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Pseudoprocesses governed by higher-order fractional differential equations*

Luisa Beghin
Dept. Statistics, Probability, Appl. Stat.
Sapienza University of Rome
p.le A.Moro 5, 00185 Rome
ITALY

Abstract

We study here a heat-type differential equation of order n greater than two, in the case where the time-derivative is supposed to be fractional. The corresponding solution can be described as the transition function of a pseudoprocess Ψ_n (coinciding with the one governed by the standard, non-fractional, equation) with a time argument $\mathcal{T}_\alpha$ which is itself random. The distribution of $\mathcal{T}_\alpha$ is presented together with some features of the solution (such as analytic expressions for its moments).

Key words: Higher-order heat-type equations, Fractional derivatives, Wright functions, Stable laws.

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1 Introduction

The study of diffusion equations with a fractional derivative component have been firstly motivated by the analysis of thermal diffusion in fractal media in Nigmatullin (1986) and Saichev and Zaslavsky (1997). This topic has been extensively treated in the probabilistic literature since the end of the Eighties: see, for examples, Wyss (1986), Schneider and Wyss (1989), Mainardi (1996), Angulo et al. (2000). Recently fractional equations of different types have been also studied, such as, for example, the Black and Scholes equation (see Wyss (2000)) and the fractional diffusion equations with stochastic initial conditions (see Anh and Leonenko (2000)).

Our aim will concern the extension, to the case of fractional time-derivative, of a class of equations which is well known in the literature, namely the *higher-order heat-type equations*. Therefore we will be interested in the solution of the following problem, for $0 < \alpha \leq 1,\ n \geq 2$,

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = k_n \frac{\partial^n}{\partial x^n} u(x,t) & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \delta(x) \end{cases} \tag{1}$$

where $\delta(\cdot)$ is the Dirac delta function, $k_n = (-1)^{q+1}$ for $n = 2q,\ q \in \mathbb{N}$, while $k_n = \pm 1$ for $n = 2q+1$. The fractional derivative appearing in (1) is meant, in the Dzherbashyan-Caputo sense, as

$$(D^\alpha f)(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(z)}{(t-z)^{1+\alpha-m}} dz, & \text{for } m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \text{for } \alpha = m \end{cases},$$

where $m - 1 = \lfloor \alpha \rfloor$ and $f \in C^m$ (see Samko et al. (1993) for a general reference on fractional calculus).

In the non-fractional case (i.e. for $\alpha = 1$) the *pseudoprocesses* $\Psi_n = \Psi_n(t)$, $t > 0$ driven by n-th order equations, i.e.

$$\frac{\partial}{\partial t} p(x,t) = k_n \frac{\partial^n}{\partial x^n} p(x,t), \qquad x \in \mathbb{R},\ t > 0, \tag{2}$$

for $n > 2$, have been introduced in the Sixties and studied so far by many authors starting from Krylov (1960), Daletsky (1969). They have been rigorously defined by constructing a signed measure Q_n on the basis of the fundamental solution $p_n = p_n(x,t)$ of (2), as we briefly recall here. Denote by $X = \{x : t \in [0,\infty) \to x(t)\}$ the space of bounded functions (the sample paths of $\Psi_n(t)$). As usual, we define the cylinder sets by

$$C = \{x : a_1 \leq x(t_1) \leq b_1, ..., a_n \leq x(t_n) \leq b_n\},$$

for a_j, b_j real numbers and $0 = t_0 < ... < t_n = t$. Now put

$$Q_n(C) = \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} \prod_{j=1}^{n} u_n(x_j - x_{j-1}; t_j - t_{j-1}) dx_j$$

where $x_j = x(t_j)$. It can be proved (see Daletsky and Fomin (1965)) that, for fixed t_j's, Q_n is a finite σ-additive measure on the Borel field generated by the cylinders C and has total variation greater than one.

The distributions of many functionals of Ψ_n have been obtained: in Hochberg and Orsingher (1994) the distribution of sojourn time on the positive half-line is presented, for n odd, while

for an arbitrary n the same topic is analyzed in Lachal (2003). For $n = 3, 4$, the case where the pseudoprocess is constrained to be zero at the end of the time interval is considered in Nikitin and Orsingher (2000) and the corresponding distribution of the sojourn time is evaluated. In Beghin et al. (2000) the distribution of the maximum is obtained under the same circumstances. In the unconditional case the maximal distribution is presented in Orsingher (1991), for n odd, while the joint distribution of the maximum and the process for diffusion of order $n = 3, 4$ is presented in Beghin et al. (2001). Lachal (2003) has extended these results to any order $n > 2$.

Some other functionals, such as the first passage time, are treated in Nishioka (1997) and Lachal (2008). Finally in Beghin and Orsingher (2005) it is proved that the local time in zero possesses a proper probability distribution which coincides with the (folded) solution of a fractional diffusion equation of order $2(n-1)/n$, $n \geq 2$.

In the fractional case under investigation (i.e. for $0 < \alpha < 1$) we prove that the process related to (1) is a pseudoprocess Ψ_n evaluated at a random time $\mathcal{T}_\alpha = \mathcal{T}_\alpha(t), t > 0$, so that we can write it as $\Psi_n(\mathcal{T}_\alpha)$. The probability law of the random time is shown to solve a fractional diffusion equation of order 2α and it can be expressed in terms of Wright functions. It is interesting to stress that the introduction of a fractional time-derivative exerts its influence only on the "temporal" argument, while the governing process is not affected and depends only on the degree n of the equation.

Moreover, in section 2, some particular cases of these results are analyzed: in the non-fractional case, $\alpha = 1$, we easily get $\mathcal{T}_\alpha(t) \stackrel{a.s.}{=} t$. For $\alpha = 1/2$ it can be verified that $\mathcal{T}_\alpha$ coincides with the reflecting Brownian motion and then the pseudoprocess governed by equation (1) reduces to $\Psi_n(|B(t)|), t > 0$ (where B denotes a standard Brownian motion).

In section 3 the moments of $\Psi_n(\mathcal{T}_\alpha)$ are obtained in two alternative ways.

