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Random walks and exclusion processes among random conductances on random infinite clusters: homogenization and hydrodynamic limit

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Abstract

We consider a stationary and ergodic random field $\{\omega(b) : b \in \mathbb{E}_d\}$ parameterized by the family of bonds in $\mathbb{Z}^d$, $d \geqslant 2$. The random variable $\omega(b)$ is thought of as the conductance of bond b and it ranges in a finite interval $[0, c_0]$. Assuming that the set of bonds with positive conductance has a unique infinite cluster $\mathscr{C}(\omega)$, we prove homogenization results for the random walk among random conductances on $\mathscr{C}(\omega)$. As a byproduct, applying the general criterion of [F] leading to the hydrodynamic limit of exclusion processes with bond–dependent transition rates, for almost all realizations of the environment we prove the hydrodynamic limit of simple exclusion processes among random conductances on $\mathscr{C}(\omega)$. The hydrodynamic equation is given by a heat equation whose diffusion matrix does not depend on the environment. We do not require any ellipticity condition. As special case, $\mathscr{C}(\omega)$ can be the infinite cluster of supercritical Bernoulli bond percolation.

Key words: disordered system, bond percolation, random walk in random environment, exclusion process, homogenization.

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1 Introduction

We consider a stationary and ergodic random field $\omega = \big(\omega(b) : b \in \mathbb{E}_d\big)$, parameterized by the set $\mathbb{E}_d$ of non–oriented bonds in $\mathbb{Z}^d$, $d \geqslant 2$, such that $\omega(b) \in [0, c_0]$ for some fixed positive constant c_0. We call ω the conductance field and we interpret $\omega(b)$ as the conductance at bond b. We assume that the network of bonds b with positive conductance has a.s. a unique infinite cluster $\mathscr{C}$, and call $\mathscr{E}$ the associated bonds. Finally, we consider the exclusion process on the graph $(\mathscr{C}, \mathscr{E})$ with generator $\mathbb{L}$ defined on local functions f as

$$\mathbb{L}f(\eta) = \sum_{b \in \mathscr{E}} \omega(b)\big(f(\eta^b) - f(\eta)\big)\,, \qquad \eta \in \{0,1\}^{\mathscr{C}}\,,$$

where the configuration η^b is obtained from η by exchanging the values of η_x and η_y, if $b = \{x, y\}$. If $\big(\omega(b) : b \in \mathscr{E}\big)$ is a family of i.i.d. random variables such that $\omega(b)$ is positive with probability p larger than the critical threshold p_c for Bernoulli bond percolation, then the above process is an exclusion process among positive random conductances on the supercritical percolation cluster. If $p = 1$, then the model reduces to the exclusion process among positive random conductances on $\mathbb{Z}^d$.

Due to the disorder, the above model is an example of non–gradient exclusion process, in the sense that the transition rates cannot be written as gradient of some local function on $\{0,1\}^{\mathscr{C}}$ [KL] (with exception of the case of constant conductances). Despite this fact, the hydrodynamic limit of the exclusion process can be proven without using the very sophisticated techniques developed for non–gradient systems (cf. [KL] and references therein), which in addition would require non trivial spectral gap estimates that fail in the case of conductances non bounded from below by a positive constant (cf. Section 1.5 in [M]). The strong simplification comes from the fact that, since the transition rates depend only on the bonds but not on the particle configuration, the function $\mathbb{L}\eta_x$, where η_x is the occupancy number at site $x \in \mathscr{C}$, is a linear combinations of occupancy numbers. Due to this degree conservation the analysis of the limiting behavior of the random empirical measure $\pi(\eta) = \sum_{x \in \mathscr{C}} \eta_x \delta_x$ is strongly simplified w.r.t. disordered models with transition rates depending both on the disorder and on the particle configuration [Q1], [FM], [Q2]. Moreover, the function $\mathbb{L}\eta_x$ can be written as $(\mathscr{L}\eta)_x$, where $\mathscr{L}$ is the generator of the random walk on $\mathscr{C}$ of a single particle among the random conductances $\omega(b)$ and $\eta \in \{0,1\}^{\mathscr{C}}$ is thought of as an observable on the state space $\mathscr{C}$ of the random walk. This observation allows to derive the hydrodynamic limit of the exclusion process on $\mathscr{C}$ from homogenization results for the random walk on $\mathscr{C}$. This reduction has been performed in [N] for the exclusion process on $\mathbb{Z}$ with bond–dependent conductances, the method has been improved and extended to the d–dimensional case in [F]. The arguments followed in [F] are very general and can be applied also to exclusion processes with bond–dependent transition rates on *general (non–oriented) graphs*, even with *non diffusive behavior* (see [FJL] for an example of application). The reduction to a homogenization problem can be performed also by means of the method of corrected empirical measure, developed in [JL] and [J]. In [JL] the authors consider exclusion processes on $\mathbb{Z}$ with bond–dependent rates, while in [J] the author proves the hydrodynamic limit for exclusion processes with bond–dependent rates on triangulated domains and on the Sierpinski gasket. Moreover, in [J] the author reobtains the hydrodynamic limit of the exclusion process on $\mathbb{Z}^d$ among conductances bounded from above and from below by positive constants. Note that this last result follows at once by applying the standard non–gradient methods (in this case, their application becomes trivial) or the discussion given in [F][Section 4]. Moreover, it is reobtained in the present paper by taking $(\omega_b : b \in \mathbb{E}_d)$ as independent strictly positive random variables.

By the above methods [N], [F], [JL], [J], the proof of the hydrodynamic limit of exclusion processes on graphs with bond–dependent rates reduces to a homogenization problem. In our contest, we solve this problem by means of the notion of *two–scale convergence*, which is particularly fruitful when dealing with homogenization problems on singular structures. The notion of two–scale convergence was introduced by G. Nguetseng [Nu] and developed by G. Allaire [A]. In particular, our proof is inspired by the method developed in [ZP] for differential operators on singular structures. Due to the ergodicity and the $\mathbb{Z}^d$–translation invariance of the conductance field, the arguments of [ZP] can be simplified: for example, as already noted in [MP], one can avoid the introduction of the Palm distribution. Moreover, despite [ZP] and previous results of homogenization of random walks in random environment (see [Ko], [Ku], [PR] for example), in the present setting we are able to avoid ellipticity assumptions.

