $(\xi_{T,t})$ but we impose an additional killing in contact with the catalyst. In other words, $(\xi'_{T,t})$ is "branching" random walk in the catalytic medium η with offspring probability $q_0 = 1$, $q_k = 0$, $k \geq 1$. Clearly we can construct the three processes $(\xi_{T,t})_{t\geq T}$, $(\xi'_{T,t})_{t\geq T}$ and $(\xi_t)_{t\geq 0}$ on one probability space such that

$$(4.27)\qquad \xi'_{T,t} \leq \xi_{T,t}, \quad t \geq T \quad \text{a.s.}$$

and

$$(4.28)\qquad \xi'_{T,t} \leq \xi_t, \quad t \geq T \quad \text{a.s.}$$

Note that $\mathcal{L}[(\xi_t, \xi_{T,t}, \xi'_{T,t})]$ is translation invariant for all $t \geq T$. From Lemma 4.1 we know that

$$(4.29)\qquad \mathbf{E}[\xi_t - \xi'_{T,t}] = \mathbf{E}[\xi_{T,t} - \xi'_{T,t}] \leq \theta_\xi(1-\sigma_T)\ell, \quad t \geq T.$$

Hence

$$(4.30)\qquad \mathbf{E}[|\xi_t - \xi_{T,t}|] \leq 2\theta_\xi(1-\sigma_T)\ell, \quad t \geq T.$$

Since $\mathbf{E}[\xi_t] = \theta_\xi \ell$ is bounded, $\{\mathcal{L}[\xi_t],\ t \geq 0\}$ is tight. Let μ be a weak limit point of $\mathcal{L}[\xi_t]$ as $t \to \infty$ and let $t_n \to \infty$ be a sequence such that $\mathcal{L}[\xi_{t_n}] \Longrightarrow \mu$, $n \to \infty$. We are done if we can show that $\mu = \mathcal{H}_{\theta_\xi}$.

For every $T > 0$ we have

$$(4.31)\qquad \mathcal{L}[\xi_{T,t_n}] \Longrightarrow \mathcal{H}_{\theta_\xi}, \quad n \to \infty.$$

Hence we can find a sequence $T_n \to \infty$, $n \to \infty$ such that

$$(4.32)\qquad \mathcal{L}[\xi_{T_n,t_n}] \Longrightarrow \mathcal{H}_{\theta_\xi}, \quad n \to \infty.$$

Since

$$(4.33)\qquad \mathbf{E}[|\xi_{t_n} - \xi_{T_n,t_n}|] \leq 2\theta_\xi(1-\sigma_{T_n}) \to 0, \quad n \to \infty$$

it is clear that $\mu = \mathcal{H}_{\theta_\xi}$.

(b) The reactant dies out locally: Proof of Theorem 2 (b)

Note first that the catalyst process $(\eta_t)_{t\geq 0}$ becomes locally extinct, i.e. $\mathcal{L}[\eta_t] \underset{t\to\infty}{\Longrightarrow} \delta_{\underline{0}}$, since it is a branching random walk where the symmetrized kernel $\hat{a}$ governing the migration is recurrent. Therefore it suffices to prove that the reactant dies out locally, that is $\lim_{T\to\infty} \mathbf{P}[\xi_T(x) > 0] = 0$. In order to cope with the randomness of the medium, we will show the equivalent statement:

$$(4.34)\qquad \limsup_{T\to\infty} \mathbf{P}[\xi_T(0) > 0] < \varepsilon \quad \text{for all } \varepsilon > 0.$$

The idea of the proof is as follows. From Lemma 2.3 in Section 2 we know that it suffices to establish (4.34) for a smaller medium. We will use as such a smaller medium one where we consider only one family of catalyst particles, which has a large population at the time T considered. Next we use a version of the Kallenberg criterion to establish the relation (4.34) above, by comparing the population $\widehat{\xi}_T(0)$, distributed for given medium according to the Palm distribution $(\mathcal{L}[\widehat{\xi_T(0)|\eta}])$, with a population generated via the Kallenberg representation (Proposition 2.2 (a)) of the Palm distribution but now for the simplified and smaller medium.

The above discussion shows that we need to work out mainly two things. (i) First we have to establish that the catalyst has within the range of the backward reactant path a site from which a family survives until times of order T and which consists over time spans of this order of order T particles. (ii) We have to analyze the behavior of the reactant population generated by the backbone in the medium consisting of the one single catalyst family, which was analyzed in the first part (i). Here are now the details.

(i) A caricature of the medium

Step 1 Recall that the reactant random walk has a drift and hence a range of order T. Therefore we would like to establish that the probability of a single ancestor catalyst family to produce a family consisting of at least δT particles during the time span say $[T/2, 2T]$ is of order T^{-1}. A classical result by Lamperti and Ney asserts that a *continuous time critical binary Galton–Watson process* $(Z_t)_{t\geq 0}$ with $Z_0 = 1$ has the property that

$$\mathcal{L}\left[T^{-1}(Z_{\alpha T})_{\alpha\in[0,2]}|Z_{2T} > 0\right] \tag{4.35}$$

converges to a limiting diffusion with positive drift which would imply the desired result. The reference is not easily available, so we give a self contained proof of the statement which we actually need.

Lemma 4.3 *Let $(Z_t)_{t\geq 0}$ be a continuous time critical binary Galton–Watson process. Then*

$$\liminf_{\delta\to 0}\ \liminf_{T\to\infty}\{T\mathbf{P}[Z_t \geq \delta T,\ t\in[T/2,2T]]\} \geq e^{-1}. \qquad \diamond \tag{4.36}$$

Proof Fix $\delta > 0$ and $\beta > \delta$. From Lemma 2.5 we know that:

$$\liminf_{T\to\infty}\{T\mathbf{P}[Z_{T/2} \geq T/4]\} = 4/e. \tag{4.37}$$

Note that by the branching property $\mathbf{P}[Z_t \geq a|Z_0 = n]$ is nondecreasing in n for every $a \geq 0$. Therefore we get from (4.37) together with the Markov property of $(Z_t)_{t\geq 0}$ that

$$\begin{aligned}&\liminf_{T\to\infty}\{T\mathbf{P}[Z_t \geq \delta T,\ t\in[T/2,2T]|Z_0 = 1]\}\\ &\quad\geq\ 4e^{-1}\liminf_{T\to\infty}\{\mathbf{P}[Z_t \geq \delta T,\ t\in[0,3T/2]|Z_0 = [T/4]]\}.\end{aligned} \tag{4.38}$$

In order to analyze the r.h.s. above we use Chebyshev's inequality to get

$$\begin{aligned}&\mathbf{P}[Z_{3T/2} \geq \beta T|Z_t \leq \delta T \text{ for some } t\in[0,3T/2],\ Z_0 = [T/4]]\\ &\qquad\leq (\beta T)^{-1}\mathbf{E}[Z_{3T/2}|Z_t \leq \delta T \text{ for some } t\in[0,3T/2],\ Z_0 = [T/4]].\end{aligned} \tag{4.39}$$

Consider now the stopping time $(3T/2) \wedge \inf\{t | Z_t \leq \delta T\}$ and use the strong Markov property to estimate the expectation on the r.h.s. of (4.39) by δT to arrive at:

$$\text{(4.40)} \quad \mathbf{P}[Z_{3T/2} \geq \beta T | Z_t \leq \delta T \text{ for some } t \in [0, 3T/2],\ Z_0 = [T/4]] \leq \delta/\beta.$$

This estimate we can rewrite as:

$$\text{(4.41)} \quad \begin{aligned} &\mathbf{P}[Z_t \leq \delta T \text{ for some } t \in [0, 3T/2] | Z_0 = [T/4]] \\ &\qquad \leq (1 - \delta/\beta)^{-1}(1 - \mathbf{P}[Z_{3T/2} \geq \beta T | Z_0 = [T/4]]). \end{aligned}$$

By the branching property we estimate the second factor on the r.h.s. of (4.41) further as follows:

$$\text{(4.42)} \quad \limsup_{T\to\infty} \mathbf{P}[Z_{3T/2} < \beta T | Z_0 = [T/4]] \leq \limsup_{T\to\infty} (\mathbf{P}[Z_{3T/2} < \beta T | Z_0 = 1])^{[T/4]}$$

By Yagloms theorem we know (recall Lemma 2.5) that $\mathbf{P}[Z_T \geq \beta T | Z_0 = 1]$ is asymptotically equivalent to $\frac{2}{T}\exp(-2\beta)$. Hence

$$\text{(4.43)} \quad \limsup_{T\to\infty} \mathbf{P}[Z_{3T/2} < \beta T | Z_0 = [T/4]] \leq \exp\left(-\frac{1}{3}\exp\left(-\frac{4}{3}\beta\right)\right).$$

Combining (4.41) - (4.43) and inserting the result in (4.38) yields:

$$\text{(4.44)} \quad \liminf_{T\to\infty} T\mathbf{P}[Z_t \geq \delta T,\ t \in [T/2, 2T]] \geq 4e^{-1}\Big(1 - \frac{\exp(-\frac{1}{3}\exp(-\frac{4}{3}\beta))}{1 - \delta/\beta}\Big).$$

Therefore

$$\text{(4.45)} \quad \limsup_{\delta\to 0}\ \limsup_{T\to\infty} T\mathbf{P}[Z_t \geq \delta T,\ t \in [T/2, 2T]] \geq 4e^{-1}\Big(1 - \exp\Big(-\frac{1}{3}\exp\Big(-\frac{4}{3}\beta\Big)\Big)\Big).$$

The r.h.s. of (4.45) converges for $\beta \to 0$ to $4e^{-1}(1 - e^{-1/3})$ which is bigger than e^{-1} as claimed.
□

Step 2 In this step we are going to define for every T the caricature $\eta^T \leq \eta$ of our medium. W.l.o.g. assume that the drift

$$h := \sum_{x=-\infty}^{\infty} b(0, x)x$$

is positive. For defining the caricature we need a subset S_T of the space–time diagram which describes for every site x the time range for which we want the catalyst in that site to be large (recall that the backward reactant path has drift $-h$):

$$\text{(4.46)} \quad S_T = \{(t, x) \in [0, \infty) \times \mathbb{Z} : (T + h^{-1}x)/2 \leq t \leq 2(T + h^{-1}x)\}.$$

Since for x close to $-hT$ or 0 this set will be narrow in the time component and hence many such sites will carry surviving catalysts we restrict the x considered. We define for $\varepsilon_1 > 0$ (which we will choose later in (4.54)):

$$\text{(4.47)} \quad I_{\varepsilon_1,T} = \{x \in \mathbb{Z} :\ -(1 - \varepsilon_1)hT \leq x \leq -\varepsilon_1 hT\}.$$

Define now for $\gamma > 0$ the events

$$\tilde{C}^\gamma_{x,T} = \{\eta_t(x) \geq \gamma T, \ t \in [(T + h^{-1}x)/2, \ 2(T + h^{-1}x)]\}. \tag{4.48}$$

$$C^\gamma_T = \bigcup_{x \in I_{\varepsilon_1,T}} \tilde{C}^\gamma_{x,T}. \tag{4.49}$$

It will be useful to decompose C^γ_T into disjoint events of the form $\tilde{C}^\gamma_{x,T}$. We define

$$C^\gamma_{x,T} = \tilde{C}^\gamma_{x,T} \cap \left(\bigcap_{\substack{y \in I_{\varepsilon_1,T} \\ y > x}} (\tilde{C}^\gamma_{y,T})^c \right), \qquad x \in I_{\varepsilon_1,T}. \tag{4.50}$$

We define now the caricature η^T of our medium η by

$$\eta^T_t(x) = \gamma T \ 1_{S_T}(t,x) \ 1_{C^\gamma_{x,T}}. \tag{4.51}$$

This should be interpreted as follows. We choose for each time T the rightmost site $x < 0$, where at time 0 a catalyst family started which survives till a late time (depending on h) and which has a large population (order T) over a time span of this order. Then the whole medium is replaced by the one generated by the single family of catalysts.

