Electronic Journal of Probability

Vol. 14 (2009), Paper no. 51, pages 1513–1531.

Journal URL
http://www.math.washington.edu/~ejpecp/

Lower estimates for random walks on a class of amenable p-adic groups

Sami Mustapha
Institut mathématique de Jussieu
175, rue du Chevaleret, 75013, Paris, France
sam@math.jussieu.fr

Abstract

We give central lower estimates for the transition kernels corresponding to symmetric random walks on certain amenable p-adic groups.

Key words: random walk; p-adic groups.

AMS 2000 Subject Classification: Primary 22E35, 60B15, 60G50.

Submitted to EJP on May 5, 2005, final version accepted April 3, 2009.

1 Introduction

Let G be a locally compact amenable group. We shall denote by $d^r g$ (resp. $d^l g$) the right (resp. left) Haar measure on G and by $\delta(g) = d^r g/d^l g$ the modular function on G normalized by $\delta(e) = 1$ where e denotes the identity of G. let $d\mu(g) = \varphi(g)d^r g \in \mathbf{P}(G)$ be a probability measure on G where $\varphi(g) \in L^\infty(G)$ is assumed to have a compact support or a fast decay at infinity. Let us assume that μ is symmetric (i.e. the involution $g \to g^{-1}$ stabilizes μ) and consider the random walk on G induced by μ, i.e. the G-valued process that evolves as follows: if $X_n = g$ is the position at time n then $X_{n+1} = gh$ where h is chosen according to μ. We shall denote by

$$d\mu^{*n}(g) = \varphi_n(g)d^r g$$

the n^{th} convolution power of μ and examine the behaviour of the decay of $\varphi_n(e)$ as $n \to \infty$.

If we restrict ourselves to unimodular real Lie groups then the answer lies in the behaviour of the volume growth of G. Let us recall that if G is a locally compact group that is generated by some symmetric compact neighbourhood $\Omega \subset G$ of the identity in G, the volume growth function is (cf. [9])

$$\gamma(t) = Vol(B_t(e)), \quad t = 1, 2, ...$$

where the volume is taken with respect to $d^r g$ (or $d^l g$) and where $B_t(e)$ is the ball of radius t centred on e defined by

$$B_t(e) = \Omega ... \Omega, \quad t \text{ times}.$$

For $x \in G$ the distance from e is defined by $|x| = \inf\{t, \ x \in B_t(e)\}$ and a left invariant distance can be defined on G by setting $d(x, y) = |y^{-1}x|$, $x, y \in G$. If Ω_1, Ω_2 are two neighbourhoods of e as above it is not difficult to check that there exists $C > 0$ such that $C^{-1} \leq |.|_2/|.|_1 \leq C$ and that the corresponding growth functions satisfy the obvious equivalence $\gamma_1 \approx \gamma_2$, i.e.

$$\gamma_1(t) \leq C\gamma_2(Ct) + C \leq C'\gamma_1(C't) + C', \quad t \geq 1.$$

For real Lie groups we have the following dichotomy (cf. [9], [13]): either

$$\gamma(t) \approx t^D$$

where $D = D(G) = 1, 2, ...$, or

$$\gamma(t) \approx e^t.$$

In the first case we say that G is of polynomial growth and in the second case we say that G is of exponential growth and the answer to our problem in the case of unimodular amenable real Lie groups was given by Varopoulos and is the following

$$\varphi_n(e) \approx n^{-D/2} \Longleftrightarrow \gamma(t) \approx t^D$$

$$\varphi_n(e) \approx e^{-n^{1/3}} \Longleftrightarrow \gamma(t) \approx e^t$$

(cf. [1], [6], [7], [10], [28]).

Varopoulos showed that the $e^{-n^{1/3}}$ versus polynomial behaviour extends to the non-unimodular amenable case depending on wether the Lie group G is (C) or (NC). This classification introduced in [24] can be expressed in terms of the roots of the ad-action of the radical of the Lie algebra of the group on its nilradical.

The discret case is more complicated (cf. [1], [11], [16], [17], [22], [23]). First there is no dichotomy in the volume growth (cf. [8]). On the other hand if we suppose that the group G is of exponential growth then one can claim only the upper bound (cf. [11])

$$\varphi_n(e) \leq C \exp(-cn^{1/3}), \quad n \geq 1.$$

In general, the matching lower bound fails. Ch. Pittet and L. Saloff-Coste showed (cf. [16]) that there are soluble groups with exponentional volume growth for which the heat kernel decays as $\exp(-cn^\alpha)$ with $\alpha \in (0,1)$ which can be taken arbitrarly close to 1. This, as mentionned above, can not happen in the case of real Lie groups.

In a recent paper (cf. [17]) Ch. Pittet and L. Saloff-Coste established the lower bound $\varphi_{2n}(e) \geq c\exp(-Cn^{1/3})$, $n = 1, 2, ...$ for the large times asymptotic behaviours of the probabilities of return to the origin at even times $2n$, for random walks associated with finite symmetric generating sets of solvable groups of finite Prüfer rank. They asked in this paper (cf. [17], §8) if a similiar lower bound is available in the case of analytic p-adic groups. An answer to this problem was given in [15] (cf. also [14]). The aim of this paper is to show that the $e^{-n^{1/3}}$ lower bound obtained in [14] can be substantially improved for a large class of amenable p-adic groups.

2 Amenable p-adic groups

In this section G will denote an algebraic connected amenable group over $\mathbb{Q}_p$ the field of p-adic numbers; $U \subset Q \subset G$ will denote the radical and the unipotent radical (cf. [3], [5]). Amenability of G is equivalent to the fact that the semi-simple group G/Q is compact (cf. [19]). Let S denote a fixed levi subgroup S of G (cf. [3]). The group G can then be written as a semi-direct product:

(1) $$G = Q \ltimes S = (U \ltimes A) \ltimes S \cong U \ltimes (A \times S)$$

where A is abelian and can be identified to the direct product of a finite group and a d-torus $T \cong (\mathbb{Q}_p^*)^d$ (cf. [3], [5]). Here $\mathbb{Q}_p^*$ denotes the multiplicative group of the field $\mathbb{Q}_p$. Since $(\mathbb{Q}_p^*)^d \cong \mathbb{Z}^d \times K$ (where K is compact, cf. [2], [4]), by considering the projection

(2) $$\pi : A \longrightarrow \mathbb{Z}^d,$$

we can fix $\pi_1, ..., \pi_d \in A$ so that each $z \in A$ admits a unique decomposition

(3) $$z = \pi_1^{n_1} ... \pi_d^{n_d} \tau, \quad n_1, ..., n_d \in \mathbb{Z}, \quad \tau \in \tilde{K},$$

where $\tilde{K}$ denotes a compact subgroup of A. Let $\mathscr{U} = Lie(U)$ denote the Lie algebra of U. Let $\overline{\mathbb{Q}}_p$ denotes a finite extension of $\mathbb{Q}_p$ which contains all the eigenvalues defined by

$$det\left(Ad(\pi_j) - \lambda I\right) = 0, \quad j = 1, ..., d.$$

The Ad-action of A on $\mathscr{U}$ extends to $\mathscr{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ and it follows from the proof of the Zassenhaus lemma (cf. [12]) that we have a decomposition of $\mathscr{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ into a direct sum

(4) $$\mathscr{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p = W_1 \oplus ... \oplus W_r$$

where the subspaces W_j $(1 \le j \le r)$ are invariant by $Ad(\pi_i)$, $i = 1, ..., d$, and such that the restriction of $Ad(\pi_i)$ to W_j is the sum of the scalar $\lambda_j(\pi_i)$ and a nilpotent endomorphism.

