Electronic Journal of Probability

Vol. 8 (2003) Paper no. 13, pages 1–20.

Journal URL
http://www.math.washington.edu/~ejpecp/

# Laplace Transforms via Hadamard Factorization

**Fuchang Gao** [1], **Jan Hannig** [2], **Tzong-Yow Lee** [3] and **Fred Torcaso** [4]

**Abstract**. In this paper we consider the Laplace transforms of some random series, in particular, the random series derived as the squared $L_2$ norm of a Gaussian stochastic process. Except for some special cases, closed form expressions for Laplace transforms are, in general, rarely obtained. It is the purpose of this paper to show that for many Gaussian random processes the Laplace transform can be expressed in terms of well understood functions using complex-analytic theorems on infinite products, in particular, the Hadamard Factorization Theorem. Together with some tools from linear differential operators, we show that in many cases the Laplace transforms can be obtained with little work. We demonstrate this on several examples. Of course, once the Laplace transform is known explicitly one can easily calculate the corresponding exact $L_2$ small ball probabilities using Sytaja Tauberian theorem. Some generalizations are mentioned.

*Key words and phrases*: Small ball probability, Laplace Transforms, Hadamard's factorization theorem.

*AMS 2000 subject classifications*: Primary 60G15

Submitted to EJP on February 24, 2003. Final version accepted on August 14 , 2003.

[1]Department of Mathematics, University of Idaho, Moscow, ID 83844-1103, fuchang@uidaho.edu,
Research of F. Gao was partially supported by NSF grant EPS-0132626 and a seed grant from the University of Idaho;

[2]corresponding author, Department of Statistics, Colorado State University, Fort Collins, CO 80523-1877, hannig@stat.colostate.edu;

[3]Department of Mathematics, University of Maryland, College Park, MD 20742, tyl@math.umd.edu;

[4]Mathematical Sciences Department, The Johns Hopkins University, Baltimore, MD 21218, torcaso@mts.jhu.edu;

# 1 Introduction

Consider the random series

$$U = \sum_{n=1}^{\infty} a_n X_n \tag{1}$$

where $\{X_n\}$ is sequence of i.i.d. random variables and $\{a_n\}$ is a sequence of numbers for which $U$ exists. The goal of this paper is to compute a closed form expression for the Laplace transform of the distribution of $U$ and, more importantly, to find conditions on the $X_n$ and $a_n$ that would make this possible.

The main observation is that the Laplace transform $L_U$ of the distribution of $U$ is the infinite product

$$L_U(s) = \prod_{n=1}^{\infty} L_{X_1}(a_n s)$$

where, of course, $L_{X_1}$ represents the Laplace transform of the identical distribution. When the Laplace transform takes the particular form

$$L_U(s) = \left( \prod_{n=1}^{\infty} \left(1 + \alpha a_n^{\beta} s^{\beta}\right) \right)^{c} \tag{2}$$

one can often recognize this infinite product in closed form via Hadamard's Factorization theorem (see Section 2). Here $\alpha, \beta > 0$, $c < 0$ and the sequence $a_n^{\beta}$ is assumed to be a summable sequence of positive real numbers.

This is especially true when $X_n = \xi_n^2$ and $\{\xi_n\}$ is an independent standard Gaussian sequence, implying (2) with $\alpha = 2$, $\beta = 1$, $c = -1/2$, and when the sequence $\{a_n\}$ is nonnegative and nice enough. Exactly how nice is the content of Theorem 2 and Corollary 2 below. In fact, these results are merely consequences of the Hadamard Factorization theorem and certain facts regarding the spectra of linear differential operators.

Our initial interest in finding a closed form expression for the Laplace transform in this situation is that through certain Tauberian theorems one can readily compute quantities of the following type:

$$P\left(U \leq \varepsilon\right)$$

for $\varepsilon$ tending to $0^+$. Indeed, computing the asymptotics of lower tail probabilities is receiving much attention (see the recent survey by Li and Shao [12]). However, a closed form expression for the Laplace transform is not only interesting in its own right, but it also carries all the information about the distribution function not just the behavior of the distribution at particular values. Indeed, in principle one can attempt to recover the distribution function by inversion.

The idea of using the Hadamard Factorization theorem is not a new one, but apparently not well-known. The authors observed the connection with Hadamard's theorem before the fourth author noticed this connection was mentioned by Terence Chan in [2]. Although the connection between the Laplace transform and Hadamard's theorem is transparent, it leads to results that are extremely powerful. In fact, many results regarding the small ball probabilities of Gaussian processes in $L_2$ norm follow easy from the results we present here. Moreover, the technique is fairly general in that it will work not only for $\xi_n^2$ but for $X_n$ having Gamma or Mittag-Leffler type distributions (see comments in Sections 3 and 4 where examples are given).

# 2 Hadamard Factorization

In this section we will present the main idea in the computation of an explicit Laplace transform and the complex-analytic tools that are necessary.

Consider the random series $U$ (see (1)) and suppose that the Laplace transform of its distribution has the form given in (2). The sequence $\{a_n\}$ necessarily tends to zero. Therefore, $z_n = 1/a_n^\beta$ for any $\beta > 0$ tends to $\infty$. Now, if one can find an entire function (that is, analytic in the entire complex plane) whose only zeros are $z_n$, then this entire function can be written as an infinite product with a very special form. The underlying tool is a classical theorem due to Hadamard. In order to state it we need to define the order of an entire function.

