Electronic Journal of Probability

Vol. 14 (2009), Paper no. 37, pages 1074–1079.

Journal URL
http://www.math.washington.edu/~ejpecp/

Erratum to “Number Variance from a probabilistic perspective, infinite systems of independent Brownian motions and symmetric α-stable processes"

B.M. Hambly and L. A. Jones
Mathematical Institute,
University of Oxford,
24-29 St Giles’,
Oxford,
OX1 3LB.
hambly@maths.ox.ac.uk, jones@maths.ox.ac.uk

Abstract

In the paper, [1], we provide an expression for the variance of the counting functions associated with the spatial particle configurations formed by infinite systems of independent symmetric α-stable processes. The formula (2.3) of the original paper, is in fact the correct expression for the expected conditional number variance. This is equal to the full variance when L is a positive integer multiple of the parameter a but, in general, the full variance has an additional bounded fluctuating term. The main results of the paper still hold for the full variance itself, although some of the proofs require modification in order to incorporate this change.

Key words: Number variance, symmetric α-stable processes, controlled variability, Gaussian fluctuations, functional limits, long memory, Gaussian processes, fractional Brownian motion.

AMS 2000 Subject Classification: Primary 60G52, 60G15; Secondary: 60F17, 15A52.

Submitted to EJP on December 4, 2008, final version accepted February 13, 2009.

The necessary changes are, referring to the numbering of the original paper, as follows.

Theorem 2.2

The theorem is true as stated for the expected conditional number variance. However, in order to incorporate the full variance, the statement of Theorem 2.2 should now read as follows

Theorem 2.2. *Fix a symmetric α-stable process $(X^{\alpha,c}(t), t \geq 0)$ on $\mathbb{R}$. Suppose we start an independent copy of this process from each of the starting positions*

$$u_j := a(j - \epsilon), \quad j \in \mathbb{Z}, \quad \text{where } a \in \mathbb{R}_+ \text{ and } \epsilon \sim \mathit{Uniform}[0,1]. \tag{1}$$

The counting function associated with the configuration of particles in space formed by this infinite system at a fixed time t and given by

$$N_t^{\alpha,c,a}[0,L] = \sum_{j=-\infty}^{\infty} \mathbb{I}[X_j^{\alpha,c}(t) + u_j \in [0,L]], \tag{2}$$

has variance

$$\mathit{Var}\big[N_t^{\alpha,c,a}[0,L]\big] = V_t^{\alpha,c,a}[L] + F^{\alpha,c,a}[L].$$

The terms of this decomposition correspond to the expected conditional number variance given by

$$V_t^{\alpha,c,a}[L] = \frac{L}{a} + \frac{2}{\pi}\int_0^\infty \frac{e^{-2ct(\theta/a)^\alpha}}{\theta^2}\Big[\cos\Big(\frac{L\theta}{a}\Big) - 1\Big]\,\mathrm{d}\theta \tag{3}$$

$$= \frac{4L}{a\pi}\int_0^\infty \frac{\sin^2(u/2)}{u^2}\big(1 - e^{-2ct(u/L)^\alpha}\big)\mathrm{d}u \tag{4}$$

and an additional fluctuating term satisfying

$$0 \leq F^{\alpha,c,a}[L] \leq \frac{2}{\pi}\int_0^\infty \frac{e^{-2ct(\theta/a)^\alpha}}{\theta^2}\Big[1 - \cos\Big(\Big(\frac{L}{a} - \Big\lfloor\frac{L}{a}\Big\rfloor\Big)\theta\Big)\Big]\,\mathrm{d}\theta \tag{5}$$

$$\leq \frac{L}{a} - \Big\lfloor \frac{L}{a}\Big\rfloor,$$

where here and throughout $\lfloor\cdot\rfloor$ denotes the floor function.

Proof. As the particles move from a uniformly perturbed starting position, the general decomposition for the number variance is

$$\mathrm{Var}\big[N_t^{\alpha,c,a}[0,L]\big] = \mathbb{E}_\epsilon\Big[\mathrm{Var}\big[N_t^{\alpha,c,a}[0,L]\big|\epsilon\big]\Big] + \mathrm{Var}_\epsilon\Big[\mathbb{E}\big[N_t^{\alpha,c,a}[0,L]\big|\epsilon\big]\Big].$$

The computation of the first term, corresponding to the expected conditional number variance is what we computed originally and remains valid. The derivation of the formula for the variance given as (2.7) in [1] implicitly assumes that the second part of the decomposition is identically zero. This is in fact not the case in general. We now remedy this by computing the additional term.

