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# THE PRINCIPLE OF LARGE DEVIATIONS FOR MARTINGALE ADDITIVE FUNCTIONALS OF RECURRENT MARKOV PROCESSES [1]

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**Abstract** We give a principle of large deviations for a generalized version of the strong central limit theorem. This generalized version deals with martingale additive functionals of a recurrent Markov process.

**Keywords** Central Limit Theorem (CLT), Large Deviations Principle (LDP), Markov Processes, Autoregressive Model (AR1), Positive Recurrent Processes, Martingale Additive Functional (MAF)

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# 1 Introduction

This paper presents a natural extension of the (ASCLT) due to Brosamler [1], [2] and Schatte [27]. In the last few years the Almost Sure Central Limit Theorem (ASCLT) has emerged as an area of probability theory in which an intensive research activity has taken place. In this context we should in particular mention the work of Lacey & Philipp [17], Berkes & Dehling [3], Csörgö & Horváth [6], Rodzik & Rychlik [26] and Touati [29].

The aim of this paper is to establish the Large Deviations Principle (LDP) for a generalized version of the (ASCLT) for Martingale Additive Functionals (MAF's). This result can be regarded as an extension of the (ASCLT) for (MAF's), proved by the second named author (see Maâouia [21]) as well as an extension of the (LDP) for the (ASCLT) for i.i.d. random variables, proved by the first named author (see Heck [14]). For a slightly weaker version of the (LDP) for the (ASCLT) for i.i.d. random variables see also March and Seppäläinen [22].

## 1.1 Notation, terminology and data

$\mathrm{X} = \{\Omega, \mathcal{F}, (\mathbb{P}_x)_{x\in \mathrm{E}}, \mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}}, (\mathrm{X}_k)_{k\in\mathbb{N}}\}$ denotes the canonical version of a homogeneous Markov process indexed by $\mathbb{N}$ (non negative integers) with values in a measurable space $(\mathrm{E}, \mathcal{E})$; $\mathbb{F}$ being its natural filtration and $\mathbb{P}_x$ its law starting from $x$.

We denote by $\Pi$ the transition probability of the Markov chain X and by $(R_p)_{p\in]0,1[}$ its resolvent:

$$\textbf{(1-1)}\ \ R_p(x,A) = \mathbb{E}_x\left(\sum_{k=1}^{\infty} p^{k-1}\,\mathbf{1}_{\{\mathrm{X}_k\in A\}}\right) = \sum_{k=1}^{\infty} p^{k-1}\Pi_k\,(x,A) \text{ for } x\in\mathrm{E},\ A\in\mathcal{E}.$$

Using Duflo's [10] and Meyn & Tweedie's [23] terminology, we call a set $C \in \mathcal{E}$ a **small set** for the Markov chain X, if there exists a probability measure $\nu$ on $\mathcal{E}$ (with $\nu(C) = 1$), $p_0 \in ]0,1[$ and $b \in ]0,1[$, such that:

$$\textbf{(1-2)}\qquad \forall\ (x,A) \in \mathrm{E}\times\mathcal{E},\ p_0\, R_{p_0}(x,A) \geq b\ 1_C(x)\,\nu(A).$$

From now on, the expression "X is a **positive recurrent Markov chain**" means that X has a small set $C$, with the following properties (1-3) and (1-4)

$$\textbf{(1-3)}\ \mathbb{E}_x\left(\limsup_{n\to\infty}\ (\mathrm{X}_n\in C)\right) = \mathbb{E}_x\left(\sum_{k=0}^{\infty}\mathbf{1}_{\{\mathrm{X}_k\in C\}} = +\infty\right) = 1,\ \forall x\in\mathrm{E},$$

$$\textbf{(1-4)}\qquad \sup_{x\in C}\ \mathbb{E}_x\,(\mathrm{T}_C) < \infty \text{ with } \mathrm{T}_C = \inf\{k\geq 1,\ \mathrm{X}_k\in C\}.$$

In this case there is a probability measure $\mu$, invariant under $\Pi$, such that X is **Harris recurrent**:

$$\textbf{(1-5)}\qquad \forall\ \mathrm{A}\in\mathcal{E} \text{ with } \mu(\mathrm{A}) > 0,\ \mathbb{E}_x\left(\overline{\lim_k}\ (\mathrm{X}_k\in\mathrm{A})\right) = 1.$$

We shall say that X is **Riemannian recurrent of order** $k$, for each $k\in\mathbb{N}$, if (1-3) and

**(1-6)** $\sup_{x \in C} \mathbb{E}_x \left(\mathrm{T}_C^k\right) < \infty$ for each $k \in \mathbb{N}$

hold.

Conversely, the existence of an invariant probability measure $\mu$ for X satisfying (1-5) implies the existence of a small set satisfying (1-3) and (1-4), if we assume that the $\sigma$-algebra $\mathcal{E}$ is countably generated (cf. Duflo [10], Meyn & Tweedie [23]).

We remind that an **additive functional** (**AF**) $\mathrm{A} = (\mathrm{A}_k)_{k \in \mathbb{N}}$ of X is an $\mathbb{F}$-adapted process, vanishing at 0, such that:

**(1-7)** $\mathrm{A}_{k+l} = \mathrm{A}_k + \mathrm{A}_l \circ \theta_k \qquad (\mathbb{P}_\nu\text{- a.s.}) \qquad \forall\, k, l \in \mathbb{N},$

for any initial law $\nu$. Here $(\theta_k)_{k \in \mathbb{N}}$ are the standard translation operators on $(\Omega\ ,\ \mathcal{F})$.

A **martingale additive functional** (**MAF**) , $\mathrm{M} = (\mathrm{M}_k)_{k \in \mathbb{N}}$ of X is an (**AF**) which is also an $(\mathbb{F},\ \mathbb{P}_\nu)$ martingale, for any initial law $\nu$ or equivalently

**(1-8)** $\mathbb{E}_x(\mathrm{M}_k) = 0\ ,\ \forall\, k \in \mathbb{N},\ \forall\, x \in \mathrm{E}.$

Next, we will use the following notation and terminology.

**(1-9)** $\mathcal{C}_0([0,1])$ is the space of continuous functions from [0,1] to $\mathbb{R}$ vanishing in 0.

**(1-10)** $\mathcal{M}_1(\mathcal{C}_0([0,1]))$ is the space of probability measures on the Borel sets of $\mathcal{C}_0([0,1])$, endowed with the topology of weak convergence.

**(1-11)** $\mathcal{N} = \mathcal{N}(0,1)$ denotes the Gaussian law with mean 0 and variance 1 on $\mathbb{R}$.

For $a \in ]0,1]$ we introduce the function

**(1-12)** $\vartheta_a\colon \mathcal{C}_0([0,1]) \longrightarrow \mathcal{C}_0([0,1])$ with $\vartheta_a(\omega)(t) = \frac{\omega(at)}{\sqrt{a}}$.

Using these functions $\vartheta_a$ we call a measure $\mathrm{Q} \in \mathcal{M}_1(\mathcal{C}_0([0,1]))$ $\vartheta$-invariant if $\mathrm{Q} = \mathrm{Q}\circ\vartheta_a^{-1}$ for all $a \in ]0,1]$. Furthermore, for two probability measures $\eta$, $\rho$ on a measurable space, we denote by $\mathrm{H}(\eta|\rho)$ the relative entropy of $\eta$ relative to $\rho$, i.e.

**(1-13)** $$\mathrm{H}(\eta|\rho) = \begin{cases} \displaystyle\int \ln\left(\frac{d\eta}{d\rho}\right) \mathrm{d}\rho & : \text{if } \eta \ll \rho, \text{ i.e. if } \eta \text{ is abs. continuous w.r.t. } \rho \\ \infty & : \text{else.} \end{cases}$$

Now we define the rate function $\mathcal{H} : \mathcal{M}_1(\mathcal{C}_0([0,1])) \longrightarrow [0,\infty]$ as follows:

**(1-14)** $$\mathcal{H}(\mathrm{Q}) = \begin{cases} \lim\limits_{a \downarrow 0} \frac{1}{\ln(a^{-1})} \mathrm{H}\left(\mathrm{Q} \circ |_{[a,1]}{}^{-1} | \mathrm{W} \circ |_{[a,1]}{}^{-1}\right) & : \text{if Q is } \vartheta\text{-invariant} \\ \infty & : \text{else} \end{cases}$$

where W is the Wiener measure on $\mathcal{C}_0([0,1])$ and $|_{[a,1]}$ denotes the restriction operator. That $\mathcal{H}$ is well defined has already been shown in Heck [14, 15].

### 1.2 ASCLT for MAF of a recurrent Markov process

The second named author proved the following general version of the (ASCLT) (see Maâouia [21]).

**Theorem A** *Let* $\mathrm{X} = (\ \Omega, \mathcal{F}, (\mathbb{P}_x)_{x\in \mathrm{E}}\ , \mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}}, (\mathrm{X}_k)_{k\in\mathbb{N}}\ )$ *be a positive recurrent Markov chain. Then every* **(MAF)** M *of* X *satisfying the assumption:*

$$\textbf{(1-15)} \qquad \mathbb{E}_x(\mathrm{M}_k^2) < \infty\ ,\ \ \forall k \in \mathbb{N}, \forall x \in \mathrm{E} \ \ \textit{and}\ \ \mathbb{E}_\mu(\mathrm{M}_1^2) = \sigma_{\mathrm{M}}^2 \ \in ]0, \infty[$$

*satisfies a functional* ASCLT (**FASCLT**) *under* $\mathbb{P}_x$ *for all initial states* $x$. *More precisely,* $\mathbb{P}_x$*-almost-surely for every* $x$*, we have the following properties:*

(**FASCLT**) *The random measures* $(W_n)_{n\in\mathbb{N}^*}$:

$$\textbf{(1-16)} \qquad W_n(\cdot) \ = \ L(n)^{-1} \sum_{k=1}^{n} k^{-1}\, \delta_{\{\Psi_k \in\ \cdot\}}$$

*converge weakly to* $W$*, the Wiener measure on* $\mathcal{C}_0([0,1])$*, where* $L(n) = \sum_{k=1}^{n} k^{-1}$ *and* $(\Psi_n)_{n\in\mathbb{N}^*}$ *is defined by:*

$$\textbf{(1-17)} \qquad \Psi_n(t) \ = \ \sigma_{\mathrm{M}}^{-1} n^{-1/2} \left\{ \mathrm{M}_{[nt]} \ + \ (nt - [nt])(\mathrm{M}_{[nt]+1} - \mathrm{M}_{[nt]}) \right\}. \quad \diamondsuit$$

## 2 Main results

Our results are stated for (**MAF**), $\mathrm{M} = (\mathrm{M}_k)_{k\in\mathbb{N}}$ of the Markov process X which satisfies the assumption (2-1) below.

$$\textbf{(2-1)} \qquad \begin{cases} \mathbb{E}_\mu\left(|\mathrm{M}_1|^\beta\right) < \infty\ ,\ \forall \beta > 0\ ; \\ \\ \mathbb{E}_\mu(\mathrm{M}_1^2) = \sigma_{\mathrm{M}}^2 \ \in ]0, \infty[\ . \end{cases}$$

For every (**MAF**) M satisfying the assumptions (2-1) we consider the processes $(\Psi_n)_n$ and the measures $(W_n)_n$, defined as in Theorem A.

**Theorem 2.1** *Let* $\mathrm{X} = (\ \Omega, \mathcal{F}, (\mathbb{P}_x)_{x\in \mathrm{E}}\ , \mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}}, (\mathrm{X}_k)_{k\in\mathbb{N}}\ )$ *be a Riemannian recurrent Markov chain of order k, for every* $k \in \mathbb{N}$ *, with a small set C. Then every* **(MAF)** M *of* X *satisfying the assumption (2-1) satisfies the* (LDP) *with constants* $(\ln(n))_{n\in\mathbb{N}^*}$ *and rate function* $\mathcal{H}$ *w.r.t.* $\mathbb{P}_x$ *for* $\mu-$ *a.a. initial states* $x \in \mathrm{E}$*; i.e. for every Borel set* $\mathrm{A} \subset \mathcal{M}_1(\mathcal{C}_0([0,1])$:

$$\textbf{(2-2)} \qquad -\inf_{\overset{o}{\mathrm{A}}} \mathcal{H} \leq \ \liminf_{n\to\infty} \ \frac{1}{\ln(n)} \ln\left(\mathbb{P}_x\left\{\mathrm{W}_n \in \ \mathrm{A}\right\}\right)$$

$$\leq \limsup_{n\to\infty} \frac{1}{\ln(n)} \ln\left(\mathbb{P}_x \{W_n \in A\}\right) \leq -\inf_{\overline{A}} \mathcal{H}$$

*for $\mu-$ a.a. initial states $x \in E$.* $\diamondsuit$

The results we present could easily be generalized to the continuous time parameter case. However for the proof of the continuous parameter case we would need rather technical oscillation estimates very similar to those used in Heck [15] in order to reduce the continuous case to the discrete time case. These lengthy technical estimates would increase the size of the paper considerably without presenting any new ideas. Therefore we decided to restrict ourselves to the discrete time parameter case.