Suppose $f(z)$ is entire and set $M(r) = \max|f(z)|$ for $|z| = r$. The order $\lambda \geq 0$ of an entire function $f(z)$ is given by

$$\lambda = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

That is, the order is the smallest number $\lambda$ such that $M(r) \leq e^{r^{\lambda+\varepsilon}}$ for any positive $\varepsilon$ and sufficiently large $r$. For any nonnegative integer $d$ let

$$P_d(z) = z + z^2/2 + \cdots + z^d/d.$$

**Theorem 1 (Hadamard's Factorization Theorem).** *Let $f(z)$ be an entire function and $\{z_k\}$ be its zeros with 0 excluded and all zeros are repeated according to their multiplicity. Suppose the order of $f(z)$ is $\lambda$, then*

$$f(z) = z^m e^{H(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{P_d(z/z_k)},$$

*where the integer $m \geq 0$ is the multiplicity of 0, $d \geq 0$ is an integer such that $d \leq \lambda < d+1$, and $H(z)$ is a polynomial of degree $\leq d$.*

For a proof see any advanced complex analysis textbook (e.g., [10] page 290). The following corollary is immediate:

**Corollary 1.** *Suppose $f$ is an entire function of order $\lambda < 1$, and $f(0) \neq 0$. Then $d = 0$, $H(z)$ is a constant, and*

$$\frac{f(z)}{f(0)} = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right).$$

Notice that $f(z)/f(0)$ is just $f$ standardized to take the value 1 at $z = 0$. If $z_n = 1/a_n^\beta$ then

$$\frac{f(z)}{f(0)} = \prod_{n=1}^{\infty} \left(1 - z a_n^\beta\right). \tag{3}$$

Therefore if the Laplace transform of interest has the form (2) then we can reconcile the infinite product with $f(z)/f(0)$ for some appropriately chosen value of $z$. Let us now demonstrate this on a simple example.

Let $X$ be a Gaussian process on $[0,1]$, and let $\{a_n\}$ be the sequence of eigenvalues, repeated according to their multiplicity, of the covariance operator of $X$. It is easy to show via the Karhunen-Loève expansion that

$$\|X\|_2^2 \equiv \int_0^1 X^2(t)\, dt = \sum_{n=1}^{\infty} a_n \xi_n^2 \tag{4}$$

where $\{\xi_n\}$ is an i.i.d. sequence of standard Gaussian random variables, and the equality is understood in distribution. The Laplace transform then becomes

$$L(s) \equiv E\left(\exp\{-s\|X\|_2^2\}\right) = \left(\prod_{n=1}^{\infty}(1 + 2sa_n)\right)^{-1/2}. \tag{5}$$

If $f$ satisfies the conditions of Corollary 1 with $z_n = 1/a_n$, then by comparing (3) and (5) we have

$$L(s) = \left(\prod_{n=1}^{\infty}(1 + 2sa_n)\right)^{-1/2} = \left(\frac{f(-2s)}{f(0)}\right)^{-1/2}.$$

Thus, at least for squared $L^2$ norms of Gaussian processes, we have the following:

**Corollary 2.** *Let $X$ be a Gaussian process whose covariance operator has nonzero eigenvalues $a_n$, repeated according to their multiplicity. Suppose there is an entire function $f(z)$ of order $\lambda < 1$, such that, $z_n = 1/a_n$, $n \geq 1$, are the only zeros, counting multiplicities, of $f(z)$. Then the Laplace transform of (4) can be expressed as*

$$E\left(\exp\{-s\|X\|_2^2\}\right) = \left(\frac{f(-2s)}{f(0)}\right)^{-1/2}.$$

The above argument hinges on the fact that we can find an entire function $f(z)$ with zeros $z_n = 1/a_n$. We demonstrate in the next section that such functions are usually obtained naturally from the Gaussian process. However, the hard part in applications is to determine that the multiplicities of the eigenvalues match those of the zeros of the entire function. We show that in many cases such a difficulty can be avoided by using results from linear differential operators.

Denote the covariance operator of $X$ by $A$ and recall that $\{a_n\}$ are the eigenvalues of $A$. Thus, they must satisfy

$$Af = af \tag{6}$$

for some eigenfunction $f$. Set $a = 1/\rho$. In many cases the eigenvalue problem (6) is equivalent, by successive differentiation, to an eigenvalue problem of a linear differential operator.

In order to present the idea clearly, instead of seeking the greatest generality, we assume that (6) is equivalent to the following eigenvalue problem:

$$Dy = \rho g(t) y, \tag{7}$$

where the function $g(t)$ is continuous on $[0,1]$ and $g(t) \neq 0$ on $(0,1)$, the linear differential operator $D$ defined on functions $y = y(t)$ for $0 \leq t \leq 1$ is given by

$$Dy = p_0(t)\frac{d^k y}{dt^k} + p_1(t)\frac{d^{k-1}y}{dt^{k-1}} + \cdots + p_{k-1}(t)\frac{dy}{dt} + p_k(t)y \tag{8}$$

and the function $y$ satisfies the boundary conditions

$$U_0(y) = U_1(y) = \cdots = U_{k-1}(y) = 0. \tag{9}$$

Here, we suppose the functions $1/p_0(t), p_1(t), \ldots, p_k(t)$ are all continuous on the interval $[0,1]$, and the boundary conditions are linear and given by:

$$U_j(y) = \sum_{i=0}^{k-1}\left(\alpha_{i,j} y^{(i)}(0) + \beta_{i,j} y^{(i)}(1)\right) \qquad \text{for } j = 0, 1, \ldots, k-1.$$

As we will see through the examples in Section 3, this assumption is reasonably general. Since the covariance operator is self-adjoint, the differential operator $D$ taken with the boundary condition (9) is also self-adjoint.

In what follows we will be interested in computing the eigenvalues of $D$ where $Dy = \rho g(t)y$ and $y$ satisfies (9). Let $y_0 = y_0(t,\rho), y_1 = y_1(t,\rho), \ldots, y_{k-1} = y_{k-1}(t,\rho)$ denote a fundamental system of solutions to the equation $Dy = \rho g(t)y$ which satisfy the initial conditions $y_j^{(i)}(0,\rho) = 0$ (1, respectively) when $j \neq i$ ($j = i$, respectively), $j, i = 0, 1, \ldots, k-1$. For fixed $t$ in the interval $[0,1]$, the functions $y_j$ are entire functions of $\rho$. The boundary value problem (7),(9) has a nontrivial solution if and only if the rank $r$ of the matrix

$$\mathcal{U}(\rho) = \begin{pmatrix} U_0(y_0) & \cdots & U_0(y_{k-1}) \\ \vdots & \ddots & \vdots \\ U_{k-1}(y_0) & \cdots & U_{k-1}(y_{k-1}) \end{pmatrix} \tag{10}$$

is less than $k$. The quantity $f(\rho) = \det \mathcal{U}(\rho)$ is called the characteristic determinant. Notice that the function $f$ is a sum of products of entire functions and therefore is itself entire.