We point out that recently the quenched central limit theorem for the random walk among constant conductances on the supercritical percolation cluster has been proven in [BB] and [MP], and previously for dimension $d \geqslant 4$ in [SS]. Afterwards, this result has been extended to the case of i.i.d. positive bounded conductances on the supercritical percolation cluster in [BP] and [M]. Their proofs are very robust and use sophisticated techniques and estimates (as heat kernel estimates, isoperimetric estimates, non trivial percolation results, ...), previously obtained in other papers. In the case of i.i.d. positive bounded conductances on the supercritical percolation cluster, we did not try to derive our homogenization results from the above quenched CLTs (this route would anyway require some technical work). Usually, the proof of the quenched CLT is based on homogenization ideas and not viceversa. And in fact, our proof of homogenization is much simpler than the above proofs of the quenched CLT. Hence, the strategy we have followed has the advantage to be self–contained, simple and to give a genuine homogenization result without ellipticity assumptions. This would have not been achieved choosing the above alternative route. In addition, our results cover more general random conductance fields, possibly with correlations, for which a proof of the quenched CLT for the associated random walk is still lacking.

The paper is organized as follows: In Section 2 we give a more detailed description of the exclusion process and the random walk among random conductances on the infinite cluster $\mathscr{C}$. In addition, we state our main results concerning the hydrodynamic behavior of the exclusion process (Theorem 2.2) and the homogenization of the random walk (Theorem 2.4 and Corollary 2.5). In Section 3 we show how to apply the results of [F] in order to derive Theorem 2.2 from Corollary 2.5, while the remaining sections are focused on the homogenization problem. In particular, the proof of Theorem 2.4 is given in Section 6, while the proof of Corollary 2.5 is given in Section 7. Finally, in the Appendix we prove Lemma 2.1, assuring that the class of random conductance fields satisfying our technical assumptions is large.

2 Models and results

2.1 The environment

The environment modeling the disordered medium is given by a stationary and ergodic random field $\omega = \big(\omega(b) \,:\, b \in \mathbb{E}_d\big)$, parameterized by the set $\mathbb{E}_d$ of non–oriented bonds in $\mathbb{Z}^d$, $d \geqslant 2$. Stationarity and ergodicity refer to the natural action of the group of $\mathbb{Z}^d$–translations. ω and $\omega(b)$ are thought of as the conductance field and the conductance at bond b, respectively. We call $\mathbb{Q}$ the law of the field ω and we assume that

$$\text{(H1)} \qquad \omega(b) \in [0, c_0], \qquad \mathbb{Q}\text{–a.s.}$$

for some fixed positive constant c_0. Hence, without loss of generality, we can suppose that $\mathbb{Q}$ is a probability measure on the product space $\Omega := [0, c_0]^{\mathbb{E}_d}$. Moreover, in order to simplify the notation, we write $\omega(x, y)$ for the conductance $\omega(b)$ if $b = \{x, y\}$. Note that $\omega(x, y) = \omega(y, x)$.

Consider the random graph $G(\omega) = \big(V(\omega), E(\omega)\big)$ with vertex set $V(\omega)$ and bond set $E(\omega)$ defined as

$$\begin{aligned} E(\omega) &:= \big\{ b \in \mathbb{E}_d \, : \, \omega(b) > 0 \big\}, \\ V(\omega) &:= \big\{ x \in \mathbb{Z}^d \, : \, x \in b \text{ for some } b \in E(\omega) \big\}. \end{aligned}$$

Due to ergodicity, the translation invariant Borel subset $\Omega_0 \subset \Omega$ given by the configurations ω for which the graph $G(\omega)$ has a unique infinite connected component (cluster) $\mathscr{C}(\omega) \subset V(\omega)$ has $\mathbb{Q}$–probability 0 or 1. We assume that

$$\text{(H2)} \qquad \mathbb{Q}(\Omega_0) = 1.$$

Below, we denote by $\mathscr{E}(\omega)$ the bonds in $E(\omega)$ connecting points of $\mathscr{C}(\omega)$ and we will often understand the fact that $\omega \in \Omega_0$.

Define $B(\Omega)$ as the family of bounded Borel functions on Ω and let $\mathscr{D}$ be the $d \times d$ symmetric matrix characterized by the variational formula

$$(a, \mathscr{D}a) = \frac{1}{m} \inf_{\psi \in B(\Omega)} \left\{ \sum_{e \in \mathscr{B}_*} \int_\Omega \omega(0, e)(a_e + \psi(\tau_e \omega) - \psi(\omega))^2 \mathbb{I}_{0, e \in \mathscr{C}(\omega)} \mathbb{Q}(d\omega) \right\}, \quad \forall a \in \mathbb{R}^d, \tag{2.1}$$

where $\mathscr{B}_*$ denotes the canonical basis of $\mathbb{Z}^d$,

$$m := \mathbb{Q}\big(0 \in \mathscr{C}(\omega)\big) \tag{2.2}$$

and the translated environment $\tau_e \omega$ is defined as $\tau_e \omega(x, y) = \omega(x + e, y + e)$ for all bonds $\{x, y\}$ in $\mathbb{E}_d$. In general, $\mathbb{I}_A$ denotes the characteristic function of A. Our last assumption on $\mathbb{Q}$ is that the matrix $\mathscr{D}$ is strictly positive:

$$\text{(H3)} \qquad (a, \mathscr{D}a) > 0, \qquad \forall a \in \mathbb{R}^d \, : \, a \neq 0.$$

In conclusion, our hypotheses on the random field ω are given by stationarity, ergodicity, (H1),(H2) and (H3). The lemma below shows that they are fulfilled by a large class of random fields. In order to state it, given $c > 0$ we define the random field $\hat{\omega}_c = \big(\hat{\omega}_c(b) \, : \, b \in \mathbb{E}_d\big)$ as

$$\hat{\omega}_c(b) = \begin{cases} 1 & \text{if } \omega(b) > c, \\ 0 & \text{otherwise}. \end{cases} \tag{2.3}$$

For $c = 0$ we simply set $\hat{\omega} := \hat{\omega}_0$.

Lemma 2.1. *Hypotheses (H2) and (H3) are satisfied if there exists a positive constant c such that the random field $\hat{\omega}_c$ stochastically dominates a supercritical Bernoulli bond percolation. In particular, if $\hat{\omega}$ itself is a supercritical Bernoulli bond percolation, then (H2) and (H3) are verified.*

If the field ω is reflection invariant or isotropic (invariant w.r.t. $\mathbb{Z}^d$ rotations by $\pi/2$), then $\mathscr{D}$ is a diagonal matrix. In the isotropic case $\mathscr{D}$ is a multiple of the identity. In particular, if ω is given by i.i.d. random conductances $\omega(b)$, then $\mathscr{D}$ is a multiple of the identity.