For the definition (4.51) to be useful we need to show that we can for given $\varepsilon > 0$ find T_0 and γ_0 such that for $T \geq T_0$ and $\gamma \leq \gamma_0$ the probability of C^γ_T is bigger than $1 - \varepsilon$. This will be carried out in the next step.

Step 3 In order to estimate $\mathbf{P}[C^\gamma_t]$ we note first that by Lemma 4.3 applied with $\delta = \gamma$, we can find $\gamma_0 > 0, \ T_0 < \infty$ such that for $T > T_0, \ 0 < \gamma < \gamma_0$:

$$\inf_{x \in I_{\varepsilon_1,T}} ((T + h^{-1}x)\mathbf{P}[\tilde{C}^\gamma_{x,T}]) > e^{-1}. \tag{4.52}$$

Assume for the moment that $\eta_0(x) = 1$ for all $x \in \mathbb{Z}$ a.s. Therefore using the independence of the $\{(\eta_t(x))_{t \geq 0}, x \in \mathbb{Z}\}$ we get:

$$\begin{aligned} \liminf_{T \to \infty} \mathbf{P}[C^\gamma_T] &\geq 1 - \limsup_{T \to \infty} \prod_{x \in I_{\varepsilon_1,T}} (1 - (T + h^{-1}x)^{-1} e^{-1}) \\ &= 1 - \limsup_{T \to \infty} \prod_{x \in I_{\varepsilon_1,T}} \left(1 - \frac{h}{x} e^{-1}\right) \\ &= 1 - \limsup_{T \to \infty} \exp\left(\sum_{x \in I_{\varepsilon_1,T}} \log(1 - \frac{h}{x} e^{-1}) \right) \\ &= 1 - \limsup_{T \to \infty} \exp\left(- \sum_{x \in I_{\varepsilon_1,T}} \frac{h}{x} e^{-1} \right) \end{aligned} \tag{4.53}$$

$$
\begin{aligned}
&= 1 - \limsup_{T\to\infty} \exp\left(-\sum_{x\in I_{\varepsilon_1,T}} \frac{h}{x} e^{-1}\right)\\
&= 1 - \exp\left(-\frac{h}{e}\int_{\varepsilon_1}^{1-\varepsilon_1} \frac{1}{x}dx\right)\\
&= 1 - \exp\left(-\frac{h}{e}(\log\varepsilon_1 - \log(1-\varepsilon_1))\right)\\
&= 1 - \left(\frac{\varepsilon_1}{1-\varepsilon_1}\right)^{h/e}.
\end{aligned}
$$

Now we choose

$$
(4.54) \qquad \varepsilon_1 < \left(1 + \left(\frac{2}{\varepsilon}\right)^{e/h}\right)^{-1}
$$

to get for $T \geq T_0(\varepsilon)$ and $\gamma \leq \gamma_0(\varepsilon)$

$$
(4.55) \qquad \mathbf{P}[C_T^\gamma] \geq 1 - \varepsilon/2.
$$

The case of general ergodic initial law for η can be handled with the ergodic theorem. We omit the details.

(ii) The reactant in the caricature of the medium

We begin by pointing out what is now needed to prove the assertion (4.34). Denote by $(\xi_t^T)_{t\geq 0}$ the reactant process evolving in the medium $(\eta_t^T)_{t\geq 0}$, which was defined in (4.51). We know from Lemma 2.3 that

$$
(4.56) \qquad \mathbf{P}[\xi_T^T(0) > 0] \geq \mathbf{P}[\xi_T(0) > 0].
$$

In view of (4.55) it suffices therefore in order to prove (4.34) to show that

$$
(4.57) \qquad \mathbf{P}[\xi_T^T(0) > 0 | C_{x,T}^\gamma] \leq \varepsilon/2 \quad \text{for all } x \in I_{\varepsilon_1,T}.
$$

For this it suffices to show that the conditional size–biased distributions diverge. We have established in Section 2, Proposition 2.3, that these distributions are stochastically larger than the Kallenberg distribution $\widehat{\xi}_T^T$ defined in (2.33). Therefore we have to show that for all $N > 0$ and T large:

$$
(4.58) \qquad \mathbf{P}[\widehat{\xi}_T^T(0) < N | C_{x,T}^\gamma] \leq \varepsilon/2 \quad \text{for all } x \in I_{\varepsilon_1,T},
$$

in order to conclude the proof of the relation (4.34).

The proof of (4.58) will now be carried out in several steps. The population $\widehat{\xi}_T^T(0)$ will be bounded below by a simpler population which we can approximate by a tractable object. This object will arise by considering a thinned out version of $\widehat{\xi}_t^T$ which will be obtained by removing reactant particles that follow ancestral paths which are difficult to treat explicitly. This will lead to an object which we are now going to study in the next step.

Step 1 Let $\mathcal{T}$ be a random tree which is generated by a critical binary Galton–Watson process. Fix a parameter $\varrho > 0$ and attach to each $v \in \mathcal{T}$ an independent (for given tree $\mathcal{T}$) Bernoulli random variable B_v^ϱ with expectation ϱ. Consider now the population Z_ϱ defined by the tree with branches removed from the first 1 on, i.e.

$$(4.59) \qquad Z_\varrho = |\{v \in \mathcal{T},\ B_v^\varrho = 1,\ B_w^\varrho = 0 \quad \text{for all } w < v\}|.$$

The distributional properties of Z_ϱ are summarized in:

Lemma 4.4

(a) The generating function $\varphi_\varrho(\lambda) = \mathbf{E}[\lambda^{Z_\varrho}],\ |\lambda| < 1$ *is given by*

$$(4.60) \qquad \varphi_\varrho(\lambda) = \frac{1}{1-\varrho}(1 - \sqrt{2\varrho(1-\varrho)(1-\lambda) + \varrho^2}).$$

(b) $\mathbf{E}[Z_\varrho] = 1, \quad \mathbf{Var}\,[Z_\varrho] = \varrho^{-1}$.

(c) $\mathbf{P}[Z_\varrho > 0] \sim \sqrt{2\varrho}$ *as* $\varrho \to 0$.

(d) Denote by $\widehat{Z}_\varrho$ *the size–biased population of* Z_ϱ*. Then the rescaled random variable converges to a Gamma(1/2) distribution.*

$$(4.61) \qquad \mathcal{L}\Big[\frac{1}{2}\varrho\widehat{Z}_\varrho\Big] \underset{\varrho \downarrow 0}{\Longrightarrow} \text{Gamma}(1/2). \qquad \diamond$$

Proof of Lemma 4.4

(a) Condition on the generation descending from the root $\emptyset$, to obtain the following equation

$$\begin{aligned}(4.62) \qquad \varphi_\varrho(\lambda) &= \mathbf{E}[\lambda^{Z_\varrho};\ B_\emptyset^\varrho = 1] + \mathbf{E}[\lambda^{Z_\varrho};\ B_\emptyset^\varrho = 0,\ \mathcal{T} \neq \{\emptyset\}] \\ &\quad + \mathbf{E}[\lambda^{Z_\varrho},\ B_\emptyset^\varrho = 0,\ \mathcal{T} = \{\emptyset\}] \\ &= \varrho\lambda + \frac{1-\varrho}{2}\varphi_\varrho^2(\lambda) + \frac{1-\varrho}{2}.\end{aligned}$$

Solving this quadratic equation in $\varphi_\varrho(\lambda)$ yields (4.60).

(b),(c) These are immediate consequences of the explicit formula for φ_ϱ.

(d) Define

$$(4.63) \qquad L_\varrho(\theta) = \mathbf{E}[e^{-\theta\widehat{Z}_\varrho}],$$

and recall that the Laplace transform of Gamma(1/2) is

$$(4.64) \qquad (1+\theta)^{-\frac{1}{2}}.$$

Furthermore note that the Laplace transform of the size–biased distribution of Z_ϱ is given in terms of Z_ϱ by

$$(4.65) \qquad L_\varrho(\theta) = \mathbf{E}[Z_\varrho \exp(-\theta Z_\varrho)] = \varphi_\varrho'(e^{-\theta})e^{-\theta} = \varrho e^{-\theta}(2\varrho(1-\varrho)(1-e^{-\theta}) + \varrho^2)^{-\frac{1}{2}}.$$

Hence

$$(4.66)\qquad \lim_{\varrho\to 0} L_\varrho(\varrho\theta/2) = \lim_{\varrho\to 0} \varrho(2\varrho(1-\varrho)(1-e^{-\varrho\theta}/2)+\varrho^2)^{-\frac{1}{2}} = (1+\theta)^{-\frac{1}{2}}. \qquad \square$$

Step 2 Now we return to the Kallenberg population $\widehat{\xi}^T_T$. We will construct a more tractable lower bound for this object, which is constructed on an event where the path of the backbone $(\bar{Y}_t)_{0\le t\le T}$ has nice properties and we will show that this event has probability close to 1 for large enough T.

Recall that the path $(\bar{Y}_t)_{t\ge 0}$ is a realization of a random walk with transition rates $\bar{b}$, that is in particular with drift $-h$. We need to control the time when the catalyst site x is reached first by the backbone $\bar{Y}_t$ and the time which is spent in this site. Here are the details.

Define

$$(4.67)\qquad \begin{aligned} \tau_x &= \inf\{t>0 : \bar{Y}_t = x\}, \quad x\le 0 \\ \tau^*_x &= \sup\{t>0 : \bar{Y}_t = x\}, \quad x\le 0. \end{aligned}$$

The sequence (recall $\bar{Y}_0 = 0$)

$$(4.68)\qquad \tau_{-1}, \tau_{-2}-\tau_{-1},\ \tau_{-3}-\tau_{-2},\dots$$

is an i.i.d. sequence. Since $\bar{Y}$ is simple random walk we can calculate explicitly that:

$$(4.69)\qquad \mathbf{E}[\tau_{-1}] = h^{-1}, \quad \mathbf{Var}\,[\tau_{-1}] = h^{-3}(1-h^2) < \infty.$$

Consider the event

$$(4.70)\qquad D^K_{x,T} = \{\tau_x \in [-xh^{-1} - K\sqrt{|x|},\ -xh^{-1}+K\sqrt{|x|}]\}.$$

By the central limit theorem we can choose both K_0 and T_0 large enough such that for $K\ge K_0$.

$$(4.71)\qquad \inf_{T\ge T_0}\ \inf_{x\in I_{\varepsilon_1,T}} \mathbf{P}[D^K_{x,T}] \ge 1-\varepsilon/4.$$

Next for $\varepsilon_2>0$ we choose $L = L(\varepsilon_2)$ large enough such that

$$(4.72)\qquad \mathbf{E}[\tau^*_x - \tau_x;\ \tau^*_x - \tau_x \ge L] \le \varepsilon_2.$$

We will in the sequel focus on the event $D^K_{x,T}$ intersected with $\{\tau^*_x - \tau_x \le L\}$ for the construction of a minorant of the Kallenberg population $\widehat{\xi}^T_t$.

Step 3 The minorant ζ^T_t of the population $\widehat{\xi}^T_t$ is constructed on the event $C^\gamma_{x,T}$ for all $x\in I_{\varepsilon_1,T}$ by the following additional rules (recall that the $C^\gamma_{x,T}$ are disjoint as family in x):

– offspring from the backbone after time $\tau_x + L$ is suppressed

– particles of ξ_t^T which visit x after time $\tau_x + 2L$ are killed.