It is clear that if $\rho_0$ is an eigenvalue of the differential operator, then $f(\rho_0) = 0$. The main issue is with the multiplicities. The geometric multiplicity of the eigenvalue $\rho_0$ is $k - \operatorname{rank} \mathcal{U}(\rho_0)$, which is the multiplicity relevant to computation of the Laplace transform. On the other hand, it is well-known that the algebraic multiplicity of the zero of $f(\rho_0)$ is always greater than or equal to the geometric multiplicity (see for example [14] page 15).

In view of self-adjointness, and the fact that our operator $D$ and boundary conditions do not depend on $\rho$, it follows from [14] pages 16-20 that the geometric and algebraic multiplicities agree.

Indeed, fix an eigenvalue $\rho_0$ and a particular eigenfunction $\phi_0$ and define $ly = Dy - \rho_0 y$ as in [14] (page 16 formulas (9), (10) there). Suppose there is a function associated with $\phi_0$. Now since the boundary conditions do not depend on $\rho$, all associated functions have to satisfy the same boundary conditions as $\phi_0$ by (9) of [14]. Let us now consider (10) of [14]. The first line $l(\phi_0) = 0$ is satisfied by definition as the eigenfunction has to satisfy the differential equation. Since $D$ does not depend on $\rho$ the second line simplifies to $l(\phi_1) = g(t)\phi_0$. Finally the self-adjoint property implies

$$0 \neq (g(t)\phi_0, \phi_0) = (l(\phi_1), \phi_0) = (\phi_1, l(\phi_0)) = 0,$$

where ( , ) means inner product. This contradiction allows us to conclude there is no function associated with $\phi_0$. Thus by [14] page 18, statement VI the geometric and algebraic multiplicities agree.

Combining this with Corollary 2 we have proved:

**Theorem 2.** *Let $X$ be a Gaussian process whose covariance operator has nonzero eigenvalues $a_n$, repeated according to their multiplicities. Suppose the covariance operator is equivalent, as discussed above, to a linear differential operator (8) associated with a set of boundary conditions (9), and the characteristic determinant $f$ is of order $\lambda < 1$. Then the Laplace transform of (4) can be expressed as*

$$E\left(\exp\{-s\|X\|_2^2\}\right) = \left(\frac{f(-2s)}{f(0)}\right)^{-1/2}.$$

*Remark* 1. Here we have sacrificed much generality and flexibility, in exchange for simplicity. Remarks will be made at the end of the examples to address this issue: for example, what if it is easier to get a basis of solutions, but not the system of fundamental solutions? And what if $1/p_0(t)$ blows up at the boundary points?

*Remark* 2. The method of converting the integral operator (6) to the differential operator (7) is routine. In a recent preprint of Nazarov and Nikitin [16] they also converted the integral equation to a boundary value problem. Their interest was to use theorems from boundary value problems to estimate the spectra. They achieve very precise asymptotics of the spectra which allow them to prove many results regarding the exact small ball probabilities mentioned in this paper (see Section 4). Our use of the boundary value theory lies in the fact that the characteristic determinant (10) is always an entire function. Thus, the main difference in the approach of this paper with that of [16] is that they are estimating the zeros of the characteristic determinant whereas we do not need to know the zeros explicitly at all.

# 3 Examples

Now, we will present some uses of Theorem 2.

First, we will compute the Laplace transforms for both a stationary Ornstein-Uhlenbeck process and an Ornstein-Uhlenbeck process starting from the origin. We shall see that the computation of the Laplace transform of the squared $L^2$ norm is very simple in light of Corollary 2. This result reaffirms the results of Dankel [4] who already obtained this result using functional integration methods.

Consider the stationary Ornstein-Uhlenbeck process $X(t)$ on the interval $[0, 1]$, that is, the centered Gaussian process determined by the covariance kernel $K(s, t) = e^{-\alpha|t-s|}/(2\alpha)$.

**Theorem 3.** *For the stationary Ornstein-Uhlenbeck process $X$ with parameter $\alpha > 0$,*

$$E\left(\exp\{-\sigma\|X\|_2^2\}\right) = e^{\alpha/2}\left(\frac{\sigma + \alpha^2}{\alpha\sqrt{\alpha^2 + 2\sigma}}\sinh(\sqrt{\alpha^2 + 2\sigma}) + \cosh(\sqrt{\alpha^2 + 2\sigma})\right)^{-1/2}.$$

*Proof.* The covariance operator takes the form $Ay(t) = \int_0^1 K(s,t)y(s)\,ds$. Therefore, the eigenvalue equation is $Ay(t) = ay(t)$. Differentiating this equation twice yields the following boundary value problem:

$$y''(t) - \left(\alpha^2 - \rho\right) y(t) = 0$$

$$y'(1) + \alpha y(1) = 0 \quad \text{and} \quad y'(0) - \alpha y(0) = 0.$$

The fundamental solutions are

$$y_0(t,\rho) = \cosh(t\sqrt{\alpha^2 - \rho}) \quad \text{and} \quad y_1(t,\rho) = \frac{1}{\sqrt{\alpha^2 - \rho}} \sinh(t\sqrt{\alpha^2 - \rho}).$$

The characteristic determinant is

$$f(\rho) = \frac{2\alpha^2 - \rho}{\sqrt{\alpha^2 - \rho}} \sinh(\sqrt{\alpha^2 - \rho}) + 2\alpha \cosh(\sqrt{\alpha^2 - \rho}),$$

which is clearly of order 1/2. Therefore, Theorem 2 implies

$$L(\sigma) = \left(\frac{f(-2\sigma)}{f(0)}\right)^{-1/2} = \sqrt{2\alpha} e^{\alpha/2} \left(\frac{2\sigma + 2\alpha^2}{\sqrt{\alpha^2 + 2\sigma}} \sinh(\sqrt{\alpha^2 + 2\sigma}) + 2\alpha \cosh(\sqrt{\alpha^2 + 2\sigma})\right)^{-1/2}.$$

This finishes the proof. □

*Remark* 3. What if we use some other basis of solutions, rather than the system of fundamental solutions, to plug in the boundary conditions and obtain a determinant $h(\rho)$ instead of the characteristic determinant $f(\rho)$? It is readily checked that

$$h(\rho) = f(\rho)p(\rho),$$

for a certain function $p(\rho)$ which never vanishes. Thus, $h(\rho)$ and $f(\rho)$ share zeros and multiplicities.