We postpone the proof of the above Lemma to the Appendix.

2.2 The exclusion process on the infinite cluster $\mathscr{C}(\omega)$

Given a realization ω of the environment, we consider the exclusion process $\eta(t)$ on the graph $\mathscr{G}(\omega) = \big(\mathscr{C}(\omega), \mathscr{E}(\omega)\big)$ with exchange rate $\omega(b)$ at bond b. This is the Markov process with paths $\eta(t)$ in the Skohorod space $D\big([0,\infty), \{0,1\}^{\mathscr{C}(\omega)}\big)$ (cf. [B]) whose Markov generator $\mathbb{L}_\omega$ acts on local functions as

$$\mathbb{L}_\omega f(\eta) = \sum_{e\in\mathscr{B}_*} \sum_{\substack{x\in\mathscr{C}(\omega):\\ x+e\in\mathscr{C}(\omega)}} \omega(x,x+e)\left(f(\eta^{x,x+e}) - f(\eta)\right), \tag{2.4}$$

where in general

$$\eta^{x,y}_z = \begin{cases} \eta_y, & \text{if } z = x\,,\\ \eta_x, & \text{if } z = y\,,\\ \eta_z, & \text{if } z \neq x, y\,. \end{cases}$$

We recall that a function f is called local if $f(\eta)$ depends only on η_x for a finite number of sites x. By standard methods [L] one can prove that the above exclusion process $\eta(t)$ is well defined.

Every configuration η in the state space $\{0,1\}^{\mathscr{C}(\omega)}$ corresponds to a system of particles on $\mathscr{C}(\omega)$ if one considers a site x occupied by a particle if $\eta_x = 1$ and vacant if $\eta_x = 0$. Then the exclusion process is given by a stochastic dynamics where particles can lie only on sites $x \in \mathscr{C}(\omega)$ and can jump from the original site x to the vacant site $y \in \mathscr{C}(\omega)$ only if the bond $\{x,y\}$ has positive conductance, i.e. x and y are connected by a bond in $\mathscr{G}(\omega)$. Roughly speaking, the dynamics can be described as follows: To each bond $b = \{x,y\} \in \mathscr{E}(\omega)$ associate an exponential alarm clock with mean waiting time $1/\omega(b)$. When the clock rings, the particle configurations at sites x and y are exchanged and the alarm clock restarts afresh. By Harris' percolation argument [D], this construction can be suitably formalized. Finally, we point out that the only interaction between particles is given by site exclusion.

We can finally describe the hydrodynamic limit of the above exclusion process among random conductances $\omega(b)$ on the infinite cluster $\mathscr{C}(\omega)$. If the initial distribution is given by the probability measure μ on $\{0,1\}^{\mathscr{C}(\omega)}$, we denote by $\mathbb{P}_{\omega,\mu}$ the law of the resulting exclusion process.

Theorem 2.2. *For $\mathbb{Q}$ almost all environments ω the following holds. Let $\rho_0 : \mathbb{R}^d \to [0,1]$ be a Borel function and let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family of probability measures on $\{0,1\}^{\mathscr{C}(\omega)}$ such that, for all $\delta > 0$ and all real functions φ on $\mathbb{R}^d$ with compact support (shortly $\varphi \in C_c(\mathbb{R}^d)$), it holds*

$$\lim_{\varepsilon\downarrow 0} \mu_\varepsilon\Big(\Big|\varepsilon^d \sum_{x\in\mathscr{C}(\omega)} \varphi(\varepsilon x)\,\eta_x - \int_{\mathbb{R}^d} \varphi(x)\rho_0(x)dx\Big| > \delta\Big) = 0\,. \tag{2.5}$$

Then, for all $t > 0$, $\varphi \in C_c(\mathbb{R}^d)$ and $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}_{\omega,\mu_\varepsilon}\Big(\Big| \varepsilon^d \sum_{x \in \mathscr{C}(\omega)} \varphi(\varepsilon x)\,\eta_x(\varepsilon^{-2} t) - \int_{\mathbb{R}^d} \varphi(x)\rho(x,t)dx \Big| > \delta \Big) = 0\,, \tag{2.6}$$

where $\rho : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ solves the heat equation

$$\partial_t \rho = \nabla \cdot (\mathscr{D} \nabla \rho) = \sum_{i,j=1}^{d} \mathscr{D}_{i,j} \partial^2_{x_i,x_j} \rho \tag{2.7}$$

with boundary condition ρ_0 at $t = 0$ and where the symmetric matrix $\mathscr{D}$ is variationally characterized by (2.1).

If a density profile ρ_0 can be approximated by a family of probability measures μ_ε on $\{0,1\}^{\mathscr{C}(\omega)}$ (in the sense that (2.5) holds for each $\delta > 0$ and $\varphi \in C_c(\mathbb{R}^d)$), then it must be $0 \leqslant \rho_0 \leqslant m$ a.s. On the other hand, if $\rho_0 : \mathbb{R}^d \to [0,m]$ is a Riemann integrable function, then it is simple to exhibit for $\mathbb{Q}$–a.a. ω a family of probability measures μ_ε on $\{0,1\}^{\mathscr{C}(\omega)}$ approximating ρ_0. To this aim we observe that, due to the ergodicity of $\mathbb{Q}$ and by separability arguments, for $\mathbb{Q}$ a.a. ω it holds

$$\lim_{\varepsilon \downarrow 0} \varepsilon^d \sum_{x \in \mathscr{C}(\omega)} \varphi(\varepsilon x) = m \int_{\mathbb{R}^d} \varphi(x) dx\,, \tag{2.8}$$

for each Riemann integrable function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with compact support. Fix such an environment ω. Then, it is enough to define μ_ε as the unique product probability measure on $\{0,1\}^{\mathscr{C}(\omega)}$ such that $\mu_\varepsilon(\eta_x = 1) = \rho_0(\varepsilon x)/m$ for each $x \in \mathscr{C}(\omega)$. Since the random variable $\varepsilon^d \sum_{x \in \mathscr{C}(\omega)} \varphi(\varepsilon x)\,\eta_x$ is the sum of independent random variables, it is simple to verify that its mean equals $\varepsilon^d \sum_{x \in \mathscr{C}(\omega)} \varphi(\varepsilon x)\rho_0(\varepsilon x)/m$ and its variance equals $\varepsilon^{2d} \sum_{x \in \mathscr{C}(\omega)} \varphi^2(\varepsilon x)[\rho_0(\varepsilon x)/m][1 - \rho_0(\varepsilon x)/m]$. The thesis then follows by means of (2.8) and the Chebyshev inequality.