# 3 The identification of an autoregressive process

In this section we shall apply our result Theorem 2.1 to autoregressive models. The latter models have a great interest in mathematical finance (for example: risk management, derivative securities like options, stochastic volatility,..., see e.g. Hull [16], section 19.6 ) .

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a sequence $\beta = (\beta_n)_{n\in\mathbb{N}^*}$ of i.i.d. real random variables with mean 0 and variance $\sigma^2 > 0$; called **white noise.** To this sequence $\beta$ and a given random variable $X_0$ we associate the first order autoregressive process (**AR1**):

**(3-1)** $$X_{n+1} = \theta\, X_n + \beta_{n+1}$$

or

**(3-2)** $$X_{n+1} = \alpha + \theta\, X_n + \beta_{n+1}$$

where $\alpha$ and $\theta$ are unknown real parameters. These parameters $\alpha$ and $\beta$ are to be estimated.

In the following we shall assume that the random variables $\beta$ satisfies the moment condition

**(3-3)** $$\mathbb{E}\left(|\beta_1|^{2\delta}\right) < \infty$$

for some $\delta > 1$.

For the (**AR1**), defined by (3-1), the least squares estimator of $\theta$:

**(3-4 a)** $$\widehat{\theta}_n = \left(\sum_{k=1}^{n} X_{k-1}^2\right)^{-1} \left(\sum_{k=1}^{n} X_{k-1}\, X_k\right) \text{ for each } n \in \mathbb{N}^*$$

satisfies

**(3-4 b)** $$\widehat{\theta}_n - \theta = \left(\sum_{k=1}^{n} \mathrm{X}_{k-1}^2\right)^{-1} \left(\sum_{k=1}^{n} \mathrm{X}_{k-1}\, \beta_k\right).$$

Under the hypothesis $\mathbb{E}\left(\beta_1^2\right) < \infty$, $\left(\widehat{\theta}_n\right)$ has the following asymptotic properties (see [10] for more details).

**(3-5)** $\left(\widehat{\theta}_n\right)_{n\in\mathbb{N}^*}$ is a strongly consistent estimator of the arbitrary unknown real parameter $\theta$.

In the **stable case** ($|\theta| < 1$), $\left(\widehat{\theta}_n\right)$ satisfies:

**(3-6)** $$\sqrt{n}\, \left(\widehat{\theta}_n - \theta\right) \underset{n\to\infty}{\Longrightarrow} \mathcal{N}\left(0\,, 1-\theta^2\right)$$

**(3-7)** $$\limsup_{n\to\infty} \sqrt{\frac{n}{\ln n}}\, \left|\widehat{\theta}_n - \theta\right| = \sqrt{1-\theta^2} \qquad \text{a.s. .}$$

Under the hypothesis (3-3) and in the stable case, the following result hold under $\mathbb{P}_x$ for all starting state $x$ :

**(3-8)** $$\mathrm{W}_n^\theta = \mathrm{L}(n)^{-1} \sum_{k=1}^{n} k^{-1}\, \delta_{\Psi_k^\theta} \underset{n\to\infty}{\Longrightarrow} \mathrm{W}$$

where $\Psi_n^\theta \in \mathcal{C}_0([0\,,\,1])$ is linear on $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ and

**(3-9)** $$\Psi_n^\theta\left(\tfrac{k}{n}\right) = \sqrt{n\,\left(1-\theta^2\right)^{-1}}\frac{k}{n}\, \left(\widehat{\theta}_k - \theta\right) \text{ for each } k \in \{1, ..., n\}$$

and "$\Longrightarrow$" denotes weak convergence.

The property (3.8) is a consequence of the FASCLT for the martingales obtained by Chaâbane [5]. It is also consequence of Theorem A above, if we assume that the noise $\beta$ satisfy (3.3) and the distribution of $\beta_1$ has a non vanishing density part. In fact, under these hypotheses, we prove the existence of a small set for the AR(1) Markov chain X (see Lemma 4.8).

The next Proposition gives the LDP associated with the property (3-8).

**Proposition 3.1** *Let* $\mathrm{X} = (\mathrm{X}_n)_{n\in\mathbb{N}}$ *be the first order stable autoregressive process* (**AR1**) *satisfying* (3-1) *constructed from a white noise* $\beta = (\beta_n)_{n\in\mathbb{N}^*}$ *with variance* $\sigma^2$ *satisfying the hypothesis* (3-3) *for all* $\delta > 1$, *such that the distribution of* $\beta_1$ *has a non vanishing density part and an unknown real parameter* $\theta \in ]-1, 1[$. *Then for all* $\mathrm{X}_0 \equiv x \neq 0$ *the following result holds for the least squares estimator.*

**(3-10)** $\left(\mathrm{W}_n^\theta\right)_n$ *satisfies the* (**PLD**) *with constants* $(\ln n)_n$ *and rate function* $\mathcal{H}$ *w.r.t.* $\mathbb{P}_x$ *; i.e. for every Borel set* $\mathrm{A} \subset \mathcal{M}_1(\mathcal{C}_0([0\,,\,1]))$:

$$
\begin{aligned}
-\inf_{\overset{o}{\mathrm{A}}}\mathcal{H} \leq & \liminf_{n\to\infty} \ \tfrac{1}{\ln n} \ln\left(\mathbb{P}_x\left\{\mathrm{W}_n^\theta \in \ \mathrm{A}\right\}\right) \\
& \leq \ \limsup_{n\to\infty} \ \tfrac{1}{\ln n} \ln\left(\mathbb{P}_x\left\{\mathrm{W}_n^\theta \in \ \mathrm{A}\right\}\right) \leq -\inf_{\overline{\mathrm{A}}}\mathcal{H}. \ \diamondsuit
\end{aligned}
$$

For the (AR1) model, defined by (3-2), we can estimate $\alpha$ and $\theta$ by:

$$
\textbf{(3-11)} \left\{ \begin{array}{ll}
\widehat{\theta}_n & = \mathrm{D}_n^{-1}\left\{\left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}\,\mathrm{X}_k\right) - \left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_k\right)\left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}\right)\right\}, \\
\widehat{\alpha}_n & = \mathrm{D}_n^{-1}\left\{\left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_k\right)\left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}^2\right) - \left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}\right)\left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}\,\mathrm{X}_k\right)\right\};
\end{array}\right.
$$

for each $n \in \mathbb{N}^*$, with $\quad \mathrm{D}_n = \left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}^2\right) - \left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}\right)^2.$

These estimators satisfy

$$
\textbf{(3-12)} \left\{ \begin{array}{ll}
\mathrm{D}_n\left(\widehat{\theta}_n - \theta\right) & = \left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}\right)\left(\frac{1}{n}\sum\limits_{k=1}^{n} \beta_k\right) - \left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}\beta_k\right), \\
\mathrm{D}_n\left(\widehat{\alpha}_n - \alpha\right) & = \left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}^2\right)\left(\frac{1}{n}\sum\limits_{k=1}^{n} \beta_k\right) - \left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}\right)\left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}\,\beta_k\right), \\
\left(\widehat{\alpha}_n - \alpha\right) & = \left(\frac{1}{n}\sum\limits_{k=1}^{n} \mathrm{X}_{k-1}\right)\left(\widehat{\theta}_n - \theta\right) + \left(\frac{1}{n}\sum\limits_{k=1}^{n} \beta_k\right);
\end{array}\right.
$$

and they have the following asymptotic properties:

**(3-13)** $\left(\widehat{\theta}_n\right)_{n\geq 1}$ and $\left(\widehat{\alpha}_n\right)_{n\geq 1}$ are strongly consistent estimators of the arbitrary unknown parameters $\theta$ and $\alpha$.

In the **stable case** ($|\theta| < 1$), $\left(\widehat{\theta}_n\right)_{n\geq 1}$ and $\left(\widehat{\alpha}_n\right)_{n\geq 1}$ satisfy:

$$
\textbf{(3-14)} \quad \left\{ \begin{array}{l}
\sqrt{n}\ \left(\widehat{\theta}_n - \theta\right) \underset{n\to\infty}{\Longrightarrow} \mathcal{N}\left(0\ ,1-\theta^2\right), \\
\sqrt{n}\ \left(\widehat{\alpha}_n - \alpha\right) \underset{n\to\infty}{\Longrightarrow} \mathcal{N}\left(0\ ,(1-\theta^2)\left(\frac{\sigma^2}{1-\theta^2} + \frac{\alpha^2}{(1-\theta)^2}\right)\right);
\end{array}\right.
$$

$$
\textbf{(3-15)} \quad \left\{ \begin{array}{l}
\limsup\limits_{n\to\infty} \sqrt{\frac{n}{\ln\ln n}}\ \left|\widehat{\theta}_n - \theta\right| = \sqrt{1-\theta^2} \qquad a.s., \\
\limsup\limits_{n\to\infty} \sqrt{\frac{n}{\ln\ln n}}\ \left|\widehat{\alpha}_n - \alpha\right| = \sqrt{(1-\theta^2)\left(\frac{\sigma^2}{1-\theta^2} + \frac{\alpha^2}{(1-\theta)^2}\right)} \qquad a.s.;
\end{array}\right.
$$

$$
\textbf{(3-16)} \quad \left\{ \begin{array}{l}
\mathrm{W}_n^\theta = \mathrm{L}(n)^{-1}\sum\limits_{k=1}^{n} k^{-1}\,\delta_{\Psi_k^\theta} \underset{n\to\infty}{\Longrightarrow} \ \mathrm{W}, \\
\mathrm{W}_n^\alpha = \mathrm{L}(n)^{-1}\sum\limits_{k=1}^{n} k^{-1}\,\delta_{\Psi_k^\alpha} \underset{n\to\infty}{\Longrightarrow} \ \mathrm{W};
\end{array}\right.
$$

where $\Psi_n^\theta$, $\Psi_n^\alpha \in \mathcal{C}_0([0\ ,\ 1])$ are linear on $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ and

$$\textbf{(3-17)}\qquad \begin{cases} \Psi_n^\theta\left(\frac{k}{n}\right) = \sqrt{\frac{n}{(1-\theta^2)}}\frac{k}{n}\left(\widehat{\theta}_k - \theta\right), \\ \Psi_n^\alpha\left(\frac{k}{n}\right) = \sqrt{\frac{n}{(1-\theta^2)\left(\frac{\sigma^2}{1-\theta^2}+\frac{\alpha^2}{(1-\theta)^2}\right)}}\frac{k}{n}\left(\widehat{\alpha}_k - \alpha\right); \forall k \in \{1, ..., n\}. \end{cases}$$

**Proposition 3.2** *Let* $\mathrm{X} = (\mathrm{X}_n)_{n\in\mathbb{N}}$ *be the (AR1) model satisfying* (3-2) *constructed from a white noise* $\beta = (\beta_n)_{n\in\mathbb{N}^*}$ *with variance* $\sigma^2$ *satisfying the hypothesis* **(**3-3**)** *for all* $\delta > 1$, *such that the distribution of* $\beta_1$ *has a non vanishing density part and an unknown real parameters* $\theta \in ]-1, 1[$ *and* $\alpha$. *Then for all* $\mathrm{X}_0 \equiv x \neq 0$ *the following result holds for the least squares estimator:*

**(3-18)** $\left(\mathrm{W}_n^\theta\right)_n$ *and* $(W_n^\alpha)_n$ *satisfy the (***LDP***) with constants* $(\ln n)_n$ *and rate function* $\mathcal{H}$ *w.r.t.* $\mathbb{P}_x$ . $\diamondsuit$

# 4 Proofs

## 4.1 An ASCLT for i.i.d. random variables

The proof of Theorem 2.1 is essentially based on a reduction to a version of the (ASCLT) for i.i.d. random variables. In order to formulate this version we introduce some notations.
For random variables $(\xi_n, \tau_n)_{n\in\mathbb{N}^*}$ as in Proposition 4.1 below we denote by $\mathrm{S}_n$ and $\mathrm{T}_n$ the corresponding partial sums, i.e. $\mathrm{S}_n = \sum_{k=1}^n \xi_k$ and $\mathrm{T}_n = \sum_{k=1}^n \tau_k$ and let for $t \geq 0$ $\mathrm{N}_t = \inf\{k \geq 0 : \mathrm{T}_{k+1} > t\}$. Further let $\mathrm{S}_n^* = \mathrm{S}_{\mathrm{N}_n}$.
As in the introduction we define random functions $\widetilde{\Psi}_n \in \mathcal{C}_0([0,1])$ by

$$\widetilde{\Psi}_n(t) = \frac{\sqrt{m}}{\sigma\sqrt{n}}\left\{\mathrm{S}_{[nt]}^* + (nt - [nt])\left(\mathrm{S}_{[nt]+1}^* - \mathrm{S}_{[nt]}^*\right)\right\}\ \forall\, t \in [0,1].$$

Finally we define random measures $\widetilde{\mathrm{W}}_n \in \mathcal{M}_1\left(\mathcal{C}_0([0,1])\right)$ by $\widetilde{\mathrm{W}}_n = \frac{1}{\mathrm{L}(n)}\sum_{k=1}^n \frac{1}{k}\,\delta_{\widetilde{\Psi}_k}$.