Let $\chi_j : A \longrightarrow \overline{\mathbb{Q}}_p^{\,*}$ defined by

$$\chi_j(z) = \lambda_j(\pi_1)^{n_1}...\lambda_j(\pi_d)^{n_d}, \;\; z = \pi_1^{n_1}...\pi_d^{n_d}\tau, \;\; j = 1, ..., r.$$

Let $|.|_p$ denote the standard p-adic norm. We shall denote by $|.|_p'$ its extension to $\overline{\mathbb{Q}}_p$. We have $|x|_p' \in \{\overline{p}^n, \; n \in \mathbb{Z}\} \cup \{0\}$, $x \in \overline{\mathbb{Q}}_p$, where $\overline{p}$ denotes a rational power of the prime p (cf. [2]). Let $\alpha_1, \alpha_2, ..., \alpha_s$ denote the different norms of the χ_j's, i.e. the different homomorphisms obtained by considering $A \longrightarrow \overline{p}^{\mathbb{Z}}$, $z \longrightarrow |\chi_j(z)|_p'$, $j = 1, ..., r$. Let $\mathscr{L} = \{\alpha_1, \alpha_2, ..., \alpha_s\}$. We shall assume that $\mathscr{L}$ is nonempty and that

$$1 \notin \mathscr{L}.$$

Here 1 denotes the homomorphism $A \to \overline{p}^{\mathbb{Z}}$ identically equal to 1. This assumption garantees the fact that the group G is compactly generated (cf. [3]). Observe that the group G is then of exponential volume growth (cf. [18]).

Let $\gamma_{j,1}, ..., \gamma_{j,d} \in \mathbb{Z}$ the integers defined by

$$\alpha_j(z) = \overline{p}^{\gamma_{j,1}n_1 + \gamma_{j,2}n_2 + ... + \gamma_{j,d}n_d}, \;\; z = \pi_1^{n_1}...\pi_d^{n_d}\tau, \;\; j = 1, ..., s. \tag{5}$$

We shall denote by L_j, $j = 1, ..., s$, the linear forms on $\mathbb{Z}^d$ defined by

$$L_j(n_1, ..., n_d) = \gamma_{j,1}n_1 + \gamma_{j,2}n_2 + ... + \gamma_{j,d}n_d, \tag{6}$$

$(n_1, ..., n_d) \in \mathbb{Z}^d$ and by $\tilde{L}_j$ the linear forms on $\mathbb{R}^d$ induced by the L_j's.

Let now $d\mu(g) = \varphi(g)d^rg \in \mathbf{P}(G)$ denote a symmetric probability measure on G. The density $\varphi(g)$ is assumed to be a continuous compactly supported function on G. To avoid unnecessary complications we shall assume that there exists $e \in \Omega = \Omega^{-1} \subset G$ such that

$$\inf\{\varphi(g), \;\; g \in \Omega\} > 0, \quad G = \bigcup_{n \ge 0} \Omega^n; \tag{7}$$

the last condition in (7) implies that $supp(\mu)$ generates the group G.

Let

$$p : G \longrightarrow G/U \longrightarrow A$$

denote the projection that we obtain from the identifications (1) and let

$$\check{\mu} = \big(\pi \check{\circ} p\big)(\mu) \in \mathbf{P}(\mathbb{Z}^d) \subset \mathbf{P}(\mathbb{R}^d), \tag{8}$$

where π denotes the canonical projection (2). It follows from (7) that there exists a choice of coordinates on $\mathbb{R}^d$ for which the covariance matrix of $\check{\mu}$ satisfies

$$\int_{\mathbb{R}^d} x_i x_j d\check{\mu}(x) = \delta_{i,j}, \quad 1 \le i, j \le d. \tag{9}$$

We shall assume that $\mathbb{R}^d$ is equipped with the Euclidean structure associated to these coordinates.

From now on we shall assume that the L_j's induce a "Weyl chamber". More precisely we suppose that

$$\Pi_{\mathscr{L}} = \{x \in \mathbb{R}^d, \quad \tilde{L}_j(x) > 0, \quad j = 1, 2, ..., s\} \subset \mathbb{R}^d$$

define a nonempty convex cone in $\mathbb{R}^d$. This condition is the analogue of the (NC)-condition introduced by Varopoulos in [24] in the setting of real amenable Lie groups. We shall prove that, under this condition, we have a lower bound of the form $\varphi_n(e) \geq cn^{-\nu}$. The argument follows the approach introduced by Varopoulos in [24].

The exact value of ν is defined as in the real case and is expressed in terms a parameter $\lambda = \lambda(d, \mathscr{L})$ that is defined as follows. In the the rank one case (i.e. $d = 1$) we shall set

$$(10^a) \qquad \lambda = 0.$$

In the case $d \geq 2$ let us denote by $\Sigma = \{x \in \mathbb{R}^d, \ |x| = 1\}$ the unit sphere in $\mathbb{R}^d$. Let Δ_Σ be the corresponding spherical Laplacian. Let $\Pi_\Sigma = \Sigma \cap \Pi_{\mathscr{L}} \subset \Sigma$. We then set

$$(10^b) \qquad \lambda = \inf\{-(\Delta_\Sigma f, f), \ \|f\|_2 = 1, \ f \in C_0^\infty(\Pi_\Sigma)\};$$

i.e. λ is the first Dirichlet eigenvalue of the region Π_Σ. The scalar product and the L^2-norm in (10^b) are taken with respect to the Euclidean volume element on Σ.

Theorem 1. *Let G and $d\mu^{*n}(g) = \varphi_n(g) d^r g$, $n = 1, 2, ...$ be as above. Then there exists $C > 0$ such that*

$$(11) \qquad \varphi_n(e) \geq \frac{1}{Cn^{1+\frac{\sqrt{(d-2)^2+4\lambda}}{2}}}, \quad n = 1, 2, ...$$

where λ is defined by (10).

The following comments may be helpful in placing the above theorem in its proper perspective.

(i) It is enough to to prove the estimate (11) when n is even since φ satisfies (7).

(ii) Observe that the group G is automatically non-unimodular, for otherwise we have (cf. [11]):

$$\varphi_n(e) \leq Ce^{-cn^{1/3}}, \quad n = 1, 2, ...$$

(iii) The upper estimate

$$\varphi_n(e) \leq \frac{C}{n^{3/2}}, \quad n = 1, 2, ...$$

is known to hold for general non-unimodular locally compact groups (cf. [26]). This shows that the index $3/2$ cannot be improved in the rank one case.

(iv) We will show (cf. §4 below) that in the case of metabelian p-adic groups, the lower bound (11) can be complemented with a similar upper bound.

Throughout the remainder of the paper C denotes a positive constant which is not always the same, even in a given line.

