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First hitting time and place, monopoles and multipoles for pseudo-processes driven by the equation $\frac{\partial}{\partial t} = \pm \frac{\partial^N}{\partial x^N}$

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Abstract

Consider the high-order heat-type equation $\partial u/\partial t = \pm\partial^N u/\partial x^N$ for an integer $N > 2$ and introduce the related Markov pseudo-process $(X(t))_{t\geqslant 0}$. In this paper, we study several functionals related to $(X(t))_{t\geqslant 0}$: the maximum $M(t)$ and minimum $m(t)$ up to time t; the hitting times τ_a^+ and τ_a^- of the half lines $(a, +\infty)$ and $(-\infty, a)$ respectively. We provide explicit expressions for the distributions of the vectors $(X(t), M(t))$ and $(X(t), m(t))$, as well as those of the vectors $(\tau_a^+, X(\tau_a^+))$ and $(\tau_a^-, X(\tau_a^-))$.

Key words: pseudo-process, joint distribution of the process and its maximum/minimum, first hitting time and place, multipoles, Spitzer's identity.

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1 Introduction

Let N be an integer greater than 2 and consider the high-order heat-type equation

$$\frac{\partial u}{\partial t} = \kappa_N \frac{\partial^N u}{\partial x^N} \tag{1.1}$$

where $\kappa_N = (-1)^{1+N/2}$ if N is even and $\kappa_N = \pm 1$ if N is odd. Let $p(t; z)$ be the fundamental solution of Eq. (1.1) and put

$$p(t; x, y) = p(t; x - y).$$

The function p is characterized by its Fourier transform

$$\int_{-\infty}^{+\infty} e^{iu\xi} p(t; \xi)\, d\xi = e^{\kappa_N t(-iu)^N}. \tag{1.2}$$

With Eq. (1.1) one associates a Markov pseudo-process $(X(t))_{t\geqslant 0}$ defined on the real line and governed by a signed measure $\mathbb{P}$, which is **not** a probability measure, according to the usual rules of ordinary stochastic processes:

$$\mathbb{P}_x\{X(t) \in dy\} = p(t; x, y)\, dy$$

and for $0 = t_0 < t_1 < \cdots < t_n$, $x_0 = x$,

$$\mathbb{P}_x\{X(t_1) \in dx_1, \ldots, X(t_n) \in dx_n\} = \prod_{i=1}^{n} p(t_i - t_{i-1}; x_{i-1} - x_i)\, dx_i.$$

Relation (1.2) reads, by means of the expectation associated with $\mathbb{P}$,

$$\mathbb{E}_x\Big(e^{iuX(t)}\Big) = e^{iux + \kappa_N t(iu)^N}.$$

Such pseudo-processes have been considered by several authors, especially in the particular cases $N = 3$ and $N = 4$. The case $N = 4$ is related to the biharmonic operator $\partial^4/\partial x^4$. Few results are known in the case $N > 4$. Let us mention that for $N = 2$, the pseudo-process considered here is a genuine stochastic process (i.e., driven by a genuine probability measure), this is the most well-known Brownian motion.

The following problems have been tackled:

- Analytical study of the sample paths of this pseudo-process: Hochberg [7] defined a stochastic integral (see also Motoo [13] in higher dimension) and proposed an Itô formula based on the correspondence $dx^4 = dt$, he obtained a formula for the distribution of the maximum over $[0, t]$ in the case $N = 4$ with an extension to the even-order case. Noteworthy, the sample paths do not seem to be continuous in the case $N = 4$;

- Study of the sojourn time spent on the positive half-line up to time t, $T(t) = \text{meas}\{s \in [0, t] : X(s) > 0\} = \int_0^t 1\!\!1_{\{X(s)>0\}}\, ds$: Krylov [10], Orsingher [19], Hochberg and Orsingher [8], Nikitin and Orsingher [15], Lachal [11] explicitly obtained the distribution of $T(t)$ (with possible conditioning on the events $\{X(t) > (\text{or} =, \text{or} <) 0\}$). Sojourn time is useful for defining local times related to the pseudo-process X, see Beghin and Orsingher [2];

- Study of the maximum and the minimum functionals

$$M(t) = \max_{0\leqslant s\leqslant t} X(s) \quad \text{and} \quad m(t) = \min_{0\leqslant s\leqslant t} X(s):$$

Hochberg [7], Beghin et al. [1, 3], Lachal [11] explicitly derived the distribution of $M(t)$ and that of $m(t)$ (with possible conditioning on some values of $X(t)$). Since the paths may not be continuous, we should write sup and inf instead. However, because of Definition 3.3, we prefer to write max and min as done in several works dealing with this matter;

- Study of the couple $(X(t), M(t))$: Beghin et al. [3] wrote out several formulas for the joint distribution of $X(t)$ and $M(t)$ in the cases $N = 3$ and $N = 4$;

- Study of the first time the pseudo-process $(X(t))_{t\geqslant 0}$ overshoots the level $a > 0$, $\tau_a^+ = \inf\{t \geqslant 0 : X(t) > a\}$: Nishioka [16, 17], Nakajima and Sato [14] adopt a distributional approach (in the sense of Schwartz distributions) and explicitly obtained the joint distribution of τ_a^+ and $X(\tau_a^+)$ (with possible drift) in the case $N = 4$. The quantity $X(\tau_a^+)$ is the first hitting place of the half-line $[a, +\infty)$. Nishioka [18] then studied killing, reflecting and absorbing pseudo-processes;

- Study of the last time before becoming definitively negative up to time t, $O(t) = \sup\{s \in [0, t] : X(s) > 0\}$: Lachal [11] derived the distribution of $O(t)$;

- Study of Equation (1.1) in the case $N = 4$ under other points of view: Funaki [6], and next Hochberg and Orsingher [9] exhibited relationships with compound processes, namely iterated Brownian motion, Benachour et al. [4] provided other probabilistic interpretations. See also the references therein.

This aim of this paper is to study the problem of the first times straddling a fixed level a (or the first hitting times of the half-lines $(a, +\infty)$ and $(-\infty, a)$):

$$\tau_a^+ = \inf\{t \geqslant 0 : X(t) > a\}, \quad \tau_a^- = \inf\{t \geqslant 0 : X(t) < a\}$$

with the convention $\inf(\emptyset) = +\infty$. In the spirit of the method developed by Nishioka in the case $N = 4$, we explicitly compute the joint "signed-distributions" (we simply shall call "distributions" throughout the paper for short) of the vectors $(X(t), M(t))$ and $(X(t), m(t))$ from which we deduce those of the vectors $(\tau_a^+, X(\tau_a^+))$ and $(\tau_a^-, X(\tau_a^-))$. The method consists of several steps:

- Defining a step-process by sampling the pseudo-process $(X(t))_{t\geqslant 0}$ on dyadic times $t_{n,k} = k/2^n$, $k \in \mathbb{N}$;

- Observing that the classical Spitzer's identity holds for any signed measure, provided the total mass equals one, and then using this identity for deriving the distribution of $(X(t_{n,k}), \max_{0\leqslant j\leqslant k} X(t_{n,j}))$ through its Laplace-Fourier transform by means of that of $X(t_{n,k})^+$ where $x^+ = \max(x, 0)$;

- Expressing time τ_a^+ (for instance) related to the sampled process $(X(t_{n,k}))_{k\in\mathbb{N}}$ by means of $(X(t_{n,k}), \max_{0\leqslant j\leqslant k} X(t_{n,j}))$;

- Passing to the limit when $n \to +\infty$.

Meaningfully, we have obtained that the distributions of the hitting places $X(\tau_a^+)$ and $X(\tau_a^-)$ are linear combinations of the successive derivatives of the Dirac distribution δ_a. In the case $N = 4$, Nishioka [16] already found a linear combination of δ_a and δ'_a and called each corresponding part "monopole" and "dipole" respectively, considering that an electric dipole having two opposite charges $\delta_{a+\varepsilon}$ and $\delta_{a-\varepsilon}$ with a distance ε tending to 0 may be viewed as one monopole with charge δ'_a. In the general case, we shall speak of "multipoles".

Nishioka [17] used precise estimates for carrying out the rigorous analysis of the pseudo-process corresponding to the case $N = 4$. The most important fact for providing such estimates is that the integral of the density p is absolutely convergent. Actually, this fact holds for any even integer N. When N is an odd integer, the integral of p is **not** absolutely convergent and then similar estimates may not be obtained; this makes the study of X very much harder in this case. Nevertheless, we have found, formally at least, remarkable formulas which agree with those of Beghin et al. [1, 3] in the case $N = 3$. They obtained them by using a Feynman-Kac approach and solving differential equations. We also mention some similar differential equations for any N. So, we guess our formulas should hold for any odd integer $N \geqslant 3$. Perhaps a distributional definition (in the sense of Schwartz distributions since the heat-kernel is locally integrable) of the pseudo-process X might provide a properly justification to comfirm our results. We shall not tackle this question here.

The paper is organized as follows: in Section 2, we write down general notations and recall some known results. In Section 3, we construct the step-process deduced from $(X(t))_{t\geqslant 0}$ by sampling this latter on dyadic times. Section 4 is devoted to the distributions of the vectors $(X(t), M(t))$ and $(X(t), m(t))$ with the aid of Spitzer's identity. Section 5 deals with the distributions of the vectors $(\tau_a^+, X(\tau_a^+))$ and $(\tau_a^-, X(\tau_a^-))$ which can be expressed by means of those of $(X(t), M(t))$ and $(X(t), m(t))$. Each section is completed by an illustration of the displayed results therein to the particular cases $N \in \{2, 3, 4\}$.

We finally mention that the most important results have been announced, without details, in a short Note [12].

2 Settings

The relation $\int_{-\infty}^{+\infty} p(t;\xi)\,d\xi = 1$ holds for all integers N. Moreover, if N is even, the integral is absolutely convergent (see [11]) and we put

$$\rho = \int_{-\infty}^{+\infty} |p(t;\xi)|\,d\xi > 1.$$

Notice that ρ does not depend on t since $p(t;\xi) = t^{-1/N} p(1;\xi/t^{1/N})$. For odd integer N, the integral of p is not absolutely convergent; in this case $\rho = +\infty$.

2.1 N^{th} roots of κ_N

We shall have to consider the N^{th} roots of κ_N (θ_l for $0 \leqslant l \leqslant N-1$ say) and distinguish the indices l such that $\Re\theta_l < 0$ and $\Re\theta_l > 0$ (one never has $\Re\theta_l = 0$). So, let us introduce the following set of indices

$$\begin{aligned} J &= \{l \in \{0,\dots,N-1\} : \Re\theta_l > 0\}, \\ K &= \{l \in \{0,\dots,N-1\} : \Re\theta_l < 0\}. \end{aligned}$$

We clearly have $J \cup K = \{0,\dots,N-1\}$, $\quad J \cap K = \emptyset$ and

$$\#J + \#K = N. \tag{2.1}$$

If $N = 2p$, then $\kappa_N = (-1)^{p+1}$, $\theta_l = e^{i[(2l+p+1)\pi/N]}$,

$$J = \{p,\dots,2p-1\} \quad \text{and} \quad K = \{0,\dots,p-1\}.$$

The numbers of elements of the sets J and K are

$$\#J = \#K = p.$$

If $N = 2p+1$, two cases must be considered:

- For $\kappa_N = +1$: $\theta_l = e^{i[2l\pi/N]}$ and

$$\begin{aligned} &J = \Big\{0,\dots,\frac{p}{2}\Big\} \cup \Big\{\frac{3p}{2}+1,\dots,2p\Big\} &&\text{and } K = \Big\{\frac{p}{2}+1,\dots,\frac{3p}{2}\Big\} &&\text{if } p \text{ is even,} \\ &J = \Big\{0,\dots,\frac{p-1}{2}\Big\} \cup \Big\{\frac{3p+3}{2},\dots,2p\Big\} &&\text{and } K = \Big\{\frac{p+1}{2},\dots,\frac{3p+1}{2}\Big\} &&\text{if } p \text{ is odd.} \end{aligned}$$

  The numbers of elements of the sets J and K are

$$\begin{array}{llll} \#J = p+1 & \text{and} & \#K = p & \text{if } p \text{ is even,} \\ \#J = p & \text{and} & \#K = p+1 & \text{if } p \text{ is odd;} \end{array}$$

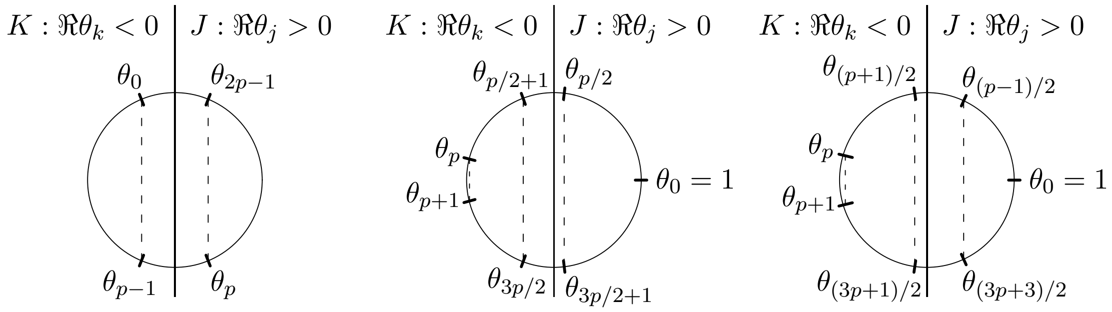
- For $\kappa_N = -1$: $\theta_l = e^{i[(2l+1)\pi/N]}$ and

$$\begin{aligned} &J = \Big\{0,\dots,\frac{p}{2}-1\Big\} \cup \Big\{\frac{3p}{2}+1,\dots,2p\Big\} &&\text{and } K = \Big\{\frac{p}{2},\dots,\frac{3p}{2}\Big\} &&\text{if } p \text{ is even,} \\ &J = \Big\{0,\dots,\frac{p-1}{2}\Big\} \cup \Big\{\frac{3p+1}{2},\dots,2p\Big\} &&\text{and } K = \Big\{\frac{p+1}{2},\dots,\frac{3p-1}{2}\Big\} &&\text{if } p \text{ is odd.} \end{aligned}$$

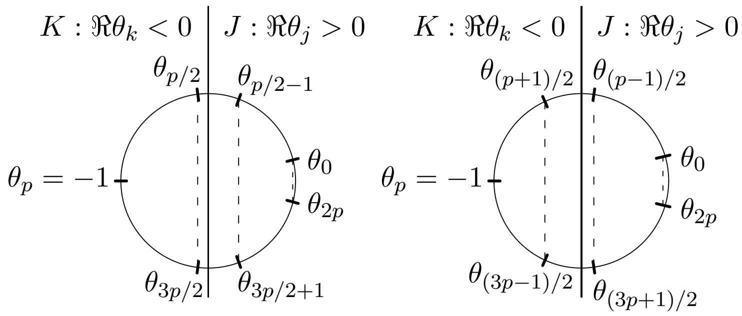
  The numbers of elements of the sets J and K are

$$\begin{array}{llll} \#J = p & \text{and} & \#K = p+1 & \text{if } p \text{ is even,} \\ \#J = p+1 & \text{and} & \#K = p & \text{if } p \text{ is odd.} \end{array}$$

Figure 1 illustrates the different cases.



Case $N = 2p$ Case $N = 2p + 1$, $\kappa_N = +1$: even p (left), odd p (right)



Case $N = 2p + 1$, $\kappa_N = -1$: even p (left), odd p (right)

Figure 1: The N^{th} roots of κ_N

2.2 Recalling some known results

We recall from [11] the expressions of the kernel $p(t;\xi)$

$$p(t;\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\xi u+\kappa_N t(-iu)^N}\,du \tag{2.2}$$

together with its Laplace transform (the so-called λ-potential of the pseudo-process $(X(t))_{t\geqslant 0}$), for $\lambda > 0$,

$$\Phi(\lambda;\xi) = \int_0^{+\infty} e^{-\lambda t}\,p(t;\xi)\,dt = \begin{cases} -\dfrac{1}{N}\,\lambda^{1/N-1}\displaystyle\sum_{k\in K}\theta_k\,e^{\theta_k\sqrt[N]{\lambda}\,\xi} & \text{for } \xi \geqslant 0,\\[2ex] \dfrac{1}{N}\,\lambda^{1/N-1}\displaystyle\sum_{j\in J}\theta_j\,e^{\theta_j\sqrt[N]{\lambda}\,\xi} & \text{for } \xi \leqslant 0.\end{cases} \tag{2.3}$$

Notice that

$$\Phi(\lambda;\xi) = \int_0^{+\infty} e^{-\lambda t}\,dt\,\mathbb{P}\{X(t)\in -d\xi\}/d\xi.$$

We also recall (see the proof of Proposition 4 of [11]):

$$\Psi(\lambda;\xi) = \int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}\{X(t)\leqslant -\xi\}\,dt = \begin{cases} \dfrac{1}{N\lambda}\displaystyle\sum_{k\in K} e^{\theta_k\sqrt[N]{\lambda}\,\xi} & \text{for } \xi \geqslant 0,\\[2ex] \dfrac{1}{\lambda}\left[1-\dfrac{1}{N}\displaystyle\sum_{j\in J} e^{\theta_j\sqrt[N]{\lambda}\,\xi}\right] & \text{for } \xi \leqslant 0.\end{cases} \tag{2.4}$$

We recall the expressions of the distributions of $M(t)$ and $m(t)$ below.

- Concerning the densities:

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t}\,dt\,\mathbb{P}_x\{M(t)\in dz\}/dz &= \frac{1}{\lambda}\,\varphi_\lambda(x-z) \quad \text{for } x\leqslant z,\\ \int_0^{+\infty} e^{-\lambda t}\,dt\,\mathbb{P}_x\{m(t)\in dz\}/dz &= \frac{1}{\lambda}\,\psi_\lambda(x-z) \quad \text{for } x\geqslant z,\end{aligned} \tag{2.5}$$

  with

$$\varphi_\lambda(\xi) = \sqrt[N]{\lambda}\sum_{j\in J}\theta_j A_j\, e^{\theta_j\sqrt[N]{\lambda}\,\xi}, \quad \psi_\lambda(\xi) = -\sqrt[N]{\lambda}\sum_{k\in K}\theta_k B_k\, e^{\theta_k\sqrt[N]{\lambda}\,\xi} \tag{2.6}$$

  and

$$A_j = \prod_{l\in J\setminus\{j\}}\frac{\theta_l}{\theta_l-\theta_j} \quad \text{for } j\in J, \quad B_k = \prod_{l\in K\setminus\{k\}}\frac{\theta_l}{\theta_l-\theta_k} \quad \text{for } k\in K.$$

- Concerning the distribution functions:

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x(M(t)\leqslant z)\,dt &= \frac{1}{\lambda}\left[1-\sum_{j\in J}A_j\, e^{\theta_j\sqrt[N]{\lambda}\,(x-z)}\right] \quad \text{for } x\leqslant z,\\ \int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x(m(t)\geqslant z)\,dt &= \frac{1}{\lambda}\left[1-\sum_{k\in K}B_k\, e^{\theta_k\sqrt[N]{\lambda}\,(x-z)}\right] \quad \text{for } x\geqslant z.\end{aligned} \tag{2.7}$$

We explicitly write out the settings in the particular cases $N\in\{2,3,4\}$ (see Fig. 2).

Example 2.1 *Case $N=2$:* we have $\kappa_2=+1$, $\theta_0=-1,\theta_1=1$, $J=\{1\},K=\{0\}$, $A_1=1$, $B_0=1$. ■

Example 2.2 *Case $N=3$:* we split this (odd) case into two subcases:

- for $\kappa_3=+1$, we have $\theta_0=1,\theta_1=e^{i\,2\pi/3},\theta_2=e^{-i\,2\pi/3}$, $J=\{0\},K=\{1,2\}$, $A_0=1$, $B_1=\frac{1}{1-e^{-i\,2\pi/3}}=\frac{1}{\sqrt{3}}\,e^{-i\,\pi/6}, B_2=\bar{B}_1=\frac{1}{\sqrt{3}}\,e^{i\,\pi/6}$;

- for $\kappa_3=-1$, we have $\theta_0=e^{i\,\pi/3},\theta_1=-1,\theta_2=e^{-i\,\pi/3}$, $J=\{0,2\},K=\{1\}$, $A_0=\frac{1}{1-e^{-i\,4\pi/3}}=\frac{1}{\sqrt{3}}\,e^{i\,\pi/6}, A_2=\bar{A}_0=\frac{1}{\sqrt{3}}\,e^{-i\,\pi/6}, B_1=1$.

■

Example 2.3 *Case $N=4$:* we have $\kappa_4=-1$, $\theta_0=e^{i\,3\pi/4},\theta_1=e^{-i\,3\pi/4},\theta_2=e^{-i\,\pi/4},\theta_3=e^{i\,\pi/4}$, $J=\{2,3\},K=\{0,1\}$, $A_2=B_0=\frac{1}{1-e^{-i\,\pi/2}}=\frac{1}{\sqrt{2}}\,e^{-i\,\pi/4}, A_3=B_1=\bar{A}_2=\frac{1}{\sqrt{2}}\,e^{i\,\pi/4}$. ■

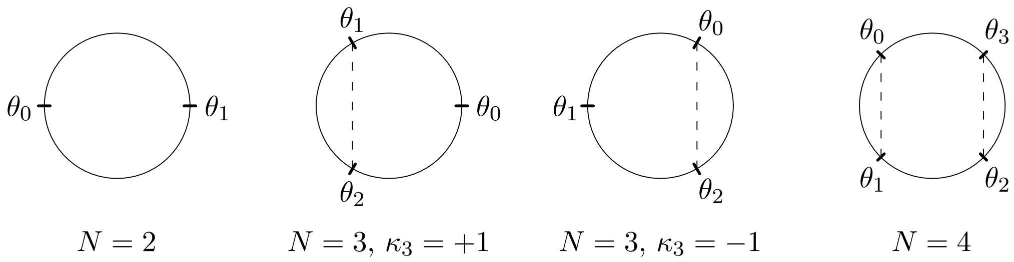


Figure 2: The N^{th} roots of κ_N in the cases $N \in \{1,2,3\}$

2.3 Some elementary properties

Let us mention some elementary properties: the relation $\prod_{l=1}^{N-1}(1-e^{i(2l\pi/N)}) = N$ entails

$$\prod_{l=0,l\neq m}^{N-1} \frac{\theta_l}{\theta_m-\theta_l} = \frac{1}{N} \quad \text{for } 0 \leqslant m \leqslant N-1. \tag{2.8}$$

The following result will be used further: expanding into partial fractions yields, for any polynomial P of degree $\deg P \leqslant \#J$,

$$\frac{P(x)}{\prod\limits_{j\in J}(1-x/\theta_j)} = \begin{cases} \displaystyle\sum_{j\in J} \frac{A_j P(\theta_j)}{1-x/\theta_j} & \text{if } \deg P \leqslant \#J-1, \\ \displaystyle\sum_{j\in J} \frac{A_j P(\theta_j)}{1-x/\theta_j} + (-1)^{\#J}\prod_{j\in J}\theta_j & \text{if } \deg P = \#J \text{ and the highest degree coefficient of } P \text{ is } 1. \end{cases} \tag{2.9}$$

- Applying (2.9) to $x=0$ and $P=1$ gives $\sum_{j\in J} A_j = \sum_{k\in K} B_k = 1$. Actually, the A_j's and B_k's are solutions of a Vandermonde system (see [11]).
- Applying (2.9) to $x=\theta_k, k\in K$, and $P=1$ gives

$$\sum_{j\in J}\frac{\theta_j A_j}{\theta_j-\theta_k} = \sum_{j\in J}\frac{A_j}{1-\theta_k/\theta_j} = \left[\prod_{j\in J}(1-\theta_k/\theta_j)\right]^{-1} = \frac{\prod\limits_{l=0,l\neq k}^{N-1}\frac{\theta_l}{\theta_l-\theta_k}}{\prod\limits_{l\in K\setminus\{k\}}\frac{\theta_l}{\theta_l-\theta_k}}$$

  which simplifies, by (2.8), into (and also for the B_k's)

$$\sum_{j\in J}\frac{\theta_j A_j}{\theta_j-\theta_k} = \frac{1}{NB_k} \quad \text{for } k\in K \text{ and } \quad \sum_{k\in K}\frac{\theta_k B_k}{\theta_k-\theta_j} = \frac{1}{NA_j} \quad \text{for } j\in J. \tag{2.10}$$

- Applying (2.9) to $P=x^p$, $p\leqslant \#J$, gives, by observing that $1/\theta_j = \bar\theta_j$,

$$\sum_{j\in J}\frac{\theta_j^p A_j}{1-\bar\theta_j x} = \begin{cases} \dfrac{x^p}{\prod\limits_{j\in J}(1-\bar\theta_j x)} & \text{if } p\leqslant \#J-1, \\ \dfrac{x^p}{\prod\limits_{j\in J}(1-\bar\theta_j x)} + (-1)^{\#J-1}\prod\limits_{j\in J}\theta_j & \text{if } p=\#J. \end{cases} \tag{2.11}$$

3 Step-process

In this part, we proceed to sampling the pseudo-process $X = (X(t))_{t\geqslant 0}$ on the dyadic times $t_{n,k} = k/2^n$, $k, n \in \mathbb{N}$ and we introduce the corresponding step-process $X_n = (X_n(t))_{t\geqslant 0}$ defined for any $n \in \mathbb{N}$ by

$$X_n(t) = \sum_{k=0}^{\infty} X(t_{n,k})\mathbb{1}_{[t_{n,k},t_{n,k+1})}(t).$$

The quantity X_n is a function of discrete observations of X at times $t_{n,k}$, $k \in \mathbb{N}$.

For the convenience of the reader, we recall the definitions of tame functions, functions of discrete observations, and admissible functions introduced by Nishioka [17] in the case $N = 4$.

Definition 3.1 *Fix $n \in \mathbb{N}$. A tame function is a function of a finite number of observations of the pseudo-process X at times $t_{n,j}$, $1 \leqslant j \leqslant k$, that is a quantity of the form $F_{n,k} = F(X(t_{n,1}), \ldots, X(t_{n,k}))$ for a certain k and a certain bounded Borel function $F : \mathbb{R}^k \longrightarrow \mathbb{C}$. The "expectation" of $F_{n,k}$ is defined as*

$$\mathbb{E}_x(F_{n,k}) = \int \cdots \int_{\mathbb{R}^k} F(x_1, \ldots, x_k)\, p(1/2^n; x, x_1) \cdots p(1/2^n; x_{k-1}, x_k)\, dx_1 \cdots dx_k.$$

We plainly have the inequality

$$|\mathbb{E}_x(F_{n,k})| \leqslant \rho^k \sup_{\mathbb{R}^k} |F|.$$

Definition 3.2 *Fix $n \in \mathbb{N}$. A function of the discrete observations of X at times $t_{n,k}$, $k \geqslant 1$, is a convergent series of tame functions: $F_{X_n} = \sum_{k=1}^{\infty} F_{n,k}$ where $F_{n,k}$ is a tame function for all $k \geqslant 1$. Assuming the series $\sum_{k=1}^{\infty} |\mathbb{E}_x(F_{n,k})|$ convergent, the "expectation" of F_{X_n} is defined as*

$$\mathbb{E}_x(F_{X_n}) = \sum_{k=1}^{\infty} \mathbb{E}_x(F_{n,k}).$$

The definition of the expectation is consistent in the sense that it does not depend on the representation of F_{X_n} as a series (see [17]): if $\sum_{k=1}^{\infty} F_{n,k} = \sum_{k=1}^{\infty} G_{n,k}$ and if the series $\sum_{k=1}^{\infty} |\mathbb{E}_x(F_{n,k})|$ and $\sum_{k=1}^{\infty} |\mathbb{E}_x(G_{n,k})|$ are convergent, then $\sum_{k=1}^{\infty} \mathbb{E}_x(F_{n,k}) = \sum_{k=1}^{\infty} \mathbb{E}_x(G_{n,k})$.

