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SMALL DIFFUSION AND FAST DYING OUT ASYMPTOTICS FOR SUPERPROCESSES AS NON-HAMILTONIAN QUASICLASSICS FOR EVOLUTION EQUATIONS

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Abstract The small diffusion and fast dying out asymptotics is calculated for nonlinear evolution equations of a class of superprocesses on manifolds, and the corresponding logarithmic limit of the solution is shown to be given by a solution of a certain problem of calculus of variations with a non-additive (and non-integral) functional.

Keywords Dawson-Watanabe superprocess, reaction diffusion equation, logarithmic limit, small diffusion asymptotics, curvilinear Ornstein-Uhlenbeck process.

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1 Introduction

As is well known, a diffusion process in $\mathbf{R}^d$ with absorption is governed by an equation of the form

$$\frac{\partial u}{\partial t} = (\mathcal{L}u)(x) \equiv \frac{1}{2}(G(x)\frac{\partial}{\partial x}, \frac{\partial}{\partial x})u - (A(x), \frac{\partial u}{\partial x}) - V(x)u, \quad x \in \mathbf{R}^d, \, t \geq 0, \tag{1.1}$$

where $G(x)$ is a symmetric non-negative $d \times d$ matrix, $A(x)$ is a vector-valued function (usually called the drift) and $V(x) \geq 0$ for all x. After the seminal papers of Varadhan [Var1]-[Var2], lots of attention in mathematical literature was given to the study of the equation

$$\frac{\partial u}{\partial t} = (\mathcal{L}_h u)(x) \equiv \frac{h}{2}(G(x)\frac{\partial}{\partial x}, \frac{\partial}{\partial x})u - (A(x), \frac{\partial u}{\partial x}) - \frac{1}{h}V(x)u, \quad x \in \mathbf{R}^d, \, t \geq 0, \tag{1.2}$$

for a small parameter $h > 0$ (see e.g. [Kol1], where one can find an extensive bibliography on this subject and also generalisations to more general Feller processes). The limit $h \to 0$ in (1.2) clearly stands for the small diffusion and fast absorption (or killing, or dying) approximation, and is often called shortly the small diffusion limit. In quantum mechanics (i.e. for the corresponding Schrödinger equation) the same limit is called the semiclassical (or quasiclassical, or WKB) approximation. This terminology is also quite justified for the diffusion equation (1.2), because the so called logarithmic limit

$$S(t,x) = \lim_{h \to 0}(-h \log u(t,x)) \tag{1.3}$$

of a solution $u(t,x)$ of (1.2) (which describes the exponential rate of decay of the solution $u(t,x)$) can be usually specified as the classical action along the solutions of the Hamiltonian system with the Hamiltonian function

$$H(x,p) = \frac{1}{2}(G(x)p,p) - (A(x),p) - V(x),$$

which represent the classical (deterministic) limit as $h \to 0$ for random trajectories of the diffusion process described by (1.2). Alternatively, this logarithmic limit can be defined as a solution of a certain problem of calculus of variations given by a Lagrangian function $L(x,v)$ that is the Legendre transform of $H(x,p)$ with respect to the second variable.

The aim of this paper is to describe similar asymptotics for a certain class of nonlinear parabolic equations (often called the reaction-diffusion equations) which appear in the study of superprocesses. Namely, the evolution equation that describes the deterministic dual process for the (non-homogeneous) Dawson-Watanabe superprocess (also called $(2,d,\beta)$ superprocess) is known to have the form (see e.g. [Et])

$$\frac{\partial u}{\partial t} = \mathcal{L}u - a(x)u^{1+\beta}, \tag{1.4}$$

where $\mathcal{L}$ is a generator of a diffusion process, given by (1.1), say, the function $a(x)$ describes the rate of branching at the point x and $\beta > 0$ specifies the branching mechanism. The value $\beta = 0$ corresponds to the case, when particles are supposed just to die at a moment of branching (in that case the non-linear equation (1.4) turns to the linear equation (1.1)). Thus, for small $h > 0$, the equation

$$\frac{\partial u}{\partial t} = \frac{h}{2}(G(x)\frac{\partial}{\partial x}, \frac{\partial}{\partial x})u - (A(x), \frac{\partial u}{\partial x}) - \frac{1}{h}V(x)u - \frac{1}{h}a(x)u^{1+h} \tag{1.5}$$

describes the (spatially non-homogeneous) Dawson-Watanabe superprocess in the approximation of small diffusion and fast dying out just as in the case of equation (1.2) for usual diffusion. The standard (homogeneous) Dawson-Watanabe superprocess in the approximation considered is described by the equation

$$\frac{\partial u}{\partial t} = \frac{h}{2}\Delta u - \frac{1}{h} u^{1+h} \tag{1.6}$$

with small $h > 0$.

In this paper we are going to characterise the deterministic process which is obtained by letting $h \to 0$ in equation (1.5) and to calculate the corresponding $h \to 0$ asymptotics of its solutions. In particular, it turns out that the logarithmic limit (1.3) of the solutions is obtained in terms of the solutions to a certain non-Hamiltonian system of ordinary differential equations. Alternatively, this limit can be described as the solution of a problem of calculus of variations with a non-additive (or non-integral) functional. Our methods are pure analytic and are based on a study of lower and upper solutions of equation (1.5) (as, for instance, in paper [Kl]) combined with the WKB type approach to the construction of asymptotic expansions.