The construction implies in particular, that on the event $C^\gamma_{x,T} \cap D^K_{x,T}$ every particle of the population $(\zeta_t^T)_{t\geq 0}$ has left the site x forever at a time

$$(4.73) \qquad t \in [-xh^{-1} - K\sqrt{|x|},\ -xh^{-1} + K\sqrt{|x|} + 2L].$$

Since after this time every particle evolves simply a random walk with transition kernel b, we conclude by the central limit theorem that the probability of every such particle to be in 0 at time T is bounded below uniformly in x and T by

$$(4.74) \qquad c/\sqrt{T} \quad \text{for some } c > 0.$$

This implies by the law of large numbers that for every $\varepsilon_3 > 0$ (recall that the dynamics of ζ_t^T depends on x through $C^\gamma_{x,T}$):

$$(4.75) \qquad \lim_{\delta\to 0} \limsup_{T\to\infty} \sup_{x\in I_{\varepsilon_1,T}} \mathbf{P}\left[\zeta_T^T(\{0\}) \leq \delta\sqrt{T} \middle| \zeta_T^T(\mathbb{Z}) \geq \varepsilon_3 T, C^\gamma_{x,T} \cap D^K_{x,T}\right] = 0.$$

Therefore if we return to the Kallenberg population $\widehat{\xi}_T^T(0)$ then we can conclude:

$$\begin{aligned}
(4.76)\ &\lim_{\delta\to 0} \limsup_{T\to\infty} \sup_{x\in I_{\varepsilon_1,T}} \mathbf{P}\left[\widehat{\xi}_T^T(0) \leq \delta\sqrt{T} \middle| C^\gamma_{x,T}\right] \\
&\leq \limsup_{T\to\infty} \sup_{x\in I_{\varepsilon_1,T}} \mathbf{P}[(D^K_{x,T})^c] + \lim_{\delta\to 0} \limsup_{T\to\infty} \sup_{x\in I_{\varepsilon_1,T}} \mathbf{P}\left[\zeta_T^T(0) \leq \delta\sqrt{T} \middle| C^\gamma_{x,T} \cap D^K_{x,T}\right] \\
&\leq \varepsilon/4 + \limsup_{T\to\infty} \sup_{x\in I_{\varepsilon_1,T}} \mathbf{P}\left[\zeta_T^T(\mathbb{Z}) \leq \varepsilon_3 T \middle| C^\gamma_{x,T} \cap D^K_{x,T}\right].
\end{aligned}$$

Hence it remains in order to conclude the proof to show that by choosing ε_3 small enough:

$$(4.77) \qquad \limsup_{T\to\infty} \sup_{x\in I_{\varepsilon_1,T}} \mathbf{P}\left[\zeta_T^T(\mathbb{Z}) \leq \varepsilon_3 T \middle| C^\gamma_{x,T} \cap D^K_{x,T}\right] \leq \varepsilon/4.$$

Step 4 We will verify the relation (4.77) by finding a majorant Z^T of ζ_T^T such that we can make the quantity $\mathbf{E}[Z^T - \zeta_T^T(\mathbb{Z})]$ small and apply to Z^T the limit theorem Lemma 4.4.

On the event $C^\gamma_{x,T} \cap D^K_{x,T}$ the population of $(\zeta_t^T)_{t\geq 0}$ does not change if we replace the catalyst $(\eta_t^T)_{t\geq 0}$ (recall 4.51) by

$$(4.78) \qquad \bar{\eta}_t^T(x) = \gamma T 1_{C^\gamma_{x,T}}, \quad t \in [0,\infty).$$

Consider the (ordinary) reactant process $(\bar{\zeta}_t^T)_{t\geq 0}$ which starts at time 0 with one particle at $\bar{Y}_T$. Since branching only happens during the time which $\bar{Y}_t$ spends in x up to time T and since b is the kernel of a transient random walk we have that the weak limit of $\bar{\zeta}_t^T(\mathbb{Z})$ exists:

$$(4.79) \qquad \mathcal{L}[\bar{\zeta}_t^T(\mathbb{Z})] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}[\bar{\zeta}_\infty^T(\mathbb{Z})].$$

Let Z^T have the size–biased distribution of $\mathcal{L}[\bar{\zeta}^T_\infty(\mathbb{Z})]$. Then by the evolution rule for $(\zeta^T_t)_{t\geq 0}$ it is clear that

$$\mathcal{L}[Z^T] \geq \mathcal{L}[\zeta^T_T(\mathbb{Z})]. \tag{4.80}$$

On the other hand the difference is due to the additional killing we have for ζ^T_t. By the definition of this additional killing and by (4.72) we estimate as follows. The probability that $\tau^*_x - \tau_x \geq L$ is at most ε_2 and in this event the bound is given by the expected total population at time T, i.e. γT. If $\tau^*_x - \tau_x \leq L$ then the expected number of particles produced and returning to x after a time which is now at least L is at most $\gamma T \varepsilon_2$. Altogether

$$\mathbf{E}[Z^T] - \mathbf{E}[\zeta^T_T(\mathbb{Z})|C^\gamma_{x,T} \cap D^K_{x,T}] \leq 2\gamma T\varepsilon_2. \tag{4.81}$$

In order to study the distribution of Z^T we note the following fact. The probability that a particle of $\bar{\zeta}^T_t$ starting in the catalyst site branches before it leaves the catalyst site x forever is given by the probability that the occupation time of $(Y_t)_{t\geq 0}$ in its starting point is larger than an independent exponentially distributed random variable with parameter γT. The occupation time is exponentially distributed (jump rate of Y is 1) with a parameter given by the probability of $(Y_t)_{t\geq 0}$ to jump to the right and never return. This probability is h. Hence the random variable Z^T has the distribution (recall (4.59) for Z_ϱ) of $\widehat{Z}_\varrho$, the size biased version of Z_ϱ with

$$\varrho = \frac{h}{h + \gamma T}. \tag{4.82}$$

Note that ϱ is independent of x. Therefore by (4.61) (uniformly in x):

$$\mathcal{L}[(h/\gamma T)Z^T] \underset{T\to\infty}{\Longrightarrow} \text{Gamma}(1/2). \tag{4.83}$$

Therefore we can choose ε_3 such that the interval

$$[0, \frac{2\gamma}{h}\varepsilon_3] \tag{4.84}$$

has under the Gamma(1/2) distribution probability at most $\varepsilon/16$. Then for $T \geq T_0$ for sufficiently large T_0 we get by combining (4.83) and (4.84):

$$\mathbf{P}[Z^T \leq 2\varepsilon_3 T] \leq \varepsilon/8. \tag{4.85}$$

With these estimates we can continue (using Chebyshev and (4.81)):

$$\begin{aligned} &\limsup_{T\to\infty} \sup_{x\in I_{\varepsilon_1,T}} \mathbf{P}[\zeta^T_T(\mathbb{Z}) \leq \varepsilon_3 T|C^\gamma_{x,T} \cap D^K_{x,T}] \\ &\leq \limsup_{T\to\infty} \sup_{x\in I_{\varepsilon_1,T}} \mathbf{P}[Z^T \leq 2\varepsilon_3 T] + (\varepsilon_3 T)^{-1}(\mathbf{E}[Z^T] - \mathbf{E}[\zeta^T_T(\mathbb{Z})|C^\gamma_{x,T} \cap D^K_{x,T}]) \\ &\leq \varepsilon/8 + \varepsilon/8 = \varepsilon/4, \end{aligned} \tag{4.86}$$

if we choose

$$\varepsilon_2 \leq \left(\frac{\varepsilon\varepsilon_3}{16\gamma}\right). \tag{4.87}$$

This proves (4.77) and hence completes the proof of Theorem 2 (b). □

5 Catalysts locally dying out: Proof of Theorem 3

The proof of Theorem 3 is based on the two key ingredients, namely Proposition 1.5 showing there are empty space time cylinders on a macroscopic scale and Proposition 1.4 where we pass to the diffusion limit. These key propositions are proved in Subsections 5(a) and 5(b) while 5(c) contains the proof of Theorem 3 up to some technical lemmas proved in 5(d).

(a) Empty space-time cylinder: Proof of Proposition 1.5 and Corollary 1.6

We begin with Proposition 1.5. The main idea is to use scaling to get the result from path properties of the diffusion limit and then transport the result to the particle system studied here by approximation. In order to be able to pass to the diffusion limit we need to control contributions from far away, which are not covered when passing to the diffusion limit in the usual topology of vague convergence on the space of measures. We break the proof into four main steps for greater transparency.

Step 1 (Decomposition of the initial state)
Let $R > 0$ and let (recall that $B(R) = [-R, R]^2 \cap \mathbb{Z}^2$):

$$M(R) = \mathcal{H}_1\Big|_{B(R)} \quad \text{and} \quad M'(R) = \mathcal{H}_1\Big|_{B(R)^c} \tag{5.1}$$

the restrictions of the Poisson field $\mathcal{H}_1$ to $B(R)$ and its complement. Because of the independence of the evolution of different families in branching systems it suffices to check (1.39) instead for the initial state $\mathcal{H}_1$ separately for the initial states $M(RT^{1/2})$ and $M'(RT^{1/2})$. We verify (1.39) for M' in Step 2 and for M in the two Steps 3 and 4 since in the latter case we have to bring into play the diffusion limit.

Step 2 (Contribution from the outside)
Now think of a system starting in $M'(RT^{1/2})$. Let $0 < 2r < R < \infty$. Let $E_{r,T}$ denote the following event

$$E_{r,T} = \Big\{ \sup_{0 \le s \le T} \eta_s(B(rT^{1/2})) > 0 \Big\}. \tag{5.2}$$

We will show that

$$\lim_{R \to \infty} \limsup_{T \to \infty} \mathbf{P}_{M'(RT^{1/2})}[E_{r,T}] = 0 \tag{5.3}$$

For $\ell \in \mathbb{N}$ and $T > 0$ define

$$\begin{aligned} p^1_{T,\ell} &= \mathbf{P}_{\delta_0}\Big[\sup_{0 \le s \le T} \eta_s(\{\pm\ell, \pm(\ell+1), \ldots\} \times \mathbb{Z}) > 0 \Big], \\ p^2_{T,\ell} &= \mathbf{P}_{\delta_0}\Big[\sup_{0 \le s \le T} \eta_s(\mathbb{Z} \times \{\pm\ell, \pm(\ell+1), \ldots\}) > 0 \Big]. \end{aligned} \tag{5.4}$$

Then for $R \geq 2r$

$$
(5.5)\qquad
\begin{aligned}
\mathbf{P}_{M'(RT^{1/2})}[E_{r,T}] &\leq \sum_{|x|\geq RT^{1/2}} \mathbf{P}_{\delta_x}[E_{r,T}] \\
&\leq 2 \sum_{\ell\geq RT^{1/2}} \ell \left(p^1_{T,\ell-rT^{1/2}} + p^2_{T,\ell-rT^{1/2}}\right) \\
&\leq 4 \sum_{\ell\geq (R-r)T^{1/2}} \ell \left(p^1_{T,\ell} + p^2_{T,\ell}\right).
\end{aligned}
$$

Now we use Kesten's result on the evolution of the rightmost particle in branching random walks (Kesten (95), Theorem 1.1) in the continuous time version (Proposition 1.3). Recall that α fulfills (1.35). By maybe making $\alpha > 6$ a little bit smaller we can assume that even (1.25) holds. Using symmetry, Proposition 1.3 implies that there exists a $C < \infty$ such that for $T \geq 1$ and $\ell \geq 2T^{1/2}$

$$
(5.6)\qquad p^i_{T,\ell} \leq \frac{C}{8}\left(T^{-1}(\ell T^{-1/2})^{-\alpha/2} + T^{(2-\alpha)/4}(\ell T^{-1/2})^{(1-\alpha)/2}\right), \qquad i = 1, 2.
$$

Hence for some $C' < \infty$,

$$
(5.7)\qquad
\begin{aligned}
\mathbf{P}_{M'(RT^{1/2})}[E_{r,T}] &\leq C \sum_{\ell\geq (R-r)T^{1/2}} \ell[T^{-1}(\ell T^{-1/2})^{-\alpha/2} + T^{(2-\alpha)/4}(\ell T^{-1/2})^{(1-\alpha)/2}] \\
&= C \sum_{\ell\geq (R-r)T^{1/2}} T^{-1+\alpha/4}\ell^{(2-\alpha)/2} + T^{1/4}\ell^{(3-\alpha)/2} \\
&\leq \frac{C'}{2}\left((R-r)^{(4-\alpha)/2} + (R-r)^{(5-\alpha)/2}\, T^{(6-\alpha)/4}\right) \\
&\leq C'(R-r)^{(4-\alpha)/2}.
\end{aligned}
$$

Hence (5.3) holds and therefore we have proved (1.39) (with $\delta = 1$) for the initial condition $M'(RT^{1/2})$, instead of $\mathcal{H}_1$. □

Step 3 (Diffusion Limit)
In order to treat now (1.39) for the contributions from inside the box that is for the initial state $M(RT^{1/2})$, we consider in this step first the corresponding property for the super Brownian motion X^{ϱ} on $\mathbb{R}^2$ and then return later to the particle problem via a theorem by Dawson, Hochberg and Vinogradov (96).

