Electronic Journal of Probability

Vol. 10 (2005), Paper no. 20, pages 691-717.

Journal URL
http://www.math.washington.edu/∼ejpecp/

Quadratic Variations along Irregular Subdivisions for Gaussian Processes

Arnaud Begyn

LSP, Université Paul Sabatier, 118 route de Narbonne
31062 TOULOUSE Cedex 04, FRANCE
E-mail: begyn@cict.fr, http://www.lsp.ups-tlse.fr/Fp/Begyn/

Abstract

In this paper we deal with second order quadratic variations along general subdivisions for processes with Gaussian increments. These have almost surely a deterministic limit under conditions on the mesh of the subdivisions. This limit depends on the singularity function of the process and on the structure of the subdivisions too. Then we illustrate the results with the example of the time-space deformed fractional Brownian motion and we present some simulations.

AMS MSC (2000): primary 60F15; 60G15; secondary 60G17; 60G18

Submitted to EJP on November 25, 2004. Final version accepted on February 24, 2005.

1 Introduction

In 1940 Paul Lévy (see [17]) proves that if W is the Brownian motion on $[0,1]$ then:

$$\lim_{n\to+\infty}\sum_{k=1}^{2^n}\left[W\left(\frac{k}{2^n}\right)-W\left(\frac{k-1}{2^n}\right)\right]^2=1,\qquad a.s..$$

Then in [2] and in [11] this result is extended to a large class of processes with Gaussian increments. In these cases the subdivisions are regular and the mesh is fixed (equal to $1/2^n$). More general subdivisions are used later in [10] and [15]. The subdivisions can be irregular and the optimal condition is that the mesh must be at most $o(1/\log n)$.

In the case of the fractional Brownian motion, the quadratic variation is used to construct some estimators of the Hurst index H. But these estimators are not asymptotically normal when $H > 3/4$ (see [12]). To solve this problem in [14], [3], [8] and [9] authors introduce generalized quadratic variations. The most common variation is the second order quadratic variation, which will be defined in the next section. But one more time subdivisions are regular.

In this paper we extend the theorem of [9] to a large class of subdivisions which may be irregular. We obtain that the limit of the second order quadratic variation depends on the structure of the sequence of subdivisions, and one more time that the mesh must be at most $o(1/\log n)$.

In the first section we state our notations. In the second one we prove our theorem. The third one is a discussion about the assumptions made on the structure of the subdivisions. In the fourth one we apply the results to the example of the time-space deformed fractional Brownian motion. In the last one we illustrate the examples with some simulations.

2 Notations

Let $(X_t)_{t\in[0,1]}$ be a square integrable process. We can define its mean function:

$$\forall t\in[0,1],\quad M_t=\mathbb{E}X_t,$$

and its covariance function:

$$\forall s,t\in[0,1],\quad R(s,t)=\mathbb{E}\big(\,(X_t-M_t)\,(X_s-M_s)\,\big).$$

We define the second order increments of R too:

$$\begin{aligned}\delta_1^{h_1,h_2}R(s,t)&=h_1R(s+h_2,t)+h_2R(s-h_1,t)-(h_1+h_2)R(s,t),\\ \delta_2^{h_1,h_2}R(s,t)&=h_1R(s,t+h_2)+h_2R(s,t-h_1)-(h_1+h_2)R(s,t).\end{aligned}$$

Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of the interval $[0,1]$. One denotes by N_n the number of subintervals of $[0,1]$ generated by π_n. We suppose that π_n can be written:

$$\pi_n = \left\{ t_0^{(n)} = 0 < t_1^{(n)} < \cdots < t_{N_n}^{(n)} = 1 \right\},$$

One sets the *upper mesh* of π_n:

$$m_n = \max\{t_{i+1}^{(n)} - t_i^{(n)}; 0 \le i \le N_n - 1\},$$

and the *lower mesh*:

$$p_n = \min\{t_{i+1}^{(n)} - t_i^{(n)}; 0 \le i \le N_n - 1\}.$$

Note that one has:

$$\forall n \in \mathbb{N}, \quad p_n \le \frac{1}{N_n} \le m_n.$$

Definition 1. *We say that the sequence of subdivisions $(\pi_n)_{n\in\mathbb{N}}$ is regular if we have:*

$$\forall n \in \mathbb{N}, \quad m_n = p_n = \frac{1}{N_n}.$$

Or equivalently:

$$\forall n \in \mathbb{N}, \forall k \in \{0;\ldots;N_n\}, \quad t_k^{(n)} = \frac{k}{N_n}.$$

For fractional processes the following second order quadratic variation has been used:

$$\mathcal{S}_{\pi_n}(X) = N_n^{1-\gamma} \sum_{k=1}^{N_n-1} \left[X_{\frac{k+1}{N_n}} + X_{\frac{k-1}{N_n}} - 2X_{\frac{k}{N_n}} \right]^2, \tag{1}$$

which is taken along the regular subdivision with mesh $1/N_n$. The real number γ is in $]0,2[$ and depends on the regularity of the process, as we will see later.

If $f : [0,1] \longmapsto \mathbb{R}$ is of class C^2 in a neighborhood of the point t then the Taylor formula yields:

$$\lim_{h\to 0} \frac{f(t+h) + f(t-h) - 2f(t)}{h^2} = f''(t). \tag{2}$$

This motivates the term $X_{\frac{k+1}{N_n}} + X_{\frac{k-1}{N_n}} - 2X_{\frac{k}{N_n}}$ in (1). If we want to use irregular subdivisions we must generalize (2). One more time the Taylor formula yields:

$$\lim_{\substack{h_1\to 0\\ h_2\to 0}} \frac{h_1 f(t+h_2) + h_2 f(t-h_1) - (h_1+h_2) f(t)}{h_1 h_2 (h_1+h_2)} = \frac{1}{2} f''(t).$$

So one defines the second order increments of X:

$$\Delta X_k = \Delta t_{k-1} X_{t_{k+1}} + \Delta t_k X_{t_{k-1}} - \left(\Delta t_{k-1} + \Delta t_k\right) X_{t_k}, \quad k = 1, \ldots, N_n - 1,$$

where (we drop the super-index in $t_k^{(n)}$ whenever it is possible):

$$\Delta t_k = t_{k+1} - t_k, \quad k = 0, \dots, N_n - 1.$$

We will see later that $\mathbb{E}\left[(\Delta X_k)^2\right]$ is asymptotically of the same order as:

$$(\Delta t_{k-1})^{\frac{3-\gamma}{2}} (\Delta t_k)^{\frac{3-\gamma}{2}} (\Delta t_{k-1} + \Delta t_k).$$

That is why we define the second order quadratic variation of X along a general subdivision by:

$$\mathcal{V}_{\pi_n}(X) = 2 \sum_{k=1}^{N_n-1} \frac{(\Delta t_k)(\Delta X_k)^2}{(\Delta t_{k-1})^{\frac{3-\gamma}{2}} (\Delta t_k)^{\frac{3-\gamma}{2}} (\Delta t_{k-1} + \Delta t_k)} \tag{3}$$

where $\gamma \in]0, +\infty[$.

Note that when the sequence $(\pi_n)_{n\in\mathbb{N}}$ is regular, one has:

$$\mathcal{V}_{\pi_n}(X) = \mathcal{S}_{\pi_n}(X).$$

Let us recall the Landau notations. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences of real numbers such that $\forall n, v_n \neq 0$. We will say that:

(i) $u_n \overset{n\to+\infty}{=} \mathcal{O}(v_n)$ if the sequence $(u_n/v_n)_{n\in\mathbb{N}}$ is bounded,

(ii) $u_n \overset{n\to+\infty}{=} o(v_n)$ if the sequence $(u_n/v_n)_{n\in\mathbb{N}}$ goes to zero when $n \to +\infty$.

To study the almost sure convergence of $\mathcal{V}_{\pi_n}(X)$ we will use the following assumption on the sequence $(\pi_n)_{n\in\mathbb{N}}$.

Definition 2. *Let $(l_k)_{1\leq k}$ be a sequence of reals in the interval $]0, +\infty[$. We say that $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of subdivisions with asymptotic ratios $(l_k)_{1\leq k}$ if it satisfies the following assumptions:*

1.

$$m_n \overset{n\to+\infty}{=} \mathcal{O}(p_n). \tag{4}$$

2.

$$\lim_{n\to+\infty} \sup_{1\leq k\leq N_n-1} \left| \frac{\Delta t_{k-1}^{(n)}}{\Delta t_k^{(n)}} - l_k \right| = 0. \tag{5}$$

The set $\mathcal{L} = \{l_1; l_2; \dots; l_k; \dots\}$ will be called the range of the asymptotic ratios of the sequence $(\pi_n)_{n\in\mathbb{N}}$.

It is clear that if the sequence $(\pi_n)_{n\in\mathbb{N}}$ is regular, then it is a sequence with asymptotic ratios $(l_k)_{1\leq k}$ where: $\forall k \geq 1,\ l_k = 1$.

Note that assumption (4) implies:

$$\exists K > 0, \forall n \in \mathbb{N}, \quad p_n \leq \frac{1}{N_n} \leq m_n \leq K p_n, \tag{6}$$

therefore $(l_k)_{1\leq k} \subset [1/K, K]$, and the closure of $\mathcal{L}$ in $]0, +\infty[$ is compact.

In the sequel we only consider process with Gaussian increments.