The proof of Theorem 2.2 is given in Section 3. As already mentioned, it is based on the general criterion for the hydrodynamic limit of exclusion processes with bond–dependent transition rates, obtained in [F] by generalizing an argument of [N], and homogenization results for the random walk on $\mathscr{C}(\omega)$ with jump rates $\omega(b)$, $b \in \mathscr{E}(\omega)$, described below.

2.3 The random walk among random conductances on the infinite cluster $\mathscr{C}(\omega)$

Given $\omega \in \Omega$ we denote by $X_\omega(t|x)$ the continuous–time random walk on $\mathscr{C}(\omega)$ starting at $x \in \mathscr{C}(\omega)$, whose Markov generator $\mathscr{L}_\omega$ acts on bounded functions $g : \mathscr{C}(\omega) \to \mathbb{R}$ as

$$\mathscr{L}_\omega g(x) = \sum_{\substack{y\,:\,y \in \mathscr{C}(\omega)\\ |x-y|=1}} \omega(x,y)\big(g(y) - g(x)\big)\,, \qquad x \in \mathscr{C}(\omega)\,. \tag{2.9}$$

The dynamics can be described as follows. After arriving at site $z \in \mathscr{C}(\omega)$, the particle waits an exponential time of parameter

$$\lambda_\omega(z) := \sum_{\substack{y\,:\,y \in \mathscr{C}(\omega)\\ |z-y|=1}} \omega(z,y)$$

and then jumps to a site $y \in \mathscr{C}(\omega)$, $|z-y|=1$, with probability $\omega(z,y)/\lambda_\omega(z)$. Since the jump rates are symmetric, the counting measure on $\mathscr{C}(\omega)$ is reversible for the random walk.

In what follows, given $\varepsilon > 0$ we will consider the rescaled random walk

$$X_{\varepsilon,\omega}(t|x) = \varepsilon X_\omega(\varepsilon^{-2}t|\varepsilon^{-1}x) \tag{2.10}$$

with starting point $x \in \varepsilon\mathscr{C}(\omega)$. We denote by μ_ω^ε the reversible rescaled counting measure

$$\mu_\omega^\varepsilon = \varepsilon^d \sum_{x \in \mathscr{C}(\omega)} \delta_{\varepsilon x}$$

and write $\mathscr{L}_\omega^\varepsilon$ for the symmetric operator on $L^2(\mu_\omega^\varepsilon)$ defined as

$$\mathscr{L}_\omega^\varepsilon g(\varepsilon x) = \varepsilon^{-2} \sum_{\substack{y\,:\, y \in \mathscr{C}(\omega) \\ |x-y|=1}} \omega(x,y)\big(g(\varepsilon y) - g(\varepsilon x)\big)\,, \qquad x \in \mathscr{C}(\omega). \tag{2.11}$$

Due to (2.8), for almost all $\omega \in \Omega$ the measure μ_ω^ε converges vaguely to the measure $m\,dx$, where the positive constant m is defined in (2.2). In what follows, $\|\cdot\|_{\mu_\omega^\varepsilon}$ and $(\cdot,\cdot)_{\mu_\omega^\varepsilon}$ will denote the norm and the inner product in $L^2(\mu_\omega^\varepsilon)$, respectively. We recall a standard definition in homogenization theory (cf. [Z], [ZP] and reference therein):

Definition 1. *Fix $\omega \in \Omega_0$. Given a family of functions $f_\omega^\varepsilon \in L^2(\mu_\omega^\varepsilon)$ parameterized by $\varepsilon > 0$ and a function $f \in L^2(m\,dx)$, f_ω^ε weakly converges to f (shortly, $f_\omega^\varepsilon \rightharpoonup f$) if*

$$\sup_\varepsilon \|f_\omega^\varepsilon\|_{\mu_\omega^\varepsilon} < \infty, \tag{2.12}$$

and

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^d} f_\omega^\varepsilon(x)\varphi(x)\mu_\omega^\varepsilon(dx) = \int_{\mathbb{R}^d} f(x)\varphi(x) m\,dx \tag{2.13}$$

for all functions $\varphi \in C_c^\infty(\mathbb{R}^d)$, while f_ω^ε strongly converges to f (shortly, $f_\omega^\varepsilon \to f$) if (2.12) holds and if

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^d} f_\omega^\varepsilon(x)\varphi^\varepsilon(x)\mu_\omega^\varepsilon(dx) = \int_{\mathbb{R}^d} f(x)\varphi(x) m\,dx\,, \tag{2.14}$$

for every family $\varphi^\varepsilon \in L^2(\mu_\omega^\varepsilon)$ weakly converging to $\varphi \in L^2(m\,dx)$.

The strong convergence $f_\omega^\varepsilon \to f$ admits the following characterization (cf. [Z][Proposition 1.1] and references therein):

Lemma 2.3. *Fix $\omega \in \Omega$ and functions $f_\omega^\varepsilon \in L^2(\mu_\omega^\varepsilon)$, $f \in L^2(mdx)$, where $\varepsilon > 0$. Then the strong convergence $f_\omega^\varepsilon \to f$ is equivalent to the weak convergence $f_\omega^\varepsilon \rightharpoonup f$ plus the relation*

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^d} f_\omega^\varepsilon(x)^2 \mu_\omega^\varepsilon(dx) = m \int_{\mathbb{R}^d} f(x)^2 dx\,. \tag{2.15}$$

We need now to isolate a Borel subset $\Omega_* \subset \Omega$ of *regular environments*. To this aim we first define Ω_1 as the set of $\omega \in \Omega_0$ (recall the definition of Ω_0 given before (H2)) such that

$$\lim_{\varepsilon \downarrow 0} \mu_\omega^\varepsilon(\Lambda_\ell) = m(2\ell)^d \,, \qquad \Lambda_\ell := [-\ell, \ell]^d \,, \tag{2.16}$$

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} \varphi(z) u\big(\tau_{z/\varepsilon}\omega\big) \mu_\omega^\varepsilon(dz) = \int_{\mathbb{R}^d} \varphi(z) dz \int_\Omega u(\omega') \mu(d\omega') \,, \tag{2.17}$$

for all $\ell > 0$, $\varphi \in C_c(\mathbb{R}^d)$, $u \in C(\Omega)$. From the the ergodicity of $\mathbb{Q}$ and the separability of $C_c(\mathbb{R}^d)$ and $C(\Omega)$, it is simple to derive that $\mathbb{Q}(\Omega_1) = 1$. The set of regular environments Ω_* will be defined in Section 4, after Lemma 4.4, since its definition requires the concept of solenoidal forms. We only mention here that $\Omega_* \subset \Omega_1$ and $\mathbb{Q}(\Omega_*) = 1$.

