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Tagged Particle Limit for a Fleming-Viot Type System

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Abstract

We consider a branching system of N Brownian particles evolving independently in a domain D during any time interval between boundary hits. As soon as one particle reaches the boundary it is killed and one of the other particles splits into two independent particles, the complement of the set D acting as a catalyst or hard obstacle. Identifying the newly born particle with the one killed upon contact with the catalyst, we determine the exact law of the tagged particle as N approaches infinity. In addition, we show that any finite number of labelled particles become independent in the limit. Both results can be seen as scaling limits of a genome population undergoing redistribution present in the Fleming-Viot dynamics.

Key words: Fleming-Viot, propagation of chaos, tagged particle, absorbing Brownian motio.

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1 Introduction

In [11], the authors prove a hydrodynamic limit for a system of Brownian motions confined to a bounded open connected set D in R^d satisfying the exterior cone condition, endowed with a branching mechanism inspired by the Fleming-Viot superprocess (see Dawson [4]). In agreement with the original model, the system conserves the number of particles $N \in \mathbb{Z}_+$. However, the generation mechanism is different. As soon as one of the particles reaches the boundary ∂D, an independent Brownian particle is created at one of the sites of the survivors, chosen with uniform probability. The model (in lattice and continuous version) is due to Burdzy, Hołyst, Ingerman and March in [2], and later studied (in its present continuous version) by some of the same authors in [3], where a law of large numbers is established for the empirical measures at fixed times. The present paper continues the investigation from [11] where a hydrodynamic limit for the joint law of the empirical measure and the average number of redistributions (boundary hits) was proven. The quantities of interest are seen as random elements in the Skorohod space. The conservation of mass present in this dynamics leads to a smooth density profile corresponding to the absorbing heat kernel conditioned on the event that a Brownian particle has not reached the boundary, as seen in equation (2.9).

We are interested in following the evolution of a finite number of particles, with fixed labels (tagged particles) as $N \to \infty$. Two questions are answered in this paper: i) we identify the tagged particle process (Theorem 2) as well as ii) we prove the so-called propagation of chaos, or the asymptotic independence of any finite collection of tagged particles (Theorem 3).

The 'particle process' point of view adopted in the following allows a fully analytic construction and description of the evolution by considering that a Brownian particle reaching the boundary will not be killed but be reborn at a different location. The Fleming-Viot and the Moran particle systems (a reference in that direction is [12]) are models for the time-evolution of the allelic profiles of a given genome population. The state space D is a set of *viable configurations* while the boundary ∂D represents a barrier of unacceptable configurations. The boundary acts as a catalyst for redistribution. Hence the Brownian motions should not be regarded as actual spatial evolutions but as a slow diffusive mutation process interrupted by the 'cloning' of a randomly chosen viable genetic profile at the time of reaching a non-admissible or non-viable profile. The adoption of the particle model perspective is more valuable at the statistical level. On one hand, Theorem 1 establishes the *macroscopic profile* of the population. This is the point estimate of the configuration evolving in time, proven as a joint law of large numbers in [11]. This law is sampling anonymous individuals and reflects the statistical distribution of the average genome. The *tagged particle* problem is to determine the evolution of a finite collection of labelled individuals (or strains), that is to study the genealogy of a collection of 'families' (a sequence of individuals and their offsprings) immersed in a vast population as $N \to \infty$. Equilibrium is of great interest in this context. A striking consequence is that even when the empirical profile is stationary, as derived in Corollary 1, no longer anonymous families or strains of a species undergo changes which can be described in a mathematically precise sense.

The answer (Theorem 2) is a mixed process of the type described in [7], more precisely a Brownian particle (profile) which is reborn (copied) in the interior of the region D after reaching the boundary (one of the non-viable profiles) and the distribution of the birth location (profile of the newborn) has density equal to the hydrodynamic limit (2.10). Since the relative mass of any finite collection of particles becomes negligible as $N \to \infty$, the updating of the branching mechanism depends essentially on the complementary set of particles only. As a consequence, any given finite collection of labelled individuals in a virtually infinite population will not 'see' each other. The propagation of chaos holds at the level of the full trajectory space (not just for marginals at a given time $t > 0$), in the sense that the joint law of the tagged particles converges weakly to the product law of the individual particles.

The question of determining the law of the tagged particle process is usually very hard (see Kipnis and Landim [15] and the references thereof) and depends on the strength of the interaction. In this case, the interaction is given by the jump (mixing) mechanism, the system being a d-dimensional family of independent Brownian motions between successive boundary hits. The diffusive nature of the limit is clear. Another helpful feature of the process is that it is essentially mean-field, in that the new location where the particle is reborn is uniformly chosen. During any finite time interval of length Δt, the system undergoes a number of boundary hits of order N, which modifies the limit of the density profile. Even though the "tracer" particle will suffer essentially one boundary hit (similar but not identical to a Poisson event with rate $\sim \Delta t$), it is its rebirth location which is completely modified during the interval. As a consequence, the tagged particle is nontrivially influenced by the cloud of mass evolving within the boundaries of the region D.

2 Notation and Results.

A few notations are needed, all consistent with [11]. For readability, these are re-introduced in the present context. Denote by $\mathbf{x} = (x_1, x_2, \dots, x_N)$ the points in $\overline{D}^N$ and let $\mathbf{x}^N(t)$ be the configuration of the process at time $t \geq 0$, indexed by a fixed N. For initial configuration $\mathbf{x}(0) = \mathbf{x}$, $P^N_{\mathbf{x}}$ or simply P^N is the law of the process. In general, we shall consider that all processes $\{\mathbf{x}^N(\cdot)\}$, for all N, are constructed on the same probability space $(\Omega, \mathcal{F}, P)$ with the same filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

For any $f \in C(\overline{D}^N)$, $\mathbf{x} \in \overline{D}^N$ and i, j two indices between 1 and N, $f^{ij}(\mathbf{x})$ will denote the $N-1$ variable function depending on $\mathbf{x}^N$ with the exception of the component x_i which is replaced by x_j, that is

$$f^{ij}(\mathbf{x}) = f(x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_N)\,. \tag{2.1}$$

Let also Δ_N be the Nd dimensional Laplacian.

The point processes $\{A^N_{ij}(\cdot)\}_{1\leq i\neq j\leq N}$, adapted to the filtration $\mathcal{F}$, denoting the number of times particle i has jumped on the location of the surviving particle j up to time $t > 0$, with right continuous paths with limit to the left (rcll) are finite and their sums $A^N_i(\cdot) = \sum_{j\neq i} A^N_{ij}(\cdot)$, $1 \leq i \leq N$ converge to infinity as $t \to \infty$ almost surely as shown in [3]. The construction of the process implies the following proposition proved in [11].

Proposition 1. *For any function $f \in C(\overline{D}^N)$, with f smooth up to the boundary, we write*

$$\mathbf{A}_f(t) = \sum_{i=1}^N \frac{1}{N-1} \int_0^t \sum_{j\neq i} \Big(f^{ij}(\mathbf{x}(s-)) - f(\mathbf{x}(s-))\Big) dA^N_i(s)\,. \tag{2.2}$$

Then,

$$f(\mathbf{x}(t)) - f(\mathbf{x}(0)) - \int_0^t \frac{1}{2}\Delta_N f(\mathbf{x}(s))ds \; - \mathbf{A}_f(t) = \mathcal{M}^{N,B}_f(t) + \mathcal{M}^{N,J}_f(t) \tag{2.3}$$

where

$$\mathcal{M}_f^{N,B}(t) = \sum_{i=1}^N \int_0^t \nabla_{x_i} f(\mathbf{x}(s)) \cdot d\mathbf{w}_i(s) \tag{2.4}$$

is the Brownian martingale and $\mathcal{M}_f^{N,J}(t)$ *is the jump martingale for which*

$$(\mathcal{M}_f^{N,J}(t))^2 - \frac{1}{N-1} \sum_{i=1}^N \int_0^t \sum_{j \neq i} \Big(f(\mathbf{x}^{ij}(s-)) - f(\mathbf{x}(s-)) \Big)^2 dA_i^N(s) \tag{2.5}$$

is a martingale. All martingales are P*-martingales with respect to the filtration* $\mathcal{F}$*.*

Remark 1: Since the support of the counting measures $\{dA_i^N(t)\}_{t \geq 0}$ is the set of hitting times of the boundary, the function $f(\mathbf{x}(s-))$ in (2.2) has the i^{th} component situated on ∂D.

Remark 2: By construction $f^{ij}(\mathbf{x}(s)) - f(\mathbf{x}(s-)) = f^{ij}(\mathbf{x}(s-)) - f(\mathbf{x}(s-))$ on the support of $dA_i^N(t)$, which makes the integrand $\mathcal{F}_{s-}$-measurable.

Definition 1. *For any* $N \in \mathbb{Z}_+$ *we define the empirical distribution process*

$$\mu^N(t, dx) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N(t)} \tag{2.6}$$

and the average number of jumps

$$A^N(t) = \frac{1}{N-1} \sum_{i=1}^N A_i^N(t)\,. \tag{2.7}$$

Let $p_{abs}(t, x, y)$ be the absorbing Brownian kernel on the set D and, for a finite measure $\mu(dx) \in \mathcal{M}(D)$ we denote by $u(t, y) = \int_D p_{abs}(t, x, y)\mu(dx)$ the solution in the sense of distributions to the heat equation with Dirichlet boundary conditions

$$\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta_x u(t, x) \qquad u(t, x)\Big|_{x \in \partial D} = 0 \qquad u(0, x) = \mu(dx)\,. \tag{2.8}$$

We also define $z(t) = \int_D u(t, x)dx > 0$ the probability of survival up to time $t > 0$ of a Brownian particle killed on the boundary ∂D starting at $t = 0$ with distribution $\mu(dx)$. The solution to the heat equation with Dirichlet boundary conditions conditional on survival up to time t is $v(t, x) = z(t)^{-1} u(t, x)$ and $\mu(t, dx) = v(t, x)dx$ is the weak solution of

$$\frac{\partial}{\partial t} v(t, x) = \frac{1}{2} \Delta_x v(t, x) - \frac{z'(t)}{z(t)} v(t, x) \qquad v(t, x)\Big|_{x \in \partial D} = 0 \qquad v(0, x) = \mu(dx)\,. \tag{2.9}$$

We shall state the hydrodynamic limit results as they appear in [11].

