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Transience of percolation clusters on wedges

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Abstract

We study random walks on supercritical percolation clusters on wedges in $\mathbb{Z}^3$, and show that the infinite percolation cluster is (a.s.) transient whenever the wedge is transient. This solves a question raised by O. Häggström and E. Mossel. We also show that for convex gauge functions satisfying a mild regularity condition, the existence of a finite energy flow on Z^2 is equivalent to the (a.s.) existence of a finite energy flow on the supercritical percolation cluster. This answers a question of C. Hoffman

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1 Introduction

For simple random walk in the $\mathbb{Z}^d$ lattice, Polya [21] showed in 1920 that the transition from recurrence to transience occurs when d increases from 2 to 3. The transition boundary is more sharply delineated by a 1983 result of T. Lyons, concerning *wedges*. For an increasing positive function h, the wedge $\mathcal{W}_h$ is the subgraph of $\mathbb{Z}^3$ induced by the vertices

$$V(\mathcal{W}_h) = \{(x, y, z) \mid x \geq 0 \text{ and } |z| \leq h(x)\} .$$

T. Lyons [17] proved that $\mathcal{W}_h$ is transient if and only if

$$\sum_{j=1}^{\infty} \frac{1}{jh(j)} < \infty. \tag{1}$$

(A locally finite graph is called transient or recurrent according to the type of simple random walk on it.) It is well-known (see, e.g., [6] or [20]) that if G is recurrent then so is any subgraph of G.

In [8] Grimmett, Kesten and Zhang proved that the infinite cluster of supercritical percolation in $\mathbb{Z}^d$ is transient when $d \geq 3$. A different proof and some extensions were given in [3]. Häggström and Mossel [10] sharpened the methods of [3] and showed that if the increasing positive function h satisfies

$$\sum_{j=1}^{\infty} \frac{1}{j\sqrt{h(j)}} < \infty \,, \tag{2}$$

then the infinite cluster of supercritical percolation in $\mathcal{W}_h$ is transient.

The condition (2) is strictly stronger than Lyons' condition (1); in particular, the function $h(j) = h_r(j) = \log^r(j)$ satisfies (2) if and only if $r > 2$, while it satisfies (1) for all $r > 1$. Häggström and Mossel asked what is the type of percolation clusters in wedges $\mathcal{W}_h$ where h satisfies Lyons' condition (1) , but does not satisfy condition (2).

Our main result answers this question:

Theorem 1. *Let h be a positive increasing function. The infinite cluster of supercritical percolation on the wedge $\mathcal{W}_h$ is transient if and only if $\mathcal{W}_h$ is transient, i.e., if and only if h satisfies (1).*

A useful notion will be that of the *core* of a subgraph. The core consists of those vertices which are in some sense far from the boundary:

Definition 2. *Let A be a subgraph of a graph G, and v_0 be a fixed vertex in G. For $C > 0$ define the C-core of A to be the subgraph of A induced by*

$$\left\{v \in A | d(v, A^C) > C \log d(v, v_0)\right\} ,$$

where d denotes the graph metric on G, and A^C is the subgraph containing all edges not in A, and their endpoints.

Theorem 1 is a consequence of the following more general statement:

Theorem 3. *Let $d \geq 3$ and let $p > p_c(\mathbb{Z}^d)$. There exists a constant $C = C_{d,p}$ with the following property: if A is a subgraph of $\mathbb{Z}^d$ s.t. (I) the C-core of A is transient, and (II) bond p-percolation on A has, a.s., a unique infinite cluster, then, a.s., the infinite p-percolation cluster of A is transient.*

We will show that for every h satisfying (1) and every $C > 0$, the C-core of $\mathcal{W}_h$ is transient. This combined with Theorem 3 provides a proof of Theorem 1.

We will restate and prove Theorem 3 as a theorem about flows. Consider each undirected edge of a graph G as two directed edges, one in each direction. Let vw be the directed edge from v to w. A **flow** F on G with source v_0 is an edge function such that $F(vw) = -F(wv)$ and such that for any vertex $v \neq v_0$: $\sum_w F(vw) = 0$. Recall the following definitions [12].

Definition 4. *Let $g : \mathbf{R} \to \mathbf{R}$ be a function. The g-**energy** of a flow F on a graph, denoted by $H_g(F)$, is defined to be*

$$\sum_{e \in E} g(|F(e)|).$$

For d and α we define $\Psi_{d,\alpha}$ to be

$$\Psi_{d,\alpha}(x) = \frac{|x|^{d/(d-1)}}{\log(1+|x|^{-1})^{\alpha}}.$$

.