**Proposition 4.1** *Let* $(\xi_n, \tau_n)_{n\in\mathbb{N}^*}$ *be independent random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *such that for some* $n_0 \in \mathbb{N}^*$

**(4-1 a)** $\mathbb{E}(\xi_n) = 0$ *for all* $n \geq 1$, $\sup_{n\geq 1}\ \mathbb{E}\left(|\xi_n|^\beta\right) < \infty$ *for all* $\beta > 0$, *and* $\mathbb{E}\left(\xi_n^2\right) = \sigma^2 > 0$ *for all* $n \geq n_0$.

**(4-1 b)** $\tau_n \geq 0$ *for all* $n \geq 1$, $\sup_{n\geq 1}\ \mathbb{E}\left(\tau_n^\beta\right) < \infty$ *for all* $\beta > 0$ *and* $\mathbb{E}(\tau_n) = m > 0$ *for all* $n \geq n_0$.

*Then* $\left(\widetilde{\mathrm{W}}_n\right)_{n\geq 1}$ *satisfies the* (LDP) *with constants* $(\ln n)_{n\geq 1}$ *and rate function* $\mathcal{H}$. $\diamondsuit$

**Remark 4.2** We shall remark that in the special case $(\xi_k)_{k\in\mathbb{N}^*}$ i.i.d. and $\tau_k \equiv 1$, i.e. $\mathrm{N}_n \equiv n$, the above proposition states the (LDP) for the (ASCLT) for i.i.d. random variables. This result is exactly the contents of Theorem 1.2 in Heck [14].

In order to prove Proposition 4.1 we shall recall for the readers convenience some simple facts:

**Lemma 4.3** *Let* $\mathrm{Y}_n$, $\mathrm{Z}_n$, $n \in \mathbb{N}^*$ *be random variables with values in a separable metric space* $(\mathrm{E}, d)$ *such that for all* $\varepsilon > 0$

$$\textbf{(4-2)} \qquad \lim_{n\to\infty} \frac{1}{\ln n} \ln\left(\mathbb{P}\left\{d(\mathrm{Y}_n, \mathrm{Z}_n) > \varepsilon\right\}\right) = -\infty \ .$$

*Then* $\left(\frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k}\, \delta_{\mathrm{Y}_k}\right)_{n\in\mathbb{N}^*}$ *and* $\left(\frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k}\, \delta_{\mathrm{Z}_k}\right)_{n\in\mathbb{N}^*}$ *are equivalent w.r.t. the* (LDP) *i.e.* $\left(\frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k}\, \delta_{\mathrm{Y}_k}\right)_{n\in\mathbb{N}^*}$ *satisfies the* (LDP) *with constants* $(\ln n)_{n\in\mathbb{N}^*}$ *if and only if* $\left(\frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k}\, \delta_{\mathrm{Z}_k}\right)_{n\in\mathbb{N}^*}$ *satisfies the* (LDP) *with constants* $(\ln n)_{n\in\mathbb{N}^*}$ *and the same rate function* $\mathcal{H}$. $\diamondsuit$

Lemma 4.3 is a minor modification of Lemma 2.7 in Heck [14]. Details shall be omitted.

**Lemma 4.4** *Let* $(\mathrm{M}_n)_{n\in\mathbb{N}}$ *be random variables with* $\mathrm{M}_0 \equiv 0$ *and let* $\eta$ *be a random variable with values in* $\mathbb{N}$.

**a)** *For all* $\beta \geq 1$ *and all* $\mathrm{p}$ , $\mathrm{q} > 1$ *with* $\frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1$ ( $\beta\mathrm{q} > 1$) *there exists* $\mathrm{C}_1 > 0$ *such that*

$$\mathbb{E}\left(|\mathrm{M}_\eta|^\beta\right) \leq \mathrm{C}_1 \sup_{k\geq 1} \left\{\mathbb{E}\left(|\mathrm{M}_k - \mathrm{M}_{k-1}|^{\mathrm{q}\beta}\right)^{1/\mathrm{q}}\right\} \mathbb{E}\left(\eta^{\mathrm{p}(\beta+2)}\right)^{1/\mathrm{p}} .$$

**b)** *If in addition* $(\mathrm{M}_n)_{n\in\mathbb{N}}$ *is a martingale then there exists* $\mathrm{C}_2 > 0$ *such that for all* $\beta \geq 2$

$$\mathbb{E}\left(|\mathrm{M}_\eta|^\beta\right) \leq \mathrm{C}_2 \sup_{k\geq 1} \left\{\mathbb{E}\left(|\mathrm{M}_k - \mathrm{M}_{k-1}|^{\mathrm{q}\beta}\right)^{1/\mathrm{q}}\right\} \mathbb{E}\left(\eta^{\mathrm{p}(\beta/2+2)}\right)^{1/\mathrm{p}} . \ \diamondsuit$$

**Proof**. Let $\mathrm{p}$ , $\mathrm{q} > 1$ such that $\frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1$. By Hölder's inequality and Chebychev's inequality

$$\textbf{(4-3)} \qquad \mathbb{E}\left(|\mathrm{M}_\eta|^\beta\right) = \sum_{k=1}^{\infty} \mathbb{E}\left(|\mathrm{M}_k|^\beta\, 1_{\{\eta=k\}}\right) \leq \sum_{k=1}^{\infty} \mathbb{E}\left(|\mathrm{M}_k|^{\beta\mathrm{q}}\right)^{\frac{1}{\mathrm{q}}} \mathbb{P}\left(\{\eta = k\}\right)^{\frac{1}{\mathrm{p}}}$$

$$\leq \sum_{k=1}^{\infty} \mathbb{E}\left(|\mathrm{M}_k|^{\beta\mathrm{q}}\right)^{\frac{1}{\mathrm{q}}} \mathbb{E}\left(\eta^{\mathrm{p}(r\beta+2)}\right)^{\frac{1}{\mathrm{p}}}\ k^{-(r\beta+2)}$$

In order to prove Part a) we shall use the following inequalities and $r = 1$ in eq. (4-3),

$$\mathbb{E}\Big(|\mathrm{M}_k|^{\beta \mathrm{q}}\Big) \leq k^{\beta \mathrm{q}} \sup_{k \geq 1} \ \mathbb{E}\Big(|\mathrm{M}_n - \ \mathrm{M}_{n-1}|^{\beta \mathrm{q}}\Big) .$$

In order to prove part b) we shall take $r = \frac{1}{2}$ in eq. (4-3), proceed as in a) and use Burkholder-Davis-Gundy inequality to estimate $\mathbb{E}\Big(|\mathrm{M}_k|^{\beta \mathrm{q}}\Big)$ for $\beta \mathrm{q} > 1$ ,

$$\mathbb{E}\Big(|\mathrm{M}_k|^{\beta \mathrm{q}}\Big) \leq \mathbb{E}\Big(\sup_{j \leq k} \ |\mathrm{M}_j|^{\beta \mathrm{q}}\Big) \leq \text{const. } \mathbb{E}\left(\left\{\sum_{j=1}^{k} (\mathrm{M}_j - \ \mathrm{M}_{j-1})^2\right\}^{\frac{\beta \mathrm{q}}{2}}\right)$$

$$\leq \text{const. } k^{\frac{\beta \mathrm{q}}{2}} \sup_{n \geq 1} \ \mathbb{E}\Big(|\mathrm{M}_n - \ \mathrm{M}_{n-1}|^{\beta \mathrm{q}}\Big) . \ \Box$$

We shall remark that one can in particular choose for $(\mathrm{M}_n)_{n \in \mathbb{N}}$ the partial sums of independent random variables with expectation 0.

**Lemma 4.5** *Let $(\tau_n)_{n \in \mathbb{N}^*}$ be independent random variables satisfying (4-1 b).*

**a)** *For every $\beta \geq 1$ there exists $\mathrm{C}_4 > 0$ such that for all $n \in \mathbb{N}^*$, $\mathbb{E}(|\mathrm{N}_n|^{\beta}) \leq \mathrm{C}_4 \, n^{\beta+1}$.*

**b)** *For all $\alpha \in ]\frac{1}{2}, 1[$ and $\gamma > 0$ there exists $\mathrm{C}_5 > 0$ such that for all sufficiently large $n \in \mathbb{N}^*$, $\mathbb{P}\{|\mathrm{N}_n - \frac{n}{m}| > n^{\alpha}\} \leq \mathrm{C}_5 \, n^{-\gamma}$.* $\diamondsuit$

**Proof**. In order to prove part a) we observe that by Lemma 4.4 and Chebychev's inequality for $k \in \mathbb{N}^*$ with $k > \frac{2}{m}n + 2n_0$

$$\mathbb{P}\{\mathrm{N}_n = \ k\} \leq \mathbb{P}\{\mathrm{T}_k \ \leq n\} \leq \mathbb{P}\left\{|\mathrm{T}_k - \mathbb{E}(\mathrm{T}_k)| \geq \tfrac{m}{2} k\right\}$$

$$\leq \left(\tfrac{2}{m}\right)^{2\beta+8} k^{-2\beta-8} \mathbb{E}\left\{|\mathrm{T}_k - \mathbb{E}(\mathrm{T}_k)|^{2\beta+8}\right\}$$

$$\leq \left(\tfrac{2}{m}\right)^{2\beta+8} \mathrm{C}_6 \, k^{\beta+6} k^{-2\beta-8} = \mathrm{C}_7 \, k^{-\beta-2}.$$

This inequality implies $\mathbb{E}(\mathrm{N}_n^{\beta}) \leq \sum_{k \leq \frac{2}{m}n+2n_0} k^{\beta} + \sum_{k > \frac{2}{m}n+2n_0} \mathrm{C}_7 \, k^{-2}$, which in turn implies part a).

For the proof of part b) we note that again Lemma 4.4 and Chebychev's inequality imply for sufficiently large $n$

$$\mathbb{P}\left\{\left|\mathrm{N}_n - \tfrac{n}{m}\right| > n^\alpha\right\} = \sum_{\left|k-\frac{n}{m}\right|>n^\alpha} \mathbb{P}\left\{\mathrm{T}_k \le n < \mathrm{T}_{k+1}\right\}$$

$$\le \sum_{k=0}^{[\frac{n}{m}-n^\alpha]} \mathbb{P}\left\{\mathrm{T}_{k+1} \ge n\right\} + \sum_{k=[\frac{n}{m}+n^\alpha]+1}^{\infty} \mathbb{P}\left\{\mathrm{T}_k \le n\right\}$$

$$\le \sum_{k=0}^{[\frac{n}{m}-n^\alpha]} \mathbb{P}\left\{\left|\mathrm{T}_{k+1} - \mathbb{E}(\mathrm{T}_{k+1})\right| \ge \tfrac{m}{2} n^\alpha\right\} + \sum_{k=[\frac{n}{m}+n^\alpha]+1}^{\infty} \mathbb{P}\left\{\left|\mathrm{T}_k - \mathbb{E}(\mathrm{T}_k)\right| \ge \mathbb{E}(\mathrm{T}_k) - n\right\}$$

$$\le \sum_{k=0}^{[\frac{n}{m}-n^\alpha]} \mathrm{C}_8\, n^{\beta+2} n^{-2\alpha\beta} + \sum_{k=[\frac{n}{m}+n^\alpha]+1}^{\infty} \mathrm{C}_9\, k^{\beta+2}(mk - mn_0 - n)^{-2\beta} \le \mathrm{C}_{10}\, n^{(1-2\alpha)\beta+3}$$

where we used that for $n \in \mathbb{N}^*$ sufficiently large and $k \le [\frac{n}{m} - n^\alpha]$

$$n - \mathbb{E}(\mathrm{T}_{k+1}) \ge n - \mathbb{E}\left(\mathrm{T}_{[\frac{n}{m}-n^\alpha]+1}\right) \ge n - m\left(\tfrac{n}{m} - n^\alpha - n_0 + 1\right) - \mathbb{E}(\mathrm{T}_{n_0}) \ge \tfrac{m}{2} n^\alpha.$$

Hence we conclude the proof of part b) by choosing $\beta$ sufficiently large. □

**Proof of Proposition 4.1** For the special case $\tau \equiv 1$ Proposition 4.1 has already been proved in Heck [14] (see Remark 4.2). Therefore, by Lemma 4.3 the proof of Proposition 4.1 is complete if we show that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with random variables $\mathcal{X} = (\mathrm{X}_i\,,\, \mathrm{x}_i)_{i\in\mathbb{N}^*}$ and $\mathcal{Y} = (\mathrm{Y}_i\,,\, \mathrm{y}_i)_{i\in\mathbb{N}^*}$ such that $\mathcal{X}$ has the same distribution as $(\xi_i, \tau_i)_{i\in\mathbb{N}^*}$ , $\mathrm{Y}_i$, $i \in \mathbb{N}^*$ are independent, $\mathcal{N}(0,1)$-distributed, $\mathrm{y}_i \equiv 1$ and

**(4-4)** $\left(\widetilde{\mathrm{W}}_n^{\mathcal{X}}\right)_{n\in\mathbb{N}^*}$ and $\left(\widetilde{\mathrm{W}}_n^{\mathcal{Y}}\right)_{n\in\mathbb{N}^*}$ are equivalent w.r.t. the LDP.