Section 4 presents some more explicit forms of the solution to equation (1) which can be obtained by splitting the interval of values for α in two different ones and treating them separately. Indeed, if we restrict ourselves to the case $\alpha \in [1/2, 1]$, the distribution of the random time $\mathcal{T}_\alpha$ coincides with $\frac{1}{\alpha}\widetilde{p}_{\frac{1}{\alpha}}(u;t)$, $u \geq 0$, where by $\widetilde{p}_{\frac{1}{\alpha}}(\cdot;t)$ we have denoted a stable law of index $1/\alpha$.

As far as the other interval is concerned (i.e. $\alpha \in (0, 1/2]$), an explicit expression of the solution can be evaluated by specifying $\alpha = 1/m$, $m \in \mathbb{N}$, $m > 2$. In this particular setting the pseudoprocess can be represented by $\Psi_n\left(G(t)\right), t > 0$, where $G(t) = \prod_{j=1}^{m-1} G_j(t), t > 0$ and $G_j(t)$, $j = 1, ..., m-1$ are independent random variables whose density is presented in explicit form.

Finally we obtain some interesting results by specifying (1) for particular values of n. For example, taking $n = 2$, we can conclude that the process related to the fractional diffusion equation

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) \qquad x \in \mathbb{R},\ t > 0, \tag{3}$$

for $0 < \alpha < 1$, is represented by $B(\mathcal{T}_\alpha)$, in accordance with the results already known on (3). In particular for $\alpha = 1/m$, equation (3) turns out to be solved by the density of the process $B\left(G(t)\right), t > 0$.

In the special case $n = 3$ the results above reduces to those presented in De Gregorio (2002), while, for $n = 4$, they represent a probabilistic alternative to the analytic approach provided by Agrawal (2000).

We note also that equation (1) can be considered as a special case of the general model analyzed by Anh and Leonenko (2001)-(2003). These authors provide the spectral theory and renormalized

solution of the so-called fractional kinetic equation; more precisely, they consider, in the two references above,

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = -\mu(-\Delta)^{\beta/2} (I-\Delta)^{\gamma/2} u(x,t), \qquad \mu, \beta > 0, \tag{4}$$

for $\alpha \in (0,1]$ and $\alpha \in (0,2]$, respectively. Equation (4) is analyzed for $x \in R^d$, under random initial conditions expressed by

$$u(x,0) = \eta(x), \qquad x \in \mathbb{R}^d,$$

where $\{\eta(x)\}$ denotes a measurable random field. The Green function of (4) is also obtained in analytic form in terms of Fox functions; in the special case where $d=1$, $\beta = 2n$ and $\gamma = 0$, this can be compared with our expression for the solution. The latter is expressed as the transition function of the composition of pseudoprocesses with a random time $\Psi_n(T_\alpha(t))$, providing an alternative interpretation.

2 First expressions for the solution

We start by considering the n-th order fractional equation and the following corresponding initial-value problem, for $0 < \alpha \leq 1,\ n \geq 2$,

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = k_n \frac{\partial^n}{\partial x^n} u(x,t) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \delta(x) \end{cases} \tag{5}$$

where $k_n = (-1)^{q+1}$ for $n = 2q$, $q \in \mathbb{N}$, while $k_n = \pm 1$ for $n = 2q+1$ and $\delta(\cdot)$ is the Dirac delta function. The first step consists in evaluating the Laplace transform of the solution $u_\alpha(x,t)$, namely

$$U_\alpha(x,s) = \int_0^\infty e^{-st} u_\alpha(x,t) dt, \tag{6}$$

and recognizing that it is related to the Laplace transform of the solution $p_n(x,t)$ of the corresponding non-fractional n-th order equation (2) (which coincides with (5) for $\alpha = 1$). The latter is usually expressed as

$$p_n(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ixz + k_n t (iz)^n} dz. \tag{7}$$

Theorem 2.1 Let $\Phi_n(x,s) = \int_0^\infty e^{-st} p_n(x,t) dt$ be the Laplace transform of the solution to (2); then (6) can be expressed as follows

$$U_\alpha(x,s) = s^{\alpha-1} \Phi_n(x, s^\alpha). \tag{8}$$

Proof

By taking the Laplace transform of (5) and considering the initial condition, we get

$$s^\alpha U_\alpha(x,s) - s^{\alpha-1} \delta(x) = k_n \frac{\partial^n}{\partial x^n} U_\alpha(x,s). \tag{9}$$

Then, by integrating (9) with respect to x in $[-\varepsilon,\varepsilon]$ and letting $\varepsilon\to 0$, we have the following condition for the $(n-1)$-th derivative

$$-s^{\alpha-1}=k_n\,\frac{\partial^{n-1}}{\partial x^{n-1}}U_\alpha(x,s)\Big|_{x=0^-}^{x=0^+},$$

which must be added to the continuity conditions in zero holding for the j-th derivatives, for $j=0,..,n-2$. Therefore our problem is reduced to the n-th order linear equations

$$\left\{\begin{array}{ll} k_n\frac{\partial^n}{\partial x^n}U_\alpha(x,s)=s^\alpha U_\alpha(x,s), & x\neq 0\\ \frac{\partial^j}{\partial x^j}U_\alpha(x,s)\Big|_{x=0^-}^{x=0^+}=0, & \text{for } j=0,1,...,n-2\\ \frac{\partial^{n-1}}{\partial x^{n-1}}U_\alpha(x,s)\Big|_{x=0^-}^{x=0^+}=-k_n s^{\alpha-1} & \end{array}\right. . \tag{10}$$

If we now impose the boundedness condition for $x\to\pm\infty$, we obtain

$$U_\alpha(x,s)=\left\{\begin{array}{ll} \sum_{k\in I}c_k e^{\theta_k s^{\alpha/n}x}, & \text{if } x>0\\ \sum_{k\in J}d_k e^{\theta_k s^{\alpha/n}x}, & \text{if } x\leq 0\end{array}\right. , \tag{11}$$

where θ_k are the n-th roots of k_n, $I=\{k:\mathbb{R}e(\theta_k)<0\}$ and $J=\{k:\mathbb{R}e(\theta_k)>0\}$. The n unknown constants $c_k, k\in I$ and $d_k, k\in J$, appearing in (11) must be determined by taking into account the matching conditions in (10), as follows:

$$\left\{\begin{array}{ll} \sum_{k\in I}c_k\theta_k^j-\sum_{k\in J}d_k\theta_k^j=0, & \text{for } j=0,...,n-2\\ \sum_{k\in I}c_k\theta_k^{n-1}-\sum_{k\in J}d_k\theta_k^{n-1}=-k_n s^{\alpha/n-1} & \end{array}\right. . \tag{12}$$