Definition 3. *A process $(X_t)_{t\in[0,1]}$ has Gaussian increments if, for any subdivision $\{t_0 = 0 < t_1 < \cdots < t_N = 1\}$ of $[0,1]$, the random vector $(X_{t_{i+1}} - X_{t_i})_{0\leq i\leq N-1}$ is Gaussian.*

In the proof of the next section we use the following notations:

$$d_{jk} = \mathbb{E}(\Delta X_j \Delta X_k), \quad j, k = 1, \ldots, N_n - 1. \tag{7}$$

And:

$$\mu_k = (\Delta t_{k-1})^{\frac{3-\gamma}{2}} (\Delta t_k)^{\frac{3-\gamma}{2}} (\Delta t_{k-1} + \Delta t_k), \quad k = 1, \ldots, N_n - 1 \tag{8}$$

We remark that:

$$\forall 1 \leq k \leq N_n - 1, \quad \frac{2p_n^4}{m_n^\gamma} \leq \mu_k \leq \frac{2m_n^4}{p_n^\gamma}. \tag{9}$$

3 The results

We prove the almost sure convergence of $\mathcal{V}_{\pi_n}(X)$ to a deterministic limit under some conditions on the covariance function of the process X and on the mesh of the subdivisions π_n.

For a function $g :]0, +\infty[\times]0, 1[\to \mathbb{R}$ we need the following assumption of continuity:

$$\forall \epsilon > 0, \exists \delta > 0;\ \forall l \in \mathcal{L}, \forall t, t^* \in]0, 1[, \quad |t - t^*| < \delta \Longrightarrow |g(l, t) - g(l, t^*)| < \epsilon. \tag{10}$$

Theorem 4. *Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions with asymptotic ratios $(l_k)_{1\leq k}$ and range of the asymptotic ratios $\mathcal{L}$. Let $(X_t)_{t\in[0,1]}$ be a square integrable process, with Gaussian increments, verifying:*

1. *$t \longmapsto M_t = \mathbb{E}X_t$ has a bounded first derivative on $[0,1]$,*
2. *the covariance function $R(s,t)$ has the following properties:*
   (a) *R is continuous on $[0,1]^2$,*

(b) *the derivative* $\frac{\partial^4 R}{\partial s^2 \partial t^2}$ *exists and is continuous on* $]0,1]^2 \backslash \{s = t\}$, *and there exists one constant* $C > 0$ *and one real* $\gamma \in]0,2[$ *such that :*

$$\forall s, t \in]0,1]^2 \backslash \{s = t\}, \left| \frac{\partial^4 R}{\partial s^2 \partial t^2}(s,t) \right| \leq \frac{C}{|s-t|^{2+\gamma}}, \tag{11}$$

(c) *We assume that there exists a function* $g :]0, +\infty[\times]0,1[\to \mathbb{R}$ *satisfying assumption (10) and such that:*

$$\forall \epsilon > 0, \exists \delta > 0, \quad \sup_{E_\delta} \left| \frac{\left(\delta_1^{h_1,h_2} \circ \delta_2^{h_1,h_2} R\right)(t,t)}{h_1^{\frac{3-\gamma}{2}} h_2^{\frac{3-\gamma}{2}} (h_1 + h_2)} - g(l,t) \right| < \epsilon, \tag{12}$$

where:

$$E_\delta = \left\{ (h_1, h_2, l, t) \in]0, +\infty[^2 \times \mathcal{L} \times]0,1[; h_1 + h_2 < \delta, h_1 \leq t \leq 1 - h_2, \left| l - \frac{h_2}{h_1} \right| < \delta \right\},$$

and such that the following limit exists and is finite:

$$\lim_{n \to +\infty} \int_0^1 g(l_n(t), t) \, dt, \tag{13}$$

where $l_n(t) = \sum_{k=1}^{N_n - 1} l_k \mathbf{1}_{]t_k^{(n)}, t_{k+1}^{(n)}[}(t)$.

3. *The lower mesh of the subdivisions* π_n *satisfy:*

$$p_n \overset{n \to +\infty}{=} o\left(\frac{1}{\log n}\right). \tag{14}$$

Then one has almost surely:

$$\lim_{n \to +\infty} \mathcal{V}_{\pi_n}(X) = 2 \lim_{n \to +\infty} \int_0^1 g(l_n(t), t) \, dt. \tag{15}$$

Remarks.

(i) *If assumptions (11), (12) is satisfied for* γ_0 *then they are satisfied for all* $\gamma > \gamma_0$ *too, but the corresponding function* g_γ *is equal to zero. When* γ_0 *is chosen as the infimum of real satisfying (11) and (12),* g_{γ_0} *can be viewed as a generalization of the singularity function of* X *(see [2]).*

(ii) *As we will see later, in most cases one is able to compute explicitly the limit* $\lim_{n \to +\infty} \int_0^1 g(l_n(t), t) \, dt$.

(iii) *We assume that the covariance function has a singularity on the set* $\{s = 0\} \cup \{t = 0\}$ *which is the case of the FBM. If the singularity is at another point than* 0, *we can use a new parameterization in order to have a the singularity at point* 0.

Proof of theorem 4. Along the proof K denotes a generic positive constant, whose value does not matter.

First we assume that the theorem is true in the case of X centered.

Because of assumption 1., one has when $n \to +\infty$:

$$\mathcal{V}_{\pi_n}(M) = 2\sum_{k=1}^{N_n-1} \frac{\Delta t_k (\Delta M_k)^2}{\mu_k} = \mathcal{O}\left(N_n \frac{m_n^\gamma}{p_n^3} m_n^4\right) = \mathcal{O}\left(m_n^\gamma\right) = o(1), \tag{16}$$

which comes from (6),(9) and assumption 1.

If X is not centered, one sets $\widetilde{X}_t = X_t - M_t$. Using Baxter's arguments (see [2]) and (16) one has:

$$\lim_{n\to+\infty} \mathcal{V}_{\pi_n}(X) = \lim_{n\to+\infty} \mathcal{V}_{\pi_n}(\widetilde{X}) \qquad a.s..$$

However the theorem is true in the case of X not centered too. So one can assume that the process X is centered without loss of generality.

First we study the asymptotic properties of $\mathbb{E}\mathcal{V}_{\pi_n}(X)$. One remarks that:

$$d_{jk} = \left(\delta_1^{\Delta t_{j-1},\Delta t_j} \circ \delta_2^{\Delta t_{k-1},\Delta t_k} R\right)(t_j, t_k)\,, \tag{17}$$

and:

$$\mathbb{E}\mathcal{V}_{\pi_n}(X) = 2\sum_{k=1}^{N_n-1} \frac{\Delta t_k d_{kk}}{\mu_k}. \tag{18}$$

Moreover assumptions (12) and (5) yield:

$$\lim_{n\to+\infty} \sup_{1\le k\le N_n-1} \left|\frac{d_{kk}}{\mu_k} - g(l_k, t_k)\right| = 0. \tag{19}$$

Therefore:

$$\begin{aligned}
&\limsup_{n\to+\infty} \left|\mathbb{E}\mathcal{V}_{\pi_n}(X) - 2\sum_{k=1}^{N_n-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} g(l_k,t)\,dt\right| \\
\le\; & 2\limsup_{n\to+\infty} \sum_{k=1}^{N_n-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \left|\frac{d_{kk}}{\mu_k} - g(l_k,t)\right|\,dt \\
\le\; & 2\limsup_{n\to+\infty}\left((N_n-1)m_n \sup_{1\le k\le N_n-1}\left|\frac{d_{kk}}{\mu_k} - g(l_k,t_k^{(n)})\right|\right) \\
& +2\limsup_{n\to+\infty} \sum_{k=1}^{N_n-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \left|g(l_k,t_k^{(n)}) - g(l_k,t)\right|\,dt.
\end{aligned}$$

Assumption (10) implies:

$$\lim_{n\to+\infty}\sum_{k=1}^{N_n-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}}\left|g(l_k,t_k^{(n)})-g(l_k,t)\right|\,dt=0.$$

This with (19) and (6) yield:

$$\limsup_{n\to+\infty}\left|\mathbb{E}\mathcal{V}_{\pi_n}(X)-2\sum_{k=1}^{N_n-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}}g(l_k,t)\,dt\right|=0. \tag{20}$$

Moreover:

$$\sum_{k=1}^{N_n-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}}g(l_k,t)\,dt=\int_0^1 g(l_n(t),t)\,dt.$$

So (13) and (20) imply:

$$\lim_{n\to+\infty}\mathbb{E}\mathcal{V}_{\pi_n}(X)=2\lim_{n\to+\infty}\int_0^1 g(l_n(t),t)\,dt. \tag{21}$$

Hence the proof is reduced to verify:

$$a.s.\quad \lim_{n\to+\infty}(\mathcal{V}_{\pi_n}(X)-\mathbb{E}\mathcal{V}_{\pi_n}(X))=0.$$

From Borel-Cantelli lemma, it is enough to find one sequence of positive real $(\epsilon_n)_{n\in\mathbb{N}}$ satisfying:

$$\lim_{n\to+\infty}\epsilon_n=0\quad\text{and}\quad\sum_{n=0}^{+\infty}\mathbb{P}(|\mathcal{V}_{\pi_n}(X)-\mathbb{E}\mathcal{V}_{\pi_n}(X)|\geq\epsilon_n)<+\infty. \tag{22}$$

For that, we will proceed like in [15] and use Hanson and Wright's bound (see [13]).