Definition 3.3 *An admissible function is a functional F_X of the pseudo-process X which is the limit of a sequence $(F_{X_n})_{n\in\mathbb{N}}$ of functions of discrete observations of X:*

$$F_X = \lim_{n\to\infty} F_{X_n},$$

such that the sequence $(\mathbb{E}_x(F_{X_n}))_{n\in\mathbb{N}}$ is convergent. The "expectation" of F_X is defined as

$$\mathbb{E}_x(F_X) = \lim_{n\to\infty} \mathbb{E}_x(F_{X_n}).$$

This definition eludes the difficulty due to the lack of σ-additivity of the signed measure $\mathbb{P}$. On the other hand, any bounded Borel function of a finite number of observations of X at any times (not necessarily dyadic) $t_1 < \cdots < t_k$ is admissible and it can be seen that, according to Definitions 3.1, 3.2 and 3.3,

$$\begin{aligned}\mathbb{E}_x[F(X(t_1),\ldots,X(t_k))] &= \int\cdots\int_{\mathbb{R}^k} F(x_1,\ldots,x_k)\,p(t_1;x,x_1)\,p(t_2-t_1;x_1,x_2)\cdots\\ &\qquad \times p(t_k-t_{k-1};x_{k-1},x_k)\,dx_1\cdots dx_k\end{aligned}$$

as expected in the usual sense.

For any pseudo-process $Z=(Z(t))_{t\geqslant 0}$, consider the functional defined for $\lambda\in\mathbb{C}$ such that $\Re(\lambda)>0$, $\mu\in\mathbb{R}$, $\nu>0$ by

$$F_Z(\lambda,\mu,\nu)=\int_0^{+\infty} e^{-\lambda t+i\mu H_Z(t)-\nu K_Z(t)} I_Z(t)\,dt \tag{3.1}$$

where H_Z, K_Z, I_Z are functionals of Z defined on $[0,+\infty)$, K_Z being non negative and I_Z bounded; we suppose that, for all $t\geqslant 0$, $H_Z(t), K_Z(t), I_Z(t)$ are functions of the continuous observations $Z(s)$, $0\leqslant s\leqslant t$ (that is the observations of Z up to time t). For $Z=X_n$, we have

$$\begin{aligned}F_{X_n}(\lambda,\mu,\nu) &= \sum_{k=0}^{\infty}\int_{t_{n,k}}^{t_{n,k+1}} e^{-\lambda t+i\mu H_{X_n}(t_{n,k})-\nu K_{X_n}(t_{n,k})} I_{X_n}(t_{n,k})\,dt\\ &= \sum_{k=0}^{\infty}\left(\int_{t_{n,k}}^{t_{n,k+1}} e^{-\lambda t}\,dt\right) e^{i\mu H_{X_n}(t_{n,k})-\nu K_{X_n}(t_{n,k})} I_{X_n}(t_{n,k})\\ &= \frac{1-e^{-\lambda/2^n}}{\lambda}\sum_{k=0}^{\infty} e^{-\lambda t_{n,k}+i\mu H_{X_n}(t_{n,k})-\nu K_{X_n}(t_{n,k})} I_{X_n}(t_{n,k}).\end{aligned} \tag{3.2}$$

Since $H_{X_n}(t_{n,k}), K_{X_n}(t_{n,k}), I_{X_n}(t_{n,k})$ are functions of $X_n(t_{n,j})=X(t_{n,j})$, $0\leqslant j\leqslant k$, the quantity $e^{i\mu H_{X_n}(t_{n,k})-\nu K_{X_n}(t_{n,k})} I_{X_n}(t_{n,k})$ is a tame function and the series in (3.2) is a function of discrete observations of X. If the series

$$\sum_{k=0}^{\infty}\left|\mathbb{E}_x\Big[e^{-\lambda t_{n,k}+i\mu H_{X_n}(t_{n,k})-\nu K_{X_n}(t_{n,k})} I_{X_n}(t_{n,k})\Big]\right|$$

converges, the expectation of $F_{X_n}(\lambda,\mu,\nu)$ is defined, according to Definition 3.2, as

$$\mathbb{E}_x[F_{X_n}(\lambda,\mu,\nu)]=\frac{1-e^{-\lambda/2^n}}{\lambda}\sum_{k=0}^{\infty}\mathbb{E}_x\Big[e^{-\lambda t_{n,k}+i\mu H_{X_n}(t_{n,k})-\nu K_{X_n}(t_{n,k})} I_{X_n}(t_{n,k})\Big].$$

Finally, if $\lim_{n\to+\infty} F_{X_n}(\lambda,\mu,\nu)=F_X(\lambda,\mu,\nu)$ and if the limit of $\mathbb{E}_x[F_{X_n}(\lambda,\mu,\nu)]$ exists as n goes to ∞, $F_X(\lambda,\mu,\nu)$ is an admissible function and its expectation is defined, according to Definition 3.3, as

$$\mathbb{E}_x[F_X(\lambda,\mu,\nu)]=\lim_{n\to+\infty}\mathbb{E}_x[F_{X_n}(\lambda,\mu,\nu)].$$

4 Distributions of $(X(t),M(t))$ and $(X(t),m(t))$

We assume that N is even. In this section, we derive the Laplace-Fourier transforms of the vectors $(X(t),M(t))$ and $(X(t),m(t))$ by using Spitzer's identity (Subsection 4.1), from which we deduce the densities of these vectors by successively inverting—three times—the Laplace-Fourier transforms (Subsection 4.2). Next, we write out the formulas corresponding to the particular cases $N \in \{2,3,4\}$ (Subsection 4.3). Finally, we compute the distribution functions of the vectors $(X(t), m(t))$ and $(X(t), M(t))$ (Subsection 4.4) and write out the formulas associated with $N \in \{2,3,4\}$ (Subsection 4.5). Although N is assumed to be even, all the formulas obtained in this part when replacing N by 3 lead to some well-known formulas in the literature.

4.1 Laplace-Fourier transforms

Theorem 4.1 *The Laplace-Fourier transform of the vectors $(X(t), M(t))$ and $(X(t), m(t))$ are given, for $\Re(\lambda) > 0, \mu \in \mathbb{R}, \nu > 0$, by*

$$\begin{aligned}\mathbb{E}_x\Big[\int_0^{+\infty} e^{-\lambda t+i\mu X(t)-\nu M(t)}\,dt\Big] &= \frac{e^{(i\mu-\nu)x}}{\prod_{j\in J}(\sqrt[N]{\lambda}-(i\mu-\nu)\theta_j)\prod_{k\in K}(\sqrt[N]{\lambda}-i\mu\theta_k)},\\ \mathbb{E}_x\Big[\int_0^{+\infty} e^{-\lambda t+i\mu X(t)+\nu m(t)}\,dt\Big] &= \frac{e^{(i\mu+\nu)x}}{\prod_{j\in J}(\sqrt[N]{\lambda}-i\mu\theta_j)\prod_{k\in K}(\sqrt[N]{\lambda}-(i\mu+\nu)\theta_k)}.\end{aligned} \tag{4.1}$$

Proof. We divide the proof of Theorem 4.1 into four parts.

• **Step 1**

Write functionals (3.1) with $H_X(t) = X(t)$, $K_X(t) = M(t)$ or $K_X(t) = -m(t)$ and $I_X(t) = 1$:

$$F_X^+(\lambda,\mu,\nu) = \int_0^{+\infty} e^{-\lambda t+i\mu X(t)-\nu M(t)}\,dt \quad\text{and}\quad F_X^-(\lambda,\mu,\nu) = \int_0^{+\infty} e^{-\lambda t+i\mu X(t)+\nu m(t)}\,dt.$$

So, putting $X_{n,k} = X(t_{n,k})$, $M_n(t) = \max_{0\leqslant s\leqslant t} X_n(s) = \max_{0\leqslant j\leqslant \lfloor 2^n t\rfloor} X_{n,j}$ where $\lfloor . \rfloor$ denotes the floor function, and next $M_{n,k} = M_n(t_{n,k}) = \max_{0\leqslant j\leqslant k} X_{n,j}$, (3.2) yields, e.g., for $F_{X_n}^+$,

$$F_{X_n}^+(\lambda,\mu,\nu) = \frac{1-e^{-\lambda/2^n}}{\lambda}\sum_{k=0}^{\infty} e^{-\lambda t_{n,k}+i\mu X_{n,k}-\nu M_{n,k}}.$$

The functional $F_{X_n}^+(\lambda,\mu,\nu)$ is a function of discrete observations of X. Our aim is to compute its expectation, that is to compute the expectation of the above series and next to take the limit as n goes to infinity. For this, we observe that, using the Markov property,

$$\begin{aligned}\Big|\mathbb{E}_x\Big[e^{-\lambda t_{n,k}+i\mu X_{n,k}-\nu M_{n,k}}\Big]\Big| &= |e^{-\lambda t_{n,k}}|\left|\sum_{j=0}^{k}\mathbb{E}_x\Big[e^{i\mu X_{n,k}-\nu X_{n,j}}\mathbb{1}_{\{X_{n,1}\leqslant X_{n,j},\dots,X_{n,k}\leqslant X_{n,j}\}}\Big]\right|\\ &\leqslant (e^{-\Re(\lambda)/2^n})^k\left|\sum_{j=0}^{k}\int\dots\int_{\{x_1\leqslant x_j,\dots,x_k\leqslant x_j\}} e^{i\mu x_k-\nu x_j}p(1/2^n;x-x_1)\cdots p(1/2^n;x_{k-1}-x_k)\,dx_1\cdots dx_k\right|\\ &\leqslant (k+1)(\rho\, e^{-\Re(\lambda)/2^n})^k.\end{aligned}$$

So, if $\Re(\lambda) > 2^n\ln\rho$, the series $\sum \mathbb{E}_x\big[e^{-\lambda t_{n,k}+i\mu X_{n,k}-\nu M_{n,k}}\big]$ is absolutely convergent and then we can write the expectation of $F_{X_n}^+(\lambda,\mu,\nu)$:

$$\mathbb{E}_x\big[F_{X_n}^+(\lambda,\mu,\nu)\big] = \frac{1-e^{-\lambda/2^n}}{\lambda}\sum_{k=0}^{\infty} e^{-\lambda t_{n,k}}\,\varphi_{n,k}^+(\mu,\nu;x) \quad \text{for } \Re(\lambda) > 2^n\ln\rho \tag{4.2}$$

with
$$\varphi^+_{n,k}(\mu,\nu;x)=\mathbb{E}_x\Big[e^{i\mu X_{n,k}-\nu M_{n,k}}\Big]=e^{(i\mu-\nu)x}\,\mathbb{E}_0\Big[e^{-(\nu-i\mu)M_{n,k}-i\mu(M_{n,k}-X_{n,k})}\Big].$$
However, because of the domain of validity of (4.2), we cannot take directly the limit as n tends to infinity. Actually, we shall see that this difficulty can be circumvented by using sharp results on Dirichlet series.

• **Step 2**

Putting $z=e^{-\lambda/2^n}$ and noticing that $e^{-\lambda t_{n,k}}=z^k$, (4.2) writes
$$\mathbb{E}_x[F^+_{X_n}(\lambda,\mu,\nu)]=\frac{1-z}{\lambda}\sum_{k=0}^{\infty}\varphi^+_{n,k}(\mu,\nu;x)\,z^k.$$
The generating function appearing in the last displayed equality can be evaluated thanks to an extension of Spitzer's identity, which we recall below.

Lemma 4.2 *Let $(\xi_k)_{k\geqslant 1}$ be a sequence of "i.i.d. random variables" and set $X_0=0$, $X_k=\sum_{j=1}^{k}\xi_j$ for $k\geqslant 1$, and $M_k=\max_{0\leqslant j\leqslant k}X_j$ for $k\geqslant 0$. The following relationship holds for $|z|<1$:*
$$\sum_{k=0}^{\infty}\mathbb{E}\big[e^{i\mu X_k-\nu M_k}\big]\,z^k=\exp\left[\sum_{k=1}^{\infty}\mathbb{E}\Big[e^{i\mu X_k-\nu X_k^+}\Big]\,\frac{z^k}{k}\right].$$
Observing that $1-z=\exp[\log(1-z)]=\exp[-\sum_{k=1}^{\infty}z^k/k]$, Lemma 4.2 yields, for $\xi_k=X_{n,k}-X_{n,k-1}$:
$$\mathbb{E}_x[F^+_{X_n}(\lambda,\mu,\nu)]=\frac{1}{\lambda}\,e^{(i\mu-\nu)x}\,\exp\left[\frac{1}{2^n}\sum_{k=1}^{\infty}\frac{e^{-\lambda t_{n,k}}}{t_{n,k}}\,\psi^+(\mu,\nu;t_{n,k})\right]\tag{4.3}$$
where
$$\begin{aligned}\psi^+(\mu,\nu;t)&=\mathbb{E}_0\Big[e^{i\mu X(t)-\nu X(t)^+}\Big]-1\\&=\mathbb{E}_0\Big[\Big(e^{i\mu X(t)}-1\Big)\,1\!\!1_{\{X(t)<0\}}\Big]+\mathbb{E}_0\Big[\Big(e^{(i\mu-\nu)X(t)}-1\Big)\,1\!\!1_{\{X(t)\geqslant 0\}}\Big]\\&=\int_{-\infty}^{0}\Big(e^{i\mu\xi}-1\Big)\,p(t;-\xi)\,d\xi+\int_{0}^{+\infty}\Big(e^{(i\mu-\nu)\xi}-1\Big)\,p(t;-\xi)\,d\xi.\end{aligned}\tag{4.4}$$
We plainly have $|\psi^+(\mu,\nu;t)|\leqslant 2\rho$, and then the series in (4.3) defines an analytical function on the half-plane $\{\lambda\in\mathbb{C}:\Re(\lambda)>0\}$. It is the analytical continuation of the function $\lambda\longmapsto\mathbb{E}_x[F^+_{X_n}(\lambda,\mu,\nu)]$ which was *a priori* defined on the half-plane $\{\lambda\in\mathbb{C}:\Re(\lambda)>2^n\ln\rho\}$. As a byproduct, we shall use the same notation $\mathbb{E}_x[F^+_{X_n}(\lambda,\mu,\nu)]$ for $\Re(\lambda)>0$. We emphasize that the rhs of (4.3) involves only **one** observation of the pseudo-processus X (while the lhs involves **several** discrete observations). This important feature of Spitzer's identity entails the convergence of the series in (4.2) with a lighter constraint on the domain of validity for λ.

• **Step 3**

In order to prove that the functional $F^+_X(\lambda,\mu,\nu)$ is admissible, we show that the series $\sum\mathbb{E}_x\big[e^{-\lambda t_{n,k}+i\mu X_{n,k}-\nu M_{n,k}}\big]$ is absolutely convergent for $\Re(\lambda)>0$. For this, we invoke a lemma of Bohr concerning Dirichlet series ([5]). Let $\sum\alpha_k e^{-\beta_k\lambda}$ be a Dirichlet series of the complex variable λ, where $(\alpha_k)_{k\in\mathbb{N}}$ is a sequence of complex numbers and $(\beta_k)_{k\in\mathbb{N}}$ is an increasing sequence of positive numbers tending to infinity. Let us denote σ_c its abscissa of convergence, σ_a its abscissa of absolute convergence and σ_b the abscissa of boundedness of the analytical continuation of its sum. If the condition $\limsup_{k\to\infty} \ln(k)/\beta_k = 0$ is fulfilled, then $\sigma_c = \sigma_a = \sigma_b$.

In our situation, we will show that the function of the variable λ in the rhs in (4.3) is bounded on each half-plane $\Re(\lambda) \geqslant \varepsilon$ for any $\varepsilon > 0$. We write it as

$$\begin{aligned}\exp\left[\sum_{k=1}^{\infty} \psi^+(\mu,\nu;t_{n,k})\,\frac{e^{-\lambda t_{n,k}}}{k}\right] &= \exp\left[\sum_{k=1}^{\infty} \frac{e^{-\lambda t_{n,k}}}{k}\,\mathbb{E}_0\Big[\big(e^{i\mu X_{n,k}}-1\big)\,\mathbb{1}_{\{X_{n,k}<0\}}\Big]\right]\\ &\quad\times\exp\left[\sum_{k=1}^{\infty} \frac{e^{-\lambda t_{n,k}}}{k}\,\mathbb{E}_0\Big[\Big(e^{(i\mu-\nu)X_{n,k}}-1\Big)\,\mathbb{1}_{\{X_{n,k}\geqslant 0\}}\Big]\right].\end{aligned}$$

For any $\alpha\in\mathbb{C}$ such that $\Re(\alpha)\leqslant 0$, we have

$$\begin{aligned}\left|\mathbb{E}_0\Big[\Big(e^{\alpha X(t)}-1\Big)\,\mathbb{1}_{\{X(t)\geqslant 0\}}\Big]\right| &= \left|\mathbb{E}_0\Big[\Big(e^{\alpha t^{1/N}X(1)}-1\Big)\,\mathbb{1}_{\{X(1)\geqslant 0\}}\Big]\right|\\ &\leqslant \int_0^{+\infty}\left|1-e^{\alpha t^{1/N}\xi}\right|\times|p(1;-\xi)|\,d\xi\\ &\leqslant 2\varrho|\alpha|t^{1/N}\end{aligned}$$

where we set $\varrho=\int_0^{+\infty}\xi\,|p(1;-\xi)|\,d\xi$ ($\varrho<+\infty$) and we used the elementary inequality $|e^{\zeta}-1|\leqslant 2|\zeta|$ which holds for any $\zeta\in\mathbb{C}$ such that $\Re(\zeta)\leqslant 0$. Similarly,

$$\left|\mathbb{E}_0\Big[\Big(e^{\alpha X(t)}-1\Big)\,\mathbb{1}_{\{X(t)<0\}}\Big]\right|\leqslant 2\varrho|\alpha|t^{1/N}.$$

Therefore,

$$\begin{aligned}&\left|\sum_{k=1}^{\infty}\frac{e^{-\lambda t_{n,k}}}{k}\,\mathbb{E}_0\Big[\Big(e^{(\alpha X_{n,k}}-1\Big)\,\mathbb{1}_{\{X_{n,k}\geqslant 0\ (\text{or}\ <0)\}}\Big]\right|\\ &\qquad\leqslant 2\varrho|\alpha|\sum_{k=1}^{\infty}\frac{e^{-\Re(\lambda)t_{n,k}}}{k}\,t_{n,k}^{1/N}=\frac{2\varrho|\alpha|}{2^n}\sum_{k=1}^{\infty}\frac{e^{-\Re(\lambda)t_{n,k}}}{t_{n,k}^{1-1/N}}\\ &\qquad\leqslant 2\varrho|\alpha|\sum_{k=1}^{\infty}\int_{t_{n,k}}^{t_{n,k+1}}\frac{e^{-\Re(\lambda)t}}{t^{1-1/N}}\,dt\leqslant 2\varrho|\alpha|\int_0^{+\infty}\frac{e^{-\Re(\lambda)t}}{t^{1-1/N}}\,dt\\ &\qquad\leqslant \frac{2\,\Gamma(1/N)\varrho|\alpha|}{\Re(\lambda)^{1/N}}.\end{aligned}\tag{4.5}$$

This proves that the rhs of (4.3) is bounded on each half-plane $\Re(\lambda)\geqslant\varepsilon$ for any $\varepsilon>0$. So, the convergence of the series in (4.2) holds in the domain $\Re(\lambda)>0$ and the functional $F_X^+(\lambda,\mu,\nu)$ is admissible.

• **Step 4**

Now, we can pass to the limit when $n\to+\infty$ in (4.3) and we obtain

$$\mathbb{E}_x[F_X^+(\lambda,\mu,\nu)]=\frac{1}{\lambda}\,e^{(i\mu-\nu)x}\,\exp\left[\int_0^{+\infty}e^{-\lambda t}\,\psi^+(\mu,\nu;t)\,\frac{dt}{t}\right]\quad\text{for }\Re(\lambda)>0.\tag{4.6}$$

A similar formula holds for F_X^-.

From (4.4), we see that we need to evaluate integrals of the form

$$\int_0^{+\infty} e^{-\lambda t}\,\frac{dt}{t}\int_0^{+\infty}(e^{\alpha\xi}-1)p(t;-\xi)\,d\xi \quad \text{for } \Re(\alpha)\leqslant 0$$

and

$$\int_0^{+\infty} e^{-\lambda t}\,\frac{dt}{t}\int_{-\infty}^{0}(e^{\alpha\xi}-1)p(t;-\xi)\,d\xi \quad \text{for } \Re(\alpha)\geqslant 0.$$

We have, for $\Re(\alpha)\leqslant 0$,

$$\begin{aligned}
&\int_0^{+\infty} e^{-\lambda t}\,\frac{dt}{t}\int_0^{+\infty}(e^{\alpha\xi}-1)p(t;-\xi)\,d\xi\\
&\quad= \int_0^{+\infty} dt\int_\lambda^{+\infty} e^{-ts}\,ds\int_0^{+\infty}(e^{\alpha\xi}-1)p(t;-\xi)\,d\xi\\
&\quad= \int_\lambda^{+\infty} ds\int_0^{+\infty}(e^{\alpha\xi}-1)\,d\xi\int_0^{+\infty} e^{-ts}\,p(t;-\xi)\,dt\\
&\quad= \int_{\sqrt[N]{\lambda}}^{+\infty} d\sigma\int_0^{+\infty}(e^{\alpha\xi}-1)\left(\sum_{j\in J}\theta_j\,e^{-\theta_j\sigma\xi}\right)d\xi \qquad (\text{by putting } \sigma=\sqrt[N]{s})\\
&\quad= \sum_{j\in J}\int_{\sqrt[N]{\lambda}}^{+\infty} d\sigma\left[\theta_j\int_0^{+\infty}\left(e^{-(\theta_j\sigma-\alpha)\xi}-e^{-\theta_j\sigma\xi}\right)d\xi\right]\\
&\quad= \sum_{j\in J}\int_{\sqrt[N]{\lambda}}^{+\infty}\left(\frac{\theta_j}{\theta_j\sigma-\alpha}-\frac{1}{\sigma}\right)d\sigma \;=\; \sum_{j\in J}\log\frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda}-\alpha\theta_j}.
\end{aligned}\tag{4.7}$$

In the last step, we used the fact that the set $\{\theta_j, j\in J\}$ is invariant by conjugating.

In the same way, for $\Re(\alpha)\geqslant 0$,

$$\int_0^{+\infty} e^{-\lambda t}\,\frac{dt}{t}\int_{-\infty}^{0}(e^{\alpha\xi}-1)p(t;-\xi)\,d\xi = \sum_{k\in K}\frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda}-\alpha\theta_k}.\tag{4.8}$$

Consequently, by choosing $\alpha=i\mu$ in (4.7) and $\alpha=i\mu-\nu$ in (4.8), and using (2.1), it comes from (4.4):

$$\exp\left[\int_0^{+\infty} e^{-\lambda t}\,\psi^+(\mu,\nu;t)\,\frac{dt}{t}\right] = \frac{\lambda}{\prod_{j\in J}(\sqrt[N]{\lambda}-(i\mu-\nu)\theta_j)\prod_{k\in K}(\sqrt[N]{\lambda}-i\mu\theta_k)}.$$

From this and (4.6), we derive the Laplace-Fourier transform of the vector $(X(t),M(t))$. In a similar manner, we can obtain that of $(X(t),m(t))$. The proof of Theorem 4.1 is now completed. ■

Remark 4.3 The formulas (4.1) can be deduced from each other by using a symmetry argument.

- For even integers N, as assumed at the beginning of the section, the obvious symmetry property $X \stackrel{dist}{=} -X$ holds and entails

$$\begin{aligned}\mathbb{E}_0\Big[e^{i\mu X(t)+\nu \min_{0\leqslant s\leqslant t} X(s)}\Big] &= \mathbb{E}_0\Big[e^{-i\mu X(t)+\nu \min_{0\leqslant s\leqslant t}(-X(s))}\Big]\\ &= \mathbb{E}_0\Big[e^{-i\mu X(t)-\nu \max_{0\leqslant s\leqslant t} X(s)}\Big].\end{aligned}$$

Observing that in this case $\{\theta_k, k \in K\} = \{-\theta_j, j \in J\}$, we have

$$\prod_{j\in J} \frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda} - i\mu\theta_j} = \prod_{k\in K} \frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda} + i\mu\theta_k}$$

and

$$\prod_{j\in J} \frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda} + (i\mu+\nu)\theta_j} = \prod_{k\in K} \frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda} - (i\mu+\nu)\theta_k},$$

which confirms the simple relationship between both expectations (4.1).

- If N is odd, although this case is not recovered by (4.1), it is interesting to note formally at least the asymmetry property $X^+ \stackrel{dist}{=} -X^-$ and $X^- \stackrel{dist}{=} -X^+$ where X^+ and X^- are the pseudo-processes respectively associated with $\kappa_N = +1$ and $\kappa_N = -1$. This would give

$$\begin{aligned}\mathbb{E}_0\Big[e^{i\mu X^+(t)+\nu \min_{0\leqslant s\leqslant t} X^+(s)}\Big] &= \mathbb{E}_0\Big[e^{-i\mu X^-(t)+\nu \min_{0\leqslant s\leqslant t}(-X^-(s))}\Big]\\ &= \mathbb{E}_0\Big[e^{-i\mu X^-(t)-\nu \max_{0\leqslant s\leqslant t} X^-(s)}\Big].\end{aligned}$$

Observing that now, with similar notations, $\{\theta_j^+, j \in J^+\} = \{-\theta_k^-, k \in K^-\}$ and $\{\theta_k^+, k \in K^+\} = \{-\theta_j^-, j \in J^-\}$, the following relations hold:

$$\prod_{j\in J^+} \frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda} - i\mu\theta_j^+} = \prod_{k\in K^-} \frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda} + i\mu\theta_k^-}$$

and

$$\prod_{j\in J^-} \frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda} + (i\mu+\nu)\theta_j^-} = \prod_{k\in K^+} \frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda} - (i\mu+\nu)\theta_k^+}.$$

Hence $(X^+(t), m^+(t))$ and $(X^-(t), -M^-(t))$ should have identical distributions, which would explain the relationship between both expectations (4.1) in this case.