By defining

$$z_k=\left\{\begin{array}{ll} c_k, & \text{if } k\in I\\ -d_k, & \text{if } k\in J\end{array}\right. , \tag{13}$$

the linear system in (12) can be rewritten as the following Vandermonde system

$$\sum_{k=0}^{n-1}z_k\theta_k^j=\left\{\begin{array}{ll} 0, & \text{for } j=0,...,n-2\\ -k_n s^{\alpha/n-1}, & \text{for } j=n-1\end{array}\right. . \tag{14}$$

Following an argument similar to Beghin and Orsingher (2005) (see p.1024-5) we get

$$\begin{aligned} z_k &= (-1)^n k_n s^{\alpha/n-1}\prod_{\substack{r=0\\ r\neq k}}^{n-1}\frac{1}{\theta_r-\theta_k} \\ &= \left\{\begin{array}{ll} -\frac{1}{n}s^{\alpha/n-1}e^{\frac{2k\pi i}{n}}, & \text{if } k_n=1\\ -\frac{1}{n}s^{\alpha/n-1}e^{\frac{(2k+1)\pi i}{n}}, & \text{if } k_n=-1\end{array}\right. ,\end{aligned} \tag{15}$$

where, in the last step, we have used formula (2.19) obtained therein. We now substitute into (11) the constants evaluated in (15), taking into account (13) and distinguishing the case of n even from the odd one. Indeed, for $n=2q+1$, the roots of k_n are respectively

$$\theta_k=\left\{\begin{array}{ll} e^{\frac{2k\pi i}{n}}, & \text{for } k_n=1\\ e^{\frac{(2k+1)\pi i}{n}}, & \text{for } k_n=-1\end{array}\right. \tag{16}$$

so that (11) becomes, in this case,

$$U_\alpha(x,s) = \begin{cases} -\frac{1}{n}s^{\alpha/n-1}\sum_{k\in I}\theta_k e^{\theta_k s^{\alpha/n}x}, & \text{for } x>0 \\ \frac{1}{n}s^{\alpha/n-1}\sum_{k\in J}\theta_k e^{\theta_k s^{\alpha/n}x}, & \text{for } x\leq 0 \end{cases} . \tag{17}$$

Analogously, for $n=2q$ and $k_n=(-1)^{q+1}$, the roots are $\theta_k = e^{\frac{(2k+q+1)\pi i}{n}}$ so that we get

$$\theta_k = \begin{cases} e^{\frac{(2k+q+1)\pi i}{n}} = e^{\frac{2k\pi i}{n}}, & \text{for } k_n=1 \\ e^{\frac{(2k+q+1)\pi i}{n}} = e^{\frac{(2k+1)\pi i}{n}}, & \text{for } k_n=-1 \end{cases} , \tag{18}$$

where, in the first line, we have used the following relationship

$$e^{(q+1)\pi i} = (-1)^{q+1} = k_n = 1,$$

while, in the second one, we have considered the fact that

$$e^{q\pi i} = (-1)k_n = 1.$$

Since (18) coincides with (16) we obtain even for $n=2q$ formula (17). The proof is completed by comparing it with formula (12) of Lachal (2003), which reads

$$\Phi_n(x,s) = \begin{cases} -\frac{1}{n}s^{1/n-1}\sum_{k\in I}\theta_k e^{\theta_k s^{1/n}x}, & \text{for } x>0 \\ \frac{1}{n}s^{1/n-1}\sum_{k\in J}\theta_k e^{\theta_k s^{1/n}x}, & \text{for } x\leq 0 \end{cases}.$$

□

By inverting the Laplace transform (8) we can obtain a first expression of the solution in terms of a fractional integral of a particular stable law. Following the notation of Samorodnitsky and Taqqu (1994), we will denote by $S_\alpha(\sigma,\beta,\mu)$ the distribution of a *stable random variable* X of index α, with characteristic function

$$Ee^{isX} = \exp\left\{-\sigma^\alpha|s|^\alpha\left(1-i\beta(sign\, s)\tan\frac{\pi\alpha}{2}\right)+i\mu s\right\}, \qquad \alpha\neq 1,\ s\in\mathbb{R}. \tag{19}$$

Moreover let $I_{(1-\alpha)}$ denote the *Riemann-Liouville fractional integral* of order $1-\alpha$, which is defined as $I_{(1-\alpha)}\left[f(w)\right](t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-w)^{-\alpha}f(w)dw$ (see Samko et al. (1993), p.33).

Theorem 2.2 Let $\overline{p}_\alpha(\cdot;u)$ be the stable distribution $S_\alpha(\sigma,1,0)$, with parameters $\sigma = (u\cos\pi\alpha/2)^{1/\alpha}, \beta = 1, \mu = 0$, then the fundamental solution to (5) can be expressed, for $0<\alpha<1$, as

$$u_\alpha(x,t) = \int_0^\infty p_n(x,u) I_{(1-\alpha)}\left[\overline{p}_\alpha(w;u)\right](t)du. \tag{20}$$

Proof

We recall that, for $0<\alpha\leq 2$ and $\alpha\neq 1$, a stable random variable $X\sim S_\alpha(\sigma,1,0)$ has Laplace transform

$$E(e^{-sX}) = e^{-\frac{\sigma^\alpha}{\cos(\pi\alpha/2)}s^\alpha}, \qquad s>0$$

(see Samorodnitsky and Taqqu (1994), p.15, for details), so that, in our case (for $\sigma = (u\cos\pi\alpha/2)^{1/\alpha}$), it reduces to $E(e^{-sX}) = e^{-s^\alpha u}$. Therefore we can rewrite (8) as

$$\begin{aligned} U_\alpha(x,s) &= s^{\alpha-1}\int_0^{+\infty} e^{-s^\alpha t}p_n(x,t)dt \\ &= s^{\alpha-1}\int_0^{+\infty}\left(\int_0^{+\infty} e^{-sz}\overline{p}_\alpha(z;u)dz\right)p_n(x,u)du \\ &= s^{\alpha-1}\int_0^{+\infty} e^{-sz}\left(\int_0^{+\infty}\overline{p}_\alpha(z;u)p_n(x,u)du\right)dz. \end{aligned} \tag{21}$$