3 Proof of Theorem 1

Let G, $d\mu(g) = \varphi(g)d^r g \in \mathbf{P}(G)$ and $d\mu^{*n}(g) = \varphi_n(g)d^r g$ be as in Theorem 1. Let $\xi_1, \xi_2, ... \in G$ be a sequence of independent equidistribued random variables of law $d\mu(g)$ and let $X_n = \xi_1\xi_2...\xi_n$, $n = 1, 2, ...$ denote the corresponding random walk starting at $X_0 = e$. Let $B \subset G$ a borel subset, we have

$$\mathbb{P}_e\left[X_n \in B\right] = \int_B \varphi_n(g)d^r g, \quad n = 1, 2, ... \tag{12}$$

The symmetry of $d\mu(g)$ implies that

$$d\mu(g) = d\mu(g^{-1}) = \varphi(g)d^r g = \varphi(g^{-1})dg = \varphi(g^{-1})\delta(g)^{-1}d^r g,$$

hence

$$\varphi(g^{-1}) = \varphi(g)\delta(g), \quad g \in G.$$

We have also

$$\varphi_n(g^{-1}) = \varphi_n(g)\delta(g), \; g \in G, \quad n = 1, 2, ...$$

On the other hand we have

$$\varphi_{2n}(e) = \int_G \varphi_n(g^{-1})\varphi_n(g)dg = \int_G \varphi_n(g)\delta(g)\varphi_n(g)dg = \int_G \varphi_n(g)^2 d^r g.$$

Schwarz inequality applied to (12) gives then

$$\varphi_{2n}(e) \geq \frac{\left(\mathbb{P}_e\left[X_n \in B\right]\right)^2}{|B|}, \quad B \subset G, \quad n = 1, 2, ... \tag{13}$$

where $|B|$ denotes the right Haar measure of B.

Let us further observe that the group G (resp. G/U) decomposes as a semi-direct (resp. direct) product

$$G = Q \ltimes S \cong U \ltimes (\mathbb{Z}^d \times \tilde{S})$$

$$G/U \cong A \times S \cong \mathbb{Z}^d \times \tilde{S}$$

where $\tilde{S}$ is compact. This follows from (1).

Let us write

$$X_n = \xi_1\xi_2...\xi_n = u_1z_1u_2z_2...u_nz_n, \quad n = 1, 2, ... \tag{14}$$

where $\xi_j = u_jz_j$ with $u_j \in U$ and $z_j \in A \times S$, $j = 1, 2, ...$ Using the interior automorphisms $x^y = yxy^{-1}$, $x, y \in G$, we rewrite (14)

$$X_n = u_1u_2^{z_1}u_3^{z_1z_2}...u_n^{z_1...z_{n-1}}z_1z_2\ldots z_n = \Gamma_n Z_n, \tag{15}$$

$$\Gamma_n \in U, \quad Z_n \in A \times S, \; n = 1, 2, ...$$

We shall use the exponential map and identify U to its Lie algebra $\mathscr{U}$ (cf. [5], [20]) and write each u_j in the above expression

$$u_i = \exp(v_i), \quad v_i \in \mathscr{U}, \quad i = 1, 2, .. \tag{16}$$

We have therefore

$$u_j^{z_1 z_2 ... z_{j-1}} = \exp\left(Ad(z_1 ... z_{j-1}) v_j\right), \quad j \geq 2. \tag{17}$$

Let us fix $e_1, e_2, ..., e_m$ a basis of $\mathscr{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ adapted to the decomposition (4). For $x = x_1 e_1 + x_2 e_2 + ... + x_m e_m \in \mathscr{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$, we set

$$\|x\| = \max_{1 \leq i \leq m} |x_i|.$$

Since μ is compactly supported we can suppose that the v_i's in (16) satisfy

$$\|v_j\| \leq C, \quad j = 1, 2, ... \tag{18}$$

where $C > 0$ is an appropriate positive constant.

Let us equip $End_{\overline{\mathbb{Q}}_p}\left(\mathscr{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p\right)$ with the norm

$$|||T||| = sup_{\|v\| \leq 1} \|Tv\|, \quad T \in End_{\overline{\mathbb{Q}}_p}\left(\mathscr{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p\right).$$

It is clear that $|||.|||$ satisfies the ultrametric property

$$|||T + T'||| \leq \max\left(|||T|||, |||T'|||\right). \tag{19}$$

Let $1 \leq l \leq r$. For

$$z = \pi_1^{n_1} ... \pi_d^{n_d} \tilde{\tau} = \tilde{z} \tilde{\tau} \in A \times S \cong \mathbb{Z}^d \times \tilde{S}$$

we have

$$Ad(z)|_{W_l} = Ad(\tilde{\tau})|_{W_l} \circ \left(\tilde{\chi}_l(z) I + \mathscr{T}(z)\right), \tag{20}$$

where $\mathscr{T}(z)$ denotes an upper triangular matrice and where

$$\tilde{\chi}_l(z) = \chi_l(\pi_1^{n_1} ... \pi_d^{n_d}) = \chi_l(\pi_1)^{n_1} ... \chi_l(\pi_d)^{n_d}.$$

On the other hand it is clear that

$$|\tilde{\chi}_l(z)|_p', \ |\tilde{\chi}_l(z)^{-1}|_p' \leq p^{C|\tilde{z}|_{\mathbb{Z}^d}}, \quad z \in A \times S, \tag{21}$$

where $|.|_{\mathbb{Z}^d}$ denotes the Euclidean norm on $\mathbb{Z}^d$. We have also

$$|||Ad(z)||| \leq C |||Ad(\pi_1)|||^{n_1} ... |||Ad(\pi_d)|||^{n_d} \leq C p^{C|\tilde{z}|_{\mathbb{Z}^d}}, \quad z \in A \times S. \tag{22}$$

Combining (20), (21), (22) we deduce that the triangular matrice $\mathscr{T}(z)$ that appears in (20) satisfies

$$|||\mathscr{T}(z)||| C \leq p^{C|\tilde{z}|_{\mathbb{Z}^d}}, \quad z \in A \times S. \tag{23}$$

Let now $z_1, z_2, ..., z_k \in A \times S$ and $1 \le l \le r$. We have

$$\begin{aligned} Ad(z_1 z_2 ... z_k)|_{W_l} &= Ad(\tilde{\tau}_1 \tilde{\tau}_2 ... \tilde{\tau}_k)|_{W_l} \circ \prod_{j=1}^{k} \left(\tilde{\chi}_l(z_j) I + \mathscr{T}(z_j) \right) \\ &= \tilde{\chi}_l(z_1 z_2 ... z_k) Ad(\tilde{\tau}_1 \tilde{\tau}_2 ... \tilde{\tau}_k)|_{W_l} \\ &\quad \circ \sum_{\alpha, i_j} \left(\tilde{\chi}_l(z_{i_1}) ... \tilde{\chi}_l(z_{i_\alpha}) \right)^{-1} \mathscr{T}(z_{i_1}) ... \mathscr{T}(z_{i_\alpha}). \end{aligned}$$

Using the fact that in the last sum all the terms corresponding to indexes $\alpha > n$ vanish and combining this with (21), (23) and the ultrametric property (19) we deduce that

$$|||Ad(z_1 z_2 ... z_k)|_{W_l}||| \le C \left| \tilde{\chi}_l(z_1 z_2 ... z_k) \right|_p' p^{C \max_{1 \le j \le k} |\tilde{z}_j|_{\mathbb{Z}^d}}.$$