■

Remark 4.4 By choosing $\nu = 0$ in (4.1), we obtain the Fourier transform of the λ-potential of the kernel p. In fact, remarking that

$$\prod_{j\in J}(\sqrt[N]{\lambda} - i\mu\theta_j) \prod_{k\in K}(\sqrt[N]{\lambda} - i\mu\theta_k) = \prod_{l=0}^{N-1}(\sqrt[N]{\lambda} - i\mu\theta_l) = \lambda - \kappa_N (i\mu)^N,$$

(4.1) yields

$$\mathbb{E}_x\left[\int_0^{+\infty} e^{-\lambda t+i\mu X(t)}\,dt\right]=\frac{e^{i\mu x}}{\lambda-\kappa_N(i\mu)^N}$$

which can be directly checked according as

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_x\left[e^{i\mu X(t)}\right]dt=\int_0^{+\infty} e^{i\mu x-\left(\lambda-\kappa_N(i\mu)^N\right)t}\,dt.$$

■

4.2 Density functions

We are able to invert the Laplace-Fourier transforms (4.1) with respect to μ and ν.

4.2.1 Inverting with respect to ν

Proposition 4.5 *We have, for $z \geqslant x$,*

$$\int_0^{+\infty} e^{-\lambda t}\,dt\,\mathbb{E}_x\left[e^{i\mu X(t)}, M(t)\in dz\right]/dz \;=\; \frac{\lambda^{(1-\#J)/N}\,e^{i\mu x}}{\prod\limits_{k\in K}(\sqrt[N]{\lambda}-i\mu\theta_k)}\sum_{j\in J}\theta_j A_j\,e^{(i\mu-\theta_j\sqrt[N]{\lambda})(z-x)},$$

and, for $z \leqslant x$,

$$\int_0^{+\infty} e^{-\lambda t}\,dt\,\mathbb{E}_x\left[e^{i\mu X(t)}, m(t)\in dz\right]/dz \;=\; -\frac{\lambda^{(1-\#K)/N}\,e^{i\mu x}}{\prod\limits_{j\in J}(\sqrt[N]{\lambda}-i\mu\theta_j)}\sum_{k\in K}\theta_k B_k\,e^{(i\mu-\theta_k\sqrt[N]{\lambda})(z-x)}. \tag{4.9}$$

Proof. Observing that $\{\theta_j, j\in J\}=\{\bar\theta_j, j\in J\}=\{1/\theta_j, j\in J\}$, we have

$$\frac{1}{\prod\limits_{j\in J}(\sqrt[N]{\lambda}-(i\mu-\nu)\theta_j)}=\frac{1}{\prod\limits_{j\in J}\left(\sqrt[N]{\lambda}-\dfrac{i\mu-\nu}{\theta_j}\right)}=\frac{\lambda^{-\#J/N}}{\prod\limits_{j\in J}\left(1-\dfrac{i\mu-\nu}{\theta_j\sqrt[N]{\lambda}}\right)}.$$

Applying expansion (2.11) to $x=(i\mu-\nu)/\sqrt[N]{\lambda}$ yields:

$$\frac{1}{\prod\limits_{j\in J}(\sqrt[N]{\lambda}-(i\mu-\nu)\theta_j)}=\lambda^{-\#J/N}\sum_{j\in J}\frac{A_j}{1-\frac{i\mu-\nu}{\theta_j\sqrt[N]{\lambda}}}=\lambda^{(1-\#J)/N}\sum_{j\in J}\frac{\theta_j A_j}{\nu-i\mu+\theta_j\sqrt[N]{\lambda}}. \tag{4.10}$$

Writing now

$$\frac{e^{-\nu x}}{\nu-i\mu+\theta_j\sqrt[N]{\lambda}}=\int_x^{+\infty} e^{-\nu z}\,e^{(i\mu-\theta_j\sqrt[N]{\lambda})(z-x)}\,dz,$$

we find that

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_x\left[e^{i\mu X(t)-\nu M(t)}\right]dt$$

$$=\frac{\lambda^{(1-\#J)/N}\,e^{i\mu x}}{\prod\limits_{k\in K}(\sqrt[N]{\lambda}-i\mu\theta_k)}\int_x^{+\infty} e^{-\nu z}\left(\sum_{j\in J}\theta_j A_j\,e^{(i\mu-\theta_j\sqrt[N]{\lambda})(x-z)}\right)dz.$$

We can therefore invert the foregoing Laplace transform with respect to ν and we get the formula (4.9) corresponding to the case of the maximum functional. That corresponding to the case of the minimum functional is obtained is a similar way. ■

Formulas (4.9) will be used further when determining the distributions of $(\tau_a^+, X(\tau_a^+))$ and $(\tau_a^-, X(\tau_a^-))$.

4.2.2 Inverting with respect to μ

Theorem 4.6 *The Laplace transforms with respect to time t of the joint density of $X(t)$ and, respectively, $M(t)$ and $m(t)$, are given, for $z \geqslant x \vee y$, by*

$$\int_0^{+\infty} e^{-\lambda t}\, dt\, \mathbb{P}_x\{X(t) \in dy, M(t) \in dz\}/dy\, dz = \frac{1}{\lambda}\, \varphi_\lambda(x-z)\, \psi_\lambda(z-y),$$

and, for $z \leqslant x \wedge y$, (4.11)

$$\int_0^{+\infty} e^{-\lambda t}\, dt\, \mathbb{P}_x\{X(t) \in dy, m(t) \in dz\}/dy\, dz = \frac{1}{\lambda}\, \psi_\lambda(x-z)\, \varphi_\lambda(z-y),$$

where the functions φ_λ and ψ_λ are defined by (2.6).

PROOF. Let us write the following equality, as in the previous subsubsection (see (4.10)):

$$\frac{1}{\prod\limits_{K \in K} (\sqrt[N]{\lambda} - i\mu\theta_k)} = -\lambda^{(1-\#K)/N} \sum_{k \in K} \frac{\theta_k B_k}{i\mu - \theta_k \sqrt[N]{\lambda}}.$$

Set

$$G(\lambda, \mu; x, z) = \int_0^{+\infty} e^{-\lambda t}\, dt\, \mathbb{E}_x\Big[e^{i\mu X(t)}, M(t) \in dz\Big]\, /dz$$

We get, by (4.9) and (2.1), for $z \geqslant x$,

$$\begin{aligned} G(\lambda, \mu; x, z) &= \frac{\lambda^{(1-\#J)/N} e^{i\mu x}}{\prod\limits_{k \in K} (\sqrt[N]{\lambda} - i\mu\theta_k)} \sum_{j \in J} \theta_j A_j\, e^{(i\mu - \theta_j \sqrt[N]{\lambda})(z-x)} \\ &= -\lambda^{(2-\#J-\#K)/N} e^{i\mu x} \sum_{k \in K} \frac{\theta_k B_k}{i\mu - \theta_k \sqrt[N]{\lambda}} \sum_{j \in J} \theta_j A_j\, e^{(i\mu - \theta_j \sqrt[N]{\lambda})(z-x)} \\ &= -\lambda^{2/N-1} \sum_{j \in J, k \in K} e^{\theta_j \sqrt[N]{\lambda}\, x} \theta_j A_j\, \theta_k B_k\, \frac{e^{(i\mu - \theta_j \sqrt[N]{\lambda})\, z}}{i\mu - \theta_k \sqrt[N]{\lambda}}. \end{aligned}$$

Writing now

$$\frac{e^{(i\mu - \theta_j \sqrt[N]{\lambda})\, z}}{i\mu - \theta_k \sqrt[N]{\lambda}} = e^{(\theta_k - \theta_j) \sqrt[N]{\lambda}\, z} \int_{-\infty}^{z} e^{(i\mu - \theta_k \sqrt[N]{\lambda})\, y}\, dy$$

gives

$$G(\lambda, \mu; x, z) = -\lambda^{2/N-1} \mathbb{1}_{\{z \geqslant x\}} \int_{-\infty}^{z} e^{i\mu y} \left[\sum_{j \in J, k \in K} \theta_j A_j\, \theta_k B_k\, e^{\sqrt[N]{\lambda}\, (\theta_j x - \theta_k y + (\theta_k - \theta_j) z)} \right] dy$$

and then

$$\int_0^{+\infty} e^{-\lambda t}\,dt\,\mathbb{P}_x\{X(t)\in dy, M(t)\in dz\}/dy\,dz$$

$$= \ -\lambda^{2/N-1}\sum_{j\in J,k\in K}\theta_j A_j\,\theta_k B_k\,e^{\sqrt[N]{\lambda}\,(\theta_j x-\theta_k y+(\theta_k-\theta_j)z)}\,\mathbb{1}_{\{z\geqslant x\vee y\}}.$$

This proves (4.11) in the case of the maximum functional and the formula corresponding to the minimum functional can be proved in a same manner. ■

Remark 4.7 Formulas (4.11) contain in particular the Laplace transforms of $X(t)$, $M(t)$ and $m(t)$ separately. As a verification, we integrate (4.11) with respect to y and z separately.

- By integrating with respect to y on $[z,+\infty)$ for $z\leqslant x$, we get

$$\int_0^{+\infty} e^{-\lambda t}\,dt\,\mathbb{P}_x\{m(t)\in dz\}/dz$$

$$\begin{aligned}
&= \ -\lambda^{2/N-1}\sum_{j\in J}\theta_j A_j\int_z^{+\infty} e^{-\theta_j\sqrt[N]{\lambda}(y-z)}\,dy\sum_{k\in K}\theta_k B_k\,e^{\theta_k\sqrt[N]{\lambda}(x-z)}\\
&= \ -\lambda^{1/N-1}\sum_{j\in J}A_j\sum_{k\in K}\theta_k B_k\,e^{\theta_k\sqrt[N]{\lambda}(x-z)}\\
&= \ -\lambda^{1/N-1}\sum_{k\in K}\theta_k B_k\,e^{\theta_k\sqrt[N]{\lambda}(x-z)} \ = \ \frac{1}{\lambda}\,\psi_\lambda(x-z).
\end{aligned}$$

  We used the relation $\sum_{j\in J}A_j=1$; see Subsection 2.3. We retrieve the Laplace transform (2.5) of the distribution of $m(t)$.

- Suppose for instance that $x\leqslant y$. Let us integrate (4.11) now with respect to z on $(-\infty,x]$. This gives

$$\int_0^{+\infty} e^{-\lambda t}\,dt\,\mathbb{P}_x\{X(t)\in dy\}/dy$$

$$\begin{aligned}
&= \ -\lambda^{2/N-1}\sum_{j\in J,k\in K}\theta_j A_j\,\theta_k B_k\,e^{\theta_k\sqrt[N]{\lambda}\,x-\theta_j\sqrt[N]{\lambda}\,y}\int_{-\infty}^{x} e^{(\theta_j-\theta_k)\sqrt[N]{\lambda}\,z}\,dz\\
&= \ \lambda^{1/N-1}\sum_{j\in J,k\in K}\frac{\theta_j A_j\,\theta_k B_k}{\theta_k-\theta_j}\,e^{\theta_j\sqrt[N]{\lambda}(x-y)}\\
&= \ \lambda^{1/N-1}\sum_{j\in J}\left(\sum_{k\in K}\frac{\theta_k B_k}{\theta_k-\theta_j}\right)\theta_j A_j\,e^{\theta_j\sqrt[N]{\lambda}(x-y)}\\
&= \ \frac{1}{N}\,\lambda^{1/N-1}\sum_{j\in J}\theta_j\,e^{\theta_j\sqrt[N]{\lambda}(x-y)},
\end{aligned}$$

  where we used (2.10) in the last step. We retrieve the λ-potential (2.3) of the pseudo-process $(X(t))_{t\geqslant 0}$ since

$$\int_0^{+\infty} e^{-\lambda t}\,dt\,\mathbb{P}_x\{X(t)\in dy\}/dy=\int_0^{+\infty} e^{-\lambda t}\,p(t;x-y)\,dt.$$

■

Remark 4.8 Consider the reflected process at its maximum $(M(t)-X(t))_{t\geqslant 0}$. The joint distribution of $(M(t),M(t)-X(t))$ writes in terms of the joint distribution of $(X(t),M(t))$, for $x=0$ (set $\mathbb{P}=\mathbb{P}_0$ for short) and $z,\zeta\geqslant 0$, as:

$$\mathbb{P}\{M(t)\in dz, M(t)-X(t)\in d\zeta\}=\mathbb{P}\{X(t)\in z-d\zeta, M(t)\in dz\}.$$

Formula (4.11) writes

$$\begin{aligned}\int_0^{+\infty}\lambda\,e^{-\lambda t}\,dt\,\mathbb{P}\{M(t)\in dz, M(t)-X(t)\in d\zeta\}/dz\,d\zeta &= \varphi_\lambda(z)\psi_\lambda(-\zeta)\\ &= \int_0^{+\infty}\lambda\,e^{-\lambda t}\,dt\,\mathbb{P}\{M(t)\in dz\}/dz\times\int_0^{+\infty}\lambda\,e^{-\lambda t}\,dt\,\mathbb{P}\{-m(t)\in d\zeta\}/d\zeta. \qquad (4.12)\end{aligned}$$

By introducing an exponentially distributed time T_λ with parameter λ which is independent of $(X(t))_{t\geqslant 0}$, (4.12) reads

$$\mathbb{P}\{M(T_\lambda)\in dz, M(T_\lambda)-X(T_\lambda)\in d\zeta\}=\mathbb{P}\{M(T_\lambda)\in dz\}\,\mathbb{P}\{-m(T_\lambda)\in d\zeta\}.$$

This relationship may be interpreted by saying that $-m(T_\lambda)$ and $M(T_\lambda)-X(T_\lambda)$ admit the same distribution and $M(T_\lambda)$ and $M(T_\lambda)-X(T_\lambda)$ are independent. ■

Remark 4.9 The similarity between both formulas (4.11) may be explained by invoking a "duality" argument. In effect, the dual pseudo-process $(X^*(t))_{t\geqslant 0}$ of $(X(t))_{t\geqslant 0}$ is defined by $X^*(t)=-X(t)$ for all $t\geqslant 0$ and we have the following equality related to the inversion of the extremities (see [11]):

$$\begin{aligned}\mathbb{P}_x\{X(t)\in dy, M(t)\in dz\}/dy\,dz &= \mathbb{P}_y\{X^*(t)\in dx, m^*(t)\in dz\}/dx\,dz\\ &= \mathbb{P}_{-y}\{X(t)\in d(-x), m(t)\in d(-z)\}/dx\,dz.\end{aligned}$$

■

Remark 4.10 Let us expand the function φ_λ as $\lambda\to 0^+$:

$$\begin{aligned}\varphi_\lambda(\xi) &= \sqrt[N]{\lambda}\sum_{j\in J}\theta_j A_j\left[\sum_{l=0}^{\#J-1}\frac{[\theta_j\sqrt[N]{\lambda}\,\xi]^l}{l!}+o\Big(\lambda^{(\#J-1)/N}\Big)\right]\\ &= \sum_{l=0}^{\#J-1}\left(\sum_{j\in J}\theta_j^{l+1}A_j\right)\frac{\lambda^{(l+1)/N}\xi^l}{l!}+o\Big(\lambda^{\#J/N}\Big).\end{aligned}$$

We have by (2.11) (for $x=0$) $\sum_{j\in J}\theta_j^{l+1}A_j=0$ for $0\leqslant l\leqslant \#J-2$ and $\sum_{j\in J}\theta_j^{\#J}A_j=(-1)^{\#J-1}\prod_{j\in J}\theta_j$. Hence, we get the asymptotics (recall that $f\sim g$ means that $\lim(f/g)=1$),

$$\varphi_\lambda(\xi)\underset{\lambda\to 0^+}{\sim}(-1)^{\#J-1}\prod_{j\in J}\theta_j\,\frac{\xi^{\#J-1}}{(\#J-1)!}\,\lambda^{\#J/N}. \qquad (4.13)$$

Similarly

$$\psi_\lambda(\xi) \underset{\lambda\to 0^+}{\sim} (-1)^{\#K} \prod_{k\in K} \theta_k \, \frac{\xi^{\#K-1}}{(\#K-1)!} \, \lambda^{\#K/N}. \tag{4.14}$$

As a result, putting (4.13) and (4.14) into (4.11) and using (2.1) and $\prod_{l=0}^{N-1}\theta_l = (-1)^{N-1}\kappa_N$ lead to

$$\int_0^{+\infty} e^{-\lambda t}\, dt \, \mathbb{P}_x\{X(t)\in dy, M(t)\in dz\}/dy\, dz \underset{\lambda\to 0^+}{\sim} \kappa_N \, \frac{(x-z)^{\#J-1}(z-y)^{\#K-1}}{(\#J-1)!\,(\#K-1)!}.$$

By integrating this asymptotic with respect to z, we derive the value of the so-called 0-potential of the absorbed pseudo-process (see [18] for the definition of several kinds of absorbed or killed pseudo-processes):

$$\int_0^{+\infty} \mathbb{P}_x\{X(t)\in dy, M(t)\leqslant a\}/dy = (-1)^{\#J-1}\kappa_N \int_{x\vee y}^{a} \frac{(z-x)^{\#J-1}(z-y)^{\#K-1}}{(\#J-1)!\,(\#K-1)!}\, dz.$$

■

4.2.3 Inverting with respect to λ

Formulas (4.11) may be inverted with respect to λ and an expression by means of the successive derivatives of the kernel p may be obtained for the densities of $(X(t), M(t))$ and $(X(t), m(t))$. However, the computations and the results are cumbersome and we prefer to perform them in the case of the distribution functions. They are exhibited in Subsection 4.4.

4.3 Density functions: particular cases

In this subsection, we pay attention to the cases $N\in\{2,3,4\}$. Although our results are not justified when N is odd, we nevertheless retrieve well-known results in the literature related to the case $N=3$. In order to lighten the notations, we set for, $\Re(\lambda)>0$,

$$\begin{aligned}
\Phi_\lambda(x,y,z) &= \int_0^{+\infty} e^{-\lambda t}\, dt\, \mathbb{P}_x\{X(t)\in dy, M(t)\in dz\}/dy\, dz,\\
\Psi_\lambda(x,y,z) &= \int_0^{+\infty} e^{-\lambda t}\, dt\, \mathbb{P}_x\{X(t)\in dy, m(t)\in dz\}/dy\, dz.
\end{aligned}$$

Example 4.11 *Case $N=2$:* using the numerical results of Example 2.1 gives

$$\varphi_\lambda(\xi) = \sqrt{\lambda}\, e^{\sqrt{\lambda}\,\xi} \quad \text{and} \quad \psi_\lambda(\xi) = \sqrt{\lambda}\, e^{-\sqrt{\lambda}\,\xi},$$

and then

$$\Phi_\lambda(x,y,z) = e^{\sqrt{\lambda}(x+y-2z)}\, \mathbb{1}_{\{z\geqslant x\vee y\}} \quad \text{and} \quad \Psi_\lambda(x,y,z) = e^{\sqrt{\lambda}(2z-x-y)}\, \mathbb{1}_{\{z\leqslant x\wedge y\}}.$$

■

Example 4.12 *Case $N=3$:* referring to Example 2.2, we have

- for $\kappa_3 = +1$:

$$\begin{aligned}
\varphi_\lambda(\xi) &= \sqrt[3]{\lambda}\, e^{\sqrt[3]{\lambda}\,\xi},\\
\psi_\lambda(\xi) &= -\frac{i\sqrt[3]{\lambda}}{\sqrt{3}}\left(e^{e^{i\,2\pi/3}\sqrt[3]{\lambda}\,\xi} - e^{e^{-i\,2\pi/3}\sqrt[3]{\lambda}\,\xi}\right) = \frac{2\sqrt[3]{\lambda}}{\sqrt{3}}\, e^{-\frac{\sqrt[3]{\lambda}}{2}\,\xi}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\,\xi\right),
\end{aligned}$$

which gives

$$\begin{aligned}
\Phi_\lambda(x,y,z) &= \frac{2}{\sqrt{3}\sqrt[3]{\lambda}}\, e^{\sqrt[3]{\lambda}(x+\frac{1}{2}\,y-\frac{3}{2}\,z)}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(z-y)\right)1\!\!1_{\{z\geqslant x\vee y\}},\\
\Psi_\lambda(x,y,z) &= \frac{2}{\sqrt{3}\sqrt[3]{\lambda}}\, e^{\sqrt[3]{\lambda}(\frac{3}{2}\,z-\frac{1}{2}\,x-y)}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(x-z)\right)1\!\!1_{\{z\leqslant x\wedge y\}};
\end{aligned}$$

- for $\kappa_3 = -1$,

$$\begin{aligned}
\varphi_\lambda(\xi) &= \frac{i\sqrt[3]{\lambda}}{\sqrt{3}}\left(e^{e^{i\,\pi/3}\sqrt[3]{\lambda}\,\xi} - e^{e^{-i\,\pi/3}\sqrt[3]{\lambda}\,\xi}\right) = -\frac{2\sqrt[3]{\lambda}}{\sqrt{3}}\, e^{\frac{\sqrt[3]{\lambda}}{2}\,\xi}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\,\xi\right),\\
\psi_\lambda(\xi) &= \sqrt[3]{\lambda}\, e^{-\sqrt[3]{\lambda}\,\xi},
\end{aligned}$$

which gives

$$\begin{aligned}
\Phi_\lambda(x,y,z) &= \frac{2}{\sqrt{3}\sqrt[3]{\lambda}}\, e^{\sqrt[3]{\lambda}(\frac{1}{2}\,x+y-\frac{3}{2}\,z)}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(z-x)\right)1\!\!1_{\{z\geqslant x\vee y\}},\\
\Psi_\lambda(x,y,z) &= \frac{2}{\sqrt{3}\sqrt[3]{\lambda}}\, e^{\sqrt[3]{\lambda}(\frac{3}{2}\,z-x-\frac{1}{2}\,y)}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(y-z)\right)1\!\!1_{\{z\leqslant x\wedge y\}}.
\end{aligned}$$

■

Example 4.13 *Case $N = 4$:* the numerical results of Example 2.3 yield

$$\begin{aligned}
\varphi_\lambda(\xi) &= -\frac{i\sqrt[4]{\lambda}}{\sqrt{2}}\left(e^{e^{-i\,\pi/4}\sqrt[4]{\lambda}\,\xi} - e^{e^{i\,\pi/4}\sqrt[4]{\lambda}\,\xi}\right) = -\sqrt{2}\sqrt[4]{\lambda}\, e^{\frac{\sqrt[4]{\lambda}}{\sqrt{2}}\,\xi}\sin\left(\frac{\sqrt[4]{\lambda}}{2}\,\xi\right),\\
\psi_\lambda(\xi) &= \frac{i\sqrt[4]{\lambda}}{\sqrt{2}}\left(e^{e^{-i\,3\pi/4}\sqrt[4]{\lambda}\,\xi} - e^{e^{i\,3\pi/4}\sqrt[4]{\lambda}\,\xi}\right) = \sqrt{2}\sqrt[4]{\lambda}\, e^{-\frac{\sqrt[4]{\lambda}}{\sqrt{2}}\,\xi}\sin\left(\frac{\sqrt[4]{\lambda}}{2}\,\xi\right),
\end{aligned}$$

which gives

$$\begin{aligned}
\Phi_\lambda(x,y,z) &= \frac{1}{\sqrt{\lambda}}\, e^{\frac{\sqrt[4]{\lambda}}{\sqrt{2}}(x+y-2z)}\left[\cos\left(\frac{\sqrt[4]{\lambda}}{\sqrt{2}}(x-y)\right) - \cos\left(\frac{\sqrt[4]{\lambda}}{\sqrt{2}}(x+y-2z)\right)\right]1\!\!1_{\{z\geqslant x\vee y\}},\\
\Psi_\lambda(x,y,z) &= \frac{1}{\sqrt{\lambda}}\, e^{\frac{\sqrt[4]{\lambda}}{\sqrt{2}}(2z-x-y)}\left[\cos\left(\frac{\sqrt[4]{\lambda}}{\sqrt{2}}(x-y)\right) - \cos\left(\frac{\sqrt[4]{\lambda}}{\sqrt{2}}(x+y-2z)\right)\right]1\!\!1_{\{z\leqslant x\wedge y\}}.
\end{aligned}$$

■

4.4 Distribution functions

In this part, we integrate (4.11) in view to get the distribution function of the vector $(X(t), M(t))$: $\mathbb{P}_x\{X(t) \leqslant y, M(t) \leqslant z\}$. Obviously, if $x > z$, this quantity vanishes. Suppose now $x \leqslant z$. We must consider the cases $y \leqslant z$ and $z \leqslant y$. In the latter, we have $\mathbb{P}_x\{X(t) \leqslant y, M(t) \leqslant z\} = \mathbb{P}\{M(t) \leqslant z\}$ and this quantity is given by (2.7). So, we assume that $z \geqslant x \vee y$. Actually, the quantity $\mathbb{P}_x\{X(t) \leqslant y, M(t) \geqslant z\}$ is easier to derive.

4.4.1 Laplace transform

Put for $\Re(\lambda) > 0$:

$$\begin{aligned} F_\lambda(x,y,z) &= \int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x\{X(t)\leqslant y, M(t)\leqslant z\}\,dt \\ \tilde{F}_\lambda(x,y,z) &= \int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x\{X(t)\leqslant y, M(t)\geqslant z\}\,dt. \end{aligned}$$

The functions F_λ and $\tilde{F}_\lambda$ are related together through

$$F_\lambda(x,y,z) + \tilde{F}_\lambda(x,y,z) = \Psi(\lambda; x-y) \tag{4.15}$$

where Ψ is given by (2.4). Using (4.11), we get

$$\begin{aligned} &\tilde{F}_\lambda(x,y,z) \\ &= \int_{-\infty}^{y}\int_{z}^{+\infty}\int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x\{X(t)\in d\xi, M(t)\in d\zeta\}\,dt \\ &= -\lambda^{2/N-1}\sum_{j\in J,k\in K}\theta_j A_j\,\theta_k B_k\,e^{\theta_j\sqrt[N]{\lambda}\,x}\int_{-\infty}^{y} e^{-\theta_k\sqrt[N]{\lambda}\,\xi}\,d\xi\int_z^{+\infty} e^{(\theta_k-\theta_j)\sqrt[N]{\lambda}\,\zeta}\,\mathbb{1}_{\{\zeta\geqslant x\vee\xi\}}\,d\zeta. \end{aligned}$$

We plainly have $\zeta\geqslant z\geqslant x\vee y\geqslant x\vee\xi$ over the integration set $(-\infty,y]\times[z,+\infty)$. So, the indicator $\mathbb{1}_{\{\zeta\geqslant x\vee\xi\}}$ is useless and we obtain the following expression for $\tilde{F}_\lambda$.