We have

$$F^{\alpha,c,a}[L] := \mathrm{Var}_\epsilon\left[\mathbb{E}\left[N_t^{\alpha,c,a}[0,L]\middle|\epsilon\right]\right].$$

Note that

$$\begin{aligned}\mathbb{E}\left[N_t^{\alpha,c,a}[0,L]\middle|\epsilon\right] &= \sum_{j=-\infty}^{\infty}\mathbb{P}\left[X_j^{\alpha,c}(t/a^\alpha)+(j-\epsilon)\in[0,L/a]|\epsilon\right]\\ &= \sum_{j=-\infty}^{\infty}\mathbb{P}\left[X_j^{\alpha,c}(t/a^\alpha)+(j-\epsilon)\in[0,\lfloor L/a\rfloor]|\epsilon\right]\\ &\quad+\sum_{j=-\infty}^{\infty}\mathbb{P}\left[X_j^{\alpha,c}(t/a^\alpha)+(j-\epsilon)\in[\lfloor L/a\rfloor,L/a]|\epsilon\right]\\ &= \left\lfloor\frac{L}{a}\right\rfloor+\sum_{j=-\infty}^{\infty}\mathbb{P}\left[X_j^{\alpha,c}(t/a^\alpha)+(j-\epsilon)\in[\lfloor L/a\rfloor,L/a]|\epsilon\right].\end{aligned}$$

Once this is noted, straightforward manipulations similar to those of the original paper yield

$$\begin{aligned}F^{\alpha,c,a}[L] &= \mathrm{Var}_\epsilon\left[\sum_{j=-\infty}^{\infty}\mathbb{P}\left[X_j^{\alpha,c}(t/a^\alpha)+j-\epsilon\in[\lfloor L/a\rfloor,L/a]|\epsilon\right]\right]\\ &= \frac{L}{a}-\left\lfloor\frac{L}{a}\right\rfloor-V_t^{\alpha,c,a}\left[a\left(\frac{L}{a}-\left\lfloor\frac{L}{a}\right\rfloor\right)\right]\\ &\quad-\sum_{j=-\infty}^{\infty}\left(\int_0^1\int_0^{L/a-\lfloor L/a\rfloor}p_{2t/a^\alpha}(j-\epsilon,z)\,\mathrm{d}z\mathrm{d}\epsilon\right)^2.\end{aligned}$$

The inequalities for the fluctuating term stated in the Theorem now follow directly from the expression for $V_t^{\alpha,c,a}$. ☐

Theorem 2.3, Corollaries 2.4, 2.5 and 2.7.

The statements of the proceeding results on the growth, saturation and divergence of the number variance (Theorem 2.3, Corollaries 2.4, 2.6 and 2.7) remain valid on replacing "number variance" with "expected conditional number variance". If we were to consider the full variance, since the additional fluctuation term is bounded by one, there is only a small change though clearly the full variance does not converge.

Our original motivation for this work was a paper of Kurt Johansson's [2] on the saturation behaviour of the expected number variance of some point processes constructed from systems of non-colliding Brownian motions. Our results, as stated, are therefore directly comparable.

Consequent Modifications

As the particles of our model are only conditionally independent given the initial displacement ϵ, the proofs that assumed full independence of particles and the statements that make use of the original expression for the variance therefore need to be modified.

Proposition 2.11 is incorrect as stated but as it is not very important for the overall development of the paper we shall not remedy this here.

Proposition 3.1 holds true though the proof needs to modified slightly as we shall elaborate below. The Proposition can also be restated in terms of the full variance. In order to do this we first recall some notation. Let $N_t^{\alpha,c,a}[0,L]$, $\mathrm{Var}\big[N_t^{\alpha,c,a}[0,L]\big]$ and $V_t^{\alpha,c,a}[L]$ denote the counting function, its variance and expected conditional number variance.

Proposition 3.1. *For $\alpha \in (0,1]$ we have that*

$$\frac{N_t^{\alpha,c,a}[0,L] - L/a}{\sqrt{Var\big[N_t^{\alpha,c,a}[0,L]\big]}} \tag{6}$$

converges in distribution to a standard normal random variable as $L \to \infty$.