Theorem 1. (from [11]) *If* $\mu^N(0, dx)$ *converges in probability in weak sense to a deterministic initial density profile* $\mu(dx) = \mu(0, dx)$ *such that* $\mu(D) = 1$*, then, for any* $T > 0$*, the joint distribution of* $(A^N(\cdot), \mu^N(\cdot, dx)) \in \mathbf{D}([0, T], \mathbb{R}_+ \times \mathcal{M}(D))$ *is tight in the Skorohod topology and the set of limit points is a delta function concentrated on the unique continuous trajectory* $(-\ln z(\cdot), \mu(\cdot, dx))$ *as defined in (2.9) and, for any* $\phi \in C^2(\overline{D})$ *and any* $\epsilon > 0$

$$\lim_{N \to \infty} P\Big(\sup_{t \in [0,T]} \Big| \frac{1}{N} \sum_{i=1}^N \phi(x_i^N(t)) - \int_D \phi(x)\mu(t, dx) \Big| > \epsilon \Big) = 0\,. \tag{2.10}$$

Let $M^N(dx)$ be the unique stationary distribution of the process $\{\mathbf{x}^N(\cdot)\}$ (the measure exists according to [3]) and $\Phi_1(x)$ be the first eigenfunction of the Laplacian with Dirichlet boundary conditions normalized such that it integrates to one over D. The next corollary is based on [2] and [11].

Corollary 1. (from [3] and [11]) *Assume the process $\{\mathbf{x}^N(\cdot)\}$ is in equilibrium at time $t = 0$. Then, the family of empirical measure processes $\{\mu^N(\cdot, dx)\}$, indexed by $N \in \mathbb{Z}_+$, is tight in the Skorohod space $\mathbf{D}([0,T], \mathcal{M}(D))$ and the unique limit point is the delta function concentrated on the constant measure $\Phi_1(x)dx$.*

We are now ready to state the main results of this paper, theorems 2 and 3 and their corollaries.

Theorem 2. *Assume that $\{x_1^N(0)\}_{N\in\mathbb{Z}_+}$ converges in probability to a random variable with probability distribution $\alpha_1(dx_1)$ such that $\alpha_1(D) = 1$ and that the other conditions of Theorem 1 are satisfied. Let $\mu(\cdot, dx)$ be the hydrodynamic limit profile and $\{\mathcal{F}_t\}_{t\geq 0}$ the filtration of the processes $\{\mathbf{x}^N(\cdot)\}$. Then $x_1^N(\cdot) \in \mathbf{D}([0,\infty), D)$ converges in distribution to a process $x_1(\cdot) \in \mathbf{D}([0,\infty), D)$ uniquely characterized by the properties:*

(i) $P(x_1(0) \in dx_1) = \alpha_1(dx_1)$,

(ii) with probability one, there exists a strictly increasing sequence $\{\tau_k\}_{k\in\mathbb{Z}_+}$ of stopping times with $\tau_0 := 0$ such that $\lim_{k\to\infty} \tau_k = \infty$ and for all $k \geq 0$, $x_1(\cdot)$ is continuous on $t \in [\tau_k, \tau_{k+1})$, with $x_1(\tau_k-) \in \partial D$ for all $k \geq 1$,

(iii) the process defined as $\{x_1((t + \tau_k) \wedge \tau_{k+1})\}_{t\geq 0}$ for $t \in [0, \tau_{k+1} - \tau_k)$ and equal to $x_1(\tau_{k+1}-)$ for $t \geq \tau_{k+1} - \tau_k$ is a Brownian motion with absorbing boundary conditions in D with respect to $\{\mathcal{F}_{t+\tau_k}\}_{t\geq 0}$ starting at $x_1(\tau_k)$, and

(iv) for any $\phi \in C^2(\overline{D})$, $E\Big[\phi(x_1(\tau_k)) \,\Big|\, \mathcal{F}_{\tau_k -}\Big] = \langle \phi, \mu(\tau_k, dx_1)\rangle$.

Remark 1: Due to the continuity of $\mu(\cdot, dx_1)$ for $t > 0$, the function $\langle \phi, \mu(\tau_k, dx_1)\rangle$ is measurable with respect to the σ - field $\mathcal{F}_{\tau_k -}$. Moreover, the redistribution probability $\mu(\tau_k, dx_1)$ depends exclusively on the boundary hitting time τ_k and not on the location $x_1(\tau_k-) \in \partial D$.

Remark 2: The initial values of $\mu(\cdot, dx_1)$ are *not* given by $\alpha_1(dx_1)$. One particle cannot influence the initial distribution alone, and the environment process may produce a limiting profile unrelated to the location of the tagged particle.

Remark 3: Essentially the characterization of the limiting process is inductive. On the time interval $[0, \tau_1)$ the process is a Brownian motion killed at the boundary ∂D starting at the limit point x_1 of the initial values $\{x_1^N(0)\}_{N\in\mathbb{Z}_+}$. At τ_1 the particle reaches the boundary and jumps to a point in D chosen randomly according to the distribution prescribed by the time-dependent deterministic measure on D evaluated at its current value $\mu(\tau_1-, dx)$. The new starting point is in the open set D and the construction is repeated indefinitely. This is the *Brownian motion with return* from [7], [8], [9], [10] in the case of a dynamic relocation distribution $\mu(\cdot, dx)$.

Corollary 2. *Under the same conditions as in Theorem 2, $\{(x_1^N(\cdot), A_1^N(\cdot))\}_{N\in\mathbb{Z}_+}$ converges weakly as $N \to \infty$ to the joint process $(x_1(\cdot), A_1(\cdot))$, with distribution given by a probability measure $Q_{\alpha_1,\mu}$ on $\mathbf{D}([0,\infty), D \times \mathbb{R}_+)$ uniquely determined by the following properties:*

(i) $Q_{\alpha_1,\mu}$ - almost surely $A_1(t)$ is a counting process, in the sense that there exists an increasing sequence of stopping times $\{\tau_k\}_{k\geq 0}$ with $\tau_0 = 0$ and $\lim_{k\to\infty}\tau_k = \infty$ such that, for all $k \geq 0$, $A_1(t)$ is constant on every $[\tau_k, \tau_{k+1})$ and has a jump of size one at every τ_k,

(ii) $A_1(t)$ is the number of times $s \in (0,t]$ such that $x_1(s-) \in \partial D$ and $x_1(\cdot)$ is continuous on $[\tau_k, \tau_{k+1})$ far all $k \geq 0$,

(iii) for any $\phi \in C^2(\overline{D})$, $Q_{\alpha_1,\mu}(x_1(0) \in dx_1) = \alpha_1(dx_1)$ and

$$\phi(x_1(t)) - \phi(x_1(0)) - \int_0^t \frac{1}{2}\Delta_d \phi(x_1(u))du - \int_0^t \Big(\langle \phi, \mu(u,dx)\rangle - \phi(x_1(u-))\Big) dA_1(u) \tag{2.11}$$

is a $Q_{\alpha_1,\mu}$- martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Here $\langle \phi, \eta\rangle$ denotes the integral of ϕ with respect to the measure $\eta(dx)$ on D.

Remark: Corollary 2 is a consequence of Theorem 2 and Proposition 12.

Theorem 3. *Under the same conditions as in Theorem 2, for $l \in \mathbb{Z}_+$ and $N >> l$ we denote by $\{x_j^N(\cdot)\}_{1\leq j\leq l}$ a collection of l tagged particles such that the starting points $x_j^N(0)$ converge in distribution to the random points $x_j \in D$ with joint probability distribution $\alpha^{(l)}(dx_1, dx_2, \ldots, dx_l) = \otimes_{j=1}^l \alpha_j(dx_j)$, where $\{\alpha_j(dx_j)\}_{1\leq j\leq l}$ are such that $\alpha_j(D) = 1$ for all $j = 1, 2, \ldots, l$. Then, the processes $\{x_j^N(\cdot)\}_{1\leq j\leq l}$ converge to a collection of l independent processes satisfying (i)-(iv) from Theorem 2 with initial distributions $\alpha_j(dx_j)$, for all $j = 1, 2, \ldots, l$.*

In general, let $Q_{\alpha,\mu}$ denote the probability law of the tagged particle process defined by Theorem 2 and Corollary 2 with initial distribution $\alpha(\cdot)$ and redistribution measure $\mu(\cdot, dx)$ on the rcll paths $\mathbf{D}([0,\infty), D)$.