Now consider the Ornstein-Uhlenbeck process $X_0(t)$ starting at 0, that is, the centered Gaussian process with the covariance kernel $K(s,t) = (e^{-\alpha|t-s|} - e^{-\alpha(t+s)})/(2\alpha)$.

**Theorem 4.** *For the Ornstein-Uhlenbeck process $X_0$ starting at 0 with parameter $\alpha \in \mathbb{R}$,*

$$E\left(\exp\{-\sigma \|X_0\|_2^2\}\right) = e^{\alpha/2} \left(\alpha\, \frac{\sinh(\sqrt{\alpha^2 + 2\sigma})}{\sqrt{\alpha^2 + 2\sigma}} + \cosh(\sqrt{\alpha^2 + 2\sigma})\right)^{-1/2}.$$

The method of the proof is the same as in the previous case, and the calculation is very similar. We omit the details.

*Remark* 4. Notice that in Theorem 4 all values of $\alpha \in \mathbb{R}$ have probabilistic interpretation. Indeed, the Ornstein-Uhlenbeck process starting at 0 can be defined as a solution to the equation

$$X_0(t) = W(t) - \alpha \int_0^t X_0(s)\, ds$$

where $W(t)$ is a standard Brownian motion. It is well known (see, for example, [17] page 379) that this equation has a strong solution for all values of $\alpha$. When $\alpha = 0$ the solution is just the ordinary Brownian motion, and Theorem 4 reduces to the Laplace transform of the integrated square of Brownian motion as it should. When $\alpha > 0$ the origin "attracts" the Brownian particle with a force proportional to the distance from the origin while $\alpha < 0$ corresponds to the origin repelling the particle in a similar fashion.

*Remark* 5. Another related example is a harmonic oscillator. In [2], Chan considered the harmonic oscillator $Z = (X, V)$ that satisfies the stochastic differential equations

$$dX(t) = V(t)dt, \qquad dV(t) = -X(t)dt + dW(t),$$

where $W(t)$ is a Brownian motion. Through some lengthy calculations, Chan obtained the Laplace transform $L(t) = E(\exp\{-t \int_0^\tau (V_s^2 - X_s^2)ds\})$. We remark that, by using Theorem 2, some calculations in [2] can be simplified. Indeed, following [2], one can find eigenvalues $a_n$ and eigenfunctions $e_n = (e_n^X, e_n^V)$ that can be used in calculating the Laplace transform in the same way as described in the introduction. By setting $u(s) = e_n^X - e_n(\tau)\cos(\tau - s)$, it is readily checked that the defining differential equation (3.17$a$)-(3.17$d$) of [2] has the same dimension of solutions as the following boundary value problem:

$$u'' + u + \rho u = 0, \qquad \sin(\tau)u(0) - \cos(\tau)u'(0) = 0, \qquad u(\tau) = 0.$$

Thus, by applying Theorem 2 we obtain the Laplace transform

$$L(t) = \begin{cases} \left(\cos(\tau)\cosh(\tau\sqrt{2t-1}) + \frac{\sin(\tau)\sinh(\tau\sqrt{2t-1})}{\sqrt{2t-1}}\right)^{-1/2} & \text{for } t \geq \frac{1}{2}, \\ \left(\cos(\tau)\cos(\tau\sqrt{1-2t}) + \frac{\sin(\tau)\sin(\tau\sqrt{1-2t})}{\sqrt{1-2t}}\right)^{-1/2} & \text{for } t < \frac{1}{2}. \end{cases}$$

The main simplification here came from the fact that we did not have to check the multiplicity of the eigenvalues.

One more remark is in order. Depending on the value of $\tau$ it is possible that some or all of the eigenvalues are negative. This is not a big problem because our discussion readily generalizes to this situation. However, one has to be careful since $L(t)$ will exist only for $t$ inside of a certain neighborhood of 0 and not for all $t > 0$ as is usually the case.

As our next example we consider the time-changed Brownian bridge. Let $B(t)$ for $0 \leq t \leq 1$ be a Brownian Bridge. Consider the time-changed process $B_\alpha(t) \equiv B(t^\alpha)$ where $\alpha > 0$. The random variable of interest here is

$$\|B_\alpha\|_2^2 \equiv \int_0^1 B^2(t^\alpha)\, dt = \int_0^1 \frac{1}{\alpha t^{1-1/\alpha}} B^2(t)\, dt$$

which can be seen as a weighted $L^2$-norm of the Brownian Bridge.

**Theorem 5.** *Let $\alpha > 0$. The Laplace transform of the squared $L^2$ norm of $B_\alpha$ is*

$$E\left(\exp\{-t\|B_\alpha\|_2^2\}\right) = \left(\frac{c\sqrt{2t}}{2}\right)^{\nu/2} \left(\Gamma(1+\nu) I_\nu\left(c\sqrt{2t}\right)\right)^{-1/2},$$

*where $I_\nu$ is the modified Bessel function of fractional order*

$$\nu = \frac{\alpha}{\alpha+1} \qquad \text{and} \qquad c = \frac{2\sqrt{\alpha}}{\alpha+1}.$$

*Proof.* We will first follow a calculation in [11]. The covariance kernel of $B_\alpha$ is $K(s,t) = s^\alpha \wedge t^\alpha - s^\alpha t^\alpha$. Differentiating the equation that defines the eigenvalues twice yields the following equivalent boundary value problem:

$$ty''(t) - (\alpha - 1)y'(t) + \alpha\rho t^\alpha y(t) = 0$$
$$y(0) = y(1) = 0.$$

Strictly speaking, Theorem 2 does not apply here since $p_0(t) = t$ and $1/p_0(t)$ blows up at $t = 0$. However, the proof of Theorem 2 can be easily modified. Indeed, notice that $y(t) = Kt^{\alpha/2}(\rho)^{-\nu/2} J_\nu(c[t^{\alpha+1}\rho]^{1/2})$ is the only solution with $y(0) = 0$ where $K$ is a constant, and $J_\nu$ is the Bessel function [1]. Thus $\rho$ is an eigenvalue if $y(1) = 0$, that is,

$$J_\nu(c\rho^{1/2}) = 0.$$

All eigenvalues have multiplicity 1 as the Bessel function $J_\nu$ has only simple positive zeros. (In particular this means that the geometric and algebraic multiplicities agree.) Since $J_\nu(c\rho^{1/2})/\rho^{\nu/2}$ is entire of order $1/2$ (see [1] formulas 9.1.10, 9.1.62 and 9.2.1), Corollary 2 now implies

$$L(t) = \left(\frac{f(-2t)}{f(0)}\right)^{-1/2} = \left(\frac{c\sqrt{2t}}{2}\right)^{\nu/2} \left(\Gamma(1+\nu) I_\nu\left(c\sqrt{2t}\right)\right)^{-1/2},$$

where the last equality follows from the fact $I_\nu(x) = e^{-\frac{1}{2}\nu\pi i} J_\nu(ix)$ for positive real $x$ (see [1] formula 9.6.3).