Definition 5. *The (d, α)-**energy** of a flow F on a graph, denoted $H_{d,\alpha}(f)$ is the $\Psi_{d,\alpha}$-energy of F, i.e.*

$$H_{d,\alpha}(f) = \sum_{e \in E} \Psi_{d,\alpha}(F(e))$$

It is well known that a graph is transient if and only if it has a flow with finite $(2, 0)$-energy (since $\Psi_{2,0}(x) = x^2$; see [17, 6, 20]). It is also known that on $\mathbb{Z}^d$ there are flows with finite $(d, 1+\varepsilon)$-energy but no flow with finite $(d, 1)$-energy (see [17] and [14]). In [14] Hoffman and Mossel (refining an earlier result of Levin and Peres [15]) proved that the same is true for the infinite cluster of supercritical percolation in $\mathbb{Z}^d$, provided that $d \geq 3$.

In [12] Hoffman proved that on the infinite cluster in $\mathbb{Z}^2$ there are flows with finite $(2, 2+\varepsilon)$-energy. For $\mathbb{Z}^2$ itself, there are flows with finite $(2, 1+\varepsilon)$-energy. In his paper, Hoffman asks whether there are flows with finite $(2, 1+\varepsilon)$-energy on the infinite percolation cluster. We prove the following:

Theorem 6. *The infinite cluster of supercritical bond percolation in $\mathbb{Z}^2$ a.s. supports a flow of finite $(2, 1+\varepsilon)$-energy.*

Theorem 6 is a corollary of a more general result.

Theorem 7. *Let $\varphi : [0, \infty] \to [0, \infty]$ be a convex function s.t.*
(I) There exists $l \in \mathbb{N}$ such that $x^{-l}\varphi(x)$ is decreasing.
(II) $\mathbb{Z}^2$ supports a flow with finite φ-energy.
Then, for every $p > \frac{1}{2}$, a.s. the infinite cluster for bond percolation (with parameter p) in $\mathbb{Z}^2$, supports a flow of finite φ-energy.

In Section 2 we give a proof of Theorem 3 and show that it implies Theorem 1. The argument is based on large deviation results by Antal-Pisztora [1].

In Section 3 we give an alternative proof of Theorem 1 for large values of the percolation parameter p. This proof relies on connectivity properties of $\mathbb{Z}^d$ instead of the Antal-Pisztora Theorem and can be extended to other graphs. This alternative approach also yields a new proof of the following theorem due to Benjamini and Schramm [4]:

Theorem 8. *Let G be the Cayley graph of a finitely generated group of polynomial growth which is not a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$. Then for p sufficiently close to 1, the infinite p-percolation cluster on G is transient.*

In Section 4 we prove Theorems 7 and 6.

2 Proof of Theorems 3 and 1

We begin by stating known lemmas that we need. First (see e.g. Chapter 2 of [19] or [20]):

Lemma 9. *Let G be a graph, and let $v_0 \in G$. Then, G is transient if and only if there exists a probability measure μ on self-avoiding paths in G starting at v_0 s.t. if P and Q are two paths chosen independently according to μ, the expected number of edges in $P \cap Q$ is finite.*

We identify μ with the flow that it induces, i.e. the expectation of a unit flow through a path chosen by μ. Note that the condition on μ is equivalent to the condition:

$$H_{2,0}(\mu) = \sum_e \mu(\{P|e \in P\})^2 < \infty. \tag{3}$$

We also need the next lemma which is proved (though not stated in this form) by Antal and Pisztora [1]. In this form it appears as Lemma 2.13 of [5].

For two vertices in the same connected component, x and y, denote by $D(x,y)$ the length of the shortest path between them.

Lemma 10. *Let $p > p_c(\mathbb{Z}^d)$. There exist $\rho = \rho(p,d)$ and $C > 0$ such that for every integer m and any $v, w \in \Lambda_m := [-m,m]^d$:*

$$\mathbf{P}_p(v \leftrightarrow w \text{ and } D(v,w) > \rho m) \le e^{-Cm}. \tag{4}$$

We will use the following equivalent form of this lemma:

Lemma 11. *If $p > p_c(\mathbb{Z}^d)$ then there exist $\rho, \theta > 0$ determined by p and d such that if v, w are both in the infinite cluster then for all $m > \rho|v-w|$*

$$\mathbf{P}_p(D(v,w) > m) \le e^{-\theta m}.$$

The idea of the proof of Theorem 3 is to construct a measure on paths in the infinite percolation cluster of A by taking the measure μ supported on paths in the core, and modifying the paths to be in the percolation cluster. A path chosen by μ will typically have gaps where it passes through closed edges or non-percolating vertices. We will replace the gap in the path by a "bridge" which will be a shortest path in the infinite cluster which connects the ends of the gap. Since more than one shortest path exists, we choose the one that is smallest in the lexicographic ordering obtained by writing the path as a list of vertices (themselves ordered by the lexicographic ordering of $\mathbb{Z}^d$). We will see that with positive probability all the gaps that are inside the C-core of G will have bridges within G.