Here $\widetilde{\mathrm{W}}_n^{\mathcal{X}}$ denotes the random measure $\widetilde{\mathrm{W}}_n$ constructed from the sequence $\mathcal{X}$. Similar we shall use the notations $\mathrm{S}_n^{*\mathcal{X}}$ and $\widetilde{\Psi}_n^{\mathcal{X}}$, to indicate that the functions are constructed from the sequence $\mathcal{X}$.

By Skorokhod's representation theorem there exits a probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$, a random variable $\widetilde{\mathrm{B}}$: $\widetilde{\Omega} \to \mathcal{C}_0\left([0,\infty[\right)$ and $\widetilde{\mathbb{P}}$ -a.s. finite stopping times $0 = \widetilde{\mathrm{R}}_0 \le \widetilde{\mathrm{R}}_1 \le \widetilde{\mathrm{R}}_2 \le$... such that

**(4-5 a)** $\widetilde{\mathrm{B}}$ is a Brownian motion

**(4-5 b)** $\left(\widetilde{\mathrm{B}}_{\widetilde{\mathrm{R}}_i} - \widetilde{\mathrm{B}}_{\widetilde{\mathrm{R}}_{i-1}}\right)_{i\in\mathbb{N}^*}$ have the same distribution as $(\xi_i)_{\,i\in\mathbb{N}^*}$

**(4-5 c)** $\widetilde{\mathrm{R}}_i - \widetilde{\mathrm{R}}_{i-1}$ are independent with $\sup_{n\ge1} \mathbb{E}\left(\left(\widetilde{\mathrm{R}}_n - \widetilde{\mathrm{R}}_{n-1}\right)^\beta\right) < \infty$ for all $\beta > 0$ and $\mathbb{E}\left(\widetilde{\mathrm{R}}_n - \widetilde{\mathrm{R}}_{n-1}\right) = \sigma^2$ for $n \ge n_0$.

(See e.g. Chapter 1, Theorem 117 in Freedman [11] and Brosamler [1], p. 570 regarding the moments for the stopping times.)

Now let $\Omega = \widetilde{\Omega} \times \mathbb{R}^{\mathbb{N}^*}$, $\mathcal{F}$ the corresponding product-$\sigma$-field and

$$\mathbb{P}\left(\mathrm{A} \times \mathrm{A}_1 \times ... \times \mathrm{A}_n \times \mathbb{R} \times ...\right) = \int_{\mathrm{A}} \prod_{i=1}^{n} \eta_i \left(\widetilde{\mathrm{B}}_{\widetilde{\mathrm{R}}_i} - \widetilde{\mathrm{B}}_{\widetilde{\mathrm{R}}_{i-1}}, \mathrm{A}_i\right) \mathrm{d}\, \widetilde{\mathbb{P}}.$$

Here $\eta_i$ denotes the conditional distribution $\eta_i\,(t,\cdot) = \mathbb{P}\left(\tau_i \in \cdot \,|\, \xi_i = t\right)$.
If we let $\mathrm{B}(\widetilde{\omega}, (t_n)_{n\in\mathbb{N}^*}) = \widetilde{\mathrm{B}}(\widetilde{\omega})$, $\mathrm{R}(\widetilde{\omega}, (t_n)_{n\in\mathbb{N}^*}) = \widetilde{\mathrm{R}}(\widetilde{\omega})$, then (4.5) still hold for $\widetilde{\mathrm{B}}$ and $\widetilde{\mathrm{R}}_i$ replaced by B and $\mathrm{R}_i$.
Hence by scaling properties of Brownian motion , if we let $\mathrm{X}_n = \mathrm{B}_{\mathrm{R}_{\;n}} - \mathrm{B}_{\mathrm{R}_{n-1}}$,
$\mathrm{x}_n\,(\widetilde{\omega}, (t_k)_{k\in\mathbb{N}^*}) = t_n$, $\mathrm{Y}_n = \frac{\sqrt{m}}{\sigma}\left(\mathrm{B}_{\frac{n\sigma^2}{m}} - \mathrm{B}_{\frac{(n-1)\sigma^2}{m}}\right)$ and finally $\mathrm{y}_n \equiv 1$ then obviously it remains to prove (4-4) for this special choice for $\mathcal{X}$ and $\mathcal{Y}$.

By Lemma 4.3 part b) the proof of (4-4) is complete if we show that for all $\varepsilon > 0$

**(4-6)** $$\lim_{n\to\infty} \frac{1}{\ln n} \ln\left(\mathbb{P}\left\{\left\|\widetilde{\Psi}_n^{\mathcal{X}} - \widetilde{\Psi}_n^{\mathcal{Y}}\right\|_\infty > \varepsilon\right\}\right) = -\infty.$$

Note that $\widetilde{\Psi}_n^{\mathcal{X}}\left(\frac{k}{n}\right) = \frac{\sqrt{m}}{\sigma\sqrt{n}}\mathrm{B}_{\mathrm{R}_{\mathrm{N}_k}}$ and $\widetilde{\Psi}_n^{\mathcal{Y}}\left(\frac{k}{n}\right) = \frac{\sqrt{m}}{\sigma\sqrt{n}}\mathrm{B}_{\frac{n\sigma^2}{m}}$. Hence the definition of $\widetilde{\Psi}_n$ via interpolation implies for $k \in \{0, 1, ...n-1\}$ and $t \in [\frac{k}{n}, \frac{k+1}{n}]$

**(4-7)** $$\left|\widetilde{\Psi}_n^{\mathcal{X}}(t) - \widetilde{\Psi}_n^{\mathcal{Y}}(t)\right| \leq \frac{\sqrt{m}}{\sigma\sqrt{n}} \max\left\{\left|\mathrm{B}_{\mathrm{R}_{\mathrm{N}_k}} - \mathrm{B}_{\frac{k\sigma^2}{m}}\right|, \left|\mathrm{B}_{\mathrm{R}_{\mathrm{N}_{\;k+1}}} - \mathrm{B}_{\frac{(k+1)\sigma^2}{m}}\right|\right\}.$$

Hence the proof of (4-6) is complete if we show that for all $\varepsilon > 0$

**(4-8)** $$\lim_{n\to\infty} \frac{1}{\ln(n)} \ln\left(\mathbb{P}\left\{\max_{k=1,...,n}\left\{\left|\mathrm{B}_{\mathrm{R}_{\mathrm{N}_k}} - \mathrm{B}_{\frac{k\sigma^2}{m}}\right|\right\} > \varepsilon\sqrt{n}\right\}\right) = -\infty.$$

Let $\varepsilon, \gamma > 0$. Fix $\mathrm{p} > 1$, $\beta > 0$ such that $\beta > 9 + 4\gamma$. By Lemma 4.4 part b), (4-5 b) and Lemma 4.5 part a) we conclude that for sufficiently large $n \in \mathbb{N}$ and $k \in \left\{1, ..., [n^{3/4}]\right\}$

**(4-9)** $$\mathbb{P}\left\{\left|\mathrm{B}_{\mathrm{R}_{\mathrm{N}_k}}\right| \geq \tfrac{\varepsilon}{2}\sqrt{n}\right\} \leq \left(\tfrac{2}{\varepsilon}\right)^{2\beta} \mathrm{C}_{12}\, n^{-\beta}\, \mathbb{E}\left(\mathrm{N}_k^{\mathrm{p}(\beta+2)}\right)^{\frac{1}{\mathrm{p}}} \leq \left(\tfrac{2}{\varepsilon}\right)^{2\beta} \mathrm{C}_{13}\, n^{-\beta}\, k^{\beta+2+\frac{1}{\mathrm{p}}}$$

**(4-10)** $$\leq \left(\tfrac{2}{\varepsilon}\right)^{2\beta} \mathrm{C}_{13}\, n^{-\frac{1}{4}\beta+\frac{9}{4}} \leq \left(\tfrac{2}{\varepsilon}\right)^{2\beta} \mathrm{C}_{13}\, n^{-\gamma}.$$

Using Lemma 4.4 part b) we conclude that for sufficiently large $n \in \mathbb{N}^*$ and $k \in \left\{1, ..., [n^{3/4}]\right\}$

$$\textbf{(4-11)} \qquad \mathbb{P}\left\{\left|\mathrm{B}_{\frac{k\sigma^2}{m}}\right| > \tfrac{\varepsilon}{2}\sqrt{n}\right\} \le \mathrm{C}_{14}\,\varepsilon^{-2\beta} n^{-\gamma}.$$

For the following we assume that $n \in \mathbb{N}^*$ is sufficiently large and $k \in \left\{[n^{1/4}], ..., n\right\}$. Observing that

$$\textbf{(4-12)}\ \mathbb{P}\left\{\left|\mathrm{B}_{\mathrm{R}_{\mathrm{N}_k}} - \mathrm{B}_{\frac{k\sigma^2}{m}}\right| > \varepsilon\sqrt{n}\right\} \le \mathbb{P}\left\{\left|\mathrm{B}_{\mathrm{R}_{\mathrm{N}_k}} - \mathrm{B}_{\frac{k\sigma^2}{m}}\right| > \varepsilon\sqrt{n}, \left|\mathrm{R}_{\mathrm{N}_k} - \tfrac{k\sigma^2}{m}\right| \le \sigma^2 k^{\frac{4}{5}}\right\}$$

$$+\,\mathbb{P}\left\{\left|\mathrm{R}_{\mathrm{N}_k} - \tfrac{k\sigma^2}{m}\right| > \sigma^2 k^{\frac{4}{5}},\ \left|\mathrm{N}_k - \tfrac{k}{m}\right| \le k^{\frac{3}{4}}\right\} + \mathbb{P}\left\{\left|\mathrm{N}_k - \tfrac{k}{m}\right| > k^{\frac{3}{4}}\right\}.$$

Obviously Lemma 4.5 part b) implies that

$$\textbf{(4-13)} \qquad \mathbb{P}\left\{\left|\mathrm{N}_k - \tfrac{k}{m}\right| > k^{\frac{3}{4}}\right\} \le \mathrm{C}_{15}\, k^{-\frac{16}{3}\gamma} \le \mathrm{C}_{15}\, n^{-\gamma}.$$

Keeping in mind that B is a Brownian motion, the symmetry properties of Brownian motion and

$$\textbf{(4-14)} \qquad \mathbb{P}\left\{\sup_{t\in[a,b]} \{|\mathrm{B}_t - \mathrm{B}_a|\} > c\right\} \le 2\,\exp\left\{-\tfrac{c^2}{2(b-a)}\right\}$$

for $c > 0,\ 0 \le a < b$ show that

$$\textbf{(4-15 a)} \qquad \mathbb{P}\left\{\left|\mathrm{B}_{\mathrm{R}_{\mathrm{N}_k}} - \mathrm{B}_{\frac{k\sigma^2}{m}}\right| > \varepsilon\sqrt{n}, \left|\mathrm{R}_{\mathrm{N}_k} - \tfrac{k\sigma^2}{m}\right| \le \sigma^2 k^{\frac{4}{5}}\right\}$$

$$\textbf{(4-15 b)} \qquad \le \mathbb{P}\left\{\sup_{t\in\left[\frac{k\sigma^2}{m} - \sigma^2 k^{\frac{4}{5}},\ \frac{k\sigma^2}{m} + \sigma^2 k^{\frac{4}{5}}\right]} \left|\mathrm{B}_t - \mathrm{B}_{\frac{k\sigma^2}{m}}\right| > \varepsilon\sqrt{n}\right\}$$

$$\textbf{(4-15 c)} \qquad \le 2\,\mathbb{P}\left\{\sup_{t\in\left[0,\sigma^2 k^{\frac{4}{5}}\right]} |\mathrm{B}_t| > \varepsilon\sqrt{n}\right\}$$

$$\textbf{(4-15 d)} \qquad \le 4\,\exp\left\{-\tfrac{\varepsilon^2}{2\sigma^2} n\, k^{-\frac{4}{5}}\right\} \le 4\,\exp\left\{-\tfrac{\varepsilon^2}{2\sigma^2} n^{\frac{1}{5}}\right\}.$$