If we use the linear forms L_l defined by (5) and (6) we then deduce

$$|||Ad(z_1 z_2 ... z_k)|_{W_l}||| \le C p^{\max_{1 \le l \le s} L_l(\tilde{z}_1 + ... + \tilde{z}_k) + C \max_{1 \le j \le k} |\tilde{z}_j|_{\mathbb{Z}^d}}.$$

By the above considerations we have finally proved that

$$|||Ad(z_1 z_2 ... z_k)||| \le C p^{\max_{1 \le l \le s} L_l(\zeta_1 + ... + \zeta_k) + C \max_{1 \le j \le n} |\zeta_j|_{\mathbb{Z}^d}},$$

$$\zeta_j = p(z_j), \quad z_j \in A \times S, \quad j = 1, ..., k, \quad k = 1, 2, ...$$

If we apply this estimate to $u_j^{z_1 z_2 ... z_{j-1}} = \exp\left(Ad(z_1 ... z_{j-1}) v_j \right)$ where the u_j's, v_j's and z_j's are as in (16), (17), (18) we then deduce

$$||Ad(z_1 ... z_{j-1}) v_j|| \le C p^{\max_{1 \le l \le s} L_l(\zeta_1 + ... + \zeta_{j-1}) + C \max_{1 \le k \le j-1} |\zeta_k|_{\mathbb{Z}^d}},$$

$$\zeta_j = p(z_j), \quad j \ge 2.$$

Observe that the measure that controls the random walk $\left(S_j \right)_{j \in \mathbf{N}}$, defined by $S_j = \zeta_0 + \zeta_1 + ... + \zeta_j$ ($\zeta_0 = 0$) is the symmetric measure $\check{\mu}$ defined by (8). The random variables ζ_j are in particular compactly supported and we have

$$||Ad(z_1 ... z_{j-1}) v_j|| \le C p^{\max_{1 \le l \le r} L_l(\zeta_1 + ... + \zeta_j)}, \ \ \zeta_j = p(z_j), \quad j \ge 2. \tag{24}$$

Let us consider, for $n = 1, 2, ...$, the event E_n defined by

$$E_n = \Bigg(L_l(\zeta_1 + ... + \zeta_j) \le C, \quad j = 1, ..., n, \quad l = 1, ..., s; \qquad \left| \zeta_1 + ... + \zeta_n \right|_{\mathbb{Z}^d} \le C \sqrt{n} \Bigg), \tag{25}$$

where $C > 0$ denotes an appropriate large constant. Using (15), Campbell-Hausdorff, (24) and the ultrametric property (19) we see that the event E_n verifies

$$E_n \subset \left[X_n \in B_n \right]$$

where B_n is defined by

$$B_n = \exp(\{u \in \mathscr{U}, \quad \|u\| \leq C \ \}).\{z \in A \times S, \quad |p(z)|_{\mathbb{Z}^d} \leq An^{1/2} \ \}.$$

It is easy to see that $d^r g$ shows that

$$|B_n| \leq Cn^{\frac{d}{2}}, \quad n = 1, 2, ... \tag{26}$$

It remains to estimate the probability of the event (25). Let Π denote the polyhedral region in $\mathbb{Z}^d$ defined by

$$\Pi = \{z \in \mathbb{Z}^d, \ \ L_j(z) \leq C, \quad j = 1, ..., s\},$$

where C denotes the same constant as in (25). Let $h_n(x, y)$, $n = 1, 2, ...$, x, $y \in \Pi$, denote the transition kernel corresponding to the random walk $\left(S_j\right)_{j \in \mathbf{N}}$ with killing outside of Π. Precise lower estimates for the kernel $h_n(x, y)$ can be obtained thanks to the results of [26] and [27]. To write down these estimates we need the following notations. Let us assume that $d \geq 2$ and let $0 < u_0(\sigma)$, $\sigma \in \Pi_\Sigma$, denote the eigenfunction corresponding to the first Dirichlet dirichlet eigenvalue of the region Π_Σ defined by (10). Let $u(x)$ be the function defined on $\overline{\Pi}_{\mathscr{L}}$ by

$$u(x) = u(r, \sigma) = r^a u_0(\sigma), \quad x = (r, \sigma) \in \mathbb{R}_+^* \times \Pi_\Sigma$$

where

$$a = \frac{\sqrt{(d-2)^2 + 4\lambda} - (d-2)}{2}, \tag{27}$$

and where (r, σ) denote the polar coordinates on $\mathbb{R}_+^* \times \Sigma$. The function u defined in this way is a positive function harmonic in $\Pi_{\mathscr{L}}$ which vanishes on $\partial \Pi_{\mathscr{L}}$ (cf. [24]). In the case where $d = 1$, the function u is defined by $u(x) \equiv x$, $x \in \mathbb{R}_+^*$.

It follows easily from [27] (cf. estimate (2) p. 359) and [26] (cf. Theorem 1 and estimate (0.3.4)) that there exists $C > 0$ such that

$$h_n(0, y) \geq \frac{u(-y)}{Cn^{a+d/2}}, \quad |y| \leq C\sqrt{n}, \quad n \geq C.$$

We have therefore

$$\mathbb{P}(E_n) \geq \frac{1}{Cn^{a+d/2}} \sum_{y \in \Pi, \ |y| \leq C\sqrt{n}} u(-y).$$

Using the homogeneity of u we deduce then that

$$\mathbb{P}(E_n) \geq \frac{1}{Cn^{a+d/2}} \int_0^{C\sqrt{n}} r^{a+d-1} dr = \frac{1}{Cn^{a/2}}. \tag{28}$$

The lower estimate (11) is an immediate consequence of (13), (26), (27) and (28). This completes the proof of Theorem 1.

4 Metabelian p-adic groups

Our aim in in this section is to show that in the case of metabelian groups the lower estimate (11) can be complemented with a similar upper bound. We keep the notation of §2. We shall denote by $\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p^*,\ \ |x|_p = 1\}$ and by $\mathbb{Z}_p = \{x \in \mathbb{Q}_p,\ \ |x|_p \leq 1\}$. Let dx denote the Haar measure on $\mathbb{Q}_p$ normalized by $dx(\mathbb{Z}_p) = 1$ and let d^*x denote the Haar measure on $\mathbb{Q}_p^*$ normalized by $d^*x(\mathbb{Z}_p^*) = 1$. Let us fix $k,\, l \geq 2$ and consider

$$G = \mathbb{Q}_p^k \ltimes_\sigma \left(\mathbb{Q}_p^*\right)^l \tag{29}$$

the semi-direct product of $\left(\mathbb{Q}_p^*\right)^l$ with the vector space $\mathbb{Q}_p^k$ where $\left(\mathbb{Q}_p^*\right)^l$ acts on $\mathbb{Q}_p^k$ by

$$x = (x_1, ..., x_k) \longrightarrow \sigma(y)x = (\chi_1(y)x_1, ..., \chi_k(y)x_k), \quad y \in \left(\mathbb{Q}_p^*\right)^l,$$

where $\chi_1, ..., \chi_k$ denote k morphisms

$$\chi_1, ..., \chi_k : \left(\mathbb{Q}_p^*\right)^l \longrightarrow \mathbb{Q}_p^*. \tag{30}$$