Proposition 4.14 *We have for $z\geqslant x\vee y$ and $\Re(\lambda)>0$:*

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x\{X(t)\leqslant y\leqslant z\leqslant M(t)\}\,dt = \frac{1}{\lambda}\sum_{j\in J,k\in K}\frac{\theta_j A_j B_k}{\theta_j-\theta_k}\,e^{\theta_j\sqrt[N]{\lambda}(x-z)+\theta_k\sqrt[N]{\lambda}(z-y)}$$

and for $z\leqslant x\wedge y$:

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x\{X(t)\geqslant y\geqslant z\geqslant m(t)\}\,dt = \frac{1}{\lambda}\sum_{j\in J,k\in K}\frac{A_j\theta_k B_k}{\theta_k-\theta_j}\,e^{\theta_j\sqrt[N]{\lambda}(z-y)+\theta_k\sqrt[N]{\lambda}(x-z)}.$$

As a result, combining the above formulas with (4.15), the distribution function of the couple $(X(t),M(t))$ emerges and that of $(X(t),m(t))$ is obtained in a similar way.

Theorem 4.15 *The distribution functions of $(X(t),M(t))$ and $(X(t),m(t))$ are respectively determined through their Laplace transforms with respect to t by*

$$\begin{aligned} &\int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x\{X(t)\leqslant y, M(t)\leqslant z\}\,dt \\ &= \begin{cases} \dfrac{1}{\lambda}\displaystyle\sum_{j\in J,k\in K}\frac{\theta_j A_j B_k}{\theta_k-\theta_j}\,e^{\theta_j\sqrt[N]{\lambda}(x-z)+\theta_k\sqrt[N]{\lambda}(z-y)} + \frac{1}{N\lambda}\sum_{k\in K} e^{\theta_k\sqrt[N]{\lambda}(x-y)} & \text{if } y\leqslant x\leqslant z, \\ \dfrac{1}{\lambda}\left[1-\dfrac{1}{N}\displaystyle\sum_{j\in J} e^{\theta_j\sqrt[N]{\lambda}(x-y)} + \sum_{j\in J,k\in K}\frac{\theta_j A_j B_k}{\theta_k-\theta_j}\,e^{\theta_j\sqrt[N]{\lambda}(x-z)+\theta_k\sqrt[N]{\lambda}(z-y)}\right] & \text{if } x\leqslant y\leqslant z, \end{cases} \end{aligned} \tag{4.16}$$

and

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x\{X(t)\geqslant y, m(t)\geqslant z\}\,dt$$

$$= \begin{cases} \dfrac{1}{\lambda}\displaystyle\sum_{j\in J,k\in K}\frac{A_j\theta_k B_k}{\theta_j-\theta_k}\,e^{\theta_j\sqrt[N]{\lambda}(x-z)+\theta_k\sqrt[N]{\lambda}(z-y)}+\frac{1}{N\lambda}\sum_{j\in J}e^{\theta_j\sqrt[N]{\lambda}(x-y)} & \textit{if } z\leqslant x\leqslant y,\\[2ex] \dfrac{1}{\lambda}\left[1-\dfrac{1}{N}\displaystyle\sum_{k\in K}e^{\theta_k\sqrt[N]{\lambda}(x-y)}+\sum_{j\in J,k\in K}\frac{A_j\theta_k B_k}{\theta_j-\theta_k}\,e^{\theta_j\sqrt[N]{\lambda}(x-z)+\theta_k\sqrt[N]{\lambda}(z-y)}\right] & \textit{if } z\leqslant y\leqslant x. \end{cases}$$

4.4.2 Inverting the Laplace transform

Theorem 4.16 *The distribution function of $(X(t),M(t))$ admits the following representation:*

$$\mathbb{P}_x\{X(t)\leqslant y\leqslant z\leqslant M(t)\}=\sum_{\substack{k\in K\\0\leqslant m\leqslant \#J-1}}a_{km}\int_0^t\!\!\int_0^s\frac{\partial^m p}{\partial x^m}(\sigma;x-z)\,\frac{I_{k0}(s-\sigma;z-y)}{(t-s)^{1-(m+1)/N}}\,ds\,d\sigma \tag{4.17}$$

where I_{k0} is given by (5.14) and

$$a_{km}=\frac{NB_k}{\Gamma(\frac{m+1}{N})}\sum_{j\in J}\frac{A_j\alpha_{jm}}{\theta_j-\theta_k},$$

the α_{jm}'s being some coefficients given by (4.18) below.

Proof. We intend to invert the Laplace transform (4.16). For this, we interpret both exponentials $e^{\theta_j\sqrt[N]{\lambda}(x-z)}$ and $e^{\theta_k\sqrt[N]{\lambda}(z-y)}$ as Laplace transforms in two different manners: one is the Laplace transform of a combination of the successive derivatives of the kernel p, the other one is the Laplace transform of a function which is closely related to the density of some stable distribution. More explicitly, we proceed as follows.

- On one hand, we start from the λ-potential (2.3) that we shall call Φ:

  $$\Phi(\lambda;\xi)=\frac{1}{N\lambda^{1-1/N}}\sum_{j\in J}\theta_j e^{\theta_j\sqrt[N]{\lambda}\,\xi}\quad\text{for } \xi\leqslant 0.$$

  Differentiating this potential $(\#J-1)$ times with respect to ξ leads to the Vandermonde system of $\#J$ equations where the exponentials $e^{\theta_j\sqrt[N]{\lambda}\,\xi}$ are unknown:

  $$\sum_{j\in J}\theta_j^{l+1}e^{\theta_j\sqrt[N]{\lambda}\,\xi}=N\lambda^{1-(l+1)/N}\,\frac{\partial^l\Phi}{\partial x^l}(\lambda;\xi)\quad\text{for } 0\leqslant l\leqslant \#J-1.$$

  Introducing the solutions α_{jm} of the $\#J$ elementary Vandermonde systems (indexed by m varying from 0 to $\#J-1$):

  $$\sum_{j\in J}\theta_j^l\alpha_{jm}=\delta_{lm},\quad 0\leqslant l\leqslant \#J-1,$$

we extract

$$\begin{aligned}\frac{\theta_j}{\lambda}\,e^{\theta_j\sqrt[N]{\lambda}\,\xi} &= N\sum_{m=0}^{\#J-1}\frac{\alpha_{jm}}{\lambda^{(m+1)/N}}\,\frac{\partial^m\Phi}{\partial x^m}(\lambda;\xi)\\ &= \int_0^{+\infty} e^{-\lambda t}\,dt\int_0^t\sum_{m=0}^{\#J-1}\frac{N\alpha_{jm}}{\Gamma(\frac{m+1}{N})}\frac{\partial^m p}{\partial x^m}(s;\xi)\,\frac{ds}{(t-s)^{1-(m+1)/N}}.\end{aligned}$$

The explicit expression of α_{jm} is

$$\alpha_{jm} = (-1)^m\frac{\sigma_{\#J-1-m}(\theta_l, l\in J\setminus\{j\})}{\prod_{l\in J\setminus\{j\}}(\theta_l-\theta_j)} = \frac{(-1)^m}{\prod_{l\in J}\theta_l}\,c_{j,\#J-1-m}\theta_j A_j \tag{4.18}$$

where the coefficients c_{jq}, $0\leqslant q\leqslant \#J-1$, are the elementary symmetric functions of the θ_l's, $l\in J\setminus\{j\}$, that is $c_{j0}=1$ and for $1\leqslant q\leqslant \#J-1$,

$$c_{jq} = \sigma_q\left(\theta_l, l\in J\setminus\{j\}\right) = \sum_{\substack{l_1,\dots,l_q\in J\setminus\{j\}\\ l_1<\dots<l_q}}\theta_{l_1}\cdots\theta_{l_q}.$$

- On the other hand, using the Bromwich formula, the function $\xi\longmapsto e^{\theta_k\sqrt[N]{\lambda}\,\xi}$ can be written as a Laplace transform. Indeed, referring to Section 5.2.2, we have for $k\in K$ and $\xi\geqslant 0$,

$$e^{\theta_k\sqrt[N]{\lambda}\,\xi} = \int_0^{+\infty} e^{-\lambda t} I_{k0}(t;\xi)\,dt$$

where I_{k0} is given by (5.14).

Consequently, the sum in Proposition 4.14 may be written as a Laplace transform which gives the representation (4.17) for the the distribution function of $(X(t), M(t))$. ■

Remark 4.17 A similar expression obtained by exchanging the roles of the indices j and k in the above discussion and slightly changing the coefficient a_{km} into another b_{jn} may be derived:

$$\mathbb{P}_x\{X(t)\leqslant y\leqslant z\leqslant M(t)\} = \sum_{\substack{j\in J\\ 0\leqslant n\leqslant \#K-1}} b_{jn}\int_0^t\int_0^s\frac{\partial^n p}{\partial x^n}(\sigma;z-y)\,\frac{I_{j0}(s-\sigma;x-z)}{(t-s)^{1-(n+1)/N}}\,ds\,d\sigma \tag{4.19}$$

where

$$b_{jn} = \frac{N\theta_j A_j}{\Gamma(\frac{n+1}{N})}\sum_{k\in K}\frac{\bar{\theta}_k B_k\beta_{kn}}{\theta_k-\theta_j}.$$

However, the foregoing result involves the same number of integrals as that displayed in Theorem 4.16. ■

4.5 Distribution functions: particular cases

Here, we write out (4.16) and (4.17) or (4.19) in the cases $N \in \{2,3,4\}$ with the same remark about the case $N = 3$ already mentioned at the beginning of Subsection 4.3. The expressions are rather simple and remarkable.

Example 4.18 *Case $N = 2$:* the double sum in (4.16) reads

$$\sum_{j \in J, k \in K} \frac{\theta_j A_j B_k}{\theta_k - \theta_j}\, e^{\theta_j \sqrt[4]{\lambda}(x-z)+\theta_k \sqrt[4]{\lambda}(z-y)} = \frac{\theta_1 A_1 B_0}{\theta_0 - \theta_1}\, e^{\theta_1 \sqrt{\lambda}(x-z)+\theta_0 \sqrt{\lambda}(z-y)}$$

with $\frac{\theta_1 A_1 B_0}{\theta_0 - \theta_1} = -\frac{1}{2}$, and then

$$F_\lambda(x,y,z) \;=\; \begin{cases} \dfrac{1}{2\lambda}\left[e^{-\sqrt{\lambda}(x-y)} - e^{\sqrt{\lambda}(x+y-2z)}\right] & \text{if } y \leqslant x \leqslant z, \\ \dfrac{1}{\lambda} - \dfrac{1}{2\lambda}\left[e^{\sqrt{\lambda}(x-y)} + e^{\sqrt{\lambda}(x+y-2z)}\right] & \text{if } x \leqslant y \leqslant z. \end{cases}$$

Formula (4.17) writes

$$\mathbb{P}_x\{X(t) \leqslant y \leqslant z \leqslant M(t)\} = a_{00} \int_0^t \int_0^s p(\sigma; x-z)\, I_{00}(s-\sigma; z-y)\, \frac{ds\, d\sigma}{\sqrt{t-s}}$$

with

$$p(t;\xi) = \frac{1}{\sqrt{\pi t}}\, e^{-\frac{\xi^2}{4t}}.$$

The reciprocal relations, which are valid for $\xi \leqslant 0$,

$$\Phi(\lambda;\xi) = \frac{e^{\sqrt{\lambda}\,\xi}}{2\sqrt{\lambda}} \quad \text{and} \quad e^{\sqrt{\lambda}\,\xi} = 2\sqrt{\lambda}\,\alpha_{10}\Phi(\lambda;\xi)$$

imply that $\alpha_{10} = 1$. Then $a_{00} = \frac{2B_0}{\Gamma(1/2)} \frac{A_1 \alpha_{10}}{\theta_1 - \theta_0} = \frac{1}{\sqrt{\pi}}$. On the other hand, we have for $\xi \geqslant 0$, by (5.14),

$$\begin{aligned} I_{00}(t;\xi) &= \frac{i\theta_0 \xi}{2\pi t}\left[i \int_0^{+\infty} e^{-t\lambda^2 + i\theta_0 \xi \lambda}\, d\lambda + i \int_0^{+\infty} e^{-t\lambda^2 - i\theta_0 \xi \lambda}\, d\lambda\right] \\ &= \frac{\xi}{2\pi t}\left[\int_0^{+\infty} e^{-t\lambda^2 + i\xi\lambda}\, d\lambda + \int_0^{+\infty} e^{-t\lambda^2 - i\xi\lambda}\, d\lambda\right] \\ &= \frac{\xi}{2\pi t} \int_{-\infty}^{+\infty} e^{-t\lambda^2 + i\xi\lambda}\, d\lambda = \frac{\xi}{2\sqrt{\pi}\, t^{3/2}}\, e^{-\frac{\xi^2}{4t}}. \end{aligned}$$

Consequently,

$$\mathbb{P}_x\{X(t) \leqslant y \leqslant z \leqslant M(t)\} = \frac{z-y}{4\pi^{3/2}} \int_0^t \int_0^s \frac{e^{-\frac{(x-z)^2}{4\sigma} - \frac{(z-y)^2}{4(s-\sigma)}}}{\sqrt{\sigma}(s-\sigma)^{3/2}\sqrt{t-s}}\, ds\, d\sigma.$$

Using the substitution $\sigma = \frac{s^2}{u+s}$ together with a well-known integral related to the modified Bessel function $K_{1/2}$, we get

$$\begin{aligned}\int_0^s \frac{e^{-\frac{(x-z)^2}{4\sigma}-\frac{(z-y)^2}{4(s-\sigma)}}}{\sqrt{\sigma}(s-\sigma)^{3/2}}\,d\sigma &= \frac{e^{-\frac{(x-z)^2+(z-y)^2}{4s}}}{\sqrt{s}}\int_0^\infty e^{-\frac{(x-z)^2}{s^2}\,u-\frac{(z-y)^2}{4u}}\,\frac{du}{u^{3/2}}\\ &= \frac{\sqrt{\pi}}{\sqrt{(z-y)s}}\,e^{-\frac{(2z-x-y)^2}{4s}}.\end{aligned}$$

Then

$$\mathbb{P}_x\{X(t)\leqslant y\leqslant z\leqslant M(t)\} = \frac{1}{2\pi}\int_0^t \frac{e^{-\frac{(2z-x-y)^2}{4s}}}{\sqrt{\sigma(t-s)}}\,ds.$$

Finally, it can be easily checked, by using the Laplace transform, that

$$\int_0^t \frac{e^{-\frac{(2z-x-y)^2}{4s}}}{\sqrt{\sigma(t-s)}}\,ds = \sqrt{\pi}\int_{2z-x-y}^\infty \frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{t}}\,dt = 2\pi\int_{2z-x-y}^\infty p(t;-\xi)\,dt.$$

As a result, we retrieve the famous reflection principle for Brownian motion:

$$\mathbb{P}_x\{X(t)\leqslant y\leqslant z\leqslant M(t)\} = \mathbb{P}\{X(t)\geqslant 2z-x-y\}.$$

■

Example 4.19 *Case $N=3$:* we have to cases to distinguish.

- Case $\kappa_3 = +1$: the sum of interest in (4.16) reads here

$$\frac{\theta_0 A_0 B_1}{\theta_1-\theta_0}\,e^{\theta_0\sqrt[3]{\lambda}(x-z)+\theta_1\sqrt[3]{\lambda}(z-y)} + \frac{\theta_0 A_0 B_2}{\theta_2-\theta_0}\,e^{\theta_0\sqrt[3]{\lambda}(x-z)+\theta_2\sqrt[3]{\lambda}(z-y)}$$

with $\frac{B_1}{\theta_1-\theta_0} = \frac{B_2}{\theta_2-\theta_0} = -\frac{1}{3}$, and then

$$F_\lambda(x,y,z) = \begin{cases} \frac{2}{3\lambda}\Big[e^{-\frac{\sqrt[3]{\lambda}}{2}(x-y)}\cos\Big(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(x-y)\Big) \\ \quad -e^{\sqrt[3]{\lambda}(x+\frac{1}{2}y-\frac{3}{2}z)}\cos\Big(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(z-y)\Big)\Big] & \text{if } y\leqslant x\leqslant z,\\ \frac{1}{\lambda}-\frac{1}{3\lambda}\Big[e^{\sqrt[3]{\lambda}(x-y)} \\ \quad +2\,e^{\sqrt[3]{\lambda}(x+\frac{1}{2}y-\frac{3}{2}z)}\cos\Big(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(z-y)\Big)\Big] & \text{if } x\leqslant y\leqslant z.\end{cases}$$

We retrieve the results (2.2) of [3]. Now, formula (4.17) writes

$$\mathbb{P}_x\{X(t)\leqslant y\leqslant z\leqslant M(t)\} = \int_0^t\!\!\int_0^s p(\sigma;x-z)\,(a_{10}I_{10}+a_{20}I_{20})(s-\sigma;z-y)\,\frac{ds\,d\sigma}{(t-s)^{2/3}}$$

where, by (2.2),

$$p(t;\xi)=\frac{1}{\pi}\int_0^{+\infty}\cos(\xi\lambda-t\lambda^3)\,d\lambda.$$

The reciprocal relations, for $\xi\leqslant 0$,

$$\Phi(\lambda;\xi)=\frac{e^{\sqrt[3]{\lambda}\,\xi}}{3\lambda^{2/3}}\quad\text{and}\quad e^{\sqrt[3]{\lambda}\,\xi}=3\lambda^{2/3}\alpha_{00}\Phi(\lambda;\xi)$$

imply that $\alpha_{00}=1$. Then

$$\begin{aligned}a_{10}&=\frac{3B_1}{\Gamma(1/3)}\,\frac{A_0\alpha_{00}}{\theta_0-\theta_1}=\frac{1}{\Gamma(1/3)},\\ a_{20}&=\frac{3B_2}{\Gamma(1/3)}\,\frac{A_0\alpha_{00}}{\theta_0-\theta_2}=\frac{1}{\Gamma(1/3)}.\end{aligned}$$

Consequently,

$$\mathbb{P}_x\{X(t)\leqslant y\leqslant z\leqslant M(t)\}=\frac{1}{\Gamma(1/3)}\int_0^t\!\!\int_0^s p(\sigma;x-z)\,q(s-\sigma;z-y)\,\frac{ds\,d\sigma}{(t-s)^{2/3}}$$

with, for $\xi\geqslant 0$, by (5.14),

$$\begin{aligned}q(t;\xi)&=(I_{10}+I_{20})(t;\xi)\\&=\frac{i\xi}{2\pi t}\left[\theta_1e^{\frac{i\pi}{3}}\int_0^{+\infty}e^{-t\lambda^3+\theta_1e^{\frac{i\pi}{3}}\xi\lambda}\,d\lambda-\theta_1e^{-\frac{i\pi}{3}}\int_0^{+\infty}e^{-t\lambda^3+\theta_1e^{-\frac{i\pi}{3}}\xi\lambda}\,d\lambda\right.\\&\qquad\left.+\,\theta_2e^{\frac{i\pi}{3}}\int_0^{+\infty}e^{-t\lambda^3+\theta_2e^{\frac{i\pi}{3}}\xi\lambda}\,d\lambda-\theta_2e^{-\frac{i\pi}{3}}\int_0^{+\infty}e^{-t\lambda^3+\theta_2e^{-\frac{i\pi}{3}}\xi\lambda}\,d\lambda\right]\\&=-\frac{i\xi}{2\pi t}\left[e^{\frac{i\pi}{3}}\int_0^{+\infty}e^{-t\lambda^3+e^{\frac{i\pi}{3}}\xi\lambda}\,d\lambda-e^{-\frac{i\pi}{3}}\int_0^{+\infty}e^{-t\lambda^3+e^{-\frac{i\pi}{3}}\xi\lambda}\,d\lambda\right]\\&=\frac{\xi}{\pi t}\int_0^{+\infty}e^{-t\lambda^3+\frac{1}{2}\xi\lambda}\sin\left(\frac{\sqrt{3}}{2}\,\xi\lambda+\frac{\pi}{3}\right)d\lambda.\end{aligned}$$

- Case $\kappa_3=-1$: the sum of interest in (4.16) reads here

$$\frac{\theta_0A_0B_1}{\theta_1-\theta_0}\,e^{\theta_0\sqrt[3]{\lambda}(x-z)+\theta_1\sqrt[3]{\lambda}(z-y)}+\frac{\theta_2A_2B_1}{\theta_1-\theta_2}\,e^{\theta_2\sqrt[3]{\lambda}(x-z)+\theta_1\sqrt[3]{\lambda}(z-y)}$$

with $\frac{\theta_0A_0}{\theta_1-\theta_0}=-\frac{1}{3}\,e^{i\,\pi/3}$ and $\frac{\theta_2A_2}{\theta_1-\theta_2}=-\frac{1}{3}\,e^{-i\,\pi/3}$, and then

$$F_\lambda(x,y,z)=\begin{cases}\dfrac{1}{3\lambda}\left[e^{-\sqrt[3]{\lambda}(x-y)}-2\,e^{\sqrt[3]{\lambda}(\frac{1}{2}x+y-\frac{3}{2}z)}\cos\left(\dfrac{\sqrt{3}}{2}\sqrt[3]{\lambda}(x-z)+\dfrac{\pi}{3}\right)\right]\\ \text{if } y\leqslant x\leqslant z,\\[2ex] \dfrac{1}{\lambda}-\dfrac{1}{3\lambda}\left[2\,e^{\frac{\sqrt[3]{\lambda}}{2}(x-y)}\cos\left(\dfrac{\sqrt{3}}{2}\sqrt[3]{\lambda}(x-y)\right)\right.\\ \left.+\,e^{\frac{\sqrt[3]{\lambda}}{2}(\frac{1}{2}x+y-\frac{3}{2}z)}\cos\left(\dfrac{\sqrt{3}}{2}\sqrt[3]{\lambda}(x-z)+\dfrac{\pi}{3}\right)\right]\\ \text{if } x\leqslant y\leqslant z.\end{cases}$$

We retrieve the results (2.2) of [3]. Next, formula (4.19) writes

$$\begin{aligned}
&\mathbb{P}_x\{X(t)\leqslant y\leqslant z\leqslant M(t)\}\\
&\qquad= \sum_{j\in\{0,2\}} b_{j0}\int_0^t\!\!\int_0^s p(\sigma;z-y)\,I_{j0}(s-\sigma;x-z)\,\frac{ds\,d\sigma}{(t-s)^{2/3}}\\
&\qquad= \int_0^t\!\!\int_0^s p(\sigma;z-y)\,(b_{00}I_{00}+b_{20}I_{20})(s-\sigma;x-z)\,\frac{ds\,d\sigma}{(t-s)^{2/3}}
\end{aligned}$$

where, by (2.2),

$$p(t;\xi)=\frac{1}{\pi}\int_0^{+\infty}\cos(\xi\lambda+t\lambda^3)\,d\lambda.$$

From the reciprocal relations, which are valid for $\xi\geqslant 0$,

$$\Phi(\lambda;\xi)=\frac{e^{-\sqrt[3]{\lambda}\,\xi}}{3\lambda^{2/3}}\quad\text{and}\quad -\,e^{-\sqrt[3]{\lambda}\,\xi}=3\lambda^{2/3}\beta_{10}\Phi(\lambda;\xi)$$

we extract the value $\beta_{10}=-1$. Therefore,

$$\begin{aligned}
b_{00} &= \frac{3\theta_0A_0}{\Gamma(1/3)}\,\frac{\bar\theta_1B_1\beta_{10}}{\theta_1-\theta_0}=\frac{e^{\frac{2i\pi}{3}}}{\Gamma(1/3)},\\
b_{20} &= \frac{3\theta_2A_2}{\Gamma(1/3)}\,\frac{\bar\theta_1B_1\beta_{10}}{\theta_1-\theta_2}=\frac{e^{-\frac{2i\pi}{3}}}{\Gamma(1/3)}.
\end{aligned}$$

Consequently,

$$\mathbb{P}_x\{X(t)\leqslant y\leqslant z\leqslant M(t)\}=\frac{1}{\Gamma(1/3)}\int_0^t\!\!\int_0^s p(\sigma;z-y)\,q(s-\sigma;x-z)\,\frac{ds\,d\sigma}{(t-s)^{2/3}}$$

where, for $\xi\leqslant 0$, by (5.14),

$$\begin{aligned}
q(t;\xi) &= (e^{\frac{2i\pi}{3}}I_{00}+e^{-\frac{2i\pi}{3}}I_{20})(t;\xi)\\
&= \frac{i\xi}{2\pi t}\left[-\theta_0\int_0^{+\infty}e^{-t\lambda^3+\theta_0e^{\frac{i\pi}{3}}\xi\lambda}\,d\lambda-\theta_0e^{\frac{i\pi}{3}}\int_0^{+\infty}e^{-t\lambda^3+\theta_0e^{-\frac{i\pi}{3}}\xi\lambda}\,d\lambda\right.\\
&\qquad\left.+\,\theta_2e^{-\frac{i\pi}{3}}\int_0^{+\infty}e^{-t\lambda^3+\theta_2e^{\frac{i\pi}{3}}\xi\lambda}\,d\lambda+\theta_2\int_0^{+\infty}e^{-t\lambda^3+\theta_2e^{-\frac{i\pi}{3}}\xi\lambda}\,d\lambda\right]\\
&= \frac{\xi}{\pi t}\left[\sqrt{3}\int_0^{+\infty}e^{-t\lambda^3+\xi\lambda}\,d\lambda+\int_0^{+\infty}e^{-t\lambda^3-\frac12\xi\lambda}\sin\left(\frac{\sqrt3}{2}\,\xi\lambda+\frac{\pi}{3}\right)d\lambda\right].
\end{aligned}$$

■

Example 4.20 *Case $N=4$:* in this case, we have

$$\sum_{j\in J,k\in K}\frac{\theta_jA_jB_k}{\theta_k-\theta_j}\,e^{\theta_j\sqrt[4]{\lambda}(x-z)+\theta_k\sqrt[4]{\lambda}(z-y)}$$

$$
\begin{aligned}
&= \frac{\theta_2 A_2 B_0}{\theta_0-\theta_2}\, e^{\theta_2\sqrt[4]{\lambda}(x-z)+\theta_0\sqrt[4]{\lambda}(z-y)} + \frac{\theta_2 A_2 B_1}{\theta_1-\theta_2}\, e^{\theta_2\sqrt[4]{\lambda}(x-z)+\theta_1\sqrt[4]{\lambda}(z-y)} \\
&\quad + \frac{\theta_3 A_3 B_0}{\theta_0-\theta_3}\, e^{\theta_3\sqrt[4]{\lambda}(x-z)+\theta_0\sqrt[4]{\lambda}(z-y)} + \frac{\theta_3 A_3 B_1}{\theta_1-\theta_3}\, e^{\theta_3\sqrt[4]{\lambda}(x-z)+\theta_1\sqrt[4]{\lambda}(z-y)}
\end{aligned}
$$

with

$$
\frac{\theta_2 A_2 B_0}{\theta_0-\theta_2} = \frac{i}{4}, \quad \frac{\theta_3 A_3 B_1}{\theta_1-\theta_3} = -\frac{i}{4}, \quad \frac{\theta_2 A_2 B_1}{\theta_1-\theta_2} = -\frac{e^{-i\pi/4}}{2\sqrt{2}}, \quad \frac{\theta_3 A_3 B_0}{\theta_0-\theta_3} = -\frac{e^{i\pi/4}}{2\sqrt{2}}.
$$