Let us introduce some notations. Let $L(x, v, s)$ be a smooth function on $\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}$, called a (generalised) Lagrangian, which is convex (perhaps non-strictly) with respect to the second variable. For a continuous piecewise smooth curve $y(.) : [0, t] \mapsto \mathbf{R}^d$ and a fixed real number λ_0, let us define the functional

$$I_t^{\lambda_0}(y(.)) = \lambda(t), \tag{1.7}$$

where $\lambda(.)$ is the solution to the Cauchy problem

$$\dot{\lambda}(\tau) = L(y(\tau), \dot{y}(\tau), \lambda(\tau)), \quad \lambda(0) = \lambda_0, \tag{1.8}$$

and the two-point function

$$S_L(t, x; x_0, \lambda_0) = \inf\{I_t^{\lambda_0}(y(.)) : \quad y(0) = x_0, \ y(t) = x\}. \tag{1.9}$$

In some cases (see section 3), one can prove that the function S_L and the corresponding minimising curve can be obtained through the solutions of the following system of ordinary differential equations

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}(x, p, s) \\ \dot{p} = -\dfrac{\partial H}{\partial x}(x, p, s) - \dfrac{\partial H}{\partial s}(x, p, s)p \\ \dot{s} = p\dfrac{\partial H}{\partial p}(x, p, s) - H(x, p, s) \end{cases} \tag{1.10}$$

with the Hamiltonian function $H(x, p, s)$ being the Legendre transform of L with respect to the second variable. This connection allows, for example, an easy method of explicit calculations of small times t and small distances $x - x_0$ asymptotic expansions of S_L. The Lagrangian L and the Hamiltonian H which are relevant to the study of equation (1.5) have the form

$$L(x, v, s) = \frac{1}{2}(G^{-1}(x)(v + A(x)), v + A(x)) + V(x) + a(x)e^{-s}, \tag{1.11}$$

$$H(x, p, s) = \frac{1}{2}(G(x)p, p) - (A(x), p) - V(x) - a(x)e^{-s}. \tag{1.12}$$

The plan of the paper is the following. In section 2 we formulate the results of the paper and then discuss their possible extensions and applications. The main Theorem 1 gives an asymptotic representation for the solution $u_\delta(t,x;x_0,h,\theta)$ of equation (1.5) with the scaled Dirac initial condition

$$u_0(x) = \theta\delta(x-x_0), \tag{1.13}$$

where θ is an arbitrary positive number. The connection between this solution and the superprocess $Z_h(t)$ corresponding to equation (1.5) is given by the following well known formula (see e.g. [Et]):

$$E_x \exp\{-\theta Z_h^x(t,x_0)\} = \exp\{-u_\delta(t,x;x_0,h,\theta)\}, \tag{1.14}$$

where E_x denotes the expectation for the superprocess with the initial measure being the unit mass $\delta_x(y) = \delta(y-x)$ at x and $Z_h^x(t,x_0)$ denotes the density (with respect to Lebesgue measure) of the corresponding measure $Z_h(t)$ at x_0. In particular,

$$E_x Z_h^x(t,x_0) = \frac{\partial}{\partial\theta} u_\delta(t,y;x_0,h,\theta)|_{\theta=0}. \tag{1.15}$$

From (1.14) and Theorem 1 one derives an asymptotic representation for the Laplace transform of the random variable $Z_h^x(t,x_0)$, and in particular, its logarithmic limit, which estimates the rate of decay (large deviations) of this random variable, see Theorem 2. In Theorem 3 we provide an exponential multiplicative asymptotics for the expectation (1.15), which (surprisingly enough) turns out to be different from what one could expect by formal differentiation of the obtained asymptotics for the characteristic function (1.14). In Theorem 4 we present similar asymptotic formulas for the Cauchy problem of equation (1.5) with smooth initial conditions of the form

$$u_0 = h^{-\alpha}\phi_0(x)\exp\{-S_0(x)/h\}, \tag{1.16}$$

where $\alpha > 0$ is a constant and ϕ_0, S_0 are smooth non-negative functions. From this asymptotics one can similarly obtain the corresponding asymptotics for the superprocess $Z_h(t)$. At the end of Section 2 we discuss the generalisations of the main Theorem 1 to the case of superprocesses on manifolds with possibly degenerate underlying diffusion (considering as the main example the curvilinear Ornstein-Uhlenbeck process) and other possible extensions of the results obtained.

Section 3 is devoted to the necessary background results on the calculus of variations for non-additive functionals. In particular, the two-point function S_L is studied there, its asymptotics is given and the main example is discussed, where S_L can be calculated explicitly. Surely, a problem of calculus of variations with non-integral functional considered here, can be reduced to a usual one by enlarging the dimension of the space. However, this procedure will destroy the nice property of non-degeneracy of Hamiltonian and Lagrangian functions, which is very convenient for proving results on the existence of global minima.

In section 4 we give a simple auxiliary result on diffusion equations having a time dependent diffusion coefficient that is singular at time zero. In section 5, a proof of Theorem 1 is provided.

2 Main results and discussion

The main result of the paper is the following.

Theorem 1. *Suppose that $G(x)$ is a positive-definite matrix, the functions G, G^{-1}, A, a, V are uniformly bounded together with their first two derivatives, and the functions V, a are non-negative. Let $u_\delta(t,x;x_0,h,\theta)$ denote a solution to problem (1.5), (1.13) (which is well known to exist globally and to be unique for small enough h). Then, there exist positive t_0, h_0, C and K such that for $t \le t_0$ and $h \le h_0$*

$$u_\delta(t,x;x_0,h,\theta) = (2\pi ht)^{-d/2}\psi_\delta(t,x;x_0,h,\theta)\exp\{-S_L(t,x;x_0,0)/h\} + O(\exp\{-M(t)/h\}), \tag{2.1}$$

where S_L is given by (1.9) with Lagrangian (1.11), ψ satisfies the two-sided estimates

$$C^{-1}\min(\theta, h^{d/2}) \le \psi_\delta(t,x;x_0,h,\theta) \le C\theta \tag{2.2}$$

uniformly for $h \le h_0$, $t \le t_0$, and

$$M(t) = \min\{S_L(t,x;x_0,0); \quad |x-x_0| = K\} - \epsilon$$

for some arbitrary $\epsilon > 0$. In particular, one has

$$\lim_{h\to 0} -h\log u(t,x;x_0,h) = S_L(t,x;x_0,0) \tag{2.3}$$

for $|x-x_0| < K$ and $t \le t_0$ and

$$\lim_{t\to 0} -ht\log u(t,x;x_0,h) = l(x,x_0),$$

where $l(x,x_0)$ is the distance between the points x, x_0 in the Riemannian metric on $\mathbf{R}^d$ defined by the matrix $G(x)^{-1}$.

