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A Matrix Representation of the Bercovici-Pata Bijection

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Abstract

Let μ be an infinitely divisible law on the real line, $\Lambda(\mu)$ its freely infinitely divisible image by the Bercovici-Pata bijection. The purpose of this article is to produce a new kind of random matrices with distribution μ at dimension 1, and with its empirical spectral law converging to $\Lambda(\mu)$ as the dimension tends to infinity. This constitutes a generalisation of Wigner's result for the Gaussian Unitary Ensemble.

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I Introduction

Wigner and Arnold have established (*cf* [2], [25]) that the empirical spectral law of matrices of the Gaussian Unitary Ensemble converges almost surely to the semi-circular law. To be more precise, let us consider W_n, a $n \times n$ Hermitian Gaussian matrix such that $\mathbb{E}\left[\exp\left(i\mathrm{tr} AW_n\right)\right] = \exp\left(-\frac{1}{n}\mathrm{tr}A^2\right)$, where tr denotes the ordinary trace on matrices; let $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ be its eigenvalues, and $\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i^{(n)}}$ be its empirical spectral law. Then

$$\mathbb{E} \circ \hat{\mu}_1 = \mathcal{N}(0,1) \text{ and } \lim_{n \to +\infty} \hat{\mu}_n = SC(0,1) \text{ a.s.}$$

where $SC(0,1)$ is the standard centered semi-circular distribution. This distribution corresponds to the Gaussian law in the framework of free probability. For instance, it arises as the asymptotic law in the free central limit theorem. Moreover, Wigner's result has been reinterpreted by Voiculescu in the early nineties, using the concept of asymptotic freeness (*cf* [24]).

A few years ago, this correspondence between Gaussian and semi-circular laws has been extended to infinitely divisible laws on $\mathbb{R}$ by Bercovici and Pata (*cf* [7]) using the Lévy-Hinçin formulas. The so-called Bercovici-Pata bijection maps any classically infinitely divisible law μ to the probability measure $\Lambda(\mu)$, as characterized by Barndorff-Nielsen and Thorbjørnsen in [4] by:

$$\forall \zeta < 0, \quad i\zeta R_{\Lambda(\mu)}(i\zeta) = \int_0^{+\infty} C^*(\mu)(\zeta x)\exp(-x)\,dx.$$

In this formula, $C^*(\mu)$ denotes the classical cumulant transform of μ, and $R_{\Lambda(\mu)}$ denotes Voiculescu's R-transform of $\Lambda(\mu)$:

$$C^*(\mu) = \ln \mathcal{F}_\mu \text{ and } R_{\Lambda(\mu)} = G_{\Lambda(\mu)}^{-1}(z) - \frac{1}{z}$$

where $\mathcal{F}_\mu$ is the Fourier transform of μ, and $G_{\Lambda(\mu)}$ the Cauchy transform of $\Lambda(\mu)$.

It turns out that $\Lambda(\mu)$ is freely infinitely divisible: for each k there exists a probability measure ν_k such that $\Lambda(\mu) = \nu_k^{\boxplus k}$, where $\boxplus$ denotes the free convolution. The Bercovici-Pata bijection has many interesting features, among them the fact that $\Lambda(\mu)$ is freely stable if μ is stable, and $\Lambda(\mu) = SC(0,1)$ if $\mu = \mathcal{N}(0,1)$. The reader who is not very familiar with free convolution or the Bercovici-Pata bijection will find in [3] a most informative exposition of the subject.

The aim of this paper is to propose a new kind of matrix ensembles; between classically and freely infinitely divisible laws connected through the Bercovici-Pata bijection, those ensembles establish a sequence similar to the Gaussian Unitary Ensemble between the Gaussian and semi-circular laws. More specifically, we will show that, for each integer n and infinitely divisible law μ, an Hermitian random matrix $X_n^{(\mu)}$ of size n can be produced such that its empirical spectral law $\hat{\mu}_n$ satisfies

$$\mathbb{E} \circ \hat{\mu}_1 = \mu \text{ and } \lim_{n \to +\infty} \hat{\mu}_n = \Lambda(\mu) \text{ a.s.} \tag{I.1}$$

This main result of our paper is stated in theorem III.2. We hope to have achieved this goal in a rather canonical way, even if $X_n^{(\mathcal{N}(0,1))}$ is not equal to W_n, but to a slight modification of it. Here are two facts which may justify this opinion:

1. For infinitely divisible measures with moments of all order, it is easy to describe the Bercovici-Pata bijection by noting that the classical cumulants of μ are equal to the free cumulants of $\Lambda(\mu)$. We will define some kind of matrix cumulants, directly inspired by Lehner's recent work ([18]), and with the property that the matrices $X_n^{(\mu)}$ have the same cumulants as μ.

2. If μ is the Cauchy law, then the classical convolution $\mu*$ and the free one $\mu\boxplus$ coincide. We will prove a simple but somehow surprising result, namely that for each n they coincide also with the convolution with respect to $X_n^{(\mu)}$.

Using these Lévy matrix ensembles as a link between classical and free frameworks, it is natural to expect to derive free properties from classical ones. For instance, there is an intimate connection between the moments of a classically infinitely divisible law and those of its Lévy measure. We shall present here how this yields an analogous result in the free framework.

The rest of the paper is organised in 4 sections. Section 2 is devoted to the definition of our matricial laws and their elementary properties. Section 3 states and proves the main theorem: the almost sure convergence of the empirical spectral laws. The proof is based on stochastic ingredients; it first establishes a concentration result and then determines the asymptotic behaviour of the Cauchy transform of the empirical spectral laws. This approach differs from the moment method used by Benaych-Georges for studying the same matrices (*cf* [5]). Section 4 presents further interesting features of these matrix ensembles, concerning the cumulants and the Cauchy convolution. And section 5 explains the application to moments of the Lévy measure.

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II New matrix ensembles

Notations

- The set of Hermitian matrices of size n will be denoted by $\mathcal{H}_n$, the subset of positive ones by $\mathcal{H}_n^+$.

- The set of the classically infinitely divisible laws on $\mathbb{R}$ (resp. $\mathcal{H}_n$) will be denoted by $\mathcal{ID}_*(\mathbb{R})$ (resp. by $\mathcal{ID}_*(\mathcal{H}_n)$). The set of the freely infinitely divisible laws on $\mathbb{R}$ will be denoted by $\mathcal{ID}_\boxplus(\mathbb{R})$.

- The normalised trace on square matrices will be denoted by tr_n (with $\mathrm{tr}_n(Id) = 1$).

II.A Definition

Let X be a real random variable with distribution $\mu \in \mathcal{ID}_*(\mathbb{R})$, and $\varphi^{(\mu)}$ be its Lévy exponent

$$\mathbb{E}\left[e^{i\lambda X}\right] = e^{\varphi^{(\mu)}(\lambda)}.$$

By the Lévy-Hinçin formula, there exist a unique finite measure $\sigma_*^{(\mu)}$ and a unique real $\gamma_*^{(\mu)}$ such that

$$\varphi^{(\mu)}(\lambda) = i\gamma_*^{(\mu)}\lambda + \int_{-\infty}^{+\infty}\left(e^{it\lambda} - 1 - \frac{it\lambda}{1+t^2}\right)\frac{1+t^2}{t^2}\sigma_*^{(\mu)}(dt),$$

with the convention that

$$\left(e^{it\lambda} - 1 - \frac{it\lambda}{1+t^2}\right)\frac{1+t^2}{t^2}\Big|_{t=0} = -\frac{\lambda^2}{2}.$$

Conversely, $\mu_*(\gamma,\sigma)$ denotes the classically infinitely divisible law determined by γ and σ, and for the sake of simplicity we shall write $\varphi^{(\gamma,\sigma)}$ instead of $\varphi^{(\mu_*(\gamma,\sigma))}$.

Let V_n be a random vector Haar distributed on the unit sphere $S_{2n-1} \subset \mathbb{C}^n$. Let us define for every Hermitian matrix $A \in \mathcal{H}_n$

$$\begin{aligned}
\varphi_n^{(\mu)}(A) &= n\mathbb{E}\left[\varphi^{(\mu)}\left(\langle V_n, AV_n\rangle\right)\right] \\
&= i\gamma_*^{(\mu)}\mathrm{tr}A \\
&\quad + \int_{-\infty}^{+\infty}\left(n\left(\mathbb{E}\left[e^{it\langle V_n, AV_n\rangle}\right] - 1\right) - \frac{it}{1+t^2}\mathrm{tr}A\right)\frac{1+t^2}{t^2}\sigma_*^{(\mu)}(dt),
\end{aligned}$$

with the convention that

$$\left(n\left(\mathbb{E}\left[e^{it\langle V_n, AV_n\rangle}\right] - 1\right) - \frac{it}{1+t^2}\mathrm{tr}A\right)\frac{1+t^2}{t^2}\Big|_{t=0} = -\frac{n\mathbb{E}\left[\langle V_n, AV_n\rangle^2\right]}{2}.$$

Due to the Lévy-Hinçin formula, $\varphi_n^{(\mu)}$ is the Lévy exponent of a classically infinitely divisible measure on $\mathcal{H}_n$, which will be denoted by $\Lambda_n(\mu)$. This means:

$$\forall A \in \mathcal{H}_n, \quad \Lambda_n(\mu)\left(\exp\left(i\mathrm{tr}A\cdot\right)\right) = \exp\varphi_n^{(\mu)}(A)$$

The following properties are obvious:

Properties II.1 *1. $\Lambda_n(\mu)$ is invariant by conjugation with a unitary matrix.*

2. If $X_n^{(\mu)}$ has distribution $\Lambda_n(\mu)$, then $\mathrm{tr}X_n^{(\mu)}$ has distribution μ^{*n}.

3. If $\mu, \nu \in \mathcal{ID}_*(\mathbb{R})$, then $\Lambda_n(\mu) * \Lambda_n(\nu) = \Lambda_n(\mu * \nu)$.

4. If D_c denotes the dilation by c, then $D_c\Lambda_n(\mu) = \Lambda_n\left(D_c\mu\right)$.

5. If $\mu = \delta_a$, then $\Lambda_n(\mu) = \delta_{aI_n}$, where I_n is the identity matrix.

6. If μ is Gaussian (resp. stable, self-decomposable, infinitely divisible) then so is $\Lambda_n(\mu)$.

II.B Explicit realisation

The following proposition describes how to construct a random matrix with distribution $\Lambda_n(\mu)$ using μ-distributed random variables:

Proposition II.2 *Let us consider a probability measure μ in $\mathcal{ID}_*(\mathbb{R})$, a positive real s, n independent random variables $\lambda_1(s), \ldots, \lambda_n(s)$ with distribution μ^{*s}, and the diagonal matrix $L(s)$ with diagonal entries $(\lambda_1(s), \ldots, \lambda_n(s))$. Let U be an independent random matrix of size n, Haar distributed on the unitary group $\mathcal{U}(n)$, and M_s be the Hermitian matrix $UL(s)U^*$. Then $\Lambda_n(\mu)$ is the weak limit of the law of $\sum_{k=1}^p M_{\frac{1}{p}}^{(k)}$ when p tends to infinity, with $\left(M_{\frac{1}{p}}^{(k)}, k = 1, \ldots, p\right)$ independent replicas of $M_{\frac{1}{p}}$.*

We would like to explain how this realisation suggests our main result (*cf* (I.1) and theorem III.2), just by letting the dimension n tend to infinity before letting the number p of replicas tend to infinity. Because of the law of large numbers, it is obvious that the empirical spectral law of each matrix $M_{\frac{1}{p}}^{(k)}$ converges a.s. to $\mu^{*\frac{1}{p}}$; since these matrices $M_{\frac{1}{p}}^{(k)}, k = 1, \ldots, p$ are also conjugated by unitary independent matrices, they are (in a way which remains to be specified) asymptotically free (*cf* proof of lemma V.4); hence, as n tends to infinity, the asymptotic law of $\sum_{k=1}^p M_{\frac{1}{p}}^{(k)}$ is likely to be $\left(\mu^{*\frac{1}{p}}\right)^{\boxplus p}$; now, let p tend to infinity; it is an immediate consequence of theorem 3.4 in [7] that $\lim_{p\to+\infty}\left(\mu^{*\frac{1}{p}}\right)^{\boxplus p} = \Lambda(\mu)$: this is precisely what is expected to be the limit of the empirical spectral law associated to $\Lambda_n(\mu)$ in theorem III.2.