Definition 12. *A* **gap** *in a percolation configuration is (the set of vertices and the set of edges of) a connected cluster in the complement of the infinite cluster.*

Clearly a gap consists of a closed cluster and anything separated by it from the infinite cluster. Therefore if in some configuration large closed clusters are rare then so are large gaps. This will be formally stated later.

We first proceed to prove Theorem 3 for p close to 1:

Lemma 13. *Let $d \geq 3$. Then there exist $p_d < 1$ and $C = C_d$ with the following property: If $p \geq p_d$, and A is a subgraph of $\mathbb{Z}^d$ s.t. (I) the C-core of A is transient, and (II) bond (resp. site) p-percolation on A has, a.s., a unique infinite cluster, then, a.s., the infinite bond (resp. site) p-percolation cluster of A is transient.*

Proof. Choose p_d close enough to 1, so that $(1 - p_d)$-percolation is sub-critical. Consider percolation on A as a restriction to A of Bernoulli percolation on all of $\mathbb{Z}^d$. Denote by I the infinite percolation cluster in A, and by $J \supseteq I$ the infinite percolation cluster in $\mathbb{Z}^d$.

A.s., $A \setminus J$ has only finite clusters. Moreover, if for a vertex x we use $C(x)$ to denote the cluster of $A \setminus J$ containing x, then there exist δ and $\alpha > 0$ such that for every x,

$$\mathbf{P}(|C(x)| > n) < \delta e^{-n^\alpha}. \tag{5}$$

For the diameter we have an exponential bound: There exists a constant γ such that

$$\mathbf{P}(\mathrm{diam}(C(x)) > n) < e^{-\gamma n}. \tag{6}$$

Let

$$\beta > \max\left(\frac{1}{\gamma}, \frac{1}{\theta}\right) \tag{7}$$

be so large that

$$1 - C_1 \sum_{R=2}^{\infty} \frac{1}{R^{1+d(\gamma\beta-1)}} - C_1 \beta d^{2d} \sum_{R=2}^{\infty} \frac{(\log R)^{2d}}{R^{1+d(\theta\beta-1)}} > 0 \tag{8}$$

where ρ and θ are from Lemma 11, γ is from (6) and $C_1 = C_1(d)$ is a constant so that there are no more than $C_1 R^{d-1}$ vertices at distance R from the origin.

We then take $C_d = l > \rho\beta d$.

The l-core of A is transient. Lemma 9 states that there is a probability measure μ, satisfying (3), on paths in the core starting at some v_0. Since transience of I is a 0-1 event, and $v_0 \in I$ with positive probability, we may condition on the event that $v_0 \in I$.

Let $P = (v_0, v_1, v_2, v_3, ...)$ be a path (chosen according to μ). A.s., P intersects J infinitely often, so we may restrict ourselves to paths that intersect J infinitely often. We modify P to get P', a path in J, as follows: If P enters a gap at x and leaves the gap at y, then we replace the part of P from x to y by the shortest open path from x to y around the gap, with ties broken in an arbitrary manner. Such replacements will be denoted as bridges over the gap. Let μ' be the resulting measure on paths (or, more precisely, if ϕ is the function which assigns P' to each P, then let $\mu' = \mu \circ \phi^{-1}$).

Clearly, μ' is supported on paths in J starting at v_0. These paths are not necessarily self-avoiding. To overcome this problem we may loop-erase the paths.

To conclude the proof of the transience of I we will prove the following two lemmas:

Lemma 14. *In the above setting, with positive probability μ' is supported on paths in I.*

Lemma 15. *In the above setting, a.s. μ' has finite energy (i.e. satisfies* (3)*).*

From these two lemmas it follows that with positive probability we have constructed a measure on paths in I satisfying (3), so with positive probability I is transient. Since transience of I is a 0-1 event we are done. □

Proof of Lemma 14. Define a point $x \in \mathbb{Z}^d$ to be α-bad if $x \notin J$, and some bridge over the gap containing x leaves the ball $B(x, \alpha)$. We show that for some constant C, with positive probability for any R there are no $(C \log R)$-bad points in $B(v_0, R)$. If this is the case, then every bridge used to fix a path in the C-core of A remains in A, and therefore the fixed path is in I.