For $0<\alpha<1$ the first term appearing in (21) can be easily inverted by considering that

$$s^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)}\int_0^{+\infty} e^{-st}t^{-\alpha}dt$$

so that the inverse Laplace transform of (21) can be written as

$$\begin{aligned} u_\alpha(x,t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-w)^{-\alpha}\left(\int_0^{+\infty}\overline{p}_\alpha(w;u)p_n(x,u)du\right)dw \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^{+\infty}\left(\int_0^t (t-w)^{-\alpha}\overline{p}_\alpha(w;u)dw\right)p_n(x,u)du. \end{aligned} \tag{22}$$

Finally we recognize in the last expression a fractional Riemann-Liouville integral $I_{(1-\alpha)}$ of order $1-\alpha$ of the stable density (where the integration is intended with respect to the first argument, since the second represents a constant in the scale parameter). □

The previous result suggests that the solution to our problem can be described as the transition function $p_n = p_n(x,u)$ of a pseudoprocess Ψ_n with a time-argument $\mathcal{T}_\alpha$ which is itself random. Only for $\alpha = 1$ we can derive from Theorem 2.1 the obvious result that $\mathcal{T}_\alpha(t) \stackrel{a.s.}{=} t$, so that the solution to (5) coincides, as expected, with $p_n(x,t)$. In all other cases the governing process coincides with the non-fractional one, while the introduction of a fractional time-derivative exerts its influence only on the time argument (as remarked above).

To check that $\mathcal{T}_\alpha$ possesses a true probability density we can observe that it is non-negative: indeed it coincides with the fractional integral of a stable density $S_\alpha(\sigma,1,0)$ with skewness parameter equal to 1 (which, by the way, for $0 < \alpha < 1$, has support restricted to $[0,\infty)$). Moreover it integrates to one, as can be ascertained by the following steps:

$$\begin{aligned}
& \int_0^\infty \frac{du}{\Gamma(1-\alpha)} \int_0^t (t-w)^{-\alpha} \overline{p}_\alpha(w;u) dw \\
= & \frac{1}{\Gamma(1-\alpha)} \int_0^\infty du \int_0^t (t-w)^{-\alpha} dw \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{sw} e^{-s^\alpha u} ds \\
= & \frac{1}{2\pi i \Gamma(1-\alpha)} \int_0^t (t-w)^{-\alpha} dw \int_0^\infty du \int_{-i\infty}^{+i\infty} e^{sw} e^{-s^\alpha u} ds \\
= & \frac{1}{2\pi i \Gamma(1-\alpha)} \int_0^t (t-w)^{-\alpha} dw \int_{-i\infty}^{+i\infty} s^{-\alpha} e^{sw} ds \\
= & \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t w^{\alpha-1} (t-w)^{-\alpha} dw = \frac{B(\alpha,1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} = 1 \, .
\end{aligned}$$

Our aim is now to explicit, by means of successive steps, the density $\overline{v}_{2\alpha} = \overline{v}_{2\alpha}(u,t)$ of $\mathcal{T}_\alpha(t), t > 0$: we first prove that it satisfies a fractional diffusion equation of order 2α and, as a consequence, it can be expressed in terms of *Wright function*. Let

$$W(x;\eta,\beta) = \sum_{k=0}^{\infty} \frac{x^k}{k!\Gamma(\eta k + \beta)}$$

be a Wright function of parameters η, β, then we state the following result.

Theorem 2.3 The fundamental solution to (5) coincides with

$$u_\alpha(x,t) = \int_0^\infty p_n(x,u) \overline{v}_{2\alpha}(u,t) du, \tag{23}$$

where

$$\overline{v}_{2\alpha}(u,t) = \frac{1}{t^\alpha} W\left(-\frac{u}{t^\alpha}; -\alpha, 1-\alpha\right), \qquad u \geq 0, \; t > 0. \tag{24}$$

Proof

It is proved in Orsingher and Beghin (2004) that, for $0 < \alpha < 1$,

$$I_{(1-\alpha)}\left[\overline{p}_\alpha(|w|;u)\right](t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-w)^{-\alpha}\overline{p}_\alpha(|w|;u)dw$$

coincides with the solution $v_{2\alpha}(u,t)$ of the following initial-value problem, for $0 < \alpha < 1$,

$$\begin{cases} \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}v(u,t) = \frac{\partial^2}{\partial u^2}v(u,t) & u \in \mathbb{R},\ t > 0 \\ v(u,0) = \delta(u) & \\ \frac{\partial}{\partial t}v(u,0) = 0 & \\ \lim_{|u|\to\infty} v(u,t) = 0 & \end{cases}, \tag{25}$$

where the second initial condition applies only for $\alpha \in (1/2, 1)$. As a consequence, formula (20) can be rewritten as (23) with

$$\overline{v}_{2\alpha}(u,t) = \begin{cases} 2v_{2\alpha}(u,t), & \text{for } u \geq 0 \\ 0, & \text{for } u < 0 \end{cases}. \tag{26}$$

Since it is known (see, among the others, Mainardi (1996)) that the solution to (25) can be expressed as

$$\begin{aligned} v_{2\alpha}(u,t) &= \frac{1}{2t^\alpha}\sum_{k=0}^{\infty}\frac{(-|u|t^{-\alpha})^k}{k!\Gamma(-\alpha k + 1 - \alpha)} \\ &= \frac{1}{2t^\alpha}W\left(-\frac{|u|}{t^\alpha}; -\alpha, 1-\alpha\right), \qquad u \in \mathbb{R},\ t > 0, \end{aligned}$$

we immediately get (24). □

Remark 2.1

By means of the previous result we can remark again that the random time $\mathcal{T}_\alpha$ possesses a true probability density, which is concentrated on the positive half line and moreover it is possible, thanks to representation (24), to evaluate the moments of any order $\delta \geq 0$ of this distribution. We recall the well known expression of the inverse of the Gamma function as integral on the Hankel contour