Proof of proposition II.2. We only need to prove the convergence of the characteristic function

$$\mathbb{E}\left[\exp\left(i\mathrm{tr}\left(A\sum_{k=1}^p M_{\frac{1}{p}}^{(k)}\right)\right)\right] = \mathbb{E}\left[\exp\left(i\mathrm{tr}\left(AM_{\frac{1}{p}}\right)\right)\right]^p.$$

Notice that

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\mathrm{tr}\left(AM_{\frac{1}{p}}\right)\right)\right] &= \mathbb{E}\left[\exp\left(i\mathrm{tr}\left(AUL\left(1/p\right)U^*\right)\right)\right] \\
&= \mathbb{E}\left[\exp\left(i\sum_{l=1}^n (U^*AU)_{l,l}\lambda_l(1/p)\right)\right] \\
&= \mathbb{E}\left[\exp\left(\frac{1}{p}\sum_{l=1}^n \varphi^{(\mu)}\left((U^*AU)_{l,l}\right)\right)\right].
\end{aligned}$$

Therefore, noting that $\varphi^{(\mu)}$ is locally bounded for an infinitely divisible law μ, we obtain:

$$\begin{aligned}
\lim_{p\to+\infty} \mathbb{E}\left[\exp\left(i\mathrm{tr}\left(A\sum_{k=1}^{p} M^{(k)}_{\frac{1}{p}}\right)\right)\right] &= \lim_{p\to+\infty} \mathbb{E}\left[\exp\left(\frac{1}{p}\sum_{l=1}^{n}\varphi^{(\mu)}\left((U^*AU)_{l,l}\right)\right)\right]^p \\
&= \exp\left(\mathbb{E}\left[\sum_{l=1}^{n}\varphi^{(\mu)}\left((U^*AU)_{l,l}\right)\right]\right) \\
&= \exp\left(n\mathbb{E}\left[\varphi^{(\mu)}\left(\langle V_n, AV_n\rangle\right)\right]\right) \\
&= \exp\left(\varphi_n^{(\mu)}(A)\right).
\end{aligned}$$

□

Remark. This explicit realisation has been used by Benaych-Georges in [5] to establish convergence results and to deal with the non-Hermitian case.

II.C Examples

- If $\mu = \mathcal{N}(0,1)$, we get:

$$\varphi_n^{(\mu)}(A) = -\frac{n}{2}\mathbb{E}\left[\langle V_n, AV_n\rangle^2\right] = -\frac{n}{2}\mathbb{E}\left[\left(\sum_{l=1}^{n} a_i|v_i|^2\right)^2\right],$$

  where $a_1,\dots,a_n$ are the eigenvalues of A, and $v_1,\dots,v_n$ the entries of the random vector V_n Haar distributed on the unit sphere. Now, using for instance [14] proposition 4.2.3, we know that:

$$\begin{aligned}
\mathbb{E}\left[|v_i|^4\right] &= \frac{2}{n(n+1)} \\
\mathbb{E}\left[|v_i|^2|v_j|^2\right] &= \frac{1}{n(n+1)} \text{ if } i\neq j.
\end{aligned}$$

  Therefore, we obtain:

$$\varphi_n^{(\mu)}(A) = -\frac{1}{2(n+1)}\sum_{l=1}^{n} a_i^2 - \frac{1}{2(n+1)}\left(\sum_{l=1}^{n} a_i\right)^2 = -\frac{1}{n+1}\frac{\mathrm{tr}\left(A^2\right)}{2} - \frac{1}{n+1}\frac{(\mathrm{tr}A)^2}{2}.$$

  This implies that $\Lambda_n(\mu)$ is the distribution of

$$\sqrt{\frac{n}{n+1}}W_n + \sqrt{\frac{1}{n+1}}gI,$$

  where W_n is the Gaussian random matrix described in the introduction, and g is a centered reduced Gaussian variable, g and W_n being independent.

- If μ is the standard Cauchy distribution

$$\mu(dx) = \frac{1}{\pi(1+x^2)}dx,$$

then $\varphi^{(\mu)}(\lambda) = -|\lambda|$, and

$$\varphi_n^{(\mu)}(A) = -n\mathbb{E}\left[|\langle V_n, AV_n\rangle|\right] = -n\mathbb{E}\left[\left|\sum_{l=1}^{n} a_i|v_i|^2\right|\right],$$

the notations being the same as in the previous example. This can be evaluated using the general method proposed by Shatashvili (*cf* [22]):

$$\begin{aligned} n\mathbb{E}\left[|\langle V_n, AV_n\rangle|\right] &= \sum_{l=1}^{n} \frac{a_l^{n-1}|a_l|}{\prod_{m\neq l}(a_l - a_m)} \\ &= \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} \\ a_1^{n-1}|a_1| & a_2^{n-1}|a_2| & \cdots & a_n^{n-1}|a_n| \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}} \end{aligned}$$

where $a_1, \ldots, a_n$ are the eigenvalues of A. There is a remarkable consequence of this formula: if we denote by $X_n^{(\mu)}$ a random matrix of law $\Lambda_n(\mu)$, then we have

$$\left(X_{n+p}^{(\mu)}(i,j)\right)_{1\leq i,j\leq n} = X_n^{(\mu)} \text{ in law.}$$

Such a property appears usually with a normalisation coefficient; but here there is none, as if μ were a Dirac mass.

- If μ is the standard Poisson distribution, then $\varphi^{(\mu)}(\lambda) = e^{i\lambda} - 1$ and

$$\varphi_n^{(\mu)}(A) = n\mathbb{E}\left[e^{i\langle V_n, AV_n\rangle} - 1\right].$$

Due to the Lévy-Ito decomposition, there is a simple explicit representation of $\Lambda_n(\mu)$. Let $(X(s), s \geq 0)$ be a standard Poisson process, and let $(V_n(p), p \in \mathbb{N})$ be a family of independent unitary Haar distributed random vectors, $(X(s), s \geq 0)$ and $(V_n(p), p \in \mathbb{N})$ being independent. Then $\Lambda_n(\mu)$ is the law of

$$\sum_{p=1}^{X(n)} V_n(p)V_n(p)^*.$$

In [19], Marçenko and Pastur studied a very similar set of random matrices, precisely of the form

$$\sum_{p=1}^{q} V_n(p)V_n(p)^*$$

with q/n converging when n tends to infinity. It appends that both families of random matrices have same asymptotic empirical spectral distribution, the Marçenko-Pastur law or free Poisson law.

III Convergence to the free Lévy laws

III.A The Bercovici-Pata bijection

Let us consider a real γ and a finite measure σ on $\mathbb{R}$, and let $\mu \equiv \mu_*(\gamma, \sigma)$ be the associated classically infinitely divisible law with Lévy exponent $\varphi^{(\gamma,\sigma)}$. Then its image through the Bercovici-Pata bijection is the probability measure on the real line $\Lambda(\mu)$ such that (*cf* [7])

$$\forall z \in \mathbb{C}^+, \quad \mathcal{R}(\Lambda(\mu))(z) = \gamma + \int_{-\infty}^{+\infty} \frac{z+t}{1-tz}\sigma(dt),$$

where $\mathcal{R}$ is Voiculescu's R-transform. The mapping Λ is a bijection between classically and freely infinitely divisible laws (*cf* [7]). Moreover, it has the following properties which we shall recall for comparison with properties II.1:

Properties III.1 *(cf [3], [7])*

1. *If μ and ν are freely infinitely divisible measures, then $\Lambda(\mu) \boxplus \Lambda(\nu) = \Lambda(\mu * \nu)$.*

2. *If D_c denotes the dilation by c, then $D_c\Lambda_n(\mu) = \Lambda(D_c\mu)$.*

3. *If $\mu = \delta_a$, then $\Lambda(\mu) = \delta_a$.*

4. *If μ is Gaussian (resp. classically stable, classically self-decomposable, classically infinitely divisible) then $\Lambda(\mu)$ is semi-circular (resp. freely stable, freely self-decomposable, freely infinitely divisible).*

For instance, if $\mu = \mathcal{N}(0,1)$, then $\Lambda(\mu) = SC(0,1)$, as has been already noticed. If μ is the Cauchy distribution, then $\Lambda(\mu) = \mu$. If μ is the Poisson distribution with parameter θ, then $\Lambda(\mu)$ is the Marçenko-Pastur distribution

$$\Lambda(\mu)(dx) = (1-\theta)^+ \delta_0(x) + \frac{\sqrt{4\theta - (x-1-\theta)^2}}{2\pi\theta x} 1_{\left[(1-\sqrt{\theta})^2, (1+\sqrt{\theta})^2\right]}(x)dx.$$

We can now state the main theorem of the article:

Theorem III.2 *For any probability measure μ in $\mathcal{ID}_*(\mathbb{R})$, let $X_n^{(\mu)}$ be a random matrix defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with distribution $\Lambda_n(\mu)$. Then its empirical spectral law $\hat{\mu}_n$ converges weakly almost surely to $\Lambda(\mu)$ as n tends to infinity.*

The proof of this theorem is based on semigroup tools. A Lévy process can be associated to each matricial Lévy law. Its semigroup is characterised by an infinitesimal generator with a rather simple explicit form. This will be a key ingredient.

III.B The infinitesimal generator

Let γ be a real, let σ be a finite measure on $\mathbb{R}$, let μ be the probability measure in $\mathcal{ID}_*(\mathbb{R})$ such that $\mu = \mu_*(\gamma, \sigma)$. Let $\Lambda_n(\mu)$ be the associated infinitely divisible law on $\mathcal{H}_n$. Then there exists a Lévy process $(X_n^{(\mu)}(s), s \geq 0)$ such that $X_n^{(\mu)}(1)$

has distribution $\Lambda_n(\mu)$ (see e.g. corollary 11.6 in [21]). The corresponding semigroup will be denoted by $P_s^{\Lambda_n(\mu)}$:

$$\forall A \in \mathcal{H}_n, \quad P_s^{\Lambda_n(\mu)}(f)(A) = \mathbb{E}\left[f\left(X_n^{(\mu)}(s) + A\right)\right]$$

The following formula is a classical result of the theory (see e.g. theorem 31.5 in [21]):

Proposition III.3 *The infinitesimal generator $\mathcal{A}_n^{(\mu)}$ associated to the semigroup $(P_s^{\Lambda_n(\mu)}, s \geq 0)$ has core $C_0^2(\mathcal{H}_n, \mathbb{C})$ - the set of twice continuously differentiable functions vanishing at infinity - and is defined by*

$$\forall f \in C_0^2(\mathcal{H}_n, \mathbb{C}),\ \forall A \in \mathcal{H}_n,$$

$$\begin{aligned}
& \mathcal{A}_n^{(\mu)}(f)(A) \\
= \;& \gamma df(A)[I_n] + \int_{-\infty}^{+\infty} \left(n\left(\mathbb{E}\left[f\left(A + tV_nV_n^*\right)\right] - f(A)\right) - \frac{t}{1+t^2} df(A)[I_n] \right) \frac{1+t^2}{t^2}\sigma(dt) \\
= \;& \gamma df(A)[I_n] + \frac{\sigma(\{0\})}{2(n+1)} \left(\sum_{i,j=1}^n d^2 f(A)[E_{i,j}][E_{j,i}] + \sum_{i=1}^n d^2 f(A)[E_{i,i}][E_{i,i}] \right) \\
& + \int_{t \in \mathbb{R}^*} \left(n\left(\mathbb{E}\left[f\left(A + tV_nV_n^*\right)\right] - f(A)\right) - \frac{t}{1+t^2} df(A)[I_n] \right) \frac{1+t^2}{t^2}\sigma(dt).
\end{aligned}$$

Remark. If a function $f \in L(\mathbb{R}, \mathbb{R})$ is differentiable, and if we extend its domain to $\mathcal{H}_n$ using spectral calculus, then $df(A)[I_n] = f'(A)$.

III.C A concentration result

The first step of the proof of theorem III.2 is to establish a concentration result for the empirical spectral law, which has some interest by himself.