We estimate the probability that a point x is α-bad: Let γ be the constant from (6) ρ, θ be as in Lemma 11. Note that w.log. we can assume $\rho > 2$. Let G be the gap containing x (G is empty if $x \in J$). The gap is unlikely to be large:

$$\mathbf{P}(\operatorname{diam}(G) > \alpha/\rho) < e^{-\gamma\alpha/\rho}.$$

Otherwise, suppose $\operatorname{diam}(G) \le \alpha/\rho < \alpha/2$. There are at most α^d vertices that could be in G, and therefore at most α^{2d} pairs of vertices that could be the endpoints of a bridge over G. Each of these pairs has $|u - v| \le \alpha/\rho$, so by Lemma 11, each bridge has length greater than α with probability at most $e^{-\theta\alpha}$.

If additionally all bridges over G have length at most α, then a point on a bridge is at distance at most $\alpha/2$ from one of the endpoints, which in turn is at distance at most $\alpha/2$ from x, and so x is not α-bad. A union bound gives for $\alpha = C \log R$:

$$\mathbf{P}(x \text{ is } \alpha\text{-bad}) \le e^{-\gamma\alpha/\rho} + (\alpha/2)^{2d} e^{-\theta\alpha} = R^{-C\gamma/\rho} + (C/2 \log R)^{2d} R^{-C\theta}.$$

Since the number of points at distance R from v_0 is $O(R^{d-1})$, for some large enough C, with high probability there are no $(C \log R)$-bad points in $B(v_0, R)$. □

Proof of Lemma 15. Define the functions

$$F(e) = \mu(\text{paths which go through e})$$

and

$$F'(e) = \mu'(\text{paths which go through e}).$$

The μ-expected (resp. μ'-expected) size of the intersection between independently chosen paths is exactly $\sum_e F(e)^2$ (resp. $\sum_e F'(e)^2$).

We show that

$$\mathbf{E}\left(\sum_e F'(e)^2\right) < \infty,$$

and thus a.s. $\sum_e F'(e)^2 < \infty$.

We say that an edge e is *projected* on an edge f (denoted $e \rightsquigarrow f$) if either $e = f$ or if some path with a gap including e has a bridge over it passing through f. It is clear that

$$F'(f) \leq \sum_{e \rightsquigarrow f} F(e).$$

We define a second relation: For edges e and f in $\mathbb{Z}^d$, we say that e is *potentially projected* on f if $e = f$ or if e is not in J and f is on a shortest path in J between two points on the boundary of the component of $\mathbb{Z}^d - J$ containing e.

Clearly, the relation $\rightarrow$ (thought of as a set) contains the relation $\rightsquigarrow$, so

$$F'(f) \leq \sum_{e \rightarrow f} F(e).$$

Another useful fact is that the (disribution of) $\rightarrow$ is invariant under all automorphisms of the graph $\mathbb{Z}^d$.

Define $S(e) = \{f | e \rightarrow f\}$ and $T(f) = \{e | e \rightarrow f\}$. Then we have the bound:

$$\begin{aligned}
\sum_{f \in E} F'(f)^2 &\leq \sum_{f \in E} \left(\sum_{e \in T(f)} F(e)\right)^2 \\
&\leq \sum_{f \in E} |T(f)| \sum_{e \in T(f)} F(e)^2 \\
&= \sum_{e \in E} \left(F(e)^2 \cdot \sum_{f \in S(e)} |T(f)|\right).
\end{aligned}$$

This separates the energy of μ' into the energy of μ, with weight determined only by the percolation configuration. If we show that $\mathbf{E} \sum_{f \in S(e)} |T(f)| < \infty$, then we are done since this expectation is the same for all e, and $\sum F(e)^2 < \infty$.

To this end, note that if $e \rightarrow f$, then the endpoints of e are $d(e,f)$-bad, and the probability of this is exponentially small in the distance:

$$\mathbf{P}(e \rightarrow f) < Ce^{-ad(e,f)}.$$

Consequently, $\mathbf{P}(|T(f)| > n) < e^{-\alpha n^\beta}$ for some constants α, β, and in particular $|T(f)|$ has all finite moments.