Let ℓ denote the Lebesgue measure on $\mathbb{R}^2$ and let $\ell_R = \ell \cdot 1_{B(R)}$, $R > 0$, the restriction of ℓ to the finite box $B(R) = [-R, R]^2 \subset \mathbb{R}^2$. Let $(\varrho_t)_{t\geq 0}$ be a super Brownian motion in $\mathbb{R}^2$ and $\mathbf{P}_\nu$ its law if $\varrho_0 = \nu$ a.s. We will prove that for $r > 0$ small enough the super catalyst satisfies

$$
(5.8)\qquad \sup_{\delta\in(0,1/4)} \mathbf{P}_{\ell_R}[\varrho_{1-2\delta}(B(2r)) > 0] < \varepsilon/4.
$$

(The appropriate choice of $\delta \in (0, 1/4)$ will be made later.)

To prove this result note that by the *scaling property* of super Brownian motion in $\mathbb{R}^2$ we get for $s > 0$ and $r > 0$,

$$
(5.9)\qquad \mathbf{P}_{\ell_R}[\varrho_s(B(r)) > 0] = \mathbf{P}_{\ell_{R/s}}[\varrho_1(B(r/s)) > 0].
$$

Hence by the monotonicity of the set function $\varrho_1(\cdot)$ and by the stochastic monotonicity of ϱ_1 in the initial state

$$(5.10) \qquad \sup_{1/2 \le s \le 1} \mathbf{P}_{\ell_R}[\varrho_s(B(r)) > 0] \le \mathbf{P}_{\ell_{2R}}[\varrho_1(B(2r)) > 0].$$

Denote by S_t the closed support of ϱ_t. From Perkins (89), Corollary 1.3, we know that for all $R > 0$ and $t > 0$

$$(5.11) \qquad \mathbf{P}_{\ell_R}[\ell(S_t) = 0] = 1.$$

Hence for ℓ-a.a. $x \in \mathbb{R}^2$

$$(5.12) \qquad \mathbf{P}_{\ell_R}[x \in S_t] = 0.$$

Let $x \in B(1)$ be such that $\mathbf{P}_{\ell_{R+1}}[x \in S_t] = 0$. Let $\ell_R^x = \ell \cdot 1_{x+B(R)}$. By translation invariance of the super Brownian motion and by monotonicity we get

$$(5.13) \qquad \mathbf{P}_{\ell_R}[0 \in S_t] = \mathbf{P}_{\ell_R^x}[x \in S_t] \le \mathbf{P}_{\ell_{R+1}}[x \in S_t] = 0.$$

By σ-continuity of $\mathbf{P}$ from above, for $R > 0$ and $\varepsilon > 0$ we find $r > 0$ such that

$$(5.14) \qquad \mathbf{P}_{\ell_{2R}}[\varrho_1(B(2r)) > 0] < \varepsilon.$$

Fix $R > 0$ and $\varepsilon > 0$. By combining (5.10) and (5.14) we know that we can choose $r > 0$ small enough so that (5.8) holds.

We come back to our original situation, i.e. the particle model. From Dawson, Hochberg and Vinogradov (96), Theorem 1.1, we know that

$$(5.15) \qquad \mathcal{L}_{M(RT^{1/2})}[(T^{-1}\eta_{tT}(T^{1/2}\ \cdot))_{t\ge 0}] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}_{\ell_R}[(\varrho_t)_{t\ge 0}]$$

in the Skorohod topology. In particular we can conclude from their result and (5.8) that for $\gamma > 0$,

$$(5.16) \qquad \limsup_{T\to\infty} \mathbf{P}_{M(RT^{1/2})}\left[\eta_{(1-2\delta)T}(B(2rT^{1/2})) > \gamma T\right] \le \mathbf{P}_{\ell_R}[\varrho_{1-2\delta}(B(2r)) > 0] < \varepsilon/4.$$

We continue the study of the contribution from inside the box $B(R)$. We have now in (5.16) a statement close to what we want except that we have on the l.h.s. the bound γT instead of 0. That is we know that the intensity of the catalyst in $B(2rT^{1/2})$ is small, but we need the box to be completely empty for T large. To deal with this problem is the content of the next step.

Step 4. (Decomposition at time $(1-2\delta)T$)
In order to improve (5.16) we decompose $(\eta_t)_{t\ge(1-2\delta)T}$ according to the positions of the particles at time $(1-2\delta)T$, namely whether they are in the large box or not. Let

$$(5.17) \qquad \eta_t = \eta_t^1 + \eta_t^2, \qquad t \ge (1-2\delta)T,$$

where both $(\eta^1_t)_{t\geq(1-2\delta)T}$ and $(\eta^2_t)_{t\geq(1-2\delta)T}$ are branching random walks on $\mathbb{Z}^2$ and such that

$$(5.18)\qquad \begin{aligned} \eta^1_{(1-2\delta)T} &= \eta_{(1-2\delta)T}1_{B(2rT^{1/2})} \\ \eta^2_{(1-2\delta)T} &= \eta_{(1-2\delta)T}1_{B(2rT^{1/2})^c}. \end{aligned}$$

Define the event $D_T = \{\|\eta^1_{(1-2\delta)T}\| \leq \frac{\varepsilon\delta}{8}T\}$. From (5.16) we know that

$$(5.19)\qquad \limsup_{T\to\infty} \mathbf{P}_{M(RT^{1/2})}[D^c_T] < \varepsilon/4.$$

On D_T however (recall Lemma 2.5) we can estimate the probability of positive total population of η^1 at time $(1-\delta)T$:

$$(5.20)\qquad \begin{aligned} \mathbf{P}_{M(RT^{1/2})}[\|\eta^1_{(1-\delta)T}\| > 0;\ D_T] &\leq \mathbf{E}_{M(RT^{1/2})}[(1+T\delta/2)^{-1}\|\eta^1_{(1-2\delta)T}\|;\ D_T]. \\ &\leq \varepsilon/4. \end{aligned}$$

Hence we can estimate η^1_s in a box for all later times:

$$(5.21)\qquad \limsup_{T\to\infty} \mathbf{P}_{M(RT^{1/2})}\left[\sup_{(1-\delta)T\leq s\leq T} \eta^1_s(B(2rT^{1/2})) > 0\right] < \varepsilon/2.$$

With the contribution η^2 from the outside (from the point of view of time $(1-\delta)T$) we proceed as in Step 2. Analogously to (5.7) we get (for $\delta > 0$ small enough)

$$(5.22)\qquad \mathbf{P}_{M(RT^{1/2})}\left[\sup_{(1-2\delta)T\leq s\leq T} \eta^2_s(B(rT^{1/2})) > 0\right] \leq 2C'\left(\frac{\sqrt{\delta}}{r}\right)^{(\alpha-4)/4} < \varepsilon/2.$$

Now we can combine (5.22) with (5.21) to get:

$$(5.23)\qquad \limsup_{T\to\infty} \mathbf{P}_{M(RT^{1/2})}\left[\sup_{(1-\delta)T\leq s\leq T} \eta_s(B(rT^{1/2})) > 0\right] < \varepsilon,$$

which completes the proof of Proposition 1.5. □

Proof of Corollary 1.6 The proof is based on Proposition 1.5 via a soft argument and an additional variance estimate.

For δ, $r > 0$ let

$$(5.24)\qquad A_{\delta,r}(T) = \left\{\sup_{(1-\delta)T\leq s\leq T} \eta_s(B(rT^{1/2})) = 0\right\}.$$

According to (1.39) in Proposition 1.5 we can choose for $\varepsilon > 0$ numbers δ, $r > 0$ such that

$$(5.25)\qquad \liminf_{T\to\infty} \mathbf{P}[A_{\delta,r}(T)] \geq 1-\varepsilon.$$

Next we come to the variance estimate. Let

$$(5.26)\qquad K_{\delta,r}(T) := \mathbf{E}\Big[2\int_0^T \sum_{x\in\mathbb{Z}^2} \eta_s(x)\, b_{T-s}(0,x)\, b_{T-s}(x,0)\, ds; A_{\delta,r}(T)\Big].$$

Note that from the definition above it is clear that

$$(5.27)\qquad K_{\delta,r}(T) \le 2\theta_\eta \int_{\delta T}^T \sum_{x\in\mathbb{Z}^2} b_s(0,x) b_s(x,0)\, ds + 2\theta_\eta \int_0^{\delta T} \sum_{x\in B(rT^{1/2})^c} b_s(0,x) b_s(x,0)\, ds.$$

It is easy to show that $K_{\delta,r} := \sup_{T\ge 1} K_{\delta,r}(T) < \infty$. In order not to interrupt the flow of the proof here the proof of this statement is deferred to Part (d) of this section (Lemma 5.1).

We combine this latter statement with (5.25) and use the second moment formula (2.13) and Chebyshev's inequality to conclude that

$$\begin{aligned}(5.28)\qquad &\limsup_{T\to\infty} \mathbf{P}[\mathbf{Var}[\xi_T(0)|\eta] \ge K_{\delta,r}/\varepsilon] \\ &= \limsup_{T\to\infty} \mathbf{P}\Big[\sum_{x\in\mathbb{Z}^2} 2\theta_\eta \int_0^T \eta_s(x)\, b_{T-s}(0,x)\, b_{T-s}(x,0)\, ds \ge K_{\delta,r}/\varepsilon\Big] \\ &\le \varepsilon + \limsup_{T\to\infty} \mathbf{P}[A_{\delta,r}(T)^c] \le 2\varepsilon.\end{aligned}$$

We continue with an argument taken from Etheridge and Fleischmann (98). Let $t_n \to \infty$ such that

$$\mathcal{L}[\mathcal{L}[\xi_{t_n}(0)|\eta]] \underset{n\to\infty}{\Longrightarrow} \mathcal{L}[\mathcal{L}[(\xi_\infty(0)|\eta]].$$

Since trivially $\mathbf{P}[\mathbf{E}[\xi_\infty(0)|\eta] \le \theta_\xi] = 1$, it suffices to prove for every $\varepsilon > 0$ that $\mathbf{P}[\mathbf{E}[\xi_\infty(0)|\eta)] \le \theta_\xi - \varepsilon] \le 2\varepsilon$ in order to get (1.40). We calculate as follows:

$$\begin{aligned}(5.29)\qquad &\mathbf{P}[\mathbf{E}[\xi_\infty(0)|\eta] \le \theta_\xi - \varepsilon] \\ &\le \mathbf{P}[\mathbf{E}[\xi_\infty(0) \wedge (2K_{\delta,r}/\varepsilon^2)|\eta] \le \theta_\xi - \varepsilon] \\ &\le \liminf_{n\to\infty} \mathbf{P}[\mathbf{E}[\xi_{t_n}(0) \wedge (2K_{\delta,r}/\varepsilon^2)|\eta] \le \theta_\xi - \varepsilon] \\ &\le \liminf_{n\to\infty} \mathbf{P}[\mathbf{E}[(\xi_{t_n}(0) - 2K_{\delta,r}/\varepsilon^2)^+|\eta] \ge \varepsilon/2] \\ &\le \liminf_{n\to\infty} \mathbf{P}[\mathbf{Var}[\xi_{t_n}(0)|\eta] \ge K_{\delta,r}/\varepsilon] \\ &\le 2\varepsilon. \quad \square\end{aligned}$$

(b) Diffusion limit: Proof of Proposition 1.4

The main idea for the proof is to compare first the particle system to super random walk (diffusion limit only of the branching mechanism) and then use moment calculations to show the convergence of super random walk quantities to the ones of the super Brownian process. The proof is presented in three steps and the proofs of some technical facts are deferred to Subsection 5(d). For notational convenience we will assume throughout the proof that $\theta_\eta = \theta_\xi = 1$.