Theorem III.4 *Let μ be a probability measure in $\mathcal{ID}_*(\mathbb{R})$, and let $(X_n^{(\mu)}(s), s \geq 0)$ be the Lévy process associated to distribution $\Lambda_n(\mu)$. Let us consider a Lipschitz function $f \in L(\mathbb{R}, \mathbb{R})$ with finite total variation, and let us define $f^{(n)} = \mathrm{tr}_n f \in L(\mathcal{H}_n, \mathcal{H}_n)$ by spectral calculus. Then, for all $\tau, \varepsilon > 0$, there exists $\delta_\varepsilon(\tau) > 0$ such that for each $n \geq 1$*

$$\mathbb{P}\left(\left|f^{(n)}\left(X_n^{(\mu)}(\tau)\right) - \mathbb{E}\left[f^{(n)}\left(X_n^{(\mu)}(\tau)\right)\right]\right| > \varepsilon\right) \leq 2e^{-n\delta_\varepsilon(\tau)}.$$

Moreover, $\delta_\varepsilon(\tau)$ is non-increasing in τ.

We need a preliminary lemma:

Lemma III.5 *Let $f \in L(\mathbb{R}, \mathbb{R})$ be a Lipschitz function with finite total variation. We shall set*

$$\|f\|'_\infty = \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right| \quad \text{and} \quad \|f\|'_1 = \sup_{\substack{n \geq 1 \\ x_1 \leq y_1 \leq x_2 \leq \cdots \leq y_n}} \sum_{i=1}^n |f(y_i) - f(x_i)|.$$

Then for all $A, B \in \mathcal{H}_n$, we have

$$|\mathrm{tr} f(A+B) - \mathrm{tr} f(A)| \leq \begin{cases} \mathrm{rg}(B)\|f\|'_1 \\ \\ \|B\|_1 \|f\|'_\infty \end{cases}$$

with $\|B\|_1 = \mathrm{tr}\,|B|$.

If we replace $\|B\|_1$ by $\sqrt{\mathrm{rg}(B)}\|B\|_2$, then the second inequality reduces to the one already established in a similar context by Guionnet and Zeitouni in lemma 1.2 of [13].

Proof. We only need to prove the lemma for B non-negative of rank one, then the result follows by an immediate induction. Let $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$ be the eigenvalues of A and $A+B$ respectively. Following Weyl's well-known inequalities (see e.g. section III.2 of [8]), we have that $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \mu_n$. Therefore, we get:

$$\begin{aligned} |\mathrm{tr} f(A+B) - \mathrm{tr} f(A)| &= \left| \sum_{i=1}^n f(\mu_i) - f(\lambda_i) \right| \\ &\leq \sum_{i=1}^n |f(\mu_i) - f(\lambda_i)| \\ &\leq \begin{cases} \|f\|'_1 \\ \\ \sum_{i=1}^n |\mu_i - \lambda_i| \cdot \|f\|'_\infty = |\sum_{i=1}^n \mu_i - \lambda_i| \cdot \|f\|'_\infty \\ = |\mathrm{tr} B| \cdot \|f\|'_\infty = \|B\|_1 \|f\|'_\infty \end{cases} \end{aligned}$$

□

Proof of theorem III.4. We will use semigroup tools to establish the concentration property. This method is well-known, see e.g. [16].

We can assume that $\tau = 1$ and $\mathbb{E}\left[f^{(n)}\left(X_n^{(\mu)}(1)\right)\right] = 0$. Let us define for any $\lambda \in \mathbb{R}$ and $s \in [0,1]$

$$\phi_\lambda(s) = P_s^{\Lambda_n(\mu)}\left(\exp\left(\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})\right)\right)(0).$$

Notice that

$$\phi_\lambda(0) = \exp\left(\lambda \mathbb{E}\left[f^{(n)}\left(X_n^{(\mu)}(1)\right)\right]\right) = 1 \text{ and } \phi_\lambda(1) = \mathbb{E}\left[\exp\left(\lambda f^{(n)}(X_n^{(\mu)}(1)\right)\right].$$

Since the semigroup $(P_s^{\Lambda_n(\mu)}, s \geq 0)$ and its generator $\mathcal{A}_n^{(\mu)}$ commute, we obtain:

$$\begin{aligned}
\phi'_\lambda(s) &= P_s^{\Lambda_n(\mu)} \circ \mathcal{A}_n^{(\mu)} \left(e^{\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})} \right)(0) \\
&\quad - P_s^{\Lambda_n(\mu)} \left(\lambda \mathcal{A}_n^{(\mu)} \circ P_{1-s}^{\Lambda_n(\mu)}(f^{(n)}) e^{\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})} \right)(0) \\
&= P_s^{\Lambda_n(\mu)} \Bigg(\gamma\lambda d \left(P_{1-s}^{\Lambda_n(\mu)}(f^{(n)}) \right)(\cdot)\,[I_n]\, e^{\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})} \\
&\qquad + \int_{-\infty}^{+\infty} \Bigg(n \left(\mathbb{E}\left[e^{\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})(\cdot + tV_nV_n^*)} \right] - e^{\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})} \right) \\
&\qquad\qquad - \lambda \frac{t}{1+t^2} d \left(P_{1-s}^{\Lambda_n(\mu)}(f^{(n)}) \right)(\cdot)\,[I_n]\, e^{\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})} \Bigg) \frac{1+t^2}{t^2} \sigma(dt) \Bigg)(0) \\
&\quad - P_s^{\Lambda_n(\mu)} \Bigg(\gamma\lambda d \left(P_{1-s}^{\Lambda_n(\mu)}(f^{(n)}) \right)(\cdot)[I_n] e^{\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})} \\
&\qquad + \int_{-\infty}^{+\infty} \Bigg(n \left(\lambda \mathbb{E}\left[P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})(\cdot + tV_nV_n^*) \right] - \lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)}) \right) e^{\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})} \\
&\qquad\qquad - \lambda \frac{t}{1+t^2} d \left(P_{1-s}^{\Lambda_n(\mu)}(f^{(n)}) \right)(\cdot)\,[I_n]\, e^{\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})} \Bigg) \frac{1+t^2}{t^2} \sigma(dt) \Bigg)(0) \\
&= P_s^{\Lambda_n(\mu)} \left(n \int_{-\infty}^{+\infty} \mathbb{E}\left[\Psi(s,t,V_n,\cdot) \right] e^{\lambda P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})} \frac{1+t^2}{t^2} \sigma(dt) \right)(0),
\end{aligned}$$

where we set

$$\Psi(s,t,V_n,\cdot) = e^{\lambda\left(P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})(\cdot + tV_nV_n^*) - P_{1-s}^{\Lambda_n(\mu)}(f^{(n)}) \right)} - 1 - \lambda \left(P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})(\cdot + tV_nV_n^*) - P_{1-s}^{\Lambda_n(\mu)}(f^{(n)}) \right)$$

Now, due to lemma III.5, we have:

$$|P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})(. + tV_nV_n^*) - P_{1-s}^{\Lambda_n(\mu)}(f^{(n)})| \leq \frac{1}{n} \left(\|f\|'_1 \wedge |t| \|f\|'_\infty \right).$$

Therefore, since $e^u - 1 - u \leq \frac{u^2}{2} e^u$ for all $u \geq 0$, we get:

$$|\Psi(s,t,V_n,\cdot)| \leq \frac{\lambda^2}{2n^2} \left(\|f\|'^2_1 \wedge t^2 \|f\|'^2_\infty \right) e^{\frac{\lambda}{n}\|f\|'_1}.$$

We can then infer that

$$\phi'_\lambda(s) \leq \frac{\lambda^2}{2n} e^{\frac{\lambda}{n}\|f\|'_1} \phi_\lambda(s) C(\|f\|'_1, \|f\|'_\infty)$$

for a function C defined by

$$C(\alpha,\beta) = \int_{-\infty}^{+\infty} \left(\alpha^2 \wedge t^2\beta^2 \right) \frac{1+t^2}{t^2} \sigma(dt).$$

By integrating $\phi'_\lambda(s)/\phi_\lambda(s)$, the preceding inequality results in:

$$\phi_\lambda(1) \leq \exp\left(\frac{\lambda^2}{2n} e^{\frac{\lambda}{n}\|f\|'_1} C(\|f\|'_1, \|f\|'_\infty) \right).$$

¿From this exponential inequality, we can deduce that, for all $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\left(f^{(n)}\left(X_n^{(\mu)}\right) \geq \varepsilon\right) &\leq \mathbb{E}\left[e^{\lambda f^{(n)}\left(X_n^{(\mu)}\right)-\lambda\varepsilon}\right] = \phi_\lambda(1)e^{-\lambda\varepsilon} \\ &\leq \exp\left(\frac{\lambda^2}{2n}e^{\frac{\lambda}{n}\|f\|_1'}C(\|f\|_1', \|f\|_\infty') - \lambda\varepsilon\right).\end{aligned}$$

But let us remark that

$$\inf_{\lambda\in\mathbb{R}}\left(\frac{\lambda^2}{2n}e^{\frac{\lambda}{n}\|f\|_1'}C(\|f\|_1', \|f\|_\infty') - \lambda\varepsilon\right) = -n\delta_\varepsilon(1),$$

where $\delta_\varepsilon(1)$ is defined by

$$\delta_\varepsilon(1) = \sup_{\lambda\in\mathbb{R}}\left(-\frac{\lambda^2}{2}e^{\lambda\|f\|_1'}C(\|f\|_1', \|f\|_\infty') + \lambda\varepsilon\right) > 0$$

Hence

$$\mathbb{P}\left(f^{(n)}\left(X_n^{(\mu)}\right) \geq \varepsilon\right) \leq e^{-n\delta_\varepsilon(1)}.$$

Furthermore, applying this inequality to $-f$, we can deduce:

$$\mathbb{P}\left(|f^{(n)}\left(X_n^{(\mu)}\right)| \geq \varepsilon\right) \leq 2e^{-n\delta_\varepsilon(1)}.$$

To conclude the proof, let us notice that

$$\delta_\varepsilon(\tau) = \sup_{\lambda\in\mathbb{R}}\left(-\tau\frac{\lambda^2}{2}e^{\lambda\|f\|_1'}C(\|f\|_1', \|f\|_\infty') + \lambda\varepsilon\right),$$

which is obviously non-increasing in τ. □

III.D Proof of theorem III.2

We need two preliminary lemmas:

Lemma III.6 *Let μ be a probability measure in $\mathcal{ID}_*(\mathbb{R})$, let $X_n^{(\mu)}(s)$ be a random matrix with distribution $\Lambda_n(\mu^{*s})$, $\lambda_i(s), i = 1, \ldots, n$ its eigenvalues, and let $(v_i, i = 1, \ldots, n)$ be an independent random vector Haar distributed on the unit complex sphere. Then, for any Lipschitz function f with bounded variations,*

$$\lim_{n\to+\infty}\sup_{s\in[0,1]}\mathbb{E}\left[\left|\sum_{i=1}^n |v_i|^2 f(\lambda_i(s)) - \mathbb{E}\left[\sum_{i=1}^n |v_i|^2 f(\lambda_i(s))\right]\right|\right] = 0.$$

This is a consequence of the concentration result of section III.C, and of the concentration property of the law of $(|v_i|^2, i = 1, \ldots, n)$.