We can finally state our main homogenization result, similar to [ZP][Theorem 6.1]:

Theorem 2.4. *Fix $\omega \in \Omega_*$. Let f_ω^ε be a family of functions with $f_\omega^\varepsilon \in L^2(\mu_\omega^\varepsilon)$ and let $f \in L^2(m dx)$. Given $\lambda > 0$, define $u_\omega^\varepsilon \in L^2(\mu_\omega^\varepsilon)$, $u^0 \in L^2(m dx)$ as the solutions of the following equations in $L^2(\mu_\omega^\varepsilon)$, $L^2(m dx)$ respectively:*

$$\lambda u_\omega^\varepsilon - \mathscr{L}_\omega^\varepsilon u_\omega^\varepsilon = f_\omega^\varepsilon \,, \tag{2.18}$$

$$\lambda u^0 - \nabla \cdot (\mathscr{D} \nabla u^0) = f \,, \tag{2.19}$$

where the symmetric matrix $\mathscr{D}$ is variationally characterized in (2.1).

(i) If $f_\omega^\varepsilon \rightharpoonup f$, then it holds

$$L^2(\mu_\omega^\varepsilon) \ni u_\omega^\varepsilon \rightharpoonup u^0 \in L^2(m dx) \,, \qquad \forall \lambda > 0 \,. \tag{2.20}$$

(ii) If $f_\omega^\varepsilon \to f$, then it holds

$$L^2(\mu_\omega^\varepsilon) \ni u_\omega^\varepsilon \to u^0 \in L^2(m dx) \,, \qquad \forall \lambda > 0 \,. \tag{2.21}$$

(iii) For each test function $f \in C_c(\mathbb{R}^d)$ and setting $f_\omega^\varepsilon := f$, it holds

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} \left| u_\omega^\varepsilon(x) - u^0(x) \right|^2 \mu_\omega^\varepsilon(dx) = 0 \,, \qquad \forall \lambda > 0 \,. \tag{2.22}$$

The proof of Theorem 2.4 will be given in Section 6. We state here an important corollary of the above result: Set $P_{t,\omega}^\varepsilon = e^{t \mathscr{L}_\omega^\varepsilon}$, $P_t = e^{t \nabla \cdot (\mathscr{D} \nabla \cdot)}$. Note that $\big(P_{t,\omega}^\varepsilon : t \geqslant 0\big)$ is the $L^2(\mu_\omega^\varepsilon)$–Markov semigroup associated to the random walk $X_{\varepsilon,\omega}(t|x)$, i.e.

$$P_{t,\omega}^\varepsilon g(x) = \mathbb{E}\left[g\left(X_{\varepsilon,\omega}(t|x) \right) \right] \,, \qquad x \in \varepsilon \mathscr{C}(\omega) \,, g \in L^2(\mu_\omega^\varepsilon) \,, \tag{2.23}$$

while P_t is the $L^2(m dx)$–Markov semigroup associated to the diffusion with generator $\nabla \cdot (\mathscr{D} \nabla \cdot)$.

As proven in Section 7 it holds:

Corollary 2.5. *For each $\omega \in \Omega_*$, given any function $f \in C_c(\mathbb{R}^d)$, it holds*

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} \left| P_{t,\omega}^\varepsilon f(x) - P_t f(x) \right|^2 \mu_\omega^\varepsilon(dx) = 0 \,. \tag{2.24}$$

In particular, for each $\omega \in \Omega_*$, given any function $f \in C_c(\mathbb{R}^d)$, it holds

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} \left| P_{t,\omega}^\varepsilon f(x) - P_t f(x) \right| \mu_\omega^\varepsilon(dx) = 0 \,. \tag{2.25}$$

3 Proof of Theorem 2.2

As already mentioned, having the homogenization result given by Corollary 2.5, Theorem 2.2 follows easily from the criterion of [F] for the hydrodynamic limit of exclusion processes with bond–dependent rates. The method discussed in [F] is an improvement of the one developed in [N] for the analysis of bulk diffusion of 1d exclusion processes with bond–dependent rates. Although in [F] we have discussed the criterion with reference to exclusion processes on $\mathbb{Z}^d$, as the reader can check the method is very general and can be applied to exclusion processes on general graphs with bond–dependent rates, also under non diffusive space–time rescaling and also when the hydrodynamic behavior is not described by heat equations (cf. [FJL] for an example).

The following proposition is the main technical tool in order to reduce the proof of the hydrodynamic limit to a problem of homogenization for the random walk performed by a single particle (in absence of other particles). Recall the definition (2.23) of the semigroup $P^{\varepsilon}_{t,\omega}$ associated to the rescaled random walk $X_{\varepsilon,\omega}$ defined in (2.10).

Proposition 3.1. *For $\mathbb{Q}$–a.a. ω the following holds. Fix $\delta, t > 0$, $\varphi \in C_c(\mathbb{R}^d)$ and let μ_ε be a family of probability measures on $\{0,1\}^{\mathscr{C}(\omega)}$. Then*

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}_{\omega,\mu_\varepsilon}\left(\left| \varepsilon^d \sum_{x\in\mathscr{C}(\omega)} \varphi(\varepsilon x)\eta_x(\varepsilon^{-2}t) - \varepsilon^d \sum_{x\in\mathscr{C}(\omega)} \eta_x(0) P^{\varepsilon}_{t,\omega}\varphi(\varepsilon x)\right| > \delta \right) = 0\,. \tag{3.1}$$