Now, the quantity we wish to bound can be written as

$$\mathbf{E} \sum_{f \in S(e)} |T(f)| = \sum_{f,g} \mathbf{P}(e \to f, g \to f).$$

For every pair of edges e, f there exist $C = 2(d-1)!$ automorphisms that maps f to e. For every such σ we have

$$\begin{aligned}\mathbf{P}(e \to f, g \to f) &= \mathbf{P}\left(\sigma(e) \to \sigma(f), \sigma(g) \to \sigma(f)\right) \\ &= \mathbf{P}\left(\sigma(e) \to e, \sigma(g) \to e\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{E} \sum_{f \in S(e)} |T(f)| &= \sum_{f,g} \mathbf{P}(e \to f, g \to f) \\ &= \frac{1}{C} \sum_{\sigma,g} \mathbf{P}(\sigma(e) \to e, \sigma(g) \to e) \\ &= \sum_{f',g'} \mathbf{P}(f' \to e, g' \to e) \\ &= \mathbf{E}\left(|T(e)|^2\right) < \infty.\end{aligned}$$

□

Proof of Theorem 3. Lemma 13 is just Theorem 3 for high enough retention probabilities, say above $\hat{p}$. To extend the result to any $p > p_c(\mathbb{Z}^d)$ we use a renormalization argument that was used by Häggström and Mossel in [10].

Let $p \in (p_c, 1]$. N is a (large) positive integer, divisible by 8, which will be determined later. Define $Q_N(v)$ to be the cube of side-length $5N/4$, centered at v. Let A be a subgraph of $\mathbb{Z}^d$ such that its CN-core is transient for $C = C_d$ as in Lemma 13. Consider the renormalized graph

$$A_N = N^{-1}\left\{v \in N\mathbb{Z}^d \middle| Q_N(v) \subset A\right\}.$$

A_N has a transient C-core, and therefore there is (a.s.) a transient infinite cluster in the $\hat{p}$-site percolation on A_n.

Apply p-percolation on A, and consider a vertex $v \in A_N$ to be open if $Q_N(v)$ contains a connected component which connects all $2d$ faces of $Q_N(v)$ but all other connected components in $Q_N(v)$ have diameter less than $N/4$ (which is the overlap between adjacent cubes). Denote the set of open vertices by $A_{N,p}$. It follows from Proposition 2.1 in Antal-Pisztora [1] that if N is large enough then $A_{N,p}$ dominates $\hat{p}$-site percolation on A_N. A connected component in $A_{N,p}$ implies a connected component in the percolation on A, and therefore p-percolation on A has a transient infinite cluster. □

Proof of Theorem 1. To deduce the Theorem from Theorem 3 it suffices to show that for any monotone function h s.t. $\mathcal{W}_h$ is transient, and any C, the C-core of $\mathcal{W}_h$ (denoted by $\mathcal{W}(h, C)$) is transient as well.

Recall Lyons' criterion: $\mathcal{W}_h$ is transient if and only if

$$\sum_{j=1}^{\infty} \frac{1}{jh(j)} < \infty$$

Lyons also proved that if the wedge $\mathcal{W}_h$ is transient, then its subgraph

$$\mathcal{V}_h = \mathcal{W}_h \cap \{(x,y,z)|x \geq 0 \text{ and } |y| \leq x\}$$

is also transient.

Now, let C be arbitrary and let h be an increasing function such that $\mathcal{W}_h$ is transient. Lyons' criterion implies that there exists x_0 s.t. for all $x > x_0$, $h(x) > 4C\log(3x)$. Indeed, if there were a sequence x_n with $h(x_n) \leq 4C\log(3x_n)$, then w.log. assume $x_n > x_{n-1}^2$, and then

$$\sum_{j=x_{n-1}}^{x_n} \frac{1}{jh(j)} \geq \sum_{j=x_{n-1}}^{x_n} \frac{1}{j \cdot 4C\log(3x_n)} > \frac{\log x_n - \log x_{n-1}}{4C\log(3x_n)} > \frac{1-o(1)}{8C},$$

and the sum over all of $\mathbb{N}$ would be infinite.