Proof. If $\psi \in L(\mathbb{C}^n, \mathbb{R})$ is a Lipschitz function, with Lipschitz constant $\|\psi\|_\infty'$, then (*cf* e.g. introduction of chapter 1 in [17]),

$$\mathbb{P}\left(|\psi(v_1, \ldots, v_n) - m_\psi| > \varepsilon\right) \leq \sqrt{\frac{\pi}{2}}\exp\left(-\frac{2n-2}{2\|\psi\|_\infty'^2}\varepsilon^2\right),$$

where m_ψ is a median of $\psi(v_1,\ldots,v_n)$. From this we can deduce that

$$|\mathbb{E}[\psi(v_1,\ldots,v_n)-m_\psi]|\leq\int_0^{+\infty}\mathbb{P}(|\psi(v_1,\ldots,v_n)-m_\psi|>\varepsilon)\,d\varepsilon\leq\frac{\pi}{2\sqrt{2n-2}}\|\psi\|'_\infty.$$

Therefore, we get the following inequality:

$$\mathbb{P}(|\psi(v_1,\ldots,v_n)-\mathbb{E}[\psi(v_1,\ldots,v_n)]|>\varepsilon)\leq C\exp\left(-\frac{n-1}{2\|\psi\|'^2_\infty}\varepsilon^2\right),$$

for some constant C. Now, for any $u=(u_1,\ldots,u_n)$ and $u'=(u'_1,\ldots,u'_n)$ on the unit complex sphere, we have:

$$\left|\sum_{i=1}^n|u_i|^2 f(\lambda_i(s))-\sum_{i=1}^n|u'_i|^2 f(\lambda_i(s))\right|\leq\sum_{i=1}^n|u_i-u'_i|\cdot|u_i+u'_i|\cdot\|f\|_\infty\leq 2\|f\|_\infty\sqrt{\sum_{i=1}^n|u_i-u'_i|^2}.$$

Therefore, $(\lambda_1(s),\ldots,\lambda_n(s))$ being fixed, the function $\sum_{i=1}^n|u_i|^2 f(\lambda_i(s))$ is Lipschitz in $(u_1,\ldots,u_n)$. Hence,

$$\mathbb{P}\left(\left|\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))-\mathbb{E}\left[\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))\Big|\lambda_1(s),\ldots,\lambda_n(s)\right]\right|>\varepsilon\Big|\lambda_1(s),\ldots,\lambda_n(s)\right)$$
$$\leq C\exp\left(-\tfrac{n-1}{8\|f\|^2_\infty}\varepsilon^2\right)$$

and

$$\mathbb{P}\left(\left|\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))-\mathbb{E}\left[\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))\Big|\lambda_1(s),\ldots,\lambda_n(s)\right]\right|>\varepsilon\right)\leq C\exp\left(-\frac{n-1}{8\|f\|^2_\infty}\varepsilon^2\right).$$

Notice that $\mathbb{E}\left[\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))\big|\lambda_1(s),\ldots,\lambda_n(s)\right]=\frac{1}{n}\sum_{i=1}^n f(\lambda_i(s))=f^{(n)}(X_n^{(\mu)}(s))$, where $f^{(n)}$ is defined as in theorem III.4. From this theorem, there exists $\delta_\varepsilon(1)>0$ such that for all $s\in[0,1]$

$$\mathbb{P}\left(\left|f^{(n)}(X_n^{(\mu)}(s))-\mathbb{E}\left[f^{(n)}(X_n^{(\mu)}(s))\right]\right|>\varepsilon\right)\leq 2\exp(-n\delta_\varepsilon(1)).$$

We can then infer that:

$$\begin{aligned}
&\sup_{s\in[0,1]}\mathbb{P}\left(\left|\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))-\mathbb{E}\left[\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))\right]\right|>2\varepsilon\right)\\
&\leq\sup_{s\in[0,1]}\mathbb{P}\left(\left|\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))-\mathbb{E}\left[\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))\Big|\lambda_1(s),\ldots,\lambda_n(s)\right]\right|>\varepsilon\right)\\
&\quad+\sup_{s\in[0,1]}\mathbb{P}\left(\left|f^{(n)}(X_n(s))-\mathbb{E}\left[f^{(n)}(X_n(s))\right]\right|>\varepsilon\right)\\
&\leq C\exp\left(-\frac{n-1}{8\|f\|^2_\infty}\varepsilon^2\right)+2\exp(-n\delta_\varepsilon(1)).
\end{aligned}$$

To conclude the proof, just remark that

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))-\mathbb{E}\left[\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))\right]\right|\right]\\
&\leq 2\varepsilon+2\|f\|_\infty\mathbb{P}\left(\left|\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))-\mathbb{E}\left[\sum_{i=1}^n|v_i|^2 f(\lambda_i(s))\right]\right|>2\varepsilon\right).
\end{aligned}$$

¿From this inequality, the result of the lemma can be deduced directly. □

Lemma III.7 *Let us consider a complex $\zeta \in \mathbb{C} \setminus \mathbb{R}$, the function $f_\zeta(x) = (\zeta - x)^{-1}$, an Hermitian matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$, a unit vector $V = (v_i, i = 1, \dots, n) \in \mathbb{C}^n$, and a real t. Then*

$$\mathrm{tr} f_\zeta(A + tVV^*) - \mathrm{tr} f_\zeta(A) = t \frac{\sum_{i=1}^n \frac{|v_i|^2}{(\zeta-\lambda_i)^2}}{1 - t\sum_{i=1}^n \frac{|v_i|^2}{\zeta-\lambda_i}}.$$

Proof. First notice that we can suppose A to be a diagonal matrix. Then, due to the resolvant equation, we obtain:

$$(\zeta - A - tVV^*)^{-1} - (\zeta - A)^{-1} = (\zeta - A)^{-1}\left(1 - tVV^*(\zeta - A)^{-1}\right)^{-1} tVV^*(\zeta - A)^{-1}.$$

Now, for t small enough, we have:

$$\begin{aligned}
\left(1 - tVV^*(\zeta - A)^{-1}\right)^{-1} &= 1 + \sum_{i \geq 1} \left(tVV^*(\zeta - A)^{-1}\right)^i \\
&= 1 + tVV^*(\zeta - A)^{-1} \sum_{i \geq 0} \left(tV^*(\zeta - A)^{-1}V\right)^i \\
&= 1 + tVV^*(\zeta - A)^{-1} \frac{1}{1 - tV^*(\zeta - A)^{-1}V}.
\end{aligned}$$

The last equality can be extended to any value of t. Therefore

$$\begin{aligned}
\mathrm{tr} f_\zeta(A + tVV^*) - \mathrm{tr} f_\zeta(A) &= \mathrm{tr}\left(tV^*(\zeta - A)^{-2}\left(1 - tVV^*(\zeta - A)^{-1}\right)^{-1} V\right) \\
&= tV^*(\zeta - A)^{-2}V \frac{1}{1 - tV^*(\zeta - A)^{-1}V} \\
&= t \frac{\displaystyle\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - \lambda_i)^2}}{1 - t\displaystyle\sum_{i=1}^n \frac{|v_i|^2}{\zeta - \lambda_i}}.
\end{aligned}$$

□

Proof of theorem III.2. Using the notations and definitions of the two previous lemmas, we are going to prove that the empirical spectral law $\hat{\mu}_n$ of $X_n^{(\mu)}(1)$ converges weakly almost surely to $\Lambda(\mu)$. Due to the concentration result of section III.C, what remains actually to be established is the convergence of $\mathbb{E} \circ \hat{\mu}_n$ to $\Lambda(\mu)$.

Step 1. For any $z = a + ib$ with $b > 0$, let us define $f_z(x) = (z - x)^{-1}$. We shall denote by $f_z^{(n)}$ the functional $\frac{1}{n}\mathrm{tr} f_z$ defined on $\mathcal{H}_n$, and by $\lambda_1^{(n)}(s), \dots, \lambda_n^{(n)}(s)$

the eigenvalues of $X_n^{(\mu)}(s)$. Due to lemma III.7, it is easy to check that

$$\begin{aligned}
&\mathcal{A}_n^{(\mu)}\left(f_z^{(n)}\right)(X_n^{(\mu)}(s))\\
&=\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(z-\lambda_i^{(n)}(s)\right)^2}+\int_{\mathbb{R}}\mathbb{E}\left[\frac{t+A(s)}{1-tA(s)}B(s)\middle|\lambda_1^{(n)}(s),\dots,\lambda_n^{(n)}(s)\right]\sigma(dt)\\
&=-\gamma\partial_z f_z^{(n)}\left(X_n^{(\mu)}(s)\right)+\int_{\mathbb{R}}\mathbb{E}\left[\frac{t+A(s)}{1-tA(s)}B(s)\middle|\lambda_1^{(n)}(s),\dots,\lambda_n^{(n)}(s)\right]\sigma(dt),
\end{aligned}$$

where $A(s)$ and $B(s)$ are defined by

$$\begin{aligned}
A(s) &= \sum_{i=1}^{n}\frac{|v_i|^2}{z-\lambda_i^{(n)}(s)}\\
B(s) &= \sum_{i=1}^{n}\frac{|v_i|^2}{\left(z-\lambda_i^{(n)}(s)\right)^2}
\end{aligned}$$

and $(v_1,\dots,v_n)$ denotes a random vector, Haar distributed on the unit sphere of $\mathbb{C}^n$.

Set $\psi^{(n)}(z,s)=P_s^{\Lambda_n(\mu)}(f_z^{(n)})(0)=\mathbb{E}\left[f_z^{(n)}\left(X_n^{(\mu)}(s)\right)\right]$. This is the Cauchy transform of the empirical spectral law of $X_n^{(\mu)}(s)$. Then

$$\begin{aligned}
\partial_s\psi^{(n)}(z,s) &= \mathbb{E}\left[\mathcal{A}_n^{(\mu)}\left(f_z^{(n)}\right)(X_n^{(\mu)}(s))\right]\\
&= -\gamma\partial_z\psi^{(n)}(z,s)+\int_{\mathbb{R}}\mathbb{E}\left[\frac{t+A(s)}{1-tA(s)}B(s)\right]\sigma(dt). \qquad \text{(III.1)}
\end{aligned}$$

Before going any further, it is time to explain intuitively the proof. The Cauchy transform $\psi(z,s)=\Lambda\left(\mu^{*s}\right)(f_z)$ of $\Lambda\left(\mu^{*s}\right)$ is characterised by $\psi(z,0)=z^{-1}$ and (*cf* [6], or lemma 3.3.9 in [14]):

$$\partial_s\psi(z,s)=-\gamma\partial_z\psi(z,s)-\int_{\mathbb{R}}\frac{t+\psi(z,s)}{1-t\psi(z,s)}\partial_z\psi(z,s)\sigma(dt). \qquad \text{(III.2)}$$

Let us denote by $\mu_s^{(\infty)}$ the (expected) limit law of $\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i(s)}$ as n tends to infinity, and by $\psi^{(\infty)}(z,s)=\mu_s^{(\infty)}(f_z)$ its Cauchy transform. Then, following lemma III.6, $A(s)$ converges almost surely to $\psi^{(\infty)}(z,s)$ and $B(s)$ to $-\partial_z\psi^{(\infty)}(z,s)$. Therefore, if we replace $A(s)$ and $B(s)$ in equation (III.1) by their limits, we remark that $\psi^{(\infty)}$ is expected to satisfy the same equation (III.2) as ψ. Hence $\psi^{(\infty)}$ ought to be equal to ψ and μ_s^{∞} equal to $\Lambda\left(\mu^{*s}\right)$.

Step 2. Following the previous heuristic hint, we would like to evaluate $\partial_s\psi(z,s)-\partial_s\psi^{(n)}(z,s)$ using equations (III.1) and (III.2):

$$\begin{aligned}
&\partial_s\psi(z,s)-\partial_s\psi^{(n)}(z,s)\\
&=-\gamma\left(\partial_z\psi(z,s)-\partial_z\psi^{(n)}(z,s)\right)\\
&+\left(\partial_z\psi(z,s)-\partial_z\psi^{(n)}(z,s)\right)\int_{\mathbb{R}}\frac{t+\psi(z,s)}{1-t\psi(z,s)}\sigma(dt)\\
&+\partial_z\psi^{(n)}(z,s)\int_{\mathbb{R}}\left(\frac{t+\psi(z,s)}{1-t\psi(z,s)}-\frac{t+\psi^{(n)}(z,s)}{1-t\psi^{(n)}(z,s)}\right)\sigma(dt)\\
&+\int_{\mathbb{R}}\left(\frac{t+\psi^{(n)}(z,s)}{1-t\psi^{(n)}(z,s)}\partial_z\psi^{(n)}(z,s)-\mathbb{E}\left[\frac{t+A(s)}{1-tA(s)}B(s)\right]\right)\sigma(dt).
\end{aligned}$$

Let us consider $z_0=a_0+ib_0$ with $b_0>0$, and let us introduce $(\zeta(s),s\geq 0)$ such that

$$\begin{aligned}
\zeta(0)&=z_0\\
\zeta'(s)&=\gamma+\int_{\mathbb{R}}\frac{t+\psi(\zeta(s),s)}{1-t\psi(\zeta(s),s)}\sigma(dt).
\end{aligned}\tag{III.3}$$

The mapping

$$z\mapsto\gamma+\int_{\mathbb{R}}\frac{t+\psi(z,s)}{1-t\psi(z,s)}\sigma(dt)$$

is locally Lipschitz on $\mathbb{C}\setminus\mathbb{R}$, as composition of locally Lipschitz functions. Indeed,

$$\left|\int_{\mathbb{R}}\frac{t+z}{1-tz}\sigma(dt)-\int_{\mathbb{R}}\frac{t+z'}{1-tz'}\sigma(dt)\right|\leq|z-z'|\cdot\sigma(1)\frac{1+|zz'|}{|\Im(z)\Im(z')|},$$