Proof. One can prove the above proposition by the same arguments used in [F][Section 3] or one can directly invoke the discussion of [F][Section 4] referred to exclusion processes on $\mathbb{Z}^d$ with non negative transition rates, bounded from above. In fact, to the probability measure μ_ε on $\{0,1\}^{\mathscr{C}(\omega)}$ one can associate the probability measure ν_ε on $\{0,1\}^{\mathbb{Z}^d}$ so characterized: ν_ε is concentrated on the event

$$\mathscr{A}_\omega := \left\{ \eta \in \{0,1\}^{\mathbb{Z}^d} \,:\, \eta_x = 0 \text{ if } x \notin \mathscr{C}(\omega) \right\},$$

and

$$\nu_\varepsilon\big(\eta_x = 1 \;\; \forall x \in \Lambda\big) = \mu_\varepsilon\big(\eta_x = 1 \;\; \forall x\in\Lambda\big)\,, \qquad \forall \Lambda \subset \mathscr{C}(\omega)\,.$$

Given $\eta(t) \in \{0,1\}^{\mathscr{C}(\omega)}$, define $\sigma(t) \in \{0,1\}^{\mathbb{Z}^d}$ as

$$\sigma_x(t) = \begin{cases} \eta_x(t) & \text{if } x \in \mathscr{C}(\omega)\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Note that, if $\eta(t)$ has law $\mathbf{P}_{\mu_\varepsilon}$, then $\sigma(t)$ is the exclusion process on $\mathbb{Z}^d$ with initial distribution ν_ε and generator $f \to \sum_{b\in\mathbb{E}_d} \omega(b)\big(f(\sigma^b) - f(\sigma)\big)$. In particular, Proposition 3.1 coincides with the limit (B.2) in [F][Section 4]. □

We can now complete the proof of Theorem 2.2. First we observe that whenever (2.5) is satisfied for functions of compact support, then it is satisfied also for functions that vanish fast at infinity. Indeed, for our purposes it is enough to show that the limit

$$\lim_{\varepsilon\downarrow 0} \mu_\varepsilon\Big(\Big| \varepsilon^d \sum_{x\in\mathscr{C}(\omega)} f(\varepsilon x)\eta_x - \int_{\mathbb{R}^d} f(x)\rho_0(x)dx \Big| > \delta \Big) = 0 \tag{3.2}$$

is valid for all functions $f \in C(\mathbb{R}^d)$ such that

$$|f(x)| \leqslant \frac{c}{1+|x|^{d+1}}, \qquad \forall x \in \mathbb{R}^d. \tag{3.3}$$

For such a function f, given $\ell > 0$ we can find $g_\ell \in C_c(\mathbb{R}^d)$ such that $g_\ell(x) = f(x)$ for all $x \in \mathbb{R}^d$ with $|x| \leqslant \ell$ and $|g_\ell(x)| \leqslant \frac{c}{1+|x|^{d+1}}$, for all $x \in \mathbb{R}^d$. Then

$$\Big|\varepsilon^d \sum_{x\in\mathscr{C}(\omega)} f(\varepsilon x)\eta_x - \varepsilon^d \sum_{x\in\mathscr{C}(\omega)} g_\ell(\varepsilon x)\eta_x\Big| \leqslant \varepsilon^d \sum_{x\in\mathbb{Z}^d:|\varepsilon x|>\ell} \frac{2c}{1+|\varepsilon x|^{d+1}} \leqslant c(\ell), \tag{3.4}$$

$$\Big|\int_{\mathbb{R}^d} f(x)\rho_0(x)dx - \int_{\mathbb{R}^d} g_\ell(x)\rho_0(x)dx\Big| \leqslant \int_{\{x\in\mathbb{R}^d:|x|>\ell\}} \frac{2c}{1+|x|^{d+1}}dx \leqslant c(\ell), \tag{3.5}$$

for a suitable positive constant $c(\ell)$ going to zero as $\ell \to \infty$. Since $g_\ell \in C_c(\mathbb{R}^d)$, by assumption (2.5) we obtain that

$$\lim_{\varepsilon\downarrow 0} \mu_\varepsilon\Big(\Big|\varepsilon^d \sum_{x\in\mathscr{C}(\omega)} g_\ell(\varepsilon x)\eta_x - \int_{\mathbb{R}^d} g_\ell(x)\rho_0(x)dx\Big| > \delta\Big) = 0. \tag{3.6}$$

The above limit together with (3.4) and (3.5) implies (3.2) for all functions $f \in C(\mathbb{R}^d)$ satisfying (3.3).

In particular, (3.3) is valid for $f = P_t\varphi$, where $P_t = e^{t\nabla\cdot(\mathscr{D}\nabla\cdot)}$ and $\varphi \in C_c(\mathbb{R}^d)$. Indeed, in this case f decays exponentially. Due to this observation, Proposition 3.1 and the fact that

$$\int_{\mathbb{R}^d} P_t\varphi(x)\rho_0(x)dx = \int_{\mathbb{R}^d} \varphi(x)P_t\rho_0(x)dx = \int_{\mathbb{R}^d} \varphi(x)\rho(x,t)dx,$$

in order to prove (2.6) it is enough to show that for $\mathbb{Q}$–a.a. ω it holds

$$\lim_{\varepsilon\downarrow 0} \mu_\varepsilon\Big(\varepsilon^d\Big|\sum_{x\in\mathscr{C}(\omega)} \eta_x P^\varepsilon_{t,\omega}\varphi(\varepsilon x) - \sum_{x\in\mathscr{C}(\omega)} \eta_x P_t\varphi(\varepsilon x)\Big| > \delta\Big) = 0 \tag{3.7}$$

for any $\varphi \in C_c(\mathbb{R}^d)$ and $\delta > 0$. Since

$$\varepsilon^d\Big|\sum_{x\in\mathscr{C}(\omega)} \eta_x P^\varepsilon_{t,\omega}\varphi(\varepsilon x) - \sum_{x\in\mathscr{C}(\omega)} \eta_x P_t\varphi(\varepsilon x)\Big| \leqslant \int_{\mathbb{R}^d} \Big|P^\varepsilon_{t,\omega}\varphi(x) - P_t\varphi(x)\Big|\mu^\varepsilon_\omega(dx),$$

(3.7) follows from (2.25) of Corollary 2.5. This concludes the proof of Theorem 2.2.

4 Square integrable forms

We now focus our attention on the proof of homogenization for the random walk on the infinite cluster. To this aim, in this section we introduce the Hilbert space of square integrable forms and show how the variational formula (2.1) can be interpreted in terms of suitable orthogonal projections inside this Hilbert space.