Step 1 (Reduction of η to $B(RT^{1/2})$)
In this step we establish for branching random walk in a catalytic medium the fact that in branching processes the population at time T in a finite window is made up of the descendents of the ancestors in $B(RT^{1/2})$ which in time T never have left the box if we make R large (compare Proposition 1.3). Hence for many arguments below we can work with a "finite system", i.e. finitely many particles on a large finite box, which is essential when we want to pass to the diffusion limit. Here are the formal arguments.

Start with the catalyst. Let $R > 0$. Recall that $B(R) = [-R, R]^2 \cap \mathbb{Z}^2$. We decompose

$$(5.30) \qquad \eta = \eta^1 + \eta^2,$$

where

$$(5.31) \qquad \eta_0^{1,R}(A) = \eta_0(A \cap B(R)), \qquad \eta_0^{2,R}(A) = \eta_0(A \cap B(R)^{\mathrm{c}}), \quad A \in \mathcal{B}(\mathbb{R}^2).$$

It is well known (see [DHV], Theorem 1) that

$$(5.32) \qquad \mathcal{L}\left[\left(T^{-1}\eta_{tT}^{1,RT^{1/2}}(T^{1/2}\cdot)\right)_{t\geq 0}\right] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}^R\left[(\varrho_t)_{t\geq 0}\right],$$

where the superscript indicates that $\mathcal{L}^R[\varrho_0] = \delta_{\ell 1_{B(R)}}$. The point in showing (a) is to deal with the infinite initial condition. We do so by showing that the contribution from $\eta^{2,RT^{1/2}}$ is negligible for large$R$.

Let

$$(5.33) \qquad E_{R,T,t} := \left\{ \sup_{0\leq s\leq tT} \eta_s^{2,RT^{1/2}}(B(RT/2)) > 0 \right\}.$$

From (5.3) we know that for $t \in (0,1)$:

$$(5.34) \qquad \lim_{R\to\infty} \limsup_{T\to\infty} \mathbf{P}[E_{R,T,t}] = 0.$$

Together with (5.32) this yields the assertion (a).

Now we turn to the proof of assertion (b). Let $(\xi_s^{1,R})_{s\geq 0}$ be the reactant process associated with the catalyst process $(\eta_s^{1,R})_{s\geq 0}$. Our aim is to show that

$$(5.35) \qquad T^{-1}\xi_{tT}^{1,RT^{1/2}}(T^{1/2}\cdot) \quad \text{and} \quad T^{-1}\xi_{tT}(T^{1/2}\cdot)$$

are close. Similarly, as in the proof of Theorem 2(a) (Step 2 in Section 4(a)) we can couple both (ξ_s) and $(\xi_s^{1,R})$ with a new reactant process $(\xi_s^{2,R})_{s\geq 0}$. In $(\xi^{2,R})$ particles branch w.r.t. the catalyst $(\eta_s 1_{B(R/2)})_{s\geq 0}$ and are killed instantaneously on $B(R/2)^{\mathrm{c}}$. This process satisfies

$$(5.36) \qquad \xi_t^{2,R} \leq \xi_t, \quad \xi_t^{2,R} \leq \xi_t^{1,R}.$$

If we let $(W_t) = ((W_t^1, W_t^2))$ be a standard Brownian motion on $\mathbb{R}^2$ and use Donsker's theorem we get for $r > 0$ and $R > \max(4r, 2t)$

$$\begin{aligned}(5.37)\qquad &\limsup_{T\to\infty} T^{-1}\mathbf{E}[\xi_{tT}(B(rT^{1/2})) - \xi_{tT}^{2,RT^{1/2}}(B(rT^{1/2}))] \\ &\qquad \le 4\theta_\xi r^2\, \mathbf{P}^0[W_s \notin [r-(R/2),(R/2)-r]^2 \text{ for some } s\in[0,t]].\end{aligned}$$

Using the reflection principle this inequality can be continued by

$$\begin{aligned}(5.38)\qquad &\le 32\theta_\xi r^2\, \mathbf{P}^0[W_t^1 \ge R/2] \\ &\le 32\theta_\xi r^2\, \exp\left(-R^2/8t\right).\end{aligned}$$

Hence

$$(5.39)\qquad \lim_{R\to\infty}\limsup_{T\to\infty}\mathbf{E}\left[|T^{-1}\xi_{tT}(B(rT^{1/2})) - T^{-1}\xi_{tT}^{2,RT^{1/2}}(B(rT^{1/2}))|\right] = 0$$

Analogously we get

$$(5.40)\qquad \lim_{R\to\infty}\limsup_{T\to\infty}\mathbf{E}\left[T^{-1}(\xi_{tT}^{1,RT^{1/2}}(B(rT^{1/2})) - \xi_{tT}^{2,RT^{1/2}}(B(rT^{1/2}));\ E_{R,T,t}^{\mathrm{c}}\right] = 0.$$

Combining (5.34),(5.39) and (5.40) we get for $\varepsilon > 0$

$$(5.41)\qquad \lim_{R\to\infty}\limsup_{T\to\infty}\mathbf{P}\left[\mathbf{E}[|T^{-1}\xi_{tT}^{1,RT^{1/2}}(B(rT^{1/2})) - T^{-1}\xi_{tT}(B(rT^{1/2}))|\eta] > \varepsilon\right] = 0.$$

A similar statement holds for X^ϱ and ϱ since the couplings can be defined on the super process level as well (see Klenke (97), Lemma 3.3). Hence it suffices to show in the next step that

$$(5.42)\qquad \mathcal{L}^{RT^{1/2}}[\mathcal{L}[T^{-1}\xi_{tT}(T^{1/2}\cdot)|\eta]] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}^R[\mathcal{L}[X^\varrho|\varrho]],$$

where the superscripts indicate that $\mathcal{L}^{RT^{1/2}}[\eta_0] = \mathcal{H}(\ell\cdot 1_{B(RT^{1/2})})$ and $\mathcal{L}^R[\varrho_0] = \delta_{\ell 1_{B(R)}}$. (Recall that ℓ is the Lebesgue measure on $\mathbb{R}^2$ respectively counting measure on $\mathbb{Z}^2$ and $\mathcal{H}(m)$ is the distribution of a Poisson point process with intensity measure m.)

Step 2 (Comparison with super random walk)
In order to go from the particle system to the super process we proceed in two steps. In the first step we perform the diffusion limit only for the branching on the reactant level that is we pass to super random walk. That is reactant particles get mass ε, the initial state gets intensity $\varepsilon^{-1}\theta_\xi\ell$ and the branching rate is $\varepsilon^{-1}\eta_t$ and then we let $\varepsilon\to 0$. This enables us to use in the next step the simpler moment formulas ((5.48) below) of super random walk to carry out the remaining diffusion limits in the particle motions and the catalytic branching.

Let $(\zeta_s)_{s\ge 0}$ be super random walk in the catalytic medium $(\eta_s)_{s\ge 0}$ started in $\mathcal{L}[\zeta_0] = \delta_\ell$. In this step we use the close connection between our model and super random walk in the catalytic medium. The latter is a system $\{(\zeta_s(x))_{s\ge 0},\ x\in\mathbb{Z}^2\}$ of interacting Feller diffusions with branching rate given by η. As initial configuration we choose $\mathcal{L}[\zeta|\eta] = \delta_\ell$ and $\mathcal{L}[\xi_0|\eta] = \mathcal{H}_1$.

In terms of Laplace transforms the relation between ζ and ξ for given η reads as follows (see, e.g., [GRW]), Lemme 1, for the non–catalytic situation).

$$\mathbf{E}[\exp(-\langle \zeta_s, 1-e^{-f}\rangle)|\eta] = \mathbf{E}[\exp(-\langle \xi_s, f\rangle)|\eta], \qquad f \in C_c^+(\mathbb{R}^2). \tag{5.43}$$

Since the law at fixed time is uniquely determined by the Laplace functional, the fact that $\|T(1-e^{-f/T}) - f\|_\infty \to 0$ as $T \to \infty$, implies via (5.43) that it suffices now to consider the relation (1.38) with ξ replaced by ζ. Since the estimates (5.39) - (5.41) hold on the super–process level it suffices in fact to show instead of (5.42):

$$\mathcal{L}^{RT^{1/2}}[\mathcal{L}[T^{-1}\zeta_{tT}(T^{1/2}\cdot)|\eta]] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}^R[\mathcal{L}[X^\varrho|\varrho]], \tag{5.44}$$

where on the l.h.s. $\mathcal{L}^{RT^{1/2}}[\zeta_0] = \delta_{\ell 1_{B(RT^{1/2})}}$.

Step 3 (Diffusion limit of super random walk)
We now need to introduce the space–time rescaling of the pair (η_t, ζ_t), that produces the diffusion limit. This is very intuitive but the proof is a bit technical.

Denote by η_s^T the rescaled catalyst process defined by $T^{-1}\eta_{sT}(T^{1/2}\cdot)$, where the mass is smeared out uniformly in the squares $T^{-1/2}(x+[-\frac{1}{2},\frac{1}{2})^2)$, $x \in \mathbb{Z}^2$, i.e.,

$$\eta_s^T(A) = T^{-1}\sum_{x\in\mathbb{Z}^2} |(T^{1/2}A) \cap (x+[-\tfrac{1}{2},\tfrac{1}{2})^2)| \cdot \eta_{sT}(\{x\}). \tag{5.45}$$

Similarly we define ζ^T and the rescaled transition probabilities $(b_s^T)_{s\geq 0}$ associated with the "motion" in ζ^T,

$$b_s^T(x,y) = T\cdot b_{sT}(x_0,y_0), \tag{5.46}$$

where $x_0, y_0 \in \mathbb{Z}^2$ are such that $x \in T^{-1/2}(x_0+[-\frac{1}{2},\frac{1}{2})^2)$ and $y \in T^{-1/2}(y_0+[-\frac{1}{2},\frac{1}{2})^2)$. (Recall that $(b_s)_{s\geq 0}$ is the transition semigroup associated with the motion part of (η_s).)

In order to show (5.44) it turns out that it is enough to show

$$\mathcal{L}^R[\mathbf{E}[\langle \zeta_t^T, f\rangle^n|\eta^T]] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}^R[\mathbf{E}[\langle X_t^\varrho, f\rangle^n|\varrho]], \qquad f\in C_c^+(\mathbb{R}^2),\ n\in\mathbb{N}. \tag{5.47}$$

Namely, we need that the conditioned moments $\langle \zeta_t^T, f\rangle^n$ are finite and that these moments for X^ϱ satisfy a growth condition in n in order to give a well posed moment problem. The finiteness of the moments is immediate from (5.44) below and the growth condition for the moments of X^ϱ is in [DF5], Section 3.3.

The proof of (5.47) will be based on the analysis of the moments and convergence results for these objects. The key to the analysis of moments is the fact that the n-th moments can be obtained from lower order moments via *recursion formulas*. For the rest of the proof we fix $f\in C_c^+(\mathbb{R}^2)$. According to [DF5], Lemma 13.

$$\mathbf{E}[\langle \zeta_t^T, f\rangle^n|\eta^T] = \sum_{k=0}^{n-1}\binom{n-1}{k}\langle \ell, u_{n-k}^T(0,t,\cdot)\rangle \mathbf{E}[\langle \zeta_t^T, f\rangle^k|\eta^T], \tag{5.48}$$

where

$$(5.49)\qquad u_n^T(s,t,x) = \begin{cases} (b_{t-s}^T f)(x) 1_{s\le t}, & n=1 \\ \dfrac{1}{2}\displaystyle\sum_{j=1}^{n-1}\binom{n}{j}\int_s^t dr \int_{\mathbb{R}^2} \eta_r^T(dy)\, b_{r-s}^T(x,y)\, u_j^T(r,t,y)\, u_{n-j}^T(r,t,y), & n\ge 2. \end{cases}$$

If we set $b_t^\infty(x,y) = (2\pi s)^{-1}\exp(-\|y-x\|^2/2s)$ and define u^∞ by replacing in (5.49) b^T by b^∞, we get the moment formulas for the super process in catalytic medium X^ϱ (see again [DF5], Lemma 13),

$$(5.50)\qquad \mathbf{E}[\langle X_t^\varrho, f\rangle^n|\varrho] = \sum_{k=0}^{n-1}\binom{n-1}{k}\langle \ell, u_{n-k}^\infty(0,t,\cdot\,)\rangle\, \mathbf{E}[\langle X_t^\varrho, f\rangle^k|\varrho].$$

The relations (5.48) and (5.50) tell us that we need to compare u_n^T and u_n^∞ as $T\to\infty$, based on information on the rescaled transition kernel $(b_s^T)_{s\ge 0}$ and the defining recursion relation (5.49). To carry this out we next give a more convenient representation of u_n^T as a kernel applied to a polynomial functional of the occupation measure of the catalyst.