Corollary 3. *Let $T > 0$ be fixed but arbitrary as above. Assume that the joint distribution of $\{x_j\}_{1\leq j\leq N}$ is symmetric and that the empirical measure $N^{-1}\sum_{i=1}^N \delta_{x_i}$ at $t = 0$ converges in distribution to a deterministic initial profile $\alpha(dx)$ concentrated on the open set D and let $\mu(t,dx)$ be the solution to (2.9) with $\mu(0,dx) = \alpha(dx)$. If Q^N is the law of the empirical process $N^{-1}\sum_{i=1}^N \delta_{x_i^N(\cdot)}$ on $\mathbf{D}([0,T], D)$, then Q^N converges weakly in probability to $Q_{\alpha,\mu}$. More precisely, if $G \in C_b(\mathbf{D}([0,T], D), \mathbb{R})$ is a bounded continuous test function on the Skorohod space, then for any $\epsilon > 0$*

$$\lim_{N\to\infty} P\left(\Big| \frac{1}{N}\sum_{i=1}^N G(x_i^N(\cdot)) - E^{Q_{\alpha,\mu}}[G(\cdot)] \Big| > \epsilon \right) = 0\,. \tag{2.12}$$

Remark: The initial distribution of the tagged particle will be the same for all particles due to symmetry, which implies it must be equal to $\alpha(dx) = \mu(0,dx)$. A simple case is when initially the particles are i.i.d. with distribution $\alpha(dx)$. The particles are coupled at any time $t > 0$ yet they satisfy the weak law of large numbers at the full path space (propagation of chaos).

3 The tagged particle.

For $r > 0$ sufficiently small we define the set

$$D_r = \Big\{ x \in D \;:\; d(x, \partial D) > r \Big\}. \tag{3.1}$$

Definition 2. *Let r_D be the inner radius of the domain D, defined as the supremum of all $r > 0$ with the properties that D_r is connected and ∂D_r is of the same regularity class as ∂D, in our case, C^2. For $r \in (0, r_D/2)$, we define the function $\gamma_r \in C^2(\overline{D})$ as a smooth version of $\mathbf{1}_{D_r^c}$ with the properties (i) $0 \le \gamma_r(x) \le 1$ if $x \in \overline{D}$, (ii) $\gamma_r(x) = 1$ if $x \in D_r^c$, (iii) $\gamma_r(x) = 0$ if $x \in D_{2r}$ and (iv) $\|\Delta \gamma_r(x)\|_\infty \le c(D) r^{-2}$ for a constant $c(D)$ determined by the domain D and independent of $r > 0$.*

The next two propositions give lower bounds for the number of particles situated *inside* the domain D, or, more precisely, in an open subset G such that $\overline{G} \subset D$. Proposition 3 is more general. Proposition 2 is proven in [11] as an intermediate step in the proof of the hydrodynamic limit. However, once this is established, the lower bound on the number of particles *inside* the domain can be generalized and can be regarded as a simple consequence of Theorem 1.

Proposition 2. *Recall $\gamma_r(x)$ from Definition 2 and define $\gamma_r^c(x) = 1 - \gamma_r(x) \ge 0$, which is smooth on $\overline{D}$ and vanishes on the boundary. Let $r_D(\mu) > 0$ be the largest radius r less than r_D such that $\mu(0, D_r) > 0$. Under the conditions of Theorem 1, for a given time interval $[0, T]$ and for any $r \le r_D(\mu)/2$ there exists a constant $C_r > 0$ and for each $N \in \mathbb{Z}_+$ an event $S_L^N(r)$ such that*

$$\begin{gathered} S_L^N(r) = \Big\{ \inf_{t \in [0,T]} \tfrac{1}{N} \textstyle\sum_{i=1}^N \gamma_r^c(x_i^N(t)) \le C_r \Big\} \quad \textit{and} \\ \limsup_{N \to \infty} P\Big(S_L^N(r) \Big) = 0 \,. \end{gathered} \tag{3.2}$$

Proposition 3. *Let $\phi \in C_c^2(D)$ be a nonnegative test function. Under the conditions of Theorem 1, for $t_0 > 0$ and any time interval $[t_0, T]$, there exists a constant $C_L(\phi) > 0$ and for each $N \in \mathbb{Z}_+$ an event $S_L^N(\phi)$ such that*

$$\begin{gathered} S_L^N(\phi) = \Big\{ \inf_{t \in [t_0,T]} \tfrac{1}{N} \textstyle\sum_{i=1}^N \phi(x_i^N(t)) \le C_L(\phi) \Big\} \quad \textit{and} \\ \limsup_{N \to \infty} P\Big(S_L^N(\phi) \Big) = 0 \,. \end{gathered} \tag{3.3}$$

Let $\phi \in C^2(\overline{D})$ and for a given index i let $f(\mathbf{x}) = \phi(x_i)$. Then the formula (2.3)-(2.2) reduces to

$$\phi(x_i^N(t)) = \phi(x_i^N(0)) + \int_0^t \frac{1}{2} \Delta_d \phi(x_i^N(s)) ds \tag{3.4}$$

$$+ \int_0^t \Big(\frac{1}{N-1} \sum_{j \ne i} \phi(x_j^N(s)) - \phi(x_i^N(s-)) \Big) dA_i^N(s) + \mathcal{M}_\phi^N(t) \,,$$

where $\mathcal{M}_\phi^N(t)$ denotes the martingale part in Ito's formula (2.3).

Lemma 1. *Let $T_r \ge 0$ be the stopping time defined as*

$$T_r = \inf\{ t > 0 \,:\, \langle \gamma_r^c(x), \mu^N(t, dx) \rangle < C_r \} \tag{3.5}$$

where $\gamma_r^c(x)$ is as in Proposition 2 and $T_r = \infty$ if the infimum is taken over the empty set. Then there exists a constant $C_1(r, T)$ independent of N such that

$$E\Big[A_1^N(T \wedge T_r)^2 \Big] \le C_1(r, T) \,. \tag{3.6}$$

Proof. We apply (3.4) for the function $\phi = \gamma_r^c$ and $i = 1$ to obtain

$$C_r\Big(A_1^N(T\wedge T_r) - A_1^N(0)\Big) \le \inf_{u\in[0,T\wedge T_r]}\Big\{\frac{1}{N}\sum_{j\neq 1}\gamma_r^c(x_j(u))\Big\}\Big(A_1^N(T\wedge T_r) - A_1^N(0)\Big) \tag{3.7}$$

$$\le \int_0^{T\wedge T_r}\Big(\frac{1}{N-1}\sum_{j\neq 1}\gamma_r^c(x_j(u)) - \gamma_r^c(x_1(u-))\Big)dA_1^N(u)$$

$$= \gamma_r^c(x_1(T\wedge T_r)) - \gamma_r^c(x_1(0)) - \int_0^{T\wedge T_r}\frac{1}{2}\Delta_d\gamma_r^c(x_1(u))du$$

$$- \int_0^{T\wedge T_r}\nabla_{x_1}\gamma_r^c(x_1(u))\cdot d\mathbf{w}_1(u) - \mathcal{M}_{\gamma_r^c}^{N,J}(T\wedge T_r)$$

observing that the martingales present in the equations remain martingales due to the optional stopping theorem. The parameter r is fixed. We divide by the constant $C_r > 0$, square both sides of the inequality, apply Schwarz's theorem on the right hand side to obtain

$$E\Big[A_1^N(T\wedge T_r)^2\Big] \le C'(r,T) + C''(r,T)E\Big[A_1^N(T\wedge T_r)\Big].$$

We denote $U^2 = E\Big[A_1^N(T\wedge T_r)^2\Big]$, apply Schwarz's inequality to the first moment of $A_1^N(T\wedge T_r)$ from the right hand side. We conclude that $U^2 \le 2C' + (C'')^2 := C_1(r,T)$ from (3.6). ☐

Proposition 4. *Under the conditions of Theorem 1, for a given time interval* $[0,T]$ *and constants* $M > 0$,

$$\lim_{M\to\infty}\limsup_{N\to\infty} P\Big(A_1^N(T) > M\Big) = 0. \tag{3.8}$$

Proof. With the notations of Lemma 1

$$P\Big(A_1^N(T) > M\Big) = P\Big(A_1^N(T) > M\,,\, T_r \le T\Big) + P\Big(A_1^N(T) > M\,,\, T_r > T\Big)$$

$$\le P\Big(S_L^N(r)\Big) + P\Big(A_1^N(T\wedge T_r) > M\,,\, T_r > T\Big)$$

$$\le P\Big(S_L^N(r)\Big) + M^{-2}E\Big[A_1^N(T\wedge T_r)^2\Big] \le P\Big(S_L^N(r)\Big) + M^{-2}C_1(r,T)$$

where we used the same notation for the set $S_L^N(r)$ as in (3.2) and applied Chebyshev's inequality based on the second moment from Lemma 1. The first probability tends to zero as $N \to 0$ and the second term vanishes as $M \to 0$. ☐

Let $\{\tau_q\}_{q\ge 1}$ be the ordered sequence of hitting times to the boundary of particle $x_1^N(\cdot)$, with the additional setting $\tau_0 := 0$. With probability one, the ordering is possible and the limit $\lim_{k\to\infty}\tau_k = \infty$ by construction (see [3], [11]).

Proposition 5. *Let* $k \ge 1$ *be fixed. Then*

$$\lim_{\eta\to 0}\limsup_{N\to\infty} P\Big(\tau_k - \tau_{k-1} \le \eta\,,\, \tau_k \le T\Big) = 0. \tag{3.9}$$

Proof. The probability of interest from (3.9) will be treated separately for $k = 1$ and $k > 1$. Write

$$\begin{aligned} p_1 &= P\Big(\tau_1 - \tau_0 \leq \eta\,,\, \tau_1 \leq T\Big)\,, \quad k = 1 \\ p_2 &= P\Big(\tau_k - \tau_{k-1} \leq \eta\,,\, \tau_k \leq T\Big)\,, \quad k > 1 \end{aligned}\,. \tag{3.10}$$

According to Theorem 2, the initial point $x_1^N \Rightarrow x_1$, where x_1 is a random variable with distribution $\alpha_1(dx_1)$ concentrated on the open set D and $\Rightarrow$ designates convergence in distribution. The distribution function of the first exit time from D for a Brownian motion $P_{x_1}(\tau_D < \eta)$ is a continuous and bounded function of the starting point x_1, implying that

$$\limsup_{N\to\infty} P\Big(A_1^N(\eta) \geq 1\Big) \leq \limsup_{N\to\infty} P_{x_1^N}\Big(\tau_D \leq \eta\Big) = \int_D P_{x_1}\Big(\tau_D \leq \eta\Big)\alpha_1(dx_1)\,. \tag{3.11}$$

The limit as $\eta \to 0$ of (3.11) is zero by dominated convergence establishing the limit of the first term p_1 in (3.10).