Let $\mathscr{M}(\mathbb{R}^d)$ be the family of Borel measures on $\mathbb{R}^d$. Given $x \in \mathbb{Z}^d$, $y \in \mathbb{R}^d$, $\omega \in \Omega$ and $v \in \mathscr{M}(\mathbb{R}^d)$, $\tau_x \omega \in \Omega$ and $\tau_y v \in \mathscr{M}(\mathbb{R}^d)$ are defined as

$$\tau_x \omega(b) = \omega(b+x) \;\; \forall b \in \mathbb{E}_d, \qquad \tau_y v(A) = v(A+y) \;\; \forall A \subset \mathbb{R}^d, \text{ Borel set}.$$

Note that the family of random measures μ_ω^ε satisfies the identity

$$\tau_{\varepsilon x} \mu_\omega^\varepsilon = \mu_{\tau_x \omega}^\varepsilon, \qquad \forall x \in \mathbb{Z}^d. \tag{4.1}$$

Let μ be the measure on Ω absolutely continuous w.r.t. $\mathbb{Q}$ such that

$$\mu(d\omega) = \mathbb{I}_{0 \in \mathscr{C}(\omega)} \mathbb{Q}(d\omega) \tag{4.2}$$

and define $\mathscr{B}$ as

$$\mathscr{B} := \{e \in \mathbb{Z}^d \,:\, |e| = 1\} = \mathscr{B}_* \cup (-\mathscr{B}_*)$$

($\mathscr{B}_*$ is the set of coordinate vectors in $\mathbb{R}^d$).

Given real functions u defined on Ω and v defined on $\Omega \times \mathscr{B}$, we define the *gradient* $\nabla^{(\omega)} u : \Omega \times \mathscr{B} \to \mathbb{R}$ and the *divergence* $\nabla^{(\omega)*} v : \Omega \to \mathbb{R}$, respectively, as follows:

$$\nabla^{(\omega)} u(\omega, e) = \hat{\omega}(0,e) \left[u(\tau_e \omega) - u(\omega) \right], \qquad u : \Omega \to \mathbb{R}, \tag{4.3}$$

$$\nabla^{(\omega)*} v(\omega) = \sum_{e \in \mathscr{B}} \omega(0,e) \left[v(\omega, e) - v(\tau_e \omega, -e) \right], \qquad v : \Omega \times \mathscr{B} \to \mathbb{R}. \tag{4.4}$$

Moreover, we endow the space $\Omega \times \mathscr{B}$ with the Borel measure M defined by

$$\int_{\Omega \times \mathscr{B}} v dM = \sum_{e \in \mathscr{B}} \int_\Omega \omega(0,e) v(\omega, e) \mu(d\omega),$$

where v is any bounded Borel function on $\Omega \times \mathscr{B}$. Note that if $u \in L^p(\mu)$ and $v \in L^p(M)$ then $\nabla^{(\omega)} u \in L^p(M)$ and $\nabla^{(\omega)*} v \in L^p(\mu)$.

The space $L^2(M)$ is called the space of *square integrable forms*. Note that M gives zero measure to the set

$$\big\{(\omega, e) \in \Omega \times \mathscr{B} \,:\, \{0, e\} \notin \mathscr{E}(\omega)\big\}.$$

Hence, given a square integrable form $v \in L^2(M)$, we can always assume that $v(\omega, e) = 0$ whenever $\{0, e\} \notin \mathscr{E}(\omega)$. We define the space of *potential forms* $L^2_{\text{pot}}(M)$ and the space of *solenoidal forms* $L^2_{\text{sol}}(M)$ as follows:

Definition 2. *The space $L^2_{pot}(M)$ is the closure in $L^2(M)$ of the set of gradients $\nabla^{(\omega)} u$ of local functions u, while $L^2_{sol}(M)$ is the orthogonal complement of $L^2_{pot}(M)$ in $L^2(M)$.*

In Lemmata 4.1, 4.2 and 4.3 below we collect some identities relating $\nabla^{(\omega)}$, $\nabla^{(\omega),*}$ and the spatial gradient $\nabla^\varepsilon_{\omega,e}$, $e \in \mathscr{B}$, defined as follows. Given a function $u : \varepsilon\mathscr{C}(\omega) \to \mathbb{R}$, the gradient $\nabla^\varepsilon_{\omega,e} u$ is the function $\nabla^\varepsilon_{\omega,e} u : \varepsilon\mathscr{C}(\omega) \to \mathbb{R}$ defined as

$$\nabla^\varepsilon_{\omega,e} u(x) = \begin{cases} \omega(x/\varepsilon, x/\varepsilon + e) \frac{u(x+\varepsilon e) - u(x)}{\varepsilon} & \text{if } x, x + \varepsilon e \in \varepsilon\mathscr{C}(\omega), \\ 0 & \text{otherwise}. \end{cases}$$

It can be written as

$$\nabla^{\varepsilon}_{\omega,e}u(x) = \tau_{x/\varepsilon}\omega(0,e)\nabla^{\varepsilon}_{e}u(x)$$

where the gradient $\nabla^{\varepsilon}_{e}u$ is defined as

$$\nabla^{\varepsilon}_{e}u(x) = \begin{cases} \frac{u(x+\varepsilon e)-u(x)}{\varepsilon} & \text{if } x, x+\varepsilon e \in \varepsilon\mathscr{C}(\omega)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Lemma 4.1 explains why $\nabla^{(\omega)*}$ is called divergence, or adjoint gradient:

Lemma 4.1. *Given functions $u \in L^2(\mu)$ and $v \in L^2(M)$, it holds*

$$\int_{\Omega\times\mathscr{B}} v\nabla^{(\omega)}u\, dM = -\int_{\Omega} \big(\nabla^{(\omega)*}v\big)\, u\, d\mu\,. \tag{4.5}$$

In particular, $\int_{\Omega\times\mathscr{B}} v dM = 0$ for any $v \in L^2_{pot}(M)$, while a square integrable form $v \in L^2(M)$ is solenoidal if and only if $\nabla^{(\omega)*}v(\omega) = 0$ for μ a.a. ω.