Proof. In the original paper we in fact consider the generating function

$$J_{N^\epsilon}(\theta) := \log \mathbb{E}_{\mathbb{P}}[\exp(i\theta N_t^{\alpha,c,a}[0,L])|\epsilon]$$

of the conditional cumulants c_k^ϵ of $N_t^{\alpha,c,a}[0,L]$ given ϵ. Note that the generating function $J_N(\theta)$ for the cumulants c_k of $N_t^{\alpha,c,a}[0,L]$ can be expressed in terms of $J_{N^\epsilon}(\theta)$ by

$$J_N(\theta) = \log\Big(\int_0^1 \exp\big(J_{N^\epsilon}(\theta)\big)\mathrm{d}\epsilon\Big).$$

In particular, it may be verified that

$$J_N(\theta) - \int_0^1 J_{N^\epsilon}(\theta)\mathrm{d}\epsilon$$

is bounded. This in turn implies that the cumulants are of the form

$$c_k = \int_0^1 c_k^\epsilon \mathrm{d}\epsilon + \text{ bounded term }. \tag{7}$$

It is easily checked that

$$\begin{aligned}
c_1 &= \frac{\mathrm{d}}{\mathrm{d}\theta} J_N(\theta)\Big|_{\theta=0} = \int_0^1 c_1^\epsilon \mathrm{d}\epsilon = L/a \\
c_2 &= \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} J_N(\theta)\Big|_{\theta=0} = \int_0^1 c_2^\epsilon \mathrm{d}\epsilon + \int_0^1 \big(c_1^\epsilon\big)^2 \mathrm{d}\epsilon - \Big(\int_0^1 c_1^\epsilon \mathrm{d}\epsilon\Big)^2 \\
&= \mathrm{Var}\big[N_t^{\alpha,c,a}[0,L]\big].
\end{aligned}$$

To prove the Proposition as stated here it is sufficient to show that in the limit as $L \to \infty$, the cumulants of the rescaled random variable

$$\frac{N_t^{\alpha,c,a}[0,L] - L/a}{\sqrt{\mathrm{Var}\big[N_t^{\alpha,c,a}[0,L]\big]}}$$

correspond to those of a standard Gaussian random variable. Following the same approach as in the original proof we therefore need to show that

$$c_k = o\big(\big(\mathrm{Var}\big[N_t^{\alpha,c,a}[0,L]\big]\big)^{k/2}\big) = o(c_2^{k/2}) \qquad \text{as } L \to \infty, \text{ for } k \geq 3.$$

In the original paper we show that

$$c_k^{\epsilon} = o((c_2^{\epsilon})^{k/2}) \qquad \text{as } L \to \infty, \text{ for } k \geq 3.$$

Since $c_2^{\epsilon} \leq \int_0^1 c_2^{\epsilon} \mathrm{d}\epsilon \leq c_2$ this implies both that

$$c_k^{\epsilon} = o((V_t^{\alpha,c,a}[L])^{k/2}) \qquad \text{as } L \to \infty, \text{ for } k \geq 3,$$

and

$$c_k^{\epsilon} = o((c_2)^{k/2}) \qquad \text{as } L \to \infty, \text{ for } k \geq 3.$$

From (7) note that this is enough to conclude the required result when scaling by the full variance. In the case where we scale by the expected conditional number variance (as originally) this gives us convergence to a Gaussian random variable with mean zero and variance

$$\lim_{L\to\infty} \frac{\mathrm{Var}\big[N_t^{\alpha,c,a}[0,L]\big]}{V_t^{\alpha,c,a}[L]} = 1$$

since the fluctuation term is bounded. $\square$

Proposition 3.3 remains true as stated. The proof may be adapted in the same vein as above. We note that

$$\begin{aligned}
\frac{\mathrm{Var}\big[N_t^{\alpha,c,a}[0,st^{1/\alpha}]\big]}{t^{1/\alpha}} &= \frac{V_t^{\alpha,c,a}[st^{1/\alpha}]}{t^{1/\alpha}} + \frac{F^{\alpha,c,a}(st^{1/\alpha})}{t^{1/\alpha}} \\
&= \frac{4s}{a\pi} \int_0^{\infty} \frac{\sin^2(u/2)}{u^2} \left(1 - e^{-2c(u/s)^{\alpha}}\right) \mathrm{d}u + \frac{F^{\alpha,c,a}(st^{1/\alpha})}{t^{1/\alpha}}.
\end{aligned}$$

and observe that since $0 \leq \frac{F^{\alpha,c,a}(st^{1/\alpha})}{t^{1/\alpha}} \leq \frac{1}{t^{1/\alpha}}$ the additional term disappears on taking the limit $t \to \infty$.