More precisely, we assume that the multiplication in G is given by

$$\begin{aligned} g.g' &= (x;y).(x';y') = \big(x + \sigma(y)x'; y.y'\big) \\ &= \Big(x_1 + \chi_1(y)x_1', x_2 + \chi_2(y)x_2', ..., x_k + \chi_k(y)x_k'; y_1.y_1', y_2.y_2', ..., y_l.y_l'\Big) \end{aligned}$$

$$g = (x, y), \quad g' = (x', y') \in G; \quad x = (x_1, ..., x_k), \;\; x' = (x_1', ..., x_k') \in \mathbb{Q}_p^k;$$

$$y = (y_1, ..., y_l), \;\; y = (y_1', ..., y_l') \in \left(\mathbb{Q}_p^*\right)^l.$$

We shall denote by

$$d^r g = dx d^* y = dx_1 ... dx_k d^* y_1 ... d^* y_l; \quad d^l g = dg = \delta(g)^{-1} d^r g$$

the right and the left invariant Haar measure on G. The modular function $\delta(g)$ is given by

$$\delta(g) = \delta(y) = |\chi_1(y)| ... |\chi_k(y)|, \quad g = (x, y) \in G. \tag{31}$$

Let $d\mu(g) = \varphi(g) d^r(g)$ denote the symmetric, compactly supported, probability measure on G defined by

$$\begin{gathered} \varphi(g) = \alpha \delta^{-1/2}(g) I_{\mathbb{Z}_p}(x_1) ... I_{\mathbb{Z}_p}(x_k) I_{\mathbb{Z}_p}(\chi_1(y)^{-1}x_1) ... I_{\mathbb{Z}_p}(\chi_k(y)^{-1}x_k) \\ \times I_{\left(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*\right)}(y_1) \dots I_{\left(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*\right)}(y_l), \\ g = (x_1, ..., x_k; y_1, ..., y_l) \in G. \end{gathered} \tag{32}$$

The notation I_A is used here to denote the indicator of a subset A and $\alpha > 0$ denotes an appropriate positive constant.

We shall assume that the χ_j's in (30) verify the condition

$$|\chi_j| \not\equiv 1, \quad j = 1, ..., k.$$

This garantees (cf. §2) that the group G defined by (29) is compactly generated and it is easy to see that $supp(\mu)$ is a generating set of the group G. We shall denote, as in §2, by L_j, $j = 1, ..., k$, the k linear forms on $\mathbb{Z}^l$ induced by the χ_j's and we shall assume that these linear forms induce a "Weyl chamber" $\Pi = \{x \in \mathbb{R}^l, \quad \tilde{L}_j(x) > 0, \quad j = 1, 2, ..., k\}$ as in §2.. We shall denote by $\Sigma = \{x \in \mathbb{R}^l, \ |x| = 1\}$ the unit sphere in $\mathbb{R}^l$, by $\Pi_\Sigma = \Sigma \cap \Pi$ and by λ be the first Dirichlet eigenvalue of the region Π_Σ. We have then the following:

Theorem 2. *Let* $G = \mathbb{Q}_p^k \ltimes_\sigma \left(\mathbb{Q}_p^*\right)^l$ *and let* $\mu \in \mathbf{P}(G)$ *be as above. Let* $d\mu^{*n}(g) = d\left(\mu * ... * \mu\right)(g) = \varphi_n(g)d^r g$ *denote the* n^{th} *convolution power of* μ*. Then there exists* $C > 0$ *such that*

$$\varphi_n(e) \leq \frac{C}{n^{1+\frac{\sqrt{(l-2)^2+4\lambda}}{2}}}, \quad n = 1, 2, ...$$

The first step to prove Theorem 2 consists in establishing an explicit formula for $\varphi_n(g)$, $g \in G$.
In what follows we shall use the notation $x_i = (x_{i,1}, ..., x_{i,k})$ (resp $y_i = (y_{i,1}, ..., y_{i,l})$) for $x_i \in \mathbb{Q}_p^k$, $i = 1, 2, ...$ (resp. $y_i \in \left(\mathbb{Q}_p^*\right)^l$, $i = 1, 2, ...$). Let us fix $g = (\xi, \zeta) = (\xi_1, ..., \xi_k; \zeta_1, ..., \zeta_l) \in G$ and $n = 1, 2, ...$ By definition of convolution product we have:

$$\begin{aligned}
\varphi_{n+1}(g) &= \int_G \varphi_n(h)\varphi_1(h^{-1}g)dh \\
&= \int_G \varphi_1(h^{-1}g)\delta(h^{-1})\varphi_n(h)d^r h \\
&= \int_G \varphi_1(h^{-1}g)\delta(h^{-1})d\mu^{*n}(h) \\
&= \int_G ... \int_G \varphi_1\Big(\left(g_1...g_n\right)^{-1} g \Big)\delta(g_1...g_n)^{-1}d\mu(g_1)...d\mu(g_n) \\
&= \int_G ... \int_G \varphi_1\Big(-\left(\sigma(y_1...y_n)^{-1}x_1 + \sigma(y_2...y_n)^{-1}x_2 + \cdots + \sigma(y_n)^{-1}x_n\right) \\
&\qquad +\sigma(y_1...y_n)^{-1}\xi, (y_1 \ldots y_n)^{-1}\zeta\Big)\varphi_1(x_1, y_1)...\varphi(x_n, y_n) \\
&\qquad \delta(y_1...y_n)^{-1}dx_1 d^* y_1 ... dx_n d^* y_n.
\end{aligned}$$

An obvious change of variables combined with Fubini gives

$$(33)\qquad \varphi_{n+1}(g) = \int_{(\mathbb{Q}_p^*)^l}\dots\int_{(\mathbb{Q}_p^*)^l}\Bigg[\int_{\mathbb{Q}_p^k}\dots\int_{\mathbb{Q}_p^k}\Bigg[\varphi_1\Bigg(-(x_1+x_2+....+x_n)$$
$$+\sigma(y_1...y_n)^{-1}\xi,(y_1...y_n)^{-1}\zeta\Bigg)\varphi_1(\sigma(y_1\dots y_n)x_1,y_1)$$
$$\varphi_1(\sigma(y_2...y_n)x_2,y_2)\dots\varphi_1(\sigma(y_n)x_n,y_n)\Bigg]dx_1...dx_n\Bigg]$$
$$\prod_{i=2}^{n}|\chi_1(y_i...y_n)|...|\chi_k(y_i...y_n)|d^*y_1...d^*y_n,$$