Applying Lemma 4.4 part b) we obtain for $\beta$ sufficiently large

$$\textbf{(4-16 a)} \qquad \mathbb{P}\left\{\left|\mathrm{R}_{\mathrm{N}_k} - \tfrac{k\sigma^2}{m}\right| > \sigma^2 k^{\frac{4}{5}},\ \left|\mathrm{N}_k - \tfrac{k}{m}\right| \le k^{\frac{3}{4}}\right\}$$

$$\textbf{(4-16 b)} \qquad \le \sum_{l=-[k^{3/4}]-1}^{[k^{3/4}]+2} \mathbb{P}\left\{\left|\mathrm{R}_{l+[\frac{k}{m}]} - \tfrac{k\sigma^2}{m}\right| > \sigma^2 k^{\frac{4}{5}}\right\}$$

$$\textbf{(4-16 c)} \qquad \le \sum_{l=-[k^{3/4}]-1}^{[k^{3/4}]+2} \mathbb{P}\left\{\left|\mathrm{R}_{l+[\frac{k}{m}]} - \mathbb{E}\left(\mathrm{R}_{l+[\frac{k}{m}]}\right)\right| > \tfrac{1}{2}\sigma^2 k^{\frac{4}{5}}\right\}$$

$$\textbf{(4-16 d)} \qquad \leq \sum_{l=-[k^{3/4}]-1}^{[k^{3/4}]+2} \mathrm{C}_{16} \frac{\left(l+[\frac{k}{m}]\right)^{\beta+2}}{k^{\frac{8}{5}\beta}}$$

$$\textbf{(4-16 e)} \qquad \leq \sum_{l=-[k^{3/4}]-1}^{[k^{3/4}]+2} \mathrm{C}'_{16}\, k^{-\frac{3}{5}\beta+2} \leq \mathrm{C}_{17}\, k^{-\frac{3}{5}\beta+3} \leq \mathrm{C}_{17}\, n^{-\gamma}.$$

Now we conclude by the inequalities (4-9) to (4-16) that

$$\textbf{(4-17)} \quad \limsup_{n\to\infty} \frac{1}{\ln n} \ln \left( \mathbb{P} \left\{ \max_{k=1,\ldots,n} \left\{ \left| \mathrm{B}_{\mathrm{R}_{\mathrm{N}_{\ k}}} - \mathrm{B}_{\frac{k\sigma^2}{m}} \right| \right\} > \varepsilon\sqrt{n} \right\} \right) \leq -\,\gamma + 1$$

and hence (4-8) by letting $\gamma \to \infty$ in (4-17). □

## 4.2 Proof of Theorem 2.1

The proof of Theorem 2.1 is divided into three steps. First we shall consider the case where the small set is also an atom for the Markov chain, second we shall prove the theorem under the additional assumption that (1-2) already holds for the transition function $\Pi$ it self instead of $\mathrm{pR_p}$. And finally in the third and last step we shall prove the general case.

We shall remark that this technique has been already used by several authors (see e.g. Duflo [10], Touati [28]). In connection with (ASCLT's) this technique was introduced by Maâouia [19-21].

First of all we shall prove

**Lemma 4-6** *Let* $\mathrm{X} = (\,\Omega, \mathcal{F}, (\mathbb{P}_x)_{x\in\mathrm{E}}\ ,\ \mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}}\,, (\mathrm{X}_k)_{k\in\mathbb{N}})$ *be a Riemannian recurrent Markov chain of order k, for each* $k \in \mathbb{N}$ *with invariant measure* $\mu$*. We have:*

**a)** *Each measurable function* $g$: $\mathrm{E} \longrightarrow [0,\infty[$ *such that* $\mu(g) < \infty$ *satisfies*

$$\sup_n \ \mathbb{E}_x\,(g(\mathrm{X}_n)) < \infty,\ \mu - a.e.$$

**b)** *Each positive random variable* $Y$ *on* $(\Omega, \mathcal{F})$ *such that* $\mathbb{E}_\mu\,(\mathrm{Y}) < \infty$ *satisfies*

$$\sup_n \ \mathbb{E}_x\,(\mathrm{Y}\circ\theta_n)) < \infty,\ \mu - a.e.. \quad \diamondsuit$$

**Proof.** To prove this lemma we shall use Theorem 2 of [24]. Indeed in order to apply Theorem 2 we have to verify that under our assumptions X is a positive Harris recurrent chain with an irreducible kernel $\Pi$, a maximal irreducible measure $\mu$ and convergence parameter 1 (see e.g. [24]). By Theorem 2.1 of [25] we know that there exist an integer $\mathrm{m}_0 \geq 1$, a positive function $s$ (called small function) satisfying $\mu(s) \in ]0,\infty[$ and a bounded positive measure $\eta$ on $(\Omega, \mathcal{F})$ (called small measure) such that the following minoration condition holds

$$\Pi_{m_0}(x, \mathrm{A}) \geq s(x)\eta(\mathrm{A}),\ \forall x \in \mathrm{E},\ \forall\ \mathrm{A} \in \mathcal{E}.$$

Then by the Theorem 2 of [25] we can see that

$$\sup_{1 \leq h \leq g} \left|\mathbb{E}_x\left(h(\mathrm{X}_n)\right) - \mu(h)\right| \underset{n \to \infty}{\longrightarrow} 0\,, \mu - a.e.\,,$$

letting $h = g$ and using the fact that $\mu(g) < \infty$ and $\mu$ is $\Pi$-invariant, we have

$$\mathbb{E}_x\left(g(\mathrm{X}_n)\right) < \infty \quad \mu - a.e., \quad \mathbb{E}_x\left(g(\mathrm{X}_n)\right) \leq \left|\mathbb{E}_x\left(g(\mathrm{X}_n)\right) - \mu(g)\right| + \mu(g)$$

and then

$$\sup_n \mathbb{E}_x\left(g(\mathrm{X}_n)\right) < \infty,\ \mu - a.e.\ .$$

So the first part of the Lemma is proved.

In order to prove the second part we simply apply part a) to the function

$$g : x \in \mathrm{E} \longmapsto \mathbb{E}_x(\mathrm{Y})\,.\ \square$$

### 4.2.1 Case I: Atomic chains

For the following we shall assume that X not only has a small set A but also that $A$ is an atom for the Markov chain X.

Let $\mathrm{T}_A$ denote the first entry time into $A$, i.e. $\mathrm{T}_A = \inf\{k > 0, \mathrm{X}_k \in A\}$ and let $\mathrm{T}_0 \equiv 0$, and $\mathrm{T}_{k+1} = \mathrm{T}_A \circ \theta_{\mathrm{T}_k} + \mathrm{T}_k$. Further for $k \in \mathbb{N}^*$ let $\xi_k = \mathrm{M}_{\mathrm{T}_k} - \mathrm{M}_{\mathrm{T}_{k-1}}$ and $\tau_k = \mathrm{T}_k - \mathrm{T}_{k-1}$. Since the chain is positive recurrent, it is well known that the invariant distribution is given by

**(4-18)** $$\mu(\cdot) = \frac{1}{\mathbb{E}_a(\mathrm{T}_A)} \mathbb{E}_a\left(\sum_{k=0}^{\mathrm{T}_A - 1} 1_{\{\mathrm{X}_k \in \cdot\}}\right) \text{ where } a \in A \text{ is arbitrary.}$$

Further, since $A$ is an atom, the Markov property implies that $\Xi = (\xi_k, \tau_k)_{k \in \mathbb{N}^*}$ is a sequence of independent random variables and $(\xi_k, \tau_k)_{k \geq 2}$ are identically distributed w.r.t. $\mathbb{P}_x$ for all $x \in \mathrm{E}$. Keeping in mind that the chain is Riemannian recurrent of order $k$ for all $k \in \mathbb{N}^*$ the Markov property shows that for $x \in \mathrm{E}$ and $\beta > 0$

$$\mathbb{E}_x\left(|\mathrm{T}_2 - \mathrm{T}_1|^\beta\right) = \mathbb{E}_a\left(|\mathrm{T}_A|^\beta\right) < \infty.$$

By Proposition 8.3.23 in Duflo [10] we conclude that $\mathbb{E}_\mu\left(\mathrm{T}_1^\beta\right) < \infty$ and hence $\mathbb{E}_x\left(\mathrm{T}_1^\beta\right) < \infty$ for $\mu - a.a.\ x \in \mathrm{E}$. This together with the identical distribution for $k \geq 2$ implies

**(4-19)** $$\sup_{n \in \mathbb{N}^*} \mathbb{E}_x\left(\tau_n^\beta\right) < \infty \text{ for } \mu - a.a.\ x \in \mathrm{E}.$$

Since $\mathrm{T}_A \geq 1$,

**(4-20)** $\qquad \mathbb{E}_x\left(\tau_k\right) = m > 0.$

Using Lemma 4.4 part a) we conclude for $\mu - a.a.\ x \in \mathrm{E}$,

$$\mathbb{E}_x\left(|\xi_1|^\beta\right) = \mathbb{E}_x\left(|\mathrm{M}_{\mathrm{T}_1}|^\beta\right) < \infty \text{ and}$$

$$\mathbb{E}_x\left(|\xi_2|^\beta\right) = \mathbb{E}_x\left(|\mathrm{M}_{\mathrm{T}_2} - \mathrm{M}_{\mathrm{T}_1}|^\beta\right) \leq 2^\beta\left\{\mathbb{E}_x\left(|\mathrm{M}_{\mathrm{T}_2}|^\beta\right) + \mathbb{E}_x\left(|\mathrm{M}_{\mathrm{T}_1}|^\beta\right)\right\} < \infty.$$

This together with the identical distribution of the $\xi_i$, $i \geq 2$ implies for $\mu - a.a.\ x \in \mathrm{E}$

**(4-21)** $$\sup_{n\in\mathbb{N}^*} \mathbb{E}_x\left(|\xi_n|^\beta\right) < \infty.$$

We have that for $\mu - a.a.\ x \in \mathrm{E}$

**(4-22)** $\qquad \mathbb{E}_x\left(\xi_1\right) = \mathbb{E}_x\left(\mathrm{M}_{\mathrm{T}_1}\right) = 0$ and $\mathbb{E}_x\left(\xi_2\right) = \mathbb{E}_a\left(\mathrm{M}_{\mathrm{T}_A}\right) = 0.$

Moreover by the Martingale property of $\mathrm{M}_n$, the Markov property of X and (4-18)

**(4-23)** $$\mathbb{E}_x\left(\xi_2^2\right) = \mathbb{E}_a\left(\mathrm{M}^2_{\mathrm{T}_A}\right) = \mathbb{E}_a\left(\sum_{k=0}^{\mathrm{T}_A-1} \mathbb{E}_{\mathrm{X}_k}\left((\mathrm{M}_1 - \mathrm{M}_0)^2\right)\right)$$
$$= \mathbb{E}_\mu\left(\mathrm{M}_1^2\right)\mathbb{E}_a\left(\mathrm{T}_A\right) = \sigma^2_{\mathrm{M}}\, m > 0.$$

(4-19) to (4-23) imply that $\Xi = (\xi_n, \tau_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Proposition 4.1. Hence Proposition 4.1 implies that $\left(\widetilde{\mathrm{W}}_n^{\Xi}\right)_{n\in\mathbb{N}^*}$ satisfies the (LDP) with the constants $(\ln n)_{n\in\mathbb{N}^*}$ and rate function $\mathcal{H}$. Therefore the proof of Theorem 2.1 is in this case complete if we show that $\left(\widetilde{\mathrm{W}}_n^{\Xi}\right)_{n\in\mathbb{N}^*}$ and $\left(\widetilde{\mathrm{W}}_n\right)_{n\in\mathbb{N}^*}$ are equivalent w.r.t. $\mathbb{P}_x$ for $\mu - a.a.\ x \in \mathrm{E}$.