Let $r > 0$ be a small but otherwise arbitrary number. Then $p_2 \leq p_{21} + p_{22}$ where

$$p_{21} = P\Big(\tau_k - \tau_{k-1} \leq \eta\,,\, \tau_k \leq T\,,\, x_1^N(\tau_{k-1}) \in \overline{D_r}\Big)$$
$$p_{22} = P\Big(\tau_k - \tau_{k-1} \leq \eta\,,\, \tau_k \leq T\,,\, x_1^N(\tau_{k-1}) \in \overline{D_r}^c\Big)\,.$$

We bound p_{21} uniformly in $N > 0$,

$$\lim_{\eta\to 0}\limsup_{N\to\infty} p_{21} \leq \lim_{\eta\to 0}\sup_{z_1\in\overline{D_r}} P\Big(\tau^D(w_{z_1}) \leq \eta\Big) = 0\,. \tag{3.12}$$

Pick $\delta/2 > \sup_{t\in[0,T]}\int_D \gamma_{2r}(x)\mu(t,dx)$. Then

$$\begin{aligned} p_{22} \leq P\Big(\Big\{x_1^N(\tau_{k-1}-) \in \partial D \text{ jumps to } x_1^N(\tau_{k-1}) \in \overline{D_r}^c\Big\}\Big) &\leq \\ E\left[(\frac{1}{N-1})\#\{j\,:\,2 \leq j \leq N\,,\, x_j^N(\tau_{k-1}) \in \overline{D_r}^c\}\right] &\leq \\ \delta + P\Big(\sup_{t\in[0,T]}\frac{1}{N}\sum_{j=2}^N \gamma_{2r}(x_j^N(t)) > \delta\Big) &\leq \\ \delta + P\Big(\sup_{t\in[0,T]}\Big|\frac{1}{N}\sum_{j=1}^N \gamma_{2r}(x_j^N(t)) - \int_D \gamma_{2r}(x)\mu(t,dx)\Big| > \delta/2 - \frac{1}{N}\Big)&\,, \end{aligned} \tag{3.13}$$

where we switched to $2r$ in order to include D_r^c in the support of the smooth function $\gamma_r(\cdot)$. In addition, we subtract $1/N$ to compensate for the particle #1 missing in the summation. Due to the hydrodynamic limit (2.10) from Theorem 1, the last probability from the bound of p_{22} vanishes as $N \to \infty$. We have proven that p_{22}, and hence the left side of (3.9), is bounded above by δ for arbitrary small $r > 0$ and $\delta/2 > \sup_{t\in[0,T]}\int_D \gamma_{2r}(x)\mu(t,dx)$. By letting $r \to 0$ and $\delta \to 0$ in that order, we obtain (3.9). □

Proposition 6. *For any* $m \in \mathbb{Z}_+$,

$$\lim_{\eta\to 0}\limsup_{N\to\infty} P\Big(\tau_m \in [T-\eta, T]\Big) = 0\,. \tag{3.14}$$

Proof. If $m = 1$ the problem is essentially independent of N since the initial point converges in distribution to a random point $x_1 \in D$ distributed according to $\alpha_1(dx_1)$ such that $\alpha_1(D) = 1$. This implies that we can partition the event into either $x_1^N \in D_r$ or $x_1^N \in D_r^c$. Since D_r^c is a closed set,

$$\lim_{r\to 0} \limsup_{N\to\infty} P\Big(x_1^N \in D_r^c\Big) \le \lim_{r\to 0} \alpha(D_r^c) = 0\,.$$

Let $m \ge 2$ and $\eta' \in (0, T-\eta)$ arbitrary otherwise. We can bound the probability from (3.14) by $p_1' + p_2'$ where

$$p_1' = P\Big(\tau_m \in [T-\eta, T)\,,\, \tau_{m-1} < T-\eta-\eta'\Big)\,, \qquad p_2' = P\Big(\tau_m\,,\, \tau_{m-1} \in [T-\eta-\eta', T)\Big)\,.$$

For any given N the random variables τ_k, $x_1(\tau_k)$, $k = 1, 2, \ldots$ have absolutely continuous distribution functions. Let r be a positive number $r < r_D$. The probability $p_2' = p_2'(\eta+\eta')$ has the same bounds as p_2 in (3.10) with η replaced by $\eta+\eta'$, since it is the probability of having two consecutive jumps in a time interval of length less than $\eta+\eta'$. Following the argument for p_2 from equation (3.12), we only need to ensure that $s \to \sup_{z_1\in\overline{D_r}} P\Big(\tau^D(w_{z_1}) \le s\Big)$ has limit zero as $s \to 0$ in order to have $\lim_{\eta'\to 0} \limsup_{\eta\to 0} p_2'(\eta+\eta') = 0$. To evaluate p_1', we bound $p_1' \le p_{11}' + p_{12}'$ with

$$p_{11}' = P\Big(T-\tau_{m-1}-\eta \le \tau_m - \tau_{m-1} < T-\tau_{m-1}\,,\, x_1(\tau_{m-1}) \in \overline{D_r}\,,\, \tau_{m-1} < T-\eta-\eta'\Big)\,,$$

$$p_{12}' = P\Big(x_1(\tau_{m-1}) \in D_r^c\Big)\,.$$

The second probability converges to zero as $r \to 0$ along the same lines of the proof for p_{22} in (3.13) in Proposition 5. It remains to bound the first term. Note that $T-\tau_{m-1}-\eta > \eta'$ and $T-\tau_{m-1} \le T$. Then

$$\begin{aligned} p_{11}' &\le P\Big(\eta' \le \tau_m - \tau_{m-1} < T\,,\, x_1(\tau_{m-1}) \in \overline{D_r}\,,\, \tau_{m-1} < T-\eta-\eta'\Big) \\ &\le P\Big(\eta' \le \tau_m - \tau_{m-1} < T\,,\, x_1(\tau_{m-1}) \in \overline{D_r}\Big) \\ &= \int_{\overline{D_r}} \left[\int_D \Big(p_{abs}(\eta', x_1, y) - p_{abs}(T, x_1, y)\Big)\, dy\right] P\Big(x_1(\tau_{m-1}) \in d\,x_1\Big) \\ &\le \sup_{x_1\in\overline{D_r}} \int_D \Big(p_{abs}(\eta', x_1, y) - p_{abs}(T, x_1, y)\Big)\, dy \\ &\le \frac{(T-\eta')}{2} \sup_{\eta'\le s\le T}\ \sup_{x_1\in\overline{D_r}} |\Delta\tilde{u}(s, x_1)|\,, \end{aligned} \tag{3.15}$$

with $\tilde{u}(s, x_1) = \int_D p_{abs}(s, x_1, y)dy$. From the symmetry of the absorbing Brownian kernel we derive that $\tilde{u}(s, x_1)$ is the solution to the heat equation on D for the half Laplacian with zero boundary conditions and uniform initial distribution. The solution is smooth away from $t = 0$ by the properties of the Brownian kernel (for example [6]). On a compact space-time subset $[\eta', T] \times \overline{D_r}$ the second derivatives of the solutions are continuous, hence bounded. With this in mind, recall that η' is an arbitrary number in $(0, T-\eta)$. Take $\eta' = T - 2\eta$ and let $\eta \to 0$ to obtain the limit for p_{11}'. □

Let X be a Polish space with norm $\|\cdot\|$ and let $\mathbf{D}([0,T],X)$ be the Skorohod space of functions with left limits and right continuous on $[0,T]$. The following are necessary and sufficient conditions for tightness in $\mathbf{D}([0,T],X)$ of the family of processes $\{y^N(\cdot)\}_{N>0} \in \mathbf{D}([0,T],X)$ (cf. [1], also [15]).

We need to define the analogue of the modulus of continuity for $\mathbf{D}([0,T],X)$. Let $\eta > 0$ and $\pi(\eta)$ the class of partitions of the interval $[0,T]$ such that there exists $l \in \mathbb{Z}_+$ and intermediate points t_i, $0 \leq i \leq l$ such that

$$0 = t_0 < t_1 < \ldots < t_l = T\,, \qquad t_i - t_{i-1} > \eta\,, \quad 1 \leq i \leq l\,.$$

For a path $y(\cdot) \in \mathbf{D}([0,T],X)$ we define

$$w'_y(\eta) = \inf_{\pi(\eta)} \max_{1\leq i\leq l} \sup_{t_{i-1}\leq s<t<t_i} |y(t) - y(s)|\,. \tag{3.16}$$

With these notations, Prohorov's theorem states that the family of processes $\{y^N(\cdot)\}_{N>0} \in \mathbf{D}([0,T],X)$ is tight in $\mathbf{D}([0,T],X)$ if and only if the following conditions are satisfied

(i) for every $t \in [0,T]$ and every $\epsilon > 0$, there is an $M > 0$ such that

$$P\Big(|y^N(t)| > M\Big) \leq \epsilon\,, \tag{3.17}$$

(ii) for any $\epsilon > 0$

$$\lim_{\eta\to 0} \limsup_{N\to\infty} P\Big(w'_{y^N}(\eta) > \epsilon\Big) = 0\,. \tag{3.18}$$

