Electronic Journal of Probability

Vol. 14 (2009), Paper no. 25, pages 633–662.

Journal URL
http://www.math.washington.edu/~ejpecp/

Characterization of maximal Markovian couplings for diffusion processes

Kazumasa Kuwada*
Graduate School of Humanities and Sciences,
Ochanomizu University,
Tokyo 112-8610, Japan
Email: kuwada@math.ocha.ac.jp

Abstract

Necessary conditions for the existence of a maximal Markovian coupling of diffusion processes are studied. A sufficient condition described as a global symmetry of the processes is revealed to be necessary for the Brownian motion on a Riemannian homogeneous space. As a result, we find many examples of a diffusion process which admits no maximal Markovian coupling. As an application, we find a Markov chain which admits no maximal Markovian coupling for specified starting points.

Key words: Maximal coupling, Markovian coupling, diffusion process, Markov chain.

AMS 2000 Subject Classification: Primary 60H30,60J10,60J60,58J65.

Submitted to EJP on June 3, 2008, final version accepted February 23, 2009.

*Partially supported by the JSPS fellowship for research abroad

1 Introduction

The concept of coupling is very useful in several areas in probability theory. Here, given two stochastic processes $\tilde{X}_t^{(1)}$ and $\tilde{X}_t^{(2)}$ on a common state space M, a stochastic process $\mathbf{X}_t = (X_t^{(1)}, X_t^{(2)})$ on $M \times M$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a coupling of $\tilde{X}_t^{(1)}$ and $\tilde{X}_t^{(2)}$ if $X^{(i)}$ and $\tilde{X}^{(i)}$ have the same law for $i = 1, 2$. A characteristic of couplings in which we are interested is the coupling time T:

$$T(\mathbf{X}) := \inf\{t > 0 \mid X_s^{(1)} = X_s^{(2)} \text{ for any } s > t\}. \tag{1.1}$$

In many applications, we would like to make $\mathbb{P}[T(\mathbf{X}) > t]$ as small as possible by taking a suitable coupling. The well-known coupling inequality provides a lower bound for this probability as follows:

$$\mathbb{P}[T(\mathbf{X}) > t] \ge \frac{1}{2} \left\| \mathbb{P}[\theta_t \tilde{X}^{(1)} \in \cdot] - \mathbb{P}[\theta_t \tilde{X}^{(2)} \in \cdot] \right\|_{\mathrm{var}}, \tag{1.2}$$

where θ_t is the shift operator (see [16] for example). We call a coupling $\mathbf{X}$ maximal if the equality holds in (1.2) at any $t > 0$. As shown in [23], a maximal coupling always exists if M is Polish and both $\tilde{X}^{(1)}$ and $\tilde{X}^{(2)}$ are cadlag processes (for discrete time Markov chains, a maximal coupling exists on more general state spaces; see [7]).

A significance of coupling methods is emphasized when we deal with couplings of Markov processes because of their deep connection with analysis (for example, see [3; 5; 6; 10; 27] and references therein). Let $(\{X_t\}_{t \ge 0}, \{\mathbb{P}_x\}_{x \in M})$ be a Markov process on M. We consider the case that $\mathbf{X}$ is a coupling of X. It means that the law of $\tilde{X}^{(i)}$ equals $\mathbb{P}_{x_i} \circ X^{-1}$ for $i = 1, 2$ for some $x_1, x_2 \in M$ with $x_1 \neq x_2$. In this case, many couplings appeared in application inherit a sort of Markov property from the original process. For example, the well-known Kendall-Cranston coupling (see [6; 10; 27]), which is a coupling of the Brownian motion on a complete Riemannian manifold, becomes a Markov process. Indeed, intuitively saying, we construct it by integrating a "coupling of infinitesimal motions" of two Brownian particles. In this paper, we formulate a Markovian nature of couplings in the following way:

Definition 1.1. We call a coupling $\mathbf{X} = (X^{(1)}, X^{(2)})$ of $(X, \mathbb{P}_{x_1})$ and $(X, \mathbb{P}_{x_2})$ Markovian when $(\theta_s \mathbf{X})$. is a coupling of $(X, \mathbb{P}_{X_s^{(1)}})$ and $(X, \mathbb{P}_{X_s^{(2)}})$ under $\mathbb{P}[\,\cdot \mid \mathbf{X}_u, 0 \le u \le s]$ for any $s \ge 0$.

This definition means that conditioning on the past trajectories preserves the property that $\mathbf{X}$ is a coupling of the original Markov process X in the future. Note that $\mathbf{X}$ is Markovian if $\mathbf{X}$ itself is a Markov process on the product space $M \times M$. Although Markovian coupling naturally appears in many cases, it is quite unclear whether Markovianity is compatible with maximality. Hence the following basic question arises; When does (or does not) a *maximal Markovian* coupling exist? Such a question has appeared repeatedly in various contexts in the literature. For example, K. Burdzy and W.S. Kendall [3] considered a similar problem in connection with estimates of a spectral gap (see Remark 2.4 below for the relation between maximal couplings and spectral gap estimates). It has been believed that maximal couplings are non-Markovian in general (see [7; 8; 18] for discrete case; see Remark 7.2 also).

The purpose of this paper is to give an answer to the question raised above for a class of Markov processes. Suppose that X is a diffusion process. Let us define the following property introduced in [13], which is closely related to the existence of a maximal Markovian coupling of a diffusion process.

Definition 1.2. For a diffusion process $(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M})$ on M and $x_1, x_2 \in M$, we say that X has a reflection structure with respect to (x_1, x_2) if there exists a continuous map $R : M \to M$ such that

(i) $R \circ R = \mathrm{id}$ and $\mathbb{P}_{x_1} \circ (RX)^{-1} = \mathbb{P}_{x_2} \circ X^{-1}$,

(ii) For $H := \{x \in M \mid Rx = x\}$, $M \setminus H = M_1 \sqcup M_2$ holds for some open sets M_1 and M_2 satisfying $R(M_1) = M_2$.

Reflection structure is a generalization of a geometric structure behind the mirror coupling of the Euclidean Brownian motion. To see it, let us suppose $M = \mathbb{R}^d$ and that X is the Brownian motion for a moment. Let R be the mirror reflection with respect to the $(d-1)$-dimensional hyperplane $H := \{z \in \mathbb{R}^d \mid |x_1 - z| = |x_2 - z|\}$ bisecting x_1 and x_2. Then the so-called mirror coupling $(X^{(1)}, X^{(2)})$ is given as follows:

$$X_t^{(2)} := \begin{cases} RX_t^{(1)} & t < \tau, \\ X_t^{(1)} & t \geq \tau, \end{cases} \tag{1.3}$$

where τ is the first hitting time of $X^{(1)}$ to H. Obviously, the mirror coupling is a strong Markov process as an $\mathbb{R}^d \times \mathbb{R}^d$-valued process. In addition, the fact $T = \tau$ implies that the mirror coupling is maximal. We can easily verify that the mirror reflection R on $\mathbb{R}^d$ carries a reflection structure. In general, the same construction of a coupling as (1.3) still works if there exists a reflection structure with respect to (x_1, x_2). We also call it the mirror coupling. We can show that the mirror coupling is a maximal Markovian coupling as well ([13], Proposition 2.2). It means that a reflection structure implies the existence of a maximal Markovian coupling.

Our main result asserts that a reflection structure is also necessary for the existence of a maximal Markovian coupling in the following framework:

Theorem 1.3. *Let M be a Riemannian homogeneous space and $(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M})$ the Brownian motion on M. Suppose that there is a maximal Markovian coupling $\mathbf{X}$ of $(X, \mathbb{P}_{x_1})$ and $(X, \mathbb{P}_{x_2})$. Then there exists a reflection structure R with respect to (x_1, x_2). Furthermore, $\mathbf{X}$ is the mirror coupling determined by R.*

To the best of the author's knowledge, such a qualitative necessary condition for the existence of a maximal Markovian coupling is not known for any Markov process until now. Moreover, this simple characterization helps us to find examples of diffusion processes which admits no maximal Markovian coupling. Actually, as we will see, there is a plenty of examples where no reflection structure exists for any pair of starting points (Theorem 6.6). Though homogeneity of the state space provides much symmetries, it is not sufficient for the existence of a reflection structure in most cases. Note that the latter part of Theorem 1.3 also asserts the uniqueness of maximal Markovian couplings. On one hand, it is shown in [13] that the mirror coupling is a unique maximal Markovian coupling when there exists a reflection structure in more general framework than Theorem 1.3 including the Brownian motion on a complete Riemannian manifold. On the other hand, Theorem 1.3 asserts the uniqueness without *a priori* assumption on the existence of a reflection structure though a stronger assumption is imposed on the state space.

As an application of Theorem 1.3, we obtain a finite state, discrete time Markov chain which admits no maximal Markovian coupling for specified starting points (Theorem 7.1). A characterization of maximal Markovian couplings given in Theorem 1.3 heavily depends on the continuity of sample paths. It does not seem to be so easy to establish a similar characterization for Markov chains. Thus we will take a different approach. We use Theorem 1.3 to show the claim by considering a sequence of Markov chains which approximates a diffusion process.

In the rest of this section, we state the organization of this paper. In section 2, we introduce an initial framework of our argument on the state space and the diffusion process on it. It is more general than what assumed in Theorem 1.3. In section 3, first we discuss some basic properties of maximal Markovian couplings on the framework introduced in section 2. Next we show in Proposition 3.11 that the existence of a maximal Markovian coupling carries a weak symmetry. It asserts that, at each time $t \in [0,\infty)$, one particle places an antipodal point of the other particle each other with respect to a set $S_t \subset M$ until they meet. We call S_t "mirror" in the sequel because it plays a role of $\{x \in M \mid Rx = x\}$ if there is a reflection structure. It should be remarked that the mirror is a non-random set while it may depend on the time parameter t. In section 4, we derive a stronger symmetry under an additional condition (Assumption 3). There we show that Assumption 3 is a sufficient condition for the mirror to be independent of t (Proposition 4.2). As a result, we obtain a homeomorphism R such that the maximal Markovian coupling satisfies (1.3) in Theorem 4.5. It leads that any maximal Markovian coupling becomes a mirror coupling in a weak sense. Note that Assumption 3 is closely related to the homogeneity of the state space imposed in Theorem 1.3. The proof of Theorem 1.3 is completed in section 5 by showing a more general assertion (Theorem 5.1). There we consider further assumptions (Assumption 4,5) which are satisfied with the Brownian motion on a Riemannian homogeneous space. Under those assumptions, we show an additional property of R corresponding to the condition (i) of Definition 1.2. Examples of a Riemannian symmetric space where the Brownian motion admits no maximal Markovian coupling are given in section 6. With the aid of Theorem 1.3, the problem is reduced to a geometric observation. In section 7, we discuss maximal Markovian couplings of Markov chains.

2 Framework

In this section, we will introduce some notations and properties that are used throughout this paper. Let (M,d) be a metric space. We review some concepts on metric geometry in order to introduce additional properties on M. We call a curve $\gamma : [0,1] \to M$ geodesic if, for each $s \in [0,1]$, there exist $\delta > 0$ such that $d(\gamma(t),\gamma(s)) = |t-s| d(\gamma(0),\gamma(1))$ holds for $|t-s| < \delta$. We call a geodesic γ minimal if the length of γ realizes the distance between its endpoints. (M,d) is called a geodesic space when there exists a minimal geodesic joining x and y for each $x, y \in M$. (M,d) is called proper when every closed metric ball of finite radius is compact. Note that properness is equivalent to local compactness on complete geodesic metric spaces by Hopf-Rinow-Cohn-Vossen Theorem (Theorem 2.5.28 in [2]). Let $\gamma, \eta : [0,1] \to M$ be minimal geodesics that has a common starting point. We say that (γ,η) is a pair of branching geodesics if $\gamma([0,1]) \cap \eta([0,1]) \setminus \{\gamma(0)\} \neq \emptyset$, $\gamma(1) \neq \eta(1)$ and neither $\gamma([0,1]) \subset \eta([0,1])$ nor $\eta([0,1]) \subset \gamma([0,1])$.

We assume M to be a complete, proper geodesic space that has no pair of branching geodesics. Note that all of these assumptions are satisfied if M is a connected complete Riemannian manifold or an Alexandrov space. In these cases, the nonbranching property is an easy consequence of the Toponogov triangle comparison theorem (see [2; 4], for example).

Let μ be a positive Borel measure on M satisfying $0 < \mu(B) < \infty$ for every metric ball B of positive radius. Note that $\mathrm{supp}[\mu] = M$ holds. Let $(\{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in M})$ be a conservative diffusion process on M. We assume that there exists a strictly positive, symmetric transition density function $p_t(x,y)$ with respect to μ. That is,

$$\mathbb{P}_x[X_t \in A] = \int_A p_t(x,y)\mu(dy)$$

holds for any $A \in \mathscr{B}(M)$. In addition, we assume that $p_t(x,y)$ is jointly continuous as a function of t and y. All of these assumptions imposed on $(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M})$ are satisfied for a broad class of symmetric diffusions including the Brownian motion on a stochastically complete, complete Riemannian manifold. In this case, μ is chosen to be the Riemannian volume measure. Note that the local parabolic Harnack inequality implies the existence and continuity of p_t (see [21; 22]). For cases enjoying the inequality, see, for example, [1; 19] and references therein. To make a connection between the behavior of X_t and the metric structure of M, we assume the following:

Assumption 1. There exists a decreasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers with $\lim_{n\to\infty} t_n = 0$ such that $d(x,z) \leq d(y,z)$ holds if $p_{t_n}(x,z) \geq p_{t_n}(y,z)$ for infinitely many $n \in \mathbb{N}$.