One remarks that $\mathcal{V}_{\pi_n}(X)$ is the square of the Euclidean norm of one (N_n-1)-dimensional Gaussian vector which components are:

$$\sqrt{\frac{2\Delta t_k}{\mu_k}}\Delta X_k,\quad 1\leq k\leq N_n-1.$$

So by the classical Cochran theorem, one can find a_n nonnegative real numbers $(\lambda_{1,n},\ldots,\lambda_{a_n,n})$ and one a_n-dimensional Gaussian vector Y_n, such that its components are independent Gaussian variables $\mathcal{N}(0,1)$ and:

$$\mathcal{V}_{\pi_n}(X)=\sum_{j=1}^{a_n}\lambda_{j,n}(Y_n^{(j)})^2. \tag{23}$$

Then the Hanson and Wright's inequality yields:

$$\forall\epsilon>0,\quad\mathbb{P}(|\mathcal{V}_{\pi_n}(X)-\mathbb{E}\mathcal{V}_{\pi_n}(X)|\geq\epsilon)\leq 2\exp\left[-\min\left(\frac{A_1\epsilon}{\lambda_n^*},\frac{A_2\epsilon^2}{\sum_{j=1}^{a_n}\lambda_{j,n}^2}\right)\right], \tag{24}$$

where A_1, A_2 are nonnegative constants, λ_n^* is defined by $\lambda_n^* = \sup_{1\le j\le a_n} \lambda_{j,n}$.

Since $\sum_{j=1}^{a_n} \lambda_{j,n} = \mathbb{E}\mathcal{V}_{\pi_n}(X)$, and $\mathbb{E}\mathcal{V}_{\pi_n}(X)$ is a convergent sequence, the sums $\sum_{j=1}^{a_n} \lambda_{j,n}$ are bounded.

Moreover, one has:

$$\sum_{j=1}^{a_n} \lambda_{j,n}^2 \le \lambda_n^* \sum_{j=1}^{a_n} \lambda_{j,n}.$$

Therefore, the inequality (24) becomes :

$$\forall 1 \ge \epsilon > 0, \quad \mathbb{P}(|\mathcal{V}_n - \mathbb{E}\mathcal{V}_n| \ge \epsilon) \le 2\exp\left(\frac{-K\epsilon^2}{\lambda_n^*}\right). \tag{25}$$

We use the following elementary result of linear algebra. Let $S = (s_{ij})_{1\le i,j\le n}$ be a $n\times n$ symmetric matrix. We note λ_{max} its higher eigenvalue. Then one has:

$$\lambda_{max} \le \max_{1\le j\le n} \sum_{i=1}^{n} |s_{ij}|.$$

This with inequality (9) yield:

$$\begin{aligned} \lambda_n^* &\le 2 \max_{1\le k\le N_n-1} \sum_{j=1}^{N_n-1} \sqrt{\frac{\Delta t_j \Delta t_k}{\mu_j \mu_k}} \, |\mathbb{E}(\Delta X_j \Delta X_k)| \\ &\le K\frac{m_n^\gamma}{p_n^3} \max_{1\le k\le N_n-1} \sum_{j=1}^{N_n-1} |d_{jk}|. \end{aligned} \tag{26}$$

So one must study the asymptotic properties of the d_{jk}. For that we proceed in three steps, according to the value of $k-j$.

- If $j=k$ then (19), (6) and inequality (9) yield:

$$\sup_{1\le k\le N_n-1} |d_{kk}| \overset{n\to+\infty}{=} \mathcal{O}\left(m_n \frac{m_n^3}{p_n^\gamma}\right) \overset{n\to+\infty}{=} \mathcal{O}\left(p_n^{4-\gamma}\right). \tag{27}$$

- If $1 \le k-j \le 2$, the Cauchy-Schwarz inequality implies: $|d_{jk}| \le \sqrt{d_{jj} d_{kk}}$. Thus:

$$\sup_{2\le k\le N_n-1} \sup_{1\le k-j\le 2} |d_{jk}| \overset{n\to+\infty}{=} \mathcal{O}\left(p_n^{4-\gamma}\right). \tag{28}$$

- If $|j-k| \ge 3$ one uses assumption (11). By symmetry of the d_{jk} one can take $j-k\ge 3$.

One has (17):

$$d_{jk} = \left(\delta_1^{\Delta t_{j-1}, \Delta t_j} \circ \delta_2^{\Delta t_{k-1}, \Delta t_k}\right) R\left(t_j, t_k\right).$$

If $f\ :]0,1]\longmapsto\mathbb{R}$ is a two times differentiable function one has for $0<h_1<t$ and $0<h_2\leq 1-t$:

$$h_1 f(t+h_2)+h_2 f(t-h_1)-(h_1+h_2)f(t)=\int_t^{t+h_2}dx\int_{t-h_1}^{t}dy\int_y^x f''(z)\,dz.$$

Hence if $j\neq 1$ and $k\neq 1$:

$$d_{jk}=\int_{t_j}^{t_{j+1}}du\int_{t_{j-1}}^{t_j}dv\int_v^u dw\int_{t_k}^{t_{k+1}}dx\int_{t_{k-1}}^{t_k}dy\int_y^x\frac{\partial^4 R}{\partial s^2\partial t^2}(w,z)\,dz.$$

Here one uses assumption (11) which yields:

$$\left|\frac{\partial^4 R}{\partial s^2\partial t^2}(w,z)\right|\leq\frac{C}{|w-z|^{2+\gamma}}.$$

And on the integration set one has:

$$|w-z|\geq t_{j-1}-t_{k+1}=\sum_{l=k+1}^{j-2}\Delta t_l\geq(j-k-2)p_n.$$

Hence one gets:

$$|d_{jk}|\leq\frac{Cm_n^4(2m_n)^2}{(j-k-2)^{\gamma+2}p_n^{\gamma+2}}=\frac{4Cm_n^6}{(j-k-2)^{\gamma+2}p_n^{\gamma+2}}.\tag{29}$$

If $j=1$ or $k=1$ one has by continuity of R:

$$d_{jk}=\lim_{\epsilon\to 0^+}d_{jk}^{(\epsilon)},$$

where for $0<\epsilon<1$:

$$d_{jk}^{(\epsilon)}=\int_{t_j}^{t_{j+1}^\epsilon}du\int_{t_{j-1}^\epsilon}^{t_j}dv\int_v^u dw\int_{t_k}^{t_{k+1}^\epsilon}dx\int_{t_{k-1}^\epsilon}^{t_k}dy\int_y^x\frac{\partial^4 R}{\partial s^2\partial t^2}(w,z)\,dz,$$

and:

$$\begin{aligned}t_{j+1}^\epsilon &= t_j+(1-\epsilon)\Delta t_j,\\ t_{j-1}^\epsilon &= t_j-(1-\epsilon)\Delta t_{j-1},\\ t_{k+1}^\epsilon &= t_k+(1-\epsilon)\Delta t_k,\\ t_{k-1}^\epsilon &= t_k-(1-\epsilon)\Delta t_{k-1}.\end{aligned}$$

Using the same techniques as above one gets for ϵ near 0^+:

$$|d_{jk}^{(\epsilon)}|\leq\frac{4Cm_n^6(1-\epsilon)^2}{((j-k-2)p_n-2\epsilon m_n))^{\gamma+2}}.$$

One makes ϵ tends to 0^+ in this inequality. It yields that (29) is still true when $j = 1$ or $k = 1$.

Therefore one has:

$$\begin{aligned}\max_{1\le k\le N_n-1} \sum_{j-k\ge 3} d_{jk} &\le K\frac{m_n^6}{p_n^{2+\gamma}} \max_{1\le k\le N_n-1} \sum_{j-k\ge 3} \frac{1}{(j-k-2)^{\gamma+2}} \\ &\le K\frac{m_n^6}{p_n^{2+\gamma}} \sum_{t=1}^{+\infty} \frac{1}{t^{\gamma+2}} \le K\frac{m_n^6}{p_n^{2+\gamma}}.\end{aligned}$$

Thanks to (6) one gets the following estimate:

$$\max_{1\le k\le N_n-1} \sum_{j-k\ge 3} d_{jk} \overset{n\to+\infty}{=} \mathcal{O}\left(\frac{m_n^6}{p_n^{\gamma+2}}\right) \overset{n\to+\infty}{=} \mathcal{O}\left(p_n^{-\gamma+4}\right).$$

So the preceding three steps and (26) yield:

$$\lambda_n^* \overset{n\to+\infty}{=} \mathcal{O}\left(p_n\right). \tag{30}$$

Hence (25) and the preceding estimates yield for $0 < \epsilon \le 1$:

$$\mathbb{P}(|\mathcal{V}_n - \mathbb{E}\mathcal{V}_n| \ge \epsilon) \le 2\exp\left(-\frac{K\epsilon^2}{p_n}\right). \tag{31}$$

Now one sets:

$$\epsilon_n^2 = \frac{2}{K} p_n \log n.$$

Then (31) becomes:

$$\mathbb{P}(|\mathcal{V}_n - \mathbb{E}\mathcal{V}_n| \ge \epsilon_n) \le 2\exp\left(-2\log n\right) = \frac{1}{n^2}.$$

So conditions (22) are satisfied.

□

Now we give briefly two cases where the limit (13) exists and can be easily computed.

(C1) If g is invariant on $\mathcal{L}$, i.e. $\forall t \in [0,1], \forall l, l^* \in \mathcal{L}, g(l,t) = g(l^*,t)$. Then it is clear that (13) is satisfied. Indeed:

$$\sum_{k=1}^{N_n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} g(l_k, t)\, dt = \int_0^1 g(l,t)\, dt,$$

for all $l \in \mathcal{L}$.