(ii) If G, G^{-1}, A, a, V are such that for $t \le t_0$ the boundary value problem for (1.10) with Hamiltonian (1.12) is uniquely solvable for all x, x_0 and thus S_L is smooth for all x, x_0 (in particular, this is the case if G, A, V and a are constants outside a compact domain), then representation (2.6) holds without the remainder term $O(\exp\{-M(t)/h\})$.

(iii) The function S_L has the following asymptotic expansion

$$S_L(t,x;x_0,s_0) = s_0 + \frac{1}{2t}(x-x_0+A(x_0)t, G(x_0)^{-1}(x-x_0+A(x_0)t))$$
$$+\frac{1}{t}\left(V(x_0)t^2 + \sum_{j=3}^{k} P_j(t,x-x_0) + O(r+t)^{k+1}\right) \tag{2.4}$$

uniformly for $t \le t_0$ and $|x-x_0| \le r$, where the P_j are polynomials in t and $x-x_0$ of degree j with coefficients depending on x_0.

(iv) In case of equation (1.6), the limit (2.3) is valid for all (t,x) and the two-point function S_L can be calculated from a certain implicit algebraic equation (see Section 3). In particular,

$$S_L(t,x;x,s_0) = \log(t+e^{s_0}) \tag{2.5}$$

and

$$S_L(t,x;x_0,s_0) = \log(2t+e^{s_0}) \tag{2.6}$$

whenever

$$|x-x_0| = \sqrt{2}e^{s_0/2}(\sqrt{1+2te^{-s_0}}-1).$$

Remark 1. The function ψ in (2.1) can be further expanded in an asymptotic series in powers of t, $x-x_0$, h, θ. *Remark 2.* The theorem can be easily generalised to include the modification of equation (1.5) where instead of nonlinear term u^{1+h} one has more general term $u^{1+h\beta(x)}$ with a bounded positive function β, which describes the spatially non-homogeneous branching mechanism. The author is grateful to T. Lyons who indicated the importance of such a modification.

Theorem 1 is proved in Section 5.

From (1.14) and Theorem 1, one can obtain asymptotic formulas describing the behavior of the corresponding superprocess $Z_h(t)$ as $h \to 0$. For example, the following is straightforward.

Corollary. *For* $|x-x_0| < K$ *and* $t \le t_0$

$$\lim_{h\to 0} h \log(-\log E_x \exp\{-\theta Z_h^x(t,x_0)\}) = -S_L(t,x;x_0,0). \tag{2.7}$$

Notice that this limit does not depend on θ and is uniform for θ from any compact interval of $(0,\infty)$. However, for $\theta \to 0$ the situation changes. In particular, one can not obtain the asymptotics for $E_x Z_h^x(t,x_0)$ from (1.15) by formal differentiation of the asymptotics (2.1). More precisely, one has the following result.

Theorem 2. *Under the assumptions of Theorem 1*

$$E_x Z_h^x(t,x_0)$$

$$= (2\pi ht)^{-d/2}(\det G(x_0))^{-1/2}(1+O(ht))\exp\{-S_{L_{red}}(t,x;x_0,0)\} + O(\exp\{-M(t)/h\}) \tag{2.8}$$

as $h \to 0$, *where* $S_{L_{red}}$ *corresponds to the reduced Lagrangian* L_{red}, *which is obtained from (1.11) by dropping the term* $a(x)e^{-s}$ *(which is responsible for non-linearity). In particular,*

$$\lim_{h\to 0} h \log E_x Z_h^x(t,x_0) = -S_{L_{red}}(t,x;x_0,0)$$

for $|x-x_0| < K$ *and* $t \le t_0$.

Proof. Due to (1.15), the l.h.s. of (2.8) is expressed in terms of the derivative of u_δ with respect to θ. Differentiating equation (1.5) with respect to θ and setting $\theta = 0$, we find that the l.h.s. of (2.8) is the solution to the equation

$$\frac{\partial u}{\partial t} = \frac{h}{2}(G(x)\frac{\partial}{\partial x}, \frac{\partial}{\partial x})u - (A(x), \frac{\partial u}{\partial x}) - \frac{1}{h}V(x)u \tag{2.9}$$

with the Dirac initial condition $u|_{t=0} = \delta(x-x_0)$, i.e. it is the fundamental solution of equation (2.9). But this equation is linear, and (2.8) follows from the well known linear theory (see e.g. [Kol1]).

The analogue of Theorem 1 for smooth initial conditions has the following form.

Theorem 3. *Let ϕ_0 and S_0 be non-negative functions such that ϕ_0 and ϕ_0^{-1} are uniformly bounded and the first and second derivatives of ϕ_0 and S_0 are uniformly bounded. Then, under the assumption of Theorem 1, there exist $t_0 > 0$ and $h_0 > 0$ such that the (unique) solution $u(t,x)$ of the problem (1.5),(1.16) has the form*

$$u(t,x) = h^{-\alpha}\phi_0(x)\psi(t,x,h)\exp\{-S(t,x)/h\},$$

for $t \le t_0$ and $h \le h_0$, where

$$S(t,x) = \inf_{\xi}\{S_L(t,x;\xi,S_0(\xi))\} \tag{2.10}$$

with S_L as above and where $\psi(t,x,h)$ has positive lower and upper bounds uniformly for all $t \le t_0$, $x \in \mathcal{R}^d$, $h \le h_0$. In particular, formula (1.3) for the logarithmic limit of the solution $u(t,x)$ holds with S given by (2.10). In the simplest case of equation (1.6) and the initial condition of form (1.15) with $S_0(x)$ being a non-negative constant S_0, the logarithmic limit exists for all (t,x) and can be calculated explicitly:

$$S(t,x) = \log(t + \exp\{S_0\}) \tag{2.11}$$

for all (t,x).