(where $\Im(z)=(z-\bar{z})/2$) and

$$|\psi(z,s)-\psi(z',s)|\leq|z-z'|\,\frac{1}{|\Im(z)\Im(z')|}.$$

Therefore, there exists a unique maximal solution to the preceding differential equation (III.3), since we have supposed $\Im(\zeta(0))=b_0>0$. Now let us remark that

$$\partial_s\left(\psi\left(\zeta(s),s\right)\right)=\partial_s(\psi)(\zeta(s),s)+\partial_z(\psi)(\zeta(s),s)\left(\gamma+\int_{\mathbb{R}}\frac{t+\psi(\zeta(s),s)}{1-t\psi(\zeta(s),s)}\sigma(dt)\right)=0$$

following equation (III.2). Hence, we can deduce that for any $s\geq 0$ such that $\zeta(s)$ is defined, then

$$\psi\left(\zeta(s),s\right)=\psi\left(\zeta(0),0\right)=\frac{1}{z_0}\text{ and }\zeta'(s)=\gamma+\int_{\mathbb{R}}\frac{1+tz_0}{z_0-t}\sigma(dt).$$

In particular,

$$\Im\left(\zeta'(s)\right)=-b_0\int_{\mathbb{R}}\frac{1+t^2}{(a_0-t)^2+b_0^2}\sigma(dt).$$

This integral vanishes when b_0 tends to $+\infty$ uniformly for a_0 in a compact set. Thus, for any $b_1 > 0$ and any compact set $F \subset \mathbb{R}$, there exists a threshold S such that for any a_0 in F, any $b_0 > S$ and any s in $[0,1]$, one has:

$$\Im\left(\zeta\left(s\right)\right) > b_1.$$

Moreover, we can infer that for any pair (b_1, b_2) with $0 < b_2 < b_1$, there exists a compact set $K \subset \mathbb{C}$ whose interior is non empty, such that for any $z_0 = a_0 + ib_0$ in K and any s in $[0,1]$

$$\Im\left(\zeta\left(s\right)\right) > b_1,\ \ \frac{1}{b_2} < \left|\Im\left(\psi(\zeta(s),s)\right)\right| = \frac{|b_0|}{a_0^2+b_0^2} < \frac{1}{b_1} \text{ and } \left|\psi(\zeta(s),s)\right| = \frac{1}{\sqrt{a_0^2+b_0^2}} < \frac{1}{b_1}.$$

For the rest of the proof, we shall suppose this property to be satisfied.

Step 3. We now focus on the evaluation of $\psi\left(\zeta(s),s\right) - \psi^{(n)}\left(\zeta(s),s\right)$:

$$\begin{aligned}
&\frac{d}{ds}\left(\psi\left(\zeta(s),s\right) - \psi^{(n)}\left(\zeta(s),s\right)\right) \\
&= \partial_z\psi^{(n)}(\zeta(s),s)\int_{\mathbb{R}}\left(\frac{t+\psi(\zeta(s),s)}{1-t\psi(\zeta(s),s)} - \frac{t+\psi^{(n)}(\zeta(s),s)}{1-t\psi^{(n)}(\zeta(s),s)}\right)\sigma(dt) \\
&\quad + \int_{\mathbb{R}}\left(\frac{t+\psi^{(n)}(\zeta(s),s)}{1-t\psi^{(n)}(\zeta(s),s)}\partial_z\psi^{(n)}(\zeta(s),s) - \mathbb{E}\left[\frac{t+A(s)}{1-tA(s)}B(s)\right]\right)\sigma(dt) \\
&= \partial_z\psi^{(n)}(\zeta(s),s)\left(\psi(\zeta(s),s) - \psi^{(n)}(\zeta(s),s)\right)\int_{\mathbb{R}}\frac{1+t^2}{\left(1-t\psi(\zeta(s),s)\right)\left(1-t\psi^{(n)}(\zeta(s),s)\right)}\sigma(dt) \\
&\quad + \int_{\mathbb{R}}\mathbb{E}\left[B(s)\left(\frac{t+\psi^{(n)}(\zeta(s),s)}{1-t\psi^{(n)}(\zeta(s),s)} - \frac{t+A(s)}{1-tA(s)}\right)\right]\sigma(dt) \\
&\quad + \int_{\mathbb{R}}\frac{t+\psi^{(n)}(\zeta(s),s)}{1-t\psi^{(n)}(\zeta(s),s)}\mathbb{E}\left[\partial_z\psi^{(n)}(\zeta(s),s) - B(s)\right]\sigma(dt).
\end{aligned}$$

But from inequalities

$$\left|\frac{1}{1-tz}\right| \le \left|\frac{z}{\Im(z)}\right| \text{ and } \left|\frac{t}{1-tz}\right| \le \frac{1}{|\Im(z)|},$$

we can establish the following upper bounds:

$$\begin{aligned}
\left|\partial_z\psi^{(n)}(\zeta(s),s)\right| &\le \frac{1}{\Im\left(\zeta(s)\right)^2} \le \frac{1}{b_1^2}, \\
\left|\frac{1+t^2}{\left(1-t\psi(\zeta(s),s)\right)\left(1-t\psi^{(n)}(\zeta(s),s)\right)}\right| &\le \frac{1+\left|\psi(\zeta(s),s)\psi^{(n)}(\zeta(s),s)\right|}{\left|\Im\left(\psi(\zeta(s),s)\right)\Im\left(\psi^{(n)}(\zeta(s),s)\right)\right|} \\
&\le \frac{1+\frac{1}{b_1^2}}{\frac{1}{b_2}\left|\Im\left(\psi^{(n)}(\zeta(s),s)\right)\right|}, \\
\left|\frac{t+\psi^{(n)}(\zeta(s),s)}{1-t\psi^{(n)}(\zeta(s),s)}\right| &\le \frac{1+\left|\psi^{(n)}(\zeta(s),s)\right|^2}{\left|\Im\left(\psi^{(n)}(\zeta(s),s)\right)\right|} \le \frac{1+\frac{1}{b_1^2}}{\left|\Im\left(\psi^{(n)}(\zeta(s),s)\right)\right|},
\end{aligned}$$

$$
\begin{aligned}
&\left|B(s)\frac{t+\psi^{(n)}(\zeta(s),s)}{1-t\psi^{(n)}(\zeta(s),s)}-\frac{t+A(s)}{1-tA(s)}\right| \\
&=\left|A(s)-\psi^{(n)}(\zeta(s),s)\right|\cdot\left|B(s)\frac{1+t^2}{(1-tA(s))\left(1-t\psi^{(n)}(\zeta(s),s)\right)}\right| \\
&\leq\left|A(s)-\psi^{(n)}(\zeta(s),s)\right|\cdot|B(s)|\,\frac{\left|A(s)\psi^{(n)}(\zeta(s),s)\right|+1}{\left|\Im(A(s))\Im\left(\psi^{(n)}(\zeta(s),s)\right)\right|} \\
&\leq\left|A(s)-\psi^{(n)}(\zeta(s),s)\right|\frac{1}{b_1\left|\Im\left(\psi^{(n)}(\zeta(s),s)\right)\right|}\left(\frac{1}{b_1^2}+1\right),
\end{aligned}
$$

the last inequality coming from

$$
|B(s)|=\left|\sum_{i=1}^{n}\frac{|v_i|^2}{\left(z-\lambda_i^{(n)}(s)\right)^2}\right|\leq\sum_{i=1}^{n}\left|\frac{|v_i|^2}{\left(z-\lambda_i^{(n)}(s)\right)^2}\right|=\frac{1}{|\Im(z)|}\,|A(s)|\,.
$$

Going back to $\psi\left(\zeta(s),s\right)-\psi^{(n)}\left(\zeta(s),s\right)$, it follows from the preceding bounds that

$$
\begin{aligned}
\left|\frac{d}{ds}\left(\psi\left(\zeta(s),s\right)-\psi^{(n)}\left(\zeta(s),s\right)\right)\right| \quad\leq\quad & \frac{C}{\left|\Im\left(\psi^{(n)}(\zeta(s),s)\right)\right|}\left|\psi(\zeta(s),s)-\psi^{(n)}(\zeta(s),s)\right| \\
&+\frac{D}{\left|\Im\left(\psi^{(n)}(\zeta(s),s)\right)\right|}\mathbb{E}\left[\left|A(s)-\psi^{(n)}(\zeta(s),s)\right|\right] \\
&+\frac{E}{\left|\Im\left(\psi^{(n)}(\zeta(s),s)\right)\right|}\mathbb{E}\left[\left|\partial_z\psi^{(n)}(\zeta(s),s)-B(s)\right|\right],
\end{aligned}
$$

where C,D,E do not depend of n nor s. From Gronwall's lemma, those inequalities provide the following bound

$$
\begin{aligned}
&\left|\psi\left(\zeta(s),s\right)-\psi^{(n)}\left(\zeta(s),s\right)\right| \qquad\qquad \text{(III.4)}\\
\leq\quad&\left(D\sup_{u\in[0,s]}\mathbb{E}\left[\left|A(u)-\psi^{(n)}(\zeta(u),u)\right|\right]+E\sup_{u\in[0,s]}\mathbb{E}\left[\left|\partial_z\psi^{(n)}(\zeta(u),u)-B(u)\right|\right]\right) \\
&\times\frac{1}{\inf_{u\in[0,s]}\left|\Im\left(\psi^{(n)}(\zeta(u),u)\right)\right|}\exp\left(s\frac{C}{\inf_{u\in[0,s]}\left|\Im\left(\psi^{(n)}(\zeta(u),u)\right)\right|}\right).
\end{aligned}
$$

Step 4. We now have to find a lower bound for $\left|\Im\left(\psi^{(n)}(\zeta(s),s)\right)\right|$. Let us consider an interval $[0,s]$ such that $\inf_{u\in[0,s]}\left|\Im\left(\psi^{(n)}(\zeta(u),u)\right)\right|>0$ (this is true in particular for $s=0$, because $\psi^{(n)}(\zeta(0),0)=1/z_0$). Since

$$
\psi^{(n)}(\zeta(u),u)=\mathbb{E}\left[A(u)\right]\ \text{and}\ \partial_z\psi^{(n)}(\zeta(u),u)=\mathbb{E}\left[B(u)\right],
$$

we can infer from inequality (III.4) and lemma III.6 that $\psi^{(n)}\left(\zeta(s),s\right)$ converges to $\psi\left(\zeta(s),s\right)$. Let us recall that

$$
\inf_{u\in[0,1]}\left|\Im\left(\psi\left(\zeta(s),s\right)\right)\right|\geq\frac{1}{b_2}>0
$$

Hence, there exists n_0 such that for any $n \geq n_0$, $\left|\Im\left(\psi^{(n)}\left(\zeta(s), s\right)\right)\right|$ is larger than $\frac{1}{2b_2}$.

Let us notice now that

$$\left|\Im\left(\psi^{(n)}(\zeta(s+h), s+h)\right)\right| \geq \left|\Im\left(\psi^{(n)}(\zeta(s), s)\right)\right| - \left|\int_s^{s+h} \left|\frac{d}{dt}\psi^{(n)}(\zeta(t), t)\right| dt\right|.$$

for all $h \geq 0$. It is easy to see that

$$\left|\frac{d}{du}\psi^{(n)}(\zeta(u), u)\right| = \left|\zeta'(u)\partial_z \psi^{(n)}(\zeta(u), u) + \mathbb{E}\left[\mathcal{A}_n^{(\mu)}\left(f_{\zeta(u)}^{(n)}\right)\left(X_n^{(\mu)}(u)\right)\right]\right|$$

is uniformly bounded in n and u by some constant c since $\Im(\zeta(u)) > b_1$. This implies that $\left|\Im\left(\psi^{(n)}(\zeta(s+h), s+h)\right)\right|$ is larger than $\frac{1}{4b_2}$ for all $h \in [0, 1/(4b_2 c)]$.

Using repeatedly this trick, one proves that $\inf_{u\in[0,1]}\left|\Im\left(\psi^{(n)}(\zeta(u), u)\right)\right| > 0$ for all n large enough. Consequently, following (III.4), $\psi^{(n)}\left(\zeta(1), 1\right)$ converges to $\psi\left(\zeta(1), 1\right)$, for all initial point $\zeta(0) = z_0$ in K.