Proposition 7. *Under the conditions of Theorem 1, the family indexed by $N \in \mathbb{Z}_+$ of the one-particle process $\{x_1^N(\cdot)\}$ on $\mathbf{D}([0,T],D)$ and the family of counting measures $\{A_1^N(\cdot)\}$ on $\mathbf{D}([0,T],\mathbb{R}_+)$ are tight in the Skorohod topology.*

Proof. Tightness of $A_1^N(\cdot)$. Condition (i) from the tightness criterion is fulfilled by Proposition 4. Due to the structure of the process $A_1^N(\cdot)$ which has only one unit increments at discrete times, we can write the probability in (3.18) as

$$P\Big(w'_{A_1^N}(\eta) \geq 1\Big) \leq P\Big(\max_{1\leq q\leq m(T)} |\tau_q - \tau_{q-1}| \leq \eta\Big) + P\Big(\tau_{m(T)} \in [T-\eta, T]\Big) \tag{3.19}$$

where $\tau_1, \tau_2, \ldots$ are the jump times of the counting process $A_1^N(\cdot)$ defined before Proposition 6. Here we define $m(T)$ such that $\tau_{m(T)} \leq T < \tau_{m(T)+1}$, which is possible a.s., since for finite N there are finitely many jumps in a finite time interval with probability one (proven in [2]). To explain the last inequality we notice that the jumps present in (3.16) are zero if the partition is taken to be exactly $l = m(T) + 1$ and $t_i = \tau_i$, $0 \leq i \leq l-1$, plus the last endpoint $\tau_l = T$, a construction we always can make unless $\max_{1\leq q\leq m(T)} |\tau_q - \tau_{q-1}| \leq \eta$ or $\tau_{m(T)} \in [T-\eta, T]$.

The second term of (3.19). For an arbitrary $M \in \mathbb{Z}_+$,

$$\begin{aligned} P\Big(\tau_{m(T)} \in [T-\eta, T]\Big) &\leq \\ P\Big(\tau_{m(T)} \in [T-\eta, T]\,,\, m(T) \leq M\Big) + P\Big(A_1^N(T) > M\Big) &\leq \\ \leq \sum_{k=1}^{M} P\Big(\tau_k \in [T-\eta, T]\Big) + P\Big(A_1^N(T) > M\Big)&\,. \end{aligned} \tag{3.20}$$

For a fixed M, in view of Proposition 6,

$$\lim_{\eta\to 0}\limsup_{N\to\infty} P\Big(\tau_{m(T)}\in[T-\eta,T]\Big)\leq \limsup_{N\to\infty} P\Big(A_1^N(T)>M\Big)\,. \tag{3.21}$$

Proposition 4 takes care of the latter probability as $M\to\infty$.

The first term of (3.19). For a given $M\in\mathbb{Z}_+$,

$$\lim_{\eta\to 0}\limsup_{N\to\infty} P\Big(\max_{1\leq q\leq m(T)}|\tau_q-\tau_{q-1}|\leq\eta\Big)\leq \tag{3.22}$$

$$\lim_{\eta\to 0}\limsup_{N\to\infty} P\Big(\max_{1\leq q\leq m(T)}|\tau_q-\tau_{q-1}|\leq\eta\,,\,m(T)\leq M\Big)+\limsup_{N\to\infty} P\Big(m(T)>M\Big)\leq$$

$$\lim_{\eta\to 0}\limsup_{N\to\infty}\Big(\sum_{k=1}^{M} P\Big(\tau_k-\tau_{k-1}\leq\eta\,,\,\tau_k\leq T\Big)\Big)+\limsup_{N\to\infty} P\Big(m(T)>M\Big)\,. \tag{3.23}$$

Since the first term of (3.23) is zero according to Proposition 5 and M is arbitrary we are done.

Tightness of the one-particle process. For an arbitrary smooth ϕ, formula (3.4) provides the variation of $\phi(x_1^N(\cdot))$ in a time interval $[s,t]$. We know that the paths of the tagged particle $x_1^N(\cdot)$ have only a finite number of discontinuities in $[0,T]$ almost surely. For convenience, we shall use a variant of the tightness condition (ii) provided in Theorems 14.4 and 15.3 in [1]. For a path $y(\cdot)\in\mathbf{D}([0,T],X)$, we define

$$w''_y(\eta)=\sup_{t_1\leq t\leq t_2\,,\,t_2-t_1\leq\eta}\min\{|y(t)-y(t_1)|\,,\,|y(t_2)-y(t)|\}\,. \tag{3.24}$$

Condition (3.18) is equivalent to

$$\begin{cases} (a) & \lim_{\eta\to 0}\limsup_{N\to\infty} P\Big(w''_y(\eta)>\epsilon\Big)=0\\ (b) & \lim_{\eta\to 0}\limsup_{N\to\infty} P\Big(\sup_{0\leq s\leq t<\eta}|y(t)-y(s))|>\epsilon\Big)=0\\ (c) & \lim_{\eta\to 0}\limsup_{N\to\infty} P\Big(\sup_{T-\eta\leq s\leq t<T}|y(t)-y(s))|>\epsilon\Big)=0\end{cases}\,. \tag{3.25}$$

Adopting the path $y(\cdot)=\phi(x_1(\cdot))$, condition (3.17) is trivially satisfied. Equation (3.4) expresses the change in $\phi(x_1^N(\cdot))$ as a sum of a continuous term with uniformly bounded integrand, a singular integral with respect to $A_1^N(\cdot)$ with uniformly bounded integrand and a martingale part (2.3) corresponding to $f(\mathbf{x})=x_1$ which will be denoted by $\mathcal{M}_1^N(\cdot)$. By subadditivity of the moduli of continuity we derive that it is enough to prove condition (3.18) of tightness for each term. The continuous part is trivial. The modulus of continuity $w'(\eta)$ corresponding to the integral with respect to $A_1^N(\cdot)$ is bounded above by $2\sup_{x_1\in D}|\phi(x_1)|w'_{A_1^N}(\eta)$ which has been shown to satisfy (3.18). We turn to the martingale part. The continuous martingale (2.4) is

$$\int_s^t \nabla\phi(x_1^N(s'))d\mathbf{w}_1(s')$$

and satisfies the maximal inequality uniformly in s,t and N. On any time interval of length less or equal to $\eta>0$, the quadratic variation will be of order η uniformly in N. The jump martingale part $\mathcal{M}_1^{N,J}(t)$ has quadratic variation (2.5)

$$(\mathcal{M}_1^{N,J}(t))^2-\frac{1}{N-1}\sum_{j\neq 1}\int_0^t\Big(\phi(x_j^N(s-)-\phi(x_1^N(s-)\Big)^2 dA_1^N(s)\,. \tag{3.26}$$

At first we shall notice that when we apply (3.25) to $y(\cdot) = \mathcal{M}_1^{N,J}(\cdot)$ we can always consider the path stopped at T_r, the stopping time defined in (3.5) for a fixed r. To make this clear, we estimate (3.25)-(a), the other two conditions (b) and (c) being treated in the same way. The condition is

$$P\Big(w''_{\mathcal{M}_1^{N,J}(\cdot)}(\eta) > \epsilon\Big) \le P\Big(w''_{\mathcal{M}_1^{N,J}(\cdot)}(\eta) > \epsilon\,,\, T_r > T\Big) + P\Big(T_r \le T\Big)$$

and because the second term is $P(S_L^N(r))$ from (3.2) we only need to look at the first term which has upper bound

$$P\Big(w''_{\mathcal{M}_1^{N,J}(\cdot\wedge T_r)}(\eta) > \epsilon\,,\, T_r > T\Big) \le P\Big(w''_{\mathcal{M}_1^{N,J}(\cdot\wedge T_r)}(\eta)\Big)\,.$$

We recall that the set of discontinuities of the martingale part is included in the set of discontinuities of $A_1^N(\cdot)$ by construction. Notice that (3.24) applied to $y(t) = \mathcal{M}_1^{N,J}(t\wedge T_r)$ is zero unless there are at least two jumps of the particle $x_1^N(\cdot)$ in the time interval,

$$(3.25b) \;\le\; P\Big(\Big\{\text{there exist two jumps } \tau' \text{ and } \tau'' \text{ in } [t_1,t_2] \text{ and } t_2 - t_1 \le \eta\;\Big\}\Big)\,, \qquad (3.27)$$

which case is taken care of by Proposition 5.

The maximal inequality estimates for (3.25)-(a) and (3.25)-(c) applied to the martingale part yield upper bounds

$$(3.25a) \;\le\; 4\|\phi\|^2\epsilon^{-2}E\Big[A_1^N(\eta\wedge T_r)\Big] \qquad (3.28)$$

$$(3.25c) \;\le\; 4\|\phi\|^2\epsilon^{-2}E\Big[A_1^N(T\wedge T_r) - A_1^N((T-\eta)\wedge T_r)\Big] \qquad (3.29)$$

To bound (3.28) we write

$$E\Big[A_1^N(\eta\wedge T_r)\Big] = E\Big[A_1^N(\eta\wedge T_r)\,\mathbf{1}_{[1,\infty)}(A_1^N(\eta\wedge T_r))\Big] \le$$
$$E\Big[(A_1^N(T\wedge T_r))^2\Big]^{\frac12}\,P\Big(A_1^N(\eta)\ge 1\Big)^{\frac12}\,. \qquad (3.30)$$

The limit as $\eta\to 0$ of (3.28) is zero with the same proof as the limit of the first term p_1 from (3.10) in Proposition 5, which concludes (3.25)-(a).