The proof of this theorem is omitted for brevity, because it is similar to the proof of Theorem 1 (in particular, the existence of the smooth function (2.10) follows from Proposition 3.5), but much simpler.

The results formulated above can be generalised in different directions. First of all, they can be easily expanded to the case of superprocesses on manifolds. Next, the results still hold for superprocesses defined by Hamiltonians (1.12) with degenerate matrix G such that the corresponding Hamiltonian is regular (in the sense of [Kol1]) for all s. This is a large class of processes parametrised by Young schemes. In particular, this class contains superdiffusions with the underlying diffusion process being the Ornstein-Uhlenbeck process. More generally, one can take a curvilinear Ornstein-Uhlenbeck process on a compact Riemannian manifold M of dimension d, say. Using arbitrary local coordinates x on M and the corresponding coordinates y on $T_x^\star M$, such a process can be defined (see e.g. [Kol1] for general analytic definition in terms of generators and/or imbeddings that we use here, and [Joe] for a geometric construction in a particular case) as the diffusion process on the cotangent bundle $T^\star M$ described by the system of equations

$$\begin{cases} \dot{x} = G(x)y \\ dy = [-\frac{1}{2}\frac{\partial}{\partial x}(G(x)y,y) + F(x,y)]\,dt + \frac{\partial}{\partial x}(r(x),dW), \end{cases} \tag{2.12}$$

where $G(x) = g^{-1}(x)$, the matrix $g(x)$ defines the Riemannian metric on M, $r : M \mapsto \mathbf{R}^m$ is an isometric embedding of M into Euclidean space of dimension $m \ge d$, W is the standard Brownian motion in $\mathbf{R}^m$ and

$$F(x,y)_i = b_i(x) + \beta_i^j(x)y_j + \frac{1}{2}\gamma_i^{kl}(x)y_k y_l$$

describes the quadratic friction force on M with $b = \{b_i\}$, $\beta = \{\beta_i^j\}$, $\gamma = \{\gamma_i^{kl}\}$ being arbitrary tensors of type $(1,0)$, $(1,1)$, $(1,2)$. If F vanishes, the corresponding process is called the stochastic geodesic flow.

The evolution equation (1.5) that corresponds to a superdiffusion with the underlying diffusion (2.12) is

$$\frac{\partial u}{\partial t} = \frac{h}{2}(g(x)\frac{\partial}{\partial x}, \frac{\partial}{\partial x})u + (G(x)y, \frac{\partial u}{\partial x})$$

$$-\frac{1}{2}(\frac{\partial}{\partial x}(G(x)y, y), \frac{\partial u}{\partial y}) + (F(x,y), \frac{\partial u}{\partial y}) - \frac{1}{h}a(x)u^{1+h}, \tag{2.13}$$

and its Hamiltonian is

$$H = \frac{1}{2}(G(x)p_y, p_y) - (G(x)y, p_x) + \frac{1}{2}\left(\frac{\partial}{\partial x}(G(x)y, y), p_y\right) - (F(x,y), p_y) - a(x)e^{-s}, \tag{2.14}$$

where $a(x)$ is a non-negative function on M specifying the branching rate. One sees by inspection that this equation is invariant under the change of local coordinates x on M.

Theorem 4. *The results of Theorems 1 -4 hold for equation (2.13) with the only modification being that instead of expansion (2.4) for the two-point function S_L one has an expansion of the form*

$$S_L(t,x,y;x_0,y_0,s_0) = t^{-3}\Sigma(t,x,y;x_0,y_0,s_0),$$

where Σ has a regular asymptotic expansion in integer powers of t, $x-x_0$, $y-y_0$ with coefficients depending on x_0, y_0, s_0.

Moreover, if M is compact, then the trace

$$\int_{T^\star M} u_\delta(t,x,y;x,y,h)\, dx\, dy \tag{2.15}$$

exists and has the form of an asymptotic expansion in integer powers of t with the multiplier $(2\pi t^3)^{-d/2}$.

In case $F=0$, one has

$$\lim_{t\to 0} -ht^3 \log u_\delta(t,x,y;x_0,y_0) = \tilde{l}(x,x_0), \tag{2.16}$$

with $\tilde{l}(x,x_0) = S_{\tilde{L}}(1,x,0;x_0,0)$, where $\tilde{L}$ corresponds to (2.14) with vanishing F and a.

The proof of this theorem is the same as for theorem 1. It is obtained by generalisation of the corresponding results from [Kol1] where the linear case $a(x)=0$ was considered. The results on the trace (2.14) for vanishing $a(x)$ are discussed in [AHK]. The limit (2.16) follow from the following scaling property of the solutions of the Hamiltonian system with H of form (2.14) with vanishing F and a: if $(x,y,p_x,p_y)(t)$ is a solution then $(x, vy, v^3p_x, v^2p_y)(vt)$ is also a solution for any positive number v.

As other situations, where it seems possible to obtain similar results, let us mention the evolution equations of superprocesses with underlying Markov processes being truncated stable or stable-like process, and also the non-linear diffusion equations describing the hydrodynamic limit of interacting Brownian particles (see e.g. [Var3]). It also seems possible to develop the corresponding asymptotics for stationary boundary value problem for equations of type (1.5) similarly to the linear case as developed e.g. in [Var1-Var2].