□

*Remark* 6. The above discussion reveals that Theorem 2 can be generalized to cover differential operators with singular boundary points of certain kinds. Indeed, differential operators with singular boundary points are well studied with many complete results obtained in the literature.

This next example is motivated by the work in [3]. In [3] they tried to extend a stochastic Fubini argument from [9] and [5] to produce a closed form expression for the Laplace transform of the squared $L^2$ norm of $m$-times integrated Brownian motion. The expression they obtained, although not in closed form for $m \geq 2$, does reduce to the result obtained in [9], that is, the case $m = 1$. In what follows we will compute not only the closed form expression in the case of $m$-times integrated Brownian motion for all $m$ but also for the slightly more difficult case of $m$-times integrated Brownian bridge.

Let $W(t)$ be a standard Brownian motion. For integer $m \geq 0$, the $m$-times integrated Brownian motion on $[0,1]$ is the Gaussian process

$$X_m(t) = \int_0^t \int_0^{s_m} \int_0^{s_{m-1}} \cdots \int_0^{s_2} W(s_1)\, ds_1\, ds_2 \cdots ds_m.$$

We will use the following notation in the rest of this section:

$$\omega_j = e^{i\frac{j}{m+1}\pi},\ \upsilon_j = e^{i\frac{2j+1}{2m+2}\pi},\ \text{and}\ \beta_j = (2t)^{1/(2m+2)} i\upsilon_j.$$

**Theorem 6.** *The Laplace transform of the squared $L^2$-norm of $m$-times integrated Brownian motion is*

$$E\left(\exp\{-t\|X_m\|_2^2\}\right) = (2m+2)^{(m+1)/2} \left|\det N_W(t)\right|^{-1/2}, \tag{11}$$

*where*

$$N_W(t) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\ \omega_0^{m+1} e^{\beta_0} & \omega_1^{m+1} e^{\beta_1} & \cdots & \omega_{2m+1}^{m+1} e^{\beta_{2m+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^{2m+1} e^{\beta_0} & \omega_1^{2m+1} e^{\beta_1} & \cdots & \omega_{2m+1}^{2m+1} e^{\beta_{2m+1}} \end{pmatrix}.$$

*Proof.* For convenience, we define two operators, $T_0$ and $T_1$, on $L_2[0,1]$:

$$T_0 f(t) = \int_0^t f(s)\, ds \quad \text{and} \quad T_1 f(t) = \int_t^1 f(s)\, ds.$$

Thus, $X_m(t) = T_0^m W(t)$. By Fubini's theorem, for any $f, g \in L_2[0,1]$, $\langle f, T_0 g\rangle = \langle T_1 f, g\rangle$. So, $T_1 = T_0^*$, that is, $T_1$ is the adjoint operator to $T_0$. It is easy to check that if a Gaussian process $X(t)$ has a covariance operator $A$ the covariance operator of $T_0 X(t)$ is $T_0 A T_1$.

It is well-known the covariance kernel of $W$ is $\min(s,t)$. Thus the covariance operator $A_0$ of $W$ is

$$A_W f(t) = \int_0^t s f(s) ds + t \int_t^1 f(s) ds = T_0 T_1 f(t).$$

From here the covariance operator of the $m$-times integrated Brownian motion $X_m(t)$ is

$$A_m f(t) = (T_0)^{m+1} (T_1)^{m+1} f(t). \tag{12}$$

Consider the eigenvalue problem $af(t) = A_m f(t)$. Upon successive differentiation and using (12), we see that the covariance kernel is associated with the following boundary value problem:

$$y^{(2m+2)}(t) = (-1)^{m+1} \rho y(t) \tag{13}$$

$$y(0) = y'(0) = \cdots = y^{(m)}(0) = y^{(m+1)}(1) = y^{(m+2)}(1) \cdots = y^{(2m+1)}(1) = 0 \tag{14}$$

where $\rho = 1/\alpha$. This is a higher order Sturm-Loiuville problem with separated boundary conditions. In particular, notice that this example satisfies the assumptions of Theorem 2. Recall $\omega_j = e^{i \frac{j}{m+1}\pi}$. For $\rho \neq 0$, denote

$$y_k(t, \rho) = \sum_{j=0}^{2m+1} \frac{\exp(i\omega_j \rho^{1/(2m+2)} t)}{(2m+2)(i\omega_j)^k \rho^{1/(2m+2)}}.$$

Since $\sum_{j=0}^{2m+1} (2m+2)^{-1} (i\omega_j)^k = 0$ for $k = \pm 1, \pm 2, \ldots, \pm(2m+1)$, we see that $y_0, \ldots, y_{2m+1}$ forms a fundamental system of solutions to the equation (13). Denote the diagonal matrix

$$D_W = \text{diagonal}\big(1, i\rho^{1/(2m+2)}, (i\rho^{1/(2m+2)})^2, \ldots, (i\rho^{1/(2m+2)})^{2m+1}\big)$$

and the Vandermonde matrix

$$V = \text{Vandermonde}\left((\omega_0 i \rho^{1/(2m+2)})^{-1}, (\omega_1 i \rho^{1/(2m+2)})^{-1}, \ldots, (\omega_{2m+1} i \rho^{1/(2m+2)})^{-1}\right). \tag{15}$$

A straightforward calculation yields

$$\mathcal{U}(\rho) = D_W \cdot M_W(\rho^{1/(2m+2)}) \cdot V^\top$$

where

$$M_W(z) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\ \omega_0^{m+1} e^{i\omega_0 z} & \omega_1^{m+1} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{m+1} e^{i\omega_{2m+1} z} \\ \omega_0^{m+2} e^{i\omega_0 z} & \omega_1^{m+2} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{m+2} e^{i\omega_{2m+1} z} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^{2m+1} e^{i\omega_0 z} & \omega_1^{2m+1} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{2m+1} e^{i\omega_{2m+1} z} \end{pmatrix}.$$