Remark 2.1. Assumption 1 is satisfied if p_t enjoys the Varadhan type short time asymptotics, i.e.

$$\lim_{t\downarrow 0} t \log p_t(x,y) = -\frac{d(x,y)^2}{2} \tag{2.1}$$

for any $x,y \in M$. This relation holds true for the Brownian motion on a Lipschitz Riemannian manifold [17]. We also state two examples having the same property. First one is a diffusion process associated with the sub-Laplacian on a nilpotent group (see [24]). The second is a canonical diffusion process on an Alexandrov space (see [15; 26]). These two cases also satisfy all other assumptions as stated above (see [14] for the latter one). For later use, we remark that the limit in (2.1) is locally uniform in $x,y \in M$ in all cases mentioned above. It should be noted that the canonical diffusion process on the Sierpinski gasket enjoys Assumption 1 while it fails (2.1) (see [12], cf. [13]). But, unfortunately, it is not included in our framework because minimal geodesics on the Sierpinski gasket can branch.

Set $D := \{(x,x) \mid x \in M\} \subset M \times M$. In the rest of this paper, we assume the following:

Assumption 2. Given $(x_1,x_2) \in M \times M \setminus D$, a coupling $\mathbf{X} = (X^{(1)}, X^{(2)})$ of $(X, \mathbb{P}_{x_1})$ and $(X, \mathbb{P}_{x_2})$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is maximal and Markovian.

The next remark concerning to Markovian couplings is essentially due to Y. Nagahata.

Remark 2.2. The following example shows that Markovianity of couplings is strictly weaker than the Markov property as an $M \times M$-valued process. Take two independent Brownian motions Y_t and $\hat{Y}_t$ on $\mathbb{R}$ with $Y_0 = 1$ and $\hat{Y}_0 = -1$. Set

$$\begin{aligned}
\tau_0 &:= \inf\left\{ t > 0 \;\middle|\; \int_0^t 1_{[1,2]}(Y_s)ds \geq 1 \right\}, \\
\tau_1 &:= \inf\left\{ t > 0 \;\middle|\; Y_t = \hat{Y}_t \right\}, \\
\tau_2 &:= \inf\left\{ t > \tau_0 \;\middle|\; Y_t = \frac{1}{2}(Y_{\tau_0} + \hat{Y}_{\tau_0}) \right\}.
\end{aligned}$$

We define a coupling $(Y_t^{(1)}, Y_t^{(2)})$ by $Y_t^{(1)} := Y_t$ and

$$Y_t^{(2)} := \begin{cases} \hat{Y}_t & t < \tau_0 \wedge \tau_1, \\ Y_t & \tau_1 < \tau_0 \text{ and } \tau_1 \le t, \\ Y_{\tau_0} + \hat{Y}_{\tau_0} - Y_t & \tau_1 \ge \tau_0 \text{ and } \tau_0 \le t < \tau_2, \\ Y_t & \tau_1 \ge \tau_0 \text{ and } \tau_2 \le t. \end{cases}$$

We can easily verify that $(Y^{(1)}, Y^{(2)})$ is Markovian. But, obviously $(Y^{(1)}, Y^{(2)})$ is not a Markov process on $M \times M$.

Before closing this section, we give a remark on the coupling inequality (1.2). The right hand side of (1.2) is given as a total variation of measures on the path space. To handle it, we show that there is a simpler expression when we consider a coupling of Markov processes. Let us define $\varphi_t(x, y)$ by

$$\varphi_t(x, y) := \frac{1}{2} \| \mathbb{P}_x \circ X_t^{-1} - \mathbb{P}_y \circ X_t^{-1} \|_{\mathrm{var}}.$$

By definition, we have

$$\varphi_t(x, y) = \frac{1}{2} \int_M |p_t(x, z) - p_t(y, z)| \mu(dz) = \sup_{E \in \mathscr{B}(M)} \int_E \left(p_t(x, z) - p_t(y, z) \right) \mu(dz).$$

Lemma 2.3. *For any $(x_1, x_2) \in M \times M \setminus D$,*

$$\frac{1}{2} \left\| \mathbb{P}_{x_1} \circ \left(\theta_t X \right)^{-1} - \mathbb{P}_{x_2} \circ \left(\theta_t X \right)^{-1} \right\|_{\mathrm{var}} = \varphi_t(x_1, x_2).$$

Proof. Let $E \in \mathscr{B}(M)$ be the positive part of a Hahn decomposition of $\mathbb{P}_{x_1} \circ X_t^{-1} - \mathbb{P}_{x_2} \circ X_t^{-1}$. It means that $\mathbb{P}_{x_1}[X_t \in A] \ge \mathbb{P}_{x_2}[X_t \in A]$ for each $A \in \mathscr{B}(M)$ with $A \subset E$ and $\mathbb{P}_{x_1}[X_t \in A] \le \mathbb{P}_{x_2}[X_t \in A]$ for each $A \in \mathscr{B}(M)$ with $A \subset E^c$. Set

$$\tilde{E} := \left\{ (w_t)_{t \ge 0} \in C([0, \infty) \to M) \;\middle|\; w_0 \in E \right\}.$$

For any $A \in \mathscr{B}(C([0, \infty) \to M))$, the Markov property implies that

$$\mathbb{P}_{x_1}[\theta_t X \in A \cap \tilde{E}] - \mathbb{P}_{x_2}[\theta_t X \in A \cap \tilde{E}] = \int_E \mathbb{P}_y[X \in A] \left(p_t(x_1, y) - p_t(x_2, y) \right) \mu(dy) \ge 0.$$

In the same way, $\mathbb{P}_{x_1}[\theta_t X \in A \cap \tilde{E}^c] \le \mathbb{P}_{x_2}[\theta_t X \in A \cap \tilde{E}^c]$ follows. Thus $\tilde{E}$ is the positive part of a Hahn decomposition of $\mathbb{P}_{x_1} \circ (\theta_t X)^{-1} - \mathbb{P}_{x_2} \circ (\theta_t X)^{-1}$. Hence the conclusion follows. □

Let $T = T(\mathbf{X})$ be the coupling time as defined in (1.1). By Lemma 2.3, (1.2) is the same as

$$\mathbb{P}[T > t] \ge \varphi_t(x_1, x_2). \tag{2.2}$$

Thus the maximality of $\mathbf{X}$ implies the equality in (2.2) for any $t > 0$.

Remark 2.4. In the same way as Lemma 2.3, we can express the notion of maximality based on (2.2) instead of (1.2) for couplings of any Markov process. With the aid of this formulation, maximal couplings of a Markov process are related to the spectral gap estimate as follows (cf. [3]). Suppose that $\mu(M) < \infty$ and $p_t(x,y)$ has the following expression:

$$p_t(x,y) = c + \mathrm{e}^{-\lambda t} g(x,y) + R(t,x,y),$$

where $c > 0$ and $\lambda > 0$ are constants, g and R are (sufficiently regular) functions and $R(t,x,y)$ decays faster than $\mathrm{e}^{-\lambda t}$ as $t \to \infty$ uniformly in x, y. The Mercer theorem guarantees that it is the case if M is a compact Riemannian manifold with or without boundary and X is the (reflecting) Brownian motion. In this case, λ is the first nonzero eigenvalue of $-\Delta/2$ (with Neumann boundary condition). By the equality in (2.2), we can easily show that any maximal coupling $\mathbf{X}$ with $\mathbf{X}_0 = (x_1, x_2)$ satisfies

$$\liminf_{t\to\infty} \left(-t^{-1} \log \mathbb{P}[T(\mathbf{X}) > t] \right) \geq \lambda. \tag{2.3}$$

It means that a maximal coupling provides an upper bound of the spectral gap by the decay rate of $\mathbb{P}[T(\mathbf{X}) > t]$. If, in addition, $g(x_1,\cdot) - g(x_2,\cdot) \neq 0$ holds, then $-\lim_{t\to\infty} t^{-1} \log \mathbb{P}[T(\mathbf{X}) > t] = \lambda$ and hence $\mathbf{X}$ is efficient in the sense of [3]. Note that, as the following example indicates, maximal couplings are not always efficient. Take $0 < a_1 < a_2$. Let $M = M_1 \times M_2$ where M_i is a circle of length a_i with a homogeneous metric. We can easily see that there is a mirror coupling $\mathbf{X}$ starting from (x,y) and (x',y) for any $x, x' \in M_1$ with $x \neq x'$ and $y \in M_2$. In this case, we have $-\lim_{t\to\infty} t^{-1} \log \mathbb{P}[T(\mathbf{X}) > t] = 2\pi^2/a_1^2$ but $\lambda = 2\pi^2/a_2^2$.

3 Existence of a mirror

We begin with basic properties of the transition density which easily follow from our assumption. The symmetry of p_t and the Schwarz inequality imply

$$\begin{aligned} p_t(x,y) &= \int_M p_{t/2}(x,z) p_{t/2}(z,y) \mu(dz) \\ &\leq \left\{ \int_M p_{t/2}(x,z) p_{t/2}(z,x) \mu(dz) \right\}^{1/2} \left\{ \int_M p_{t/2}(y,z) p_{t/2}(z,y) \mu(dz) \right\}^{1/2} \\ &= p_t(x,x)^{1/2} p_t(y,y)^{1/2} \end{aligned} \tag{3.1}$$

for $x, y \in M$.

Lemma 3.1. *The equality holds in* (3.1) *if and only if* $x = y$.

Proof. It suffices to show "only if" part. The equality in (3.1) implies $p_{t/2}(x,z) = p_{t/2}(y,z)$ for any $z \in M$ since both of $p_{t/2}(x,\cdot)$ and $p_{t/2}(y,\cdot)$ are L^1-normalized, positive and continuous. In particular, $p_{t/2}(x,y) = p_{t/2}(x,x) = p_{t/2}(y,y)$ holds. By applying the same argument iteratively, we obtain $p_{t/2^n}(x,z) = p_{t/2^n}(y,z)$ for any $z \in M$ and $n \in \mathbb{N}$. It yields $\mathbb{E}_x[f(X_{2^{-n}t})] = \mathbb{E}_y[f(X_{2^{-n}t})]$ for any bounded continuous function f. Thus, by letting $n \to \infty$, we obtain $f(x) = f(y)$. Since f is arbitrary, $x = y$ follows. □

Lemma 3.2. *For each* $t > 0$, $\varphi_t(\cdot,\cdot)$ *is continuous on* $M \times M$.

Proof. Take a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ in $M \times M$ so that it converges to $(x, y) \in M \times M$. By the triangle inequality, $\left|\varphi_t(x_n, y_n) - \varphi_t(x, y)\right| \le \varphi_t(x_n, x) + \varphi_t(y_n, y)$ holds. Since we have

$$|p_t(z,w) - p_t(z',w)| = p_t(z,w) + p_t(z',w) - 2p_t(z,w) \wedge p_t(z',w),$$

the dominated convergence theorem together with the conservativity of X implies

$$\lim_{n\to\infty} \varphi_t(x_n, x) = \lim_{n\to\infty} \left(1 - \int_M p_t(x_n, z) \wedge p_t(x, z)\mu(dz)\right) = 0$$

because $p_t(z,w) \wedge p_t(z',w) \le p_t(z',w)$ holds. Hence the conclusion follows. $\square$

Lemma 3.3. *For each $s, t, u > 0$, the following hold:*

(i) $\mathbb{E}[\varphi_s(\mathbf{X}_t)] = \varphi_{t+s}(x_1, x_2)$,

(ii) $\mathbb{E}\left[\varphi_s(\mathbf{X}_{t+u}) \mid \mathbf{X}_q, 0 \le q \le u\right] = \varphi_{s+t}(\mathbf{X}_u)$ $\mathbb{P}$*-a.s..*

Note that Lemma 3.2 guarantees that the above expectations are well-defined.

Proof. Set $\mathscr{F}_t := \sigma(\mathbf{X}_s, 0 \le s \le t)$. By the maximality of $\mathbf{X}$ and the definition of T,

$$\varphi_{t+s}(x_1, x_2) = \mathbb{P}[T > t + s] = \mathbb{E}[\mathbb{P}[T \circ \theta_t > s \mid \mathscr{F}_t]].$$

Since $\mathbf{X}$ is Markovian, the coupling inequality for $\mathbb{P}[T \circ \theta_t > s \mid \mathscr{F}_t]$ yields

$$\mathbb{P}[T \circ \theta_t > s \mid \mathscr{F}_t] \ge \varphi_s(\mathbf{X}_t). \tag{3.2}$$

Thus we obtain $\varphi_{t+s}(x_1, x_2) \ge \mathbb{E}[\varphi_s(\mathbf{X}_t)]$. Take $E \in \mathscr{B}(M)$. By the definition of φ_s, we have

$$\varphi_s(\mathbf{X}_t) \ge \int_E \left(p_s(X_t^{(1)}, z) - p_s(X_t^{(2)}, z)\right) \mu(dz)$$

and hence we have

$$\begin{aligned}
\mathbb{E}[\varphi_s(\mathbf{X}_t)] &\ge \mathbb{E}\left[\int_E \left(p_s(X_t^{(1)}, z) - p_s(X_t^{(2)}, z)\right) \mu(dz)\right] \\
&= \int_E \left(p_{s+t}(x_1, z) - p_{s+t}(x_2, z)\right) \mu(dz).
\end{aligned}$$

By taking a supremum on $E \in \mathscr{B}(M)$, we obtain $\mathbb{E}[\varphi_s(\mathbf{X}_t)] \ge \varphi_{s+t}(x_1, x_2)$ and hence (i) holds. Now we have

$$\mathbb{P}[T \circ \theta_t > s \mid \mathscr{F}_t] = \varphi_s(\mathbf{X}_t) \text{ } \mathbb{P}\text{-a.s.} \tag{3.3}$$

since equality must hold in (3.2) $\mathbb{P}$-a.s. by the above argument. The equality (3.3) yields

$$\mathbb{E}\left[\varphi_s(\mathbf{X}_{t+u}) \mid \mathscr{F}_u\right] = \mathbb{E}\left[\mathbb{P}\left[T \circ \theta_{t+u} > s \mid \mathscr{F}_{t+u}\right] \mid \mathscr{F}_u\right] = \mathbb{P}\left[T \circ \theta_{t+u} > s \mid \mathscr{F}_u\right] \text{ } \mathbb{P}\text{-a.s..}$$

Since $\{T \circ \theta_{t+u} > s\} = \{T \circ \theta_u > s + t\}$ holds, (3.3) again yields

$$\mathbb{P}\left[T \circ \theta_{t+u} > s \mid \mathscr{F}_u\right] = \mathbb{P}\left[T \circ \theta_u > s + t \mid \mathscr{F}_u\right] = \varphi_{s+t}(\mathbf{X}_u) \text{ } \mathbb{P}\text{-a.s..}$$

Hence (ii) follows. $\square$

Remark 3.4. The argument in the proof of Lemma 3.3 implies

$$\mathbb{E}\left[\varphi_s(\mathbf{X}_t)\right] = \inf_{\nu} \int_{M\times M} \varphi_s(x,y)\nu(dxdy),$$

where the infimum is taken on any probability measure ν on $M \times M$ satisfying $\nu(A\times M) = \mathbb{P}_{x_1}[X_t \in A]$ and $\nu(M \times A) = \mathbb{P}_{x_2}[X_t \in A]$ for each $A \in \mathscr{B}(M)$. It means that the law of maximal Markovian coupling solves the Monge-Kantorovich problem for $\mathbb{P}_{x_1} \circ X_t^{-1}$ and $\mathbb{P}_{x_2} \circ X_t^{-1}$ with the cost function φ_s (for the Monge-Kantorovich problem, see [25] and references therein, for example).