An interesting problem is to obtain and justify the multiplicative exponential asymptotics for equation (1.5) globally, i.e. for all t, x including focal points. In particular, one can expect

(2.3), (2.7) to hold globally and with essentially weaker assumptions on the smoothness of the coefficients of equation (1.5). The well known proofs for the case of vanishing $a(x)$ depend on linearity and do not work here directly. On the other hand, the methods of [Ma2] are developed for nonlinear equations with unitary nonlinearities and give a method for justifying additive, but not multiplicative asymptotics.

3 On calculus of variations for non-additive functionals

Let the functions H and L of form (1.11), (1.12) be fixed, where $G(x)$ is symmetric positive matrix for all x and G, G^{-1}, A, a, V are all uniformly bounded together with their derivatives (in fact, the existence of only the first two bounded derivatives is essential for the most of results given below), and functions $V(x)$ and $a(x)$ are non-negative. We shall denote by $X(t, x_0, p_0, s_0)$, $P(t, x_0, p_0, s_0)$, $S(t, x_0, p_0, s_0)$ the solution of the Cauchy problem for system (1.10) with initial conditions (x_0, p_0, s_0) at time zero. In this section we collect the necessary results on the solutions to the boundary value problem for system (1.10) and give the interpretation of these solutions as minimising curves for a certain optimisation problem. Proofs are omitted, because they are obtained as more or less straightforward generalisations of the corresponding proofs given in the book [Kol1] (Sections 3.1,3.2) for the case of H of form (1.12) with vanishing $a(x)$. At the end of the section we consider a particular example where all solutions can be calculated explicitly.

Proposition 3.1. Local resolution of the boundary value problem. *For any sufficiently small $c > 0$, there exist positive constants r, t_0 such that for all (t, x, x_0, s_0) from the domain*

$$D = \{t \in (0, t_0], \quad |x - x_0| < r, \quad s_0 \geq 0\}$$

there exists a unique $p_0 = p_0(t, x; x_0, s_0)$ such that $\|p_0\| \leq c/t$ and

$$X(t, x_0, p_0(t, x; x_0, s_0), s_0) = x. \tag{3.1}$$

The function $p_0(t, x; x_0, s_0)$ is continuously differentiable in D. Moreover,

$$\frac{\partial X}{\partial p_0}(t, x_0, p_0, s_0) = tG(x_0)(1 + O(t)). \tag{3.2}$$

In particular, if G, A, a, V are constants outside a compact set, then $p_0(t, x; x_0, s_0)$ is defined for $0 \leq t_0$ and all x, x_0, i.e. we have spatially global uniqueness. This proposition allows to define smooth functions S and P on D by the formulas

$$P(t, x; x_0, s_0) = P(t, x_0, p_0(t, x; x_0, s_0), s_0), \quad S(t, x; x_0, s_0) = S(t, x_0, p_0(t, x; x_0, s_0), s_0). \tag{3.3}$$

The function $S(t, x; x_0, s_0)$ will be called the two-point function (or the action) for the Hamiltonian (1.12).

The following statement presents the Hamiltonian formalism for the calculus of variations with non-additive functionals.

Proposition 3.2. Local properties of the action.

(i) The following boundary conditions hold

$$\frac{\partial S}{\partial x}(t,x;x_0,s_0)=P(t,x;x_0,s_0),\quad \frac{\partial S}{\partial x_0}(t,x;x_0,s_0)=-\frac{\partial S}{\partial s_0}(t,x;x_0,s_0)p_0(t,x;x_0,s_0) \qquad (3.4)$$

(ii) As a function of (t,x) the function $S(t,x;x_0,s_0)$ satisfies the first order equation

$$\frac{\partial S}{\partial t}+H(x,\frac{\partial S}{\partial x},S)=0, \qquad (3.5)$$

(iii) the curve $y(\tau)=X(\tau,x_0,p_0(t,x;x_0,s_0),s_0)$ is the unique minimising curve for problem (1.9), (1.11); in particular,

$$S(t,x;x_0,s_0)=S_L(t,x;x_0,s_0).$$

and has the asymptotic representation (2.4).

Remark. The asymptotic expansion (2.4) does not depend on s_0, because it gives exponentially small (for small t) corrections to the action.

We say that the pair (t,x) is a regular point with respect to a chosen initial values x_0,s_0, if there exists a unique p_0 such that $X(t,x_0,p_0,s_0)=x$ and the curve $y(\tau)=X(\tau,x_0,p_0,s_0)$ furnishes a minimum to problem (1.9), which is not degenerate in the sense that the matrix $\frac{\partial X}{\partial p_0}(t,x_0,p_0,s_0)$ is not degenerate. We shall denote by $Reg(x_0,s_0)$ the set of regular points with respect to x_0,s_0.

Proposition 3.3. Global resolution of the boundary value problem for (1.10). *(i) For arbitrary $t>0$ and x_0,x, there exists a p_0 such that $X(t,x_0,p_0,s_0)=x$ and the curve $y(\tau)=X(\tau,x_0,p_0,s_0)$ furnishes a minimum to problem (1.9). (ii) For arbitrary x_0,s_0, the set $Reg(x_0,s_0)$ is an open connected and everywhere dense set in $\mathbf{R}_+\times\mathbf{R}^d$.*

Proposition 3.4. Global properties of the two-point function. *(i) The function $S_L(t,x;x_0,s_0)$ is an everywhere finite and continuous function. (ii) For all $(t,x;x_0,s_0)$ there exists a p_0 such that $S_L(t,x;x_0,s_0)=S(t,x_0,p_0,s_0)$, (iii) $S_L(t,x;x_0,s_0)$ is smooth and satisfies equation (3.5) on the set $Reg(x_0,s_0)$.*

Remark . One can prove (see [Kol2]) that $S_L(t,x;x_0,s_0)$ is the unique generalised solution of the Cauchy problem for (3.5) with discontinuous initial data: $S(0,x_0;x_0,s_0)=s_0$ and $S(0,x;x_0,s_0)=+\infty$ for $x\neq x_0$ (generalised solutions being defined as idempotent semilinear distributions (see [KM])). The function S defines an idempotent measure and a maxingale (in the sense of [Pu]).