Therefore, the characteristic determinant is

$$f(\rho) \equiv \det \mathcal{U}(\rho) = \frac{\det M_W(\rho^{1/(2m+2)})}{(-i)^m (2m+2)^{m+1}}.$$

The function $f(\rho)$ is of order $1/(2m+2)$ and $f(0) = 1$. Thus, Theorem 2 implies

$$L(t) = (f(-2t)/f(0))^{-1/2}.$$

To finish the proof notice that the calculation of $f(-2t)$ involves $(-2t)^{1/(2m+2)}$, where $t \in \mathbb{R}$. There are $2m+2$ different choices of $(-1)^{1/(2m+2)}$. For $t > 0$, choose $(-2t)^{1/(2m+2)} = \upsilon_0 (2t)^{1/(2m+2)}$. Then $(-2t)^{1/(2m+2)} i\omega_j = (2t)^{1/(2m+2)} i\upsilon_j = \beta_j$ and the equation (11) follows from the fact that the Laplace transform is always a non-negative real function. $\square$

Let us now consider the $m$-times integrated Brownian bridge. Let $B(t)$ be a standard Brownian bridge. For integer $m \geq 0$, the $m$-times integrated Brownian bridge on $[0,1]$ is the Gaussian process

$$Y_m(t) = \int_0^t \int_0^{s_m} \int_0^{s_{m-1}} \cdots \int_0^{s_2} B(s_1)\, ds_1\, ds_2 \cdots ds_m.$$

**Theorem 7.** *The Laplace transform of the squared $L^2$-norm of $m$-times integrated Brownian bridge is*

$$E\left(\exp\{-t\|Y_m\|_2^2\}\right) = (2t)^{1/(4m+4)} (2m+2)^{(m+1)/2} \left|\det N_B(t)\right|^{-1/2}, \tag{16}$$

*where*

$$N_B(t) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\ \omega_0^m e^{\beta_0} & \omega_1^m e^{\beta_1} & \cdots & \omega_{2m+1}^m e^{\beta_{2m+1}} \\ \omega_0^{m+2} e^{\beta_0} & \omega_1^{m+2} e^{\beta_1} & \cdots & \omega_{2m+1}^{m+2} e^{\beta_{2m+1}} \\ \omega_0^{m+3} e^{\beta_0} & \omega_1^{m+3} e^{\beta_1} & \cdots & \omega_{2m+1}^{m+3} e^{\beta_{2m+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^{2m+1} e^{\beta_0} & \omega_1^{2m+1} e^{\beta_1} & \cdots & \omega_{2m+1}^{2m+1} e^{\beta_{2m+1}} \end{pmatrix}.$$

The method of the proof is the same as in the previous case. The only difference is that the boundary condition (14) is replaced by

$$\begin{aligned} y(0) = y'(0) = \cdots = y^{(m-1)}(0) = y^{(m)}(0) = y^{(m)}(1) \\ = y^{(m+2)}(1) = y^{(m+3)}(1) \cdots = y^{(2m+1)}(1) = 0. \end{aligned}$$

This leads to the characteristic determinant

$$\det \mathcal{U}(\rho) = \det \left( D_B \cdot M_B(\rho^{1/(2m+2)}) \cdot V^\top \right),$$

where $V$ was defined in (15),

$$\begin{aligned} D_B = \operatorname{diagonal}\big(1, i\rho^{1/(2m+2)}, (i\rho^{1/(2m+2)})^2, \ldots, (i\rho^{1/(2m+2)})^m, \\ (i\rho^{1/(2m+2)})^m, (i\rho^{1/(2m+2)})^{m+2}, (i\rho^{1/(2m+2)})^{m+3}, \ldots, (i\rho^{1/(2m+2)})^{2m+1}\big), \end{aligned}$$

and

$$M_B(z) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^{m-1} & \omega_1^{m-1} & \cdots & \omega_{2m+1}^{m-1} \\ \omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\ \omega_0^m e^{i\omega_0 z} & \omega_1^m e^{i\omega_1 z} & \cdots & \omega_{2m+1}^m e^{i\omega_{2m+1} z} \\ \omega_0^{m+2} e^{i\omega_0 z} & \omega_1^{m+2} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{m+2} e^{i\omega_{2m+1} z} \\ \omega_0^{m+3} e^{i\omega_0 z} & \omega_1^{m+3} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{m+3} e^{i\omega_{2m+1} z} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^{2m+1} e^{i\omega_0 z} & \omega_1^{2m+1} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{2m+1} e^{i\omega_{2m+1} z} \end{pmatrix}.$$

The rest of the calculation is very similar and we omit the details.

*Remark* 7. In the case of $m = 1, 2$ the formulas (11) and (16) simplify to:

$$
\begin{aligned}
E\left(\exp\{-t\|X_1\|_2^2\}\right) &= \left(\frac{4}{2+\cos(2^{3/4}t^{1/4})+\cosh(2^{3/4}t^{1/4})}\right)^{1/2}\\
E\left(\exp\{-t\|Y_1\|_2^2\}\right) &= \left(\frac{2^{7/4}t^{1/4}}{\sin(2^{3/4}t^{1/4})+\sinh(2^{3/4}t^{1/4})}\right)^{1/2}\\
E\left(\exp\{-t\|X_2\|_2^2\}\right) =& 6\Big(9+16\cos(3^{1/2}\,2^{-5/6}\,t^{1/6})\,\cosh(2^{-5/6}\,t^{1/6})+8\cosh(2^{1/6}\,t^{1/6})\\
&\qquad +2\cos(2^{1/6}\,3^{1/2}\,t^{1/6})\,\cosh(2^{1/6}\,t^{1/6})+\cosh(2^{7/6}\,t^{1/6})\Big)^{-1/2}\\
E\left(\exp\{-t\|Y_2\|_2^2\}\right) =& 3\sqrt{2}(2t)^{1/12}\Big(4\sqrt{3}\,\cosh(2^{-5/6}\,t^{1/6})\,\sin(3^{1/2}\,2^{-5/6}\,t^{1/6})\\
&\qquad +\sqrt{3}\,\cosh(2^{1/6}\,t^{1/6})\,\sin(2^{1/6}\,3^{1/2}\,t^{1/6})\\
&\qquad +4\,\cos(3^{1/2}\,2^{-5/6}\,t^{1/6})\,\sinh(2^{-5/6}\,t^{1/6})+4\sinh(2^{1/6}\,t^{1/6})\\
&\qquad +\cos(2^{1/6}\,3^{1/2}\,t^{1/6})\,\sinh(2^{1/6}\,t^{1/6})\\
&\qquad +2\cosh(2^{1/6}\,t^{1/6})\,\sinh(2^{1/6}\,t^{1/6})\Big)^{-1/2}
\end{aligned}
$$

In particular, we see that in the case of once integrated Brownian motion our formula recovers the result of [3] and [9] as it should.