Let us define a measure μ_t^D on M by

$$\mu_t^D(A) := \int_A p_t(x_1,z) \wedge p_t(x_2,z)\mu(dz).$$

We define an embedding $\iota : M \to M \times M$ by $\iota(x) = (x,x)$. Let us define measures μ_t and $\mu_{0,t}$ on $M \times M$ by

$$\begin{aligned}\mu_t &:= \mathbb{P} \circ \mathbf{X}_t^{-1},\\ \mu_{0,t} &:= \mu_t - \mu_t^D \circ \iota^{-1}.\end{aligned}$$

By Lemma 3.3 (i), we can show the following as in the proof of Proposition 3.5 in [13]:

Proposition 3.5. $\mu_t|_D = \mu_t^D \circ \iota^{-1}$.

Note that $\{T \le t\} \subset \{\mathbf{X}_t \in D\}$ obviously holds. In addition, by the maximality and Proposition 3.5, $\mathbb{P}[T \le t] = \mathbb{P}[\mathbf{X}_t \in D]$. Thus Proposition 3.5 yields the following:

Corollary 3.6. $\mu_{0,t}(E) = \mathbb{P}[\mathbf{X}_t \in E \setminus D] = \mathbb{P}[\mathbf{X}_t \in E, T > t]$.

The following lemma asserts that $\mu_{0,t}$ is nondegenerate.

Lemma 3.7. *For any $t > 0$, $\mu_{0,t} \not\equiv 0$.*

Proof. Suppose $\mu_{0,t} \equiv 0$ for some $t > 0$. Then we have

$$\begin{aligned}\mu_{0,t}(M \times M) &= \int_M \left(p_t(x_1,z) - p_t(x_1,z) \wedge p_t(x_2,z)\right)\mu(dz)\\ &= \int_M \left(p_t(x_2,z) - p_t(x_1,z) \wedge p_t(x_2,z)\right)\mu(dz)\\ &= 0\end{aligned}$$

and hence $p_t(x_1,z) = p_t(x_2,z)$ holds for every $z \in M$. Thus Lemma 3.1 asserts $x_1 = x_2$. But it contradicts with the choice of x_1 and x_2. □

For $x, y \in M$, we define $E_0(x,y)$, $E_0^*(x,y)$ and $H_0(x,y)$ by

$$\begin{aligned}E_0(x,y) &:= \{z \in M \mid d(x,z) < d(y,z)\},\\ E_0^*(x,y) &:= \{z \in M \mid d(x,z) > d(y,z)\},\\ H_0(x,y) &:= \{z \in M \mid d(x,z) = d(y,z)\}.\end{aligned}$$

Lemma 3.8. *For any $(x,y) \in M \times M \setminus D$ and $z \in H_0(x,y)$, z is an accumulation point of both $E_0(x,y)$ and $E_0^*(x,y)$. In particular, $\overline{E_0(x,y)} \cap \overline{E_0^*(x,y)} = H_0(x,y)$.*

Proof. Take $z \in H_0(x,y)$. Let γ be a minimal geodesic joining x and z and γ^* a minimal geodesic joining y and z. Take w on γ^* with $w \neq y,z$. Then the triangle inequality asserts

$$d(x,z) - d(z,w) \le d(x,w).$$

Since $x \neq y$ and geodesics on M cannot branch, the equality cannot hold in the above inequality. Thus the fact $z \in H_0(x,y)$ implies

$$d(y,w) = d(y,z) - d(z,w) < d(x,w).$$

Hence $w \in E_0^*(x,y)$ holds. Since we can take w as close to z as possible, z is an accumulation point of $E_0^*(x,y)$. By the same argument, z is also an accumulation point of $E_0(x,y)$. These arguments imply $H_0(x,y) \subset \overline{E_0(x,y)} \cap \overline{E_0^*(x,y)}$. The converse inclusion obviously holds. $\square$

For $x,y \in M$ and $t > 0$, let us define $E_t(x,y)$, $E_t^*(x,y)$ and $H_t(x,y)$ as follows:

$$\begin{aligned} E_t(x,y) &:= \{z \in M \mid p_t(x,z) > p_t(y,z)\}, \\ E_t^*(x,y) &:= \{z \in M \mid p_t(x,z) < p_t(y,z)\}, \\ H_t(x,y) &:= \{z \in M \mid p_t(x,z) = p_t(y,z)\}. \end{aligned}$$

For $x,y \in M$ and $t \ge 0$, let us define $F_t(x,y)$ and $F_t^*(x,y)$ as follows:

$$\begin{aligned} F_t(x,y) &:= \liminf_{n\to\infty} E_{t+t_n}(x,y), \\ F_t^*(x,y) &:= \liminf_{n\to\infty} E_{t+t_n}^*(x,y). \end{aligned}$$

Recall that $\{t_n\}_{n\in\mathbb{N}}$ is given in Assumption 1. For simplicity, we denote $E_t(x_1,x_2)$, $F_t(x_1,x_2)$, etc. by E_t, F_t, etc. respectively. Note that the continuity of $p_t(x,y)$ implies $E_t(x,y) \subset F_t(x,y)$ and $E_t^*(x,y) \subset F_t^*(x,y)$.

Proposition 3.9. *$\overline{E_0(x,y)} = \overline{F_t}$ and $\overline{E_0^*(x,y)} = \overline{F_t^*}$ hold for $\mu_{0,t}$-a.e.(x,y).*

Proof. It suffices to show the former equality because the latter is shown in the same manner. First we consider the case $t = 0$. By Assumption 1, we have

$$E_0(x,y) \subset F_0(x,y) \subset E_0(x,y) \cup H_0(x,y) = \overline{E_0(x,y)} \tag{3.4}$$

for any $x,y \in M$. Here the last equality follows from Lemma 3.8. It implies $\overline{E_0} = \overline{F_0}$. For $t,s > 0$, Lemma 3.3 (i) and the definition of φ_s yield

$$\begin{aligned} &\int_{M\times M} \left(\int_M (p_s(x,z) - p_s(y,z))\, 1_{E_s(x,y)\cup H_s(x,y)}(z)\mu(dz) \right) \mu_t(dxdy) \\ &= \int_{M\times M} \varphi_s(x,y)\mu_t(dxdy) \\ &= \varphi_{s+t}(x_1,x_2) \\ &= \int_M (p_{s+t}(x_1,z) - p_{s+t}(x_2,z))\, 1_{E_{s+t}}(z)\mu(dz) \\ &= \int_{M\times M} \left(\int_M (p_s(x,z) - p_s(y,z))\, 1_{E_{s+t}}(z)\mu(dz) \right) \mu_t(dxdy). \end{aligned} \tag{3.5}$$

Since $E_s(x,y) \cup H_s(x,y)$ is the positive part of a Hahn decomposition of $(p_s(x,\cdot) - p_s(y,\cdot))d\mu$,

$$\mu\left(E_{s+t} \setminus \left(E_s(x,y) \cup H_s(x,y)\right)\right) = 0 \tag{3.6}$$

holds for $\mu_{0,t}$-a.e.(x,y). Note that $H_s(x,y)$ in (3.6) cannot be omitted because $\mu(E_{s+t} \cap H_s(x,y))$ may be positive. By a similar argument, we also obtain

$$\mu\left(E_s(x,y) \setminus E_{s+t}\right) = 0 \tag{3.7}$$

for $\mu_{0,t}$-a.e.(x,y). First we observe what follows from (3.6). Because $E_{s+t} \setminus (E_s(x,y) \cup H_s(x,y))$ is open and μ has a positive measure on every metric ball of positive radius, (3.6) implies $E_{s+t} \setminus (E_s(x,y) \cup H_s(x,y)) = \emptyset$ and hence $E_{s+t} \subset E_s(x,y) \cup H_s(x,y)$. It implies

$$F_t \subset \liminf_{n\to\infty}(E_{t_n}(x,y) \cup H_{t_n}(x,y)) \tag{3.8}$$

for $\mu_{0,t}$-a.e.(x,y). By Assumption 1,

$$\liminf_{n\to\infty}(E_{t_n}(x,y) \cup H_{t_n}(x,y)) \subset E_0(x,y) \cup H_0(x,y) = \overline{E_0(x,y)} \tag{3.9}$$

holds. Combining (3.8) with (3.9), we obtain

$$F_t \subset \overline{E_0(x,y)} \tag{3.10}$$

for $\mu_{0,t}$-a.e.(x,y). Next we observe what follows from (3.7). The first inclusion in (3.4) implies

$$\begin{aligned} E_0(x,y) \setminus F_t &\subset \bigcup_{n\in\mathbb{N}} \left(\bigcap_{m\geq n} E_{t_m}(x,y) \cap \bigcap_{k\in\mathbb{N}} \left(\bigcup_{l\geq k} E^c_{t+t_l} \right) \right) \\ &\subset \bigcup_{n\in\mathbb{N}} \bigcup_{l\geq n} \left(\bigcap_{m\geq n} E_{t_m}(x,y) \cap E^c_{t+t_l} \right) \\ &\subset \bigcup_{n\in\mathbb{N}} \bigcup_{l\geq n} \left(E_{t_l}(x,y) \cap E^c_{t+t_l} \right). \end{aligned}$$

Here the first inclusion follows from $\bigcap_{k\in\mathbb{N}} \left(\bigcup_{l\geq k} E^c_{t+t_l} \right) \subset \bigcup_{l\geq n} E^c_{t+t_l}$ and the second follows from $\bigcap_{m\geq n} E_{t_m}(x,y) \subset E_{t_l}(x,y)$ for $l \geq n$. Thus (3.7) yields

$$\mu\left(E_0(x,y) \setminus F_t\right) \leq \sum_{n\in\mathbb{N}} \sum_{l\geq n} \mu(E_{t_l}(x,y) \setminus E_{t+t_l}) = 0 \tag{3.11}$$

for $\mu_{0,t}$-a.e.(x,y). Since $E_0(x,y)$ is open, (3.11) implies

$$E_0(x,y) \subset \overline{F_t} \tag{3.12}$$

for $\mu_{0,t}$-a.e.(x,y). Hence $\overline{E_0(x,y)} = \overline{F_t}$ follows from (3.10) and (3.12). □

The following corollary will be used in the next section.

Corollary 3.10. *For each $t,u > 0$, $\overline{E_0(\mathbf{X}_{t+u})} = \overline{F_t(\mathbf{X}_u)}$ and $\overline{E^*_0(\mathbf{X}_{t+u})} = \overline{F^*_t(\mathbf{X}_u)}$ holds $\mathbb{P}$-a.s. on $\{T > t+u\}$.*

We can prove Corollary 3.10 by a similar argument as in the proof of Proposition 3.9 based on Lemma 3.3 (ii) instead of Lemma 3.3 (i).

By Proposition 3.9, there exists $\Omega_0 \in \mathscr{F}$ with $\mathbb{P}[\Omega_0] = 1$ such that, for each $\omega \in \Omega_0$ and $t \in [0,\infty) \cap \mathbb{Q}$,

$$\overline{E_0(\mathbf{X}_t(\omega))} = \overline{F_t}, \qquad \overline{E_0^*(\mathbf{X}_t(\omega))} = \overline{F_t^*} \tag{3.13}$$

hold if $T(\mathbf{X}(\omega)) > t$. Let us define $S_t \subset M$ and $\mathscr{A} \subset C([0,\infty) \to M \times M)$ by $S_t := \overline{F_t} \cap \overline{F_t^*}$ and

$$\mathscr{A} := \left\{ \gamma = \{(\gamma_t^{(1)}, \gamma_t^{(2)})\}_{t \geq 0} \;\middle|\; d(\gamma_t^{(1)}, z) = d(\gamma_t^{(2)}, z) \text{ for all } t \geq 0 \text{ and } z \in S_t \right\}.$$

Note that $S_0 = H_0$ holds.