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i}\int_{Ha} e^\tau \tau^{-x} d\tau,$$

which implies the representation of the Wright function as

$$\begin{aligned} W(x;\eta,\beta) &= \sum_{k=0}^{\infty}\frac{x^k}{k!\Gamma(\eta k + \beta)} \\ &= \frac{1}{2\pi i}\int_{Ha} e^\tau \sum_{k=0}^{\infty}\frac{x^k\tau^{-\eta k-\beta}}{k!}d\tau \\ &= \frac{1}{2\pi i}\int_{Ha}\tau^{-\beta}e^{\tau + x\tau^{-\eta}}d\tau. \end{aligned}$$

Therefore we can show that

$$\begin{aligned} & \int_0^{+\infty} u^\delta \overline{v}_{2\alpha}(u,t)du \qquad (27) \\ = & \int_0^{\infty} \frac{u^\delta}{t^\alpha} W\left(-\frac{u}{t^\alpha}; -\alpha, 1-\alpha\right) du \\ = & \int_0^{\infty} \frac{u^\delta}{t^\alpha} \frac{du}{2\pi i} \int_{Ha} e^{y-\frac{u}{t^\alpha}y^\alpha} y^{\alpha-1} dy \\ = & \frac{1}{2\pi i} \int_{Ha} e^y y^{\alpha-1} dy \frac{1}{t^\alpha} \int_0^{+\infty} e^{-\frac{u}{t^\alpha}y^\alpha} u^\delta du \\ = & \frac{t^{\alpha\delta}}{2\pi i} \int_{Ha} e^y y^{-\alpha\delta-1} dy \int_0^{+\infty} e^{-z} z^\delta dz \\ = & \frac{\Gamma(1+\delta)t^{\alpha\delta}}{\Gamma(1+\alpha\delta)} = \frac{t^{\alpha\delta}\Gamma(\delta)}{\alpha\Gamma(\alpha\delta)} \,. \end{aligned}$$

From (27) it is again evident that $\int_0^{+\infty} \overline{v}_{2\alpha}(u,t)du = 1$ by choosing $\delta = 0$.

Remark 2.2

It is interesting to analyze the particular case obtained for $\alpha = 1/2$: indeed, from the previous results, we can show that the process governed by $\frac{\partial^{1/2}}{\partial t^{1/2}} u(x,t) = k_n \frac{\partial^n}{\partial x^n} u(x,t)$, $x \in \mathbb{R}$, $t > 0$, can be represented as $\Psi_n(|B(t)|), t > 0$, where $B(t), t > 0$ denotes a standard Brownian motion. This can be seen by noting that $S_{1/2}\left(\frac{u^2}{2}, 1, 0\right)$ coincides with the Lévy distribution, so that the fractional integral in (20) reduces to

$$\begin{aligned} I_{(1/2)}\left[\overline{p}_{1/2}(w;u)\right](t) &= \frac{1}{\Gamma(1/2)} \int_0^t \frac{u e^{-u^2/4w}}{2\sqrt{\pi(t-w)w^3}} dw \qquad (28) \\ &= \frac{e^{-u^2/4t}}{\sqrt{\pi t}} \,, \qquad u > 0, t > 0, \end{aligned}$$

where the second step follows by applying formula n.3.471.3, p.384 of Gradshteyn and Rhyzik (1994), for $\mu = 1/2$. Formula (28) represents the density of a Brownian motion with reflecting barrier in $u = 0$. This result is confirmed by noting that equation (25), for $\alpha = 1/2$, reduces to the heat equation $\frac{\partial}{\partial t} v(x,t) = \frac{\partial^2}{\partial x^2} v(x,t)$ and then the corresponding process coincides with a Brownian motion with $\sigma^2 = 2t$. Alternatively, from (24), by applying some known properties of

the Gamma function, we can write

$$\begin{aligned}\overline{v}_1(u,t) &= \frac{1}{\sqrt{t}}\sum_{k=0}^{\infty}\frac{(-ut^{-1/2})^k}{k!\Gamma\left(1-\frac{k+1}{2}\right)} \\ &= \frac{1}{\sqrt{t}}\sum_{\substack{k=0\\ k\ \text{even}}}^{\infty}\frac{(-1)^{k/2}(ut^{-1/2})^k\Gamma\left(\frac{k+1}{2}\right)}{\pi k!} \\ &= \frac{1}{\pi\sqrt{t}}\sum_{\substack{k=0\\ k\ \text{even}}}^{\infty}\frac{(-1)^{k/2}(ut^{-1/2})^k\Gamma\left(k+1\right)\sqrt{\pi}2^{1-(k+1)}}{k!\Gamma\left(\frac{k}{2}+1\right)} \\ &= \frac{1}{\sqrt{\pi t}}\sum_{j=0}^{\infty}\frac{(-1)^j u^{2j}(4t)^{-j}}{j!} = \frac{e^{-u^2/4t}}{\sqrt{\pi t}}\,.\end{aligned} \tag{29}$$

3 On the moments of the solution

We are now interested in evaluating the moments of the solution of equation (5), that is the moments of the pseudoprocess $\Psi_n(\mathcal{T}_\alpha(t)), t > 0$: as we will see, they can be obtained in two alternative ways.

By using the representation of the solution derived in (23) and thanks to the independence of the leading process from the (random) temporal argument, we can write the r-th order moments as

$$\begin{aligned}&E\left(\Psi_n^r(\mathcal{T}_\alpha(t))\right) \\ &= \int_0^\infty E\Psi_n^r(s)\overline{v}_{2\alpha}(s,t)ds,\end{aligned} \tag{30}$$

for $r \in \mathbb{N}, t > 0$. The moments of the non-fractional n-th order pseudoprocess can be evaluated by means of the Fourier transform of the solution of equation (2) which can be expressed as follows

$$\begin{aligned}E\left(e^{i\beta\Psi_n(t)}\right) &= \int_{-\infty}^{+\infty}e^{i\beta x}p_n(x,t)dx = e^{(-i\beta)^n k_n t} \\ &= \sum_{j=0}^{\infty}\frac{(i\beta)^{nj}}{(nj)!}\frac{(-1)^{nj}k_n^j t^j (nj)!}{j!}\,.\end{aligned} \tag{31}$$

Therefore we get

$$E\Psi_n^r(t) = \begin{cases}\frac{(-1)^r(k_n t)^{r/n}r!}{(r/n)!} & r = nj,\ j = 1,2,... \\ 0 \qquad r \neq nj\end{cases},$$

which, inserted together with (27) into (30), gives, for $r = nj$, $j = 1, 2, ...$

$$\begin{aligned}&E\left(\Psi_n^r(\mathcal{T}_\alpha(t))\right) \\ &= \frac{(-1)^{nj}k_n^j(nj)!}{j!}\int_0^\infty s^j\overline{v}_{2\alpha}(s,t)ds \\ &= (-1)^{nj}k_n^j t^{\alpha j}\frac{\Gamma(nj+1)}{\Gamma(\alpha j+1)}\,,\end{aligned} \tag{32}$$

while it is equal to zero for $r \neq nj$.