Step 5. The set of non-negative measures with total mass less than 1 is compact for the topology associated to the set $C_0(\mathbb{R})$ of continuous functions vanishing at infinity. Let $\left(\mathbb{E}\circ\left(\frac{1}{n_p}\sum_{i=1}^{n_p}\delta_{\lambda_i^{(n_p)}(1)}\right), p \geq 0\right)$ be a convergent sub-sequence of $\left(\mathbb{E}\circ\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i^{(n)}(1)}\right), n \geq 0\right)$, with limit ν; then the Cauchy transform $\psi^{(n_p)}(\zeta(1), 1)$ of $\mathbb{E}\circ\left(\frac{1}{n_p}\sum_{i=1}^{n_p}\delta_{\lambda_i^{(n_p)}(1)}\right)$ is convergent to the Cauchy transform $\nu\left(f_{\zeta(1)}\right)$ of ν. Therefore $\nu\left(f_{\zeta(1)}\right)$ is equal to the Cauchy transform $\psi(\zeta(1), 1)$ of $\Lambda(\mu)$. Let us remark now that

$$\zeta(1) = z_0 + \gamma + \int_{\mathbb{R}} \frac{1 + tz_0}{z_0 - t}\sigma(dt)$$

As $z_0 = \zeta(0)$ runs over K, $\zeta(1)$ runs over a set whose interior is non empty. Due to the maximum principle, this implies that $\nu\left(f_{z'}\right) = \psi(z', 1)$ for all $z' \in \mathbb{C}$ with $\Im\left(z'\right) > 0$. Hence, ν is equal to $\Lambda(\mu)$. This gives that there is only one accumulation point for the sequence $\left(\mathbb{E}\circ\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i^{(n)}(1)}\right), n \geq 1\right)$, hence this sequence is convergent w.r.t. the topology associated to $C_0(\mathbb{R})$. Since this topology coincides with the weak topology, we can conclude that $\left(\mathbb{E}\circ\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i^{(n)}(1)}\right), n \geq 1\right)$ converges weakly to $\Lambda(\mu)$.

Using the concentration result in theorem III.4, and the Borel-Cantelli lemma, it is now easy to check $\frac{1}{n}\sum_{i=1}^{n} f_z\left(\lambda_i^{(n)}(1)\right)$ converges a.s. to $\Lambda(\mu)(f_z)$ for all z in a fixed dense denumerable set in $\mathbb{C}$. Following same arguments as before, this implies that $\left(\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i^{(n)}(1)}, n \geq 1\right)$ converges weakly a.s. to $\Lambda(\mu)$. $\square$

IV Some other properties

IV.A Matrix cumulants

If a probability measure μ in $\mathcal{ID}_*(\mathbb{R})$ has moments of all orders, so has $\Lambda(\mu)$. Moreover, there is a simple characterisation of the relation between μ and $\Lambda(\mu)$, first remarked by Anshelevich in [1]. If we denote by $\left(c_p^*, p \geq 1\right)$ and $\left(c_p^{\boxplus}, p \geq 1\right)$ the classical and the free cumulants respectively, then

$$\forall p \geq 1, \quad c_p^*(\mu) = c_p^{\boxplus}(\Lambda(\mu)).$$

Our aim in this section is to show that this property can be extended to $\Lambda_n(\mu)$. We first need to define matrix cumulants, and in so doing we shall be inspired by a recent paper by Lehner. In 1975, Good remarked that

$$c_p^*(\mu) = \frac{1}{p}\mathbb{E}\left[\left(\sum_{k=1}^{p} X^{(k)} e^{i\frac{2\pi k}{p}}\right)^p\right]$$

where the $X^{(k)}, k = 1, \ldots, p$ are i.i.d. random variables with distribution μ (*cf* [11]). Lehner has established the corresponding formula for the free case (*cf* [18]):

$$c_p^{\boxplus}(\mu) = \frac{1}{p}\tau\left[\left(\sum_{k=1}^{p} X^{(k)} e^{i\frac{2\pi k}{p}}\right)^p\right],$$

where the $X^{(k)}, k = 1, \ldots, p$ are free random variables with distribution μ. This suggests the following definition of a matrix cumulant of order p for a distribution μ on $\mathcal{H}_n$ which is invariant by unitary conjugation:

$$c_p^{(n)}(\mu) = \frac{1}{p}\mathbb{E}\left[\operatorname{tr}_n\left(\sum_{k=1}^{p} X^{(k)} e^{i\frac{2\pi k}{p}}\right)^p\right]$$

where the $X^{(k)}, k = 1, \ldots, p$ are i.i.d. random matrices with distribution μ.

Remark. Unlike the classical and free cases, these matrix cumulants do not characterise μ, nor its empirical spectral law.

Proposition IV.1 *Let μ be a probability measure in $\mathcal{ID}_*(\mathbb{R})$ with moments of all orders. Then*

$$\forall n, p \geq 1, \quad c_p^*(\mu) = c_p^{(n)}(\Lambda_n(\mu)) = c_p^{\boxplus}(\Lambda(\mu)).$$

Proof of proposition IV.1. The explicit realization of law $\Lambda_n(\mu)$ (*cf* section II.B) could have been used with some combinatorics, but it is easier to adapt the simple proof of Good's result due to Groeneveld and van Kampen (*cf* [12]). Recall that

$$\mu(\exp(i\lambda\cdot)) = \exp\left(\varphi^{(\mu)}(\lambda)\right) = \exp\left(\sum_{q\geq 1} \frac{i^q \lambda^q}{q!} c_q^*(\mu)\right).$$

Let us consider p random matrices $X_n^{(k)}, k = 1, \ldots, p$ i.i.d. with distribution $\Lambda_n(\mu)$. We can then infer that

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\mathrm{tr}\left(A\sum_{k=1}^{p} X_n^{(k)} e^{i\frac{2\pi k}{p}}\right)\right)\right] &= \prod_{k=1}^{p} \mathbb{E}\left[\exp\left(ie^{i\frac{2\pi k}{p}}\mathrm{tr}\left(AX_n^{(1)}\right)\right)\right] \\
&= \prod_{k=1}^{p} \exp\left(\varphi_n^{(\mu)}\left(ie^{i\frac{2\pi k}{p}}A\right)\right) \\
&= \prod_{k=1}^{p} \exp\left(n\mathbb{E}\left[\varphi^{(\mu)}\left(ie^{i\frac{2\pi k}{p}}\langle V_n, AV_n\rangle\right)\right]\right) \\
&= \prod_{k=1}^{p} \exp\left(n\sum_{q\geq 1}\frac{i^q e^{i\frac{2\pi k}{p}q}}{q!}c_q^*(\mu)\mathbb{E}\left[\langle V_n, AV_n\rangle^q\right]\right) \\
&= \exp\left(n\sum_{q\geq 1}\frac{i^q}{q!}c_q^*(\mu)\mathbb{E}\left[\langle V_n, AV_n\rangle^q\right]\left(\sum_{k=1}^{p} e^{i\frac{2\pi k}{p}q}\right)\right),
\end{aligned}$$

where V_n is a unitary uniformly distributed random vector on $\mathbb{C}^n$. Now $\sum_{k=1}^{p} e^{i\frac{2\pi k}{p}q} = p$ if q is a multiple of p, and it vanishes otherwise. Therefore

$$\mathbb{E}\left[\exp\left(i\mathrm{tr}\left(A\sum_{k=1}^{p} X_n^{(k)} e^{i\frac{2\pi k}{p}}\right)\right)\right] = \exp\left(np\sum_{q\geq 1}\frac{i^{pq}}{pq!}c_{pq}(\mu)\mathbb{E}\left[\langle V_n, AV_n\rangle^{pq}\right]\right).$$

But, denoting by $a_{i,j}$ the entries of the matrix A, we obtain:

$$\begin{aligned}
&\frac{i^p}{p!}\mathbb{E}\left[\mathrm{tr}\left(\sum_{k=1}^{p} X_n^{(k)} e^{i\frac{2\pi k}{p}}\right)^p\right] \\
&= \sum_{i_1,\ldots,i_p=1,\ldots,n} \partial_{a_{i_1,i_2}}\circ\cdots\circ\partial_{a_{i_p,i_1}}\Big|_{A=0}\mathbb{E}\left[\exp\left(i\mathrm{tr}\left(A\sum_{k=1}^{p} X_n^{(k)} e^{i\frac{2\pi k}{p}}\right)\right)\right] \\
&= \sum_{i_1,\ldots,i_p=1,\ldots,n} \partial_{a_{i_1,i_2}}\circ\cdots\circ\partial_{a_{i_p,i_1}}\Big|_{A=0}\exp\left(np\sum_{q\geq 1}\frac{i^{pq}}{pq!}c_{pq}^*(\mu)\mathbb{E}\left[\langle V_n, AV_n\rangle^{pq}\right]\right) \\
&= np\frac{i^p}{p!}c_p^*(\mu)\sum_{i_1,\ldots,i_p=1,\ldots,n} \partial_{a_{i_1,i_2}}\circ\cdots\circ\partial_{a_{i_p,i_1}}\mathbb{E}\left[\langle V_n, AV_n\rangle^{p}\right] \\
&= np\frac{i^p}{p!}c_p^*(\mu)\sum_{i_1,\ldots,i_p=1,\ldots,n}\mathbb{E}\left[|v_{i_1}|^2\ldots|v_{i_p}|^2\right] = np\frac{i^p}{p!}c_p^*(\mu).
\end{aligned}$$

In other terms, we have established that

$$c_p^{(n)}\left(\Lambda_n(\mu)\right) = \frac{1}{p}\mathbb{E}\left[\mathrm{tr}_n\left(\sum_{k=1}^{p} X_n^{(k)} e^{i\frac{2\pi k}{p}}\right)^p\right] = c_p^*(\mu).$$

□

IV.B The Cauchy matrix

The Cauchy distribution is a meeting point between classical and free probabilities. It is 1-stable in both cases, and it has the same convolution kernel. Let us develop the latter point. Let C be a real even Cauchy variable with parameter ε, and let U be another random variable. Then for each Borelian bounded function f, we have

$$\mathbb{E}\left[f(C+U)|U\right] = P_\varepsilon\left(f\right)\left(U\right)$$

whether C and U are independent or free, if $\mathbb{E}\left[\cdot|U\right]$ denotes the orthogonal projector onto $L^2(U)$ and P_ε the convolution kernel such that $P_\varepsilon\left(f\right)\left(x\right) = \int_{\mathbb{R}} f(x-y)\frac{\varepsilon}{\pi(\varepsilon^2+y^2)}dy$.

Due to this property, the Cauchy distribution is very useful for regularizing in free probability, since ordinary free convolution kernels are much more complicated (*cf* [9]). The question arised whether there were a family of "Cauchy" matrices with the same property, being a link between the classical case (in one dimension) and the free case (in infinite dimension). This is the original motivation of our introduction of the new matrix ensembles $\Lambda_n(\mu)$.

Theorem IV.2 *Let μ_ε be the even Cauchy distribution with parameter ε*

$$\mu_\varepsilon(dx) = \frac{\varepsilon}{\pi(\varepsilon^2+x^2)}dx$$

and let C_n be a random Hermitian matrix of law $\Lambda_n\left(\mu_\varepsilon\right)$. Then, for each Hermitian matrix A and each real Borelian bounded function f, we have

$$\mathbb{E}\left[f(A+C_n)\right] = P_\varepsilon(f)(A).$$

Lemma IV.3 *Let $M = (M_{i,j})_{i,j=1}^n$ be a square matrix of size n, such that $M = A+iB$ for some $A \in \mathcal{H}_n$ and $B \in \mathcal{H}_n^+$. Then*

$$\Im\left(\frac{\det(M)}{\det(M^{(1,1)})}\right) > 0,$$

where $M^{(1,1)}$ denotes $(M_{i,j})_{i,j=2}^n$.