We bound (3.29) by

$$E\Big[A_1^N(T\wedge T_r) - A_1^N((T-\eta)\wedge T_r)\Big] \le$$
$$E\Big[A_1^N(T\wedge T_r) - A_1^N((T-\eta)\wedge T_r)\mathbf{1}_{[1,\infty)}(A_1^N(T\wedge T_r) - A_1^N((T-\eta)\wedge T_r))\Big] \le$$
$$E\Big[(A_1^N(T\wedge T_r))^2\Big]^{\frac12}\,P\Big(A_1^N(T) - A_1^N(T-\eta)\ge 1\Big)^{\frac12}\,, \qquad (3.31)$$

and use Proposition 6 to conclude the proof for (3.29).

□

4 Properties of the limiting process.

The subset of paths in $\mathbf{D}([0,T],\mathbb{R}_+)$ which are piecewise constant, nondecreasing, integer valued, with finite number of discontinuities and with all jumps of size exactly one will be called the subset of *counting paths* and will be denoted by F.

Proposition 8. *Let $(x_1(\cdot), A_1(\cdot))$ be a limit point in $\mathbf{D}([0,T], D\times\mathbb{R}_+)$ of the tight family of joint processes $\{x_1^N(\cdot), A_1^N(\cdot)\}_{N\in\mathbb{Z}_+}$. Then, almost surely with respect to the law of the limit point,*

a) $A_1(T) < \infty$,

b) $x_1(t)$, $A_1(t)$ are continuous at any point $t \in [0,T]$ where $x_1(t-) \in D$,

c) $A_1(\cdot)$ belongs to F.

Proof. a) This is a direct consequence of condition (i) for tightness given in (3.17).

b) Denote $y(t) = (x(t), a(t))$, $0 \le t \le T$ a path in $\mathbf{D}([0,T], D\times\mathbb{R}_+)$ and F_1 the set of paths continuous at any time t where $y(t-) \in D\times\mathbb{R}_+$. Notice that $D\times\mathbb{R}_+$ is an open set on $\mathbb{R}^d\times\mathbb{R}_+$. It is easy to check that the complement F_1^c of the set is open. For fixed N, if P^N denotes the joint law of $(x_1^N)(\cdot), A_1^N(\cdot))$, then $P^N(F_1) = 1$. By the properties of weak convergence, $P(F_1) \ge \limsup_{N\to\infty} P^N(F_1) = 1$.

c) *Step 1.* We shall prove that F is a closed set in $\mathbf{D}([0,T],\mathbb{R}_+)$. Let $a_m(\cdot)$ be a sequence of elements of F converging to $a(\cdot)$. At any continuity point t of $a(\cdot)$, $\lim_{m\to\infty} a_m(t) = a(t)$. We can see that $a(\cdot)$ takes values in $\mathbb{Z}_+$ due to the right continuity. Let t_0 be a discontinuity point of $a(\cdot)$. Clearly $t_0 > 0$. There exists an increasing sequence $\{t'_l\}_{l\ge1}$ and a decreasing sequence $\{t''_l\}_{l\ge1}$ of continuity points of $a(\cdot)$, converging to t_0. This implies that

$$a(t_0) - a(t_0-) = \lim_{l\to\infty} a(t''_l) - a(t'_l) \tag{4.1}$$

and

$$a(t''_l) - a(t'_l) = \lim_{m\to\infty} a_m(t''_l) - a_m(t'_l) \tag{4.2}$$

The terms on the left hand side of (4.2) are nonnegative, hence $a(\cdot)$ is nondecreasing. Also, since the right hand side of (4.2) is a nonnegative integer, the same is true about the limiting path $a(\cdot)$. In general, for any $f \in D([0,T],R)$, let

$$J(f) = \sup_{t\in[0,T]} |f(t) - f(t-)| \tag{4.3}$$

be the maximum jump size of f on $[0,T]$. It is known that J is a continuous functional on the Skorohod space (see [1]). This implies that the jumps of $a(\cdot)$ are at most of size one. If it had an infinite number of discontinuities, they would be bounded below by the constant one and the path $a(\cdot)$ would be unbounded, which is impossible. This shows the last requirement for $a(\cdot)$ needed to belong to F is fulfilled.

Step 2: The limit $A_1(\cdot)$ is concentrated on counting paths. Since F^c is an open negligible set for any finite $N \in \mathbb{Z}_+$, it is negligible in the limit, that is,

$$0 = \liminf_{N\to\infty} P\Big(A_1^N(\cdot) \in F^c\Big) \ge P\Big(A_1(\cdot) \in F^c\Big).$$

□

Proposition 9. *Under the conditions of Theorem 1, let $x_1(\cdot)$ and $A_1(\cdot)$ be limit points of the tight families of processes indexed by $N \in \mathbb{Z}_+$ $\{x_1^N(\cdot)\} \in \mathbf{D}([0,T],D)$ and $\{A_1^N(\cdot)\} \in \mathbf{D}([0,T],\mathbb{R}_+)$, respectively. Then, for any $\phi \in C^2(\overline{D})$*

$$\mathcal{M}_{1,\phi}(t) = \phi(x_1(t)) - \phi(x_1(0)) - \int_0^t \frac{1}{2}\Delta_d \phi(x_1(u))du \tag{4.4}$$

$$- \int_0^t \Big(\langle \phi, \mu(u,dx)\rangle - \phi(x_1(u-))\Big) dA_1(u)$$

is a P - local square integrable martingale (class $\mathcal{M}^2_{loc}$) with respect to the filtration of the process $\mathcal{F}$ continuous between the increasing sequence of boundary hitting times $\{\tau_k\}_{k\geq 0}$ where $x_1(\tau_k-) \in \partial D$ such that, for any $k = 0, 1, \dots$

$$\mathcal{M}_{1,\phi}((t\vee\tau_k)\wedge\tau_{k+1})^2 - \int_{\tau_k}^{(t\vee\tau_k)\wedge\tau_{k+1}} \|\nabla\phi(x_1(s))\|^2 ds \tag{4.5}$$

is also a martingale.

Proof. Step 1. Localization. For any $m \in \mathbb{Z}_+$ and for any paths $(\tilde{x}_1(\cdot), \tilde{A}_1(\cdot))$, let $\xi_m = \inf\{t > 0 : \tilde{A}_1(t) > m\}$ and define the operator Ξ_m on $\mathbf{D}([0,T],D) \times \mathbf{D}([0,T],\mathbb{R}_+)$

$$[\Xi_m(\tilde{x}_1(\cdot), \tilde{A}_1(\cdot))](t) = (\tilde{x}_1(t\wedge\xi_m), \tilde{A}_1(t\wedge\xi_m)) \tag{4.6}$$

with the understanding that the paths remain constant after ξ_m. The mapping Ξ_m is linear and bounded, hence the pair of processes $\{\Xi_m(x_1^N(\cdot), A_1^N(\cdot))\} \in \mathbf{D}([0,T],D) \times \mathbf{D}([0,T],\mathbb{R}_+)$ is tight and $\Xi_m(x_1(\cdot), A_1(\cdot))$ is one of its limit points.

We start by writing formula (3.4) for a fixed N, before stopping at ξ_m, which in the limit should yield (4.4). This is Itô formula (2.3) for functions depending on only one variable $x_1 \in D$. Both sides of (2.3) remain martingales if we stop at ξ_m. The equation reads as

$$\phi(x_1^N(t\wedge\xi_m)) - \phi(x_1^N(0)) - \int_0^{t\wedge\xi_m} \frac{1}{2}\Delta\phi(x_1^N(u))du \tag{4.7}$$

$$- \int_0^{t\wedge\xi_m} \frac{1}{N-1}\sum_{i\neq 1} \phi(x_i^N(u)) - \phi(x_1^N(u-))dA_1^N(u) =$$

$$\int_0^{t\wedge\xi_m} \nabla\phi(x_1^N(u))\cdot \mathbf{w}_1(u) + \mathcal{M}^{N,J}_{\phi(x_1)}(t)\,. \tag{4.8}$$

We are interested in showing that the left-hand side of (4.4) is a martingale (4.5), that is, the conclusion holds for the limit processes stopped at ξ_m. First we notice that the mapping Ξ_m is continuous and bounded. Let $0 \leq s \leq t \leq T$ and $\Psi(\omega)$ be a $\mathcal{F}_s$-measurable smooth bounded function. The functional $\Gamma_{\phi,s,t,m}$ on the Skorohod space $\mathbf{D}([0,T],D) \times \mathbf{D}([0,T],\mathbb{R}_+)$ defined as the product of $\Psi(\omega)$ and the difference of the values at times t and s of (4.4) is bounded and continuous. This functional differs from the analogue difference for (4.7) by an error term

$$\Big[\int_{s\wedge\xi_m}^{t\wedge\xi_m} \Big(\frac{1}{N-1}\sum_{i\neq 1}\phi(x_i^N(u)) - \langle\phi, \mu(u,dx)\rangle\Big)dA_1^N(u)\Big]\Psi(\omega) \tag{4.9}$$

which tends to zero uniformly as $N \to 0$ due to the localization of $A_1^N(t)$ to $A_1^N(t \wedge \xi_m) \le m$. The Brownian martingale part (4.8) remains uniformly bounded in the square norm. The jump martingale part $\mathcal{M}^{N,J}_{\phi(x_1)}(t)$ has bounded quadratic variation as well, uniformly in N, again due to the fact that $A_1^N(T \wedge \xi_m) \le m$. Proposition 8 b), c) provides a characterization of the limiting processes $(x_1(\cdot), A_1(\cdot))$ showing that they are continuous in the random time intervals $[\tau_k, \tau_{k+1})$. We recall that $\{\tau_k\}_{k\ge 0}$ are stopping times. For each N, the expression (4.4) differs from the continuous martingale $\mathcal{M}_1^{N,B}((t \vee \tau_k) \wedge \tau_{k+1})$ from (2.4) by a term which is uniformly bounded and also converges to zero in probability.