The expression describing the covariance structure of the fluctuation process needs to be corrected to incorporate the additional fluctuation terms. Note in particular that the covariance structure does now in fact depend on t. Proposition 4.1 should now read

Proposition 4.1. $\{Z_t^{\alpha,c,a}(s); s \in [0,\infty)\}$ *has covariance structure*

$$\begin{aligned}
Cov\big(Z_t^{\alpha,c,a}(r), Z_t^{\alpha,c,a}(s)\big) &= \frac{f^{\alpha,c,a}(s) + f^{\alpha,c,a}(r) - f^{\alpha,c,a}(|r-s|)}{2} \\
&\quad + \frac{F^{\alpha,c,a}(st^{1/\alpha}) + F^{\alpha,c,a}(rt^{1/\alpha}) - F^{\alpha,c,a}(|r-s|t^{1/\alpha})}{2t^{1/\alpha}}.
\end{aligned}$$

Proof. The method of the proof is exactly the same but the variance terms are replaced with the new expression and this leads to the additional terms. $\square$

Proposition 4.2 remains true as stated with a suitable modification of the proof. Since

$$\frac{\left(F^{\alpha,c,a}(st^{1/\alpha}) + F^{\alpha,c,a}(rt^{1/\alpha}) - F^{\alpha,c,a}(|r-s|t^{1/\alpha})\right)}{2t^{1/\alpha}} \to 0$$

as $t \to \infty$ the covariance structure of $Z_t^{\alpha,c,a}$ and $\{G^{\alpha,c,a}(s) : s \in [0,\infty)\}$ coincide in the limit. Due to the incorrect assumption of full independence of particles, the proof needs to be adapted. By an analogous argument to the above, proving that the cumulants of the conditional two dimensional joint distribution of $(Z_t^{\alpha,c,a}(s_1), Z_t^{\alpha,c,a}(s_2))$ given ϵ converge to the cumulants of a two-dimensional Gaussian distribution is sufficient to deduce that in fact the cumulants of the two dimensional joint distribution of $(Z_t^{\alpha,c,a}(s_1), Z_t^{\alpha,c,a}(s_2))$ also converge to Gaussian cumulants. Hence

$$(Z_t^{\alpha,c,a}(s_1), Z_t^{\alpha,c,a}(s_2)) \to \text{MultivariateNormal}(0, \Sigma^{\alpha,c,a}) \tag{8}$$

in distribution as $t \to \infty$, where $\Sigma^{\alpha,c,a}$ is the 2×2 covariance matrix

$$\begin{pmatrix} f^{\alpha,c,a}(s_1) & \frac{1}{2}\left(f^{\alpha,c,a}(s_1) + f^{\alpha,c,a}(s_2) - f^{\alpha,c,a}(|s_1 - s_2|)\right) \\ \frac{1}{2}\left(f^{\alpha,c,a}(s_1) + f^{\alpha,c,a}(s_2) - f^{\alpha,c,a}(|s_1 - s_2|)\right) & f^{\alpha,c,a}(s_2) \end{pmatrix}.$$

as before.

Lemma 4.9 remains true as stated for $t \geq 1$, with only a small modification of the proof. The application of Chebyshev's inequality to obtain the first upper bound of the proof needs to be edited to incorporate the true variance. The inequality should in fact read

$$\begin{aligned} \mathbb{P}\left[|Z_t^{\alpha,c,a}(r_{k_j+v}) - Z_t^{\alpha,c,a}(r_{k_j+u})| \geq \gamma\right] &\leq \frac{f^{\alpha,c,a}(|r_{k_j+v} - r_{k_j+u}|)}{\gamma^2} \\ &\quad + \frac{F^{\alpha,c,a}[|r_{k_j+v} - r_{k_j+u}|]}{t^{1/\alpha}} \\ &\leq 2\frac{(v-u)t^{-1/\alpha}}{a\gamma^2} \end{aligned}$$

for $t \geq 1$. The additional factor of two is absorbed by the constant κ in the statement of the Lemma.

As far as we are aware other statements and proofs are unaffected by the error.

Acknowledgements

We would like to thank T.Bojdecki, L.G. Gorostiza and A. Talarczyk for drawing our attention to this error and for outlining their thoughts on possibilities for the rectification of this mistake.

References

[1] Ben Hambly and Liza Jones. Number variance from a probabilistic perspective: infinite systems of independent Brownian motions and symmetric α-stable processes. *Electron. J. Probab.*, 12:no. 30, 862–887 (electronic), 2007. MR2318413

[2] K. Johansson. Determinantal processes with number variance saturation. *Comm. Math. Phys.*, 252(1-3):111–148, 2004. MR2103906