Proof. The eigenvalues of M have a positive imaginary part. Hence M is inversible, and the eigenvalues of M^{-1} have a negative imaginary part. Therefore

$$\frac{\det(M^{(1,1)})}{\det(M)} = \left(M^{-1}\right)_{1,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} M^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a negative imaginary part, which proves the lemma. □

Lemma IV.4 *Let $\lambda_1, \ldots, \lambda_n$ be n real independent even Cauchy random variables with parameter 1, let Λ be the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ as diagonal entries, and let U be an independent random matrix Haar distributed on $\mathcal{U}(n)$. Then*

$$\mathbb{E}\left[\left(A+iB-cU\Lambda U^*\right)^{-1}\right] = \left(A+iB+i|c|I\right)^{-1}$$

for all $A \in \mathcal{H}_n$, $B \in \mathcal{H}_n^+$, and $c \in \mathbb{R}$.

Proof. This is an elementary consequence of the following result: if $\Im(z) > 0$, then we have

$$\mathbb{E}\left[\frac{1}{z - c\lambda_1}\right] = \frac{1}{z + i|c|}.$$

More generally, if $f(X)$ is a rational fonction with numerator's degree less or equal to denominator's degree, and whose singularities are all in $\mathbb{C}_+ = \{z \in \mathbb{C}/\Im(z) > 0\}$, then

$$\mathbb{E}\left[f\left(\lambda_1\right)\right] = f(-i).$$

Let us introduce

$$R(U, \lambda_1, \ldots, \lambda_n) = \left(A + iB - cU\Lambda U^*\right)^{-1} = U\left(U^*AU + iU^*BU - c\Lambda\right)^{-1}U^*.$$

For symmetry reasons, we can assume $c > 0$. Define $M = U^*AU + iU^*BU - c\Lambda^{\vee_{1,1}}$ where $\Lambda^{\vee_{1,1}}$ is the diagonal matrix whose upper left entry has been cancelled. Then the entries of $\left(A + iB - cU\Lambda U^*\right)^{-1}$ are rational fonctions in λ_1, with $\det\left(A + iB - cU\Lambda U^*\right) = \det(M) - c\lambda_1 \det(M^{(1,1)})$ as denominator, and with pole $\frac{1}{c}\frac{\det(M)}{\det(M^{(1,1)})}$, whose imaginary part is positive, following lemma IV.3. We can then infer that:

$$\mathbb{E}\left[R(U, \lambda_1, \ldots, \lambda_n)|U, \lambda_2, \ldots, \lambda_n\right] = R(U, -i, \lambda_2, \ldots, \lambda_n).$$

Repeating this argument, we can deduce

$$\begin{aligned}
\mathbb{E}\left[R(U, \lambda_1, \ldots, \lambda_n)\right] &= \mathbb{E}\left[R(U, -i, \ldots, -i)\right] \\
&= \mathbb{E}\left[U\left(U^*AU + iU^*BU + icI\right)^{-1}U^*\right] = \left(A + iB + icI\right)^{-1}.
\end{aligned}$$

□

Using the same tricks, we can establish the following generalisation of the previous lemma:

Lemma IV.5 *Let $\Lambda_1, \ldots, \Lambda_m$ be independent replicas of the preceding matrix Λ, let $U_1, \ldots, U_m$ be independent random matrices Haar distributed on $\mathcal{U}(n)$, let $A_1, \ldots, A_q$ be Hermitian matrices, let $B_1, \ldots, B_q$ be positive Hermitian matrices, and let $\alpha_{j,k}, j = 1, \ldots, m; k = 1, \ldots, q$ be reals. Then*

$$\mathbb{E}\left[\prod_{k=1}^{q}\left(A_k + iB_k - \sum_{j=1}^{m}\alpha_{j,k}U_j\Lambda_jU_j^*\right)^{-1}\right] = \prod_{k=1}^{q}\left(A_k + iB_k + i\sum_{j=1}^{m}|\alpha_{j,k}|I\right)^{-1}.$$

Proof of theorem IV.2. Due to linearity and density arguments, we only need to check the desired property for the function $f_z(x) = \frac{1}{z-x}$. We can suppose $\Im(z) > 0$. Now, due to lemma IV.4, with its notations, we have for all Hermitian matrix A

$$\mathbb{E}\left[\left(zI - A - \frac{\varepsilon}{m}\sum_{j=1}^{m}U_j\Lambda_jU_j^*\right)^{-1}\right] = \left(zI - A + i\varepsilon I\right)^{-1}.$$

Therefore, since C_n is the limit in law of $\varepsilon\frac{1}{m}\sum_{j=1}^{m}U_j\Lambda_jU_j^*$ when $m \to +\infty$ (*cf* section II.B), we can deduce that:

$$\mathbb{E}\left[f_z(A + C_n)\right] = \mathbb{E}\left[\left(zI - A - C_n\right)^{-1}\right] = \left((z + i\varepsilon)I - A\right)^{-1} = P_\varepsilon(f_z)(A).$$

□

V Application: moments of the Lévy laws

Using matrix ensembles, it is tempting to try to transport classical properties to the free framework. We shall present here a simple example, dealing with the relation between the generalised moments of a probability measure μ in $\mathcal{ID}_*(\mathbb{R})$ and those of its associated Lévy measure. Let us first recall Kruglov's standard result:

Definition. A function g: $\mathbb{R}_+ \to \mathbb{R}_+$ is said to be sub-multiplicative if there exists a non-negative real K such that

$$\forall x, y \in \mathbb{R}, \quad g\left(|x+y|\right) \leq K g\left(|x|\right) g\left(|y|\right).$$

Examples include $1 + x^\beta$ with $\beta > 0$, $\exp x$, $\ln(e + x)$, and so on.

Theorem V.1 *(cf [15] or section 25 of [21]) Let μ be a probability measure in $\mathcal{ID}_*(\mathbb{R})$, and let g be a sub-multiplicative locally bounded function. Then*

$$\int_{|x|>1} g\left(|x|\right) \sigma_*^{(\mu)}(dx) < +\infty \Longleftrightarrow \int g\left(|x|\right) \mu(dx) < +\infty.$$

Our aim is to adapt the proof given in [21], and, using the Lévy matrices, to establish a similar, although partial, result for the free infinitely divisible laws. But we shall first introduce a new definition. Remark that a function g is sub-multiplicative if and only if there exists K in $\mathbb{R}_+$ such that for any probability measures μ and ν we have

$$\mu * \nu\left(g\left(|\cdot|\right)\right) \leq K \mu\left(g\left(|\cdot|\right)\right) \nu\left(g\left(|\cdot|\right)\right).$$

This may suggest the following definition:

Definition. A function g: $\mathbb{R}_+ \to \mathbb{R}_+$ is said to be freely sub-multiplicative if there exists a non-negative real K such that for any probability measures μ and ν we have

$$\mu \boxplus \nu\left(g\left(|\cdot|\right)\right) \leq K \mu\left(g\left(|\cdot|\right)\right) \nu\left(g\left(|\cdot|\right)\right).$$

Of course, if g is freely sub-multiplicative, then it is sub-multiplicative (take $\mu, \nu = \delta_x, \delta_y$).

Proposition V.2 *Functions $1 + x^\beta$ with $\beta > 0$, $\exp x$ and $\ln(e + x)$ are freely sub-multiplicative.*

Lemma V.3 *A function g: $\mathbb{R}_+ \to \mathbb{R}_+$ is freely sub-multiplicative if there exists K in $\mathbb{R}_+$ such that for any probability measures μ and ν with bounded support we have*

$$\mu \boxplus \nu\left(g\left(|\cdot|\right)\right) \leq K \mu\left(g\left(|\cdot|\right)\right) \nu\left(g\left(|\cdot|\right)\right).$$

Proof. Let μ,ν be probability measures, and define $\mu_A(\cdot) = \mu\,(\cdot|\,[-A,A])$ and $\nu_A(\cdot) = \nu\,(\cdot|\,[-A,A])$ for $A > 0$ large enough. Due to the monotone convergence theorem, $\lim_{A\to+\infty} \mu_A\,(g\,(|\cdot|)) = \mu\,(g\,(|\cdot|))$ and $\lim_{A\to+\infty} \nu_A\,(g\,(|\cdot|)) = \nu\,(g\,(|\cdot|))$. Since μ_A (resp. ν_A) converges weakly to μ (resp. ν) as A tends to infinity, so does $\mu_A \boxplus \nu_A$ to $\mu \boxplus \nu$ (*cf* theorem 4.13 in [6]). Dealing with the fact that g is not bounded presents no difficulty, which allows us to conclude the proof. □

Lemma V.4 *A function g is freely sub-multiplicative if there exists K in $\mathbb{R}_+$ such that*

$$\forall n \geq 1,\quad \forall A,B \in \mathcal{H}_n,\quad \mathbb{E}\left[\mathrm{tr}_n\left(g\left(|A + UBU^*|\right)\right)\right] \leq K\mathrm{tr}_n\left(g\left(|A|\right)\right)\mathrm{tr}_n\left(g\left(|B|\right)\right),$$

where $|M|$ denotes $\sqrt{M^2}$ and U is a random matrix which is Haar distributed on $\mathcal{U}(n)$.

Proof. Let μ (resp. ν) be a probability measure with bounded support, and let $(A_n, n \geq 1)$ (resp. $(B_n, n \geq 1)$) be a family of uniformly bounded Hermitian matrices whose empirical spectral law converges to μ (resp. ν). Then

$$\lim_{n\to+\infty} \mathrm{tr}_n\left(g\left(|A_n|\right)\right) = \mu\left(g\left(|\cdot|\right)\right) \text{ and } \lim_{n\to+\infty} \mathrm{tr}_n\left(g\left(|B_n|\right)\right) = \nu\left(g\left(|\cdot|\right)\right).$$

Moreover, due to Voiculescu's results ([24]), which have been recently improved by Xu ([26]) and Collins ([10]), if $(U_n, n \geq 1) \in \prod_{n\geq1} \mathcal{U}(n)$ is a family of unitary Haar distributed random matrices, then $(A_n, U_n B_n U_n^*)$ are asymptotically free. This implies that

$$\lim_{n\to+\infty} \mathbb{E}\left[\mathrm{tr}_n\left(g\left(|A_n + U_n B_n U_n^*|\right)\right)\right] = \mu \boxplus \nu\left(g\left(|\cdot|\right)\right).$$

The desired result of the lemma is now obvious. □

Lemma V.5 *A function g is freely sub-multiplicative if there exists K in $\mathbb{R}_+$ such that*

$$\forall n \geq 1,\quad \forall A,B \in \mathcal{H}_n,\quad \mathrm{tr}_n\left(g\left(|A+B|\right)\right) \leq K\mathrm{tr}_n\left(g\left(|A|\right)g\left(|B|\right)\right).$$

Proof. We only need to prove that inequality in lemma V.4 holds; this is a direct consequence of the two following facts:

1. $g\left(|UBU^*|\right) = g\left(U|B|U^*\right) = Ug\left(|B|\right)U^*$ for any unitary matrix U;
2. $\mathbb{E}\left[\mathrm{tr}(AUBU^*)\right] = \frac{1}{n}\mathrm{tr}(A)\mathrm{tr}(B)$ if U is Haar distributed on $\mathcal{U}(n)$.

□

Proof of proposition V.2. We only need to prove that inequality in lemma V.5 is satisfied by the functions listed in the proposition. Let $A, B \in \mathcal{H}_n$.

- The inequality:

$$\operatorname{tr}\left(e^{|A+B|}\right) \leq \operatorname{tr}\left(e^{A+B}\right) + \operatorname{tr}\left(e^{-A-B}\right)$$

is obvious. Let $\lambda_1 \leq \cdots \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$ be the eigenvalues of $A+B$ and $|A|+|B|$ respectively. Since $|A|+|B|-A-B$ is a non-negative matrix, we infer from already mentioned Weyl's inequalities that $\lambda_1 \leq \mu_1, \ldots, \lambda_n \leq \mu_n$. Therefore $\operatorname{tr}\left(e^{A+B}\right)$ is less or equal to $\operatorname{tr}\left(e^{|A|+|B|}\right)$; of course, same inequality holds for $\operatorname{tr}\left(e^{-A-B}\right)$. Now, using the Golden-Thompson inequality (*cf* chapter 8 of [23]), which states that $\operatorname{tr}\left(e^{C+D}\right) \leq \operatorname{tr}\left(e^C e^D\right)$ for any Hermitian matrices C and D, we deduce:

$$\operatorname{tr}\left(e^{|A+B|}\right) \leq 2\operatorname{tr}\left(e^{|A|}e^{|B|}\right)$$

Hence, function $\exp x$ is freely sub-multiplicative.