Claim 16. *We may assume, w.log., that $h(x+1) - h(x) \in \{0,1\}$ for every x.*

For such h, we conclude the proof by showing that the C core of $\mathcal{W}_h$ contains all but finitely many vertices of $\mathcal{V}_{h/2}$. Indeed, assume $a = (x,y,z) \in \mathcal{V}_{h/2}$, and that $x > x_0$. Since $y \leq x$ and $z \leq x$ we have $|a| \leq 3x$. For vertex $a + \Delta a$ at distance at most $C\log 3x$ from a we have:

$$\begin{aligned} z &\leq h(x)/2, \\ 2C\log 3x &\leq h(x)/2, \\ |\Delta x| + \Delta z &\leq 2C\log 3x, \\ h(x) - |\Delta x| &\leq h(x + \Delta x). \end{aligned}$$

Summing up these bounds gives

$$z + \Delta z \leq h(x + \Delta x),$$

and so $a + \Delta a \in \mathcal{W}_h$. □

Proof of Claim 16. From the definition of wedges, clearly we can assume that h is integer valued. Given h, define f inductively by: $f(0) = h(0)$ and

$$f(n+1) = \min(h(n+1), f(n)+1).$$

Let $A = \{j : f(j) = h(j)\}$ and $B = \{j | f(j) = f(j-1)+1\}$. For indices in A we have

$$\sum_{j\in A} \frac{1}{jf(j)} = \sum_{j\in A} \frac{1}{jh(j)} \leq \sum_{j=1}^{\infty} \frac{1}{jh(j)} < \infty. \tag{9}$$

Let $B = \{b_1, b_2, \ldots\}$. Clearly $b_j \geq j$. Since $\{b_k\}$ are the increase points of f and so $f(b_k) = f(0) + k \geq k$. Thus

$$\sum_{j\in B} \frac{1}{jf(j)} = \sum_{k=1}^{\infty} \frac{1}{b_k f(b_k)} \leq \sum_{k=1}^{\infty} k^{-2} < \infty. \tag{10}$$

Combining the two sums, since $A \cup B = \mathbb{N}$, we find that $\sum (jf(j))^{-1} < \infty$, and therefore $\mathcal{W}_f$ is transient.

If Theorem 1 holds for $\mathcal{W}_f$, then (since $\mathcal{W}_f \subseteq \mathcal{W}_g$) the Theorem holds for $\mathcal{W}_g$ as well. $\square$

3 General Graphs

If the percolation parameter p is close enough to 1, specifically if $p > 1 - p_c$, then a.s. the closed edges compose only finite closed clusters. If $p > p_s$ for some (larger) critical p_s then the closed clusters will not only be finite, but they will also be isolated from each other, and then a path that passes through a gap C can be bridged without leaving its boundary ∂C. This notion will lead us to another proof for "nice" graphs.

Definition 17. *The **k-boundary** of a set A in a graph G consists of all vertices outside A but at distance at most k from A:*

$$B_k(A) = \{v \big| d(v, A) < k\} \setminus A.$$

*The **inner k-boundary** of a set A in a graph G is the k-boundary of $G - A$, i.e. $\{x \in A : d(x, G \setminus A) < k\}$.*

Definition 18. *A graph G is said to have **connected k-boundaries** if for any connected set of vertices A s.t. $G \setminus A$ is also connected, the subgraph spanned by its k-boundary is connected.*

The property of having connected k-boundaries is rather general and holds for many graphs. For example, if a planar graph G which can be embedded in the plane so that all faces have at most $2k + 1$ edges, then G has connected k-boundaries, since if a set A includes a vertex from a face, then all other vertices of that face will be within distance k from A. (See, for example, Lemma 4.4 of [11]).

The lattice $\mathbb{Z}^d$ has connected 3-boundaries. This follows from a special case of a result in [2] that tells us that minimal cut sets in $\mathbb{Z}^d$ are half-connected (i.e. together with their neighbors they form connected sets).

We say that an edge is k-**strongly open** if it is open and so are all the edges at distance up to k from it. An edge is k-**weakly closed** if it is not k-strongly open. A similar definition is used for vertices in site percolation. $C(v)$ denotes the k-weakly closed cluster of v.

Lemma 19. *If a graph G has degrees bounded by d, there exists $p_s = p_s(k)$ such that for $p > p_s$, the weakly closed clusters $C(v)$ are a.s. finite. Moreover, for appropriate $\gamma = \gamma(p)$,*

$$\mathbf{P}(diam(C(v)) > n) < e^{-\gamma n},$$

and $\gamma(p) \to \infty$ as $p \to 1$.