Following the same steps as in the proof of Proposition 4.1 we see that in order to verify this equivalence it suffices to prove that for $\mu - a.a.\ x \in \mathrm{E}$

**(4-24)** $$\lim_{n\to\infty} \frac{1}{\ln n}\ln\left(\mathbb{P}\left\{\max_{k=1,\ldots,n}\left\{\left|\mathrm{M}_k - \mathrm{S}_k^{*\,\Xi}\right|\right\} > \varepsilon\sqrt{n}\right\}\right) = -\infty.$$

If we let $\mathrm{Y}_n = \sup_{\mathrm{T}_n \leq t < \mathrm{T}_{n+1}} |\mathrm{M}_t - \mathrm{M}_{\mathrm{T}_n}|$, and as in the prior section $\mathrm{N}_n = \inf\{k \geq 0,\ \mathrm{T}_{k+1} > n\}$, then it is easy to see that

**(4-25)** $\qquad \left|\mathrm{M}_n - \mathrm{S}_n^{*\,\Xi}\right| \leq \mathrm{Y}_{\mathrm{N}_n}.$

By Doob's inequality and (4-21) for $\mu - a.a.\ x \in \mathrm{E}$ and $\beta > 1$

$$\textbf{(4-26)}\ \sup_{k\geq 0}\ \mathbb{E}_x\left(\mathrm{Y}_k^{\beta}\right) \leq \mathrm{C}_{18}\ \sup_{k\geq 0}\ \mathbb{E}_x\left(\left|\mathrm{M}_{\mathrm{T}_{k+1}} - \ \mathrm{M}_{\mathrm{T}_k}\right|^{\beta}\right) = \mathrm{C}_{18}\ \sup_{k\geq 1}\ \mathbb{E}_x\left(|\xi_k|^{\beta}\right).$$

So by Lemma 4.5 part a)

$$\textbf{(4-27 a)}\ \mathbb{E}_x\left(\mathrm{Y}_{\mathrm{N}_k}^{\beta}\right) = \sum_{l=0}^{\infty}\ \mathbb{E}_x\left(\mathrm{Y}_l^{\beta}1_{\{\mathrm{N}_k=l\}}\right) \leq \sum_{l=0}^{\infty}\ \mathbb{E}_x\left(\mathrm{Y}_l^{2\beta}\right)^{1/2}\ \mathbb{P}_x\left(\mathrm{N}_k = l\right)^{1/2}$$

$$\textbf{(4-27 b)}\qquad \leq \mathrm{C}_{19}\ \sum_{l=0}^{\infty} \frac{\mathbb{E}_x\left(\mathrm{N}_k^4\right)^{1/2}}{l^2} \leq \mathrm{C}_{20}\, k^3\,.$$

This together with Chebychev's inequality and (4-25) implies

$$\textbf{(4-28)}\ \mathbb{P}\left\{\ \max_{k=1,\ldots,n}\ \left\{\left|\mathrm{M}_k\text{-}\mathrm{S}_k^{*\,\Xi}\right|\right\} > \varepsilon\sqrt{n}\right\}$$

$$\leq \sum_{k=0}^{n} \mathbb{P}_x\left(|\mathrm{Y}_{\mathrm{N}_k}| > \varepsilon\sqrt{n}\right) \leq \sum_{k=0}^{n} \mathrm{C}_{21}\frac{k^3}{n^{\beta+6}} \leq \mathrm{C}_{22}n^{-\beta}.$$

This concludes the proof of Theorem2.1 for the special case of atoms. □

### 4.2.2 Case II: Chains with minoration property

We shall proof in this section Theorem 2.1 under the additional assumption, that there exist a set $C \in \mathcal{E}$, b$\in ]0,1[$ and a probability measure $\nu \in \mathcal{M}_1(\mathrm{E})$ with $\nu(C) = 1$ such that

$$\textbf{(4-29)}\qquad \Pi(x,\cdot) \geq \mathrm{b}\ 1_C(x)\ \nu(\cdot).$$

We shall remark that in particular $C$ is a small set (see e.g. Duflo [10] p. 286). Using this small set we construct (as in [21] for example) a new chain called split chain, i.e. a canonical version of a homogeneous Markov process

$$\overline{\mathrm{X}} = \left(\overline{\Omega},\, \overline{\mathcal{F}},\, (\mathbb{P}_{\overline{x}})_{\overline{x}\in\overline{\mathrm{E}}},\, \overline{\mathbb{F}} = (\overline{\mathcal{F}}_k)_{k\in\mathbb{N}}\,,\ (\overline{\mathrm{X}}_k)_{k\in\mathbb{N}}\right)$$

with values in $\overline{\mathrm{E}} = \mathrm{E}\times\{0,1\}$ and transition probability

$$\textbf{(4-30)}\ \overline{\Pi}\left((x,i), \mathrm{A}\times\mathrm{B}\right) = \begin{cases} \nu(\mathrm{A})\left\{(1-\mathrm{b}1_{\mathrm{C}}(x)\,)\delta_1(\mathrm{B}) + \mathrm{b}1_{\mathrm{C}}(x)\,\delta_0(\mathrm{B})\right\} & \text{if } i=0, \\ Q(x,\mathrm{A})\left\{(1-\mathrm{b}1_{\mathrm{C}}(x))\,\delta_1(\mathrm{B}) + \mathrm{b}1_{\mathrm{C}}(x)\,\delta_0(\mathrm{B})\right\} & \text{if } i=1; \end{cases}$$

where $Q(x,A) = (1-\mathrm{b}1_C(x))^{-1}\ (\Pi(x,A) - \mathrm{b}1_C(x)\nu(A)\,)$ . It is well known that

**Remarks 4.7**

**a)** $\mathrm{E}\times\{0\}$ is an atom for $\overline{\mathrm{X}}$ and

**(4-31)** $\sup_{\overline{x}\in \mathrm{E}\times\{0\}} \quad \sup_{\overline{x}\in \mathrm{E}\times\{0\}} \overline{\mathbb{E}}_{\overline{x}}\left(\mathrm{T}^k_{\mathrm{E}\times\{0\}}\right) \leq \mathrm{C}_{22} \sup_{\overline{x}\in C} \mathbb{E}_x\left(\mathrm{T}^k_C\right) < \infty$ for all $k \geq 1$.

**b)** If we denote the invariant distribution (which obviously exists by part a) of this remark) by $\overline{\mu}$, then $\overline{\mu}$ is related to the invariant distribution $\mu$ of the original Markov chain through $\overline{\mu}(A\times\{0\}) = \mathrm{b}\mu(A\cap C)$ and $\overline{\mu}(A\times\{1\}) = (1-\mathrm{b})\mu(A\cap C) + \mu(A\cap C^c)$.

**c)** Part b) of the remark implies easily that for all $\mathrm{Z}\in L^1(\mu)$, $\overline{\mathbb{E}}_{\overline{\mu}}\left(\overline{\mathrm{Z}}\right) = \mathbb{E}_\mu(\mathrm{Z})$ and $\mathbb{E}_x(\mathrm{Z}) = \mathrm{b}\, 1_{\{x\in C\}}\overline{\mathbb{E}}_{(x,0)}\left(\overline{\mathrm{Z}}\right) + (1-\mathrm{b}1_{\{x\in C\}})\overline{\mathbb{E}}_{(x,1)}\left(\overline{\mathrm{Z}}\right)$ for all $x\in\mathrm{E}$. Here $\overline{\mathrm{Z}}$ denotes the natural lift of Z from $\Omega$ to $\overline{\Omega} = \Omega\times\{0,1\}^{\mathbb{N}}$, i.e. $\overline{\mathrm{Z}}(\omega,(x_n)_{n\geq 0}) = \mathrm{Z}(\omega)$.

For details on the above construction and the remark we refer to Duflo [10], section 8.2.4.

By Remark 4.3 b) we conclude

**(4-32)** $\overline{\mathbb{E}}_{\overline{\mu}}\left(\overline{\mathrm{M}}_1^2\right) = \mathbb{E}_\mu\left(\mathrm{M}_1^2\right) = \sigma^2_{\mathrm{M}}$ and $\overline{\mathbb{E}}_{\overline{\mu}}\left(\left|\overline{\mathrm{M}}_1\right|^\beta\right) = \mathbb{E}_\mu\left(\left|\mathrm{M}_1\right|^\beta\right) < \infty$.

Since Theorem 2.1 has already been proved for chains with atoms, we conclude by (4-31) and (4-32) that $\left(\overline{\mathrm{W}}_n^{\mathrm{M}}\right)_{n>0}$ satisfies the (LDP) with constants $(\ln n)_{n>0}$ and rate function $\mathcal{H}$ w.r.t. $\overline{\mathbb{P}}_{\overline{x}}$ for $\overline{\mu}-a.a.\ \overline{x}\in\overline{\mathrm{E}}$.
Here $\overline{\mathrm{W}}_n^{\mathrm{M}}$ denote the empirical measure defined as in (1-16) with $(\mathrm{M}_k)_{k>0}$ replaced by $\left(\overline{\mathrm{M}}_k\right)_{k>0}$. It is not hard to see that $\overline{\mathrm{W}}_n^{\mathrm{M}}$ is the lift of $\mathrm{W}_n$. We therefore conclude by Remark 4.7 part c) for $x\in\mathrm{E}\setminus C$ $\mathbb{P}_x\{\mathrm{W}_n\in\cdot\} = \overline{\mathbb{P}}_{(x,1)}\left\{\overline{\mathrm{W}}_n^{\mathrm{M}}\in\cdot\right\}$ and for $x\in C$ $\mathbb{P}_x\{\mathrm{W}_n\in\cdot\} = \mathrm{b}\overline{\mathbb{P}}_{(x,0)}\left\{\overline{\mathrm{W}}_n^{\mathrm{M}}\in\cdot\right\} + (1-\mathrm{b})\overline{\mathbb{P}}_{(x,1)}\left\{\overline{\mathrm{W}}_n^{\mathrm{M}}\in\cdot\right\}$ and hence $(\mathrm{W}_n)_{n>0}$ satisfies the (LDP) with constants $(\ln n)_{n>0}$ and rate function $\mathcal{H}$ w.r.t. $\mathbb{P}_x$ for $\mu-a.a.\ x\in\mathrm{E}$.$\square$

### 4.2.3 Case III: General case

In this section will shall finish the proof of Theorem 2.1.
By enlarging the space if necessary, we may assume without loss of generality that there exits a sequence of i.i.d. random variables $(\rho_k)_{k>0}$ with $\mathbb{P}_x(\rho_1 = 0) = \mathrm{p}_0$ and $\mathbb{P}_x(\rho_1 = k) = (1-\mathrm{p}_0)^2\, \mathrm{p}_0^{k-1}$ for $k\in\mathbb{N}^*$ and $x\in\mathrm{E}$ which in addition are independent of the Markov chain. Then $\mathbb{E}_x(\rho_1) = 1$. Now let $\mathcal{R}_0 \equiv 0$, $\mathcal{R}_n = \sum_{k=1}^n \rho_k$ and $\widehat{\mathrm{X}}_n = \mathrm{X}_{\mathcal{R}_n}$.

Then $\widehat{\mathrm{X}} = \left(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathbb{P}}_x)_{x\in\mathrm{E}}, \widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_k)_{k\in\mathbb{N}}, (\widehat{\mathrm{X}}_k)_{k\in\mathbb{N}}\right)$ is a Markov chain with transition probability $\widehat{\Pi}(x,dy) = \mathrm{p}_0\delta_x(dy) + (1-\mathrm{p}_0)^2\ \mathrm{R}_{\mathrm{p}_0}(x,dy)$.

Hence, since $C$ is a small set for X (i.e.(1-2) holds) $C$ is a small set for $\widehat{X}$ which satisfies in addition (4-29). Further $\mu$ is also the invariant distribution for $\widehat{X}$ .

Moreover, if we let $\widehat{M}_n = M_{\mathcal{R}_n}$ then $\widehat{M} = \left(\widehat{M}_n, \, \sigma\left(\widehat{X}_k, \, k \leq n\right)\right)_n$ is a Martingale additive functional.

We shall show now that $\widehat{X}$ and $\widehat{M}$ satisfy the assumption of Theorem 2.1. By Lemma 4.4 part a) and the fact that $\mathbb{E}_\mu\left(|M_1|^\beta\right) = \mathbb{E}_\mu\left(|M_n - M_{n-1}|^\beta\right) < \infty$ we obtain

$$\mathbb{E}_\mu\left(\left|\widehat{M}_1\right|^\beta\right) = \mathbb{E}_\mu\left(|M_{\mathcal{R}_1}|^\beta\right) < \infty, \, \mathbb{E}_\mu\left(\widehat{M}_1^2\right) = \mathbb{E}_\mu\left(M_1^2\right)\mathbb{E}(\mathcal{R}_1) = \sigma_M^2 .$$

Furthermore if we let $\widehat{T}_C = \inf\left\{k \geq 1, \, \widehat{X}_k \in C\right\}$ then

$$\textbf{(4-33)} \qquad \sup_{x\in C} \mathbb{E}_x\left(\widehat{T}_C^\beta\right) \leq C_{23} \sup_{x\in C} \mathbb{E}_x\left(T_C^\beta\right).$$

For $\beta = 1$ this is exactly part 4) of Proposition 8.2.13 in Duflo [10]. The general case is proved by a straight forward modification of the proof for $\beta = 1$ given in Duflo. Details shall be omitted.

We therefore obtain from the previous part of the proof of Theorem 2.1 (case II) that $\left(W_n^{\widehat{M}}\right)_{n\geq 1}$ satisfies the (LPD) with constants $(\ln(n))_{n\geq 1}$ and rate function $\mathcal{H}$ w.r.t. $\widehat{\mathbb{P}}_x$ for $\mu - a.a.\ x \in E$.