We can conclude that, for the limit processes $\Xi_m(x_1(\cdot), A_1(\cdot))$, the expression (4.4) is a square-integrable martingale corresponding to the limit of (4.8) with quadratic variation equal to $\int_0^{t\wedge\xi_m} \|\nabla\phi(x_1(u))\|^2 du$ on every time interval $[\tau_k, \tau_{k+1})$.

Step 2. Proposition 8 guarantees that $\xi_m \to \infty$ with probability one with respect to any limiting joint law of the processes. □

Proposition 10. *Under the same conditions as in Proposition 9, with probability one with respect to the limiting distribution of $(x_1^N(\cdot), A_1^N(\cdot))$, for any $t \in [0,T]$, if $x_1(t)$ has a jump at t, then $A_1(t)$ has a jump at t.*

Proof. We consider

$$k(\omega) = k = \min\{j \ge 1 \,:\, x_1(\tau_j-) \in \partial D\,,\, A_1(\tau_j) - A_1(\tau_j-) = 0\}\,.$$

Let $\Psi(\omega) = \phi(x_1(s \wedge \tau_k))$ be a $\mathcal{F}_{\tau_k\wedge s}$ - measurable function along the same lines as in the proof of Proposition 9 and take the expected value of its product with the martingale (4.4). We obtain

$$\lim_{s\uparrow\infty} E\Big[\Big(\phi(x_1(\tau_k)) - \phi(x_1(\tau_k \wedge s)) - \int_{\tau_k\wedge s}^{\tau_k} \frac{1}{2}\Delta_d\phi(x_1(u))du\Big)\Psi(\omega)\Big] \tag{4.10}$$

$$= \lim_{s\uparrow\infty} E\Big[\Big(\int_{\tau_k\wedge s}^{\tau_k} \Big(\langle\phi, \mu(u,dx)\rangle - \phi(x_1(u-))\Big)dA_1(u)\Big)\Psi(\omega)\Big] = 0 \tag{4.11}$$

by the choice of τ_k. Since the continuous part of the time integral has uniformly bounded integrand, we derive that $\lim_{s\uparrow\infty} E\Big[\Big(\phi(x_1(\tau_k)) - \phi(x_1(s\wedge\tau_k))\Big)\phi(x_1(s\wedge\tau_k))\Big] = 0$ or equivalently

$$E\Big[\Big(\phi(x_1(\tau_k)) - \phi(x_1(\tau_k-))\Big)\phi(x_1(\tau_k-))\Big] = 0 \tag{4.12}$$

by dominated convergence. The function $\phi(x_1)$ is arbitrary in $C^2(\overline{D})$. If we choose $\phi(x_1) = \gamma_r(x_1)$ as in Definition 2, a function equal to one on the boundary ∂D with support included in D_{2r}^c and such that $\gamma_r(x_1) \le 1$ if $x_1 \in D$ we notice that $\phi(x_1(\tau_k)) = 1$ almost surely or $x_1(\tau_k) \in \partial D$ almost surely which contradicts the construction of $x_1(\cdot) \in \mathbf{D}([0,T], D)$ taking values in D with probability one. □

Proposition 11. *Under the same conditions as in Proposition 9, the process $A_1(t)$ is equal to the number of visits to the boundary ∂D of $\{x_1(s-)\}$, where $s \in [0,t]$.*

Proof. This proposition is a consequence of Propositions 8 and 10. The only remaining fact to prove is that if $x_1(t-) \in \partial D$, then $x_1(\cdot)$ has a jump at t. Let $x_1(t-) \in \partial D$. The path $x_1(\cdot)$ is in the support of the limiting distribution of the processes $(x_1^N(\cdot), A_1^N(\cdot))$. The proof of tightness implies that the path is D-valued almost surely. If $x_1(t-) \in \partial D$ and $x_1(t) = x_1(t-)$ (to ensure continuity) we violate this condition. Consequently, with probability one, the conclusion holds. □

5 Proof of the main results.

Proof of Theorem 2.

Part (i). This is an immediate consequence of the assumptions of the theorem.

Part (ii). According to Proposition 8 c), for any finite $T > 0$ the number of boundary visits on $[0, T]$ is finite with probability one. On the other hand, $\lim_{k\to\infty} \tau_k < \infty$ only in the case that there would be infinitely many jumps in $[0, T]$, a contradiction, or if there would simply be a finite number of jumps. The smoothness of $\mu(t, dx) = v(t, x)dx$ for $t > 0$ implies that this latter possibility occurs with probability zero. The rest follows from Proposition 11.

Part (iii). For any $k \in \mathbb{Z}_+$, the optional sampling theorem applied to the martingale (4.4) together with the characterization for the structure of the discontinuities from Proposition 11 prove that

$$\phi(x_1((t+\tau_k) \wedge \tau_{k+1})) - \phi(x_1(\tau_k)) - \int_{\tau_k}^{(t+\tau_k)\wedge\tau_{k+1}} \frac{1}{2}\Delta_d \phi(x_1(u))du \tag{5.1}$$

is a continuous martingale. Since the martingale problem is well posed for the half Laplacian on $\mathbb{R}^d$, the *localization theorem* (in [5], Chapter 6) indicates that the stopped martingale problem given by equation (5.1) has a unique solution starting at $x_1(\tau_k^D)$, which proves (iii) from the theorem.

Part (iv). We have shown that $\{x_1^N(\cdot)\}_{N\in\mathbb{Z}_+}$ and $\{A_1^N(\cdot)\}_{N\in\mathbb{Z}_+}$ are tight in the Skorohod spaces $\mathbf{D}([0, T], D)$ and $\mathbf{D}([0, T], \mathbb{R}_+)$. The mechanism of the proof is to assume we have a limit point $(x_1(\cdot), A_1(\cdot))$ taken over a subsequence $N' \to \infty$ and show that it is unique and satisfies (i) and (ii) of Theorem 2. Let $\phi \in C^2(\overline{D})$. The martingale defined in (4.4) is continuous between the ordered sequence of boundary hits $\{\tau_k\}_{k\in\mathbb{Z}_+}$, that is the stopping times such that $x_1(\tau_k-) \in \partial D$ and has quadratic variation (4.5). The counting process $A_1(t)$ can be regarded as a function with bounded variation on the time interval $[0, T]$ almost surely, for arbitrary $T > 0$. The structure of $A_1(t)$ is given by Proposition 11. Repeating the argument given in Proposition 10 in equations (4.10), (4.11) and (4.12) we obtain that

$$E\Big[\phi(x_1(\tau_k)) - \phi(x_1(\tau_k-)) \,\Big|\, \mathcal{F}_{\tau_k-}\Big] = E\Big[\langle \phi, \mu(\tau_k, dx)\rangle - \phi(x_1(\tau_k-)) \,\Big|\, \mathcal{F}_{\tau_k-}\Big]$$

or simply

$$E\Big[\phi(x_1(\tau_k^D)) \,\Big|\, \mathcal{F}_{\tau_k-}\Big] = E\Big[\langle \phi, \mu(\tau_k^D, dx)\rangle \,\Big|\, \mathcal{F}_{\tau_k-}\Big] = \langle \phi, \mu(\tau_k^D-, dx)\rangle \tag{5.2}$$

by the continuity of the asymptotic profile $\mu(\cdot, dx)$, concluding (i) from Theorem 2. This concludes the theorem. □

Proof of Corollary 2.

Part 1. The expression (2.11) in Corollary 2 is a local square-integrable martingale.

As in the proof of Theorem 2, let $(x_1(\cdot), A_1(\cdot))$ be a limit point of the tight sequence $\{x_1^N(\cdot), A_1^N(\cdot)\}_{N\in\mathbb{Z}_+}$. Condition (i) from the corollary is exactly Proposition 11 meanwhile condition (ii) was done in the proof of Theorem 2 and coincides with the fact that (2.11) stopped at ξ_m is equal to the martingale (4.4). Notice that $\lim_{m\to\infty}\xi_m = \infty$ due to the fact that the jumps are of size one and part (ii) of Theorem 2. Conditions (i)-(iii) imply uniqueness once again due to the localization theorem for the stopped martingale problem mentioned in the preceding proof.

Part 2. For any $T > 0$, $E[A_1(T)] < \infty$.

Up to ξ_m, the expression (2.11) in Corollary 2 is a square-integrable martingale and $\lim_{m\to\infty}\xi_m = \infty$ with probability one. If $\Phi_1(x)$ is the first eigenfunction of the Dirichlet Laplacian on D with eigenvalue $\lambda_1 < 0$, let

$$C_{\Phi_1,T} = \inf_{t\in[0,T]} \langle \Phi_1\,,\,\mu(t,dx)\rangle > 0\,. \tag{5.3}$$

Notice that even though $\mu(0,dx)$ might be singular, the integral with the eigenfunction Φ_1 is continuous on $[0,T]$. Since it is a positive function for any $t\in[0,T]$, the infimum is also positive. The local martingale $\mathcal{M}_{\Phi_1}(T\wedge\xi_m)$ defined by (2.11) from the *Part 1* of the proof gives

$$C_{\Phi_1,T}A_1(T\wedge\xi_m) \le \int_0^{T\wedge\xi_m} \langle\Phi_1, \mu(s,dx)\rangle dA_1(s) = \tag{5.4}$$

$$\Phi_1(x_1(T\wedge\xi_m)) - \Phi_1(x_1(0)) - \int_0^{T\wedge\xi_m} \frac{1}{2}\Delta_d\Phi_1(x_1(s))ds + \mathcal{M}_{\Phi_1}(T\wedge\xi_m)\,.$$

The expected value of the left side of (5.4) is uniformly bounded by $(2 + (|\lambda_1|/2)T)\|\Phi_1\|$. Let $m\to\infty$ and the monotone convergence theorem concludes that $\mathcal{M}_{\Phi_1}(t\wedge\xi_m)$ defined by (2.11) is uniformly integrable and implies that $\mathcal{M}_{\Phi_1}(t)$ is a martingale, the proof of the corollary. □

5.1 Proof of Theorem 3.

We shall establish first an l-dimensional analogue to the formula (2.11).