(C2) If the sequence of functions $(l_n(t))_{n\in\mathbb{N}}$ converges uniformly to $l(t)$ on the interval $[0,1]$, then:

$$\lim_{n\to+\infty} \int_0^1 g(l_n(t),t)\, dt = \int_0^1 g(l(t),t)\, dt,$$

thanks to the Riemann theorem. So assumption (13) is fulfilled.

In the next section we give a method to construct irregular subdivisions of which range of asymptotic ratios can be chosen infinite.

4 Construction of irregular sequences of subdivisions

We give a necessary and sufficient condition to find a sequence of subdivisions $(\pi_n)_{n\in\mathbb{N}}$ which has a given sequence of asymptotic ratios $(l_k)_{k\geq 1} \subset]0, +\infty[$.

Proposition 5. *Let $(l_k)_{k\geq 1}$ be a sequence of real numbers in $]0, +\infty[$ and $(N_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integer numbers such that:*

$$\exists D > 1, \forall L \in \mathbb{N}^*, \frac{1}{D} < \prod_{j=1}^{L} l_j < D, \tag{32}$$

$$\lim_{n\to+\infty} \sum_{j=1}^{N_n-2} \prod_{i=j+1}^{N_n-1} l_i = +\infty. \tag{33}$$

Then $(l_k)_{k\geq 1}$ is the sequence of asymptotic ratios of a sequence of subdivisions $\left(\pi_n = \left\{t_0^{(n)} = 0 < t_1^{(n)} < \cdots < t_{N_n}^{(n)} = 1\right\}\right)_{n\in\mathbb{N}}$. This sequence is unique under the condition:

$$\forall 1 \leq k \leq N_n - 1, l_k = \frac{\Delta t_{k-1}^{(n)}}{\Delta t_k^{(n)}}. \tag{34}$$

The converse is true: if $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of subdivisions, $(N_n)_{n\in\mathbb{N}}$ is the associated sequence of number of subintervals and $(l_k)_{k\geq 1}$ is the sequence of the asymptotic ratios of $(\pi_n)_{n\in\mathbb{N}}$, then the sequences $(l_k)_{k\geq 1}$ and $(N_n)_{n\in\mathbb{N}}$ satisfy the conditions (32) and (33).

Proof of proposition 5. We begin with the proof of the converse. We take $(\pi_n)_{n\in\mathbb{N}}$, $(l_k)_{k\geq 1}$ and $(N_n)_{n\in\mathbb{N}}$ as in the second part of the proposition.

One has: $m_n \overset{n\to+\infty}{=} \mathcal{O}(p_n)$. Therefore there exists a constant $K > 0$ such that:

$$\frac{1}{K} \leq \frac{p_n}{m_n} \leq \frac{m_n}{p_n} \leq K.$$

Moreover, for all $L \in \mathbb{N}^*$:

$$\prod_{j=1}^{L} l_j = \lim_{p\to+\infty} \prod_{j=1}^{L} \frac{\Delta t_{j-1}^{(p)}}{\Delta t_j^{(p)}} = \lim_{p\to+\infty} \frac{\Delta t_0^{(p)}}{\Delta t_L^{(p)}}.$$

And:

$$\frac{p_n}{m_n} \leq \lim_{p\to+\infty} \frac{\Delta t_0^{(p)}}{\Delta t_L^{(p)}} \leq \frac{m_n}{p_n},$$

which shows that condition (32) is satisfied with $D = K$.

Likewise one has:

$$\prod_{i=j+1}^{N_n-1} l_i = \lim_{p\to+\infty} \frac{\Delta t_j^{(p)}}{\Delta t_{N_n-1}^{(p)}} \geq \frac{p_n}{m_n} \geq \frac{1}{K}.$$

Therefore:

$$\sum_{j=1}^{N_n-2}\prod_{i=j+1}^{N_n-1} l_i \geq \sum_{j=1}^{N_n-2}\frac{1}{K}=\frac{N_n-2}{K},$$

which implies (33).

Now we prove the first part of the proposition. We take two sequences $(l_k)_{k\geq 1}$ and $(N_n)_{n\in\mathbb{N}}$ satisfying conditions (32) and (33).

For $n\in\mathbb{N}$, we define the subdivision $\pi_n=\left\{t_0^{(n)}=0<t_1^{(n)}<\cdots<t_{N_n}^{(n)}=1\right\}$ in a recursive way:

$$\begin{cases} t_{N_n}^{(n)}=1, \\ t_{N_n-1}^{(n)}=\frac{\sum_{j=1}^{N_n-2}\prod_{i=j+1}^{N_n-1} l_i}{1+\sum_{j=1}^{N_n-2}\prod_{i=j+1}^{N_n-1} l_i} \\ t_k^{(n)}=t_{k+1}^{(n)}-\frac{\prod_{i=k+1}^{N_n-1} l_i}{1+\sum_{j=1}^{N_n-2}\prod_{i=j+1}^{N_n-1} l_i}, \forall 1\leq k\leq N_n-2, \\ t_0^{(n)}=0. \end{cases}$$

One gets a sequence of subdivisions with asymptotic ratios $(l_k)_{k\geq 1}$. Indeed:

$$\forall 1\leq k\leq N_n-1, \frac{\Delta t_{k-1}^{(n)}}{\Delta t_k^{(n)}}=l_k,$$

so the sequence satisfies (5).

We note k_n and j_n the integer such that:

$$m_n=\Delta t_{k_n}^{(n)} \qquad \text{and} \qquad p_n=\Delta t_{j_n}^{(n)}.$$

We envisage three cases. First case: $k_n\geq j_n$. Then:

$$\frac{m_n}{p_n}=\frac{\Delta t_{k_n}^{(n)}}{\Delta t_{j_n}^{(n)}}=\frac{\Delta t_{j_n+1}^{(n)}}{\Delta t_{j_n}^{(n)}}\frac{\Delta t_{j_n+2}^{(n)}}{\Delta t_{j_n+1}^{(n)}}\cdots\frac{\Delta t_{k_n}^{(n)}}{\Delta t_{k_n-1}^{(n)}}=\frac{1}{\prod_{i=j_n+1}^{k_n} l_i}.$$

One sets $\forall L\in\mathbb{N}^*, P_L=\prod_{i=1}^{L} l_i$ and $P_0=1$. One has:

$$\frac{m_n}{p_n}=\frac{P_{j_n}}{P_{k_n}},$$

and because of (32):

$$\forall L\in\mathbb{N}^*, \frac{1}{D}\leq P_L\leq D.$$

Therefore:

$$\frac{m_n}{p_n}\leq D^2.$$

Second case: $k_n < j_n$. We use the same arguments. One has:

$$\frac{m_n}{p_n} = \prod_{i=k_n}^{j_n-1} l_i = \frac{P_{j_n-1}}{P_{k_n-1}} \leq D^2.$$

Third case: $k_n = j_n$. Then:

$$\frac{m_n}{p_n} = 1 \leq D^2.$$

Hence in all cases:

$$\frac{m_n}{p_n} \leq D^2,$$

so the sequence $(\pi_n)_{n\in\mathbb{N}}$ satisfies (4) too.

Now we must show the unicity of $(\pi_n)_{n\in\mathbb{N}}$ under the assumptions of the proposition. Necessarily one has $\sum_{k=0}^{N_n-1} \Delta t_k^{(n)} = 1$ which yields:

$$\left(1 + \sum_{j=1}^{N_n-2} \prod_{i=j+1}^{N_n-1} \frac{\Delta t_{i-1}^{(n)}}{\Delta t_i^{(n)}}\right) \Delta t_{N_n-1}^{(n)} = 1.$$

With (34) one gets:

$$\left(1 + \sum_{j=1}^{N_n-2} \prod_{i=j+1}^{N_n-1} l_i\right) \Delta t_{N_n-1}^{(n)} = 1. \tag{35}$$

And with (35) and (34), one gets the recursive procedure used to define $(\pi_n)_{n\in\mathbb{N}}$. Since this procedure defines a unique sequence, $(\pi_n)_{n\in\mathbb{N}}$ is the unique solution of our problem.

□

Example. One sets $N_n = n$ and $l_k = 1 + \frac{1}{k^2}$. Note that the sequence (P_L) is increasing and lower bounded by 1. So condition (32) is satisfied with $D = \prod_{j=1}^{+\infty} \left(1 + \frac{1}{j^2}\right)$.
Moreover:

$$\prod_{i=j}^{N_n-1} l_i \geq l_j = 1 + \frac{1}{j^2} \geq 1,$$

so condition (33) is satisfied too.

In next section we illustrate the conclusion of theorem 4 with one example: the *time-space deformed fractional Brownian motion.*

5 Example: time-space deformed fractional Brownian motion

5.1 The fractional Brownian motion

The fractional Brownian motion (FBM) with Hurst's index $H \in]0,1[$ is the centered Gaussian process $\left(B_t^H\right)_{t\in\mathbb{R}}$, vanishing at the origin, with covariance function given by:

$$\forall s,t \in \mathbb{R}, \quad R(s,t) = \mathrm{Cov}\left(B_s^H, B_t^H\right) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right).$$

For $H = 1/2$, $\left(B_t^H\right)_{t\in\mathbb{R}}$ is the Brownian motion.

The process B^H is *self-similar with index H*:

$$\forall \epsilon > 0, \left\{B^H(\epsilon t); t \in \mathbb{R}\right\} \stackrel{(\mathcal{L})}{=} \left\{\epsilon^H B^H(t); t \in \mathbb{R}\right\},$$

and its increments are stationary.