Proposition 3.11. *$\Omega_0 \subset \{\mathbf{X} \in \mathscr{A}\}$. In particular, $\mathbb{P}[\mathbf{X} \in \mathscr{A}] = 1$.*

Proof. Take $\omega \in \Omega_0$. By Lemma 3.8, (3.13) yields $H_0(\mathbf{X}_t(\omega)) = S_t$ for $t \in [0,\infty) \cap \mathbb{Q}$. It implies $d(X_t^{(1)}(\omega), z) = d(X_t^{(2)}(\omega), z)$ for every $z \in S_t$ and $t \in [0,\infty) \cap \mathbb{Q}$. Take $t \in [0,\infty) \setminus \mathbb{Q}$ and $z \in S_t$ arbitrary. We claim

$$d(X_t^{(1)}(\omega), z) = d(X_t^{(2)}(\omega), z). \tag{3.14}$$

It suffices to consider the case $T(\mathbf{X}(\omega)) > t$. By the definition of S_t, there exist sequences $\{z_n\}_{n\in\mathbb{N}} \subset F_t$ and $\{z_n^*\}_{n\in\mathbb{N}} \subset F_t^*$ such that $\lim_{n\to\infty} z_n = \lim_{n\to\infty} z_n^* = z$. By the definition of F_t and F_t^*, for each $n \in \mathbb{N}$, there exists a strictly increasing sequence $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $z_n \in E_{t+t_{k_n}}$ and $z_n^* \in E_{t+t_{k_n}}^*$ holds. Since the transition density is continuous in time, there exists $s_n < t_{k_n}$ satisfying $t + s_n \in \mathbb{Q}$ and $(p_{t+s_n}(x_1, z_n) - p_{t+s_n}(x_2, z_n)) \wedge (p_{t+s_n}(x_2, z_n^*) - p_{t+s_n}(x_1, z_n^*)) > 0$. It implies $z_n \in E_{t+s_n}$ and $z_n^* \in E_{t+s_n}^*$. Since $t + s_n \in \mathbb{Q}$ and $\omega \in \Omega_0$, (3.13) yields

$$E_{t+s_n} \subset F_{t+s_n} \subset \overline{E_0(\mathbf{X}_{t+s_n}(\omega))}, \qquad E_{t+s_n}^* \subset F_{t+s_n}^* \subset \overline{E_0^*(\mathbf{X}_{t+s_n}(\omega))}.$$

Thus any minimal geodesic joining z_n and z_n^* must intersect $H_0(\mathbf{X}_{t+s_n}(\omega))$. Take w_n from the intersection. Then we have $d(z_n, w_n) \vee d(z_n^*, w_n) \leq d(z_n, z_n^*)$ and hence

$$\lim_{n\to\infty} d(z, w_n) = \lim_{n\to\infty} d(z_n, w_n) = \lim_{n\to\infty} d(z_n^*, w_n) = 0.$$

By the continuity of the sample path $(X_\cdot^{(1)}(\omega), X_\cdot^{(2)}(\omega))$, we have

$$d(X_t^{(1)}(\omega), z) = \lim_{n\to\infty} d(X_{t+s_n}^{(1)}(\omega), w_n) = \lim_{n\to\infty} d(X_{t+s_n}^{(2)}(\omega), w_n) = d(X_t^{(2)}(\omega), z).$$

Therefore (3.14) follows. □

Lemma 3.12. *Let τ and τ' be defined by*

$$\tau := \inf\left\{ t > 0 \;\middle|\; \lim_{s\uparrow t} d(X_s^{(1)}, S_s) = \lim_{s\uparrow t} d(X_s^{(2)}, S_s) = 0 \right\},$$
$$\tau' := \inf\left\{ t > 0 \;\middle|\; X_t^{(1)} = X_t^{(2)} \right\}.$$

Then $\tau' \leq \tau \leq T$ a.s..

Proof. Take $\omega \in \Omega_0$. Then (3.13) implies $X_t^{(1)}(\omega) \in \overline{F}_t$ and $X_t^{(2)}(\omega) \in \overline{F_t^*}$ for $t \in [0, T(\omega)) \cap \mathbb{Q}$. In addition, $H_0(\mathbf{X}_t(\omega)) = S_t$ implies

$$d(\mathbf{X}_t(\omega)) = d(X_t^{(1)}(\omega), S_t) + d(X_t^{(2)}(\omega), S_t). \tag{3.15}$$

First let us take an increasing sequence $\{s_n\}_{n\in\mathbb{N}}$ in $[0,\infty) \cap \mathbb{Q}$ with $\lim_{n\to\infty} s_n = T(\omega)$. Then, by (3.15) for $t = s_n$ and the continuity of sample path,

$$0 = d(\mathbf{X}_{T(\omega)}) = \lim_{n\to\infty} d(\mathbf{X}_{s_n}) = \lim_{n\to\infty} \left(d(X_{s_n}^{(1)}, S_{s_n}) + d(X_{s_n}^{(2)}, S_{s_n}) \right)$$

holds. Therefore $\tau(\omega) \leq T(\omega)$ follows. Next let us take an increasing sequence $\{s_n'\}_{n\in\mathbb{N}}$ in $[0,\infty) \cap \mathbb{Q}$ with $\lim_{n\to\infty} s_n' = \tau(\omega)$. Since $s_n' \leq T(\omega)$, (3.15) for $t = s_n'$ implies

$$\lim_{n\to\infty} \left(d(X_{s_n'}^{(1)}, S_{s_n'}) + d(X_{s_n'}^{(2)}, S_{s_n'}) \right) = \lim_{n\to\infty} d(\mathbf{X}_{s_n'}) = 0.$$

Hence $\tau'(\omega) \leq \tau(\omega)$ and the conclusion follows. □

Remark 3.13. In Lemma 3.12, perhaps $\tau' \neq T$ occurs with a positive probability. But, if $(X^{(1)}, X^{(2)})$ is a strong Markov process on $M \times M$, then $\tau' = T$ holds $\mathbb{P}$-a.s.. Indeed, let us define $(\hat{X}^{(1)}, \hat{X}^{(2)})$ by $\hat{X}^{(1)} := X^{(1)}$ and

$$\hat{X}_t^{(2)} := \begin{cases} X_t^{(2)} & t < \tau', \\ X_t^{(1)} & t \geq \tau'. \end{cases}$$

Then $(\hat{X}^{(1)}, \hat{X}^{(2)})$ is actually a coupling of $(X, \mathbb{P}_{x_1})$ and $(X, \mathbb{P}_{x_2})$ because τ' is a Markov time. Since Lemma 3.12 asserts $\mathbb{P}[T > t] \geq \mathbb{P}\left[\tau' > t\right]$, the maximality of $(X^{(1)}, X^{(2)})$ yields $\mathbb{P}[T > t] = \mathbb{P}[\tau' > t]$. Hence $T = \tau'$ holds almost surely.

4 A weak characterization

At the beginning, we introduce the following additional condition:

Assumption 3. $p_t(x,y) \leq p_t(x,x)$ holds for any $t > 0$ and $x, y \in M$. Furthermore, the equality holds if and only if $x = y$.

Remark 4.1. The Brownian motion on a Riemannian homogeneous space satisfies Assumption 3. Indeed, for any isometry $g : M \to M$, $p_t(x,y) = p_t(gx, gy)$ holds for $x, y \in M$. Since the action of the isometry group is transitive, $p_t(x,x) = p_t(y,y)$ holds. Hence (3.1) and Lemma 3.1 yield Assumption 3. The above argument indicates that a hypoelliptic symmetric diffusion process on a homogeneous space generated by invariant vector fields also satisfies Assumption 3. A basic example is the diffusion process on a Heisenberg group associated with the sub-Laplacian.

Proposition 4.2. *Under Assumption 3, $S_{t+u} = S_u = H_0$ holds for any $t, u > 0$.*

For the proof, we show the following auxiliary lemma.

Lemma 4.3. *For any $s, q > 0$ and measurable $A \subset M \times M$,*

$$\mathbb{P}[\mathbf{X}_q \in A, T > s+q] = \int_A \varphi_s(x,y)\mu_{0,q}(dxdy). \tag{4.1}$$

In particular, $\operatorname{supp}[\mathbb{P}|_{\{T>s+q\}} \circ (X_q^{(1)})^{-1}] = \overline{E_q}$.

Proof. Note that we have

$$\{T(\mathbf{X}) > s+q\} = \{T(\theta_q \mathbf{X}) > s\} = \{T(\theta_q \mathbf{X}) > s\} \cap \{\mathbf{X}_q \in D^c\}.$$

Thus (3.3) and Corollary 3.6 yield

$$\begin{aligned}
\mathbb{P}[\mathbf{X}_q \in A, T > s+q] &= \mathbb{P}\left[\mathbf{X}_q \in A \setminus D, T(\theta_q \mathbf{X}) > s\right] \\
&= \mathbb{E}\left[1_{\{\mathbf{X}_q \in A \setminus D,\}} \mathbb{P}\left[T(\theta_q \mathbf{X}) > s \mid \mathbf{X}_{q'}, 0 \le q' \le q\right]\right] \\
&= \mathbb{E}\left[\varphi_s(\mathbf{X}_q)\, ;\, \mathbf{X}_q \in A \setminus D\right] \\
&= \int_A \varphi_s(x,y) d\mu_{0,q}(dxdy).
\end{aligned}$$

Thus (4.1) holds. Note that, by Lemma 3.1, $\varphi_s(x,y) > 0$ holds if and only if $(x,y) \notin D$. By virtue of Corollary 3.6, we can easily show that the support of the measure $E \mapsto \mu_{0,q}(E \times M)$ equals $\overline{E_q}$. Thus the conclusion follows. □

Proof of Proposition 4.2. We may assume $t > t_1$ without loss of generality. Take $q, s > 0$. Note that Lemma 4.3 implies $\mathbb{P}[\mathbf{X}_q \in D, T > s+q] = 0$. If $X_q^{(1)} \notin E_s(\mathbf{X}_q)$, then we have

$$p_s(X_q^{(1)}, X_q^{(1)}) \le p_s(X_q^{(2)}, X_q^{(1)}) \tag{4.2}$$

and hence $X_q^{(1)} = X_q^{(2)}$ holds by Assumption 3. The same argument also works for $X_q^{(2)}$ and $E_s^*(\mathbf{X}_q)$ instead of $X_q^{(1)}$ and $E_s(\mathbf{X}_q)$. Thus $X_q^{(1)} \in E_s(\mathbf{X}_q)$ and $X_q^{(2)} \in E_s^*(\mathbf{X}_q)$ hold on $\{T > s+q\}$ $\mathbb{P}$-a.s.. Therefore Proposition 3.9 and Corollary 3.10 yield

$$\begin{aligned}
X_q^{(1)} &\in \overline{F_s(\mathbf{X}_q)} = \overline{E_0(\mathbf{X}_{s+q})} = \overline{F_{s+q}}, \\
X_q^{(2)} &\in \overline{F_s^*(\mathbf{X}_q)} = \overline{E_0^*(\mathbf{X}_{s+q})} = \overline{F_{s+q}^*}
\end{aligned}$$

$\mathbb{P}$-a.s. on $\{T > s+q\}$. Thus Lemma 4.3 yields $E_q \subset \overline{F_{s+q}}$. By applying this inclusion in the case $(q,s) = (u+t_n, t-t_n)$,

$$F_u = \liminf_{n\to\infty} E_{u+t_n} \subset \liminf_{n\to\infty} \overline{F_{(t-t_n)+(u+t_n)}} = \overline{F_{t+u}}$$

and hence $\overline{F_u} \subset \overline{F_{t+u}}$. By the same argument, we obtain $\overline{F_u^*} \subset \overline{F_{t+u}^*}$. Thus $S_u \subset S_{t+u}$ holds.

In order to show $S_{t+u} = S_u$, suppose $S_{t+u} \setminus S_u \neq \emptyset$ and take $z \in S_{t+u} \setminus S_u$. Then either $z \notin \overline{F_u}$ or $z \notin \overline{F_u^*}$ holds. We only deal with the case $z \notin \overline{F_u}$ because the other one will be treated in the same way. Take $\delta > 0$ so small that $B_\delta(z) \cap \overline{F_u} = \emptyset$ holds. For $q > 0$, we can take $(x,y) \in E_q \times E_q^*$ so that it satisfies

$$\overline{E_0(x,y)} = \overline{F_q}, \qquad \overline{E_0^*(x,y)} = \overline{F_q^*}. \tag{4.3}$$

Note that such a pair (x, y) exists by Proposition 3.9 and Lemma 3.7. The expression (4.3) in the case $q = u$ yields $\overline{F_u}^c \subset \overline{F_u^*}$ and hence $B_\delta(z) \subset \overline{F_u^*}$ holds. It implies

$$B_\delta(z) \subset \overline{F_{t+u}^*} \tag{4.4}$$

since we have obtained $\overline{F_u^*} \subset \overline{F_{t+u}^*}$. Take $(x, y) \in E_{t+u} \times E_{t+u}^*$ so that it satisfies (4.3) in the case $q = t + u$. Then Lemma 3.8 yields $S_{t+u} = H_0(x, y)$. Since $z \in S_{t+u}$, there is a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $E_0(x, y)$ such that z_n converges to z. Then clearly $z_n \notin \overline{E_0^*(x, y)} = \overline{F_{t+u}^*}$ for any $n \in \mathbb{N}$, but it contradicts with (4.4). Thus we obtain $S_{t+u} = S_u$.