Define the space–time catalytic occupation measure

$$(5.51)\qquad \kappa^T(dy,ds) = \eta_s^T(dy)ds.$$

We can write for $n\ge 2$, u_n^T as:

$$(5.52)\qquad u_n^T(s,t,x) = \int_{(\mathbb{R}^2\times[0,t])^{n-1}} K_n^T(s,t,x;\mathbf{y},\mathbf{r})\,(\kappa^T)^{\otimes(n-1)}(d\mathbf{y},d\mathbf{r}).$$

For this relation to be true we have to define $K_n^T(s,t,x,\mathbf{y},\mathbf{r})$, $\mathbf{y} = (y_1,\dots,y_{n-1})\in(\mathbb{R}^2)^{n-1}$, $\mathbf{r} = (r_1,\dots,r_{n-1})\in[0,1]^{n-1}$, inductively in terms of $(b_s^T)_{s\ge 0}$ as follows:

$$(5.53)\qquad K_1^T(s,t,x) = (b_{t-s}^T f)(x)1_{s\le t},$$

$$\begin{aligned}(5.54)\qquad & K_n^T(s,t,x;(y_1,\dots,y_{n-1}),(r_1,\dots,r_{n-1})) \\ &= b_{r_1-s}^T(x,y_1)\sum_{j=1}^{n-1}\binom{n}{j}K_j^T(r_1,t,y_1,(y_2,\dots,y_j),(r_2,\dots,r_j)) \\ &\qquad\qquad \times K_{n-j}^T(r_1,t,y_1;(y_{j+1},\dots,y_{n-1}),(r_{j+1},\dots,r_{n-1})).\end{aligned}$$

Furthermore abbreviate:

$$(5.55)\qquad \widetilde{K}_n^T(s,t;\mathbf{y},\mathbf{r}) = \langle \ell, K_n^T(s,t;\cdot,\mathbf{y},\mathbf{r})\rangle.$$

Analogously we define $K_n^\infty(s,t,x;\mathbf{y},\mathbf{r})$ and $\widetilde{K}_n^\infty(s,t;\mathbf{y},\mathbf{r})$ based on the Brownian transition probabilities.

It can be show that (see Lemma 5.2 in Part (d) of this section)

$$(5.56)\qquad \sup\{|(b_t^T f)(x) - (b_t^\infty f)(x)|;\ t \geq 0,\ x \in \mathbb{R}^2\} \underset{T\to\infty}{\longrightarrow} 0,$$

(recall that f is bounded and uniformly continuous). In other words, (recall (5.53)) $\widetilde{K}_1^T$ and K_1^T converge to $\widetilde{K}_1^\infty$ and K_1^∞. An induction (carried out in Part (d) of this section in Lemma 5.2 and 5.3) yields that for every $n \geq 2$

$$(5.57)\qquad K_n^T(s,t,x;\cdot,\cdot) \underset{T\to\infty}{\longrightarrow} K_n^\infty(s,t,x;\cdot,\cdot) \quad \text{in } L^\infty((\mathbb{R}^2)^{n-1} \times [0,\infty)^{n-1})$$

and

$$(5.58)\qquad \widetilde{K}_n^T(s,t,x;\cdot,\cdot) \underset{T\to\infty}{\longrightarrow} \widetilde{K}_n^\infty(s,t,x;\cdot,\cdot) \quad \text{in } L^\infty((\mathbb{R}^2)^{n-1} \times [0,\infty)^{n-1}).$$

In order to transfer this information to the moment measures, note that by (5.48) and (5.52) we can define for $i = 0, \ldots, n-1$ multinomials of $\widetilde{K}_j^T(0,t)$, which we call $J_{n,i}^T(t;\mathbf{y},\mathbf{r})$, $\mathbf{y} \in (\mathbb{R}^2)^i$, $\mathbf{r} \in [0,\infty)^i$ such that

$$(5.59)\qquad \mathbf{E}[\langle \zeta_t^T, f\rangle^n | \eta^T] = \sum_{i=0}^{n-1} \int J_{n,i}^T(t;\mathbf{y},\mathbf{r})\,(\kappa^T)^{\otimes i}(d\mathbf{y}, d\mathbf{r}).$$

(We interpret $\int J_{n,0}(t) d(\kappa^T)^{\otimes 0}$ as equal to $J_{n,0}(t) \in \mathbb{R}$.) A similar statement is true for X^ϱ with integral kernels $J_{n,i}^\infty(t)$. To get the convergence of the n-th moments as claimed in (5.47) we have to estimate two things, namely how far both ingredients J^T and κ^T are from their respective limits.

We start with J^T. By (5.58) clearly,

$$(5.60)\qquad J_{n,i}^T(t) \underset{T\to\infty}{\longrightarrow} J_{n,i}^\infty(t) \quad \text{in } L^\infty((\mathbb{R}^2)^i \times [0,\infty)^i).$$

Note that for the space–time occupation measure $\mathbf{E}^R[\|\kappa^T\|] = 4R^2 t$, $T \in (0,\infty)$. Hence for $\varepsilon > 0$ we obtain with Chebyshev's inequality that

$$(5.61)\qquad \begin{aligned} \mathbf{P}^R\Big[\int |J_{n,i}^T(t) - J_{n,i}^\infty(t)|\, d(\kappa^T)^{\otimes i} > \varepsilon\Big] &\leq \mathbf{P}^R[\|J_{n,i}^T(t) - J_{n,i}^\infty(t)\|_\infty \cdot \|\kappa^T\|^i > \varepsilon] \\ &\leq 4R^2 t \varepsilon^{-1/i} \|J_{n,i}^T(t) - J_{n,i}^\infty(t)\|_\infty^{1/i} \underset{T\to\infty}{\longrightarrow} 0. \end{aligned}$$

Using the same estimate we see that for each $n \in \mathbb{N}$ and $i \leq n$,

$$(5.62)\qquad \Big\{\int J_{n,i}^\infty(t) d(\kappa^T)^{\otimes i},\ T \geq 1\Big\} \text{ is uniformly integrable w.r.t. } \mathbf{P}^R.$$

Next comes κ^T out and we have to bring into play the convergence of the process η^T as $T \to \infty$. According to [DHV] we have the *convergence of* η^T *to* ϱ *in path space*. This we use in the following way. Define in view of (5.59) the following functional of the catalyst process: For $\gamma \in D([0,t], \mathcal{M}_f(\mathbb{R}^d))$ (= space of càdlàg functions equipped with the Skorohod topology) let

$$(5.63) \qquad m_n(\gamma) := \sum_{i=0}^{n-1} \int ds_1 \cdots ds_{i-1}\, J^\infty_{n,i}(t; (y_1, \ldots, y_i), (s_1, \ldots, s_i))\ \gamma_{s_1}(dy_1) \cdots \gamma_{s_i}(dy_i).$$

Note that m_n is a *continuous* functional of γ. Thus using (5.62) we get from [DHV, 1997], Theorem 1.1, that

$$(5.64) \qquad \mathcal{L}^R[m_n(\eta^T)] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}^R[m_n(\varrho)].$$

Together with (5.61) this implies (5.47). Hence the proof of Proposition 1.4 is complete. □

(c) The reactant is asymptotically Mixed Poisson: Proof of Theorem 3

The idea of the proof is to use the empty space–time cylinder (Proposition 1.5) to construct a coupling of the reactant $(\xi_t)_{t\geq(1-\delta)T}$ with a system of independent random walks starting in $\xi_{(1-\delta)T}$. The latter systems have Poisson states as extremal equilibria. Hence by rescaling the Proposition 1.4 (diffusion limit) will yield the conclusion. Here are the details.

We start with a random walk estimate. Recall that $a_t(\cdot,\cdot)$ is the transition function of the random walk X_t. For $R > 0$ and $x, y \in B(R)$ define the transition function of random walk with killing at the complement of $B(R)$,

$$(5.65) \qquad \widetilde{a}_{t,R}(x,y) = \mathbf{P}^x[X_t = y\ :\ X_s \in B(R),\ 0 \leq s \leq t].$$

Using Donsker's theroem, $a_{\delta T}(x,y) - \widetilde{a}_{\delta T, rT^{1/2}}(x,y)$ is asymptoticallly as $T \to \infty$ the probability that a standard two–dimensional Brownian motion (W_t), started in $x/\sqrt{\delta T}$ hits $(y+[0,1]^2)/\sqrt{\delta T}$ at time δT without leaving $[-r,r]^2$ Hence using symmerty and the reflection principle we get

$$(5.66) \qquad \begin{aligned} &\limsup_{T\to\infty} \left[T \sup \left\{ a_{\delta T}(x,y) - \widetilde{a}_{\delta T, rT^{1/2}}(x,y)\ :\ x,y \in B((r/2)T^{1/2}) \right\} \right] \\ &\qquad = \sup_{x,y\in[-r/2,r/2]^2} \mathbf{P}^x[W_\delta \in dy;\ W_s \notin [-r,r]^2 \text{ for some } s \in (0,\delta)]/dy \\ &\qquad \leq \frac{2}{\pi\delta} \exp\left(-r^2/2\delta\right). \end{aligned}$$

We introduce the following coupling between three processes $(\xi^i_t)_{t\geq 0}$, $i = 1,2,3$, which all start in a configuration living on $B(2rT^{1/2})$ which later we choose to be the configuration $(\xi_{(1-\delta)T})$ restricted to $B(rT^{1/2})$. Let $(\xi^1_t)_{t\geq 0}$ be branching random walk on $\mathbb{Z}^2$ with branching rate given by the catalyst $(\eta_{(1-\delta)T+t})_{t\geq 0}$. Let $(\xi^2_t)_{t\geq 0}$ be independent simple random walks on $\mathbb{Z}^2$. Finally we define $(\xi^3_t)_{t\geq 0}$ by introducing in $(\xi^2_t)_{t\geq 0}$ for each particle a killing at the boundary of $B(rT^{1/2})$. Denote by τ the first killing time of ξ^3_t,

$$\tau = \inf\{t \geq 0,\ \xi^3_t < \xi^2_t\}.$$

For initial state μ (concentrated in $B(rT^{1/2})$) we can construct $(\xi_t^1, \xi_t^2, \xi_t^3)_{t\geq 0}$ on one probability space $\mathbf{P}^\mu$ such that $\mathbf{P}^\mu[\xi_0^1 = \xi_0^2 = \xi_0^3 = \mu] = 1$ and such that for $0 \leq t \leq \delta T$ (recall the definition of $A_{\delta,r}(T)$ from (5.24)),

$$A_{\delta,r}(T) \cap \{\tau > t\} \subset \{\xi_t^1 = \xi_t^2 = \xi_t^3\}. \tag{5.67}$$

Note that for $x, y \in B((r/2)T^{1/2})$ and $0 \leq t \leq \delta$,

$$\begin{aligned} \mathbf{E}^{\delta_x}[|\xi_t^1(y) - \xi_t^2(y)|;\, A_{\delta,r}(T)] &\leq \mathbf{E}^{\delta_x}[|\xi_t^1(y) - \xi_t^3(y)|;\, A_{\delta,r}(T)] + \mathbf{E}^{\delta_x}[|\xi_t^3(y) - \xi_t^2(y)|] \\ &\leq 2\left[a_t(x,y) - \widetilde{a}_{t,rT^{1/2}}(x,y)\right]. \end{aligned} \tag{5.68}$$