Moreover the Hölder critical exponent of its sample paths is equal to H (see [1] th.8.3.2 and th.2.2.2), in the following sense:

Definition 6. *Let $\beta \in]0,1[$. A process $(X_t)_{t\in\mathbb{R}}$ is said to have Hölder critical exponent β whenever it satisfies the two following properties:*

- *for any $\beta^* \in]0,\beta[$, the sample paths of X satisfy a.s. a uniform Hölder condition of order β^* on any compact set, i.e; for any compact set K of $\mathbb{R}$, there exists a positive r.v. A such that a.s.:*
$$\forall s,t \in K, |X_s - X_t| \leq A|s-t|^{\beta^*};$$
- *for any $\beta^* \in]\beta,1[$, a.s. the sample paths of X fails to satisfy any uniform Hölder condition of order β^*.*

In [9] authors proved that almost surely:

$$\lim_{n\to+\infty} n^{2H-1} \sum_{k=1}^{n-1} \left[B^H_{\frac{k+1}{n}} + B^H_{\frac{k-1}{n}} - 2B^H_{\frac{k}{n}}\right]^2 = 4 - 2^{2H}. \tag{36}$$

We are able now to sharpen this result. Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions with asymptotic ratios $(l_k)_{k\geq 1}$.

On the one hand we compute the derivative $\frac{\partial^4 R}{\partial s^2 \partial t^2}$. One has:

$$\forall s,t \in]0,1]^2 \backslash \{s=t\}, \frac{\partial^4 R}{\partial s^2 \partial t^2}(s,t) = \frac{2H(2H-1)(2H-2)(2H-3)}{|s-t|^{2+\gamma}},$$

where $\gamma = 2 - 2H$.

So (11) is satisfied with $C = 2H(2H-1)(2H-2)(2H-3)$.

On the other hand we must compute the singularity function of B^H. We use the notations of the theorem 4. One has:

$$\frac{\left(\delta_1^{h_1,h_2} \circ \delta_2^{h_1,h_2} R\right)(t,t)}{h_1^{\frac{3-\gamma}{2}} h_2^{\frac{3-\gamma}{2}} (h_1+h_2)} = \frac{1+\lambda^{2H-1}-(1+\lambda)^{2H-1}}{\lambda^{H-\frac{1}{2}}}, \tag{37}$$

where $\lambda = h_2/h_1$.

To check (12) one sets $g(l,t) = \frac{1+l^{2H-1}-(1+l)^{2H-1}}{l^{H-\frac{1}{2}}}$ and one remarks:

$$\sup_{E_\delta} \left| \frac{\left(\delta_1^{h_1,h_2} \circ \delta_2^{h_1,h_2} R\right)(t,t)}{h_1^{\frac{3-\gamma}{2}} h_2^{\frac{3-\gamma}{2}} (h_1+h_2)} - g(l,t) \right| = \sup_{|\lambda - l|<\delta} |\phi(\lambda) - \phi(l)|,$$

where $\phi(\lambda) = \frac{1+\lambda^{2H-1}-(1+\lambda)^{2H-1}}{\lambda^{H-\frac{1}{2}}}$.

The function ϕ is continuous on $]0,+\infty[$, so it is uniformly continuous on compact subsets of $]0,+\infty[$. This property proves the limit (12). Moreover it is obvious that g satisfies assumption (10). Therefore theorem 4 can be applied to B^H.

By example, for a regular sequence of subdivisions one has a.s.:

$$\lim_{n\to+\infty} \mathcal{V}_{\pi_n}(B^H) = \lim_{n\to+\infty} N_n^{2H-1} \sum_{k=1}^{N_n-1} \left[B^H_{\frac{k+1}{N_n}} + B^H_{\frac{k-1}{N_n}} - 2B^H_{\frac{k}{N_n}} \right]^2 = 4 - 2^{2H}, \tag{38}$$

with N_n such that $\frac{1}{N_n} = o\left(\frac{1}{\log n}\right)$.

We give as well an example of non regular subdivision. We set:

$$N_n = 2n,$$

and:

$$\begin{aligned} \forall 0 \le k \le n, t_{2k}^{(n)} &= \frac{3k}{3n}, \\ \forall 0 \le k \le n-1, t_{2k+1}^{(n)} &= \frac{3k+1}{3n}. \end{aligned}$$

This is a sequence of subdivisions with asymptotic ratios $(l_k)_{1\le k}$ where:

$$\begin{aligned} \forall k \ge 1, l_{2k} &= 2, \\ \forall k \ge 0, l_{2k+1} &= \frac{1}{2}. \end{aligned}$$

Its range of asymptotic ratios is $\mathcal{L} = \{1/2; 2\}$ and $g(l, t)$ is invariant in l on this set (which is condition (C1)), so one is able to compute the limit (13). Therefore theorem 4 yields:

$$\lim_{n \to +\infty} \mathcal{V}_{\pi_n}(B^H) = \frac{1 + 2^{2H-1} - 3^{2H-1}}{2^{H-3/2}}. \tag{39}$$

Note that when $H \neq 1/2$, the value of this limit is different from (38). Indeed its depends not only on the singularity function of the process but on the asymptotic ratios of the subdivisions too.

For $H = 1/2$ the singularity function g does not more depend on l. Indeed:

$$\forall t \in]0, 1[, \forall l \in]0, +\infty[, g(l, t) = 2.$$

It is classical that the standard quadratic variation of the Brownian motion does not depend on the sequence of subdivisions; we obtain the same result for the second order quadratic variation.

5.2 Time-space deformed fractional Brownian motion

Let σ and ω be two functions from $\mathbb{R}$ to $\mathbb{R}$. We define the (σ, ω)-time-space deformed fractional Brownian motion $(Z_t^H)_{t \in \mathbb{R}}$ by the formula:

$$\forall t \in \mathbb{R}, Z_t^H = \sigma(t) B^H(\omega(t)). \tag{40}$$

This is a centered Gaussian process with covariance function:

$$\forall s, t \in \mathbb{R}, \mathbb{E}(Z_s^H Z_t^H) = \frac{\sigma(s)\sigma(t)}{2} \left(|\omega(s)|^{2H} + |\omega(t)|^{2H} - |\omega(s) - \omega(t)|^{2H} \right).$$

We want to study this process in the following way: does it have the same properties as the FBM? It is clear that it has no more stationary increments. In the sequel, we study its properties of self-similarity and Hölder regularity, and we apply theorem 4 to this process.

The following lemma will be useful:

Lemma 7. *Let I, J, K be three subintervals of $\mathbb{R}$ and α, β two real numbers in [0,1]. We will say that a function $f : I \longrightarrow J$ is α-Hölderian on I if:*

$$\exists K > 0, \forall x, y \in I, |f(x) - f(y)| \leq K|x - y|^\alpha.$$

1. *Let $f : I \longrightarrow J$ and $g : I \longrightarrow K$ be two functions. Assume that f is α-Hölderian and bounded on I, and g is β-Hölderian and bounded on I. Then the function $fg : I \longrightarrow \mathbb{R}$ is $\min(\alpha, \beta)$-Hölderian.*

2. *Let $f : I \longrightarrow J$ and $g : J \longrightarrow K$ be two functions. Assume that f is α-Hölderian an g is β-Hölderian. Then the function $f \circ g : I \longrightarrow K$ is $\alpha\beta$-Hölderian.*

¿From this lemma we can deduce immediately the regularity of the sample paths of Z^H.

Proposition 8. *We assume that σ is Σ-Hölderian and ω is Ω-Hölderian on a compact interval I of $\mathbb{R}$ with $\Sigma, \Omega \in [0,1]$. Then for all $H' < H$ the sample paths of Z^H are a.s. $\min(\Sigma, \Omega H')$-Hölderian on I.*

Proof of proposition 8. It is a straightforward consequence of lemma 7.

$\square$

Now we show a property of self-similarity of the process Z^H. We begin with the definition of this property (see [4]) in the case of a Gaussian field.

Definition 9. *Let $d \in \mathbb{N}^*$ and $\beta > 0$. A process $(X(t))_{t\in\mathbb{R}^d}$ is locally asymptotically self-similar (l.a.s.s.) of order β at point $t_0 \in \mathbb{R}^d$ if the finite dimensional distributions of the field*

$$\left\{ \frac{X(t_0+\epsilon t) - X(t_0)}{\epsilon^\beta} ; t \in \mathbb{R}^d \right\}$$

converge to the finite dimensional distributions of a non trivial Gaussian field when $\epsilon \to 0^+$. The limit field is called the tangent field at point t_0.

We give assumptions on the functions σ, ω so that the process Z^H is l.a.s.s..

Proposition 10. *Let $t_0 \in \mathbb{R}$ and $\Sigma \in]H, 1]$. Assume that:*

1. *ω is differentiable at point t_0 and $\omega'(t_0) \neq 0$,*
2. *σ is Σ-Hölderian on a neighborhood of t_0 and $\sigma(t_0) \neq 0$.*

Then the process Z^H is l.a.s.s. at point t_0 of order H and the tangent process is:

$$\left(T_t^{(t_0)}\right)_{t\in\mathbb{R}} = \sigma(t_0)|\omega'(t_0)|^H \left(B_t^H\right)_{t\in\mathbb{R}}.$$

Remark.
Under these assumptions proposition 8 yields that for all $H' < H$ the sample paths of Z^H are a.s. H'-Hölderian. Proposition 10 yields that they are not H'-Hölderian in the neighborhood of t_0 for all $H' > H$. Therefore the Hölder critical exponent of Z^H is equal to H.

Proof of proposition 10. Along the proof K will denote a generic positive constant, whose value does not matter.