Proof. Since whether an edge (or site) is k-weakly closed depends only on those edges up to distance k from it, by the results of [16] the configuration of k-strongly open edges dominates bond-percolation with parameter $q = q(p)$, and, moreover, $q(p) \to 1$ as $p \to 1$. If $p > p_s$ such that $q(p_s) > 1 - d^{-1}$, then the weakly closed clusters are dominated by sub-critical Galton-Watson trees, and so

$$\mathbf{P}(\mathrm{diam}(C(v)) > n) < e^{-\gamma n}. \tag{11}$$

If q is large, the Galton-Watson trees become smaller, and γ can be made arbitrarily large. $\square$

Lemma 20. *Let G be a graph with connected k-boundaries. Let $\bar{B}$ be a connected component of the complement of the k-strongly open cluster. Let U be the inner k-boundary of $\bar{B}$. Then*
(A) U is in the (regular) open cluster.
(B) U is connected.

Proof. (A) follows from the definition of the inner boundary and of k-strongly open edges. (B) follows from the fact that U is the k-boundary of $G - \bar{B}$ and the fact that G has connected k-boundaries. □

Theorem 21. *Let G be a transient graph with bounded degrees and connected k-boundaries. Assume also that for p close enough to 1, the p-percolation on G has, a.s., a unique infinite cluster. Under those conditions, if p is close enough to 1, then the infinite p-percolation cluster of G is transient.*

Note that Theorem 1 follows as a corollary, since transient wedges in $\mathbb{Z}^3$ satisfy the requirements of the Theorem.

Proof. As before, the argument involves bridging gaps in paths. Take $p > p_s$ where p_s is as defined in Lemma 19. Denote by I the infinite percolation cluster in G. If $C(v)$ is the k-weakly closed cluster containing v, then we know that $\mathbf{P}(\mathrm{diam}(C(v)) > n) < e^{-\gamma n}$.

Lemma 9 states that there is a probability measure μ, satisfying (3), on paths in G starting at some v_0. Since transience of I is a 0-1 event, and $v_0 \in I$ with positive probability, we may assume that $v_0 \in I$.

Let $P = (v_0, v_1, v_2, v_3, ...)$ be a path (chosen according to μ). A.s., P intersects I infinitely many times, so we may restrict ourselves to such paths. Now, we modify P to get P', a path in I, as follows: At any time at which P enters a gap B, we consider the k-weakly closed extension of the gap, $\bar{B}$, and we replace the part of the path from the first time P reaches $\bar{B}$ until the last time it leaves it, by a shortest path in the inner k-boundary of $\bar{B}$. Such a bridge exists by Lemma 20 (B). If ϕ is the function which assigns P' to each P, then let $\mu' = \mu \circ \phi^{-1}$. By part (A) of the lemma, μ' is supported on paths in I.

In place of the Antal-Pisztora bound on the length of bridges, we now use the trivial bound that the length of a bridge over a gap is bounded by the size of the gap's k-boundary.

We now need to show that almost surely μ' has finite energy. To do that, we repeat and slightly modify the calculation from the proof of Lemma 15. Recall the notations $S(e) = \{f|e \to f\}$ and $T(f) = \{e|e \to f\}$. We've seen that

$$\sum_{f \in E} F'(f)^2 \leq \sum_{e \in E} \left(F(e)^2 \cdot \sum_{f \in S(e)} |T(f)| \right).$$

It now suffices to show that for p large enough, for any edge e,

$$\mathbf{E} \sum_{f \in S(e)} |T(f)| < D < \infty,$$

for some uniform bound D.

In order for $e \to f$ to occur, f must be in the k-boundary of $C(e)$, and therefore $\mathrm{diam}(C(e)) \geq d(e,f) - k$. By the exponential bound on the diameters of weakly closed clusters,

$$\mathbf{P}(e \to f) \leq Ce^{-\gamma d(e,f)}.$$

As before we write

$$\mathbf{E} \sum_{f \in S(e)} |T(f)| = \sum_{f,g} \mathbf{P}(e \to f, g \to f),$$

and use the exponential bound:

$$\mathbf{P}(e \to f, g \to f) \leq exp(-\gamma \max(d(e,f), d(g,f))).$$

Since there are at most Cd^k edges f at distance k from e, and at most Cd^l edges g at distance l from any of these f's, we have:

$$\begin{aligned}\sum_{f,g} \mathbf{P}(e \to f, g \to f) &\leq \sum_{k,l} C^2 d^{l+k} e^{-\gamma \max(k,l)} \\ &\leq 2 \sum_{k<l} C^2 d^{l+k} e^{-\gamma l} \\ &\leq 2 \sum_{l} C^2 l d^{2l} e^{-\gamma l}.\end{aligned}$$

This is bounded, uniformly for all edge e, whenever $d^2 e^{-\gamma}$. Lemma 19, guarantees this holds for p large enough. $\square$

Now we can prove Theorem 8.