Proposition 3.5. *Let S_0 be a positive differentiable function which is bounded together with its first two derivatives. Then there exists a t_0 such that the mapping*

$$x_0\mapsto X(t,x_0,\frac{\partial S_0}{\partial x_0}(x_0),S_0(x_0)) \qquad (3.6)$$

is a diffeomrphism $\mathbf{R}^d\mapsto\mathbf{R}^d$ for all $t\in[0,t_0]$. Moreover, if $x_0(t,x)$ denotes the solution of equation (3.6), then

$$S(t,x_0,\frac{\partial S_0}{\partial x_0}(x_0),S_0(x_0))|_{x_0=x_0(t,x)}=\min_\xi\{S_L(t,x;\xi,S_0(\xi))\}$$

and this function gives a unique (classical) solution to the Cauchy problem of equation (3.3) with the initial condition $S_0(x)$.

We now give the results of the calculation of S_L in the simplest case of Hamiltonian (1.12) not depending on x, i.e. for the Hamiltonian

$$H(x,p,s) = \frac{1}{2}(p,p) - e^{-s}, \quad L(x,v,s) = \frac{1}{2}(v,v) + e^{-s} \tag{3.7}$$

First we observe the following simple general fact.

Proposition 3.6. *Suppose a Hamiltonian function $H(x,p,s)$ does not depend explicitly on the first variable x. Then the system (1.10) admits the vector-valued integral $I(p,s) = Hp/(p,p)$. i.e. the function I is constant along any solutions of (1.10). In particular, the direction of p is also an integral.*

Proof. Direct verification.

Turning to a special case given by (3.7), we note now that since the direction of p is invariant, the integration of (1.10) is effectively reduced to the solution of the system

$$\begin{cases} \dot{r} = e^{-s} r \\ \dot{s} = r^2/2 + e^{-s} \end{cases}$$

where $r = |p|$. Since this system has an integral $I = H/r = r/2 - e^{-s}/r$, it can be integrated. In particular, for the cases when $p = 0$ or $I = H = 0$, explicit formulas can be easily obtained and the following holds.

Proposition 3.7. *In case (3.7), formulas (2.5),(2.6) are valid.* The last statement concerns a generalisation of results above to the case of degenerate Hamiltonians.

Proposition 3.8. *The results of Proposition 3.2, 3.3 (and 3.1 with a different choice of domain D) also hold for degenerate quadratic Hamiltonians of the form*

$$H(x,y,p_x,p_y,s) = \frac{1}{2}(g(x)p_y,p_y) - (b(x,y),p_x) - c(x,y)p_y - V(x,y) - a(x)e^{-s}$$

depending on two group of variables x and y, which are regular in the sense of [Kol1] for all s. For example, Hamiltonian (2.14) corresponding to Ornstein-Uhlenbeck process (even curvilinear) considered in section 2 fall in this class.

4 On reaction-diffusion equations with a singularity at time zero

Here we prove a simple auxiliary result on equations of the form

$$\frac{\partial \phi}{\partial t} = \alpha(t)\mathcal{L}(t)\phi(t,x) + f(t,x,\phi(t,x)), \quad t \geq 0, \quad x \in \mathbf{R}^d, \tag{4.1}$$

where $\mathcal{L}(t)$ are second order parabolic operators of the form

$$(\mathcal{L}(t)u)(x) = \frac{1}{2}(G(t,x)\frac{\partial}{\partial x}, \frac{\partial}{\partial x})u - (A(t,x), \frac{\partial u}{\partial x})$$

with uniformly bounded $G(t,x)$, $G^{-1}(t,x)$, $A(t,x)$ and their derivatives, f is a smooth function of its arguments and $\alpha(t)$ is a positive smooth function of $t>0$ such that $\alpha(t)\to\infty$ as $t\to 0$.

Remark. Actually, we are interested in the case when the integral $\int_0^t \alpha(s)\,ds$ diverges (otherwise the standard properties of parabolic equations still hold for (4.1)).

A simplest example of (4.1) is given by the equation

$$\frac{\partial\phi}{\partial t}=t^{-\beta}\Delta\phi+f(t,x,\phi), \tag{4.2}$$

which exibits a remarkable property that the corresponding Cauchy problem may not be solvable in $C_0(\mathbf{R}^d)$. Namely, if $\beta>1$ and $f\equiv 0$, then for arbitrary $\phi_0\in C_0(\mathbf{R}^d)$ and $t_0>0$, there is no function $\phi(t,x)$ such that (i) $\phi\in C_0(\mathbf{R}^d)$ for all $t\in[0,t_0]$ and converges to ϕ_0 as $t\to 0$, and (ii) ϕ satisfies (4.2) for $t\in(0,t_0)$. In fact, if such ϕ exists, then for all $0<\tau<t<t_0$

$$\phi(t,.)=\exp\{\Delta\int_\tau^t s^{-\beta}\,ds\}\phi(\tau,.).$$

In particular,

$$\phi(t,.)=\lim_{\tau\to 0}\exp\{\frac{1}{\beta-1}(\tau^{-(\beta-1)}-t^{-(\beta-1)})\Delta\}\phi(\tau,.).$$

However, the limit on the r.h.s. of this equation vanishes (since a solution to a standard heat equation tends to zero as time goes to infinity), which contradicts the initial condition.

It turns out however that the solutions of (4.1) with constant initial conditions may well exist. To tackle this problem, let us introduce the familly of operators $U(t,\tau)$ for $0<\tau<t$ that give the (unique) solution to the Cauchy problem of equation (4.1) with vanishing f and with initial conditions at $t=\tau$. Clearly, $U(t,\tau)$ are well defined contractions for all $0<\tau<t$.