Let $t_0 \in \mathbb{R}$. We denote by R (resp. ρ) the covariance function of the process Z^H (resp. B^H). Since we work with centered Gaussian process it is enough to show that:

$$\forall s,t \in \mathbb{R}, \lim_{\epsilon\to 0^+} \frac{1}{\epsilon^{2H}} Cov\left(Z^H(t_0+\epsilon s)) - Z^H(t_0), Z^H(t_0+\epsilon t) - Z^H(t_0)\right) \\ = \sigma(t_0)^2 |\omega'(t_0)|^{2H} \rho(s,t). \quad (41)$$

One sets for $h_1, h_2 > 0$:

$$\Lambda^{h_1,h_2} R(t_0) = R(t_0 + h_1, t_0 + h_2) - R(t_0 + h_1, t_0) - R(t_0, t_0 + h_2) + R(t_0, t_0).$$

Equality (41) is equivalent to:

$$\forall s, t \in \mathbb{R}, \lim_{\epsilon \to 0^+} \frac{\Lambda^{\epsilon s, \epsilon t} R(t_0)}{\epsilon^{2H}} = \sigma(t_0)^2 |\omega'(t_0)|^{2H} \rho(s,t). \tag{42}$$

One takes $s, t \in \mathbb{R}$. One has:

$$R(s,t) = \frac{\sigma(s)\sigma(t)}{2} \left(|\omega(s)|^{2H} + |\omega(t)|^{2H} - |\omega(s) - \omega(t)|^{2H} \right).$$

Therefore:

$$\begin{aligned}
\Lambda^{\epsilon s, \epsilon t} R(t_0) = {} & \frac{\sigma(t_0 + \epsilon s)\sigma(t_0 + \epsilon t)}{2} \Big(|\omega(t_0 + \epsilon s)|^{2H} + |\omega(t_0 + \epsilon t)|^{2H} - |\omega(t_0 + \epsilon s) \\
& \qquad - \omega(t_0 + \epsilon t)|^{2H} \Big) \\
& - \frac{\sigma(t_0 + \epsilon s)\sigma(t_0)}{2} \left(|\omega(t_0 + \epsilon s)|^{2H} + |\omega(t_0)|^{2H} - |\omega(t_0 + \epsilon s) - \omega(t_0)|^{2H} \right) \\
& - \frac{\sigma(t_0 + \epsilon t)\sigma(t_0)}{2} \left(|\omega(t_0 + \epsilon t)|^{2H} + |\omega(t_0)|^{2H} - |\omega(t_0 + \epsilon t) - \omega(t_0)|^{2H} \right) \\
& + \sigma(t_0)^2 |\omega(t_0)|^{2H}.
\end{aligned}$$

One sets:

$$\Lambda^{\epsilon s, \epsilon t} R(t_0) = \Phi_1(\epsilon) + \Phi_2(\epsilon),$$

where:

$$\begin{aligned}
\Phi_1(\epsilon) = {} & \frac{\sigma(t_0 + \epsilon t) - \sigma(t_0)}{2} \left(|\omega(t_0 + \epsilon s)|^{2H} \sigma(t_0 + \epsilon s) - |\omega(t_0)|^{2H} \sigma(t_0) \right) \\
& + \frac{\sigma(t_0 + \epsilon s) - \sigma(t_0)}{2} \left(|\omega(t_0 + \epsilon t)|^{2H} \sigma(t_0 + \epsilon t) - |\omega(t_0)|^{2H} \sigma(t_0) \right),
\end{aligned}$$

and:

$$\begin{aligned}
\Phi_2(\epsilon) = {} & \frac{\sigma(t_0 + \epsilon s)\sigma(t_0)}{2} |\omega(t_0 + \epsilon s) - \omega(t_0)|^{2H} \\
& + \frac{\sigma(t_0 + \epsilon t)\sigma(t_0)}{2} |\omega(t_0 + \epsilon t) - \omega(t_0)|^{2H} \\
& - \frac{\sigma(t_0 + \epsilon t)\sigma(t_0 + \epsilon s)}{2} |\omega(t_0 + \epsilon t) - \omega(t_0 + \epsilon s)|^{2H}.
\end{aligned}$$

First we show that:

$$\lim_{\epsilon \to 0^+} \frac{\Phi_1(\epsilon)}{\epsilon^{2H}} = 0. \tag{43}$$

By assumption σ is Σ-Hölderian on a neighborhood of t_0 and ω is differentiable at t_0, so it is clear that ω is 1-Hölderian on a neighborhood of t_0. Moreover the function $x \mapsto |x|^{2H}$

is $\min(1, 2H)$-Hölderian on a neighborhood of t_0. Therefore lemma 7 allows us to conclude that the function $x \mapsto |\omega(x)|^{2H}\sigma(x)$ is $\min(\Sigma, 2H)$-Hölderian on a neighborhood of t_0.

Hence there exists $K > 0$ such that for ϵ enough small:

$$|\Phi_1(\epsilon)| \le K\epsilon^{\Sigma+\min(\Sigma,2H)}.$$

Moreover one has assumed that $\Sigma > H$, so $\Sigma + \min(\Sigma, 2H) > 2H$. This shows (43).

Now we show that:

$$\lim_{\epsilon\to 0^+} \frac{\Phi_2(\epsilon)}{\epsilon^{2H}} = \sigma(t_0)^2|\omega'(t_0)|^{2H}\rho(s,t). \tag{44}$$

If $s = 0$ or $t = 0$ equality (44) is obvious. So one can assume that $s \neq 0$ and $t \neq 0$. Since ω is differentiable at point t_0 one has:

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \frac{|\omega(t_0+\epsilon s) - \omega(t_0)|^{2H}}{\epsilon^{2H}} &= s^{2H}|\omega'(t_0)|^{2H}, \\
\lim_{\epsilon\to 0^+} \frac{|\omega(t_0+\epsilon t) - \omega(t_0)|^{2H}}{\epsilon^{2H}} &= t^{2H}|\omega'(t_0)|^{2H}, \\
\lim_{\epsilon\to 0^+} \frac{|\omega(t_0+\epsilon s) - \omega(t_0+\epsilon t)|^{2H}}{\epsilon^{2H}} &= |s-t|^{2H}|\omega'(t_0)|^{2H}.
\end{aligned}$$

This shows equality (44). Since (43) and (44) are sufficient conditions for (42) the proposition is proved.

$\square$

Now we apply theorem 4 in order to generalize previous results about the FBM. We make additional assumptions on σ and ω.

Proposition 11. *Assume that:*

1. *σ is of class C^3 and ω of class C^2 on $[0,1]$,*
2. *$\omega(0) \ge 0$,*
3. *$\inf_{t\in[0,1]} \omega'(t) = \beta > 0$.*

Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions with asymptotic ratios $(l_k)_{k\ge 1}$. Assume too that:

$$p_n \overset{n\to+\infty}{=} o\left(\frac{1}{\log n}\right).$$

Then theorem 4 can be applied to Z^H with $\gamma = 2 - 2H$ and:

$$g(l,t) = \sigma(t)^2|\omega'(t)|^{2H}\frac{1 + l^{2H-1} - (1+l)^{2H-1}}{l^{H-1/2}}. \tag{45}$$

Proof of proposition 11. Along the proof K denotes a generic positive constant, whose value does not matter.

We denote by R the covariance function of Z^H. One has:

$$R(s,t)=\frac{\sigma(s)\sigma(t)}{2}\left(|\omega(s)|^{2H}+|\omega(t)|^{2H}-|\omega(s)-\omega(t)|^{2H}\right).$$

We must show that the assumptions of theorem 4 are satisfied. For 1 and 2.(a) it is obvious.

For assumption 2.(b): since $\forall t>0,\omega(t)>\omega(0)=0$ and $\forall s\neq t,\omega(s)\neq\omega(t)$ it is clear that $\frac{\partial^4 R}{\partial s^2\partial t^2}$ exists and is continuous on $]0,1]^2\backslash\{s=t\}$.

Moreover the functions σ,ω and their first and second derivatives are bounded on $[0,1]$. The computation of $\frac{\partial^4 R}{\partial s^2\partial t^2}$ shows that there exists $K>0$ such that:

$$\forall s,t\in]0,1]^2\backslash\{s=t\},\left|\frac{\partial^4 R}{\partial s^2\partial t^2}(s,t)\right|\leq K\sum_{\mathbf{i}\in\mathcal{I}}\left(|\omega(s)|^{2H-i_1}+|\omega(t)|^{2H-i_2}-|\omega(s)-\omega(t)|^{2H-i_3}\right),$$

where $\mathbf{i}=(i_1,i_2,i_3)$ and $\mathcal{I}$ is a subset of $\{0;1;2;3;4\}^3$.

The functions $(s,t)\mapsto|s-t|^{2+\gamma}|\omega(s)|^{2H-i_1}$ and $(s,t)\mapsto|s-t|^{2+\gamma}|\omega(t)|^{2H-i_2}$ are bounded on $]0,1]^2\backslash\{s=t\}$ since they are continuous on $[0,1]^2$.

Likewise $(s,t)\mapsto|s-t|^{2+\gamma}|\omega(s)-\omega(t)|^{2H-i_3}$ is bounded on $]0,1]^2\backslash\{s=t\}$ when $2H-i_3\geq 0$.

If $2H-i_3<0$ one uses the assumption on ω'. One gets:

$$\forall s,t\in[0,1],|\omega(s)-\omega(t)|\geq\beta|s-t|,$$

and consequently:

$$\forall s,t\in]0,1]^2\backslash\{s=t\},|\omega(s)-\omega(t)|^{2H-k}\leq\beta^{2H-k}|s-t|^{2H-k}.$$

Moreover there exists $K>0$ such that:

$$\forall s,t\in]0,1]^2\backslash\{s=t\},|s-t|^{2H-k}\leq\frac{K}{|s-t|^{2+\gamma}}.$$

Therefore assumption 2.(b) is fulfilled.