We can alternatively derive the moments of the pseudoprocesses by evaluating them directly from the characteristic function of the solution. The latter can be obtained by performing successively the Fourier and Laplace transforms of equation (5) as follows: let us denote by $\widetilde{u}_\alpha(\beta,t)$ the Fourier transform of the solution, i.e.

$$\widetilde{u}_\alpha(\beta,t) = \int_{-\infty}^{+\infty} e^{i\beta x} u_\alpha(x,t)dx, \qquad \beta, t > 0,$$

then, from (5), we get

$$\frac{\partial^\alpha \widetilde{u}_\alpha}{\partial t^\alpha}(\beta,t) = k_n(-i\beta)^n \widetilde{u}_\alpha(\beta,t). \tag{33}$$

By applying now the Laplace transform to (33) we get

$$s^\alpha \widetilde{U}_\alpha(\beta,s) - s^{\alpha-1} = k_n(-i\beta)^n \widetilde{U}_\alpha(\beta,s),$$

so that the Fourier-Laplace transform of the solution can be written as

$$\widetilde{U}_\alpha(\beta,s) = \frac{s^{\alpha-1}}{s^\alpha - k_n(-i\beta)^n}\ . \tag{34}$$

Now recall that for the *Mittag-Leffler function*

$$\mathrm{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

the Laplace transform (for $\beta = 1$) is equal to

$$\int_0^\infty e^{-sz} \mathrm{E}_{\alpha,1}(cz^\alpha)dz = \frac{s^{\alpha-1}}{s^\alpha - c}$$

(see Podlubny (1999), formula (1.80) p. 21, for $k = 0$, $\beta = 1$); hence from (34) we get the following expression for the characteristic function of the solution

$$\widetilde{u}_\alpha(\beta,t) = \mathrm{E}_{\alpha,1}(k_n(-i\beta)^n t^\alpha) \tag{35}$$

and for the solution itself

$$u_\alpha(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ix\beta} \mathrm{E}_{\alpha,1}(k_n(-i\beta)^n t^\alpha)d\beta. \tag{36}$$

In the particular case $\alpha = 1$ the Mittag-Leffler function reduces to the exponential so that (35) coincides with the Fourier transform of the solution to the n-th order equation, reported in (31), as it should be in the non-fractional case. Analogously, from (36) we get the usual expression of $p_n(x,t)$ reported in (7). On the other hand, in the fractional case ($\alpha \neq 1$) formula (35) reduces, for $n = 2$, to the well-known Fourier transform of the solution to equation (3).

Finally we can evaluate the moments of the solution by rewriting formula (35) as

$$\widetilde{u}_\alpha(\beta,t) = \sum_{j=0}^{\infty} \frac{(i\beta)^{nj}}{(nj)!} \frac{(-1)^{nj} k_n^j t^{\alpha j}}{\Gamma(\alpha j + 1)} \Gamma(nj+1),$$

so that we get again expression (32).

4 More explicit forms of the solution

In order to obtain a more explicit form of the solution to (5), in terms of known densities, we need to distinguish between two intervals of values for α.

(i) Case $1/2 \leq \alpha < 1$

If we restrict ourselves to the case $\alpha \in [1/2, 1)$, so that $1 \leq 2\alpha < 2$, it is possible to apply a result obtained in Fujita (1990), which expresses the solution to a time-fractional diffusion equation in terms of a stable density of appropriate index. By adapting that result to our case, we can conclude that the solution to (25) coincides with

$$v_{2\alpha}(u,t) = \frac{1}{2\alpha}\widetilde{p}_{1/\alpha}(|u|;t), \qquad u \in \mathbb{R},\, t > 0,$$

where $\widetilde{p}_{1/\alpha}(\cdot;t)$ denotes a stable density of index $1/\alpha \in [1,2)$ with parameters $\sigma = (t\cos(\pi - \frac{\pi}{2\alpha}))^{\alpha}$, $\beta = -1$, $\mu = 0$ (for brevity $S_{1/\alpha}(\sigma, -1, 0)$).
Therefore the density of $\mathcal{T}_{\alpha}(t), t > 0$ is proportional to the positive branch of a stable density, as the following expression shows:

$$\overline{v}_{2\alpha}(u,t) = \frac{1}{\alpha}\widetilde{p}_{1/\alpha}(u;t), \qquad u > 0,\, t > 0. \tag{37}$$

Remark 4.1

It is possible to recognize, in the previous expression, a known density, by resorting to results on the supremum of stable processes (see, for example, Bingham (1973)). More precisely, let us define $Y(t) = \sup_{0\leq s\leq t} X_{1/\alpha}(s)$ where $X_{1/\alpha}(t), t > 0$ is a stable process of index $1/\alpha$ and with characteristic function

$$E(e^{isX_{1/\alpha}(t)}) = \exp\left\{-t|s|^{1/\alpha}\left(1 + i\tan\frac{\pi}{2\alpha}\frac{s}{|s|}\right)\right\}, \qquad t, s > 0.$$