Proposition 12. *Let $\{(x_j(\cdot), A_j(\cdot))\}_{1\le j\le l}$ be weak limits along a subsequence of the tight family of processes $\{(x_j^N(\cdot), A_j^N(\cdot)\}_{1\le j\le l}$ and let $F\in C^2(\overline{D}^l)$. Then*

$$\mathcal{M}_F(t) = F(x_1(t),\ldots,x_l(t)) - F(x_1(0),\ldots,x_l(0)) - \int_0^t \frac{1}{2}\Delta F(x_1(u),\ldots,x_l(u))du \tag{5.5}$$

$$-\sum_{j=1}^l \int_0^t \Big(\langle F(x_1(u),\ldots,x_l(u)), \mu(u,dx_j)\rangle - F(x_1(u-),\ldots,x_l(u-))\Big)dA_j(u) \tag{5.6}$$

is a local square-integrable martingale (of class $\mathcal{M}^2_{loc}$), continuous between the increasing joint sequence of boundary hitting times $\{\tau_k^{(l)}\}_{k\ge0}$ such that, for any $k = 0,1,2,\ldots$

$$\Big(\mathcal{M}_F((t\vee\tau_k^{(l)})\wedge\tau_{k+1}^{(l)})\Big)^2 - \sum_{j=1}^l \int_{\tau_k^{(l)}}^{(t\vee\tau_k^{(l)})\wedge\tau_{k+1}^{(l)}} \|\nabla_{x_j}F(x_j(u))\|^2 du \tag{5.7}$$

is also a local martingale. In the expression $\langle F(x_1(u),\ldots,x_l(u)),\mu(u,dx_j)\rangle$ we understand that variables i not equal to j are the processes $x_i(u)$ while the variable on the position j is deterministic and integrated against the measure $\mu(u,dx_j)$. Also, Δ represents the ld dimensional Laplacian.

Proof. The space $\overline{D}^l$ is compact such that it is sufficient to prove (5.5 - 5.6) for cylinder functions $F(x_1,\ldots,x_l)=\Pi_{j=1}^l f_j(x_j)$ and pass to the limit in the supremum norm over $\overline{D}^l$. Set $\xi_m^{(l)}$ the minimum of all times when the point processes $A_j(\cdot)$ exceed $m\in\mathbb{Z}_+$ and T.

Step 1. In analogy with (4.7), for a fixed $N\in\mathbb{Z}_+$, we shall calculate the Itô formula for the product $F(x_1,\ldots,x_l)=\Pi_{j=1}^l f_j(x_j)$. The reasoning is analogue to the proof of Proposition 9 with the exception that the errors contain product functions. However, all the factors $f_j(x_j)$ are bounded in the uniform norm. The uniform limit in probability (2.10) from Theorem 1 together with the localization $A_j^N(t\wedge\xi_m^{(l)})\le m$ for all $j=1,2,\ldots l$ imply that the error terms of type (4.9) are uniformly approaching zero as $N\to\infty$. Once again, for the limit of the continuous martingales (5.7), with uniformly bounded quadratic variation.

Step 2. We make the observation that the limit point of $\sum_{j=1}^l A_j^N(t)$ has jumps of size one only (almost surely). This property is true as long as N is fixed. On the other hand, the maximum jump size is a continuous functional on the Skorohod space (see (4.3) and the comment thereof). This implies that all the $A_j(\cdot)$ are mutually singular. In addition, the counting processes $A_j(\cdot)$ are piecewise constant with a finite number of discontinuities on any finite time interval. This line of reasoning has been presented in more detail in the proof of Theorem 2.

Step 3. The last steps are inductive. If $l=1$, formula (5.5)-(5.6) is exactly (4.4)-(4.5). Let's assume the formula is valid for any $k\le l-1$ and we want to prove it for l. At the former step we have proven (5.5)-(5.6) for the special cylinder functions. Any $C^2(\overline{D}^l)$ function can be approximated in the uniform norm by finite linear combinations of cylinder functions. As long as we keep the processes stopped at ξ_m, in order to have a bound on the expected value of the total variation of $A_j(\cdot)$, for all j, we can carry out the limit over the approximating multi-variable test functions. After doing the algebra, formula (5.5)-(5.6) is proven for arbitrary $l\in\mathbb{Z}_+$ and $m\in\mathbb{Z}_+$.

Step 4. From Step 2 we know that no two particles x_j will reach the boundary at the same time, hence there are no simultaneous jumps. This enables us to order the set of all hitting times of the boundary ∂D by either one of the l particles. The optional stopping theorem together with the localization theorem (in [5], Chapter 6) applied to the half Laplacian in dimension $l\,d$ on the open set D^l ensure that between jump times the joint motion of the l particles is independent Brownian motion. From tightness, the paths of all particles are concentrated on right continuous paths with left limit. This proves the uniqueness of the joint process, since the jumps are completely defined by $\mu(t,dx)$ according to (5.2), and the filtration is not just the filtration associated to a specific particle but the original process filtration up to the current boundary hitting time $\tau_k^{(l)}$. In the same fashion as in the proof of Theorem 2, we have shown that the limit points $(x_j(\cdot),A_j(\cdot))$, for all $j=1,2,\ldots l$, stopped at ξ_m satisfy (5.5)-(5.7).

Step 5. The limit processes stopped at ξ_m and a family of l independent processes defined in Theorem 2 stopped at ξ_m coincide. By construction, the stopped process is also unique. The number of tagged particles l is finite. Proposition 8 is sufficient to show that $P(\lim_{k\to\infty}\tau_k^{(l)}\to\infty)=1$. □

Proof of Theorem 3. The reasoning assumes that we identify a joint limit point of the tight family of processes $\{(x_1^N(\cdot), A_1^N(\cdot)), \ldots, (x_l^N(\cdot), A_l^N(\cdot))\}_{N\in\mathbb{Z}_+}$. Once uniqueness is established, we have proven that the joint distribution converges in distribution to a specific measure on $[\mathbf{D}([0,\infty), D) \times \mathbf{D}([0,\infty), \mathbb{R}_+)]^{\otimes l}$. Proposition 12 is essentially a reformulation of Theorem 3. In fact, we can even limit ourselves to the version of the proposition for cylinder functions only. Due to the structure Proposition 11 and the localization theorem (with respect to the boundary hits τ_k^D, we can identify that the process is an independent Brownian motion in D^l between boundary hits. At τ_k^D, conditional upon which particle hits the boundary, the redistribution function is $\mu(\tau_k^D, dx)$, similarly to (5.2). This occurs independently from the other particles, as seen from the factorization of the cylinder functions. □

5.2 Proof of Corollary 3.

Proof. Let $\phi \in C^2(\bar(D))$. The symmetry of the joint law of $\{x_i^N\}_{1\le i\le N}$ implies that $E[\phi(x_i^N)] = E[\phi(x_j^N)]$ for all pairs (i,j). Since $E[N^{-1}\sum_{i=1}^N \phi(x_i^N)] = E[\phi(x_1^N)]$ converges to $\int_D \phi(x)\alpha(dx)$ as $N\to\infty$ we conclude that the initial distribution of the tagged particle is $\alpha_1(dx_1) = \alpha(dx_1)$. We have shown that $Q_{\alpha_j,\mu} = Q_{\alpha,\mu}$, for all $1 \le j \le N$. Let G be a continuous bounded functional on the space $\mathbf{D} = \mathbf{D}([0,T], D)$. Without loss of generality we can assume $G(x(\cdot))$ is a cylinder function, meaning that there exists a finite number $l \in \mathbb{Z}_+$, an increasing collection of times $\{t_q\}_{0\le q\le l}$ such that $t_0 = 0$, $t_l \le T$ and a function $\mathbf{g} \in C_b^1(D^{l+1})$ such that $G(x(\cdot)) = \mathbf{g}(x(t_0), x(t_1), \ldots, x(t_l))$. We calculate

$$E\Big[\Big|\frac{1}{N}\sum_{i=1}^N G(x_i^N(\cdot)) - \int_{\mathbf{D}} G(x(\cdot))dQ_{\alpha,\mu}\Big|^2\Big] \le S_1^N(G) + S_2^N(G)$$

such that

$$\lim_{N\to\infty} S_1^N(G) = \lim_{N\to\infty} \frac{1}{N^2}\sum_{i=1}^N E\Big[\Big(G(x_i^N(\cdot)) - \int_{\mathbf{D}} G(x(\cdot))dQ_{\alpha,\mu}\Big)^2\Big] = 0$$

and

$$S_2^N(G) = \frac{1}{N^2}\sum_{1\le i\ne j\le N} E\Big[\Big(G(x_i^N(\cdot)) - \int_{\mathbf{D}} G(x(\cdot))dQ_{\alpha,\mu}\Big)\Big(G(x_j^N(\cdot)) - \int_{\mathbf{D}} G(x(\cdot))dQ_{\alpha,\mu}\Big)\Big].$$

The symmetry of the initial distribution of the process allows us to write

$$\lim_{N\to\infty} S_2^N(G) = \lim_{N\to\infty} E\Big[\Big(G(x_1^N(\cdot)) - \int_{\mathbf{D}} G(x(\cdot))dQ_{\alpha,\mu}\Big)\Big(G(x_2^N(\cdot)) - \int_{\mathbf{D}} G(x(\cdot))dQ_{\alpha,\mu}\Big)\Big] = 0$$

as a consequence of Theorem 3 and the observation made at the beginning of this proof. □

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