Proposition 4.1. *(i) If $f(t,x,\phi)$ is a smooth bounded (for ϕ from compact intervals and for all x) function, then a bounded solution $\phi(t,x)=\phi(t,x;\phi_0)$ to (4.1) with a constant initial condition ϕ_0 is unique, exists at least locally, i.e. for $t\le t_0$ with some $t_0>0$, and has the form $\phi_0+O(t)$ for small t. (ii) In particular, if $f(t,x,\phi)=f(t,x)$ does not depend explicitly on ϕ, then this solution is defined globally and is given by the formula*

$$\phi(t,.)=\phi_0+\int_0^t U(t,s)f(s,.)\,ds. \tag{4.3}$$

(iii) Suppose there exist two smooth function $f_1(t,\phi)$, $f_2(t,\phi)$ such that

$$f_1(t,\phi)\le f(t,x,\phi)\le f_2(t,\phi)$$

for all $t>0$, x, ϕ. Denote by $\phi_j(t;\phi_0)$, $j=1,2$, the solutions to the ordinary differential equations $\dot\phi=f_j(t,\phi)$ with the initial condition ϕ_0. Then

$$\phi_1(t;\phi_0)\le\phi(t,x;\phi_0)\le\phi_2(t;\phi_0). \tag{4.4}$$

In particular, if ϕ_j, $j=1,2$, are globally defined, then $\phi(t,x;\phi_0)$ is globally defined as well.

Proof. First, suppose that f does not depend on ϕ. If a solution ϕ exists, then $\phi-\phi_0$ is a solution to (4.1) with vanishing initial conditions (here we use the assumption that ϕ_0 is a constant), which implies that for all $0 < \tau < t$

$$\phi(t,.) = \phi_0 + U(t,\tau)(\phi(\tau,.) - \phi_0) + \int_\tau^t U(t,s) f(s,.)\, ds.$$

Taking the limit $\tau \to 0$ yields (4.3). For the general case, it follows from (4.3) that if a solution ϕ exists, it satisfies the integral equation

$$\phi(t,.) = \phi_0 + \int_0^t U(t,s) f_{\phi_0}(s,.,\phi(s,.))\, ds, \tag{4.5}$$

where $f_{\phi_0}(s,x,\phi) = f(s,x,\phi - \phi_0)$ and all required properties follow as in the case of semilinear parabolic equations without singularities (see e.g. [Sm]), i.e. the uniqueness and local existence follow from Gronwall's lemma and the contraction mapping principle respectively, and estimates (4.4) follow from the comparison principle.

5 Proof of Theorem 1

Inserting a function of the form

$$u = u(t,x,h) = \phi(t,x,h) \exp\{-S(t,x)/h\} \tag{5.1}$$

in (1.5) one concludes that (1.5) holds if S satisfies (3.5) with H given by (1.12) and the following transport equation holds for S and ϕ:

$$\begin{aligned} &\frac{\partial \phi}{\partial t} + (G(x)\frac{\partial S}{\partial x}, \frac{\partial \phi}{\partial x}) + \frac{1}{2} tr\, (G(x)\frac{\partial^2 S}{\partial x^2})\phi - (A(x), \frac{\partial \phi}{\partial x}) \\ &-\frac{h}{2}(G(x)\frac{\partial}{\partial x}, \frac{\partial}{\partial x})\phi + a(x)\frac{\phi^h - 1}{h}\phi e^{-S} = 0. \end{aligned} \tag{5.2}$$

Due to Proposition 3.2, one can find a t_0 and K such that S_L is smooth and satisfies (3.5) for $t \le t_0$ and $|x - x_0| \le K$. To simplify the situation, we observe that it is enough to prove only part (ii) of the theorem. In fact, changing smoothly the coefficients G, A and a of our equation outside a compact neighborhood K of x_0 in such a way that they become constants for large x and using the theorem for this case, we obtain a function $\tilde{u}_\delta$ that satisfies initial condition (1.13) precisely and equation (1.5) up to a term of order $O(\exp\{-M(t)/h\})$ which, moreover, vanishes inside K. Then standard arguments of the theory of semilinear equations (see [Sm]) prove that the difference between u_δ and $\tilde{u}_\delta$ is also of order $O(\exp\{-M(t)/h\})$. Thus (i) follows from (ii). Statements (iii) and (iv) follow from Propositions 3.2 and 3.7 respectively.

Thus we reduced the discussion to the case when S_L is given by (3.3) and is a smooth solution to (3.5) for all x, x_0. Therefore, looking for u_δ in the form

$$u_\delta(t,x;x_0,h,\theta) = \phi(t,x;x_0,h,\theta) \exp\{-S_L(t,x;x_0,0)/h\},$$

we conclude that it satisfies (1.5), (1.13) whenever ϕ satisfies (5.2) with $S = S_L$ and has the form

$$\phi(t,x,h) = (2\pi h t)^{-d/2} (\det G(x_0))^{-1/2} \theta (1 + o(1)), \quad t \to 0.$$

Notice that equation (5.2) for ϕ has a singular drift $\frac{\partial S_L}{\partial x}$ which is of order $(x-x_0)/t$. To get rid of it, we shall rewrite this equation in the characteristic representation. Namely, we introduce new spatial coordinates $y=y(t,x)=p_0(t,x;x_0,0)$ defined by (3.1). In these coordinates the time derivative of a function corresponds to the operator

$$\frac{\partial}{\partial t}+\frac{\partial x(t,y)}{\partial t}\frac{\partial}{\partial x}=\frac{\partial}{\partial t}+\frac{\partial H}{\partial p}(x(t,y),\frac{\partial S_L}{\partial x}(t,x(t,y)))\frac{\partial}{\partial x}$$

in old coordinates. Therefore, from (1.12) one concludes that in the new coordinate equation (5.2) takes the form