In what follows, we will prove $S_u = H_0$. By definition, $\overline{F_t} \subset E_t \cup H_t$ and $\overline{F_t^*} \subset E_t^* \cup H_t$ hold. Hence $S_t \subset H_t$ for each $t > 0$. Thus Assumption 1 implies

$$S_u = \bigcap_{n\in\mathbb{N}} S_{t_n} \subset \bigcap_{n\in\mathbb{N}} H_{t_n} \subset H_0.$$

We turn to the converse inclusion. Assumption 1 guarantees that $x_1 \in E_{t_n}$ and $x_2 \in E_{t_n}^*$ hold for sufficiently large n. Take such n and $(x, y) \in E_{t_n} \times E_{t_n}^*$ so that it satisfies (4.3) in the case $q = t_n$. Then, Lemma 3.8 yields

$$\begin{aligned} x_1 &\in E_{t_n} \subset \overline{F_{t_n}} = \overline{E_0(x, y)} = E_0(x, y) \cup H_0(x, y), \\ x_2 &\in E_{t_n}^* \subset \overline{F_{t_n}^*} = \overline{E_0^*(x, y)} = E_0^*(x, y) \cup H_0(x, y). \end{aligned}$$

Since $H_0(x, y) = S_{t_n} \subset H_0$, the fact $x_1, x_2 \notin H_0$ implies $x_1 \in E_0(x, y)$ and $x_2 \in E_0^*(x, y)$. Suppose $H_0 \setminus S_{t_n} \neq \emptyset$ and take $w \in H_0 \setminus S_{t_n}$. Take a minimal geodesic γ joining x_1 and w and γ' joining w and x_2. We define a path $\tilde{\gamma}$ by concatenating γ and γ' at w. Then, the discussion in the proof of Lemma 3.8 implies

$$\gamma \cap H_0 = \gamma' \cap H_0 = \{w\} = \tilde{\gamma} \cap H_0. \tag{4.5}$$

Here we identify each geodesic with the set of its trajectory. Since $H_0(x, y) = S_{t_n} \subset H_0$, we obtain $\tilde{\gamma} \cap H_0(x, y) = \emptyset$. It contradicts with the fact that the endpoints x_1 and x_2 of $\tilde{\gamma}$ belong to $E_0(x, y)$ and $E_0^*(x, y)$ respectively. Hence $H_0 = S_{t_n} = S_u$ follows. □

Remark 4.4. The mirror S_t may depend on time parameter t in general. To see it, we observe the following simple example. Take $x_1, x_2 \in \mathbb{R}^d$ with $x_1 \neq x_2$ and $v \in \mathbb{R}^d$. Set $H := \{z \in \mathbb{R}^d \mid |x_1 - z| = |x_2 - z|\}$ and $S_t := tv + H$. Let R_t be the mirror reflection with respect to S_t. Let us define two process $Y_t^{(1)}$ and $Y_t^{(2)}$ by $Y_t^{(1)} := x_1 + B_t + vt$ and

$$Y_t^{(2)} := \begin{cases} R_t Y_t^{(1)} & t < \tau, \\ Y_t^{(1)} & t \geq \tau, \end{cases}$$

where B_t is the standard Brownian motion on $\mathbb{R}^d$ and $\tau := \inf\{t > 0 \mid Y_t^{(1)} \in S_t\}$. We can easily verify that $(Y^{(1)}, Y^{(2)})$ is a maximal Markovian coupling of two Brownian motions with the drift v. Strictly speaking, this is not the case because the symmetry of p_t fails. The author does not know that such a example exists in the class of symmetric diffusions.

The following theorem provides a weak characterization of maximal Markovian couplings.

Theorem 4.5. *Under Assumption 3, there exists a continuous map $R : M \to M$ satisfying the following:*

(i) $R \circ R = \mathrm{id}$ *and* $Rx = x$ *if and only if* $x \in H_0$,

(ii) $\mathbf{X}_t = (X_t^{(1)}, X_t^{(2)})$ *is written as follows* $\mathbb{P}$*-almost surely:*

$$X_t^{(2)} = \begin{cases} RX_t^{(1)} & t < T, \\ X_t^{(1)} & t \geq T. \end{cases} \tag{4.6}$$

Before proving Theorem 4.5, we show the following auxiliary lemma.

Lemma 4.6. *If* $(x, y) \in M \times M \setminus D$ *satisfies*

$$d(x,z) = d(y,z) \text{ for every } z \in H_0, \tag{4.7}$$

then $(x, y) \in E_0 \times E_0^* \cup E_0^* \times E_0$ *holds. In particular, for* $x \in M$*, a point* $y \in M \setminus \{x\}$ *satisfying* (4.7) *is unique if it exists.*

Proof. Suppose that $y \in M \setminus \{x\}$ satisfies (4.7). If $x \in H_0$, then (4.7) obviously fails when $z = x$. Thus the cases $x \in H_0$ and $y \in H_0$ are excluded. Suppose $x, y \in E_0$. Let γ be a minimal geodesic joining x_2 and x. Take $z_0 \in \gamma \cap H_0$. Then we have

$$d(x, x_2) = d(x, z_0) + d(z_0, x_2) = \inf_{z \in H_0} \left(d(x,z) + d(z, x_2) \right).$$

By the same argument, $d(y, x_2) = \inf_{z \in H_0} \left(d(y,z) + d(z,x_2) \right)$ follows. Thus (4.7) implies $d(x, x_2) = d(y, x_2) = d(y, z_0) + d(z_0, x_2)$. Since $x \neq y$, we can take a minimal geodesic joining x_2 and y that branches from γ at z_0. It contradicts with our assumption. In the same way, we can exclude the case $x, y \in E_0^*$. Hence the former assertion follows.

Let us turn to the latter assertion. We consider the case that (4.7) holds for $(x, y) = (x', y')$ and $(x, y) = (x', y'')$ for $x' \in M$ and $y', y'' \in M \setminus \{x'\}$. Then the former assertion implies $(y', y'') \in E_0 \times E_0 \cup E_0^* \times E_0^*$. Since (4.7) holds for $(x, y) = (y', y'')$, we obtain $y' = y''$ by using the former assertion again. □

Proof of Theorem 4.5. Let us define a set $A \subset M$ as follows:

$$A := \left\{ x \in M \mid \text{ there exists } y \in M \setminus \{x\} \text{ such that (4.7) holds } \right\}.$$

For $x \in A$, we define $Rx := y$, where y is a point satisfying (4.7). Lemma 4.6 guarantees that R is well-defined. For $x \in H_0$, we define $Rx := x$. Set $\hat{A} = A \cup H_0$. First we show that $\hat{A}$ is closed and that R is continuous on $\hat{A}$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $\hat{A}$ that converges to $x \in M$. Take $z_0 \in H_0$. Since $d(z_0, x_n) = d(z_0, Rx_n)$ holds for any $n \in \mathbb{N}$, $\{d(z_0, Rx_n)\}_{n \in \mathbb{N}}$ is bounded. Thus the properness of M yields that $\{Rx_n\}_{n \in \mathbb{N}}$ has an accumulation point y. Note that $y \neq x$ holds if $x \in E_0 \cup E_0^*$. Indeed, if $x \in E_0$, then $x_n \in E_0$ for sufficiently large n and Lemma 4.6 implies $Rx_n \in E_0^*$ for such n. Choose a subsequence $\{Rx_{n_k}\}_{k \in \mathbb{N}}$ that converges to y. Then we have

$$d(y, z) = \lim_{k \to \infty} d(Rx_{n_k}, z) = \lim_{k \to \infty} d(x_{n_k}, z) = d(x, z)$$

holds for any $z \in H_0$. Thus $x \in \hat{A}$ and $y = Rx$. Since the choice of an accumulation point of $\{Rx_n\}_{n \in \mathbb{N}}$ is arbitrary, the above argument also implies the continuity of R on $\hat{A}$.

Next we show $\hat{A} = M$. Proposition 4.2 and Proposition 3.11 assert that, for $\omega \in \Omega_0$ and $t \in (0, T(\mathbf{X}))$, $X_t^{(1)}(\omega), X_t^{(2)}(\omega) \in \hat{A}$ and $RX_t^{(1)}(\omega) = X_t^{(2)}(\omega)$ hold. Take $x \in E_0$. Then Assumption 1 yields $x \in E_{t_n}$ for sufficiently large $n \in \mathbb{N}$. For such $n \in \mathbb{N}$, Corollary 3.6 implies that $\{X_{t_n}^{(1)}(\omega) \mid \omega \in \Omega_0,\ T(\mathbf{X}(\omega)) > t_n\}$ is dense in E_{t_n}. Since $\hat{A}$ is closed, $x \in \hat{A}$ follows. In the same way, we obtain $E_0^* \subset \hat{A}$ and hence $\hat{A} = M$. Now the conditions (i) and (ii) obviously follow and the proof is completed. $\square$

Note that Theorem 4.5 implies that $\mathbf{X}$ is a strong Markov process on $M \times M$. Thus Remark 3.13 together with Proposition 4.2 yields the following:

Corollary 4.7. *T equals the first hitting time τ_0 of $X_\cdot^{(1)}$ to H_0 $\mathbb{P}$-almost surely.*

5 Riemannian homogeneous spaces

In this section, we derive a stronger characterization of maximal Markovian couplings under the following assumptions:

Assumption 4. (2.1) holds locally uniformly in $x, y \in M$.

Assumption 5. If the conclusion of Theorem 4.5 holds, then $\mu \circ R^{-1} = \mu$.

Theorem 5.1. *Assume Assumption 3,4,5. Suppose that there is a maximal Markovian coupling $\mathbf{X}$ of $(X, \mathbb{P}_{x_1})$ and $(X, \mathbb{P}_{x_2})$. Then there exists a reflection structure R with respect to (x_1, x_2). Furthermore, $\mathbf{X}$ is the mirror coupling determined by R.*

Remark 5.2. We have mentioned a class of processes satisfying Assumption 4 in Remark 2.1. As we will show in Lemma 5.3, Theorem 4.5 together with Assumption 4 implies that R is isometry. Thus Assumption 5 is satisfied if μ is invariant under isometry. In particular, Assumption 4 and Assumption 5 hold under the assumption in Theorem 1.3. Hence Theorem 5.1 implies Theorem 1.3.

In the rest of this section, we use the notation in Theorem 4.5. To complete the proof of Theorem 5.1, it suffices to show that the reflection R makes the process invariant under Assumption 4 and Assumption 5.

Lemma 5.3. *Under Assumption 4, R is an isometry on M.*

Proof. Since $Rx = x$ holds for $x \in H_0$, $d(x, y) = d(Rx, Ry)$ trivially holds for $x, y \in H_0$. When $x \notin H_0$ and $y \in H_0$, $d(x, y) = d(Rx, y) = d(Rx, Ry)$ follows directly from the definition of R. For $x \in E_0$ and $y \in E_0^*$, we have

$$d(x, y) = \inf_{z \in H_0} \big(d(x, z) + d(z, y) \big) = \inf_{z \in H_0} \big(d(Rx, z) + d(z, Ry) \big) = d(Rx, Ry)$$

since every curve joining x and y must intersect H_0. Finally we consider the case $x, y \in E_0$. Take $s > 0$ so small that $x \in E_s$ and $Rx \in E_s^*$. Take $\delta > 0$ so small that

$$\overline{B_\delta(x)} \subset E_0 \cap E_s, \quad \overline{B_\delta(y)} \subset E_0, \quad \overline{B_\delta(Rx)} \subset E_0^* \cap E_s^*, \quad \overline{B_\delta(Ry)} \subset E_0^*$$

hold. Set $V_1 := B_\delta(x) \cap R(B_\delta(Rx))$ and $V_2 := B_\delta(y) \cap R(B_\delta(Ry))$. The fact $V_2 \subset E_0$ together with Theorem 4.5 implies

$$\{X_t^{(2)} \in V_2\} \subset \{X_t^{(1)} \in V_2\},$$
$$\{X_t^{(1)} \in V_2\} \setminus \{X_t^{(2)} \in V_2\} = \{X^{(1)} \in V_2, T > t\}.$$

Thus, the strong Markov property, Theorem 4.5, Corollary 4.7 and Corollary 3.6 yield

$$\begin{aligned}
&\mathbb{P}\left[X_{s+t}^{(1)} \in V_2, X_s^{(1)} \in V_1,\ T > s+t\right] \\
&= \mathbb{E}\left[\mathbb{P}_{X_s^{(1)}}\left[X_t^{(1)} \in V_2,\ T > t\right] 1_{\{X_s^{(1)} \in V_1,\ T > s\}}\right] \\
&= \mathbb{E}\left[\left(\mathbb{P}_{X_s^{(1)}}\left[X_t^{(1)} \in V_2\right] - \mathbb{P}_{X_s^{(2)}}\left[X_t^{(2)} \in V_2\right]\right) 1_{\{X_s^{(1)} \in V_1,\ T > s\}}\right] \\
&= \int_{V_1} \left(p_s(x_1, z) - p_s(x_2, z)\right) \left\{\int_{V_2} \left(p_t(z, w) - p_t(Rz, w)\right) \mu(dw)\right\} \mu(dz).
\end{aligned} \tag{5.1}$$

In the same way, we have

$$\begin{aligned}
&\mathbb{P}\left[X_{s+t}^{(2)} \in RV_2, X_s^{(2)} \in RV_1,\ T > s+t\right] \\
&= \int_{RV_1} \left(p_s(x_2, z) - p_s(x_1, z)\right) \left\{\int_{RV_2} \left(p_t(z, w) - p_t(Rz, w)\right) \mu(dw)\right\} \mu(dz).
\end{aligned} \tag{5.2}$$

Now we claim that, if $z, w \in E_0$ or $z, w \in E_0^*$,

$$d(z, w) < d(Rz, w). \tag{5.3}$$

Let γ be a minimal geodesic joining w and Rz and take $z_0 \in \gamma \cap H_0$. Since $d(z, z_0) = d(Rz, z_0)$, we have

$$d(z, w) \le d(z, z_0) + d(z_0, w) = d(Rz, w).$$

If the equality holds in the above inequality, then we can take a minimal geodesic joining w and z that branches from γ at z_0. Hence the claim follows.