Hence,

$$
F_\lambda(x,y,z) = \begin{cases}
\dfrac{1}{2\lambda}\Bigg[e^{-\frac{\sqrt[4]{\lambda}}{\sqrt{2}}(x-y)} \cos\left(\dfrac{\sqrt[4]{\lambda}}{\sqrt{2}}(x-y)\right) - e^{\frac{\sqrt[4]{\lambda}}{\sqrt{2}}(x+y-2z)} \\
\times\left(\cos\left(\dfrac{\sqrt[4]{\lambda}}{\sqrt{2}}(x-y)\right) - \sin\left(\dfrac{\sqrt[4]{\lambda}}{\sqrt{2}}(x-y)\right) - \sin\left(\dfrac{\sqrt[4]{\lambda}}{\sqrt{2}}(x+y-2z)\right)\right)\Bigg] \\
\text{if } y \leqslant x \leqslant z, \\[2ex]
\dfrac{1}{\lambda} - \dfrac{1}{2\lambda}\Bigg[e^{\frac{\sqrt[4]{\lambda}}{\sqrt{2}}(x-y)} \cos\left(\dfrac{\sqrt[4]{\lambda}}{\sqrt{2}}(x-y)\right) + e^{\frac{\sqrt[4]{\lambda}}{\sqrt{2}}(x+y-2z)} \\
\times\left(\cos\left(\dfrac{\sqrt[4]{\lambda}}{\sqrt{2}}(x-y)\right) - \sin\left(\dfrac{\sqrt[4]{\lambda}}{\sqrt{2}}(x-y)\right) - \sin\left(\dfrac{\sqrt[4]{\lambda}}{\sqrt{2}}(x+y-2z)\right)\right)\Bigg] \\
\text{if } x \leqslant y \leqslant z.
\end{cases}
$$

We retrieve the results (3.2) of [3]. Now, formula (4.17) writes

$$
\begin{aligned}
\mathbb{P}_x\{X(t) \leqslant y \leqslant z \leqslant M(t)\} &= \int_0^t\!\!\int_0^s p(\sigma; x-z)\,(a_{00}I_{00}+a_{10}I_{10})(s-\sigma; z-y)\,\frac{ds\,d\sigma}{(t-s)^{3/4}} \\
&\quad + \int_0^t\!\!\int_0^s \frac{\partial p}{\partial x}(\sigma; x-z)\,(a_{01}I_{00}+a_{11}I_{10})(s-\sigma; z-y)\,\frac{ds\,d\sigma}{\sqrt{t-s}}.
\end{aligned}
$$

where, by (2.2),

$$
p(t;\xi) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\lambda^4}\cos(\xi\lambda)\,d\lambda.
$$

Let us consider the system

$$
\begin{cases}
\theta_2\, e^{\theta_2\sqrt[4]{\lambda}\,\xi} + \theta_3\, e^{\theta_3\sqrt[4]{\lambda}\,\xi} = 4\lambda^{3/4}\Phi(\lambda;\xi) \\
\theta_2^2\, e^{\theta_2\sqrt[4]{\lambda}\,\xi} + \theta_3^2\, e^{\theta_3\sqrt[4]{\lambda}\,\xi} = 4\sqrt{\lambda}\,\dfrac{\partial\Phi}{\partial\xi}(\lambda;\xi)
\end{cases}
$$

which can be conversely written

$$
\begin{cases}
\theta_2\, e^{\theta_2\sqrt[4]{\lambda}\,\xi} = \dfrac{4}{\theta_3-\theta_2}\left(\theta_3\lambda^{3/4}\Phi(\lambda;\xi) - \sqrt{\lambda}\,\dfrac{\partial\Phi}{\partial\xi}(\lambda;\xi)\right) \\
\theta_3\, e^{\theta_3\sqrt[4]{\lambda}\,\xi} = \dfrac{4}{\theta_3-\theta_2}\left(-\theta_2\lambda^{3/4}\Phi(\lambda;\xi) + \sqrt{\lambda}\,\dfrac{\partial\Phi}{\partial\xi}(\lambda;\xi)\right)
\end{cases}
$$

or, by means of the coefficients $\alpha_{20}, \alpha_{21}, \alpha_{30}, \alpha_{31}$,

$$\begin{cases} \theta_2\, e^{\theta_2 \sqrt[4]{\lambda}\,\xi} & = & 4\Big(\alpha_{20}\lambda^{3/4}\Phi(\lambda;\xi) + \alpha_{21}\sqrt{\lambda}\,\dfrac{\partial \Phi}{\partial \xi}(\lambda;\xi)\Big), \\ \theta_3\, e^{\theta_3 \sqrt[4]{\lambda}\,\xi} & = & 4\Big(\alpha_{30}\lambda^{3/4}\Phi(\lambda;\xi) + \alpha_{31}\sqrt{\lambda}\,\dfrac{\partial \Phi}{\partial \xi}(\lambda;\xi)\Big). \end{cases}$$

Identifying the two above systems yields the coefficients we are looking for:

$$\begin{aligned} \alpha_{20} &= \frac{\theta_3}{\theta_3 - \theta_2} = A_2 = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}, \\ \alpha_{30} &= -\frac{\theta_2}{\theta_3 - \theta_2} = A_3 = \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}, \\ \alpha_{21} &= -\frac{1}{\theta_3 - \theta_2} = -\theta_2 A_2 = \frac{i}{\sqrt{2}}, \\ \alpha_{31} &= \frac{1}{\theta_3 - \theta_2} = -\theta_3 A_3 = -\frac{i}{\sqrt{2}}, \end{aligned}$$

and next:

$$\begin{aligned} a_{00} &= \frac{4B_0}{\Gamma(1/4)}\left[\frac{A_2\alpha_{20}}{\theta_2 - \theta_0} + \frac{A_3\alpha_{30}}{\theta_3 - \theta_0}\right] = \frac{1}{\sqrt{2}\,\Gamma(1/4)}, \\ a_{10} &= \frac{4B_1}{\Gamma(1/4)}\left[\frac{A_2\alpha_{20}}{\theta_2 - \theta_1} + \frac{A_3\alpha_{30}}{\theta_3 - \theta_1}\right] = \frac{1}{\sqrt{2}\,\Gamma(1/4)}, \\ a_{01} &= \frac{4B_0}{\Gamma(1/2)}\left[\frac{A_2\alpha_{21}}{\theta_2 - \theta_0} + \frac{A_3\alpha_{31}}{\theta_3 - \theta_0}\right] = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2\pi}}, \\ a_{11} &= \frac{4B_1}{\Gamma(1/2)}\left[\frac{A_2\alpha_{21}}{\theta_2 - \theta_1} + \frac{A_3\alpha_{31}}{\theta_3 - \theta_1}\right] = \frac{e^{\frac{i\pi}{4}}}{\sqrt{2\pi}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{P}_x\{X(t) \leqslant y \leqslant z \leqslant M(t)\} &= \int_0^t\!\!\int_0^s p(\sigma; x-z)\, q_1(s-\sigma; z-y)\, \frac{ds\, d\sigma}{(t-s)^{3/4}} \\ &\quad + \int_0^t\!\!\int_0^s \frac{\partial p}{\partial x}(\sigma; x-z)\, q_2(s-\sigma; z-y)\, \frac{ds\, d\sigma}{\sqrt{t-s}} \end{aligned}$$

with, for $\xi \geqslant 0$, by (5.14),

$$q_1(t;\xi) = (a_{00}I_{00} + a_{10}I_{10})(t;\xi) = \frac{1}{\sqrt{2}\,\Gamma(1/4)}\,(I_{00} + I_{10})(t;\xi)$$

and

$$\begin{aligned} q_2(t;\xi) &= (a_{01}I_{00} + a_{11}I_{10})(t;\xi) = \frac{1}{\sqrt{2\pi}}\,(e^{-\frac{i\pi}{4}} I_{00} + e^{\frac{i\pi}{4}} I_{10})(t;\xi) \\ &= \frac{1}{2\pi}\,(I_{00} + I_{10})(t;\xi) - \frac{i}{2\pi}\,(I_{00} - I_{10})(t;\xi). \end{aligned}$$

Let us evaluate the intermediate quantities $(I_{00} \pm I_{10})(t;\xi)$:

$$\begin{aligned}(I_{00}+I_{10})(t;\xi) &= \frac{i\xi}{2\pi t}\left[\theta_0 e^{\frac{i\pi}{4}} \int_0^{+\infty} e^{-t\lambda^4+\theta_0 e^{\frac{i\pi}{4}}\xi\lambda}\,d\lambda - \theta_0 e^{-\frac{i\pi}{4}} \int_0^{+\infty} e^{-t\lambda^4+\theta_0 e^{-\frac{i\pi}{4}}\xi\lambda}\,d\lambda\right.\\ &\quad \left. + \theta_1 e^{\frac{i\pi}{4}} \int_0^{+\infty} e^{-t\lambda^4+\theta_1 e^{\frac{i\pi}{4}}\xi\lambda}\,d\lambda - \theta_1 e^{-\frac{i\pi}{4}} \int_0^{+\infty} e^{-t\lambda^4+\theta_1 e^{-\frac{i\pi}{4}}\xi\lambda}\,d\lambda\right]\\ &= \frac{\xi}{\pi t}\int_0^{+\infty} e^{-t\lambda^4}\cos(\xi\lambda)\,d\lambda\end{aligned}$$

and

$$(I_{00}-I_{10})(t;\xi) = -\frac{i\xi}{\pi t}\left[\int_0^{+\infty} e^{-t\lambda^4-\xi\lambda}\,d\lambda - \int_0^{+\infty} e^{-t\lambda^4}\sin(\xi\lambda)\,d\lambda\right].$$

Then

$$\begin{aligned}q_1(t;\xi) &= \frac{\xi}{\pi\sqrt{2}\,\Gamma(1/4)\,t}\int_0^{+\infty} e^{-t\lambda^4}\cos(\xi\lambda)\,d\lambda,\\ q_2(t;\xi) &= \frac{\xi}{2\pi^2\,t}\int_0^{+\infty} e^{-t\lambda^4}\left(\cos(\xi\lambda)+\sin(\xi\lambda)-e^{-\xi\lambda}\right)d\lambda.\end{aligned}$$

■

4.6 Boundary value problem

In this part, we show that the function $x \longmapsto F_\lambda(x,y,z)$ solves a boundary value problem related to the differential operator $\mathcal{D}_x = \kappa_N \frac{d^N}{dx^N}$. Fix $y < z$ and set $F(x) = F_\lambda(x,y,z)$ for $x \in (-\infty, z]$.

Proposition 4.21 *The function F satisfies the differential equation*

$$\mathcal{D}_x F(x) = \begin{cases} \lambda F(x) - 1 & \text{for } x \in (-\infty, y), \\ \lambda F(x) & \text{for } x \in (y,z), \end{cases} \tag{4.20}$$

together with the conditions

$$F^{(l)}(z^-) = 0 \quad \text{for } 0 \leqslant l \leqslant \#J-1, \tag{4.21}$$

$$F^{(l)}(y^+) - F^{(l)}(y^-) = 0 \quad \text{for } 0 \leqslant l \leqslant N-1. \tag{4.22}$$

Proof. The differential equation (4.20) is readily obtained by differentiating (4.16) with respect to x. Let us derive the boundary condition (4.21):

$$\begin{aligned}F^{(l)}(z^-) &= \frac{\lambda^{l/N}}{\lambda}\sum_{j\in J,k\in K}\frac{\theta_j^{l+1}A_jB_k}{\theta_k-\theta_j}\,e^{\theta_k\sqrt[N]{\lambda}(z-y)} + \frac{\lambda^{l/N}}{N\lambda}\sum_{k\in K}\theta_k^{l+1}\,e^{\theta_k\sqrt[N]{\lambda}(z-y)}\\ &= \lambda^{l/N-1}\sum_{k\in K}\left[\sum_{j\in J}\frac{\theta_j^{l+1}A_j}{\theta_k-\theta_j}B_k + \frac{\theta_k^{l+1}}{N}\right]e^{\theta_k\sqrt[N]{\lambda}(z-y)} = 0\end{aligned}$$

where the last equality comes from (2.11) with $x = \theta_k$ which yields $\displaystyle\sum_{j\in J}\frac{\theta_j^{l+1}A_j}{\theta_k-\theta_j} = -\frac{\theta_k^{l+1}}{NB_k}$. Condition (4.22) is quite easy to check. ■

Remark 4.22 Condition (4.22) says that the function F is regular up to the order $n-1$. It can also be easily seen that $F^{(N)}(y^+) - F^{(N)}(y^-) = \kappa_N$ which says that the function $F^{(N)}$ has a jump at point y. On the other hand, the boundary value problem (4.20)–(4.21)–(4.22) (the differential equation together with the $N + \#J$ conditions) augmented of a boundedness condition on $(-\infty, y)$ may be directly solved by using Vandermonde determinants. ■

5 Distributions of $(\tau_a^+, X(\tau_a^+))$ and $(\tau_a^-, X(\tau_a^-))$

The integer N is again assumed to be even. Recall we set $\tau_a^+ = \inf\{t \geqslant 0 : X(t) > a\}$ and $\tau_a^- = \inf\{t \geqslant 0 : X(t) < a\}$. The aim of this section is to derive the distributions of the vectors $(\tau_a^+, X(\tau_a^+))$ and $(\tau_a^-, X(\tau_a^-))$. For this, we proceed in three steps: we first compute the Laplace-Fourier transform of, e.g., $(\tau_a^+, X(\tau_a^+))$ (Subsection 5.1); we next invert the Fourier transform (with respect to μ, Subsubsection 5.2.1) and we finally invert the Laplace transform (with respect to λ, Subsubsection 5.2.2). We have especially obtained a remarkable formula for the densities of $X(\tau_a^+)$ and $X(\tau_a^-)$ by means of multipoles (Subsection 5.4).

5.1 Laplace-Fourier transforms

We have a relationship between the distributions of $(\tau_a^+, X(\tau_a^+))$ and $(X(t), M(t))$, and between those of $(\tau_a^-, X(\tau_a^-))$ and $(X(t), m(t))$.

Lemma 5.1 *The Laplace-Fourier transforms of the vectors $(\tau_a^+, X(\tau_a^+))$ and $(\tau_a^-, X(\tau_a^-))$ are related to the distributions of the vectors $(X(t), M(t))$ and $(X(t), m(t))$ according as, for $\Re(\lambda) > 0$ and $\mu \in \mathbb{R}$,*

$$\begin{aligned} \mathbb{E}_x\Big[e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}\Big] &= \big(\lambda - \kappa_N (i\mu)^N\big) \int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}_x\Big[e^{i\mu X(t)}, M(t) > a\Big]\, dt \quad \text{for } x \leqslant a, \\ \mathbb{E}_x\Big[e^{-\lambda\tau_a^- + i\mu X(\tau_a^-)}\Big] &= \big(\lambda - \kappa_N (i\mu)^N\big) \int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}_x\Big[e^{i\mu X(t)}, m(t) < a\Big]\, dt \quad \text{for } x \geqslant a. \end{aligned} \tag{5.1}$$

Proof. We divide the proof of Lemma 5.1 into five steps.

• **Step 1**

For the step-process $(X_n(t))_{t\geqslant 0}$, the corresponding first hitting time $\tau_{a,n}^+$ is the instant $t_{n,k}$ with k such that $X(t_{n,j}) \leqslant a$ for all $j \in \{0, \dots, k-1\}$ and $X(t_{n,k}) > a$, or, equivalently, such that $M_{n,k-1} \leqslant a$ and $M_{n,k} > a$ where $M_{n,k} = \max_{0\leqslant j\leqslant k} X_{n,j}$ and $X_{n,k} = X(t_{n,k})$ for $k \geqslant 0$ and $M_{n,-1} = -\infty$. We have, for $x \leqslant a$,

$$\begin{aligned} e^{-\lambda\tau_{a,n}^+ + i\mu X_n(\tau_{a,n}^+)} &= \sum_{k=0}^{\infty} e^{-\lambda t_{n,k} + i\mu X_{n,k}} 1\!\!1_{\{M_{n,k-1} \leqslant a < M_{n,k}\}} \\ &= \sum_{k=0}^{\infty} e^{-\lambda t_{n,k} + i\mu X_{n,k}} \Big[1\!\!1_{\{M_{n,k} > a\}} - 1\!\!1_{\{M_{n,k-1} > a\}}\Big]. \end{aligned} \tag{5.2}$$

Let us apply classical Abel's identity to sum (5.2). This yields, since $1\!\!1_{\{M_{n,-1}>a\}} = 0$ and $\lim_{k\to+\infty} e^{-\lambda t_{n,k}+i\mu X_{n,k}} 1\!\!1_{\{M_{n,k}>a\}} = 0$, for $\Re(\lambda) > 0$:

$$e^{-\lambda\tau_{a,n}^+ + i\mu X_n(\tau_{a,n}^+)} = \sum_{k=0}^{\infty}\left[e^{-\lambda t_{n,k}+i\mu X_{n,k}} - e^{-\lambda t_{n,k+1}+i\mu X_{n,k+1}}\right] 1\!\!1_{\{M_{n,k}>a\}}.$$

The functional $e^{-\lambda\tau_{a,n}^+ + i\mu X_n(\tau_{a,n}^+)}$ is a function of discrete observations of X.

• **Step 2**

In order to evaluate the expectation of the foregoing functional, we need to check that the series

$$\sum_{k=0}^{\infty}\mathbb{E}_x\Big[\Big(e^{-\lambda t_{n,k}+i\mu X_{n,k}} - e^{-\lambda t_{n,k+1}+i\mu X_{n,k+1}}\Big) 1\!\!1_{\{M_{n,k}>a\}}\Big]$$

is absolutely convergent. For this, we use the Markov property and derive the following estimate:

$$\begin{aligned}
\Big|\mathbb{E}_x\Big[e^{-\lambda t_{n,k}+i\mu X_{n,k}} 1\!\!1_{\{M_{n,k}\leqslant a\}}\Big]\Big| &= \Big|e^{-\lambda t_{n,k}}\mathbb{E}_x\Big[e^{i\mu X_{n,k}} 1\!\!1_{\{X_{n,1}\leqslant a,\dots,X_{n,k}\leqslant a\}}\Big]\Big| \\
&\leqslant |e^{-\lambda/2^n}|^k \left|\int_{-\infty}^a \cdots \int_{-\infty}^a e^{i\mu x_k} p(1/2^n; x-x_1)\cdots p(1/2^n; x_{k-1}-x_k)\,dx_1\cdots dx_k\right| \\
&\leqslant (\rho\, e^{-\Re(\lambda)/2^n})^k.
\end{aligned}$$

We recall that in the last inequality $\rho = \int_{-\infty}^{+\infty}|p(t;z)|\,dz < +\infty$. Similar computations yield the inequality $\big|\mathbb{E}_x\big[e^{-\lambda t_{n,k}+i\mu X_{n,k}}\big]\big| \leqslant (\rho\, e^{-\Re(\lambda)/2^n})^k$. Because of the identity $1\!\!1_{\{M_{n,k}>a\}} = 1 - 1\!\!1_{\{M_{n,k}\leqslant a\}}$, we plainly get

$$\Big|\mathbb{E}_x\Big[e^{-\lambda t_{n,k}+i\mu X_{n,k}} 1\!\!1_{\{M_{n,k}>a\}}\Big]\Big| \leqslant 2(\rho\, e^{-\Re(\lambda)/2^n})^k.$$

Upon adding one integral more in the above discussion, it is easily seen that

$$\Big|\mathbb{E}_x\Big[e^{-\lambda t_{n,k}+i\mu X_{n,k+1}} 1\!\!1_{\{M_{n,k}>a\}}\Big]\Big| \leqslant 2(\rho\, e^{-\Re(\lambda)/2^n})^{k+1}.$$

As a result, when choosing λ such that $\Re(\lambda) > 2^n \ln\rho$, we have

$$\sum_{k=0}^{\infty}\Big|\mathbb{E}_x\Big[\Big(e^{-\lambda t_{n,k}+i\mu X_{n,k}} - e^{-\lambda t_{n,k+1}+i\mu X_{n,k+1}}\Big) 1\!\!1_{\{M_{n,k}>a\}}\Big]\Big| \leqslant \frac{2(1+\rho\, e^{-\Re(\lambda)/2^n})}{1-\rho\, e^{-\Re(\lambda)/2^n}} < +\infty.$$

• **Step 3**

Therefore, we can evaluate the expectation of $e^{-\lambda\tau_{a,n}^+ + i\mu X_n(\tau_{a,n}^+)}$. By the Markov property we get, for $\Re(\lambda) > 2^n\ln\rho$,

$$\begin{aligned}
&\mathbb{E}_x\Big[e^{-\lambda\tau_{a,n}^+ + i\mu X_n(\tau_{a,n}^+)}\Big] \\
&\quad = \sum_{k=0}^{\infty} e^{-\lambda t_{n,k}}\mathbb{E}_x\Big[e^{i\mu X_{n,k}} 1\!\!1_{\{M_{n,k}>a\}}\Big(1 - e^{-\lambda/2^n}\, e^{i\mu(X_{n,k+1}-X_{n,k})}\Big)\Big] \\
&\quad = \sum_{k=0}^{\infty} e^{-\lambda t_{n,k}}\mathbb{E}_x\Big[e^{i\mu X_{n,k}} 1\!\!1_{\{M_{n,k}>a\}}\Big(1 - e^{-\lambda/2^n}\,\mathbb{E}_{X_{n,k}}\Big(e^{i\mu(X_{n,1}-X_{n,0})}\Big)\Big)\Big].
\end{aligned}$$

Since $\mathbb{E}_{X_{n,k}}\big(e^{i\mu(X_{n,1}-X_{n,0})}\big)=e^{\kappa_N(i\mu)^N/2^n}$ we obtain, for $\Re(\lambda)>2^n\ln\rho$,

$$\begin{aligned}\mathbb{E}_x\Big[e^{-\lambda\tau_{a,n}^{+}+i\mu X_n(\tau_{a,n}^{+})}\Big] &= 2^n\Big(1-e^{-\left(\lambda-\kappa_N(i\mu)^N\right)/2^n}\Big)\\ &\quad\times\frac{1}{2^n}\sum_{k=0}^{\infty}e^{-\lambda t_{n,k}}\mathbb{E}_x\Big[e^{i\mu X_{n,k}}1\!\!1_{\{M_{n,k}>a\}}\Big].\end{aligned}\tag{5.3}$$

• **Step 4**

In order to take the limit of (5.3) as n tends to ∞, we have to check the validity of (5.3) for any λ such that $\Re(\lambda)>0$. For this, we first consider its Laplace transform with respect to a:

$$\begin{aligned}&\int_x^{+\infty}e^{-\nu a}\mathbb{E}_x\Big[e^{-\lambda\tau_{a,n}^{+}+i\mu X_n(\tau_{a,n}^{+})}\Big]\,da\\ &\qquad=\Big(1-e^{-\left(\lambda-\kappa_N(i\mu)^N\right)/2^n}\Big)\sum_{k=0}^{\infty}e^{-\lambda t_{n,k}}\int_x^{+\infty}e^{-\nu a}\mathbb{E}_x\Big[e^{i\mu X_{n,k}}1\!\!1_{\{M_{n,k}>a\}}\Big]\,da.\end{aligned}$$

The sum in the above equality writes, using the settings of Subsection 4.1,

$$\begin{aligned}&\sum_{k=0}^{\infty}e^{-\lambda t_{n,k}}\int_x^{+\infty}e^{-\nu a}\mathbb{E}_x\Big[e^{i\mu X_{n,k}}1\!\!1_{\{M_{n,k}>a\}}\Big]\,da\\ &\quad=\sum_{k=0}^{\infty}e^{-\lambda t_{n,k}}\mathbb{E}_x\Big[e^{i\mu X_{n,k}}\int_x^{M_{n,k}}e^{-\nu a}\,da\Big]\\ &\quad=\sum_{k=0}^{\infty}\frac{e^{-\lambda t_{n,k}}}{\nu}\Big[e^{-\nu x}\mathbb{E}_x\big(e^{i\mu X_{n,k}}\big)-\mathbb{E}_x\big(e^{i\mu X_{n,k}-\nu M_{n,k}}\big)\Big]\\ &\quad=\frac{1}{\nu}\left[e^{(i\mu-\nu)x}\sum_{k=0}^{\infty}e^{-(\lambda-\kappa_N(i\mu)^N)t_{n,k}}-\sum_{k=0}^{\infty}e^{-\lambda t_{n,k}}\mathbb{E}_x\big(e^{i\mu X_{n,k}-\nu M_{n,k}}\big)\right]\\ &\quad=\frac{1}{\nu}\left[\frac{e^{(i\mu-\nu)x}}{1-e^{-(\lambda-\kappa_N(i\mu)^N)/2^n}}-\frac{\lambda}{1-e^{-\lambda/2^n}}\,\mathbb{E}_x\big(F_{X_n}^{+}(\lambda,\mu,\nu)\big)\right]\\ &\quad=\frac{e^{(i\mu-\nu)x}}{\nu}\left[\frac{1}{1-e^{-(\lambda-\kappa_N(i\mu)^N)/2^n}}-\frac{1}{1-e^{-\lambda/2^n}}\exp\left(\frac{1}{2^n}\sum_{k=1}^{\infty}\frac{e^{-\lambda t_{n,k}}}{t_{n,k}}\,\psi^{+}(\mu,\nu;t_{n,k})\right)\right].\end{aligned}$$

We then obtain

$$\begin{aligned}&\int_x^{+\infty}e^{-\nu a}\mathbb{E}_x\Big[e^{-\lambda\tau_{a,n}^{+}+i\mu X_n(\tau_{a,n}^{+})}\Big]\,da\\ &\qquad=\frac{e^{(i\mu-\nu)x}}{\nu}\left[1-\frac{1-e^{-(\lambda-\kappa_N(i\mu)^N)/2^n}}{1-e^{-\lambda/2^n}}\exp\left(\frac{1}{2^n}\sum_{k=1}^{\infty}\frac{e^{-\lambda t_{n,k}}}{t_{n,k}}\,\psi^{+}(\mu,\nu;t_{n,k})\right)\right].\end{aligned}$$

Inverting the Laplace transform yields, noting that the function $a\longmapsto\mathbb{E}_x\Big[e^{-\lambda\tau_{a,n}^{+}+i\mu X_n(\tau_{a,n}^{+})}\Big]$ is right-continuous,

$$\mathbb{E}_x\Big[e^{-\lambda\tau_{a,n}^{+}+i\mu X_n(\tau_{a,n}^{+})}\Big]=\frac{1}{2i\pi}\lim_{\varepsilon\to 0^+}\int_{c-i\infty}^{c+i\infty}e^{(i\mu-\nu)x+\nu(a+\varepsilon)}$$

$$\times \left[1 - \frac{1-e^{-(\lambda-\kappa_N(i\mu)^N)/2^n}}{1-e^{-\lambda/2^n}}\, \exp\left(\frac{1}{2^n}\sum_{k=1}^{\infty}\frac{e^{-\lambda t_{n,k}}}{t_{n,k}}\,\psi^+(\mu,\nu;t_{n,k})\right)\right]\frac{d\nu}{\nu}.$$

Putting

$$\psi^+(\mu,\nu;t) = \psi_1(i\mu;t) + \psi_2(i\mu-\nu;t)$$

with

$$\psi_1(\alpha;t) = \mathbb{E}_0\Big[\Big(e^{\alpha X(t)}-1\Big)\,1\!\!1_{\{X(t)<0\}}\Big], \quad \psi_2(\alpha;t) = \mathbb{E}_0\Big[\Big(e^{\alpha X(t)}-1\Big)\,1\!\!1_{\{X(t)\geqslant 0\}}\Big],$$

the exponential within the last displayed integral writes

$$\begin{aligned}
&\exp\left(\frac{1}{2^n}\sum_{k=1}^{\infty}\frac{e^{-\lambda t_{n,k}}}{t_{n,k}}\,\psi^+(\mu,\nu;t_{n,k})\right)\\
&\qquad = \exp\left(\frac{1}{2^n}\sum_{k=1}^{\infty}\frac{e^{-\lambda t_{n,k}}}{t_{n,k}}\,\psi_1(i\mu;t_{n,k})\right)\exp\left(\frac{1}{2^n}\sum_{k=1}^{\infty}\frac{e^{-\lambda t_{n,k}}}{t_{n,k}}\,\psi_2(i\mu-\nu;t_{n,k})\right).
\end{aligned}$$

Noticing that

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{(i\mu-\nu)x+\nu a}\frac{d\nu}{\nu} = e^{i\mu x},$$

we get

$$\begin{aligned}
\mathbb{E}_x\Big[e^{-\lambda\tau_{a,n}^+ + i\mu X_n(\tau_{a,n}^+)}\Big] &= e^{i\mu x}\left[1 - \frac{1-e^{-(\lambda-\kappa_N(i\mu)^N)/2^n}}{1-e^{-\lambda/2^n}}\exp\left(\frac{1}{2^n}\sum_{k=1}^{\infty}\frac{e^{-\lambda t_{n,k}}}{t_{n,k}}\,\psi_1(i\mu;t_{n,k})\right)\right.\\
&\qquad \left.\times\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{(a-x)\nu}\exp\left(\frac{1}{2^n}\sum_{k=1}^{\infty}\frac{e^{-\lambda t_{n,k}}}{t_{n,k}}\,\psi_2(i\mu-\nu;t_{n,k})\right)\frac{d\nu}{\nu}\right].
\end{aligned}$$

By imitating the method used by Nishioka (Appendix in [17]) for deriving subtil extimates, it may be seen that this last expression is bounded over the half-plane $\Re(\lambda) \geqslant \varepsilon$ for any $\varepsilon > 0$. Hence, as in the proof of the validity of (4.2) for $\Re(\lambda) > 0$, we see that (5.3) is also valid for $\Re(\lambda) > 0$. It follows that the functional $e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}$ is admissible.