*Remark* 8. The method presented here also works for many other processes, e.g., any generalized integrated Brownian motion (see [8] for the definition of generalized integrated Brownian motions and some results), the so-called bridged integrated Brownian motion and integrated centered Brownian motion of [16]. Since the calculations are very similar to the calculations presented here we will not include the details.

# 4 Small Ball Rates

The revelation of this paper came about when the authors were attempting to obtain an exact form for the small deviation probability of the random variable (4). As a consequence of Hadamard's factorization, if one is dealing with the $L_2$ norm of Gaussian processes then this exact asymptotic follows from the Sytaja Tauberian theorem [18]:

$$
P\left(\sum_{n=1}^{\infty} a_n\xi_n^2 \le \varepsilon^2\right) \sim (-2\pi t^2 h''(t))^{-1/2}\exp\{t\varepsilon^2 - h(t)\} \tag{17}
$$

where $h(t) = -\log L(t)$ where $L(t)$ is the Laplace transform and $t = t(\epsilon)$ satisfies

$$\frac{t\varepsilon^2 - th'(t)}{\sqrt{-t^2 h''(t)}} \to 0. \tag{18}$$

It is important to point out that the Sytaja's result (17) has been extended by Lifshits [13] (see also Dunker, Lifshits and Linde [6]) to more general sums $\sum_{n=1}^{\infty} a_n Z_n$, where $\{Z_n\}$ is a sequence of i. i. d. non-negative random variables with finite second moment and whose distribution satisfies some weak regularity conditions, and $\{a_n\}$ is a summable sequence of positive constants. Thus, the method of this paper in principle generalizes to other cases mentioned in the introduction. However, we wish to emphasize that the exact asymptotics of the small ball probability including *all* constants are completely determined by the approach we propose.

After we submitted this paper, we were made aware of a preprint by Nazarov [15] where he computes the exact small ball probabilities for many of the processes we include in this section. Nazarov entends the work of [16] to compute exact constants for many of the processes considered there using slightly different methods. In fact, some exact constants for the integrated Brownian motions have already appeared in [7]. Nevertheless, although our work was done independently, in order to avoid overlaps with [15] we omit most of the proofs of the following corollaries which are simple consequences of the theorems proved in section 3 and the Sytaja Tauberian theorem.

**Corollary 3.** *Let $X$ be the stationary Ornstein-Uhlenbeck process with parameter $\alpha > 0$, Then*

$$P\left(\|X\|_2 \leq \varepsilon\right) \sim 8\sqrt{\frac{\alpha}{\pi}}\, e^{\alpha/2} \varepsilon^2 \exp\{-\frac{1}{8}\varepsilon^{-2}\} \quad as \quad \varepsilon \to 0.$$

*Let $X_0$ be the Ornstein-Uhlenbeck process starting at 0 with parameter $\alpha \in \mathbb{R}$, then*

$$P\left(\|X_0\|_2 \leq \varepsilon\right) \sim \frac{4}{\sqrt{\pi}} e^{\alpha/2} \varepsilon \exp\{-\frac{1}{8}\varepsilon^{-2}\} \quad as \quad \varepsilon \to 0.$$

The constant $\sqrt{\alpha}$ reflects the degeneracy of the measure when $\alpha$ tends to 0. The reader will also notice the small ball probability for $X_0$ reduces to that of Brownian motion when $\alpha = 0$ as it should. Finally, the apparent discrepancy between Corollary 3 and the results in [16] and [15] comes from the fact that we use a different covariance kernel to define the Ornstein-Uhlenbeck process (see the sentence preceding Theorem 3). This result also differs from [15] in that it shows the explicit dependence on the parameter $\alpha$.

To show how simply the calculations follow from the Laplace transform we include a proof for the stationary Ornstein-Uhlenback process. The proofs of all other corollaries below are similar and we leave the details to the reader.

*Proof.* Let $h(t) = -\log L(t)$ as given in (17). We need to compute the asymptotic behavior of $h(t)$, $th'(t)$ and $t^2h''(t)$ as $t \to \infty$. Using Theorem 3 we get:

$$\begin{aligned} h(t) &= -\frac{\alpha}{2} + \frac{1}{2}\log\frac{\sqrt{t}}{\sqrt{2}\alpha} + \frac{\sqrt{t}}{\sqrt{2}} - \frac{1}{2}\log 2 + o(1), \\ th'(t) &= \frac{\sqrt{t}}{2\sqrt{2}} + \frac{1}{4} + o(\sqrt{1}), \\ -2\pi t^2 h''(t) &= \frac{\pi\sqrt{t}}{2\sqrt{2}} + o(\sqrt{t}), \end{aligned}$$

and therefore (18) allows us to choose

$$\sqrt{t} = \frac{\varepsilon^{-2}}{2\sqrt{2}}\,.$$

Substituting these into (17) yields the result. $\square$

**Corollary 4.** *Let $B_\alpha(t) = B(t^\alpha)$ be the time changed Brownian bridge. Then*

$$P\left(\|B_\alpha\|_2 \le \varepsilon\right) \sim c_\alpha \varepsilon^{1/2-\nu} \exp\left\{-\frac{\nu}{2(\alpha+1)}\varepsilon^{-2}\right\}, \quad as \quad \varepsilon \to 0_+$$

*where*

$$c_\alpha = 2(\pi)^{-1/4}\left(\frac{\nu}{\alpha+1}\right)^{\frac{\nu}{2}-\frac{1}{4}}\left(\Gamma\left(\nu+1\right)\right)^{-1/2} \quad and \quad \nu = \frac{\alpha}{\alpha+1}$$

This example was considered in [11]. The reader will notice when $\alpha = 1$ the constant in front becomes $\sqrt{8}/\sqrt{\pi}$ as is well-known for the Brownian bridge.