It corresponds, for any fixed t, to the stable law $\widetilde{p}_{1/\alpha}(\cdot;t)$ defined above and, for t varying, to a spectrally negative process, which has no positive jumps (since, for $\beta = -1$, the Lévy-Khinchine measure assigns zero mass to $(0,\infty)$, see Samorodnitsky and Taqqu (1994), p.6). Under these circumstances and for $1/\alpha \in [1,2)$, it is known that the Laplace transform of $Y(t)$ is equal, for any $s, t > 0$, to

$$E(e^{-sY(t)}) = \mathrm{E}_{\alpha,1}(-st^{\alpha}),$$

where $\mathrm{E}_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined above. Since it is also well-known that

$$\int_0^{\infty} e^{-su}\widetilde{p}_{1/\alpha}(u;t)du = \alpha\mathrm{E}_{\alpha,1}(-st^{\alpha}), \qquad t, s > 0,$$

we can conclude that

$$E(e^{-sY(t)}) = \int_0^{\infty} e^{-su}\frac{1}{\alpha}\widetilde{p}_{1/\alpha}(u;t)du.$$

Alternatively it can be shown, by adapting the result of Bingham (1973), that the density of $Y(t)$ can be written as

$$\begin{aligned} P\left\{Y(t) \in du\right\} &= \frac{t^{-\alpha}}{\alpha\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sin\left(\pi n\alpha\right) \Gamma\left(1+n\alpha\right) \left(\frac{u}{t^{\alpha}}\right)^{n-1} du \\ &= \frac{1}{\alpha} \widetilde{p}_{1/\alpha}(u;t) du, \qquad u>0,\, t>0 \end{aligned}$$

which coincides with (37).

Formula (37) shows that, for $1/2 \leq \alpha < 1$,

$$I_{(1-\alpha)}\left[\overline{p}_{\alpha}(w;u)\right](t) = \frac{1}{\alpha} \widetilde{p}_{1/\alpha}(u;t), \qquad u>0,\, t>0.$$

Then, as a result of the fractional integration of the stable density $\overline{p}_{\alpha}(\cdot;t)$, which is totally skewed to the right (with support $[0,\infty)$), we obtain the positive (normalized) branch of a new stable density $\widetilde{p}_{1/\alpha}(\cdot;t)$ (defined on the whole real axes, since it is $1/\alpha \in (1,2]$), which represents the distribution of the maximum of a stable process of index $1/\alpha$.

(ii) Case $0 < \alpha \leq 1/2$

We turn now to the other interval of values for α, i.e. $(0,1/2]$, so that, in this case, it is $0 < 2\alpha \leq 1$. An explicit expression of the solution can be evaluated by specifying $\alpha = 1/m$, $m \in \mathbb{N}$, $m > 2$. In this particular setting, problem (25) becomes

$$\begin{cases} \frac{\partial^{2/m}}{\partial t^{2/m}} v(u,t) = \frac{\partial^2}{\partial u^2} v(u,t), \qquad u \in \mathbb{R},\ t>0 \\ v(u,0) = \delta(u) \\ \lim_{|u|\to\infty} v(u,t) = 0 \end{cases} \tag{38}$$

so that it can be considered as a special case of the fractional telegraph equation studied in Beghin and Orsingher (2003), for $\lambda = 0$ and $c = 1$. By applying formula (2.11) of that paper, the solution to (38) can be expressed, for $u \in \mathbb{R}$, $t > 0$, as

$$\begin{aligned} & v_{2/m}(u,t) \qquad\qquad (39) \\ = \ & \left(\frac{m}{2\pi}\right)^{\frac{m-1}{2}} \frac{1}{2\sqrt{t}} \int_0^{\infty} dw_1 ... \int_0^{\infty} dw_{m-1} \cdot \\ & \cdot e^{-\frac{w_1^m + ... + w_{m-1}^m}{\sqrt[m-1]{m^m t}}} w_2 \cdots w_{m-1}^{m-2} \left[\delta(u - w_1 \cdots w_{m-1}) + \delta(u + w_1 \cdots w_{m-1}\right]. \end{aligned}$$

By taking, as before,

$$\overline{v}_{2/m}(u,t) = \begin{cases} 2v_{2/m}(u,t), & \text{for } u \geq 0 \\ 0, & \text{for } u < 0 \end{cases}, \tag{40}$$

the solution to our problem (5) can be expressed, in this case, as

$$\begin{aligned} u_{1/m}(x,t) &= \int_0^{\infty} p_n(x,u) \overline{v}_{2/m}(u,t) du \\ &= \int_0^{\infty} p_n(x,u) p_{G(t)}(u) du \end{aligned}$$

where $G(t) = \prod_{j=1}^{m-1} G_j(t), t > 0$ and $G_j(t)$, $j = 1, ..., m-1$ are independent random variables with the following probability law

$$p_{G_j(t)}(w) = \frac{1}{m^{\frac{j}{m-1}-1} t^{\frac{j}{m(m-1)}} \Gamma(\frac{j}{m})} \exp\left(-\frac{w^m}{\sqrt[m-1]{m^m t}}\right) w^{j-1} \quad w > 0. \tag{41}$$

We can check the independence by noting that

$$\begin{aligned} & \prod_{j=1}^{m-1} p_{G_j(t)}(w_j) \qquad (42) \\ = & \prod_{j=1}^{m-1} \frac{1}{m^{\frac{j}{m-1}-1} t^{\frac{j}{m(m-1)}} \Gamma(\frac{j}{m})} \exp\left(-\frac{w_j^m}{\sqrt[m-1]{m^m t}}\right) w_j^{j-1} \\ = & \left(\frac{m}{2\pi}\right)^{\frac{m-1}{2}} \frac{1}{\sqrt{t}} \exp\left(-\frac{\sum_{j=1}^{m-1} w_j^m}{\sqrt[m-1]{m^m t}}\right) \prod_{j=1}^{m-1} w_j^{j-1}, \end{aligned}$$

where, in the second step, we have applied the multiplication formula of the Gamma function

$$\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)...\Gamma\left(z+\frac{m-1}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz} \Gamma(mz),$$

for $z = 1/m$. The last expression in (42) coincides with the joint density of the variables $G_j(t)$ given in formula (1.7) of Beghin and Orsingher (2003). Therefore the corresponding pseudoprocess is represented, in this case, as $\Psi_n(G(t)), t > 0$.

Remark 4.2

We can check the previous results, obtained separately for the two intervals, by choosing $\alpha = 1/2$. From both cases we obtain again that the pseudoprocess governed by our equation can be represented by $\Psi_n(|B(t)|), t > 0$.