• **Step 5**

Now, we can let n tend to $+\infty$ in (5.3). For $\Re(\lambda) > 0$, we obviously have

$$\lim_{n\to+\infty} 2^n\Big(1 - e^{-\big(\lambda-\kappa_N(i\mu)^N\big)/2^n}\Big) = \lambda - \kappa_N(i\mu)^N,$$

and we finally obtain the relationship (5.1) corresponding to τ_a^+. The proof of that corresponding to τ_a^- is quite similar. ■

Theorem 5.2 *The Laplace-Fourier transforms of the vectors $(\tau_a^+, X(\tau_a^+))$ and $(\tau_a^-, X(\tau_a^-))$ are determined, for $\Re(\lambda) > 0$ and $\mu \in \mathbb{R}$, by*

$$\begin{aligned}
\mathbb{E}_x\Big[e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}\Big] &= \sum_{j\in J} A_j \prod_{l\in J\setminus\{j\}}\left(1 - \frac{i\mu}{\sqrt[N]{\lambda}}\,\bar\theta_l\right) e^{\theta_j\sqrt[N]{\lambda}\,(x-a)}\,e^{i\mu a} \quad \textit{for } x \leqslant a,\\
\mathbb{E}_x\Big[e^{-\lambda\tau_a^- + i\mu X(\tau_a^-)}\Big] &= \sum_{k\in K} B_k \prod_{l\in K\setminus\{k\}}\left(1 - \frac{i\mu}{\sqrt[N]{\lambda}}\,\bar\theta_l\right) e^{\theta_k\sqrt[N]{\lambda}\,(x-a)}\,e^{i\mu a} \quad \textit{for } x \geqslant a.
\end{aligned} \tag{5.4}$$

PROOF. Using (4.9) gives

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_x\Big[e^{i\mu X(t)}, M(t) > a\Big]\,dt$$
$$= \frac{\lambda^{(1-\#J)/N}\,e^{i\mu x}}{\prod_{k\in K}(\sqrt[N]{\lambda} - i\mu\theta_k)} \sum_{j\in J}\theta_j A_j \int_a^{+\infty} e^{(i\mu-\theta_j\sqrt[N]{\lambda})(z-x)}\,dz. \tag{5.5}$$

Plugging the following equality

$$\lambda - \kappa_N(i\mu)^N = \prod_{l=0}^{N-1}(\sqrt[N]{\lambda} - i\mu\theta_l) = \prod_{j\in J}(\sqrt[N]{\lambda} - i\mu\theta_j) \times \prod_{k\in K}(\sqrt[N]{\lambda} - i\mu\theta_k)$$

into (5.5) and remarking that the set $\{\theta_j, j \in J\}$ is invariant by conjugating yield

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_x\Big[e^{i\mu X(t)}, M(t) > a\Big]\,dt$$
$$= \lambda^{(1-\#J)/N} e^{i\mu x}\,\frac{\prod_{j\in J}(\sqrt[N]{\lambda} - i\mu\theta_j)}{\lambda - \kappa_N(i\mu)^N} \sum_{j\in J}\frac{A_j}{\sqrt[N]{\lambda} - i\mu\bar{\theta}_j}\,e^{-(i\mu-\theta_j\sqrt[N]{\lambda})(x-a)}$$
$$= \frac{e^{i\mu x}}{\lambda - \kappa_N(i\mu)^N}\sum_{j\in J} A_j \prod_{l\in J\setminus\{j\}}\left(1 - \frac{i\mu}{\sqrt[N]{\lambda}}\,\bar{\theta}_l\right) e^{-(i\mu-\theta_j\sqrt[N]{\lambda})(x-a)}. \tag{5.6}$$

Consequently, by putting (5.6) into (5.1), we obtain (5.4). ■

Remark 5.3 Choosing $\mu = 0$ in (5.4) supplies the Laplace transforms of τ_a^+ and τ_a^-:

$$\mathbb{E}_x\Big[e^{-\lambda\tau_a^+}\Big] = \sum_{j\in J} A_j\, e^{\theta_j\sqrt[N]{\lambda}\,(x-a)} \quad \text{for } x \leqslant a,$$
$$\mathbb{E}_x\Big[e^{-\lambda\tau_a^-}\Big] = \sum_{k\in K} B_k\, e^{\theta_k\sqrt[N]{\lambda}\,(x-a)} \quad \text{for } x \geqslant a.$$

■

Remark 5.4 An alternative method for deriving the distribution of $(\tau_a^+, X(\tau_a^+))$ consists of computing the joint distribution of $\big(X(t), \mathbb{1}_{(-\infty,a)}(M(t))\big)$ instead of that of $(X(t), M(t))$ and next to invert a certain Fourier transform. This way was employed by Nishioka [17] in the case $N = 4$ and may be applied to the general case *mutatis mutandis*. ■

Remark 5.5 The following relationship issued from fluctuation theory holds for Levy processes: if $x \leqslant a$,

$$\mathbb{E}_x\Big[e^{-\lambda\tau_a^- + i\mu X(\tau_a^-)}\Big] = e^{i\mu a}\,\frac{\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_x\big[e^{i\mu(M(t)-a)}, M(t) \geqslant a\big]\,dt}{\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_0\big[e^{i\mu M(t)}\big]\,dt}. \tag{5.7}$$

Let us check that (5.7) also holds, at least formally, for the pseudo-process X. We have, by (2.5),

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_x\Big[e^{i\mu(M(t)-a)}, M(t) \geqslant a\Big]\,dt$$

$$
\begin{aligned}
&= \int_a^{+\infty} e^{i\mu(z-a)}\,dz \int_0^{+\infty} e^{-\lambda t}\,dt\,\mathbb{P}_x\{M(t)\in dz\}/dz \\
&= \int_a^{+\infty} \lambda^{1/N-1} \sum_{j\in J} \theta_j A_j\, e^{i\mu(z-a)-\theta_j \sqrt[N]{\lambda}\,(z-x)}\,dz \\
&= \frac{1}{\lambda}\sum_{j\in J} \frac{\theta_j A_j}{\theta_j - \frac{i\mu}{\sqrt[N]{\lambda}}}\, e^{\theta_j \sqrt[N]{\lambda}\,(x-a)}. \qquad (5.8)
\end{aligned}
$$

For $x = a$, this yields, by (2.11),

$$
\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_0\Big[e^{i\mu M(t)}\Big]\,dt = \frac{1}{\lambda}\sum_{j\in J}\frac{\theta_j A_j}{\theta_j - \frac{i\mu}{\sqrt[N]{\lambda}}} = \frac{1}{\lambda}\left[\prod_{j\in J}\left(1-\frac{i\mu}{\sqrt[N]{\lambda}}\,\bar{\theta}_j\right)\right]^{-1}. \qquad (5.9)
$$

As a result, by plugging (5.8) and (5.9) into (5.7), we retrieve (5.4). ■

Example 5.6 *Case $N = 2$:* we simply have

$$
\begin{aligned}
\mathbb{E}_x\Big[e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}\Big] &= e^{i\mu a+\sqrt{\lambda}\,(x-a)} \quad \text{for } x\leqslant a, \\
\mathbb{E}_x\Big[e^{-\lambda\tau_a^- + i\mu X(\tau_a^-)}\Big] &= e^{i\mu a-\sqrt{\lambda}\,(x-a)} \quad \text{for } x\geqslant a.
\end{aligned}
$$

■

Example 5.7 *Case $N = 3$:*

- In the case $\kappa_3 = +1$, we have, for $x \leqslant a$,

$$
\sum_{j\in J} A_j \prod_{l\in J\setminus\{j\}}\left(1-\frac{i\mu}{\sqrt[3]{\lambda}}\,\bar{\theta}_l\right) e^{\theta_j\sqrt[3]{\lambda}\,(x-a)} = A_0\, e^{\theta_0\sqrt[3]{\lambda}\,(x-a)} = e^{\sqrt[3]{\lambda}\,(x-a)},
$$

and, for $x \geqslant a$,

$$
\begin{aligned}
&\sum_{k\in K} B_k \prod_{l\in K\setminus\{k\}}\left(1-\frac{i\mu}{\sqrt[3]{\lambda}}\,\bar{\theta}_l\right) e^{\theta_k\sqrt[3]{\lambda}\,(x-a)} \\
&\quad= B_1\left(1-\frac{i\mu}{\sqrt[3]{\lambda}}\,\bar{\theta}_2\right) e^{\theta_1\sqrt[3]{\lambda}\,(x-a)} + B_2\left(1-\frac{i\mu}{\sqrt[3]{\lambda}}\,\bar{\theta}_1\right) e^{\theta_2\sqrt[3]{\lambda}\,(x-a)} \\
&\quad= \frac{1}{\sqrt{3}}\, e^{-\frac12\sqrt[3]{\lambda}\,(x-a)}\left[\left(e^{-i\pi/6}+\frac{\mu}{\sqrt[3]{\lambda}}\right) e^{i\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\,(x-a)} + \left(e^{i\pi/6}-\frac{\mu}{\sqrt[3]{\lambda}}\right) e^{-i\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\,(x-a)}\right].
\end{aligned}
$$

Therefore, (5.4) writes

$$
\begin{aligned}
\mathbb{E}_x\Big[e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}\Big] &= e^{i\mu a+\sqrt[3]{\lambda}\,(x-a)} \quad \text{for } x\leqslant a, \\
\mathbb{E}_x\Big[e^{-\lambda\tau_a^- + i\mu X(\tau_a^-)}\Big] &= \frac{2}{\sqrt{3}}\, e^{i\mu a-\frac12\sqrt[3]{\lambda}\,(x-a)}\left[\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\,(x-a)-\frac{\pi}{6}\right)\right. \\
&\qquad \left. + \frac{i\mu}{\sqrt[3]{\lambda}}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\,(x-a)\right)\right] \quad \text{for } x\geqslant a.
\end{aligned}
$$

- In the case $\kappa_3 = -1$, we similarly have that

$$\begin{aligned}\mathbb{E}_x\Big[e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}\Big] &= \frac{2}{\sqrt{3}}\, e^{i\mu a + \frac{1}{2}\sqrt[3]{\lambda}\,(x-a)} \left[\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\,(x-a) + \frac{\pi}{6}\right)\right. \\ &\qquad \left. - \frac{i\mu}{\sqrt[3]{\lambda}} \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\,(x-a)\right)\right] \quad \text{for } x \leqslant a, \\ \mathbb{E}_x\Big[e^{-\lambda\tau_a^- + i\mu X(\tau_a^-)}\Big] &= e^{i\mu a - \sqrt[3]{\lambda}\,(x-a)} \quad \text{for } x \geqslant a.\end{aligned}$$

■

Example 5.8 *Case $N = 4$:* we have, for $x \leqslant a$,

$$\begin{aligned}&\sum_{j\in J} A_j \prod_{l\in J\setminus\{j\}} \left(1 - \frac{i\mu}{\sqrt[4]{\lambda}}\,\bar{\theta}_l\right) e^{\theta_j \sqrt[4]{\lambda}\,(x-a)} \\ &\quad = A_2\left(1 - \frac{i\mu}{\sqrt[4]{\lambda}}\,\bar{\theta}_3\right) e^{\theta_2\sqrt[4]{\lambda}\,(x-a)} + A_3\left(1 - \frac{i\mu}{\sqrt[4]{\lambda}}\,\bar{\theta}_2\right) e^{\theta_3\sqrt[4]{\lambda}\,(x-a)} \\ &\quad = \frac{1}{\sqrt{2}}\, e^{\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a)} \left[\left(e^{-i\pi/4} - \frac{\mu}{\sqrt[4]{\lambda}}\right) e^{-i\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a)} + \left(e^{i\pi/4} + \frac{\mu}{\sqrt[4]{\lambda}}\right) e^{i\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a)}\right],\end{aligned}$$

and, for $x \geqslant a$,

$$\begin{aligned}&\sum_{k\in K} B_k \prod_{l\in K\setminus\{k\}} \left(1 - \frac{i\mu}{\sqrt[4]{\lambda}}\,\bar{\theta}_l\right) e^{\theta_k \sqrt[4]{\lambda}\,(x-a)} \\ &\quad = B_0\left(1 - \frac{i\mu}{\sqrt[4]{\lambda}}\,\bar{\theta}_1\right) e^{\theta_0\sqrt[4]{\lambda}\,(x-a)} + B_1\left(1 - \frac{i\mu}{\sqrt[4]{\lambda}}\,\bar{\theta}_0\right) e^{\theta_1\sqrt[4]{\lambda}\,(x-a)} \\ &\quad = \frac{1}{\sqrt{2}}\, e^{-\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a)} \left[\left(e^{-i\pi/4} + \frac{\mu}{\sqrt[4]{\lambda}}\right) e^{i\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a)} + \left(e^{i\pi/4} - \frac{\mu}{\sqrt[4]{\lambda}}\right) e^{-i\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a)}\right].\end{aligned}$$

Therefore, (5.4) becomes

$$\begin{aligned}\mathbb{E}_x\Big[e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}\Big] &= \sqrt{2}\, e^{i\mu a + \frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a)} \left[\cos\left(\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a) + \frac{\pi}{4}\right)\right. \\ &\qquad \left. + \frac{i\mu}{\sqrt[4]{\lambda}} \sin\left(\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a)\right)\right] \quad \text{for } x \leqslant a, \\ \mathbb{E}_x\Big[e^{-\lambda\tau_a^- + i\mu X(\tau_a^-)}\Big] &= \sqrt{2}\, e^{i\mu a - \frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a)} \left[\cos\left(\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a) - \frac{\pi}{4}\right)\right. \\ &\qquad \left. + \frac{i\mu}{\sqrt[4]{\lambda}} \sin\left(\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}\,(x-a)\right)\right] \quad \text{for } x \geqslant a.\end{aligned}$$

We retrieve formula (8.3) of [17]. ■

5.2 Density functions

We invert the Laplace-Fourier transform (5.4). For this, we proceed in two stages: we first invert the Fourier transform with respect to μ and next invert the Laplace transform with respect to λ.

5.2.1 Inverting with respect to μ

Let us expand the product $\prod_{l\in J\setminus\{j\}}\left(1-\bar{\theta}_l x\right)$ as

$$\prod_{l\in J\setminus\{j\}}\left(1-\bar{\theta}_l x\right)=\sum_{q=0}^{\#J-1}\bar{c}_{jq}(-x)^q \tag{5.10}$$

where the coefficients c_{jq}, $0\leqslant q\leqslant \#J-1$, are the elementary symmetric functions of the θ_l's, $l\in J\setminus\{j\}$, that is, more explicitly, $c_{j0}=1$ and for $1\leqslant q\leqslant \#J-1$,

$$c_{jq}=\sigma_q\left(\theta_l,l\in J\setminus\{j\}\right)=\sum_{\substack{l_1,\dots,l_q\in J\setminus\{j\}\\ l_1<\dots<l_q}}\theta_{l_1}\cdots\theta_{l_q}.$$

In a similar way, we also introduce $d_{k0}=1$ and for $1\leqslant q\leqslant \#K-1$,

$$d_{kq}=\sigma_q\left(\theta_l,l\in K\setminus\{k\}\right)=\sum_{\substack{l_1,\dots,l_q\in K\setminus\{k\}\\ l_1<\dots<l_q}}\theta_{l_1}\cdots\theta_{l_q}.$$

By applying expansion (5.10) to $x=i\mu/\sqrt[N]{\lambda}$, we see that (5.4) can be rewritten as

$$\begin{aligned}\mathbb{E}_x\!\left[e^{-\lambda\tau_a^+ +i\mu X(\tau_a^+)}\right] &= \sum_{j\in J}A_j\sum_{q=0}^{\#J-1}\bar{c}_{jq}\left(-\frac{i\mu}{\sqrt[N]{\lambda}}\right)^q e^{\theta_j\sqrt[N]{\lambda}\,(x-a)}\,e^{i\mu a}\\ &= \sum_{q=0}^{\#J-1}\frac{1}{\lambda^{q/N}}\left[\sum_{j\in J}\bar{c}_{jq}A_j e^{\theta_j\sqrt[N]{\lambda}\,(x-a)}\right](-i\mu)^q\,e^{i\mu a}.\end{aligned}$$

Now, observe that $(-i\mu)^q\,e^{i\mu a}$ is nothing but the Fourier transform of the q^{th} derivative of the Dirac distribution viewed as a tempered Schwartz distribution:

$$(-i\mu)^q\,e^{i\mu a}=\int_{-\infty}^{+\infty}e^{i\mu z}\,\delta_a^{(q)}(z)\,dz. \tag{5.11}$$

Hence, we have obtained the following intermediate result for the distribution of $(\tau_a^+,X(\tau_a^+))$ and also for that of $(\tau_a^-,X(\tau_a^-))$.

Proposition 5.9 *We have, for $\Re(\lambda)>0$,*

$$\begin{aligned}\mathbb{E}_x\!\left[e^{-\lambda\tau_a^+},X(\tau_a^+)\in dz\right]/dz &= \sum_{q=0}^{\#J-1}\lambda^{-q/N}\left[\sum_{j\in J}\bar{c}_{jq}A_j\,e^{\theta_j\sqrt[N]{\lambda}\,(x-a)}\right]\delta_a^{(q)}(z)\quad \text{for } x\leqslant a,\\ \mathbb{E}_x\!\left[e^{-\lambda\tau_a^-},X(\tau_a^-)\in dz\right]/dz &= \sum_{q=0}^{\#K-1}\lambda^{-q/N}\left[\sum_{k\in K}\bar{d}_{kq}B_k\,e^{\theta_k\sqrt[N]{\lambda}\,(x-a)}\right]\delta_a^{(q)}(z)\quad \text{for } x\geqslant a.\end{aligned} \tag{5.12}$$

The appearance of the successive derivatives of δ_a suggests to view the distribution of $(\tau_a^+,X(\tau_a^+))$ as a tempered Schwartz distribution (that is a Schwartz distribution acting on the space $\mathcal{S}$ of the $\mathcal{C}^\infty$-functions exponentially decreasing together with their derivatives characterized by

$$\forall\varphi,\psi\in\mathcal{S},\ \iint\varphi(t)\psi(z)\,\mathbb{P}_x\{\tau_a^+\in dt,X(\tau_a^+)\in dz\}=\mathbb{E}_x\!\left[\varphi(\tau_a^+)\psi(X(\tau_a^+))\right].$$

5.2.2 Inversion with respect to λ

In order to extract the densities of $(\tau_a^+, X(\tau_a^+))$ and $(\tau_a^-, X(\tau_a^-))$ from (5.12), we search functions I_{lq}, $0 \leqslant q \leqslant \max(\#I-1, \#J-1)$, such that, for $\Re(\theta_l \xi) < 0$,

$$\int_0^{+\infty} e^{-\lambda t} I_{lq}(t;\xi)\, dt = \lambda^{-q/N} e^{\theta_l \sqrt[N]{\lambda}\, \xi}. \tag{5.13}$$

The rhs of (5.13) seems closed to the Laplace transform of the probability density function of a completely asymmetric stable random variable, at least for $q=0$. Nevertheless, because of the presence of the complex term θ_l within the rhs of (5.13), we did not find any precise relationship between the function I_{lq} and stable processes. So, we derive below an integral representation for I_{lq}.

Invoking Bromwich formula, the function I_{lq} writes

$$\begin{aligned} I_{lq}(t;\xi) &= \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \lambda^{-q/N} e^{t\lambda + \theta_l \xi \sqrt[N]{\lambda}}\, d\lambda \;=\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} (i\lambda)^{-\frac{q}{N}} e^{it\lambda + \theta_l \xi \sqrt[N]{i\lambda}}\, d\lambda \\ &= \frac{1}{2\pi} \left[e^{-\frac{i\pi q}{2N}} \int_0^{+\infty} \lambda^{-\frac{q}{N}} e^{it\lambda + \theta_l e^{\frac{i\pi}{2N}} \xi \sqrt[N]{\lambda}}\, d\lambda + e^{\frac{i\pi q}{2N}} \int_0^{+\infty} \lambda^{-\frac{q}{N}} e^{-it\lambda + \theta_l e^{-\frac{i\pi}{2N}} \xi \sqrt[N]{\lambda}}\, d\lambda \right]. \end{aligned}$$

The substitution $\lambda \longmapsto \lambda^N$ yields

$$\begin{aligned} I_{lq}(t;\xi) &= \frac{N}{2\pi} \left[e^{-\frac{i\pi q}{2N}} \int_0^{+\infty} \lambda^{N-q-1} e^{it\lambda^N + \theta_l e^{\frac{i\pi}{2N}} \xi \lambda}\, d\lambda \right. \\ &\quad \left. + e^{\frac{i\pi q}{2N}} \int_0^{+\infty} \lambda^{N-q-1} e^{-it\lambda^N + \theta_l e^{-\frac{i\pi}{2N}} \xi \lambda}\, d\lambda \right] \end{aligned}$$

and the substitutions $\lambda \longmapsto e^{\pm \frac{i\pi}{2N}} \lambda$ together with the residues theorem provide

$$\begin{aligned} I_{lq}(t;\xi) &= \frac{Ni}{2\pi} \left[e^{-\frac{i\pi q}{N}} \int_0^{+\infty} \lambda^{N-q-1} e^{-t\lambda^N + \theta_l e^{\frac{i\pi}{N}} \xi \lambda}\, d\lambda \right. \\ &\quad \left. - e^{\frac{i\pi q}{N}} \int_0^{+\infty} \lambda^{N-q-1} e^{-t\lambda^N + \theta_l e^{-\frac{i\pi}{N}} \xi \lambda}\, d\lambda \right]. \end{aligned}$$

In particular, for $q=0$ we have, by integration by parts,

$$I_{l0}(t;\xi) = \frac{i\theta_l \xi}{2\pi t} \left[e^{\frac{i\pi}{N}} \int_0^{+\infty} e^{-t\lambda^N + \theta_l e^{\frac{i\pi}{N}} \xi \lambda}\, d\lambda - e^{-\frac{i\pi}{N}} \int_0^{+\infty} e^{-t\lambda^N + \theta_l e^{-\frac{i\pi}{N}} \xi \lambda}\, d\lambda \right]. \tag{5.14}$$

Remark 5.10 The following relation holds between all the functions I_{lq}'s:

$$\frac{\partial^m I_{lq}}{\partial \xi^m}(t;\xi) = \theta_l^m I_{l\,q-m}(t;\xi) \quad \text{for } 0 \leqslant m \leqslant q.$$

■

Hence, (5.12) can be rewritten as an explicit Laplace transform with respect to λ:

$$\mathbb{E}_x\!\left[e^{-\lambda \tau_a^+}, X(\tau_a^+) \in dz\right]/dz = \int_0^{+\infty} e^{-\lambda t}\, dt \left[\sum_{q=0}^{\#J-1} \left(\sum_{j \in J} \bar{c}_{jq} A_j\, I_{jq}(t; x-a) \right) \delta_a^{(q)}(z) \right].$$

We are able to state the main result of this part.