By applying Assumption 4 to (5.1) and (5.2) together with (5.3),

$$\lim_{t \downarrow 0} 2t \log\left(\mathbb{P}\left[X_{s+t}^{(1)} \in V_2, X_s^{(1)} \in V_1,\ T > s+t\right]\right) = -\inf_{\substack{z \in V_1 \\ w \in V_2}} d(z, w)^2,$$
$$\lim_{t \downarrow 0} 2t \log\left(\mathbb{P}\left[X_{s+t}^{(2)} \in RV_2, X_s^{(2)} \in RV_1,\ T > s+t\right]\right) = -\inf_{\substack{z \in RV_1 \\ w \in RV_2}} d(z, w)^2$$

for sufficiently small $\delta > 0$. Since the left hand side of (5.1) equals that of (5.2) by Theorem 4.5 (ii), we obtain

$$\inf_{\substack{z \in V_1 \\ w \in V_2}} d(z, w) = \inf_{\substack{z \in V_1 \\ w \in V_2}} d(Rz, Rw).$$

Hence $d(x, y) = d(Rx, Ry)$ follows as δ tends to 0. □

Proposition 5.4. *Suppose that R is isometry and Assumption 5 holds. Then $p_t(x,y) = p_t(Rx,Ry)$ for $x,y \in M$.*

Remark 5.5. If we consider the Brownian motion on a Riemannian homogeneous space, the conclusion of Proposition 5.4 directly follows from Lemma 5.3. In this sense, Proposition 5.4 is not so essential since, at this moment, we have no example satisfying Assumption 3 without invariance of the transition density under isometries.

Proof. For $x,y \in H_0$, it is trivial. First we consider the case $x \in E_0$ and $y \in H_0$. By virtue of Proposition 3.9, Corollary 3.10 and Proposition 4.2, we obtain

$$H_0 = S_{t+u} = \overline{F_t(\mathbf{X}_u)} \cap \overline{F_t^*(\mathbf{X}_u)} \subset H_t(\mathbf{X}_u) \ \mathbb{P}\text{-a.s..}$$

It means

$$p_t(x,y) = p_t(Rx,y) \quad \text{for } y \in H_0 \tag{5.4}$$

for $x = X_u^{(1)}$ $\mathbb{P}$-a.s.. As we did in the last part of the proof of Theorem 4.5, we can extend (5.4) for any $x \in E_0$.

Next we consider the case $x = x_1$ and $y \in E_0 \cup E_0^*$. Take $\delta > 0$ so small that $(B_\delta(y) \cup B_\delta(Ry)) \cap H_0 = \emptyset$. Note that Lemma 5.3 yields $R(B_\delta(y)) = B_\delta(Ry)$. Thus, for $z \in H_0$, (5.4) and Assumption 5 imply

$$\int_{B_\delta(y)} p_t(z,w)\mu(dw) = \int_{R(B_\delta(y))} p_t(z,Rw)\mu(dw) = \int_{B_\delta(Ry)} p_t(z,w)\mu(dw). \tag{5.5}$$

When $y \in E_0^*$, the strong Markov property for $X^{(1)}$ and $X^{(2)}$ together with (5.5) implies

$$\begin{aligned}
\mathbb{P}\left[X_t^{(1)} \in B_\delta(y)\right] &= \mathbb{P}\left[X_t^{(1)} \in B_\delta(y),\ T < t\right] \\
&= \mathbb{E}\left[1_{\{T<t\}} \int_{B_\delta(y)} p_{t-T}(X_T^{(1)}, z)\mu(dz)\right] \\
&= \mathbb{E}\left[1_{\{T<t\}} \int_{B_\delta(Ry)} p_{t-T}(X_T^{(2)}, z)\mu(dz)\right] \\
&= \mathbb{P}\left[X_t^{(2)} \in B_\delta(Ry),\ T < t\right] \\
&= \mathbb{P}\left[X_t^{(2)} \in B_\delta(Ry)\right].
\end{aligned} \tag{5.6}$$

Dividing both side of (5.6) by $\mu(B_\delta(y))$ and letting $\delta \downarrow 0$, we obtain

$$p_t(x_1,y) = p_t(x_2,Ry). \tag{5.7}$$

Here we used Assumption 5. When $y \in E_0$, (5.6) and Theorem 4.5 (ii) imply

$$\begin{aligned}
\mathbb{P}\left[X_t^{(1)} \in B_\delta(y)\right] &= \mathbb{P}\left[X_t^{(1)} \in B_\delta(y),\ T < t\right] + \mathbb{P}\left[X_t^{(1)} \in B_\delta(y),\ T \geq t\right] \\
&= \mathbb{P}\left[X_t^{(2)} \in B_\delta(y),\ T < t\right] + \mathbb{P}\left[X_t^{(2)} \in R(B_\delta(y)),\ T \geq t\right] \\
&= \mathbb{P}\left[X_t^{(1)} \in B_\delta(Ry),\ T < t\right] + \mathbb{P}\left[X_t^{(2)} \in B_\delta(Ry),\ T \geq t\right] \\
&= \mathbb{P}\left[X_t^{(2)} \in B_\delta(Ry)\right].
\end{aligned}$$

Hence (5.7) also follows as we did after (5.6) had been obtained.

Finally we consider the case $x \in E_0$ and $y \in E_0 \cup E_0^*$. Take $s > 0$ so small that $x \in E_s$. Take $\delta > 0$ sufficiently small. Now we have

$$\begin{aligned}\mathbb{P}\left[X_s^{(1)} \in B_\delta(x), X_{s+t}^{(1)} \in B_\delta(y)\right] = \mathbb{P}&\left[X_s^{(1)} \in B_\delta(x), X_{s+t}^{(1)} \in B_\delta(y),\ T < s\right] \\ &+ \mathbb{P}\left[X_s^{(1)} \in B_\delta(x), X_{s+t}^{(1)} \in B_\delta(y), s \le T < s+t\right] \\ &+ \mathbb{P}\left[X_s^{(1)} \in B_\delta(x), X_{s+t}^{(1)} \in B_\delta(y), s+t \le T\right]. \end{aligned} \tag{5.8}$$

By Theorem 4.5 (ii) and Lemma 5.3,

$$\begin{aligned}&\mathbb{P}\left[X_s^{(1)} \in B_\delta(x), X_{s+t}^{(1)} \in B_\delta(y), s+t \le T\right] \\ &\qquad = \mathbb{P}\left[X_s^{(2)} \in B_\delta(Rx), X_{s+t}^{(2)} \in B_\delta(Ry), s+t \le T\right]. \end{aligned} \tag{5.9}$$

In a similar way as in (5.6),

$$\begin{aligned}&\mathbb{P}\left[X_s^{(1)} \in B_\delta(x), X_{s+t}^{(1)} \in B_\delta(y), s \le T < s+t\right] \\ &= \mathbb{E}\left[1_{\{X_s^{(1)} \in B_\delta(x)\}\cap\{s \le T < s+t\}} \int_{B_\delta(y)} p_{t+s-T}(X_T^{(1)}, z)\mu(dz)\right] \\ &= \mathbb{P}\left[X_s^{(1)} \in B_\delta(x), X_{s+t}^{(2)} \in B_\delta(Ry), s \le T < s+t\right] \\ &= \mathbb{P}\left[X_s^{(2)} \in B_\delta(Rx), X_{s+t}^{(2)} \in B_\delta(Ry), s \le T < s+t\right]. \end{aligned} \tag{5.10}$$

By replacing $X^{(1)}$, x and y with $X^{(2)}$, Rx and Ry in (5.8), we obtain a corresponding decomposition. Combining it and (5.8) with (5.9) and (5.10), we obtain

$$\begin{aligned}\mathbb{P}\left[X_s^{(1)} \in B_\delta(x), X_{s+t}^{(1)} \in B_\delta(y)\right] &- \mathbb{P}\left[X_s^{(1)} \in B_\delta(x), X_{s+t}^{(1)} \in B_\delta(y),\ T < s\right] \\ &= \mathbb{P}\left[X_s^{(2)} \in B_\delta(Rx), X_{s+t}^{(2)} \in B_\delta(Ry)\right] \\ &\quad - \mathbb{P}\left[X_s^{(2)} \in B_\delta(Rx), X_{s+t}^{(2)} \in B_\delta(Ry),\ T < s\right]. \end{aligned} \tag{5.11}$$

Here we have

$$\begin{aligned}&\mathbb{P}\left[X_s^{(1)} \in B_\delta(x), X_{s+t}^{(1)} \in B_\delta(y),\ T < s\right] \\ &\qquad = \mathbb{E}\left[1_{\{T<s\}} \int_{B_\delta(x)} p_{s-T}(X_T^{(1)}, w)\left(\int_{B_\delta(y)} p_t(w,z)\mu(dz)\right)\mu(dw)\right]. \end{aligned}$$

Thus, dividing both side of (5.11) by $\mu(B_\delta(x))\mu(B_\delta(y))$ and tending δ to 0, we obtain

$$\begin{aligned}&\left(p_s(x_1, x) - \mathbb{E}\left[1_{\{T<s\}} p_{s-T}(X_T^{(1)}, x)\right]\right) p_t(x,y) \\ &\qquad = \left(p_s(x_2, Rx) - \mathbb{E}\left[1_{\{T<s\}} p_{s-T}(X_T^{(2)}, Rx)\right]\right) p_t(Rx, Ry). \end{aligned} \tag{5.12}$$

Note that Corollary 3.6 implies

$$
\begin{aligned}
&p_s(x_1,x) - \mathbb{E}\left[1_{\{T<s\}} p_{s-T}(X_T^{(1)},x)\right] \\
&\qquad = \lim_{\delta\downarrow 0} \frac{1}{\mu(B_\delta(x))} \left(\int_{B_\delta(x)} p_s(x_1,z)\mu(dz) - \mathbb{E}\left[1_{\{T<s\}} \int_{B_\delta(x)} p_{s-T}(X_T^{(1)},z)\mu(dz) \right] \right) \\
&\qquad = \lim_{\delta\downarrow 0} \frac{1}{\mu(B_\delta(x))} \mathbb{P}\left[X_s^{(1)} \in B_\delta(x),\ T \geq s \right] \\
&\qquad = \lim_{\delta\downarrow 0} \frac{1}{\mu(B_\delta(x))} \int_{B_\delta(x)} \left(p_s(x_1,z) - p_s(x_2,z) \right) \mu(dz) \\
&\qquad = p_s(x_1,x) - p_s(x_2,x) > 0. \qquad (5.13)
\end{aligned}
$$

By the same argument, we have

$$
\begin{aligned}
p_s(x_2,x) - \mathbb{E}\left[1_{\{T<s\}} p_{s-T}(X_T^{(2)},x)\right] &= p_s(x_2,Rx) - p_s(x_1,Rx) \\
&= p_s(x_1,x) - p_s(x_2,x). \qquad (5.14)
\end{aligned}
$$

Here the last equality follows from (5.7). By substituting (5.13) and (5.14) into (5.12), the desired result follows. □

Proof of Theorem 5.1. It suffices to show that the map R defined in Theorem 4.5 carries a reflection structure with respect to (x_1, x_2). By the argument in the proof of Theorem 4.5, (ii) of Definition 1.2 follows with $H = H_0$, $M_1 = E_0$ and $M_2 = E_0^*$. Proposition 5.4 together with Assumption 5 implies that the finite dimensional distributions of $\mathbb{P}_{x_1} \circ (RX)^{-1}$ and $\mathbb{P}_{x_2} \circ X^{-1}$ are equal. It yields (i) of Definition 1.2. □

6 Examples: Riemannian symmetric spaces

In this section, we consider some examples of the Brownian motion on a Riemannian symmetric spaces. Since any Riemannian symmetric space is homogeneous, we can apply Theorem 1.3. Thus a maximal Markovian coupling exists if and only if there is a reflection structure. The following three examples indicate that the existence of a reflection structure imposes a strong restriction on the underlying space. d_M denotes the distance function on a metric space M.

Example 6.1. ($\mathbb{S}^d, \mathbb{R}^d, \mathbb{H}^d$) First we review the cases that M is simply connected and has a constant curvature, That is, M is either a sphere $\mathbb{S}^d$, a Euclidean space $\mathbb{R}^d$ or a hyperbolic space $\mathbb{H}^d$ corresponding to the signature of the curvature. As studied in Example 4.6 in [13], there is a reflection structure with respect to (x_1, x_2) for any $(x_1, x_2) \in M \times M \setminus D$.

Remark 6.2. We give a basic observation used in the following examples. Suppose that there exists a reflection structure on a Riemannian homogeneous space M. Then the argument in the proof of Theorem 4.5 implies that $H_0 = \{z \in M \mid d_M(x_1,z) = d_M(x_2,z)\}$ equals the set of the fixed points of the induced map R. Since R is an isometry by Lemma 5.3, each connected component of H_0 must be a totally geodesic submanifold of M (see [11] p.61, for example).

Example 6.3. (Non-constant curvatures) Assume that M is an irreducible global symmetric space. Note that an involutive isometry whose fixed points form a submanifold of codimension 1 exists if and only if M is of constant curvature (see [9]). Now suppose that there exists a reflection structure on M. The induced map R is an involutive isometry. Moreover, the fixed points H_0 must be of codimension 1 since H_0 separates M into two disjoint open sets. Thus, if M has a non-constant curvature, then there is no reflection structure with respect to (x_1, x_2) for any $x_1, x_2 \in M$.