Indeed from the first case, i.e. for $1/2 \leq \alpha < 1$, we get, by means of (37), that the density of $\mathcal{T}_\alpha(t), t > 0$, for $\alpha = 1/2$, coincides with the folded normal. More precisely, $S_2(\sqrt{t}, -1, 0)$ coincides with $N(0, 2t)$ and then

$$\overline{v}_1(u,t) = 2\widetilde{p}_2(u;t) = \frac{e^{-u^2/4t}}{\sqrt{\pi t}} \tag{43}$$

for $u > 0$, $t > 0$.

On the other hand, if we consider the expression of the density of $\mathcal{T}_\alpha$ obtained for $0 < \alpha \leq 1/2$, we get, for $\alpha = 1/2$ and $m = 2$, from (42) that again

$$p_{G_1(t)}(u) = \frac{e^{-u^2/4t}}{\sqrt{\pi t}}. \tag{44}$$

Moreover both (43) and (44) coincide with (28) derived above, as expected.

An interesting application of our results can be obtained by specializing Theorems 2.2 and 2.3 to the particular case $n = 2$. In this situation the pseudoprocess $\Psi_n(t), t > 0$ reduces to the Brownian motion (with variance $2t$) $B(t), t > 0$ and therefore the solution of (5) coincides with the transition density of the process $B(T_\alpha(t)), t > 0$ obtained by the composition of B with the random time T_α. We state this last result as follows

Corollary 4.1

The solution of the problem

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \delta(x) \end{cases} \tag{45}$$

for $0 < \alpha \leq 1$, is represented by the transition function of $B(\mathcal{T}_\alpha)$. The density of the random time $\mathcal{T}_\alpha = \mathcal{T}_\alpha(t), t > 0$ is the folded solution of the time-fractional equation

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} v(u,t) = \frac{\partial^2}{\partial u^2} v(u,t) \qquad u \in \mathbb{R},\ t > 0$$

and is given in (24).

We can prove that this is in accordance with what is already known on (45): for $n = 2$ we can substitute in (23) the transition function of the Brownian motion, so that we get:

$$\begin{aligned} u_\alpha(x,t) &= \frac{1}{t^\alpha} \int_0^\infty \frac{e^{-x^2/4u} du}{\sqrt{4\pi u}} W\left(-\frac{u}{t^\alpha}; -\alpha, 1-\alpha\right) \\ &= \frac{1}{t^\alpha} \int_0^\infty \frac{e^{-x^2/4u} du}{\sqrt{4\pi u}} \frac{1}{2\pi i} \int_{Ha} \frac{e^{y - \frac{u}{t^\alpha} y^\alpha}}{y^{1-\alpha}} dy \\ &= \frac{1}{4it^\alpha \sqrt{\pi^3}} \int_{Ha} \frac{e^y}{y^{1-\alpha}} dy \int_0^\infty \frac{e^{-\frac{x^2}{4u} - \frac{u}{t^\alpha} y^\alpha}}{\sqrt{u}} du. \end{aligned} \tag{46}$$

If we show now that

$$\int_0^\infty \frac{e^{-\frac{x^2}{4u} - \frac{u}{t^\alpha} y^\alpha}}{\sqrt{u}} du = \sqrt{\pi} t^{\alpha/2} y^{-\alpha/2} e^{-\frac{|x|}{t^{\alpha/2}} y^{\alpha/2}} \tag{47}$$

and substitute (47) into (46), we finally get the known result

$$\begin{aligned} u_\alpha(x,t) &= \frac{1}{2t^{\alpha/2}} \frac{1}{2\pi i} \int_{Ha} \frac{e^{y - \frac{|x|}{t^{\alpha/2}} y^{\alpha/2}}}{y^{1-\alpha/2}} dy \\ &= \frac{1}{2t^{\alpha/2}} W\left(-\frac{|x|}{t^{\alpha/2}}; -\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right). \end{aligned}$$

In order to verify formula (47) we use the following relationship, known for the Laplace transform of the first-passage time of Brownian motion,

$$e^{-|x|\sqrt{s}} = \int_0^\infty e^{-su} \frac{|x|}{2\sqrt{\pi}\sqrt{u^3}} e^{-\frac{|x|^2}{4u}} du,$$

which, integrated with respect to x gives (47), for $s = y^\alpha/t^\alpha$. Alternatively, we can apply formula n.3.471.9, p.384 of Gradshteyn and Ryzhik (1994), for $\beta = x^2/4$, $\gamma = y^\alpha/t^\alpha$, $\nu = 1/2$ (noting that $K_{1/2}(z) = \sqrt{\pi/2z}e^{-z}$, see Gradshteyn and Ryzhik (1994), n.8469.3, p.978).

We can also compare the result of Corollary 4.1 with formula (3.1) of Anh and Leonenko (2001), which, in our notation, reads

$$u_\alpha(x,t) = \frac{1}{\alpha}\int_0^{+\infty} \frac{t}{u^{1+\frac{1}{\alpha}}} \frac{e^{-x^2/4ut^\alpha}}{\sqrt{4\pi u t^\alpha}} \overline{p}_\alpha\left(\frac{t}{u^{1/\alpha}};1\right) du, \tag{48}$$

where again $\overline{p}_\alpha(\cdot;)$ is the stable distribution $S_\alpha(\sigma,1,0)$, with parameters $\sigma = (u\cos\pi\alpha/2)^{1/\alpha}, \beta = 1, \mu = 0$. Formula (48) coincides with the density of $B(T_\alpha)$; indeed, by the change of variable $z = ut^\alpha$, we can rewrite

$$u_\alpha(x,t) = \frac{1}{\alpha}\int_0^{+\infty} \frac{1}{u^{1+\frac{1}{\alpha}}} \frac{e^{-x^2/4ut^\alpha}}{\sqrt{4\pi z}} \overline{p}_\alpha\left(\frac{t}{u^{1/\alpha}};1\right) du\ .$$

If we restrict ourselves to the case $\alpha \in [1/2, 1)$, the density of $\mathcal{T}_\alpha(t), t > 0$ is again proportional to the positive branch of a stable density, as expressed in (37).

On the other hand, for $\alpha \in (0, 1/2]$ and in particular $\alpha = 1/m,\ m \in \mathbb{N}$, it is represented by the law presented in (39) and (40), so that the process governed by equation (45) is, in this case, $B(\prod_{j=1}^{m-1} G_j(t)), t > 0$.

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