where we used (31). Let us consider the product

$$\prod_{i=1}^{n}\varphi_1(\sigma(y_i...y_n)x_i,y_i)$$

that appears in equation (33). By (32), this product is equal to

$$\alpha^n\prod_{i=1}^{n}\delta^{-1/2}(y_i)\prod_{i=1}^{n}\Bigg(I_{\mathbb{Z}_p}(\chi_1(y_i...y_n)x_{i,1})\dots I_{\mathbb{Z}_p}(\chi_k(y_i...y_n)x_{i,k})\Bigg)$$
$$\times\prod_{i=1}^{n-1}\Bigg(I_{\mathbb{Z}_p}(\chi_1(y_{i+1}...y_n)x_{i,1})\dots I_{\mathbb{Z}_p}(\chi_k(y_{i+1}...y_n)x_{i,k})\Bigg)$$
$$\times I_{\mathbb{Z}_p}(x_{n,1})\dots I_{\mathbb{Z}_p}(x_{n,k})\prod_{i=1}^{n}I_{\left(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*\right)}(y_{i,1})\dots I_{\left(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*\right)}(y_{i,l})$$
$$=\alpha^n\prod_{i=1}^{n}\delta^{-1/2}(y_i)\prod_{i=1}^{n-1}\Bigg(I_{|\chi_1(y_{i+1}...y_n)|\max(1,|\chi_1(y_i)|)\mathbb{Z}_p}(x_{i,1})$$
$$\dots I_{|\chi_k(y_{i+1}...y_n)|\max(1,|\chi_k(y_i)|)\mathbb{Z}_p}(x_{i,k})\Bigg)$$
$$I_{\max(1,|\chi_1(y_n)|)\mathbb{Z}_p}(x_{n,1})\dots I_{\max(1,|\chi_k(y_n)|)\mathbb{Z}_p}(x_{n,k})$$
$$\prod_{i=1}^{n}I_{\left(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*\right)}(y_{i,1})\dots I_{\left(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*\right)}(y_{i,l}).$$

We also have

$$\begin{aligned}&\varphi_1(-(x_1+x_2+....+x_n)+\sigma(y_1...y_n)^{-1}\xi,(y_1...y_n)^{-1}\zeta)\\ &=\alpha\delta^{-1/2}((y_1...y_n)^{-1}\zeta)\\ &\quad\times I_{\max(1,|\chi_1(y_1...y_n)\zeta^{-1}|)\mathbb{Z}_p}\left(x_{1,1}+x_{2,1}+....+x_{n,1}-\chi_1(y_1...y_n)^{-1}\xi_1\right)\\ &\quad\dots I_{\max(1,|\chi_k(y_1...y_n)^{-1}\zeta^{-1}|)\mathbb{Z}_p}\left(x_{1,k}+x_{2,k}+....+x_{n,k}-\chi_k(y_1...y_n)^{-1}\xi_k\right)\\ &\quad\times I_{\left(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*\right)}((y_{1,1}...y_{n,1})^{-1}\zeta_1)\dots I_{\left(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*\right)}((y_{1,l}...y_{n,l})^{-1}\zeta_l).\end{aligned}$$

On the other hand we shall set

$$m(y)=\alpha\delta^{-1/2}(y)\prod_{j=1}^{k}\min\left(1,|\chi_j(y)|\right)I_{\left(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*\right)}(y_1)\dots I_{\left(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*\right)}(y_l), \tag{34}$$

$$y=(y_1,...,y_l)\in\left(\mathbb{Q}_p^*\right)^l.$$

What motivates this notation is the fact that

$$\int_{\mathbb{Q}_p^k}\varphi(x,y)dx=m(y),\quad y\in(\mathbb{Q}_p^*)^l. \tag{35}$$

Setting

$$\begin{aligned}A(y_1,y_2,...,y_n)&=m\left((y_1...y_n)^{-1}\zeta\right)\prod_{j=1}^{k}\max\left(1,|\chi_j(y_1...y_n)\zeta^{-1}|\right)\\&\times\prod_{i=1}^{n}\max\left(1,|\chi_1(y_i)|^{-1}\right)\dots\max\left(1,|\chi_k(y_i)|^{-1}\right)m(y_i),\end{aligned} \tag{36}$$

$$\begin{aligned}&B_j(x_1,x_2,...,x_n;y_1,y_2,...,y_n)=B_j(x_1,x_2,...,x_n)\\&=I_{\max(1,|\chi_j(y_1...y_n)\zeta^{-1}|)\mathbb{Z}_p}\Big(x_{1,j}+x_{2,j}+...\\&+x_{n,j}-\chi_j(y_1...y_n)^{-1}\xi_j\Big)I_{\max(1,|\chi_j(y_n)|)\mathbb{Z}_p}(x_{n,j})\\&\times\prod_{i=1}^{n-1}I_{|\chi_j(y_{i+1}...y_n)|\max(1,|\chi_j(y_i)|)\mathbb{Z}_p}(x_{i,j}),\\&j=1,2,...k,\end{aligned} \tag{37}$$

we rewrite (33) as

$$\begin{aligned}\varphi_{n+1}(g)\;=\;&\int_{(\mathbb{Q}_p^*)^l}\cdots\int_{(\mathbb{Q}_p^*)^l}\left[\int_{\mathbb{Q}_p^k}\cdots\int_{\mathbb{Q}_p^k}\prod_{j=1}^{k}B_j\left(x_1,x_2,...,x_n\right)dx_1...dx_n\right]\\&A\left(y_1,y_2,...,y_n\right)\prod_{i=2}^{n}|\chi_1(y_i...y_n)|\dots|\chi_k(y_i...y_n)|\\&d^*y_1...d^*y_n.\end{aligned} \tag{38}$$

The next step is to calculate

$$\int_{\mathbb{Q}_p}\cdots\int_{\mathbb{Q}_p}B_j(x_1,x_2,...,x_n)dx_{1,j}dx_{2,j}...dx_{n,j},\quad j=1,...,k.$$

Towards this we observe that:

$$\int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} I_{\max(1,|\chi_j(y_1\dots y_n)\zeta^{-1}|)\mathbb{Z}_p}(x_{1,j}+\dots+x_{n,j}-\chi_j(y_1\dots y_n)^{-1}\xi_j)$$

$$I_{\max(1,|\chi_j(y_n)|)\mathbb{Z}_p}(x_{n,j})\prod_{i=1}^{n-1} I_{|\chi_j(y_{i+1}\dots y_n)|\max(1,|\chi_j(y_i)|)\mathbb{Z}_p}(x_{i,j})dx_{1,j}dx_{2,j}\dots dx_{n,j}$$

$$=\int_{\mathbb{Q}_p}\left(I_{|\chi_j(y_2\dots y_n)|\max(1,|\chi_j(y_1)|)\mathbb{Z}_p} * \cdots * I_{|\chi_j(y_n)|\max(1,|\chi_j(y_{n-1})|)\mathbb{Z}_p} * I_{\max(1,|\chi_j(y_n)|)\mathbb{Z}_p}\right)(x)$$

$$I_{\max(1,|\chi_j(y_1\dots y_n)\zeta^{-1}|)\mathbb{Z}_p}(x-\chi_j(y_1\dots y_n)^{-1}\xi_j)dx$$

where $*$ denotes the usual convolution in $\mathbb{Q}_p$ (cf. [21]). Taking Fourier transform (cf. [21]) we obtain

$$\begin{aligned}
&\int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} B_j(x_1,x_2,\dots,x_n)dx_{1,j}dx_{2,j}\dots dx_{n,j}\\
&= Min\left[\min_{1\le i\le n-1}\left(|\chi_j(y_{i+1}\dots y_n)|\max(1,|\chi_j(y_i)|)\right);\max(1,|\chi_j(y_n)|)\right]\\
&\min\left(1,|\chi_j(y_n)|^{-1}\right)\prod_{i=1}^{n-1}|\chi_j(y_{i+1}\dots y_n)|^{-1}\min\left(1,|\chi_j(y_i)|^{-1}\right)\\
&\times\Big(I_{Min\left[\min_{1\le i\le n-1}\left(|\chi_j(y_{i+1}\dots y_n)|\max(1,|\chi_j(y_i)|)\right);\max(1,|\chi_j(y_i)|)\right]\mathbb{Z}_p} *\\
&I_{\max(1,|\chi_j(y_1\dots y_n)\zeta|^{-1})\mathbb{Z}_p}\Big)(\chi_j(y_1\dots y_n)^{-1}\xi_j).
\end{aligned}$$