Example 6.4. (Real projective spaces) Under the canonical metric, the real projective space $\mathbb{RP}^d$ becomes an irreducible global symmetric space of positive constant curvature. We claim that there is no reflection structure on $\mathbb{RP}^d$ if $d \geq 2$. As we have seen in Example 6.3, having a constant curvature is necessary for the existence of a reflection structure. This case implies that it is *not* sufficient. Actually, there exists a reflection map in the sense of [9] but it does not divide $\mathbb{RP}^d$ into two components.

Now we turn to show the claim. Take $(x_1, x_2) \in M \times M \setminus D$ arbitrary. By taking an appropriate chart, we may assume

$$x_1 = [y_1 : y_2 : 0 : \cdots : 0],$$
$$x_2 = [-y_1 : y_2 : 0 : \cdots : 0]$$

for some $y_1, y_2 \in \mathbb{R} \setminus \{0\}$ without loss of generality. For simplicity, we assume $y_1^2 + y_2^2 = 1$. For $(z_1, \ldots, z_{d+1}) \in \mathbb{R}^{d+1}$ with $\sum_{i=1}^{d+1} z_i^2 = 1$,

$$\cos d_{\mathbb{RP}^d}(x_1, [z_1 : \cdots : z_{d+1}]) = \left(y_1 z_1 + y_2 z_2\right) \vee \left\{-(y_1 z_1 + y_2 z_2)\right\} = |y_1 z_1 + y_2 z_2|.$$

In the same way, we obtain

$$\cos d_{\mathbb{RP}^d}(x_2, [z_1 : \cdots : z_{d+1}]) = |y_1 z_1 - y_2 z_2|.$$

These observations yield

$$H_0 = \left\{ [z_1 : \cdots : z_{d+1}] \;\middle|\; z_1 = 0 \text{ or } z_2 = 0 \right\}.$$

Note that H_0 is not a manifold since it has a singularity at $[0 : 0 : z_3 : \cdots : z_{d+1}]$. Thus there is no reflection structure by Remark 6.2.

Example 6.5. (Tori) Let us consider the d-dimensional torus $\mathbb{T}^d$ for $d \geq 2$. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We endow $\mathbb{T}$ with a flat metric induced from $\mathbb{R}$ and $\mathbb{T}^d$ the product metric. Take $(x_1, x_2) \in \mathbb{T}^d \times \mathbb{T}^d \setminus D$. Let us denote them by $x_1 = (x_{11}, \ldots, x_{1d}), x_2 = (x_{21}, \ldots, x_{2d})$ for $x_{ij} \in \mathbb{T}$. We claim that there exists a reflection structure if and only if there is $k \in \{1, \ldots, d\}$ such that $x_{1j} = x_{2j}$ for any $j \neq k$.

First we show the "if" part. For simplicity, we assume $k = 1$. Then we can easily verify that a map R defined by

$$R(y_1, \ldots, y_d) = (x_1 + x_2 - y_1, y_2, \ldots, y_d)$$

carries a reflection structure with respect to (x_1, x_2). Next we show the "only if" part. It suffices to show that, for each $j_1, j_2 \in \{1, \ldots, d\}$ with $j_1 \neq j_2$, $x_{1j_1} = x_{2j_1}$ or $x_{1j_2} = x_{2j_2}$ must hold. By symmetry, we may assume $(j_1, j_2) = (1, 2)$ without loss of generality. For a subset $M_0 \subset M$, we endow M_0 with the geodesic metric inherited from M. It means that, for $x, y \in M_0$, $d_{M_0}(x, y)$ is the infimum of the length of all rectifiable curve joining x and y in M_0. For $k = 3, \ldots, d$, take $z_k \in \mathbb{T}$ so that

$d_{\mathbb{T}}(x_{1k}, z_k) = d_{\mathbb{T}}(x_{2k}, z_k)$ holds. Set $\hat{H} := \left\{(y_1, \ldots, y_d) \in \mathbb{T}^d \;\middle|\; y_k = z_k \text{ for } k = 3, \ldots, d\right\}$. Note that $\hat{H}$ is isometric to $\mathbb{T}^2$. By the assumption, we have an isometry R the set of whose fixed points equals H_0. We set $\tilde{H} = H_0 \cap \hat{H}$. Take $w_1, w_2 \in \tilde{H}$ and suppose that w_1 and w_2 are connected by a path $\tilde{\gamma}$ in $\tilde{H}$. By replacing w_2 with another point on $\tilde{\gamma}$ if necessary, we may assume that $d_{H_0}(w_1, w_2) < 1/2$. Since w_1 and w_2 is connected in H_0 and each connected component of H_0 is totally geodesic in M by Remark 6.2, there exists a minimal geodesic γ in H_0 joining w_1 and w_2. Since γ is a geodesic also in $\mathbb{T}^d$, γ is locally a line segment. We identify the tangent space $T_{\gamma_0}\hat{H}$ with a corresponding subspace of $T_{\gamma_0}\mathbb{T}^d$. If $\dot{\gamma}_0 \notin T_{\gamma_0}\hat{H}$, the length of γ becomes greater than 1. Thus $\dot{\gamma}_0 \in T_{\gamma_0}\hat{H}$ must hold by the minimality of γ. Since γ is locally a line segment, $\gamma \subset \hat{H}$ holds. These observations yield $\gamma \subset \tilde{H}$ and $d_{H_0}(w_1, w_2) = d_{\tilde{H}}(w_1, w_2)$.

Now we reduce the problem to the case $d = 2$. Let $x_1^{(2)} := (x_{11}, x_{12})$ and $x_2^{(2)} := (x_{21}, x_{22})$ be elements in $\mathbb{T}^2$. Let us define $H^{(2)}$ by

$$H^{(2)} = \left\{ y \in \mathbb{T}^2 \;\middle|\; d_{\mathbb{T}^2}(x_1^{(2)}, y) = d_{\mathbb{T}^2}(x_2^{(2)}, y) \right\}.$$

Take $w_1, w_2 \in H^{(2)}$ with $d_{H^{(2)}}(w_1, w_2) < 1/2$. Note that $H^{(2)}$ is isometric to $\tilde{H}$. Thus the minimal geodesic γ in $\mathbb{T}^2$ joining w_1 and w_2 is contained in $H^{(2)}$. By the observation in Example 4.8 in [13], such an assertion holds true if and only if either $x_{11} = x_{21}$ or $x_{21} = x_{22}$ holds. In fact, if neither of them holds, then $H^{(2)}$ has a singular point. Hence the conclusion follows.

Combining these examples with Theorem 1.3, Example 6.3-6.5 is summarized as follows:

Theorem 6.6. *Let M be an irreducible global symmetric space with* $\dim M \geq 2$.

(i) *If M has a non-constant curvature, then no maximal Markovian coupling of the Brownian motion exists on M*

(ii) *Suppose $M = \mathbb{RP}^d$. Then no maximal Markovian coupling of the Brownian motion exists on M.*

(iii) *Suppose $M = \mathbb{T}^d$. Then a maximal Markovian coupling starting from distinct points $(x_{11}, \ldots, x_{1d})$ and $(x_{21}, \ldots, x_{2d})$ exists if and only if there exists $k \in \{1, \ldots, d\}$ such that $x_{1j} = x_{2j}$ for any $j \neq k$.*

7 A case for Markov chains

The goal of this section is to show the following:

Theorem 7.1. *There exists a discrete time Markov chain on a finite state space where maximal Markovian coupling does not exist with respect to specified starting points.*

Remark 7.2. In the class of continuous time Markov chains on a finite state space, an example discussed in [3] (Example 2.12) admits no maximal coupling which is a Markov process on the product space for any pair of distinct starting points. In [3], they showed that any coupling $\mathbf{X}$ of the Markov chain which is a Markov process on the product space satisfies

$$\lim_{t \to \infty} -\frac{1}{t} \log \mathbb{P}[T(\mathbf{X}) > t] < \lambda,$$

where λ is the first nonzero eigenvalue of the Markov chain. As observed in Remark 2.4, any maximal coupling satisfies (2.3). Thus no maximal coupling can be a Markov process on the product space.

For the proof of Theorem 7.1, we construct an approximating sequence of couplings $\mathbf{W}^{(m)}$ of Markov chains that converges in law to a coupling of two Brownian motions on $\mathbb{T}^d$. Let $\{Z_{n,i}\}_{n\in\mathbb{N}, i\in\{1,\ldots,d\}}$ be $\mathbb{R}$-valued, independent and identically distributed random variables defined by

$$\mathbb{P}[Z_{1,1}=1]=\mathbb{P}[Z_{1,1}=-1]=\frac{1}{4}, \quad \mathbb{P}[Z_{1,1}=0]=\frac{1}{2}.$$

Then $Z_n=(Z_{n,1},\ldots,Z_{n,d})$ $(n=1,2,\ldots)$ are $\mathbb{R}^d$-valued, independent and identically distributed random variables. Let $\pi:\mathbb{R}^d\to\mathbb{T}^d$ be the canonical projection. For $x\in m^{-1}\mathbb{Z}^d$, let us define $\tilde{Y}_n^{(m)}(x)$ and $Y_n^{(m)}(x)$ by

$$\tilde{Y}_n^{(m)}(x):=x+\frac{1}{m}\left(Z_1+\cdots+Z_n\right)$$

and $Y_n^{(m)}(x):=\pi(\tilde{Y}_n^{(m)}(x))$. Then $\{Y_n^{(m)}(x)\}_{n=0}^{\infty}$ is an irreducible Markov chain on $\pi(m^{-1}\mathbb{Z})$. Let us denote the n-step transition probability of $\tilde{Y}_\cdot^{(m)}$ from x to y by $\tilde{p}_n^{(m)}(x,y)$. In the same manner, $p_n^{(m)}(x,y)$ denotes the transition probability of $Y_\cdot^{(m)}$. We show the following auxiliary lemma which asserts the local central limit theorem on $\mathbb{T}$.

Lemma 7.3. *Assume that m' satisfies $m'/2m^2=\sigma+O(m^{-2})$ as $m\to\infty$ for some $\sigma>0$. Then*

$$\lim_{m\to\infty}\left(\sqrt{2\pi\sigma}m\,\mathbb{P}\left[\sum_{l=1}^{m'}Z_{l,1}\in y+m\mathbb{Z}\right]-\sum_{k\in\mathbb{Z}}\exp\left(-\frac{1}{2\sigma}\left(\frac{y}{m}+k\right)^2\right)\right)=0$$

holds uniformly in $y\in\mathbb{Z}$.

Lemma 7.3 seems to follow easily from the local central limit theorem for $\tilde{p}_n(x,y)$, but we need to estimate that fluctuations are so small as to be negligible. Our proof is based on the arguments in Chapter 2 of [20]. Though such an extension may be well-known, we will give a proof for completeness.

Proof. We set $\varphi(\xi):=\mathbb{E}[\mathrm{e}^{i\xi Z_{1,1}}]=1-(1-\cos\xi)/2$. Then orthogonality of trigonometric functions yields

$$\mathbb{P}\left[\sum_{l=1}^{m'}Z_{l,1}=y\right]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(\xi)^{m'}\mathrm{e}^{-iy\xi}d\xi. \tag{7.1}$$

Take $y_m\in\{0,1,\ldots,m-1\}$ so that $y-y_m\in m\mathbb{Z}$ holds. Take $N\in\mathbb{N}$ satisfying $N>2\sigma$. Then, for sufficiently large m, (7.1) yields

$$\begin{aligned}\sqrt{2\pi\sigma}m\,\mathbb{P}\left[\sum_{l=1}^{m'}Z_{l,1}\in y+m\mathbb{Z}\right]&=\sqrt{2\pi\sigma}m\sum_{k=-Nm}^{Nm}\mathbb{P}\left[\sum_{l=1}^{m'}Z_{l,1}\in y_m+mk\right]\\&=\sqrt{\frac{\sigma}{2\pi}}\sum_{k=-Nm}^{Nm}\int_{-m\pi}^{m\pi}\varphi\left(\frac{\theta}{m}\right)^{m'}\mathrm{e}^{-i(y_m/m+k)\theta}d\theta.\end{aligned} \tag{7.2}$$

Here the first equality follows from the fact $|\sum_{l=1}^{m'} Z_{l,1}| \le m'$. We decompose the right hand side of (7.2) as follows:

$$\sqrt{\frac{\sigma}{2\pi}} \sum_{k=-Nm}^{Nm} \int_{-m\pi}^{m\pi} \varphi\left(\frac{\theta}{m}\right)^{m'} \mathrm{e}^{-i(y_m/m+k)\theta} d\theta$$
$$= \sqrt{\frac{\sigma}{2\pi}} \left(\sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{e}^{-\sigma\theta^2/2} \mathrm{e}^{-i(y_m/m+k)\theta} d\theta + I_1 + I_2 + I_3 + I_4 \right),$$

where

$$I_1 := \sum_{k=-Nm}^{Nm} \int_{\{m^{1/3} \le |\theta| \le m\pi\}} \varphi\left(\frac{\theta}{m}\right)^{m'} \mathrm{e}^{-i(y_m/m+k)\theta} d\theta,$$
$$I_2 := \sum_{k=-Nm}^{Nm} \int_{-m^{1/3}}^{m^{1/3}} \left(\varphi\left(\frac{\theta}{m}\right)^{m'} - \mathrm{e}^{-\sigma\theta^2/2} \right) \mathrm{e}^{-i(y_m/m+k)\theta} d\theta,$$
$$I_3 := -\sum_{k=-Nm}^{Nm} \int_{\{|\theta| \ge m^{1/3}\}} \mathrm{e}^{-\sigma\theta^2/2} \mathrm{e}^{-i(y_m/m+k)\theta} d\theta,$$
$$I_4 := -\sum_{\substack{k\in\mathbb{Z} \\ |k|>Nm}} \int_{-\infty}^{\infty} \mathrm{e}^{-\sigma\theta^2/2} \mathrm{e}^{-i(y_m/m+k)\theta} d\theta.$$