**Corollary 5.** *Let $X_m(t)$ be the $m$-times integrated Brownian motion and $Y_m(t)$ be the $m$-times integrated Brownian bridge. Then*

$$P\left(\|X_m\|_2 \le \varepsilon\right) \sim C_m^W \varepsilon^{\frac{1}{2m+2}} \exp\{-D_m \varepsilon^{-\frac{2}{2m+1}}\} \quad as \quad \varepsilon \to 0_+$$

*and*

$$P\left(\|Y_m\|_2 \le \varepsilon\right) \sim C_m^B \exp\{-D_m \varepsilon^{-\frac{2}{2m+1}}\} \quad as \quad \varepsilon \to 0_+,$$

*where*

$$D_m = \frac{2m+1}{2}\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-\frac{2m+2}{2m+1}},$$

$$C_m^W = \frac{(2m+2)^{(m+1)/2}}{|\det U|}\left(\frac{2m+2}{(2m+1)\pi}\right)^{1/2}\left[(2m+2)\sin\frac{\pi}{2m+2}\right]^{(m+1)/(2m+1)},$$

$$C_m^B = \left(\frac{(2m+2)^{m+3}\sin\frac{\pi}{2m+2}}{(2m+1)\pi|\det U\det V|}\right)^{1/2},$$

*and*

$$U = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_m \\ \omega_0^2 & \omega_1^2 & \cdots & \omega_m^2 \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_m^m \end{pmatrix}, \qquad V = \begin{pmatrix} \omega_{m+1}^m & \omega_{m+2}^m & \cdots & \omega_{2m+1}^m \\ \omega_{m+1}^{m+2} & \omega_{m+2}^{m+2} & \cdots & \omega_{2m+1}^{m+2} \\ \omega_{m+1}^{m+3} & \omega_{m+2}^{m+3} & \cdots & \omega_{2m+1}^{m+3} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{m+1}^{2m+1} & \omega_{m+2}^{2m+1} & \cdots & \omega_{2m+1}^{2m+1} \end{pmatrix}. \tag{19}$$

We would like to remark when $m = 1$ we obtain the value $C_1^W = 8\sqrt{2}/\sqrt{3\pi}$, which is the value quoted in [16] page 4, i.e., $C_{\text{dist}} = \sqrt{2}$. In fact, for generalized integrated Brownian motions, the exact small ball asymptotics were obtained in [7] by different methods.

# 5 Acknowledgement

The authors thank Leon Greenberg, David Hamilton, Wenbo Li for useful conversations. The third named author is grateful for the warm hospitality of the National Center for Theoretical Sciences in Taiwan, R.O.C., where part of this work was done. We also wish to thank Professor Yakov Nikitin who pointed out some typos and important references, Professor Alexander Nazarov who sent us his wonderful preprint, and to the referee who offered valuable comments on the first version of this work.

# References

[1] M. Abramowitz and Stegun I. A. *Hanbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover, 1970.

[2] Terence Chan. Indefinite quadratic functionals of Gaussian processes and least-action paths. *Ann. Inst. H. Poincaré Probab. Statist.*, 27(2):239–271, 1991.

[3] X. Chen and W. Li. Quadradic Functionals and Small Ball Probabilities for the $m$-fold Integrated Brownian Motion. *Ann. Probab. (to appear).*

[4] Thad Dankel, Jr. On the distribution of the integrated square of the Ornstein-Uhlenbeck process. *SIAM J. Appl. Math.*, 51(2):568–574, 1991.

[5] C. Donati-Martin and M. Yor. Fubini's theorem for double Wiener integrals and the variance of the Brownian path. *Ann. Inst. H. Poincaré Probab. Statist.*, 27(2):181–200, 1991.

[6] T. Dunker, M. A. Lifshits, and W. Linde. Small deviation probabilities of sums of independent random variables. In *High dimensional probability (Oberwolfach, 1996)*, volume 43 of *Progr. Probab.*, pages 59–74. Birkhäuser, Basel, 1998.

[7] F. Gao, J. Hannig, T.-Y. Lee, and F. Torcaso. Exact $L^2$ small balls of Gaussian processes. *Journal of Theoretical Probability (to appear)*, 2003.

[8] F. Gao, J. Hannig, and F. Torcaso. Integrated Brownian motions and Exact $L_2$-small balls. *Ann. Probab. (to appear).*

[9] Davar Khoshnevisan and Zhan Shi. Chung's law for integrated Brownian motion. *Trans. Amer. Math. Soc.*, 350(10):4253–4264, 1998.

[10] Serge Lang. *Complex analysis.* Springer-Verlag, New York, second edition, 1985.

[11] Wenbo V. Li. Comparison results for the lower tail of Gaussian seminorms. *J. Theoret. Probab.*, 5(1):1–31, 1992.

[12] Wenbo V. Li and Qi-Man Shao. A note on the Gaussian correlation conjecture. In *High dimensional probability, II (Seattle, WA, 1999)*, pages 163–171. Birkhäuser Boston, Boston, MA, 2000.

[13] M. A. Lifshits. On the lower tail probabilities of some random series. *Ann. Probab.*, 25(1):424–442, 1997.

[14] M. A. Naimark. *Linear differential operators. Part I: Elementary theory of linear differential operators.* Frederick Ungar Publishing Co., New York, 1967.

[15] A. I. Nazarov. On the Sharp Constant in the Small Ball Asymptotics of some Gaussian processes under $L_2$-norm. preprint, 2003.

[16] A. I. Nazarov and Ya. Yu. Nikitin. Exact small ball behavior of integrated Gaussian processes under $L_2$-norm and spectral asymptotics of boundary value problems. tech report, Università Bocconi, Studi Statistici, No. 70., 2003.

[17] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion.* Springer-Verlag, Berlin, third edition, 1999.

[18] G. N. Sytaja. On some representations of the Gaussian measure in Hilbert space (in Russian). In *Theory of Stochastic Processes, Publication 2 (Ukranian Academy of Science)*, pages 93–104. 1974.