Hence by (5.66)

$$\begin{aligned} &\limsup_{T\to\infty}\left[T \sup\left\{\mathbf{E}^{\delta_x}\left[|\xi_{\delta T}^1(y) - \xi_{\delta T}^2(y)|;\, A_{\delta,r}(T)\right] : x,y \in B((r/2)T^{1/2})\right\}\right] \\ &\qquad\qquad \leq \frac{2}{\pi\delta}\exp(-r^2/2\delta). \end{aligned} \tag{5.69}$$

On the other hand for $y \in B((r/4)T^{1/2})$ we get from the central limit theorem that for $r > 0$ fixed and T sufficiently large that

$$\begin{aligned} \sum_{x\in(B((r/2)T^{1/2}))^{\mathrm{c}}} \mathbf{E}^{\delta_x}\left[|\xi_{\delta T}^1(y) - \xi_{\delta T}^2(y)|\right] &\leq \sum_{x\in(B((r/2)T^{1/2}))^{\mathrm{c}}} 2a_{\delta T}(x,y) \\ &\leq \sum_{x\in B((r/4)T^{1/2})^{\mathrm{c}}} 2a_{\delta T}(0,x) \\ &\leq 4\mathbf{P}^0[W_\delta \notin [-r/4, r/4]^2] \\ &\leq 4\exp(-r^2/32\delta). \end{aligned} \tag{5.70}$$

Now we apply the above estimates to our situation, that is we assume $\xi_0^1 = \xi_0^2 = \xi_{(1-\delta)T}$ and $\xi_{(1-\delta)T+t} = \xi_t^1$. Hence for $f : \mathbb{Z}^2 \to \mathbb{R}$ with finite support, for T large enough, (5.69) and (5.70) give

$$\begin{aligned} &\mathbf{E}_{\theta_\eta,\theta_\xi}[|\langle \xi_T - \xi_{\delta T}^2, f\rangle|] \\ &\quad = \mathbf{E}_{\theta_\eta,\theta_\xi}[|\langle \xi_{\delta T}^1 - \xi_{\delta T}^2, f\rangle|;\, A_{\delta,r}(T)] + 2\theta_\xi \mathbf{P}_{\theta_\eta}[(A_{\delta,r}(T))^{\mathrm{c}}] \cdot \|f\|_1 \\ &\quad \leq 4\theta_\xi \exp\left(-\frac{r^2}{32\delta}\right)\|f\|_1 \\ &\qquad + 2\theta_\xi \|f\|_1 \left(\frac{1}{\pi\delta}\exp(-r^2/2\delta)T^{-1}|B((r/2)T^{1/2})| + \mathbf{P}_{\theta_\eta}[(A_{\delta,r}(T))^{\mathrm{c}}]\right). \end{aligned} \tag{5.71}$$

Thus (e.g., for $r = \delta^{1/4}$),

$$\limsup_{T\to\infty} \mathbf{E}_{\theta_\eta,\theta_\xi}\left[|\langle \xi_T - \xi_{\delta T}^2, f\rangle|\right] = \|f\|_1 \cdot o(\delta). \tag{5.72}$$

Recall Proposition 2.1 on Poisson equilibria and note that for $y \in \mathbb{Z}$,

$$\mathbf{E}[\xi^2_{\delta T}(y)|\xi_{(1-\delta)T}] = (p_{\delta T} * \xi_{(1-\delta)T})(y). \tag{5.73}$$

Hence for $T \to \infty$ (by Proposition 2.1) the distribution of $\xi^2_{\delta T}$ given $\xi_{(1-\delta)T}$ is close to a Poisson field with intensity measure $p_{\delta T} * \xi_{(1-\delta)T}$ (which of course depends on η). Hence by the diffusion limit Proposition 1.4 (and the central limit theorem for $p_{\delta T}$),

$$\mathcal{L}_{\theta_\eta,\theta_\xi}\left[\mathcal{L}_{\theta_\eta,\theta_\xi}\left[\xi^2_{\delta T}|\eta\right]\right] \underset{T\to\infty}{\Longrightarrow} \mathcal{L}_{\theta_\eta,\theta_\xi}\left[\mathbf{E}_{\theta_\eta,\theta_\xi}\left[\mathcal{H}\Big((p^B_\delta * X^\varrho_{1-\delta})(0)\Big)|(\varrho)\right]\right]. \tag{5.74}$$

However by (5.72) and (1.32) we can let $\delta \to 0$ on both sides to obtain the assertion (1.36). $\square$

(d) Some technical lemmas

In this section we give further details for technical parts from the moment calculations in the proof of Theorem 3. Recall that $B(R) = [-R, R]^2 \cap \mathbb{Z}^2$.

Lemma 5.1 *For $0 < \delta < 1$ and $r > 0$ the following quantity is bounded for $T \geq 1$*

$$\int_{\delta T}^{T} \sum_{x\in\mathbb{Z}^2} b_s(0,x)b_s(x,0)\,ds + \int_0^{\delta T} \sum_{x\in B(rT^{1/2})^c} b_s(0,x)b_s(x,0)\,ds. \qquad \diamond \tag{5.75}$$

Proof For the first term in (5.76) note that $\sum_{x\in\mathbb{Z}^2} b_s(0,x)b_s(x,0)$ is the probability that the difference of two indepepndent random walks according to (b_t) hit 0 at time s. Hence the local central limit theorem gives

$$\lim_{T\to\infty} \int_{\delta T}^{T} \sum_{x\in\mathbb{Z}^2} b_s(0,x)b_s(x,0)\,ds = \frac{1}{4\pi}\log(\delta^{-1}). \tag{5.76}$$

The other integral in (5.75) can be estimated as follows. First we make use of the following standard estimate for non–degenerate two–dimensional random walk

$$C := \sup_{s\geq 1}\sup_{x\in\mathbb{Z}^2} s b_s(0,x) < \infty. \tag{5.77}$$

Next we apply Chebyshev's inequality to the coordinates of $x = (x_1, x_2)$

$$\sum_{x\in B(rT^{1/2})^c} b_s(0,x) \leq \frac{2s}{r^2 T}. \tag{5.78}$$

Putting together (5.77) and (5.78) we get

$$\int_0^{\delta T} \sum_{x\in B(rT^{1/2})^c} b_s(0,x)b_s(x,0)\,ds \leq \frac{2C\delta}{r^2} + 1. \tag{5.79}$$

$\square$

Recall from (5.46) that b^T is the rescaled simple random walk kernel and that b^∞ is the heat kernel.

Lemma 5.2 *Let $\mathcal{F} = \{f_i,\, i \in I\}$ be a bounded uniformly continuous family of real valued functions, indexed by some index set I, i.e., for $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for $x, y \in \mathbb{R}^d$ with $|x - y| < \delta$ and for all $i \in I$ we have*

$$|f_i(x) - f_i(y)| < \varepsilon. \tag{5.80}$$

Then $\mathcal{F}' := \{(b_t^\infty f_i),\, t \geq 0,\, i \in I\}$ is a bounded uniformly continuous family and

$$\sup_{t,i,x} |(b_t^T f_i - b_t^\infty f_i)(x)| \underset{T\to\infty}{\longrightarrow} 0. \qquad \diamond \tag{5.81}$$

Proof Clearly $\mathcal{F}'$ is bounded. Also $\mathcal{F}'$ is uniformly continuous with the same $\delta(\varepsilon)$ as $\mathcal{F}$.

Now we show that (5.81) holds. Let $\varepsilon > 0$ and define

$$M := \sup_{i,x} |f_i(x)| < \infty.$$

Note that by Chebyshev's inequality, e.g., we can find $t_0 > 0$ such that

$$b_t^T 1_{([-\delta,\delta]^2)^c}(0) \leq \frac{\varepsilon}{2M}, \quad t \in [0, t_0],\, T \in [1, \infty]. \tag{5.82}$$

Hence for $t \leq t_0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} |(b_t^T f_i - b_t^\infty f_i)(x)| &\leq \|f_i\|_\infty \cdot \big((b_t^T + b_t^\infty) 1_{([-\delta,\delta]^2)^c}(0)\big) + \sup_{y:|y-x|<\delta} |f_i(y) - f_i(x)| \\ &\leq 2\varepsilon \end{aligned} \tag{5.83}$$

On the other hand for $t \geq t_0$ by the central limit theorem (and uniform continuity of $\mathcal{F}$)

$$\sup_{t>t_0,\, i\in I} \|b_t^T f_i - b_t^\infty f_i\|_\infty \underset{T\to\infty}{\longrightarrow} 0. \qquad \square \tag{5.84}$$

In order to state the next lemma let $f \in C_c^+(\mathbb{R}^d)$ and recall the definition of K^T and K^∞ from (5.53) and (5.54).

Lemma 5.3 *For $n \geq 1$,*

$$\mathcal{F}_n := \left\{K_n^\infty(s, t, \cdot\,; \mathbf{y}, \mathbf{r});\; s, t \geq 0,\, \mathbf{y} \in (\mathbb{R}^2)^{n-1},\, \mathbf{r} \in [0, \infty)^{n-1}\right\} \tag{5.85}$$

is a bounded uniformly continuous family. Furthermore, if we let

$$\alpha_n^T = \sup_{s,t,x,\mathbf{y},\mathbf{r}} |K_n^T(s, t, x; \mathbf{y}, \mathbf{r}) - K_n^\infty(s, t, x; \mathbf{y}, \mathbf{r})|, \tag{5.86}$$

$$\widetilde{\alpha}_n^T = \sup_{s,t,\mathbf{y},\mathbf{r}} |\widetilde{K}_n^T(s, t; \mathbf{y}, \mathbf{r}) - \widetilde{K}_n^\infty(s, t; \mathbf{y}, \mathbf{r})|, \tag{5.87}$$

then

$$\alpha_n^T \underset{T\to\infty}{\longrightarrow} 0 \tag{5.88}$$

and

$$\widetilde{\alpha}_n^T \underset{T\to\infty}{\longrightarrow} 0. \qquad \diamond \tag{5.89}$$

Proof From Lemma 5.2 (with $\mathcal{F} = \{f\}$) we know that the claim of the lemma holds for $n = 1$.

Now assume that $n \geq 2$ and that the claim of the lemma has been shown for $n' < n$. Note that

$$\begin{aligned}(5.90)\quad \Big\{ \sum_{j=1}^{n-1} \binom{n}{j} K_j^\infty(s,t,\cdot\,;(y_2,\dots,y_j),(r_2,\dots,r_j))\; K_{n-j}^\infty(s,t,\cdot\,;(y_{j+1},\dots,y_{n-1}),(r_{j+1},\dots,r_{n-1}));\\ s,t>0,\ \mathbf{y}\in(\mathbb{R}^2)^{n-1},\ \mathbf{r}\in[0,\infty)^{n-1}\Big\}\end{aligned}$$

is a bounded uniformly continuous family. Hence, by Lemma 5.2, $\mathcal{F}_n$ is a bounded and uniformly continuous family. Let $\beta_n = \sup\{\|k\|_\infty,\, k \in \mathcal{F}_n\}$. Then

$$\begin{aligned}&\sup_{t,\mathbf{r},\mathbf{y}} \Big| K_j^T(r_1,t,y_1;(y_2,\dots,y_j),(r_2,\dots,r_j)) K_{n-j}^T(r_1,t,y_1;(y_{j+1},\dots,y_{n-1}),(r_{j+1},\dots,r_{n-1}))\\ (5.91)\quad &\qquad -K_j^\infty(r_1,t,y_1;(y_2,\dots,y_j),(r_2,\dots,r_j)) K_{n-j}^\infty(r_1,t,y_1;(y_{j+1},\dots,y_{n-1}),(r_{j+1},\dots,r_{n-1}))\Big|\\ &\leq (\alpha_j^T+\beta_j)\alpha_{n-j}^T + \alpha_j^T\beta_{n-j} \underset{T\to\infty}{\longrightarrow} 0.\end{aligned}$$

Hence

$$\begin{aligned}(5.92)\quad &\Big| K_n^T(s,t,x;\mathbf{y},\mathbf{r}) - K_n^\infty(s,t,x;\mathbf{y},\mathbf{r})\Big|\\ &\leq \Big|(b_{r_1-s}^T(x,y_1) - b_{r_1-s}^\infty(x,y_1)) \sum_{j=1}^{n-1}\binom{n}{j} K_j^\infty((r_1,t,y_1;(y_2,\dots,y_j),(r_2,\dots,r_j))\\ &\qquad\qquad \times K_{n-j}^\infty(r_1,t,y_1;(y_{j+1},\dots,y_{n-1}),(r_{j+1},\dots,r_{n-1}))\Big|\\ &\quad + (b_{r_1,s}^T(x,y_1) + b_{r_1,s}^\infty(x,y_1)) \sum_{j=1}^{n-1}\big((\alpha_j^T + \beta_j\alpha_{n-j}^T + \alpha^T\beta_{n-j}\big).\end{aligned}$$

By Lemma 5.2 (and the induction hypothesis) and by (5.91) the r.h.s. of (5.92) tends to 0 as $T \to \infty$, uniformly in $s, t, x, \mathbf{y}$ and $\mathbf{r}$, which takes care of (5.88). Note that the sum in the last summand of the r.h.s. of (5.92) is also an upper bound for $\widetilde{\alpha}_n^T$. Hence also (5.89) follows. □

6 Appendix

In this appendix we prove Proposition 1.3.