Since we have

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\sigma\theta^2/2} \mathrm{e}^{-i(y_m/m+k)\theta} d\theta = \sqrt{\frac{2\pi}{\sigma}} \exp\left(-\frac{1}{2\sigma} \left(\frac{y_m}{m} + k \right)^2 \right), \tag{7.3}$$

the conclusion follows once we show $\lim_{m\to\infty} I_j = 0$ uniformly in y for $j = 1, 2, 3, 4$. First, (7.3) yields

$$|I_4| \le 2\sqrt{\frac{2\pi}{\sigma}} \sum_{\substack{k\in\mathbb{N} \\ k>Nm}} \exp\left(-\frac{1}{2\sigma}(k-1)^2 \right)$$

and hence $\lim_{m\to\infty} I_4 = 0$. Second, we have

$$|I_3| \le (2Nm+1)\mathrm{e}^{-\sigma m^{2/3}/4} \int_{-\infty}^{\infty} \mathrm{e}^{-\sigma\theta^2/4} d\theta$$

and hence $\lim_{m\to\infty} I_3 = 0$. Next we deal with I_1. By elementary inequalities $1 + x \le e^x$ for $x \in \mathbb{R}$

and $1-\cos x \ge x^2/4$ when $|x|$ is small, we have, for $m^{1/3} \le |\theta| \le m\pi$,

$$\begin{aligned}
0 \le \varphi\left(\frac{\theta}{m}\right)^{m'} &= \left(1-\frac{1}{2}\left(1-\cos\left(\frac{\theta}{m}\right)\right)\right)^{m'} \\
&\le \exp\left(-\frac{m'}{2}\left(1-\cos\left(\frac{\theta}{m}\right)\right)\right) \\
&\le \exp\left(-\frac{m'}{2}\left(1-\cos\left(m^{-2/3}\right)\right)\right) \\
&\le \exp\left(-\frac{m'}{8m^{4/3}}\right).
\end{aligned}$$

It yields

$$|I_1| \le 2m\pi(2Nm+1)\exp\left(-\frac{m'}{8m^{4/3}}\right).$$

Since $m' \approx 2\sigma m^2$ for large m, the right hand side of the above inequality converge to 0 as $m \to \infty$. Finally we give an estimate to I_2. To achieve it, we give an upper and lower estimate of $\varphi(\theta/m)^{m'}$. Take m sufficiently large and $|\theta| \le m^{1/3}$. The elementary inequality $1-\cos x \ge x^2/2 - x^4/24$ for $x \in \mathbb{R}$ yields

$$\varphi\left(\frac{\theta}{m}\right)^{m'} \le \exp\left(-\frac{m'}{2}\left(\frac{\theta^2}{2m^2}-\frac{\theta^4}{24m^4}\right)\right) = \exp\left(-\frac{m'\theta^2}{4m^2}+\frac{m'\theta^4}{48m^4}\right). \tag{7.4}$$

On the other hand, elementary inequalities $1-\cos x \le x^2/2$ for $x \in \mathbb{R}$ and $\log(1-x) \ge -x-x^2$ when $|x|$ is small yield

$$\begin{aligned}
\varphi\left(\frac{\theta}{m}\right)^{m'} &= \exp\left(m'\log\left(1-\frac{1}{2}\left(1-\cos\left(\frac{\theta}{m}\right)\right)\right)\right) \\
&\ge \exp\left(m'\log\left(1-\frac{\theta^2}{4m^2}\right)\right) \\
&\ge \exp\left(m'\left(-\frac{\theta^2}{4m^2}-\frac{\theta^4}{16m^4}\right)\right) \\
&= \exp\left(-\frac{m'\theta^2}{4m^2}-\frac{m'\theta^4}{16m^4}\right).
\end{aligned} \tag{7.5}$$

Note that, by the assumption on m', $\theta^2(\sigma - m'/m^2) \approx 0$ and $m'\theta^4/m^4 \approx 0$ holds. Since $|\mathrm{e}^x - 1| \le 2|x|$ for $x \approx 0$, (7.4) and (7.5) yield

$$\left|\varphi\left(\frac{\theta}{m}\right)^{m'} - \mathrm{e}^{-\sigma\theta^2/2}\right| \le \left(\theta^2\left|\sigma-\frac{m'}{2m^2}\right| + \frac{m'\theta^4}{8m^4}\right)\mathrm{e}^{-\sigma\theta^2/2}.$$

Therefore the above inequality implies

$$|I_2| \le (2Nm+1)\left(\left|\sigma-\frac{m'}{2m^2}\right|\int_{-\infty}^{\infty}\theta^2\mathrm{e}^{-\sigma\theta^2/2}d\theta + \frac{m'}{8m^4}\int_{-\infty}^{\infty}\theta^4\mathrm{e}^{-\sigma\theta^2/2}d\theta\right).$$

By the assumption on m', the right hand side of the above inequality converges to 0 as $m \to \infty$. Since uniformity in y obviously holds, the proof is completed. □

For $i = 1, 2$, take $x_i \in \mathbb{T}^d$ and $\tilde{x}_i \in \pi^{-1}(x_i)$. Take $\tilde{x}_i^{(m)} \in m^{-1}\mathbb{Z}$ for each $m \in \mathbb{N}$ so that they satisfy $\lim_{m\to\infty} \tilde{x}_i^{(m)} = \tilde{x}_i$ for $i = 1, 2$. By [8], there exists a maximal coupling $\mathbf{Y}^{(m)}$ of $Y^{(m)}(\tilde{x}_1^{(m)})$ and $Y^{(m)}(\tilde{x}_2^{(m)})$. It means

$$\mathbb{P}[T(\mathbf{Y}^{(m)}) > n] = \frac{1}{2} \sum_{z\in\pi(m^{-1}\mathbb{Z}^d)} \left| p_n^{(m)}(x_1^{(m)}, z) - p_n^{(m)}(x_2^{(m)}, z) \right| \tag{7.6}$$

for every $n \in \mathbb{N}$. For $x \in \mathbb{R}$, we set $\lfloor x \rfloor := \sup\{k \in \mathbb{Z} \mid x - k \geq 0\}$. Set

$$\mathbf{W}_t^{(m)} = (W_t^{(m,1)}, W_t^{(m,2)}) := \mathbf{Y}_{\lfloor 2m^2 t\rfloor}^{(m)}.$$

To show Theorem 7.1, it suffices to show the following:

Proposition 7.4. *There exists $m \in \mathbb{N}$ such that $\mathbf{W}^{(m)}$ is not Markovian.*

Proof. Let $\tilde{\mathbf{W}}_t^{(m)} = (\tilde{W}_t^{(m,1)}, \tilde{W}_t^{(m,2)})$ be the natural lift of $\mathbf{W}_t^{(m)}$ to $\mathbb{R}^d \times \mathbb{R}^d$ with $\tilde{\mathbf{W}}_0^{(m)} = (\tilde{x}_1^{(m)}, \tilde{x}_2^{(m)})$. By the invariance principle, as $m \to \infty$, $\{\tilde{W}_t^{(m,i)}\}_{t\geq 0}$ converges in law to the Brownian motion $\{B_t^{(i)}\}_{t\geq 0}$ on $\mathbb{R}^d$ starting at $\tilde{x}_i \in \mathbb{R}^d$ for $i = 1, 2$. Hence $\tilde{\mathbf{W}}^{(m)}$ is tight in $D([0,\infty) \to \mathbb{R}^d \times \mathbb{R}^d)$. Thus there exists a subsequence $\{\tilde{\mathbf{W}}^{(m_l)}\}_{l\in\mathbb{N}}$ such that it converges in law to a process $\tilde{\mathbf{W}}$. Since $\{\tilde{\mathbf{W}}_t\}_{t\geq 0}$ is a coupling of $B^{(1)}$ and $B^{(2)}$, we obtain a coupling $\mathbf{W}$ of two Brownian motions on $\mathbb{T}^d$ starting at (x_1, x_2) by $\mathbf{W}_t := \pi(\tilde{\mathbf{W}}_t)$.

We will show $\mathbf{W}$ maximal. Once we have shown it, the conclusion holds in the following way: Suppose $\mathbf{W}^{(m_l)}$ to be Markovian for all $l \in \mathbb{N}$. Then so is $\mathbf{W}$. But, we can choose $x_1, x_2 \in \mathbb{T}^d$ appropriately so that there exists no reflection structure with respect to (x_1, x_2) by Example 6.5. In this case, the Markovianity of $\mathbf{W}$ contradicts with the maximality by Theorem 1.3.

Now let us turn to show the maximality of $\mathbf{W}$. We claim that the coupling time T is lower semi-continuous on $D([0,\infty) \to \mathbb{T}^d \times \mathbb{T}^d)$. To show it, take $(\omega_1^{(n)}, \omega_2^{(n)}) \in D([0,\infty) \to \mathbb{T}^d \times \mathbb{T}^d)$ satisfying $T((\omega_1^{(n)}, \omega_2^{(n)})) \leq t$ for all $n \in \mathbb{N}$ and assume that $(\omega_1^{(n)}, \omega_2^{(n)})$ converges to $(\omega_1, \omega_2) \in D([0,\infty) \to \mathbb{T}^d \times \mathbb{T}^d)$ as n tends to ∞. Then $\omega_1^{(n)}(u) = \omega_2^{(n)}(u)$ holds for any $u > t$ and hence the definition of the Skorokhod topology implies $\omega_1(s) = \omega_2(s)$ for any $s > t$. It means $T((\omega_1, \omega_2)) \leq t$ and therefore the claim follows. The fact that $\{(\omega_1, \omega_2) \mid T(\omega_1, \omega_2) > t\}$ is open yields

$$\liminf_{l\to\infty} \mathbb{P}\left[T(\mathbf{W}^{(m_l)}) > t\right] \geq \mathbb{P}\left[T(\mathbf{W}) > t\right]. \tag{7.7}$$

Since we have $\{T(\mathbf{W}^{(m_l)}) > t\} = \{T(\mathbf{Y}^{(m_l)}) > \lfloor 2m_l^2 t\rfloor\}$, (7.6) implies

$$\mathbb{P}\left[T(\mathbf{W}^{(m_l)}) > t\right] = \frac{1}{2} \sum_{z\in\pi(m^{-1}\mathbb{Z}^d)} \left| p_{\lfloor 2m_l^2 t\rfloor}^{(m_l)}(x_1^{(m_l)}, z) - p_{\lfloor 2m_l^2 t\rfloor}^{(m_l)}(x_2^{(m_l)}, z) \right|. \tag{7.8}$$

Set $\tilde{x}_i^{(m)} =: (\tilde{x}_{i1}^{(m)}, \ldots, \tilde{x}_{id}^{(m)})$ and take $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_d) \in m^{-1}\mathbb{Z}^d$. Then the transition probability is expressed as follows:

$$p_{\lfloor 2m^2 t\rfloor}^{(m)}(x_i^{(m)}, \pi(\tilde{z})) = \prod_{j=1}^{d} \mathbb{P}\left[Z_{1,j} + \cdots + Z_{\lfloor 2m^2 t\rfloor, j} \in m(\tilde{z}_j - \tilde{x}_{ij}^{(m)}) + m\mathbb{Z}\right].$$

Thus Lemma 7.3 yields

$$\begin{aligned}
&\lim_{l\to\infty}\sum_{z\in\pi(m_l^{-1}\mathbb{Z}^d)}\left|p^{(m_l)}_{\lfloor 2m_l^2 t\rfloor}(x_1^{(m_l)},z)-p^{(m_l)}_{\lfloor 2m_l^2 t\rfloor}(x_2^{(m_l)},z)\right|\\
&\quad=\lim_{l\to\infty}\sum_{\substack{\tilde z_j\in m_l^{-1}\mathbb{Z}\cap[0,1)\\ j=1,\dots,d}}\frac{1}{m_l^d\sqrt{2\pi t}^d}\left|\prod_{j=1}^d\left\{\sum_{k\in\mathbb{Z}}\exp\left(-\frac{1}{2t}\left(\tilde x_{1j}^{(m_l)}-\tilde z_j+k\right)^2\right)\right\}\right.\\
&\qquad\qquad\qquad\left.-\prod_{j=1}^d\left\{\sum_{k\in\mathbb{Z}}\exp\left(-\frac{1}{2t}\left(\tilde x_{2j}^{(m_l)}-\tilde z_j+k\right)^2\right)\right\}\right|\\
&\quad=\int_{\mathbb{T}^d}\left|p_t(x_1,y)-p_t(x_2,y)\right|\mu(dy).
\end{aligned}\tag{7.9}$$

Here μ is the normalized Haar measure and $p_t(x,y)$ is the transition density of the Brownian motion on $\mathbb{T}^d$ given by

$$p_t(\pi(x),\pi(y))=\sum_{\mathbf{k}\in\mathbb{Z}^d}\frac{1}{\sqrt{2\pi t}^d}\exp\left(-\frac{\left|x-y-\mathbf{k}\right|^2}{2t}\right).$$

Therefore, substituting (7.8) and (7.9) into (7.7), we obtain

$$\frac{1}{2}\int_{\mathbb{T}^d}\left|p_t(x_1,y)-p_t(x_2,y)\right|\mu(dy)\ge\mathbb{P}\left[T(\mathbf{W})>t\right].$$

Thus $\mathbf{W}$ is maximal and the proof is completed. □

Acknowledgment. The author would tell my gratitude to Professor Yukio Nagahata for his valuable comments, and to anonymous referees for improvement of this paper.

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