Let us denote by $S_j^{\zeta}(y_1,\dots,y_n)\subset\mathbb{Q}_p$, $y_1,\dots,y_n\in\mathbb{Q}_p^*$ the subset defined by

$$S_j^{\zeta}(y_1,\dots,y_n) = Min\left[\min_{1\le i\le n-1}\left(|\chi_j(y_{i+1}\dots y_n)|\max(1,|\chi_j(y_i)|)\right);\right.$$

$$\left.\max(1,|\chi_j(y_n)|)\right]\wedge\max(1,|\chi_j(y_1\dots y_n)\zeta^{-1}|)\mathbb{Z}_p.$$

With this notation we have

$$\begin{aligned}
(39)\qquad &\int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} B_j(x_1,x_2,\dots,x_n)dx_{1,j}dx_{2,j}\dots dx_{n,j}\\
&= \min\Bigg(1;Min\Bigg(\min_{1\le i\le n-1}\left(|\chi_j(y_{i+1}\dots y_n)|\max(1,|\chi_j(y_i)|)\right);\\
&\max(1,|\chi_j(y_n)|)\Bigg)\min(1,|\chi_j(y_1\dots y_n)^{-1}\zeta|)\Bigg)\min\left(1,|\chi_j(y_n)|^{-1}\right)\\
&\prod_{i=1}^{n-1}|\chi_j(y_{i+1}\dots y_n)|^{-1}\min\left(1,|\chi_j(y_i)|^{-1}\right)I_{S_j^{\zeta}(y_1,\dots,y_n)}(\chi_j(y_1\dots y_n)^{-1}\xi_j),
\end{aligned}$$

$$j = 1, ..., k.$$

Putting together (33), (36), (37), (38) and (39) and taking into account the fact that

$$\prod_{j=1}^{k} \min\left(1, |\chi_j(y_n)|^{-1}\right) \prod_{i=1}^{n-1} \left(\prod_{j=1}^{k} \min\left(1, |\chi_j(y_i)|^{-1}\right) \right)$$

$$\times \prod_{i=1}^{n} \max\left(1, |\chi_1(y_i)|^{-1}\right) \dots \max\left(1, |\chi_k(y_i)|^{-1}\right)$$

$$= \prod_{j=1}^{k} |\chi_j(y_1 \dots y_n)|^{-1} = \delta^{-1}(y_1 \dots y_n),$$

and that

$$\delta^{-1}(y_1 \dots y_n) \prod_{j=1}^{k} \max\left(1, |\chi_j(y_1 \dots y_n)\zeta^{-1}|\right) m\left((y_1 \dots y_n)^{-1}\zeta\right)$$

$$= \alpha\delta^{-1/2}(y_1 \dots y_n)\delta^{-1/2}(\zeta) I_{\left(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*\right)}((y_{1,1} \dots y_{n,1})^{-1}\zeta_1)$$

$$\dots I_{\left(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*\right)}((y_{1,l} \dots y_{n,l})^{-1}\zeta_l),$$

we finally deduce the formula

$$\begin{aligned}(40) \quad \varphi_{n+1}(g) \quad = \quad & \alpha\delta^{-1/2}(\zeta) \int_{(\mathbb{Q}_p^*)^l} \dots \int_{(\mathbb{Q}_p^*)^l} \prod_{j=1}^{k} Min\Bigg(1; \\ & min\Bigg(\min_{1 \le i \le n-1} \left(|\chi_j(y_{i+1} \dots y_n)| \max(1, |\chi_j(y_i)|) \right); \\ & \max(1, |\chi_j(y_n)|)\Bigg) \min(1, |\chi_j(y_1 \dots y_n)^{-1}\zeta|)\Bigg) \\ & \delta^{-1/2}(y_1 \dots y_n) \prod_{j=1}^{k} I_{S_j^{\zeta}(y_1, \dots, y_n)}(\chi_j(y_1 \dots y_n)^{-1}\xi_j) \\ & \prod_{j=1}^{l} I_{\left(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*\right)}((y_{1,j} \dots y_{n,j})^{-1}\zeta_l) \\ & m(y_1) \dots m(y_n) d^*y_1 \dots d^*y_n.\end{aligned}$$

The next step is to give a probabilistic interpretation of (40). Towards that let us consider a sequence of independent identically $\left(\mathbb{Q}_p^*\right)^l$-valued random variables $Y_1, Y_2, \dots$ with distribution on $\left(\mathbb{Q}_p^*\right)^l$ given by

$$(41) \qquad \mathbb{P}[Y_j \in dy] = m(y)d^*y$$

where $m(y)$ is as in (34). Observe that by (35) we have $\int m(y)d^*y = 1$. The formula (40) can be rewritten

$$\begin{aligned}\varphi_{n+1}(g) \quad = \quad & \alpha\delta^{-1/2}(\zeta)\mathbb{E}\Bigg[\prod_{j=1}^{k} Min\Bigg(1; min\Bigg(\min_{1\le i\le n-1}\Big(|\chi_j(Y_{i+1}...Y_n)|\max(1,|\chi_j(Y_i)|)\Big);\\ & \max(1,|\chi_j(Y_n)|)\Bigg)\min(1,|\chi_j(Y_1...Y_n)^{-1}\zeta|)\Bigg)\\ & \delta^{-1/2}(Y_1\dots Y_n)\prod_{j=1}^{k} I_{S_j^{\zeta}(Y_1,\dots,Y_n)}(\chi_j(Y_1...Y_n)^{-1}\xi_j)\\ & \prod_{j=1}^{l} I_{\left(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*\right)}((Y_{1,j}\dots Y_{n,j})^{-1}\zeta_j)\Bigg].\end{aligned}$$

In the case $g = e$ the above formula simplifies considerably and we obtain

$$\begin{aligned}\varphi_{n+1}(e) \le C\mathbb{E}\Bigg(&\prod_{j=1}^{k}\min\left[1,\min_{1\le i\le n}\left|\chi_j(Y_i...Y_n)\right|\right]\\ &\times\prod_{j=1}^{l} I_{\left(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*\right)}((Y_{1,j}\dots Y_{n,j})^{-1})\Bigg).\end{aligned}$$