Therefore the proof of Theorem 2.1 is complete, if we show that $\left(W_n^{\widehat{M}}\right)_{n\geq 1}$ and $\left(W_n^M\right)_{n\geq 1}$ are equivalent w.r.t. the (LDP). The proof of the equivalence however is a straight forward modification of the proof of (4-4). For the readers convenience we shall sketch the proof below.

As for (4-4) the proof can be reduced to

$$\textbf{(4-34)} \qquad \lim_{n\to\infty} \frac{1}{\ln n} \ln\left(\mathbb{P}_x\left\{\max_{k=1,\ldots,n} \left|\widehat{M}_k - M_k\right| > \varepsilon\sqrt{n}\right\}\right) = -\infty$$

for $\mu - a.a.\ x \in E$ (compare (4-8)). Observe that for $\mu - a.a.\ x \in E$

$$\textbf{(4-35)} \qquad C_{24} = \sup_n \mathbb{E}_x\left(|M_n - M_{n-1}|^\beta\right) < \infty$$

and hence by Lemma 4.4

$$\textbf{(4-36)} \qquad \sup_n \mathbb{E}_x\left(\left|\widehat{M}_n - \widehat{M}_{n-1}\right|^\beta\right) = \sup_n \mathbb{E}_x\left(\left|M_{\mathcal{R}_n} - M_{\mathcal{R}_{n-1}}\right|^\beta\right) < \infty.$$

Fix such an $x \in E$ and $\gamma > 0$, Using (4-35) we obtain as in (4-11) and (4-12) for sufficiently large $n \geq 1$ and $k \in \left\{1, ..., [n^{3/4}]\right\}$

$$\textbf{(4-37)} \qquad \mathbb{P}_x\left\{|M_k| > \varepsilon\sqrt{n}\right\} \leq C_{25}\ n^{-\gamma} \text{ and } \mathbb{P}_x\left\{\left|\widehat{M}_k\right| > \varepsilon\sqrt{n}\right\} \leq C_{26}\ n^{-\gamma} .$$

By Lemma 4.4 and Chebychev's inequality we conclude for $k \in \{[n^{3/4}], ..., n\}$

**(4-38)** $\quad \mathbb{P}_x \left\{ |\mathcal{R}_k - k| > k^{3/4} \right\} \leq \mathrm{C}_{27}\ k^{-\frac{4}{3}\gamma} \leq \mathrm{C}_{28}\ n^{-\gamma}.$

Now by Doob's maximal inequality, the Burkholder-Davis-Gundy inequality and Lemma 4.4 for sufficiently large $n \in \mathbb{N}^*$

**(4-39 a)** $\quad \mathbb{P}_x \left\{ \left| \widehat{\mathrm{M}}_k - \mathrm{M}_k \right| > \varepsilon\sqrt{n},\ |\mathcal{R}_k - k| \leq k^{3/4} \right\}$

**(4-39 b)** $\quad \leq \mathbb{P}_x \left\{ \max_{l \in \{1,...,2[k^{3/4}]+1\}} \left| \mathrm{M}_{k-[k^{3/4}]} - \mathrm{M}_{l+k-[k^{3/4}]} \right| > \frac{\varepsilon}{2}\sqrt{n} \right\}$

**(4-39 c)** $\quad \leq \mathrm{C}_{29}\, \mathbb{E}_x \left( \left| \mathrm{M}_{k+[k^{3/4}]} - \mathrm{M}_{k-[k^{3/4}]} \right|^{2\beta} \right) n^{-\beta}$

**(4-39 d)** $\quad \leq \mathrm{C}_{30}\ k^{\frac{3}{4}(\beta+2)}\ n^{-\beta}\ \leq \mathrm{C}_{30}\, n^{-\gamma}.$

Using these estimates we conclude the proof of (4-34) in the same way as the proof of (4-8). This completes the proof of Theorem 2.1. □

## 4.3 Proof of Proposition 3.1 and Proposition 3.2

We shall prove only Proposition 3.1 because the proof of Proposition 3.2 is a straight forward modification of the proof of Proposition 3.1 and contains no new ideas.
We shall denote by $\mathrm{X}^x = (\mathrm{X}_k^x)_{k \in \mathbb{N}}$ the (AR1) given through (3-1) with $\mathrm{X}_0 \equiv x$.
We observe first that if $\mathrm{X} = (\ \Omega, \mathcal{F}, (\ \mathbb{P}_x\ )_{x \in \mathrm{E}}\ ,\ \mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}}, (\mathrm{X}_k)_{k \in \mathbb{N}}\ )$ is a standard Markov chain on $\mathbb{R}$ with transition probability $\Pi(x, \cdot) = \mathbb{P}(\theta x + \beta_1 \in \cdot)$, then:

**(4-40)** The distribution of $\mathrm{X}^x$ under $\mathbb{P}$ is equal to that of X under $\mathbb{P}_x$.

It is well known that in the stable case the Markov chain has an invariant measure $\mu$, which is equal to the distribution of $\sum_{k=1}^{\infty} \theta^{k-1}\beta_k$.
We shall prove next

**Lemma 4.8**

**a)** *For every $x \in \mathbb{R}$ and $\delta > 0$,* $\sup_{n \in \mathbb{N}^*} \ \mathbb{E}_x \left( |\mathrm{X}_n|^\delta \right) < \infty.$

**b)** *Let $C = ]a, b[$ with $a < b$ , $\mu(C) > 0$ and $\varepsilon > 0$. Then* $\sup_{x \in \mathrm{C}_\varepsilon} \ \mathbb{E}_x \left( (\mathrm{T}_{\mathrm{C}_\varepsilon})^\delta \right) < \infty$ *for every $\delta > 0$, where $\mathrm{C}_\varepsilon = ]a - \varepsilon, b + \varepsilon[$.* ♢

**Proof**. For the proof of part a) we may assume without loss of generality that $\delta \in 2\mathbb{N}^*$. Using Hölder's inequality and the identical distribution of the random variables $(\beta_n)_{n\in\mathbb{N}^*}$ and letting $\beta_0 = x$ we obtain

$$\mathbb{E}_x\left(|\mathrm{X}_n|^{\delta}\right) \le \sum_{l_1,\dots,l_\delta=0}^{n} |\theta|^{l_1}\dots|\theta|^{l_\delta}\,\mathbb{E}_x\left(|\beta_{l_1}|\dots|\beta_{l_\delta}|\right) \le (1-|\theta|)^{-\delta}\,\mathbb{E}_x\left((|\beta_0|+\dots+|\beta_\delta|)^{\delta}\right).$$

In order to prove part b) it obviously suffices to show that there exist $m \in \mathbb{N}^*$, $\mathrm{q} \in ]0,1[$ and $\mathrm{C}_{31} > 0$ such that for all $n \in \mathbb{N}^*$

**(4-41)** $$\sup_{x\in \mathrm{C}_\varepsilon} \mathbb{P}_x\left(\mathrm{T}_{\mathrm{C}_\varepsilon} > m\,n\right) \le \mathrm{C}_{31}\,\mathrm{q}^n.$$

Using (3-1) we obtain inductively for $k$, $n \in \mathbb{N}^*$ with $k < n$

**(4-42)** $$\mathrm{X}_n = \theta^{n-k}\,\mathrm{X}_k + \sum_{l=1}^{n-k} \theta^{n-k-l}\beta_{l+k} = \theta^{n-k}\,\mathrm{X}_k + \mathrm{Z}_{k,n}$$

For $m \in \mathbb{N}^*$ with $4|\theta|^m\,(|a|+|b|+1) < \varepsilon$ we conclude for $x \in \mathrm{C}_\varepsilon$

$$\begin{aligned}\mathbb{P}_x\left(\mathrm{T}_{\mathrm{C}_\varepsilon} > m\,n\right) &\le \mathbb{P}_x\left(\bigcap_{k=1}^{n}\{\mathrm{X}_{k\,m} \notin \mathrm{C}_\varepsilon\}\right) = \mathbb{P}_x\left(\bigcap_{k=1}^{n}\left\{\theta^{k\,m}x + \mathrm{Z}_{0,km} \notin \mathrm{C}_\varepsilon\right\}\right)\\ &\le \mathbb{P}_x\left(\bigcap_{k=1}^{n}\left\{\mathrm{Z}_{0,k\,m} \notin \mathrm{C}_{\frac{\varepsilon}{2}}\right\}\right) = \alpha_n(0).\end{aligned}$$

We dropped the parameter $x$ in $\alpha_n(0)$, since the distribution of $\mathrm{Z}_{0,i\,m}$ , $i \in \mathbb{N}^*$ under $\mathbb{P}_x$ is independent of $x$. In the following we fix $x \in \mathbb{R}$. Analogously to (4-42) we obtain

**(4-43)** $$\mathrm{Z}_{0,i\,m} = \theta^m\,\mathrm{Z}_{0,(i-1)\,m} + \mathrm{Z}_{(i-1)m,i\,m}.$$

We observe next that $\mathrm{Z}_{(i-1)m,i\,m}$, $i \in \mathbb{N}^*$ are i.i.d. and that the distribution of $\mathrm{Z}_{0,m}$ converges (for $m \to \infty$) weakly to the invariant measure $\mu$. Hence by the Portmanteau Lemma $\limsup_{m\to\infty} \mathbb{P}_x\left(\mathrm{Z}_{0,m} \notin C\right) \le \mu\left(\mathbb{R}\backslash C\right) \le 1-\mu\left(C\right) < 1$.

For the following fix $r \in ]\mu\left(\mathbb{R}\backslash C\right),1[$ and $m \in \mathbb{N}^*$ such that $\mathbb{P}_x\left(\mathrm{Z}_{0,m} \notin C\right) \le r$, $4|\theta|^m\,(|a|+|b|+1) < \varepsilon$ and $\mathrm{q} = r + 2|\theta|^m < 1$.
Letting $\alpha_n(i) = \mathbb{P}_x\left(\bigcap_{k=1}^{n}\left\{\mathrm{Z}_{0,k\,m} \notin \left[-\left(\frac{|\theta|^{-m}}{2}\right)^i\varepsilon,\ \left(\frac{|\theta|^{-m}}{2}\right)^i\varepsilon\right]\right\}\right)$ for $i \in \mathbb{N}^*$. We shall see that for $i \in \mathbb{N}$ and $n \ge 2$

**(4-44)** $$\alpha_n(i) \le r\,\alpha_{n-1}(i) + \alpha_{n-1}(i+1)$$

Indeed, using (4-43) and the independence of the $\mathrm{Z}_{(i-1)\,m,i\,m}$, $i \in \mathbb{N}^*$, we conclude

$$\alpha_n(0) = \mathbb{P}_x\left(\bigcap_{k=1}^{n}\left\{\mathrm{Z}_{0,k\,m} \notin \mathrm{C}_{\frac{\varepsilon}{2}}\right\}\right)$$

$$\leq \mathbb{P}_x\left(\begin{array}{c}\bigcap_{k=1}^{n-1}\left\{\mathrm{Z}_{0,k\,m} \notin \mathrm{C}_{\frac{\varepsilon}{2}}\right\} \cap \left\{\mathrm{Z}_{0,(n-1)\,m} \in [-(2|\theta|^m)^{-1}\varepsilon,\ (2|\theta|^m)^{-1}\varepsilon]\right\} \\ \cap\left\{|\theta|^m\,\mathrm{Z}_{0,(n-1)\,m} + \mathrm{Z}_{(n-1)\,m\,,\,n\,m} \notin\ \mathrm{C}_{\frac{\varepsilon}{2}}\right\}\end{array}\right)$$

$$+\mathbb{P}_x\left(\bigcap_{k=1}^{n-1}\left\{\mathrm{Z}_{0,km} \notin \mathrm{C}_{\frac{\varepsilon}{2}}\right\} \cap \left\{\mathrm{Z}_{0,(n-1)\,m} \notin [-(2|\theta|^m)^{-1}\varepsilon,\ (2|\theta|^m)^{-1}\varepsilon]\right\}\right)$$

$$\leq \mathbb{P}_x\left(\bigcap_{k=1}^{n-1}\left\{\mathrm{Z}_{0,km} \notin \mathrm{C}_{\frac{\varepsilon}{2}}\right\} \cap \left\{\mathrm{Z}_{(n-1)\,m,nm} \notin C\right\}\right) +\ \alpha_{n-1}(1)$$

$$\leq\ \alpha_{n-1}(0)\mathbb{P}_x\left(\mathrm{Z}_{(n-1)\,m,\,n\,m} \notin C\right) +\ \alpha_{n-1}(1) = r\,\alpha_{n-1}(0) +\ \alpha_{n-1}(1).$$