Now we verify assumption 2.(c). For $f:[0,1]\longrightarrow\mathbb{R}$ one sets:

$$\Delta^{h_1,h_2}f(t)=h_1f(t+h_2)+h_2f(t-h_1)-(h_1+h_2)f(t),$$

where $h_1\leq t\leq 1-h_2$.

One has:

$$\delta_2^{h_1,h_2}R(s,t) = \frac{\sigma(s)}{2}|\omega(s)|^{2H}\Delta^{h_1,h_2}\sigma(t) + \frac{\sigma(s)}{2}\Delta^{h_1,h_2}(\sigma|\omega|^{2H})(t) - \frac{\sigma(s)}{2}\Delta^{h_1,h_2}(\sigma|\omega(\cdot)-\omega(s)|^{2H})(t).$$

Consequently:

$$\begin{aligned}\delta_1^{h_1,h_2}\circ\delta_2^{h_1,h_2}R(s,t) &= \frac{1}{2}\Delta^{h_1,h_2}(\sigma|\omega|^{2H})(s)\Delta^{h_1,h_2}\sigma(t) + \frac{1}{2}\Delta^{h_1,h_2}(\sigma|\omega|^{2H})(t)\Delta^{h_1,h_2}\sigma(s)\\ &\quad -\frac{h_1}{2}\Delta^{h_1,h_2}\left(\sigma(\cdot)\sigma(t+h_2)|\omega(t+h_2)-\omega(\cdot)|^{2H}\right)(s)\\ &\quad -\frac{h_2}{2}\Delta^{h_1,h_2}\left(\sigma(\cdot)\sigma(t-h_1)|\omega(t-h_1)-\omega(\cdot)|^{2H}\right)(s)\\ &\quad +\frac{h_1+h_2}{2}\Delta^{h_1,h_2}\left(\sigma(\cdot)\sigma(t)|\omega(t)-\omega(\cdot)|^{2H}\right)(s).\end{aligned}$$

Therefore:

$$\begin{aligned}\delta_1^{h_1,h_2}\circ\delta_2^{h_1,h_2}R(t,t) &= \Delta^{h_1,h_2}(\sigma|\omega|^{2H})(t)\Delta^{h_1,h_2}\sigma(t)\\ &\quad +h_1(h_1+h_2)\sigma(t)\sigma(t+h_2)|\omega(t+h_2)-\omega(t)|^{2H}\\ &\quad +h_2(h_1+h_2)\sigma(t)\sigma(t-h_1)|\omega(t-h_1)-\omega(t)|^{2H}\\ &\quad -h_1h_2\sigma(t-h_1)\sigma(t+h_2)|\omega(t+h_2)-\omega(t-h_1)|^{2H}.\end{aligned}$$

One sets:

$$\Psi^*(h_1,h_2,t) = \frac{\left(\delta_1^{h_1,h_2}\circ\delta_2^{h_1,h_2}R\right)(t,t)}{h_1h_2(h_1+h_2)h_1^{2H-1}} = \lambda^{H-1/2}\frac{\left(\delta_1^{h_1,h_2}\circ\delta_2^{h_1,h_2}R\right)(t,t)}{h_1^{\frac{3-\gamma}{2}}h_2^{\frac{3-\gamma}{2}}(h_1+h_2)},$$

with $\lambda = \frac{h_2}{h_1}$.

One has:

$$\Psi^*(h_1,h_2,t) = \Psi_1(h_1,h_2,t) + \Psi_2(h_1,h_2,t) + \Psi_3(h_1,h_2,t) - \Psi_4(h_1,h_2,t),$$

where:

$$\begin{aligned}\Psi_1(h_1,h_2,t) &= \frac{\Delta^{h_1,\lambda h_1}(\sigma|\omega|^{2H})(t)}{h_1^{H-1/2}}\frac{\Delta^{h_1,\lambda h_1}\sigma(t)}{\lambda(1+\lambda)h_1^3},\\ \Psi_2(h_1,h_2,t) &= \sigma(t)\sigma(t+\lambda h_1)\left|\frac{\omega(t+\lambda h_1)-\omega(t)}{\lambda h_1}\right|^{2H}\lambda^{2H-1},\\ \Psi_3(h_1,h_2,t) &= \sigma(t)\sigma(t-h_1)\left|\frac{\omega(t-h_1)-\omega(t)}{h_1}\right|^{2H},\\ \Psi_4(h_1,h_2,t) &= \sigma(t-h_1)\sigma(t+\lambda h_1)\left|\frac{\omega(t+\lambda h_1)-\omega(t-h_1)}{(1+\lambda)h_1}\right|^{2H}(1+\lambda)^{2H-1}.\end{aligned}$$

Let $\epsilon > 0$. We take E_δ for a $\delta > 0$ as in theorem 4. One sets for $(l,t) \in]0,+\infty[\times]0,1[$:

$$\begin{aligned} g_1(l,t) &= 0, \\ g_2(l,t) &= \sigma(t)^2|\omega'(t)|^{2H} l^{2H-1}, \\ g_3(l,t) &= \sigma(t)^2|\omega'(t)|^{2H}, \\ g_4(l,t) &= \sigma(t)^2|\omega'(t)|^{2H}(1+l)^{2H-1} \end{aligned}$$

Firstly we show that:

$$\exists \delta_1 > 0, \sup_{E_{\delta_1}} |\Psi_1(h_1,h_2,t) - g_1(l,t)| \le \epsilon. \tag{46}$$

Applying Taylor formula it is easy to see that the term $\frac{\Delta^{h_1,\lambda h_1}\sigma(t)}{\lambda(1+\lambda)h_1^3}$ is uniformly bounded on E_δ for all $\delta > 0$ (thanks to the facts that σ is C^3 and $\mathcal{L}$ has a compact closure in $]0,+\infty[$). Moreover, since $\mathcal{L}$ has a compact closure and $\sigma|\omega|^{2H}$ is continuous, there exists $K > 0$ such that $\forall 0 < \delta < 1$:

$$\frac{\Delta^{h_1,\lambda h_1}(\sigma|\omega|^{2H})(t)}{h_1^{H-1/2}} \le K h_1^{3/2-H} \qquad \text{on} \qquad E_\delta.$$

This proves (46).

Secondly we prove:

$$\exists \delta_2 > 0, \sup_{E_{\delta_2}} |\Psi_2(h_1,h_2,t) - g_2(l,t)| + \sup_{E_{\delta_3}} |\Psi_3(h_1,h_2,t) - g_3(l,t)| + \sup_{E_{\delta_4}} |\Psi_4(h_1,h_2,t) - g_4(l,t)| \le \epsilon. \tag{47}$$

Likewise Taylor formula and compacity of the closure of $\mathcal{L}$ imply that the terms $\frac{\omega(t+\lambda h_1)-\omega(t)}{\lambda h_1}$, $\frac{\omega(t-h_1)-\omega(t)}{h_1}$ and $\frac{\omega(t+h_2)-\omega(t-\lambda h_2)}{(1+\lambda)h_2}$ have for limit $\omega'(t)$ on E_δ. Then use the fact that σ and $x \mapsto |x|^{2H}$ are uniformly continuous on compact sets and ω is of class C^2. One gets (47).

Then (46) and (47) imply that assumption (12) is fulfilled with $g(l,t) = \frac{1}{l^{H-1/2}}(g_1(l,t) + g_2(l,t) + g_3(l,t) + g_4(l,t))$. This function satisfies (10) thanks to the compacity of $\mathcal{L}$ and the uniform continuity of $\sigma|\omega|^{2H}$.

Therefore assumption 2.(c) is fulfilled.

$\square$

Example: the fractional Ornstein-Uhlenbeck process. We consider the Lamperti transform (see [16]) of the FBM: we take $\sigma(t) = e^{-\lambda t}$ and $\omega(t) = \alpha e^{\frac{\lambda t}{H}}$ where $\lambda, \alpha > 0$. This new process is stationary. If $H = 1/2$ it is the Ornstein-Uhlenbeck process with parameters $\sqrt{2\alpha\lambda}$ and λ. So for $H \in]0,1[$ it is called the fractional Ornstein-Uhlenbeck process (see [6]) denoted by O^H. All the results given above can be applied.

If one uses proposition 11 one gets that the singularity function of the process O^H is equal to $g(l,t) = \left(\frac{\alpha\lambda}{H}\right)^{2H} \frac{1+l^{2H-1}-(1+l)^{2H-1}}{l^{H-1/2}}$. If one takes a regular sequence of subdivisions such that $1/N_n = o(1/\log n)$ one gets a.s.:

$$\lim_{n\to+\infty} \mathcal{V}_{\pi_n}(O^H) = \lim_{n\to+\infty} N_n^{2H-1} \sum_{k=1}^{N_n-1} \left[O^H_{\frac{k+1}{N_n}} + O^H_{\frac{k-1}{N_n}} - 2O^H_{\frac{k}{N_n}}\right]^2 = (4-2^{2H})\left(\frac{\alpha\lambda}{H}\right)^{2H}.$$

Like in the case of the FBM it yields a strongly consistent estimator of the parameter H given by:

$$\widehat{H}_n = \frac{1}{2} - \frac{\log\left(\sum_{k=1}^{N_n-1}\left[O^H_{\frac{k+1}{N_n}} + O^H_{\frac{k-1}{N_n}} - 2O^H_{\frac{k}{N_n}}\right]^2\right)}{2\log N_n}.$$

In the last section we illustrate the result obtained about the FBM with some simulations.