Theorem 5.11 *The joint "distributional densities" of the vectors* $(\tau_a^+, X(\tau_a^+))$ *and* $(\tau_a^-, X(\tau_a^-))$ *are given by*

$$\begin{aligned} \mathbb{P}_x\{\tau_a^+ \in dt, X(\tau_a^+) \in dz\}/dt\,dz &= \sum_{q=0}^{\#J-1} \mathcal{J}_q(t; x-a)\,\delta_a^{(q)}(z) \quad \textit{for } x \leqslant a, \\ \mathbb{P}_x\{\tau_a^- \in dt, X(\tau_a^-) \in dz\}/dt\,dz &= \sum_{q=0}^{\#K-1} \mathcal{K}_q(t; x-a)\,\delta_a^{(q)}(z) \quad \textit{for } x \geqslant a, \end{aligned} \tag{5.15}$$

where

$$\mathcal{J}_q(t;\xi) = \sum_{j\in J} \bar{c}_{jq} A_j\, I_{jq}(t;\xi) \quad \textit{and} \quad \mathcal{K}_q(t;\xi) = \sum_{k\in K} \bar{d}_{kq} B_k\, I_{kq}(t;\xi).$$

Remark 5.12 Another expression for $\mathcal{J}_q(t;\xi)$, for instance, may be written. Indeed, for $\xi \leqslant 0$ and $0 \leqslant q \leqslant \#J-1$,

$$\begin{aligned} \mathcal{J}_q(t;\xi) &= \frac{Ni}{2\pi}\Bigg[e^{-\frac{i\pi q}{N}} \int_0^{+\infty} \Bigg(\sum_{j\in J} \bar{c}_{jq} A_j\, e^{\theta_j e^{\frac{i\pi}{N}}\xi\lambda}\Bigg) \lambda^{N-q-1} e^{-t\lambda^N}\,d\lambda \\ &\quad - e^{\frac{i\pi q}{N}} \int_0^{+\infty} \Bigg(\sum_{j\in J} \bar{c}_{jq} A_j\, e^{\theta_j e^{-\frac{i\pi}{N}}\xi\lambda}\Bigg) \lambda^{N-q-1} e^{-t\lambda^N}\,d\lambda\Bigg]. \end{aligned} \tag{5.16}$$

The second integral displayed in (5.16) is the conjugate of the first one. In effect, by introducing the symmetry $\sigma : j \in J \longmapsto \sigma(j) \in J$ such that $\theta_{\sigma(j)} = \bar{\theta}_j$, we can see that

$$A_{\sigma(j)} = \prod_{l\in J\setminus\{\sigma(j)\}} \frac{\theta_l}{\theta_l - \theta_{\sigma(j)}} = \prod_{l\in J\setminus\{j\}} \frac{\theta_{\sigma(l)}}{\theta_{\sigma(l)} - \theta_{\sigma(j)}} = \prod_{l\in J\setminus\{j\}} \frac{\bar{\theta}_l}{\bar{\theta}_l - \bar{\theta}_j} = \bar{A}_j$$

and

$$c_{\sigma(j)q} = \sigma_q\left(\theta_l, l\in J\setminus\{\sigma(j)\}\right) = \sigma_q\left(\theta_{\sigma(l)}, l\in J\setminus\{j\}\right) = \sigma_q\left(\bar{\theta}_l, l\in J\setminus\{j\}\right) = \bar{c}_{jq}.$$

So, the sum within the second integral in (5.16) writes

$$\sum_{j\in J} \bar{c}_{\sigma(j)q} A_{\sigma(j)}\, e^{\theta_{\sigma(j)} e^{-\frac{i\pi}{N}}\xi\lambda} = \sum_{j\in J} c_{jq} \bar{A}_j\, e^{\bar{\theta}_j e^{-\frac{i\pi}{N}}\xi\lambda} = \Bigg(\sum_{j\in J} \bar{c}_{jq} A_j\, e^{\theta_j e^{\frac{i\pi}{N}}\xi\lambda}\Bigg)^{-}.$$

As a result,

$$\mathcal{J}_q(t;\xi) = -\frac{N}{\pi}\,\Im\Bigg[e^{-\frac{i\pi q}{N}} \int_0^{+\infty} \Bigg(\sum_{j\in J} \bar{c}_{jq} A_j\, e^{\theta_j e^{\frac{i\pi}{N}}\xi\lambda}\Bigg) \lambda^{N-q-1} e^{-t\lambda^N}\,d\lambda\Bigg].$$

In particular, $\mathcal{J}_q(t;\xi)$ is real and for $q=0$ we have, since $c_{j0}=1$ and $\sum_{j\in J} A_j = 1$,

$$\begin{aligned} \mathcal{J}_0(t;\xi) &= -\frac{N}{\pi}\,\Im\Bigg[\int_0^{+\infty} \Bigg(\sum_{j\in J} A_j\, e^{\theta_j e^{\frac{i\pi}{N}}\xi\lambda}\Bigg) \lambda^{N-1} e^{-t\lambda^N}\,d\lambda\Bigg] \\ &= -\frac{\xi}{\pi t}\,\Im\Bigg[e^{\frac{i\pi}{N}} \int_0^{+\infty} \Bigg(\sum_{j\in J} \theta_j A_j\, e^{\theta_j e^{\frac{i\pi}{N}}\xi\lambda}\Bigg) e^{-t\lambda^N}\,d\lambda\Bigg] \end{aligned}$$

which is nothing but $\mathbb{P}_x\{\tau_a^+ \in dt\}/dt$. ■

5.3 Distribution of the hitting places

We now derive the distribution of the hitting places $X(\tau_a^+)$ and $X(\tau_a^-)$. To do this for $X(\tau_a^+)$ for example, we integrate (5.15) with respect to t:

$$\begin{aligned}
\mathbb{P}_x\{X(\tau_a^+) \in dz\}/dz &= \int_0^{+\infty} \mathbb{P}_x\{\tau_a^+ \in dt, X(\tau_a^+) \in dz\}/dz \\
&= \sum_{q=0}^{\#J-1} \left[\int_0^{+\infty} \mathcal{J}_q(t; x-a)\, dt\right] \delta_a^{(q)}(z) \\
&= -\frac{N}{\pi} \sum_{q=0}^{\#J-1} |x-a|^q \left[\int_0^{+\infty} \Im\left(e^{-\frac{i\pi q}{N}} \sum_{j\in J} \bar{c}_{jq} A_j\, e^{-\theta_j e^{\frac{i\pi}{N}} \lambda}\right) \frac{d\lambda}{\lambda^{q+1}}\right] \delta_a^{(q)}(z). \qquad (5.17)
\end{aligned}$$

We need two lemmas for carrying out the integral in (5.17).

Lemma 5.13 *For any integers m, n such that $1 \leqslant n \leqslant m-1$ and any complexes $a_1, \dots, a_m$ and $b_1, \dots, b_m$ such that $\Re(b_j) \geqslant 0$ and $\Im\Big(\sum_{j=1}^m a_j b_j^l\Big) = 0$ for $0 \leqslant l \leqslant n-1$,*

$$\int_0^{+\infty} \Im\left(\sum_{j=1}^m a_j e^{-b_j\lambda}\right) \frac{d\lambda}{\lambda^n} = \frac{(-1)^n}{(n-1)!} \Im\left(\sum_{j=1}^m a_j b_j^{n-1} \log b_j\right).$$

Proof. We proceed by induction on n.
For $n = 1$, because of the condition $\Im\Big(\sum_{j=1}^m a_j\Big) = 0$, we can replace $\Im(a_m)$ by $-\Im\Big(\sum_{j=1}^{m-1} a_j\Big)$. This gives

$$\int_0^{+\infty} \Im\left(\sum_{j=1}^m a_j e^{-b_j\lambda}\right) \frac{d\lambda}{\lambda} = \int_0^{+\infty} \Im\left[\sum_{j=1}^{m-1} a_j \left(e^{-b_j\lambda} - e^{-b_m\lambda}\right)\right] \frac{d\lambda}{\lambda}.$$

The foregoing integral involves the elementary integral below:

$$\int_0^{+\infty} \Im\left(e^{-b_j\lambda} - e^{-b_m\lambda}\right) \frac{d\lambda}{\lambda} = \Im\left(\log b_m - \log b_j\right).$$

Therefore,

$$\int_0^{+\infty} \Im\left(\sum_{j=1}^m a_j e^{-b_j\lambda}\right) \frac{d\lambda}{\lambda} = \Im\left[\sum_{j=1}^{m-1} a_j (\log b_m - \log b_j)\right] = -\Im\left[\sum_{j=1}^m a_j \log b_j\right]$$

which proves Lemma 5.13 in the case $n = 1$.

Assume now the result of the lemma valid for an integer $n \geqslant 1$. Let m be an integer such that $m \geqslant n+2$ and $a_1, \dots, a_m$ and $b_1, \dots, b_m$ be complex numbers such that $\Re(b_j) \geqslant 0$ and $\Im\Big(\sum_{j=1}^m a_j b_j^l\Big) = 0$ for $0 \leqslant l \leqslant n$. By integration by parts, we have

$$\int_0^{+\infty} \Im\left(\sum_{j=1}^m a_j e^{-b_j\lambda}\right) \frac{d\lambda}{\lambda^{n+1}} = \left[-\frac{1}{n\lambda^n} \Im\left(\sum_{j=1}^m a_j e^{-b_j\lambda}\right)\right]_0^\infty - \frac{1}{n} \int_0^{+\infty} \Im\left(\sum_{j=1}^m a_j b_j e^{-b_j\lambda}\right) \frac{d\lambda}{\lambda^n}.$$

Applying L'Hôpital's rule n times, we see, using the condition $\Im\Big(\sum_{j=1}^m a_j b_j^l\Big) = 0$ for $0 \leqslant l \leqslant n$, that $\left[-\frac{1}{n\lambda^n}\Im\left(\sum_{j=1}^m a_j e^{-b_j\lambda}\right)\right]_0^\infty = 0$. Putting $\tilde{a}_j = a_j b_j$, we get

$$\int_0^{+\infty} \Im\left(\sum_{j=1}^m a_j e^{-b_j\lambda}\right)\frac{d\lambda}{\lambda^{n+1}} = -\frac{1}{n}\int_0^{+\infty} \Im\left(\sum_{j=1}^m \tilde{a}_j e^{-b_j\lambda}\right)\frac{d\lambda}{\lambda^n}.$$

We have $\Im\Big(\sum_{j=1}^m \tilde{a}_j b_j^l\Big) = \Im\Big(\sum_{j=1}^m a_j b_j^{l+1}\Big) = 0$ for $0 \leqslant l \leqslant n-1$. Then, invoking the recurrence hypothesis, the intermediate integral writes

$$\int_0^{+\infty} \Im\left(\sum_{j=1}^m \tilde{a}_j e^{-b_j\lambda}\right)\frac{d\lambda}{\lambda^n} = \frac{(-1)^n}{(n-1)!}\Im\left(\sum_{j=1}^m \tilde{a}_j b_j^{n-1}\log b_j\right)$$

and thus

$$\int_0^{+\infty} \Im\left(\sum_{j=1}^m a_j e^{-b_j\lambda}\right)\frac{d\lambda}{\lambda^{n+1}} = \frac{(-1)^{n+1}}{n!}\Im\left(\sum_{j=1}^m a_j b_j^n\log b_j\right)$$

which achieve the proof of Lemma 5.13. ■

Lemma 5.14 *We have, for $0 \leqslant p \leqslant q \leqslant \#J-1$,*

$$\sum_{j\in J}\bar{c}_{jq}\theta_j^p A_j = \begin{cases} 0 & \text{if } p \leqslant q-1, \\ (-1)^q & \text{if } p = q. \end{cases}$$

Proof. Consider the following polynomial:

$$\begin{aligned}\sum_{q=0}^{\#J-1}\left(\sum_{j\in J}\bar{c}_{jq}\theta_j^p A_j\right)(-x)^q &= \sum_{j\in J}\theta_j^p A_j\sum_{q=0}^{\#J-1}\bar{c}_{jq}(-x)^q \\ &= \sum_{j\in J}\theta_j^p A_j\prod_{l\in J\setminus\{j\}}(1-\bar{\theta}_l x) \\ &= \prod_{l\in J}(1-\bar{\theta}_l x)\sum_{j\in J}\frac{\theta_j^p A_j}{1-\bar{\theta}_j x}.\end{aligned}$$

We then obtain, due to (2.11), if $p \leqslant \#J-1$,

$$\sum_{q=0}^{\#J-1}\left(\sum_{j\in J}\bar{c}_{jq}\theta_j^p A_j\right)(-x)^q = x^p$$

which entails the result by identifying the coefficients of both polynomials above. ■

Now, we state the following remarkable result.

Theorem 5.15 *The "distributional densities" of $X(\tau_a^+)$ and $X(\tau_a^-)$ are given by*

$$\begin{aligned}\mathbb{P}_x\{X(\tau_a^+)\in dz\}/dz &= \sum_{q=0}^{\#J-1}(-1)^q\frac{(x-a)^q}{q!}\,\delta_a^{(q)}(z) \quad \text{for } x \leqslant a, \\ \mathbb{P}_x\{X(\tau_a^-)\in dz\}/dz &= \sum_{q=0}^{\#K-1}(-1)^q\frac{(x-a)^q}{q!}\,\delta_a^{(q)}(z) \quad \text{for } x \geqslant a.\end{aligned} \tag{5.18}$$

It is worth that the distributions of $X(\tau_a^+)$ and $X(\tau_a^-)$ are linear combinations of the successive derivatives of the Dirac distribution δ_a. This noteworthy fact has already been observed by Nishioka [16, 17] in the case $N=4$ and the author spoke of "monopoles" and "dipoles" respectively related to δ_a and δ'_a (see also [18] for more account about relationships between monopoles/dipoles and different kinds of absorbed/killed pseudo-processes). More generally, (5.18) suggests to speak of "multipoles" related to the $\delta_a^{(q)}$'s.

In the case of Brownian motion ($N=2$), the trajectories are continuous, so $X(\tau_a^{\pm})=a$ and then we classically write $\mathbb{P}_x\{X(\tau_a^{\pm})\in dz\}=\delta_a(dz)$ where δ_a is viewed as the Dirac probability measure. For $N\geqslant 4$, it emerges from (5.18) that the distributional densities of $X(\tau_a^{\pm})$ are concentrated at the point a through a sequence of successive derivatives of δ_a where δ_a is now viewed as a Schwartz distribution. Hence, we could guess in (5.18) a curious and unclear kind of continuity. In Subsection 5.6, we study the distribution of $X(\tau_a^{\pm}-)$ which will reveal itself to coincide with that of $X(\tau_a^{\pm})$. This will confirm this idea of continuity.

Proof. Let us evaluate the integral in (5.17). We have, thanks to Lemma 5.14,

$$e^{-\frac{i\pi q}{N}}\sum_{j\in J}\bar{c}_{jq}A_j\left(\theta_j e^{\frac{i\pi}{N}}\right)^l=e^{\frac{i\pi}{N}(l-q)}\sum_{j\in J}\bar{c}_{jq}A_j\theta_j^l=0 \text{ if } l\leqslant q-1.$$

Therefore, the conditions of Lemma 5.13 are fulfilled and we get

$$\begin{aligned}
&\mathbb{P}_x\{X(\tau_a^+)\in dz\}/dz\\
&\quad=\sum_{q=0}^{\#J-1}\frac{(-1)^qN}{\pi q!}\,|x-a|^q\left[\sum_{j\in J}\Im\left(\bar{c}_{jq}A_j\theta_j^q\log\left(\theta_je^{\frac{i\pi}{N}}\right)\right)\right]\delta_a^{(q)}(z)\\
&\quad=\sum_{q=0}^{\#J-1}\frac{(-1)^qN}{\pi q!}\,|x-a|^q\left[\Re\left(\sum_{j\in J}\bar{c}_{jq}A_j\theta_j^q\arg(\theta_j)\right)+\frac{\pi}{N}\Re\left(\sum_{j\in J}\bar{c}_{jq}A_j\theta_j^q\right)\right]\delta_a^{(q)}(z).
\end{aligned}$$

The second sum within the brackets is equal, by Lemma 5.14, to $(-1)^q$. The first one vanishes: indeed, by using the symmetry $\sigma: j\in J\longmapsto\sigma(j)\in J$ such that $\theta_{\sigma(j)}=\bar{\theta}_j$,

$$\begin{aligned}
\Re\left(\sum_{j\in J}\left(\bar{c}_{jq}A_j\theta_j^q\right)\arg(\theta_j)\right)&=\frac{1}{2}\left(\sum_{j\in J}\bar{c}_{jq}A_j\theta_j^q\arg(\theta_j)+\sum_{j\in J}c_{jq}\bar{A}_j\bar{\theta}_j^q\arg(\theta_j)\right)\\
&=\frac{1}{2}\left(\sum_{j\in J}\bar{c}_{jq}A_j\theta_j^q\arg(\theta_j)+\sum_{j\in J}c_{\sigma(j)q}\bar{A}_{\sigma(j)}\bar{\theta}_{\sigma(j)}^q\arg(\theta_{\sigma(j)})\right).
\end{aligned}$$

The terms of the second last sum are the opposite of those of the first sum since

$$c_{\sigma(j)q}\bar{A}_{\sigma(j)}\bar{\theta}_{\sigma(j)}^q=\bar{c}_{jq}A_j\theta_j^q\qquad\text{and}\qquad\arg(\theta_{\sigma(j)})=-\arg(\theta_j)$$

which proves the assertion. As a result, we get (5.18). ■

5.4 Fourier transforms of the hitting places

By using (5.18) and (5.11), it is easy to derive the Fourier transforms of the hitting places $X(\tau_a^+)$ and $X(\tau_a^-)$.

Proposition 5.16 *The Fourier transforms of $X(\tau_a^+)$ and $X(\tau_a^-)$ are given by*

$$\begin{aligned}\mathbb{E}_x\Big[e^{i\mu X(\tau_a^+)}\Big] &= e^{i\mu a}\sum_{q=0}^{\#J-1}\frac{(x-a)^q}{q!}\,(i\mu)^q \quad \textit{for } x\leqslant a,\\ \mathbb{E}_x\Big[e^{i\mu X(\tau_a^-)}\Big] &= e^{i\mu a}\sum_{q=0}^{\#K-1}\frac{(x-a)^q}{q!}\,(i\mu)^q \quad \textit{for } x\geqslant a.\end{aligned} \tag{5.19}$$

In this part, we suggest to retrieve (5.19) by letting λ tend to 0^+ in (5.4). We rewrite (5.4), for instance for $x\leqslant a$, as

$$\begin{aligned}\mathbb{E}_x\Big[e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}\Big] &= e^{i\mu a}\prod_{l\in J}\left(1-\frac{i\mu}{\sqrt[N]{\lambda}}\,\bar\theta_l\right)\sum_{j\in J}\frac{A_j}{1-\frac{i\mu}{\sqrt[N]{\lambda}}\,\bar\theta_j}\,e^{\theta_j\sqrt[N]{\lambda}\,(x-a)}\\ &= \left(-\frac{i\mu}{\sqrt[N]{\lambda}}\right)^{\#J-1}\left(\prod_{j\in J}\bar\theta_j\right)e^{i\mu a}\\ &\quad\times\prod_{j\in J}\left(1-\frac{\theta_j\sqrt[N]{\lambda}}{i\mu}\right)\sum_{j\in J}\frac{\theta_j A_j}{1-\frac{\theta_j\sqrt[N]{\lambda}}{i\mu}}\,e^{\theta_j\sqrt[N]{\lambda}\,(x-a)}.\end{aligned} \tag{5.20}$$

Using the elementary expansions, as $\lambda\to 0^+$,

$$\begin{aligned}\frac{1}{1-\frac{\theta_j\sqrt[N]{\lambda}}{i\mu}} &= \sum_{p=0}^{\#J-1}\left(\frac{\theta_j\sqrt[N]{\lambda}}{i\mu}\right)^p + o\left(\lambda^{(\#J-1)/N}\right),\\ e^{\theta_j\sqrt[N]{\lambda}\,(x-a)} &= \sum_{q=0}^{\#J-1}\frac{1}{q!}\left(\theta_j\sqrt[N]{\lambda}\,(x-a)\right)^q + o\left(\lambda^{(\#J-1)/N}\right),\end{aligned}$$

yields

$$\begin{aligned}\sum_{j\in J}\frac{\theta_j A_j}{1-\frac{\theta_j\sqrt[N]{\lambda}}{i\mu}}\,e^{\theta_j\sqrt[N]{\lambda}\,(x-a)} &= \sum_{j\in J}\theta_j A_j\left[\sum_{r=0}^{\#J-1}\left(\sum_{q=0}^{r}\frac{(x-a)^q}{q!(i\mu)^{r-q}}\right)\left(\theta_j\sqrt[N]{\lambda}\right)^r\right] + o\left(\lambda^{(\#J-1)/N}\right)\\ &= \sum_{r=0}^{\#J-1}\left(\sum_{j\in J}\theta_j^{r+1}A_j\right)\left(\sum_{q=0}^{r}\frac{(x-a)^q}{q!(i\mu)^{r-q}}\right)\lambda^{r/N} + o\left(\lambda^{(\#J-1)/N}\right).\end{aligned}$$

On the other hand, applying (2.11) to $x=0$ gives

$$\sum_{j\in J}\theta_j^{r+1}A_j = \begin{cases} 0 & \text{if } r\leqslant \#J-2,\\ (-1)^{\#J-1}\prod_{j\in J}\theta_j & \text{if } r=\#J-1.\end{cases}$$

Therefore,

$$\sum_{j\in J}\frac{\theta_j A_j}{1-\frac{\theta_j\sqrt[N]{\lambda}}{i\mu}}\,e^{\theta_j\sqrt[N]{\lambda}\,(x-a)} \underset{\lambda\to 0^+}{\sim} (-1)^{\#J-1}\left(\prod_{j\in J}\theta_j\right)\left(\sum_{q=0}^{\#J-1}\frac{(x-a)^q}{q!(i\mu)^{\#J-1-q}}\right)\lambda^{(\#J-1)/N}. \tag{5.21}$$

Consequently, the limit of $\mathbb{E}_x\Big[e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}\Big]$ as $\lambda \to 0^+$ ensues. The constant arising when combining (5.20) and (5.21) is

$$(-1)^{\#J-1}\left(\prod_{j\in J}\theta_j\right)\times\left(\prod_{j\in J}\bar{\theta}_j\right)(-i\mu)^{\#J-1}e^{i\mu a} = (i\mu)^{\#J-1}e^{i\mu a}.$$

In view of (5.19), we have proved the equality

$$\lim_{\lambda\to 0^+}\mathbb{E}_x\Big[e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}\Big] = \mathbb{E}_x\Big[e^{i\mu X(\tau_a^+)}\Big].$$

Remark 5.17 The distribution of $X(\tau_a^+)$ may also be deduced from the joint distribution of $(\tau_a^+, X(\tau_a^+))$ through (5.12). Indeed, by letting λ tend to 0 in (5.12) and using elementary expansions together with Lemma 5.14,

$$\begin{aligned}
\sum_{j\in J}\bar{c}_{jq}A_j\,e^{\theta_j\sqrt[N]{\lambda}(x-a)} &= \sum_{j\in J}\bar{c}_{jq}A_j\sum_{p=0}^{q}\frac{(\theta_j(x-a))^p}{p!}\,\lambda^{p/N} + o\left(\lambda^{q/N}\right)\\
&= \sum_{p=0}^{q}\left(\sum_{j\in J}\bar{c}_{jq}A_j\theta_j^p\right)\frac{(x-a)^p}{p!}\,\lambda^{p/N} + o\left(\lambda^{q/N}\right)\\
&\underset{\lambda\to 0^+}{\sim} (-1)^q\frac{(x-a)^q}{q!}\,\lambda^{q/N},
\end{aligned}$$

which, with (5.12), confirms (5.18). ■

5.5 Strong Markov property for $\tau_a^\pm$

We roughly state a strong Markov property related to the hitting times $\tau_a^\pm$.

Theorem 5.18 *For suitable functionals F and G, we have*

$$\begin{aligned}
&\mathbb{E}_x\Big[F\Big((X(t))_{0\leqslant t<\tau_a^\pm}\Big)\,G\big((X(t+\tau_a^\pm))_{t\geqslant 0}\big)\Big]\\
&\qquad = \mathbb{E}_x\Big[F\Big((X(t))_{0\leqslant t<\tau_a^\pm}\Big)\,\mathbb{E}_{X(\tau_a^\pm)}\left[G((X(t))_{t\geqslant 0})\right]\Big], \qquad (5.22)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_x\big[G\big((X(t+\tau_a^+))_{t\geqslant 0}\big)\big] &= \sum_{q=0}^{\#J-1}\frac{(x-a)^q}{q!}\,\frac{\partial^q}{\partial z^q}\,\mathbb{E}_z[G((X(t))_{t\geqslant 0})]\Big|_{z=a} \quad \text{if } x\geqslant a,\\
\mathbb{E}_x\big[G\big((X(t+\tau_a^-))_{t\geqslant 0}\big)\big] &= \sum_{q=0}^{\#K-1}\frac{(x-a)^q}{q!}\,\frac{\partial^q}{\partial z^q}\,\mathbb{E}_z[G((X(t))_{t\geqslant 0})]\Big|_{z=a} \quad \text{if } x\leqslant a.
\end{aligned} \qquad (5.23)$$

Proof. We first consider the step-process X_n and we use the notations of Subsection 5.1. On the set $\{\tau_{a,n}^+ = k/2^n\}$, the quantities $F\Big((X_n(t))_{0\leqslant t<\tau_{a,n}^+}\Big)$ and $G\big((X_n(t+\tau_{a,n}^+))_{t\geqslant 0}\big)$ depend respectively on $X_{n,0}, X_{n,1}, \ldots, X_{n,k-1}$ and $X_{n,k}, X_{n,k+1}, \ldots$ So we can set, if $\tau_{a,n}^+ = k/2^n$,

$$\begin{aligned}
F\Big((X_n(t))_{0\leqslant t<\tau_{a,n}^+}\Big) &= F_k(X_{n,0}, X_{n,1}, \ldots, X_{n,k-1}) \;=\; F_{n,k-1},\\
G\big((X_n(t+\tau_{a,n}^+))_{t\geqslant 0}\big) &= G_k(X_{n,k}, X_{n,k+1}, \ldots) \;=\; G_{n,k}.
\end{aligned}$$

Therefore,

$$F\Big((X_n(t))_{0\leqslant t<\tau_{a,n}^+}\Big)\,G\big((X_n(t+\tau_{a,n}^+))_{t\geqslant 0}\big)=\sum_{k=1}^{\infty}F_{n,k-1}G_{n,k}\mathbb{1}_{\{M_{n,k-1}<a\leqslant M_{n,k}\}}.$$

Taking the expectations, we get for $x\leqslant a$:

$$\begin{aligned}
&\mathbb{E}_x\Big[F\Big((X_n(t))_{0\leqslant t<\tau_{a,n}^+}\Big)\,G\big((X_n(t+\tau_{a,n}^+))_{t\geqslant 0}\big)\Big]\\
&\qquad=\sum_{k=1}^{\infty}\mathbb{E}_x\Big[F_{n,k-1}\mathbb{1}_{\{M_{n,k-1}<a\leqslant M_{n,k}\}}\mathbb{E}_{X_{n,k}}(G_{n,0})\Big]\\
&\qquad=\mathbb{E}_x\Big[F\Big((X_n(t))_{0\leqslant t<\tau_{a,n}^+}\Big)\,\mathbb{E}_{X(\tau_{a,n}^+)}\left[G((X_n(t))_{t\geqslant 0})\right]\Big] \qquad (5.24)
\end{aligned}$$

and (5.22) ensues by taking the limit of (5.24) as n tends to $+\infty$ in the sense of Definition 3.3. In particular, choosing $F=1$, (5.22) writes for $x\leqslant a$

$$\mathbb{E}_x\big[G\big((X(t+\tau_a^+))_{t\geqslant 0}\big)\big]=\int_{-\infty}^{+\infty}\mathbb{P}_x\{X(\tau_a^+)\in dz\}\mathbb{E}_z[G((X(t))_{t\geqslant 0})]$$

which, by (5.18), immediately yields (5.23). ■

The argument in favor of discontinuity evoked in [11] should fail since, in view of (5.13), a term is missing when applying the strong Markov property.