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}tr\,(G(x)\frac{\partial^2 S}{\partial x^2})\phi$$
$$-\frac{h}{2}(G(x)\frac{\partial y}{\partial x}\frac{\partial}{\partial y},\frac{\partial y}{\partial x}\frac{\partial}{\partial y})\phi+a(x)\frac{\phi^h-1}{h}\phi e^{-S}=0.$$

Here and further on the variable x means $x(t,y)=X(t,x_0,y,0)$, where we use the notation introduced at the beginning of section 3. Next, by the standard application of the Liouville theorem (widely used in the theory of WKB approximation, see e.g. [Ma1], [Kol1]) we conclude that the Jacobian

$$J(t,y;x_0)=\det(\frac{\partial X}{\partial y})(t,x_0,y)$$

of the solution $X(t,x_0,y,0)$ of the Cauchy problem for equation (1.10) satisfies the equation

$$\frac{\partial}{\partial t}J^{-1/2}+\frac{1}{2}J^{-1/2}\,tr\,\left(\frac{\partial^2 H}{\partial p\partial x}+\frac{\partial^2 H}{\partial p^2}\frac{\partial^2 S_L}{\partial x^2}\right)=0$$

(with H of form (1.12) as usual). Therefore, changing the unknown function ϕ to the function

$$\psi(t,x,h)=(2\pi h)^{d/2}\phi(t,x;x_0,h,\theta)J^{1/2}(t,y;x_0)^{-1}$$

we conclude that ψ satisfies the equation

$$\frac{\partial \psi}{\partial t}-\frac{1}{2}tr\,\frac{\partial}{\partial x}(G(x)p-A(x))\psi$$
$$-\frac{h}{2}\gamma^{-1}(G(x)\frac{\partial y}{\partial x}\frac{\partial}{\partial y},\frac{\partial y}{\partial x}\frac{\partial}{\partial y})(\gamma\psi)+a(x)\frac{(\psi\gamma(2\pi h)^{-d/2})^h-1}{h}\psi e^{-S}=0. \tag{5.5}$$

Due to (3.2), the initial condition for ϕ corresponds to the condition

$$\psi(t,x,h)=\theta+o(1),\quad t\to 0,$$

for ψ. From (3.2) we conclude that the second order part of the operator

$$\gamma^{-1}(G(x)\frac{\partial y}{\partial x}\frac{\partial}{\partial y},\frac{\partial y}{\partial x}\frac{\partial}{\partial y})\gamma \tag{5.6}$$

has the form $t^{-2}(\tilde{G}(t,y)\frac{\partial}{\partial y},\frac{\partial}{\partial y})$, where $\tilde{G}$ and its inverse $\tilde{G}^{-1}$ are uniformly bounded. By inspection of the other term of the operator (5.6), one concludes that equation (5.5) has the form

$$\frac{\partial \psi}{\partial t}=t^{-2}\mathcal{L}(t)\psi+b(t,y,h)\psi-a(x)\frac{((2\pi h)^{-d/2}J^{-1/2}\psi)^h-1}{h}e^{-S}\psi \tag{5.7}$$

with a uniformly bounded $b(t,y,h)$ and $\mathcal{L}$ of the same form as in (4.1). Applying Proposition 4.1 we get the local existence and uniqueness of the solution to this equation with the constant initial condition $\psi|_{t=0} = \theta$ for any h small enough. Moreover, since for large ψ the nonlinear term in (5.7) becomes negative, the required global upper bound for the solution follows trivially. So, we need only to prove that the solution never approaches zero and find a positive lower bound for it. Due to Proposition 4.1 and the estimate (3.2), this lower bound is given by the solution $\psi(t) = \psi(t;\theta,h)$ of the ordinary differential equation

$$\dot{\psi} = -\frac{c}{h}((t^{-d/2}\psi)^h - 1)\psi \tag{5.8}$$

with the initial condition $\psi|_{t=0} = \theta$, where c is a positive constant (we neglect the linear term in (5.7), because it clearly does not change anything). To get a lower bound for $\psi(t)$, we observe that the set of stationary points (zeros of the r.h.s. of (5.8)) is given by the curve $\psi = t^{d/2}$. Then we deduce that the solution $\psi(t)$ has always (for all positive h and θ) exactly one intersection with this curve $\psi = t^{d/2}$, and this intersection point gives a unique local and global minimum for $\psi(t)$. To estimate this intersection point, we observe that it can be estimated from below by the unique point of intersection of this curve with the solution $\tilde{\psi}(t) = \tilde{\psi}(t;\theta,h)$ of the equation $\dot{\psi} = -c(t^{-d/2}\psi)^h\psi/h$. Integrating this equation explicitly yields

$$\tilde{\psi}(t)^{-h} = \theta^{-h} + ct^{1-dh/2}/(1-dh/2),$$

and for the point of intersection y of this curve with the curve $\psi = t^{d/2}$ we get the equation

$$y^{-h} = \theta^{-h} + \frac{c}{1-dh/2}y^{-h+2/d}.$$

In terms of $v = (y/\theta)^{2/d}$ it takes the form

$$v^{hd/2} + c\theta^{2/d}v/(1-dh/2) - 1 = 0,$$

and by monotonicity one can estimate its solution from below by the unique positive solution of the equation

$$v^{hd/2} + \tilde{c}(\theta)v - 1 = 0,$$

with $\tilde{c}(\theta) = c/(1-dh_0/2)\theta^{2/d}$. From this equation one finds $h = 2\log(1-\tilde{c}(\theta)v)/(d\log v)$, which implies the estimate $v \geq \kappa\min(1, h/2\tilde{c}(\theta))$ with some constant κ. It follows that $y \geq \kappa\min(\theta, h^{d/2})$, which in its turn implies the required lower bound for $\psi(t;\theta,h)$. The proof is therefore complete.

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