6 Simulations

We illustrate the results with some simulations in the case of the FBM. To simulate the FBM we use the method of the circulant matrix (see [5] and [7]) and Matlab©. We use two sequences of subdivisions: the first one is regular and the second one is not (we use the irregular sequence defined in section 5.1). If one uses the method of the circulant matrix to simulate the FBM one obtains a discretized path along a regular subdivision. Therefore for simulations we use the fact that the irregular sequence of section 5.1 can be refined in a regular sequence of subdivisions.

On figure 1 we have represented the values of $\mathcal{V}_{\pi_n}(B^H)$ against the values of n, when grows up to 1500, in the case of a regular sequence of subdivisions. We have made this for three values of H: 0.3, 0.5 and 0.8. On the figure we can see the convergence of $\mathcal{V}_{\pi_n}(B^H)$ claimed by (38). The value of the limit is respectively equal to 2.4843, 2 and 0.9686.

On figure 2 we have replaced the sequence of regular subdivisions with the irregular sequence constructed in section 5.1. Its range of asymptotic ratios is $\mathcal{L} = \{1/2; 2\}$. On the figure we can see the convergence of $\mathcal{V}_{\pi_n}(B^H)$ claimed by (39). The value of the limit is respectively equal to 2.5581, 2 and 0.9463.

Figure 3 represents the histograms of the value of $\mathcal{V}_{\pi_n}(B^H)$ for 1500 simulations with $n = 1500$ and $H = 0.3$, 0.5 and 0.8, in the case of a regular sequence of subdivisions. In each case the mean is respectively equal to 2.4821, 1.9986 and 0.9659, and the standard deviation is equal to 0.1192, 0.0912 and 0.0393.

On figure 4 we do the same but we replace the regular sequence of subdivisions with the irregular sequence constructed in section 5.1. In each case the mean is respectively equal to 2.5535, 1.9966 and 0.9455, and the standard deviation is equal to 0.0880, 0.0650 and 0.0300.

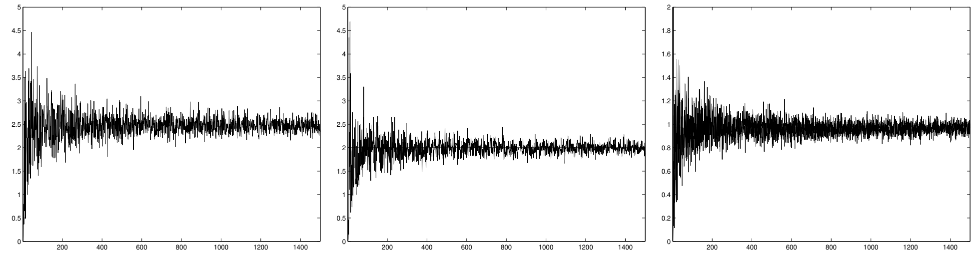

Figure 1: Convergence of $\mathcal{V}_{\pi_n}(B^H)$ for $H = 0.3$, 0.5 and 0.8 when n grows up to 1500 and the subdivisions are regular.

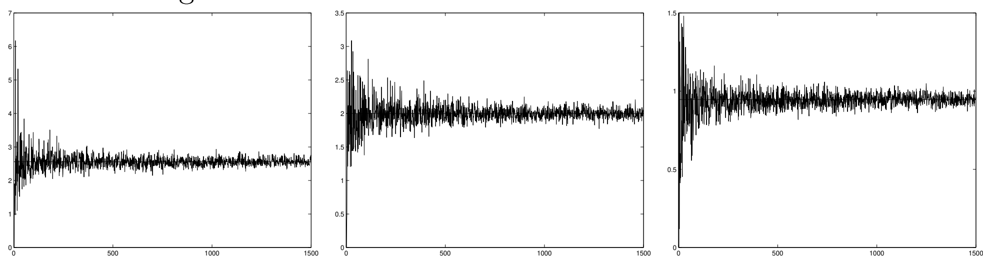

Figure 2: Convergence of $\mathcal{V}_{\pi_n}(B^H)$ for $H = 0.3$, 0.5 and 0.8 when n grows up to 1500 and the subdivisions are irregular with range of asymptotic ratios equal to $\mathcal{L} = \{1/2; 2\}$.

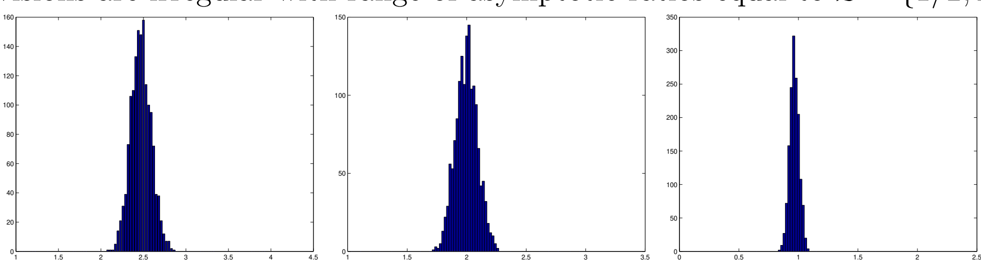

Figure 3: Histograms of 1500 simulations of $\mathcal{V}_{\pi_n}(B^H)$ for $H = 0.3$, 0.5 and 0.8 when $n = 1500$ and the subdivisions are regular.

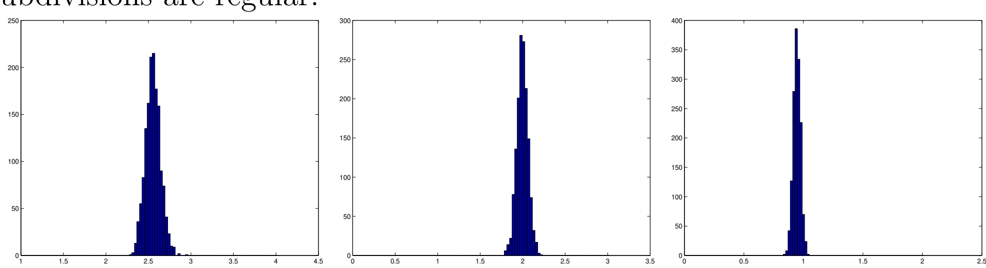

Figure 4: Histograms of 1500 simulations of $\mathcal{V}_{\pi_n}(B^H)$ for $H = 0.3$, 0.5 and 0.8 when $n = 1500$ and the subdivisions are irregular with range of asymptotic ratios equal to $\mathcal{L} = \{1/2; 2\}$.

We can see on all the histograms that $\mathcal{V}_{\pi_n}(B^H)$ seems to be asymptotically normal. We know it for the case of the FBM and regular subdivisions. We will try to show it in the general case in a future paper.

References

[1] A. Adler. *The geometry of random fields.* Ed. John Wiley and Sons, 1981.

[2] G. Baxter. A strong limit theorem for Gaussian processes. *Proc. Amer. Soc.*, 7:522–527, 1956.

[3] A. Benassi, S. Cohen, J. Istas, and S. Jaffard. Identification of filtered white noises. *Stoch. Prob. Appl.*, 75:31–49, 1998.

[4] A. Benassi, S. Jaffard, and D. Roux. Elliptic Gaussian random processes. *Rev. Math. Iboamerica*, 13:7–18, 1997.

[5] G. Chan and A.T.A. Wood. Simulation of a stationary Gaussian process in $[0,1]^d$. *Journ. of Comput. and Graph. Statistics.*, 3:409–432, 1994.

[6] P. Cheredito, H. Kawaguchi, and M. Maejima. Fractional Ornstein-Uhlenbeck processes. *Elect. Journ. of Proba.*, 8(3):1–14, 2003.

[7] J.F. Coeurjolly. *Inférence statistique pour les mouvements Brownien fractionnaires et multifractionnaires. (French).* PhD thesis, Université Joseph Fourier Grenoble I., 2000.

[8] J.F. Coeurjolly. Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths. *Stat. Inference Stoch. Process.*, 4(2):199–227, 2001.

[9] S. Cohen, X. Guyon, O. Perrin, and M. Pontier. Singularity functions for fractional processes, and application to fractional Brownian sheet. *To appear in Annales de l'I.H.P.*

[10] R.M. Dudley. Sample functions of the Gaussian process. *Ann. Probability*, 1:66–103, 1973.

[11] E.G. Gladyshev. A new limit theorem for stochastic processes with Gaussian increments. *Theor. Prob. Appl.*, 6(1):52–61, 1961.

[12] X. Guyon and J. León. Convergence in law of the H-variations of a stationary Gaussian process in $\mathbb{R}$. *Ann. Inst. H. Poincaré Probab. Statist.*, 25(3):265–282, 1989.

[13] D.L. Hanson and F.T. Wright. A bound on tail probabilities for quadratic forms in indepedent random variables. *Ann. Math. Statist.*, 42:1079–1083, 1971.

[14] J. Istas and G. Lang. Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Annales de l'I.H.P.*, 33:407–436, 1997.

[15] R. Klein and E. Gine. On quadratic variations of processes with Gaussian increments. *Annals of Probability.*, 3(4):716–721, 1975.

[16] J.W. Lamperti. Semi-stable stochastic processes. *Trans. Amer. Math. Soc.*, 104:62–78, 1962.

[17] P. Lévy. Le mouvement Brownien plan. (french). *Amer. J. Math.*, 62:487–550, 1940.