Proof of Theorem 8. By Gromov's theorem (see [9]), any finitely generated group of polynomial growth is a finite extension of a finitely presented group. Corollary 4 of [2] tells us that the Cayley graph of such a group has connected k-boundaries for some k (depending on the lengths of the relations). Corollary 10 of the same paper tells us that for p close enough to 1, the infinite cluster is unique. Since G is not a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$, by a theorem of Varopoulos, G is transient (see [22]). Thus all the conditions of Theorem 21 are satisfied. $\square$

4 Flows on percolation clusters in $\mathbb{Z}^2$

We first restate and prove Theorem 7.

Theorem 7. *Let $\varphi : [0,\infty] \to [0,\infty]$ be a convex function s.t.*
(I) There exists $l \in \mathbb{N}$ such that $x^{-l}\varphi(x)$ is decreasing.
(II) $\mathbb{Z}^2$ supports a flow with finite φ-energy.
Then, for every $p > \frac{1}{2}$, a.s. the infinite percolation cluster supports a flow with finite φ-energy.

Proof. First, we note that φ is increasing, because $\varphi(0) = 0$, $\varphi(x) \geq 0$ for every x and φ is convex. Throughout the proof the term energy will refer to φ-energy. We begin with a flow F on $\mathbb{Z}^2$ with finite energy with the source at 0. Such a flow exists by condition (II) of the theorem. The function φ is increasing and convex in the flow on each edge, so the flow F can be made acyclic (i.e. a flow s.t. there is no cycle s.t. all of its edges get positive flow) without increasing its energy. Normalize the flow so that the total flow out of 0 is 1 and the energy is finite. Since F is an acyclic flow with source at 0, it induces a probability measure μ on self-avoiding paths starting at 0, see e.g. [18]. The μ-measure of the set of paths passing through e is exactly $|F(e)|$.

As before, note that the existence of a flow with finite energy on the percolation cluster is a 0-1 event, so we can assume that 0 belongs to the infinite cluster. Since in $\mathbb{Z}^2$ we have $p_c = \frac{1}{2}$ our condition $p > 1 - p_c$ is equivalent to $p > p_c$. We construct as before a new measure μ', by using a shortest bridge over any gap in a path. We wish to show that μ' a.s. has finite energy.

The function φ is increasing and convex in $[0,1]$, and, moreover,

$$\begin{aligned} \varphi\left(\sum_{i=1}^{n} x_i\right) &= \varphi\left(n^{-1}\sum_{i=1}^{n} n x_i\right) \\ &\leq n^{-1}\sum_{i=1}^{n}\varphi(n x_i) \\ &\leq n^{-1}\sum_{i=1}^{n} n^l \varphi(x_i) \\ &= n^{l-1}\sum_{i=1}^{n}\varphi(x_i). \end{aligned} \tag{12}$$

We now proceed to estimate the energy of F'. Using the notation from the proof of Lemma 15:

$$\begin{aligned} \varphi(F'(f)) &\leq \varphi(\sum_{e\to f} F(e)) \\ &\leq |S(f)|^{l-1}\sum_{e\to f}\varphi(F(e)). \end{aligned}$$

Summing over f we get:

$$\begin{aligned} H_\varphi(F') &= \sum_f \varphi(F'(f)) \\ &\leq \sum_f |S(f)|^{l-1}\sum_{e\to f}\varphi(F(e)) \\ &= \sum_e \varphi(F(e)) \times \left(\sum_{f\in S(e)} |T(f)|^{l-1}\right) \end{aligned}$$

As in the proof of Lemma 15 (where l was 1), we have by transitivity

$$\mathbf{E}\sum_{f\in S(e)} |T(f)|^{l-1} = \mathbf{E}|T(f)|^l.$$

Since $|T(f)|$ has all finite moments, we have

$$\mathbf{E}H_\varphi(F') \leq H_\varphi(F)\mathbf{E}|T(f)|^l < \infty,$$

and thus a.s. F' has finite energy. □

Proof of Theorem 6. This follows from Theorem 7 once we notice that the function $\Psi_{2,1+\varepsilon} = \frac{|x|^2}{\log(1+|x|^{-1})^{1+\varepsilon}}$ corresponding to $(2, 1+\varepsilon)$-energy satisfies the conditions of the Theorem with $l = 4$. $\square$

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