- Following Hölder-McCarthy inequalities (*cf* theorems 2.7 and 2.8 in [20]),

$$\operatorname{tr}|A+B|^\beta \leq K\left(\operatorname{tr}|A|^\beta + \operatorname{tr}|B|^\beta\right)$$

where $K=1$ if $\beta \in]0,1]$ and $K = 2^{\beta-1}$ if $\beta \in [1,+\infty[$. This yields

$$\begin{aligned} \operatorname{tr}_n\left(I+|A+B|^\beta\right) &\leq K\left(1+\operatorname{tr}_n|A|^\beta + \operatorname{tr}_n|B|^\beta\right) \\ &\leq K\operatorname{tr}_n\left(\left(I+|A|^\beta\right)\left(I+|B|^\beta\right)\right). \end{aligned}$$

Therefore, function $1+x^\beta$ is freely sub-multiplicative.

- Let $U \in \mathcal{U}(n)$ such that $A+B = U|A+B|$. From Rotfel'd inequality (*cf* corollary 8.7 [23]), we can infer that

$$\begin{aligned} \det\left(I+|A+B|\right) &= \det\left(I+U^*A+U^*B|\right) \\ &\leq \det\left(I+|U^*A|\right)\det\left(I+|U^*B|\right) \\ &= \det\left(I+|A|\right)\det\left(I+|B|\right). \end{aligned}$$

Applying the logarithm to this inequality we can deduce that

$$\operatorname{tr}\left(\ln\left(I+|A+B|\right)\right) \leq \operatorname{tr}\left(\ln\left(I+|A|\right)+\ln\left(I+|B|\right)\right).$$

Consequently,

$$\begin{aligned} \operatorname{tr}_n\left(\ln\left(eI+|A+B|\right)\right) &= 1+\operatorname{tr}_n\left(\ln\left(I+\frac{1}{e}|A+B|\right)\right) \\ &\leq 1+\operatorname{tr}_n\left(\ln\left(I+\frac{1}{e}|A|\right)+\ln\left(I+\frac{1}{e}|B|\right)\right) \\ &\leq \operatorname{tr}_n\left(\left(I+\ln\left(I+\frac{1}{e}|A|\right)\right)\left(I+\ln\left(I+\frac{1}{e}|B|\right)\right)\right) \\ &= \operatorname{tr}_n\left(\ln\left(eI+|A|\right)\ln\left(eI+|B|\right)\right). \end{aligned}$$

¿From which we can conclude that function $\ln(e+x)$ is freely sub-multiplicative.

□

Remark. Despite these examples, we believe that free sub-multiplicativity is strictly stronger than (ordinary) sub-multiplicativity.

We shall now introduce the main theorem of this section.

Theorem V.6 *Let ν be a probability measure in $\mathcal{ID}_{\boxplus}(\mathbb{R})$, and g be a freely sub-multiplicative locally bounded function. Then*

$$\int g(|x|)\,\sigma_{\boxplus}^{(\nu)}(dx) < +\infty \Longrightarrow \int g(|x|)\,\nu(dx) < +\infty.$$

Remark. Hiai and Petz have already established that ν has compact support if and only if $\sigma_{\boxplus}^{(\nu)}$ has (*cf* theorem 3.3.6 in [14]).

Lemma V.7 *Let ν be a probability measure in $\mathcal{ID}_{\boxplus}(\mathbb{R})$ such that the support of $\sigma_{\boxplus}^{(\nu)}$ is bounded. Then, for all $c > 0$, $\nu(\exp(c|\cdot|))$ is finite.*

Proof. Consider $\mu = \Lambda^{-1}(\nu) \in \mathcal{ID}_*(\mathbb{R})$. We shall prove that for all $c > 0$ there exists $C > 0$ such that

$$\forall n \geq 1, \quad \Lambda_n(\mu)[\mathrm{tr}_n \exp(c|\cdot|)] < C.$$

Letting n go into infinity, this indeed implies the desired result since $\Lambda_n(\mu) \circ \mathrm{tr}_n$ converges to ν.

Since we have

$$\mathrm{tr}_n(\exp(c|A|)) \leq \mathrm{tr}_n(\exp(cA)) + \mathrm{tr}_n(\exp(-cA))$$

for any Hermitian matrix A, we only need to prove that $\Lambda_n(\mu)[\exp(c\cdot)]$ is uniformly bounded in n. Define

$$\psi_n(s) = \mathbb{E}\left[\mathrm{tr}_n\left(\exp\left(cX_n^{(\mu)}(s)\right)\right)\right],$$

where $(X_n^{(\mu)}(s), s \geq 0)$ denotes the Lévy process associated to $\Lambda_n(\mu)$. It follows from lemmas 25.6, 25.7 and from the proof of theorem 25.3 in [21], that ψ_n is locally bounded. Therefore ψ_n can be differentiated to obtain

$$\begin{aligned}
\psi_n'(s) &= \mathbb{E}\left[\mathcal{A}_n^{(\mu)}(\mathrm{tr}_n \circ \exp(c\cdot))\left(X_n^{(\mu)}(s)\right)\right] \\
&= \mathbb{E}\left[c\gamma \mathrm{tr}_n\left(\exp\left(cX_n^{(\mu)}(s)\right)\right)\right] \\
&\quad + \int_{-\alpha}^{\alpha} \mathbb{E}\Bigg[\mathrm{tr}\exp\left(cX_n^{(\mu)}(s) + ctV_nV_n^*\right) - \mathrm{tr}\left(\exp\left(cX_n^{(\mu)}(s)\right)\right) \\
&\qquad\qquad - \frac{ct}{1+t^2}\mathrm{tr}_n\left(\exp\left(cX_n^{(\mu)}(s)\right)\right)\Bigg]\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt),
\end{aligned}$$

where α is such that $\sigma_{\boxplus}^{(\nu)}([-\alpha,\alpha]) = 1$. Now, following Golden-Thompson inequalities (*cf* proof of proposition V.2), we can infer that

$$\begin{aligned}
\mathbb{E}\left[\mathrm{tr}\exp\left(cX_n^{(\mu)}(s) + ctV_nV_n^*\right)\right] &\leq \mathbb{E}\left[\mathrm{tr}\exp\left(cX_n^{(\mu)}(s)\right)\exp\left(ctV_nV_n^*\right)\right] \\
&= \mathbb{E}\left[\mathrm{tr}_n\exp\left(cX_n^{(\mu)}(s)\right)\right]\mathbb{E}\left[\mathrm{tr}\exp\left(ctV_nV_n^*\right)\right],
\end{aligned}$$

since $(X_n^{(\mu)}(s), V_n V_n^*)$ and $(X_n^{(\mu)}(s), U(V_n V_n^*)U)$ have same distribution, if U is an independent unitary random matrix which is Haar distributed (*cf* proof of lemma V.5). Therefore

$$\begin{aligned}
\psi_n'(s) &\leq c\gamma\psi_n(s) + \psi_n(s)\int_{-\alpha}^{\alpha}\left(\mathbb{E}\left[\operatorname{tr}\exp\left(ctV_nV_n^*\right)\right] - \operatorname{tr}I - \frac{ct}{1+t^2}\right)\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt) \\
&= \psi_n(s)\left(c\gamma + \int_{-\alpha}^{\alpha}\left(\mathbb{E}\left[\operatorname{tr}\left(I + V_nV_n^*(e^{ct}-1)\right)\right] - \operatorname{tr}I - \frac{ct}{1+t^2}\right)\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt)\right) \\
&= \psi_n(s)\left(c\gamma + \int_{-\alpha}^{\alpha}\left(e^{ct} - 1 - \frac{ct}{1+t^2}\right)\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt)\right).
\end{aligned}$$

Noting that $\psi_n(0) = 1$, we can deduce that

$$\forall s \geq 0, \quad \psi_n(s) \leq \exp\left(s\left(c\gamma + \int_{-\alpha}^{\alpha}\left(e^{ct} - 1 - \frac{ct}{1+t^2}\right)\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt)\right)\right),$$

from which we conclude that $\psi_n(1)$ is indeed uniformly bounded in n. □

Proof of theorem V.6. We shall adapt the proof of theorem 25.3 in [21]. Let us first define ν_0 and ν_1 by

$$\begin{aligned}
R_{\nu_0} &= \gamma' + \int_{-1}^{1}\frac{t+z}{1-tz}\sigma_{\boxplus}^{(\nu)}(dt) \\
R_{\nu_1} &= \int_{|t|>1}\left(\frac{t+z}{1-tz} + \frac{1}{t}\right)\sigma_{\boxplus}^{(\nu)}(dt) = \int_{|t|>1}\frac{t}{1-tz}\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt),
\end{aligned}$$

where γ' is such that $\nu_0 \boxplus \nu_1 = \nu$. Then

$$\nu\left(g\left(|\cdot|\right)\right) = \nu_0 \boxplus \nu_1\left(g\left(|\cdot|\right)\right) \leq K\nu_0\left(g\left(|\cdot|\right)\right)\nu_1\left(g\left(|\cdot|\right)\right). \tag{V.1}$$

Since g is a sub-multiplicative locally bounded function, there exist b and c such that $g(x) \leq be^{c|x|}$ for all x in $\mathbb{R}$ (*cf* lemma 25.5 [21]). Therefore, due to lemma V.7, $\nu_0\left(g\left(|\cdot|\right)\right)$ is finite.

Consider now $\mu_1 = \Lambda^{-1}(\nu_1) \in \mathcal{ID}_*(\mathbb{R})$. Then it is a consequence of the already mentioned theorem 3.4 of [7] that ν_1 is the weak limit of $\left(\mu_1^{*\frac{1}{p}}\right)^{\boxplus p}$ as p tends to infinity.

Since g is freely sub-multiplicative, we have:

$$\left(\mu_1^{*\frac{1}{p}}\right)^{\boxplus p}\left(g\left(|\cdot|\right)\right) \leq K^{p-1}\left(\mu_1^{*\frac{1}{p}}\left(g\left(|\cdot|\right)\right)\right)^p.$$

But, since the Fourier transform of μ_1 is equal to

$$\mu_1\left(e^{i\lambda\cdot}\right) = \exp\left(\int_{|t|>1}\left(e^{i\lambda t} - 1\right)\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt)\right),$$

μ_1 is a compound Poisson distribution, and one gets (*cf* theorem 4.3 in [21]):

$$\begin{aligned}
\mu_1^{*\frac{1}{p}}\left(g\left(|\cdot|\right)\right) &= e^{-\int_{|t|>1}\frac{1}{p}\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt)} \\
&\times \sum_{m=0}^{\infty}\frac{1}{m!}\int_{|t_1|>1}\cdots\int_{|t_m|>1}g\left(|t_1 + \cdots + t_m|\right)\prod_{j=1}^{m}\left(\frac{1}{p}\frac{1+t_j^2}{t_j^2}\sigma_{\boxplus}^{(\nu)}(dt_j)\right).
\end{aligned}$$

Now, g is freely sub-multiplicative, therefore sub-multiplicative. This gives

$$\begin{aligned}
\mu_1^{*\frac{1}{p}}\left(g\left(|\cdot|\right)\right) &\leq e^{-\int_{|t|>1}\frac{1}{p}\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt)}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{K^{m-1}}{p^m}\left(\int_{|t|>1}g\left(|t|\right)\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt)\right)^m \\
&\leq \frac{1}{K}\exp\left(\frac{K}{p}\int_{|t|>1}g\left(|t|\right)\frac{1+t^2}{t^2}\sigma_{\boxplus}^{(\nu)}(dt)\right) \\
&\leq \frac{1}{K}\exp\left(\frac{2K}{p}\sigma_{\boxplus}^{(\nu)}\left(g\left(|\cdot|\right)\right)\right).
\end{aligned}$$

Hence, it holds that

$$\left(\mu_1^{*\frac{1}{p}}\right)^{\boxplus p}\left(g\left(|\cdot|\right)\right) \leq \frac{1}{K}\exp\left(2K\sigma_{\boxplus}^{(\nu)}\left(g\left(|\cdot|\right)\right)\right).$$

Letting p go into infinity, we deduce that

$$\nu_1\left(g\left(|\cdot|\right)\right) \leq \frac{1}{K}\exp\left(2K\sigma_{\boxplus}^{(\nu)}\left(g\left(|\cdot|\right)\right)\right),$$

which concludes the proof with inequality (V.1). □

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