Proof First note that statement (1.26) is an immediate consequence of (1.27).

The idea of the proof of (1.27) is to relate (ϕ_t) to its embedded genealogical tree and to make use of recent results of Kesten (95) for discrete time branching random walks.

We settle the scene for applying Kesten's result by introducing the embedded tree. Let $\mathcal{T}$ be an ordinary (critical binary) Galton–Watson tree. For $v \in \mathcal{T}$ let $|v|$ denote the generation of v, i.e. the distance to the root $\emptyset \in \mathcal{T}$, and write $\mathcal{T}_n = \{v \in \mathcal{T} : |v| \leq n\}$ for the restriction of $\mathcal{T}$ to the first n generations.

We recover the continuous–time tree $\mathcal{T}^c$ belonging to ϕ from $\mathcal{T}$ by attaching independent exp(1) distributed lifetimes $L(v)$ to all individuals v of $\mathcal{T}$. The random displacement $X(v)$ of an individual v during its lifetime has the distribution

$$\mathcal{L}[X(v)] = \mathcal{L}\Big[\sum_{i=1}^{G_v} Z_{v,i}\Big],$$

where G_v has geometric distribution with parameter $\frac{1}{2}$,

$$\mathbf{P}[Z_{v,i} = x] = a(0,x), \qquad x \in \mathbb{Z},$$

and the G_v and $Z_{v,i}$ are independent.

The "real time" $\tau(v)$ and position $S(v)$ at the end of the life of individual v arise as

$$\tau(v) := \sum_{w \le v} L(w)$$

and

$$S(v) := \sum_{w \le v} X(v),$$

where "$w \le v$" stands for "w is an ancestor of v in $\mathcal{T}$". As in Kesten (95), we write

$$\mathcal{M}_n := \max_{v \in \mathcal{T}_n} S(v).$$

Moreover, let us denote by Δ_v the maximal displacement of individual v during its lifetime, and note that

$$\mathcal{L}[\Delta_v] = \mathcal{L}\Big[\max_{j=1,\dots,G_v} \Big|\sum_{i=1}^{j} Z_{v,i}\Big|\Big].$$

Finally, let us abbreviate

$$\mathcal{M}^T := \sup_{0 \le s \le T} \operatorname{supp} \phi_s.$$

We write ζ for the generation number at which $\mathcal{T}$ goes extinct.

Kesten (95) shows in his Theorem 1.1 that the family $\{\mathcal{L}[n^{-1/2}\mathcal{M}_n | \zeta > n],\ n \in \mathbb{N}\}$ is tight (and converges weakly) and he gives uniform upper bounds for the tails of distributions. We will make use of these bounds only. Using that our random variables X_v have moments of order $\alpha > 4$, we get from Kesten's Theorem 1.1 that there exists a $C > 0$ such that for all $z > 0$ and $n \in \mathbb{N}$:

$$\mathbf{P}[n^{-1/2}\mathcal{M}_n > z | \zeta \ge n] \le C z^{-\alpha}. \tag{6.1}$$

Note that the same bound holds if we condition on $\{\zeta = n\}$:

$$\text{(6.2)} \qquad \mathbf{P}[n^{-1/2}\mathcal{M}_n > z|\zeta = n] \le Cz^{-\alpha}.$$

We will need Kesten's result also to get estimates on the lifetimes of the individuals. For this purpose we attach to each v a real valued (mean zero) displacement $X'(v) = 1 - L(v)$. Since $X'(v)$ has all moments, Kesten's theorem applies and we get that there exists C such that for $z > 0$ and $n \in \mathbb{N}$.

$$\text{(6.3)} \qquad \mathbf{P}[\max_{v \in \mathcal{T}_n} |v| - \tau(v) \ge zn^{1/2}|\zeta = n] \le Cz^{-\alpha}.$$

We are now ready to estimate the tail of $\mathcal{M}^T$. Fix $z > 0$. In the sequel C, C' and so on denote constants that may depend on α but not on z or T. We decompose the event $\{\mathcal{M}^T \ge zT^{1/2}\}$:

$$\text{(6.4)} \qquad \begin{aligned} \mathbf{P}[\mathcal{M}^T \ge zT^{1/2}] &= \sum_{n=1}^{\infty} \mathbf{P}[\zeta = n,\ \mathcal{M}^T \ge zT^{1/2}] \\ &\le \sum_{n=1}^{[zT]} \mathbf{P}\Big[\zeta = n,\ \mathcal{M}_n \ge \frac{z}{2}T^{1/2}\Big] + \sum_{n=1}^{[zT]} \mathbf{P}\Big[\zeta = n,\ \max_{v \in \mathcal{T}_n} \Delta_v > \frac{z}{2}T^{1/2}\Big] \\ &\quad + \mathbf{P}\Big[\zeta > zT,\ \mathcal{M}_{[zT]} \ge \frac{z}{2}T^{1/2}\Big] + \mathbf{P}\Big[\zeta > zT,\ \max_{v \in \mathcal{T}_{[zT]}} \Delta_v > \frac{z}{2}T^{1/2}\Big] \\ &\quad + \mathbf{P}[\zeta > [zT],\ \exists v \in \mathcal{T} \setminus \mathcal{T}_{[zT]} : \tau(v) \le T] \\ &=: A_1 + A_2 + A_3 + A_4 + A_5. \end{aligned}$$

We start with estimating A_1. Here and later we will tacitly make use of the fact

$$\text{(6.5)} \qquad \sup\{n^2 \mathbf{P}[\zeta = n],\ n \in \mathbb{N}\} < \infty.$$

(See e.g. Athreya and Ney (72), Corollary I.9.1, for the stronger result $n^2\mathbf{P}[\zeta = n] \to 2,\ n \to \infty$).

Using (6.2) and (6.5) we get for $\alpha > 0$

$$\text{(6.6)} \qquad \begin{aligned} A_1 = \sum_{n=1}^{[zT]} \mathbf{P}\Big[\mathcal{M}_n \ge \frac{z}{2}T^{1/2},\ \zeta = n\Big] &\le C \sum_{n=1}^{[zT]} \frac{1}{n^2} \mathbf{P}\Big[\mathcal{M}_n \ge \frac{z}{2}(n^{-1}T)^{1/2} n^{1/2} | \zeta = n\Big] \\ &\le C' \sum_{n=1}^{[zT]} \Big(\frac{z}{2}(n^{-1}T)^{1/2}\Big)^{-\alpha} n^{-2} \\ &\le C'' \frac{1}{T} z^{-1-\alpha/2}. \end{aligned}$$

In order to estimate A_2, we first note that

$$\mathbf{E}[|\mathcal{T}_n| \Big| \zeta = n] \le \mathbf{E}[|\mathcal{T}_n| \Big| \zeta \ge n] \le n^2,$$

hence

$$\mathbf{P}[|\mathcal{T}_n| \ge kn^2 \Big| \zeta = n] \le \frac{1}{k} \qquad \text{for all } k > 0.$$

Note that by the maximum inequality, Δ_v has a moment of order α. Hence

$$
\begin{aligned}
(6.7) \qquad A_2 &= \sum_{n=1}^{[zT]} \mathbf{P}[\zeta = n,\ \max_{v \in \mathcal{T}_n} \Delta_v > \frac{z}{2} T^{1/2}] \\
&\leq \sum_{n=1}^{[zT]} \Big(\mathbf{P}[\zeta = n,\ |\mathcal{T}_n| > kn^2] + \mathbf{P}[\zeta = n] \cdot kn^2 \mathbf{P}[\Delta \geq \frac{z}{2} T^{1/2}] \Big) \\
&\leq C \sum_{n=1}^{[zT]} \Big(\frac{1}{n^2} \frac{1}{k} + \frac{1}{n^2} kn^2 C' z^{-\alpha} T^{-\frac{\alpha}{2}} \Big) \\
&\leq C'' \Big(\frac{1}{k} + kz^{-\alpha+1} T^{-\frac{\alpha}{2}+1} \Big).
\end{aligned}
$$

With the choice $k = z^{(\alpha-1)/2}\, T^{(\alpha-2)/4}$, we get

$$
(6.8) \qquad A_2 \leq C'' z^{(1-\alpha)/2}\, T^{(2-\alpha)/4}.
$$

Next we estimate A_3.

$$
\begin{aligned}
(6.9) \qquad A_3 &= \mathbf{P}[\zeta > zT,\ \mathcal{M}_{[zT]} \geq \frac{z}{2} T^{1/2}] \\
&= \mathbf{P}[\mathcal{M}_{[zT]} \geq \frac{z^{1/2}}{2} (zT)^{1/2} | \zeta > zT] \mathbf{P}[\zeta > zT] \\
&\leq C z^{-\frac{\alpha}{2}} \cdot (zT)^{-1},
\end{aligned}
$$

where we used Kesten (95), Theorem 1.1, in the last inequality.

Turning to A_4, arguing as for A_2 (with $k = z^{(\alpha-2)/2}\, T^{(\alpha-4)/4}$) we obtain

$$
\begin{aligned}
(6.10) \qquad A_4 &= \mathbf{P}[\zeta > zT;\ \max_{v \in \mathcal{T}_{[zT]}} \Delta_v \geq \frac{z}{2} T^{1/2}] \\
&\leq \mathbf{P}[\zeta > zT;\ |\mathcal{T}_{[zT]}| > k \cdot z^2 \cdot T^2] + \mathbf{P}[\zeta > zT] \cdot k \cdot z^2 T^2 C z^{-\alpha} T^{-\alpha/2} \\
&\leq \frac{1}{zT} \Big[\frac{1}{k} + kz^{2-\alpha} T^{2-\alpha/2} \Big] \\
&\leq z^{-\alpha/2}\, T^{-\alpha/4}.
\end{aligned}
$$

Finally, in order to estimate A_5, we make use of (6.3). Assume that $z \geq 2$ and note that for $v \in \mathcal{T}_{[zT]+1} \setminus \mathcal{T}_{[zT]}$ the inequality $\tau(v) \leq T$ implies $|v| - \tau(v) \geq \frac{zT}{2}$. Hence using (6.3) we get

$$
\begin{aligned}
(6.11) \qquad A_5 &= \mathbf{P}[\zeta > zT,\ \exists v \in \mathcal{T}_{[zT]+1} \setminus \mathcal{T}_{[zT]} : \tau(v) \leq T] \\
&\leq \mathbf{P}\left[\zeta > zT,\ \exists v \in \mathcal{T}_{[zT]+1} :\ |v| - \tau(v) \geq \frac{1}{2} \sqrt{zT}\,\sqrt{zT} \right] \\
&\leq C'(zT)^{-\frac{\alpha}{2}+1}.
\end{aligned}
$$

Putting together (6.6) - (6.11) we get $C > 0$ such that for $z \geq 2$ and $T \geq 1$

$$
\mathbf{P}[\mathcal{M}^R \geq zT^{1/2}] \leq C(z^{-\alpha/2}\, T^{-1} + z^{(1-\alpha)/2}\, T^{(2-\alpha)/4}). \quad \square
$$

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