5.6 Just before the hitting time

In order to lighten the notations, we simply write $\tau_a^{\pm}=\tau_a$ and we introduce the jump $\Delta_a X=X(\tau_a)-X(\tau_a-)$.

Proposition 5.19 *The Laplace-Fourier transform of the vector $(\tau_a,X(\tau_a-),\Delta_a X)$ is related to those of the vectors $(\tau_a,X(\tau_a-))$ and $(\tau_a,X(\tau_a))$ according as, for $\Re(\lambda)>0$ and $\mu,\nu\in\mathbb{R}$,*

$$\mathbb{E}_x\Big[e^{-\lambda\tau_a+i\mu X(\tau_a-)+i\nu\Delta_a X}\Big]=\mathbb{E}_x\Big[e^{-\lambda\tau_a+i\mu X(\tau_a-)}\Big]=\mathbb{E}_x\Big[e^{-\lambda\tau_a+i\mu X(\tau_a)}\Big]. \qquad (5.25)$$

Proof. The proof of Proposition 5.19 is similar to that of Lemma 5.1. So, we outline the main steps with less details. We consider only the case where $\tau_a=\tau_a^+$ and $x\leqslant a$, the other one is quite similar.

• **Step 1**

Recall that for the step-process $(X_n(t))_{t\geqslant 0}$, the first hitting time $\tau_{a,n}^+$ is the instant $t_{n,k}$ with k such that $M_{n,k-1}\leqslant a$ and $M_{n,k}>a$, and then $X(\tau_{a,n}-)=X_{n,k-1}$ and $X(\tau_{a,n})=X_{n,k}$. Set $\Delta_{n,k}=X_{n,k}-X_{n,k-1}$. We have, for $x\leqslant a$,

$$\begin{aligned}
&e^{-\lambda\tau_{a,n}+i\mu X_n(\tau_{a,n}-)+i\nu\Delta_a X_n}\\
&\qquad=\sum_{k=1}^{\infty}e^{-\lambda t_{n,k}+i\mu X_{n,k-1}+i\nu\Delta_{n,k}}\mathbb{1}_{\{M_{n,k-1}\leqslant a<M_{n,k}\}}\\
&\qquad=e^{-\lambda t_{n,1}+i\mu X_{n,0}+i\nu(X_{n,1}-X_{n,0})}\\
&\qquad\quad+\sum_{k=1}^{\infty}\Big[e^{-\lambda t_{n,k+1}+i\mu X_{n,k}+i\nu\Delta_{n,k+1}}-e^{-\lambda t_{n,k}+i\mu X_{n,k-1}+i\nu\Delta_{n,k}}\Big]\mathbb{1}_{\{M_{n,k}\leqslant a\}}. \qquad (5.26)
\end{aligned}$$

• **Step 2**

We take the expectation of (5.26):

$$\mathbb{E}_x\Big[e^{-\lambda\tau_{a,n}+i\mu X_n(\tau_{a,n}-)+i\nu\Delta_a X_n}\Big] = e^{-\lambda/2^n+i\mu x+\kappa_N(i\nu)^N/2^n}$$
$$+\sum_{k=1}^{\infty} e^{-\lambda t_{n,k}}\,\mathbb{E}_x\Big[e^{i\mu X_{n,k-1}}1\!\!1_{\{M_{n,k-1}\leqslant a\}}\left(e^{-\lambda/2^n+i\mu\Delta_{n,k}+i\nu\Delta_{n,k+1}}-e^{i\nu\Delta_{n,k}}\right)1\!\!1_{\{X_{n,k}\leqslant a\}}\Big].$$

The expectation in the rhs of the foregoing equality can be evaluated as follows:

$$\begin{aligned}
&\mathbb{E}_x\Big[e^{i\mu X_{n,k-1}}1\!\!1_{\{M_{n,k-1}\leqslant a\}}\left(e^{-\lambda/2^n+i\mu\Delta_{n,k}+i\nu\Delta_{n,k+1}}-e^{i\nu\Delta_{n,k}}\right)1\!\!1_{\{X_{n,k}\leqslant a\}}\Big]\\
&\qquad= \int_{-\infty}^{a} e^{i\mu y}\,\mathbb{P}_x\{X_{n,k-1}\in dy, M_{n,k-1}\leqslant a\}\\
&\qquad\quad\times\mathbb{E}_0\Big[\left(e^{-\lambda/2^n+i\mu\Delta_{n,1}+i\nu\Delta_{n,2}}-e^{i\nu\Delta_{n,1}}\right)1\!\!1_{\{\Delta_{n,1}\leqslant a-y\}}\Big]\\
&\qquad= \int_{-\infty}^{a} e^{i\mu y}\mathbb{P}_x\{X_{n,k-1}\in dy, M_{n,k-1}\leqslant a\}\\
&\qquad\quad\times\Big[e^{-\lambda/2^n}\,\mathbb{E}_0\big(e^{i\mu X_{n,1}}1\!\!1_{\{X_{n,1}\leqslant a-y\}}\big)\,\mathbb{E}_0\big(e^{i\nu X_{n,1}}\big)-\mathbb{E}_0\big(e^{i\nu X_{n,1}}1\!\!1_{\{X_{n,1}\leqslant a-y\}}\big)\Big].
\end{aligned}$$

For computing the term within brackets, we need the following quantities:

$$\mathbb{E}_0\Big(e^{i\mu(\text{or }\nu)X_{n,1}}1\!\!1_{\{X_{n,1}\leqslant a-y\}}\Big)=\int_{-\infty}^{a-y} e^{i\mu(\text{or }\nu)z}\,p(1/2^n;-z)\,dz,\quad \mathbb{E}_0\big(e^{i\nu X_{n,1}}\big)=e^{\kappa_N(i\nu)^N/2^n}.$$

With these relations at hand, we get

$$\begin{aligned}
&\mathbb{E}_x\Big[e^{-\lambda\tau_{a,n}+i\mu X_n(\tau_{a,n}-)+i\nu\Delta_a X_n}\Big]\\
&\quad= e^{-(\lambda-\kappa_N(i\nu)^N)/2^n+i\mu x}+\frac{1}{2^n}\sum_{k=1}^{\infty} e^{-\lambda t_{n,k}}\int_{-\infty}^{a} e^{i\mu y}\,\mathbb{P}_x\{X_{n,k-1}\in dy, M_{n,k-1}\leqslant a\}\\
&\qquad\times 2^n\Big[e^{-(\lambda-\kappa_N(i\nu)^N)/2^n}\int_{-\infty}^{a-y} e^{i\mu z}\,p(1/2^n;-z)\,dz-\int_{-\infty}^{a-y} e^{i\nu z}\,p(1/2^n;-z)\,dz\Big].
\end{aligned}\tag{5.27}$$

• **Step 3**

We now take the limit of (5.27) as n tends to infinity:

$$\begin{aligned}
&\mathbb{E}_x\Big[e^{-\lambda\tau_a+i\mu X(\tau_a-)+i\nu\Delta_a X}\Big]\\
&\qquad= e^{i\mu x}+\int_0^{\infty} e^{-\lambda t}\,dt\int_{-\infty}^{a} e^{i\mu y}\,\varphi(\lambda,\mu,\nu;y)\,\mathbb{P}_x\{X(t)\in dy, M(t)\leqslant a\}
\end{aligned}$$

where we set, for $y<a$,

$$\varphi(\lambda,\mu,\nu;y)=\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\left[e^{-(\lambda-\kappa_N(i\nu)^N)\varepsilon}\int_{-\infty}^{a-y} e^{i\mu z}\,p(\varepsilon;-z)\,dz-\int_{-\infty}^{a-y} e^{i\nu z}\,p(\varepsilon;-z)\,dz\right].\tag{5.28}$$

• **Step 4**

For evaluating the above function φ, we need two lemmas.

Lemma 5.20 *For $0 \leqslant p \leqslant N$, we have*

$$\mathbb{E}_0[X(t)^p] = \begin{cases} 1 & \textit{for } p = 0, \\ 0 & \textit{for } 1 \leqslant p \leqslant N-1, \\ \kappa_N N!\, t & \textit{for } p = N. \end{cases}$$

PROOF. By differentiating k times the identity $\mathbb{E}_0\left(e^{iuX(t)}\right) = e^{\kappa_N (iu)^N t}$ with respect to u and next substituting $u = 0$, we have that

$$\mathbb{E}_0[X(t)^k] = (-i)^k \, \frac{\partial^k}{\partial u^k} \left[e^{\kappa_N (iu)^N t} \right] \Big|_{u=0}.$$

Fix a complex number $\alpha \neq 0$. It can be easily seen by induction that there exists a family of polynomials $(P_k)_{k \in \mathbb{N}}$ such that, for all $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial u^k}\left(e^{\alpha u^N}\right) = P_k(u)\, e^{\alpha u^N}. \tag{5.29}$$

In particular, we have $P_0(u) = 1$ and $P_1(u) = N\alpha u^{N-1}$. Using the Leibniz rule, we obtain

$$\begin{aligned} P_k(u) &= e^{-\alpha u^N} \frac{\partial^k}{\partial u^k}\left(e^{\alpha u^N}\right) = e^{-\alpha u^N} \frac{\partial^{k-1}}{\partial u^{k-1}}\left(N\alpha\, u^{N-1} e^{\alpha u^N}\right) \\ &= N!\, \alpha \sum_{j=\max(0,k-N)}^{k-1} \binom{k-1}{j} \frac{u^{N+j-k}}{(N+j-k)!}\, P_j(u). \end{aligned}$$

This ascertains the aforementioned induction and gives, for $u = 0$,

$$P_k(0) = \begin{cases} 0 & \text{if } 1 \leqslant k \leqslant N-1, \\ N!\, \alpha\, P_0(0) = N!\, \alpha & \text{if } k = N. \end{cases}$$

Choosing $\alpha = \kappa_N i^N t$ and $u = 0$ in (5.29), we immediately complete the proof of Lemma 5.20. ■

Lemma 5.21 *For $\alpha < 0 < \beta$, the following expansion holds as $\varepsilon \to 0^+$:*

$$\int_\alpha^\beta e^{i\mu z}\, p(\varepsilon; -z)\, dz = 1 + \kappa_N (i\mu)^N \varepsilon + o(\varepsilon). \tag{5.30}$$

PROOF. Performing a simple change of variables and using some asymptotics of [11], we get

$$\begin{aligned} \int_\alpha^\beta e^{i\mu z}\, p(\varepsilon; -z)\, dz &= \int_{\alpha/\varepsilon^{1/N}}^{\beta/\varepsilon^{1/N}} e^{i\mu\varepsilon^{1/N} z}\, p(1; -z)\, dz = \int_{-\infty}^{+\infty} e^{i\mu\varepsilon^{1/N} z}\, p(1; -z)\, dz + o(\varepsilon) \\ &= \sum_{p=0}^{\infty} \frac{(i\mu)^p}{p!}\, \varepsilon^{p/N} \int_{-\infty}^{+\infty} z^p\, p(1; -z)\, dz + o(\varepsilon). \end{aligned}$$

Observing that $\int_{-\infty}^{+\infty} z^p\, p(1; -z)\, dz = \mathbb{E}_0(X(1)^p)$, we immediately derive from Lemma 5.20 the expansion (5.30). ■

• **Step 5**

Now, plugging (5.30) into (5.28), it comes

$$\begin{aligned}\varphi(\lambda,\mu,\nu;y) &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left[\left(1-\left(\lambda-\kappa_N(i\nu)^N\right)\varepsilon+o(\varepsilon)\right)\left(1+\kappa_N(i\mu)^N\varepsilon+o(\varepsilon)\right)\right.\\ &\quad\left.-\left(1+\kappa_N(i\nu)^N\varepsilon+o(\varepsilon)\right)\right] = -\lambda+\kappa_N(i\mu)^N.\end{aligned}$$

Therefore,

$$\mathbb{E}_x\!\left[e^{-\lambda\tau_a+i\mu X(\tau_a-)+i\nu\Delta_a X}\right] = e^{i\mu x}-\left(\lambda-\kappa_N(i\mu)^N\right)\int_0^\infty e^{-\lambda t}\,\mathbb{E}_x\!\left[e^{i\mu X(t)},M(t)\leqslant a\right]dt.$$

Writing finally

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}\,\mathbb{E}_x\!\left[e^{i\mu X(t)},M(t)\leqslant a\right]dt\\ &\qquad= \int_0^\infty e^{-\lambda t}\,\mathbb{E}_x\!\left[e^{i\mu X(t)}\right]dt-\int_0^\infty e^{-\lambda t}\,\mathbb{E}_x\!\left[e^{i\mu X(t)},M(t)>a\right]dt\\ &\qquad= \frac{1}{\lambda-\kappa_N(i\mu)^N}-\int_0^\infty e^{-\lambda t}\,\mathbb{E}_x\!\left[e^{i\mu X(t)},M(t)>a\right]dt,\end{aligned}$$

we obtain (5.25) by invoking the relationship (5.1) and by noting that the result does not depend on ν (and hence we can choose $\nu=0$). ■

Choosing $\mu=0$ or $\nu=0$, we obtain the corollary below.

Corollary 5.22 *We have, for $\Re(\lambda)>0$ and $\mu,\nu\in\mathbb{R}$,*

$$\mathbb{E}_x\!\left[e^{-\lambda\tau_a+i\mu X(\tau_a-)}\right]=\mathbb{E}_x\!\left[e^{-\lambda\tau_a+i\mu X(\tau_a)}\right]\quad\text{and}\quad\mathbb{E}_x\!\left[e^{-\lambda\tau_a+i\nu\Delta_a X}\right]=\mathbb{E}_x\!\left[e^{-\lambda\tau_a}\right].$$

From Corollary 5.22, we expect that $\Delta_a X=X(\tau_a)-X(\tau_a-)=0$ in the following sense: $\forall\varphi\in\mathcal{S}$, $\mathbb{E}_0[\varphi(\Delta_a X)]=\varphi(0)$. This provides a new argument in favor of continuity.

5.7 Particular cases

Example 5.23 *Case $N=3$:*

- In the case $\kappa_3=+1$, densities (5.15) write

$$\mathbb{P}_x\{\tau_a^+\in dt,X(\tau_a^+)\in dz\}/dt\,dz=\mathcal{J}_0(t;x-a)\,\delta_a(z)\text{ for }x\leqslant a$$

  and

$$\mathbb{P}_x\{\tau_a^-\in dt,X(\tau_a^-)\in dz\}/dt\,dz=\mathcal{K}_0(t;x-a)\,\delta_a(z)+\mathcal{K}_1(t;x-a)\,\delta'_a(z)\text{ for }x\geqslant a.$$

  Here, we have $d_{11}=\theta_2=\bar\theta_1$, $d_{21}=\theta_1=\bar\theta_2$ and

$$\begin{aligned}\mathcal{J}_0(t;\xi) &= -\frac{\xi}{\pi t}\,\Im\!\left[e^{\frac{i\pi}{3}}\int_0^{+\infty}\theta_0A_0\,e^{\theta_0 e^{\frac{i\pi}{3}}\xi\lambda}\,e^{-t\lambda^3}\,d\lambda\right]\\ &= -\frac{\xi}{\pi t}\,\Im\!\left[e^{\frac{i\pi}{3}}\int_0^{+\infty}e^{e^{\frac{i\pi}{3}}\xi\lambda-t\lambda^3}\,d\lambda\right]\\ &= -\frac{\xi}{\pi t}\int_0^{+\infty}e^{\frac12\xi\lambda-t\lambda^3}\sin\left(\frac{\sqrt3}{2}\,\xi\lambda+\frac{\pi}{3}\right)d\lambda;\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_0(t;\xi) &= -\frac{\xi}{\pi t}\,\Im\!\left[e^{\frac{i\pi}{3}}\int_0^{+\infty}\left(\theta_1 B_1\, e^{\theta_1 e^{\frac{i\pi}{3}}\xi\lambda}+\theta_2 B_2\, e^{\theta_2 e^{\frac{i\pi}{3}}\xi\lambda}\right)e^{-t\lambda^3}\,d\lambda\right]\\
&= -\frac{\xi}{\pi\sqrt{3}\,t}\,\Re\!\left[e^{\frac{i\pi}{3}}\int_0^{+\infty}\left(e^{-\xi\lambda}-e^{e^{-\frac{i\pi}{3}}\xi\lambda}\right)e^{-t\lambda^3}\,d\lambda\right]\\
&= -\frac{\xi}{\pi\sqrt{3}\,t}\int_0^{+\infty}\left[\frac{1}{2}\,e^{-\xi\lambda}-e^{\frac{1}{2}\xi\lambda}\cos\left(\frac{\sqrt{3}}{2}\,\xi\lambda-\frac{\pi}{3}\right)\right]e^{-t\lambda^3}\,d\lambda;
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_1(t;\xi) &= -\frac{3}{\pi}\,\Im\!\left[e^{-\frac{i\pi}{3}}\int_0^{+\infty}\left(\bar{d}_{11} B_1\, e^{\theta_1 e^{\frac{i\pi}{3}}\xi\lambda}+\bar{d}_{21} B_2\, e^{\theta_2 e^{\frac{i\pi}{3}}\xi\lambda}\right)\lambda\,e^{-t\lambda^3}\,d\lambda\right]\\
&= -\frac{\sqrt{3}}{\pi}\,\Re\!\left[e^{-\frac{i\pi}{3}}\int_0^{+\infty}\left(e^{-\xi\lambda}-e^{e^{-\frac{i\pi}{3}}\xi\lambda}\right)\lambda\,e^{-t\lambda^3}\,d\lambda\right]\\
&= -\frac{\sqrt{3}}{\pi}\int_0^{+\infty}\left[\frac{1}{2}\,e^{-\xi\lambda}-e^{\frac{1}{2}\xi\lambda}\cos\left(\frac{\sqrt{3}}{2}\,\xi\lambda+\frac{\pi}{3}\right)\right]\lambda\,e^{-t\lambda^3}\,d\lambda.
\end{aligned}$$

- In the case $\kappa_3=-1$, densities (5.15) write

$$\mathbb{P}_x\{\tau_a^+\in dt, X(\tau_a^+)\in dz\}/dt\,dz=\mathcal{J}_0(t;x-a)\,\delta_a(z)+\mathcal{J}_1(t;x-a)\,\delta_a'(z)\text{ for }x\leqslant a$$

and

$$\mathbb{P}_x\{\tau_a^-\in dt, X(\tau_a^-)\in dz\}/dt\,dz=\mathcal{K}_0(t;x-a)\,\delta_a(z)\text{ for }x\geqslant a.$$

In this case, we have $c_{01}=\bar{\theta}_2=\theta_0$, $c_{21}=\bar{\theta}_0=\theta_2$ and

$$\begin{aligned}
\mathcal{J}_0(t;\xi) &= -\frac{\xi}{\pi\sqrt{3}\,t}\int_0^{+\infty}\left[\frac{1}{2}\,e^{\xi\lambda}+e^{-\frac{1}{2}\xi\lambda}\cos\left(\frac{\sqrt{3}}{2}\,\xi\lambda+\frac{\pi}{3}\right)\right]e^{-t\lambda^3}\,d\lambda\\
\mathcal{J}_1(t;\xi) &= -\frac{\sqrt{3}}{\pi}\int_0^{+\infty}\left[\frac{1}{2}\,e^{\xi\lambda}-e^{-\frac{1}{2}\xi\lambda}\cos\left(\frac{\sqrt{3}}{2}\,\xi\lambda-\frac{\pi}{3}\right)\right]\lambda\,e^{-t\lambda^3}\,d\lambda\\
\mathcal{K}_0(t;\xi) &= -\frac{\xi}{\pi t}\int_0^{+\infty}e^{-\frac{1}{2}\xi\lambda-t\lambda^3}\sin\left(\frac{\sqrt{3}}{2}\,\xi\lambda-\frac{\pi}{3}\right)d\lambda.
\end{aligned}$$

Let us point out that the functions $\mathcal{J}_0$, $\mathcal{J}_1$, $\mathcal{K}_0$ and $\mathcal{K}_1$ may be expressed by means of Airy functions.

■

Example 5.24 *Case $N=4$:* formulas (5.15) read here

$$\mathbb{P}_x\{\tau_a^+\in dt, X(\tau_a^+)\in dz\}/dt\,dz=\mathcal{J}_0(t;x-a)\,\delta_a(z)+\mathcal{J}_1(t;x-a)\,\delta_a'(z)\text{ for }x\leqslant a$$

and

$$\mathbb{P}_x\{\tau_a^-\in dt, X(\tau_a^-)\in dz\}/dt\,dz=\mathcal{K}_0(t;x-a)\,\delta_a(z)+\mathcal{K}_1(t;x-a)\,\delta_a'(z)\text{ for }x\geqslant a.$$

We have $c_{21} = \theta_3 = \bar{\theta}_2$, $c_{31} = \theta_2 = \bar{\theta}_3$, $d_{01} = \theta_1 = \bar{\theta}_0 = -\bar{\theta}_2$, $d_{11} = \theta_0 = \bar{\theta}_1 = -\bar{\theta}_3$ and

$$\begin{aligned}
\mathcal{J}_0(t;\xi) &= -\frac{\xi}{\pi t}\,\Im\bigg[e^{\frac{i\pi}{4}}\int_0^{+\infty}\Big(\theta_2 A_2\, e^{\theta_2 e^{\frac{i\pi}{4}}\xi\lambda} + \theta_3 A_3\, e^{\theta_3 e^{\frac{i\pi}{4}}\xi\lambda}\Big)\, e^{-t\lambda^4}\,d\lambda\bigg] \\
&= \frac{\xi}{\pi\sqrt{2}\,t}\,\Re\bigg[e^{\frac{i\pi}{4}}\int_0^{+\infty}(e^{\xi\lambda} - e^{i\xi\lambda})\, e^{-t\lambda^4}\,d\lambda\bigg] \\
&= \frac{\xi}{2\pi t}\int_0^{+\infty}\Big[e^{\xi\lambda} - \sqrt{2}\,\cos\Big(\xi\lambda + \frac{\pi}{4}\Big)\Big]\, e^{-t\lambda^4}\,d\lambda \\
&= \frac{\xi}{2\pi t}\int_0^{+\infty}\Big[e^{\xi\lambda} - \cos(\xi\lambda) + \sin(\xi\lambda)\Big]\, e^{-t\lambda^4}\,d\lambda;
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_1(t;\xi) &= -\frac{4}{\pi}\,\Im\bigg[e^{-\frac{i\pi}{4}}\int_0^{+\infty}\Big(\bar{c}_{21} A_2\, e^{\theta_2 e^{\frac{i\pi}{4}}\xi\lambda} + \bar{c}_{31} A_3\, e^{\theta_3 e^{\frac{i\pi}{4}}\xi\lambda}\Big)\,\lambda^2\, e^{-t\lambda^4}\,d\lambda\bigg] \\
&= \frac{2\sqrt{2}}{\pi}\,\Re\bigg[e^{-\frac{i\pi}{4}}\int_0^{+\infty}(e^{\xi\lambda} - e^{i\xi\lambda})\,\lambda^2\, e^{-t\lambda^4}\,d\lambda\bigg] \\
&= -\frac{2}{\pi}\int_0^{+\infty}\Big[e^{\xi\lambda} - \sqrt{2}\,\cos\Big(\xi\lambda - \frac{\pi}{4}\Big)\Big]\,\lambda^2\, e^{-t\lambda^4}\,d\lambda \\
&= \frac{2}{\pi}\int_0^{+\infty}\Big[\cos(\xi\lambda) + \sin(\xi\lambda) - e^{\xi\lambda}\Big]\,\lambda^2\, e^{-t\lambda^4}\,d\lambda
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathcal{K}_0(t;\xi) &= \frac{\xi}{2\pi t}\int_0^{+\infty}\Big[\cos(\xi\lambda) + \sin(\xi\lambda) - e^{-\xi\lambda}\Big]\, e^{-t\lambda^4}\,d\lambda; \\
\mathcal{K}_1(t;\xi) &= \frac{2}{\pi}\int_0^{+\infty}\Big[e^{-\xi\lambda} - \cos(\xi\lambda) + \sin(\xi\lambda)\Big]\,\lambda^2\, e^{-t\lambda^4}\,d\lambda.
\end{aligned}$$

We retrieve formulas (8.17) and (8.18) of [17]. ∎

5.8 Boundary value problem

We end up this work by exhibiting a boundary value problem satisfied by the Laplace-Fourier transform $U(x) = \mathbb{E}_x\Big[e^{-\lambda\tau_a^+ + i\mu X(\tau_a^+)}\Big]$, $x \in (-\infty, a)$.

Proposition 5.25 *The function U satisfies the differential equation*

$$\mathcal{D}_x U(x) = \lambda U(x) \quad \textit{for } x \in (-\infty, a) \tag{5.31}$$

together with the conditions

$$U^{(l)}(a^-) = (i\mu)^l e^{i\mu a} \quad \textit{for } 0 \leqslant l \leqslant \#J - 1. \tag{5.32}$$

Proof. The differential equation (5.31) is readily obtained by differentiating (5.4) with respect to x. Let us derive the boundary conditions (5.32): by (5.4),

$$U^{(l)}(a^-) = \lambda^{l/N}\prod_{j\in J}\Big(1 - \frac{i\mu}{\sqrt[N]{\lambda}}\,\bar{\theta}_j\Big)\Bigg(\sum_{j\in J}\frac{\theta_j^l A_j}{1 - \frac{i\mu}{\sqrt[N]{\lambda}}\,\bar{\theta}_j}\Bigg)e^{i\mu a}.$$

By (2.11) we see that

$$\sum_{j\in J}\frac{\theta_j^l A_j}{1-\frac{i\mu}{\sqrt[N]{\lambda}}\,\bar{\theta}_j}=\frac{(i\mu)^l}{\lambda\prod_{j\in J}\left(1-\frac{i\mu}{\sqrt[N]{\lambda}}\,\bar{\theta}_j\right)}$$

which proves Condition (5.32). ∎

We also refer the reader to [18] for a very detailed account on PDE's with various boundary conditions and their connections with different kinds of absorbed/killed pseudo-processes.

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