Now we project the random walk $Y_1...Y_n$ controlled by (41) on $\mathbb{Z}^l\subset\mathbb{R}^l$ and we assume that $\mathbb{R}^l$ is equipped with the Euclidean structure ensuring (9). We shall denote by $S_n = X_1 + X_2 + ... + X_n$ ($n \ge 1$) the random walk obtained via this projection. We have

$$\varphi_{n+1}(e) \le C\mathbb{E}\left(\prod_{j=1}^{k}\min_{1\le i\le n} p^{L_j(S_n-S_i)},\quad S_n\in K\right),\quad n = 1,2,...$$

where K denotes a neighbourhood of the origin in $\mathbb{Z}^l$. It follows then that

$$\varphi_{n+1}(e) \le C\mathbb{E}\left(\prod_{j=1}^{k} p^{-\max_{1\le i\le n} L_j(S_i)},\quad S_n\in K\right),\quad n = 1,2,...$$

and therefore

(42)
$$\begin{aligned}\varphi_{n+1}(e) \le C\sum_{\lambda_1\ge 0,\dots,\lambda_k\ge 0} p^{-\sum_{i=1}^{k}\lambda_i}\mathbb{P}\Bigg(&\max_{0\le i\le n} L_1(S_i) = \lambda_1,\dots,\\ &\max_{0\le i\le n} L_k(S_i) = \lambda_k;\quad S_n\in K\Bigg),\quad n = 1,2,...\end{aligned}$$

Let us now fix $x_0\in\Pi$ such that

$$L_j(x_0) > 0,\quad j = 1,...,k.$$

Let

$$\lambda_0 = \min_{1\le j\le k} L_j(x_0).$$

We have

$$L_j(u_0) = L_j(x_0/\lambda_0) \geq 1.$$

It follows then form (42) that

$$\begin{aligned}\varphi_{n+1}(e) &\leq C \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} p^{-\sum_{i=1}^k \lambda_i} \mathbb{P}\Big(L_1(S_i) \leq \lambda_1 L_1(u_0), \ldots, L_k(S_i) \leq \lambda_k L_k(u_0), \\ &\qquad 0 \leq i \leq n; \quad S_n \in K \Big) \\ &\leq C \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} p^{-\sum_{i=1}^k \lambda_i} \mathbb{P}\Big(L_1(S_i - (\max_{1 \leq i \leq k} \lambda_i + C) u_0) \leq 0, \ldots, \\ &\qquad L_k(S_i - (\max_{1 \leq i \leq k} \lambda_i + C) u_0) \leq 0, \quad 0 \leq i \leq n; \quad S_n \in K \Big).\end{aligned}$$

Choosing $C > 0$ large enough, we can apply Theorem 4 of [26] and we deduce

$$\begin{aligned}\varphi_{n+1}(e) &\leq \frac{C}{n^{1+\frac{\sqrt{(l-2)^2+4\lambda}}{2}}} \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} \left(\max_{1 \leq i \leq k} \lambda_i + 1 \right)^C p^{-\sum_{i=1}^k \lambda_i} \\ &\leq \frac{C}{n^{1+\frac{\sqrt{(l-2)^2+4\lambda}}{2}}} \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} (\sum_{i=1}^k \lambda_i + 1)^C p^{-\sum_{i=1}^k \lambda_i} \\ &\leq \frac{C}{n^{1+\frac{\sqrt{(l-2)^2+4\lambda}}{2}}}.\end{aligned}$$

This completes the proof of Theorem 2.

References

[1] G. Alexopoulos. A lower estimate for central probabilities on polycyclic groups. *Canad. J. Math.* **44** (1992), 897-910. MR1186471

[2] G. Bachman. *Introduction to p-Adic Numbers and Valuation Theory*. Academic Press, 1964. MR0169847

[3] A. Borel, J. Tits. Groupes réductifs. *Publications Math. IHES* **27** (1965), 55-150. MR0207712

[4] J. W. S. Cassels. *Local Fields*. Camb. Univ. Press, 1986. MR0861410

[5] C. Chevalley. *Théorie des Groupes de Lie, tome III*. Hermann, 1955. MR0068552

[6] Th. Coulhon, A. Grigor'yan. On diagonal lower bounds for heat kernels on non-compact manifolds and Markov chains. *Duke Math. J.* **89** (1997), 133-199. MR1458975

[7] Th. Coulhon, A. Grigor'yan, Ch. Pittet. A geometric approach to on-diagonal heat kernel lower bounds on groups. *Ann. Inst. Fourier* **51** (2001), 1763-1827. MR1871289

[8] R.I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR, Ser. Mat* **48** (1984), 939-985. MR0764305

[9] Y. Guivarc'h. Croissance polynomiale et périodes des fonctions harmoniques. *Bull. Soc. Math. France* **101** (1973), 333-379. MR0369608

[10] W. Hebisch. On heat kernels on Lie groups. *Math. Zeit.* **210** (1992), 593-605. MR1175724

[11] W. Hebisch, L. Saloff-Coste. Gaussian estimates for Markov chains and random walks on groups. *Annals of Probability* **21** (1993), 673-709. MR1217561

[12] N. Jacobson. *Lie Algebras*, Interscience Publishers, 1962. MR0143793

[13] J. Jenkins. Growth of connected locally compact groups. *J. Funct. Anal.* **12** (1973), 113-127. MR0349895

[14] S. Mustapha. Random walks on unimodular p-adic groups. *Stochastic Process and their Applications* **115**, (2005) 927-937. MR2134485

[15] S. Mustapha. Bornes inférieures pour les marches aléatoires sur les groupes p-adiques moyennables. *Ann. Inst. H. Poincaré Probab. Statist.* **42** (2006), 81-88. MR2196972

[16] Ch. Pittet, L. Saloff-Coste. On random walks on wreath products. *Annals of Probability*, **30** (2002), 948-977. MR1905862

[17] Ch. Pittet, Saloff-Coste. Random walks on finite rank solvable groups. *Jour. Eur. Math. Soc.* **4** (2003), 313-342. MR2017850

[18] C. R. E. Raja. On classes of p-adic Lie groups. *New York J. Math* **5** (1999), 101-105. MR1703206

[19] H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups*. Oxford, Math. Monograph, 1968. MR1802924

[20] J.-P. Serre. *Lie Algebras and Lie Groups*. Benjamin, New York, 1965. MR0218496

[21] M. H. Taibleson. *Fourier analysis on local fields*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1975. MR0487295

[22] N. Th. Varopoulos. Random walks on soluble groups. *Bull. Sci. Math.* **107** (1983), 337-344. MR0732356

[23] N. Th. Varopoulos. Groups of superpolynomial growth. *Harmonic analysis* (Sendai, 1990), ICM Satellite Conference Proceedings, Springer, 1991. MR1261441

[24] N. Th. Varopoulos. Diffusion on Lie groups I, II, III. *Canad. J. Math.* **46** (1994), 438-448, 1073-1092; **48** (1996), 641-672. MR1271225, MR1295132, MR1402333

[25] N. Th. Varopoulos. Wiener-Hopf theory and nonunimodular groups. *J. Func. Analysis* **120** (1994), 467-483. MR1266317

[26] N. Th. Varopoulos. Potential theory in conical domains. *Math. Proc. Cambridge Philos. Soc.* **125** (1999), 335-384. MR1643806

[27] N. Th. Varopoulos. Marches aléatoires et diffusions dans les domaines lipschitziens. *C. R. Acad. Sci. Paris Sér. I Math* **332** (2001), 357-360. MR1821476

[28] N. Th. Varopoulos, L. Saloff-Coste, Th. Coulhon. *Analysis and geomety on groups*. Cambridge tracts in mathematics, 102, 1993. MR1218884