We used also the fact that (by the choice of $m$) $\mathrm{C}_{\frac{\varepsilon}{2}} \subseteq [-(2|\theta|^m)^{-1}\varepsilon,\ (2|\theta|^m)^{-1}\varepsilon]$.
The case $i \geq 1$ is proved analogously.
Using (4-44) an easy induction argument shows that

$$\alpha_n(0) \leq \mathrm{C}_{32} \sum_{i=0}^{n-1} \binom{n-1}{i}\ r^{n-1-i}\ \alpha_1(i)$$

Observing that by part a) and Chebychev's inequality $\alpha_1(i) \leq \mathrm{C}_{32}\,(2|\theta|^m)^i$ for some $\mathrm{C}_{32} > 0$, we conclude

$$\alpha_n(0) \leq \mathrm{C}_{32} \sum_{i=0}^{n-1} \binom{n-1}{i}\ r^{n-1-i}\ \ (2|\theta|^m)^i = \mathrm{C}_{32}\ (r + 2|\theta|^m)^{n-1}\ = \frac{\mathrm{C}_{32}}{\mathrm{q}}\,\mathrm{q}^n.\ \ \square$$

Next we shall show that

**Lemma 4.9** *There exists a small set $C$ such that the Markov chain is Riemannian of any order $k$.* $\diamondsuit$

**Proof**. Since the distribution of $\beta_i$ has a non vanishing density part, the distribution of $\beta_i+\theta\beta_{i-1}$ has a non vanishing density part with a continuous density say $h$. Hence the exists $a < b$ such that $\inf_{]a,b[} h\ > 0$.
A simple application of Borel-Cantelli Lemma shows that there exists a $n_0 \in \mathbb{N}^*$ such that for all $n \geq n_0$ and $x \in \mathbb{R}$

**(4-45)** $$\mathbb{P}_x\left(\sum_{k=n+1}^{\infty}\ |\,\beta_k\,\theta^k| < \varepsilon\right) \geq \tfrac{1}{2}.$$

Let $\varepsilon \in ]0,1[$ such that $\frac{a}{2(1-\theta^2)} + 2\varepsilon < \frac{b}{2(1-\theta^2)} - 2\varepsilon$ and let $n \geq n_0$ such that $|\theta|^n \left(2 - \frac{|a|+|b|}{1-\theta^2}\right) < \varepsilon$. Now observe that for $n \in \mathbb{N}^*$

$$\textbf{(4-46)} \qquad \mathrm{X}_{2n} = \theta^{2n}\ \mathrm{X}_0 + \sum_{i=1}^{n} \theta^{2n-2i}\ (\beta_{2i} + \theta\beta_{2i-1}) = \theta^{2n}\ \mathrm{X}_0 + \mathrm{Z}_{0,2n}$$

and hence the distribution of $\mathrm{X}_{2n}$ w.r.t. $\mathbb{P}_x$ has a non vanishing density part with a continuous density, say $f_{2n}^x$, such that

$$\textbf{(4-47)} \qquad \inf_{x \in \left] \frac{a}{1-\theta^2},\ \frac{b}{1-\theta^2} \right[} \ \inf_{\left]a \frac{1-\theta^{2n}}{1-\theta^2} + \varepsilon,\ b \frac{1-\theta^{2n}}{1-\theta^2} - \varepsilon\right[} f_{2n}^x > 0.$$

Moreover since the invariant measure $\mu$ is equal to the distribution of $\mathrm{Z}_{0,2n} + \sum_{k=2n+1}^{\infty} \theta^{k-1}\beta_k$, it is easy to see that by (4-45) and (4-47) $\mu$ has a non vanishing density part say $g$ with $\inf_{\left]a \frac{1-\theta^{2n}}{1-\theta^2} + 2\varepsilon,\ b \frac{1-\theta^{2n}}{1-\theta^2} - 2\varepsilon\right[} g > 0$ .

By (4-47) we see that $[a \frac{1-\theta^{2n}}{1-\theta^2} + 2\varepsilon,\ b \frac{1-\theta^{2n}}{1-\theta^2} - 2\varepsilon]$ is a small set, and by Lemma 4.8 the Markov chain is Riemannian recurrent of any order $k$. $\square$

**Lemma 4.10** *For every $x \in \mathbb{R}$ and $\delta > 0$ there exists $\mathrm{C}_{33} > 0$ such that*

$$\textbf{(4-48)} \qquad \mathbb{E}_x\left(\left|\sum_{k=1}^{n} \mathrm{X}_{k-1}^2 - n\frac{\sigma^2}{1-\theta^2}\right|^{\delta}\right) \leq \mathrm{C}_{33}\, n^{\delta/2}.\ \diamondsuit$$

**Proof**. We remark that

$$\textbf{(4-49)} \qquad \sum_{k=1}^{n} \mathrm{X}_{k-1}^2 - n\frac{\sigma^2}{1-\theta^2} = \frac{1}{1-\theta^2}\left(\sum_{k=1}^{n} \left\{ \mathrm{X}_k^2 - \mathbb{E}_x\left(\mathrm{X}_k^2 \ /\ \mathcal{F}_{k-1}\right)\right\} - \mathrm{X}_n^2 + \mathrm{X}_0^2\right)$$

hence we conclude the proof by applying the Burkholder-Davis-Gundy inequality to the martingale $\left(\sum_{k=1}^{n} \left\{ \mathrm{X}_k^2 - \mathbb{E}_x\left(\mathrm{X}_k^2 \ /\ \mathcal{F}_{k-1}\right)\right\}\right)_n$ and the part a) of Lemma 4.8. $\square$

**Proof of Proposition 3.1**. We observe first that $\mathrm{M}_n = \sum_{k=1}^{n} \mathrm{X}_{k-1}\beta_k$, $n \in \mathbb{N}^*$ is a (MFA) with $\sigma_{\mathrm{M}}^2 = \mathbb{E}_\mu\left(\mathrm{M}_1^2\right) = \frac{\sigma^4}{1-\theta^2}$. Hence, by Lemma 4.9 we can apply Theorem 2.1 to M and the Markov chain X.

Letting $\mathrm{M}_n^x = \sum_{k=1}^{n} \mathrm{X}_{k-1}^x \beta_k$ we conclude by Theorem 2.1 and (4-40) that $\left(\Psi_n^{\mathrm{M}^x}\right)_{n\in\mathbb{N}^*}$ satisfies the (LDP) with constants $(\ln(n))_{n\in\mathbb{N}^*}$ and rate function $\mathcal{H}$ w.r.t. $\mathbb{P}_x$ for $\mu - a.a.\ x \in \mathbb{R}$.

Furthermore by (4-42) $\mathrm{M}_n^x - \mathrm{M}_n^y = (x-y)\sum\limits_{k=1}^{n} \theta^{k-1}\beta_k$. Since with $\mathrm{M}_n^x$ and $\mathrm{M}_n^y$ also $\mathrm{M}_n^x - \mathrm{M}_n^y$ is a martingale, Chebychev's inequality and Doob's maximal inequality imply easily for $\gamma \in \mathbb{N}^*$

$$\mathbb{P}\left\{ \max_{k=1,\dots,n} |\mathrm{M}_n^x - \mathrm{M}_n^y| > \varepsilon\sqrt{n} \right\} \le \mathrm{C}_{34}\, \varepsilon^{-2\gamma}\, n^{-\gamma}\, |x-y|^{2\gamma}\, \mathbb{E}_x\left( \left| \sum_{k=1}^{n} \theta^{k-1}\beta_k \right|^{2\gamma} \right)$$

$$\le \mathrm{C}_{35}\, n^{-\gamma}.$$

This in turn implies that $\left(\mathrm{W}_n^{\mathrm{M}^x}\right)_{n\in\mathbb{N}^*}$ and $\left(\mathrm{W}_n^{\mathrm{M}^y}\right)_{n\in\mathbb{N}^*}$ are equivalent w.r.t. the (LDP) (see the proof of (4-4) and in particular (4-6) and (4-8)).

Therefore it remains to prove that for all initial states $x \ne 0$ $\left(\mathrm{W}_n^{\theta}\right)_{n\in\mathbb{N}^*}$ and $\left(\mathrm{W}_n^{\mathrm{M}^x}\right)_{n\in\mathbb{N}^*}$ are equivalent w.r.t. (LDP).

The proof of the equivalence is again very similar to that of (4-42), so that it suffices to give only a sketch of the proof.

Fix $x \ne 0$. A simple calculation show that

$$\left|\Psi_n^{\theta}\left(\tfrac{k}{n}\right) - \Psi_n^{\mathrm{M}^x}\left(\tfrac{k}{n}\right)\right| = \left| \sqrt{\tfrac{n}{1-\theta^2}}\left(\tfrac{k}{n}\right)\left(\sum_{k=1}^{k} \mathrm{X}_{k-1}^2\right)^{-1} \mathrm{M}_k^x - \sqrt{\tfrac{1-\theta^2}{n\sigma^4}}\mathrm{M}_k^x \right|$$

$$= \sqrt{\tfrac{1-\theta^2}{n\sigma^4}}|\mathrm{V}_k^x|\,|\mathrm{M}_k^x|\,|\mathrm{U}_k^x|^{-1}$$

where $\mathrm{U}_n^x = \sum\limits_{k=1}^{n} \left(\mathrm{X}_{k-1}^x\right)^2$ and $\mathrm{V}_n^x = n\frac{\sigma^2}{1-\theta^2} - \mathrm{U}_n^x$.

Using again the same arguments as in the proof of (4-4) it remains to verify

$$\textbf{(4-50)} \qquad \limsup_{n\to\infty}\ \mathbb{P}\left\{ \max_{k=1,\dots,n} |\mathrm{V}_k^x|\,|\mathrm{M}_k^x|\,|\mathrm{U}_k^x|^{-1} > \varepsilon\sqrt{n} \right\} = -\infty$$

Observing that $\mathrm{U}_n^x \ge (\mathrm{X}_0^x)^2 = x^2 > 0$ we obtain by Chebychev's inequality, Hölder's inequality and Lemma 4.10 and Lemma 4.4 analogously to (4-9) that for $\gamma > 0$, $n$ sufficiently large and $k \le n^{1/8}$

$$\textbf{(4-51)} \qquad \mathbb{P}\left\{ |\mathrm{V}_k^x|\,|\mathrm{M}_k^x|\,|\mathrm{U}_k^x|^{-1} > \varepsilon\sqrt{n} \right\} \le \mathrm{C}_{36}\,\left(x^2\varepsilon\right)^{-2\gamma} n^{\frac{\gamma}{4}}\, n^{-\frac{\gamma}{2}} \le \mathrm{C}_{37}\, n^{-\frac{\gamma}{4}}$$

and for $k \in \left\{[n^{1/8}],\dots,n\right\}$

$$\textbf{(4-52)} \qquad \mathbb{P}\left\{ |\mathrm{V}_k| > k^{\frac{3}{4}} \right\} \le \mathrm{C}_{38}\, n^{-\frac{\gamma}{4}}.$$

Finally again by Chebychev's inequality and Lemma 4.4 we conclude for $n$ sufficiently large and $k \in \left\{[n^{1/8}],\dots,n\right\}$

$$\textbf{(4-53 a)} \qquad \mathbb{P}\Big\{ |\mathrm{V}_k^x|\,|\mathrm{M}_k^x|\,|\mathrm{U}_k^x|^{-1} > \varepsilon\sqrt{n},\ \ |\mathrm{V}_k^x| \le k^{\frac{3}{4}} \Big\}$$

$$\textbf{(4-53 b)} \qquad \le\ \mathbb{P}\Big\{ k^{\frac{3}{4}}\,|\mathrm{M}_k^x|\, 2\left(\sigma^2(1-\theta^2)^{-1}k\right)^{-1} > \varepsilon\sqrt{n} \Big\}$$

$$\textbf{(4-53 c)} \qquad \le\ \mathbb{P}\Big\{ |\mathrm{M}_k^x| > \varepsilon\, n^{\frac{17}{32}} \Big\} \le \mathrm{C}_{39}\, n^{-\gamma}.$$

Now (4-50) is immediate consequence of (4-51) to (4-53), since $\gamma > 0$ was arbitrary. This concludes the proof of Proposition 3.1. □

## 5 Remarks

We shall conclude this paper with some remarks.

**a)** Theorem 2.1 implies (LDP) for further a.s. limit theorems, like the (ASCLT) on the real line, a.s. versions of arcsine law (Compare Corollary 2.10 and Examples 2.11 in Heck [14]).

**b)** The (LDP) for (ASCLT) implies in particular easily the (ASCLT) itself. Therefore, for random variables satisfying the assumption of Theorem 2.1, Theorem 2.1 can be regarded as an generalization of Theorem A (see Corollary 2.12 in Heck [14]).

**c)** The moment assumptions like (1-6) or (2-1) are necessary compare to the corresponding assumptions in Heck [14] and March and Seppäläinen [22]. Lifshits & Stankevich [18] prove the necessity of these hypotheses.

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