Proof. By definition

$$\begin{aligned}\int_{\Omega\times\mathscr{B}} v\nabla^{(\omega)}u\, dM &= \sum_{e\in\mathscr{B}}\int_{\Omega} \mu(d\omega)\omega(0,e)v(\omega,e)\big(u(\tau_e\omega)-u(\omega)\big) \\ &= -\sum_{e\in\mathscr{B}}\int_{\Omega} \mu(d\omega)\omega(0,e)v(\omega,e)u(\omega) + \sum_{e\in\mathscr{B}}\int_{\Omega} \mu(d\omega)\omega(0,e)v(\omega,e)u(\tau_e\omega)\,. \end{aligned}\tag{4.6}$$

Since

$$\mu(d\omega)\omega(0,e) = \mathbb{Q}(d\omega)\mathbb{I}_{0\in\mathscr{C}(\omega)}\omega(0,e) = \mathbb{Q}(d\omega)\mathbb{I}_{e\in\mathscr{C}(\omega)}\omega(0,e) = \mathbb{Q}(d\omega)\mathbb{I}_{0\in\mathscr{C}(\tau_e\omega)}\tau_e\omega(0,-e),$$

$v(\omega,e) = v\big(\tau_{-e}(\tau_e\omega),e\big)$ and $\mathbb{Q}(d\omega) = \mathbb{Q}(d\tau_e\omega)$, we can conclude that

$$\begin{aligned}\int_{\Omega} \mu(d\omega)\omega(0,e)v(\omega,e)u(\tau_e\omega) &= \int_{\Omega} \mathbb{Q}(d\tau_e\omega)\mathbb{I}_{0\in\mathscr{C}(\tau_e\omega)}\tau_e\omega(0,-e)v\big(\tau_{-e}(\tau_e\omega),e\big)u(\tau_e\omega) \\ &= \int_{\Omega} \mathbb{Q}(d\omega)\mathbb{I}_{0\in\mathscr{C}(\omega)}\omega(0,-e)v(\tau_{-e}\omega,e)u(\omega) = \int_{\Omega} \mu(d\omega)\omega(0,-e)v(\tau_{-e}\omega,e)u(\omega)\,.\end{aligned}$$

Hence the last sum in (4.6) can be rewritten as

$$\sum_{e\in\mathscr{B}}\int_{\Omega} \mu(d\omega)\omega(0,e)v(\omega,e)u(\tau_e\omega) = \sum_{e\in\mathscr{B}}\int_{\Omega} \mu(d\omega)\omega(0,e)v(\tau_e\omega,-e)u(\omega)\,.$$

The above identity and (4.6) allows to conclude the proof of (4.5), while the second part of the lemma follows easily from (4.5). □

We point out another integration by parts formula.

Lemma 4.2. *Let $e \in \mathscr{B}$, $u \in L^1(\mu^\varepsilon_\omega)$, $\psi \in B(\Omega)$ and define $v : \Omega \times \mathscr{B} \to \mathbb{R}$ as*

$$v(\omega, e') := \psi(\omega)\delta_{e,e'} .$$

Then,

$$\int_{\mathbb{R}^d} \nabla^\varepsilon_{\omega,e} u(x)\psi(\tau_{x/\varepsilon}\omega)\mu^\varepsilon_\omega(dx) = -\varepsilon^{-1} \int_{\mathbb{R}^d} u(x)\big(\nabla^{(\omega)*}v\big)(\tau_{x/\varepsilon}\omega)\mu^\varepsilon_\omega(dx) . \tag{4.7}$$

Proof. By definition of v, we have

$$\nabla^{(\omega)*}v(\omega) = \omega(0,e)\psi(\omega) - \omega(0,-e)\psi(\tau_{-e}\omega) . \tag{4.8}$$

Moreover, since $\mathbb{I}_{z\in\mathscr{C}(\omega)}\omega(z,z+e) = \mathbb{I}_{z+e\in\mathscr{C}(\omega)}\omega(z,z+e)$, we can write

$$\sum_{z\in\mathscr{C}(\omega)} \omega(z,z+e)\,(u(\varepsilon z+\varepsilon e) - u(\varepsilon z))\,\psi(\tau_z\omega) = \\ - \sum_{z\in\mathscr{C}(\omega)} u(\varepsilon z)\,\big(\omega(z,z+e)\psi(\tau_z\omega) - \omega(z,z-e)\psi(\tau_{z-e}\omega)\big) . \tag{4.9}$$

Identities (4.8) and (4.9) allow to conclude the proof of (4.7). □

Finally, we point out the simple identities

$$\nabla^\varepsilon_{\omega,e}(a(x)b(x)) = \Big(\nabla^\varepsilon_{\omega,e}a(x)\Big)\, b(x+\varepsilon e) + a(x)\,\Big(\nabla^\varepsilon_{\omega,e}b(x)\Big) , \tag{4.10}$$

$$\nabla^\varepsilon_{e}(a(x)b(x)) = \Big(\nabla^\varepsilon_{e}a(x)\Big)\, b(x+\varepsilon e) + a(x)\,\Big(\nabla^\varepsilon_{e}b(x)\Big) , \tag{4.11}$$

valid for all functions $a, b : \varepsilon\mathscr{C}(\omega) \to \mathbb{R}^d$. In what follows, (4.10) and (4.11) will be frequently used without explicit mention.

Lemma 4.3. *Let $u \in L^2(\mu)$. Suppose that for all functions $\psi \in C(\Omega)$ and for all $e \in \mathscr{B}$ it holds*

$$\int_\Omega u(\omega)\nabla^{(\omega)*}v(\omega)\mu(d\omega) = 0 , \qquad v(\omega, e') := \psi(\omega)\delta_{e,e'} . \tag{4.12}$$

Then, u is constant μ–almost everywhere.

Proof. Due to Lemma 4.1, for all functions $\psi \in C(\Omega)$ and for all $e \in \mathscr{B}$ it holds

$$0 = \int_{\Omega\times\mathscr{B}} \big(\nabla^{(\omega)}u\big)v\,dM = \int_\Omega \mu(d\omega)\omega(0,e)\,\big(u(\tau_e\omega) - u(\omega)\big)\,\psi(\omega) . \tag{4.13}$$

Hence,

$$\mathbb{I}_{\{0,e\}\in\mathscr{E}(\omega)}\,\big(u(\tau_e\omega) - u(\omega)\big) = 0 , \qquad \mathbb{Q}\text{–}a.s. \tag{4.14}$$

Due the translation invariance of $\mathbb{Q}$ we conclude that

$$\mathbb{I}_{\{x,x+e\}\in\mathscr{E}(\omega)}\,\big(u(\tau_{x+e}\omega) - u(\tau_x\omega)\big) = 0 , \qquad \forall x \in \mathbb{Z}^d ,\ \mathbb{Q}\text{–}a.s. \tag{4.15}$$

Since $\mathscr{C}(\omega)$ is connected, (4.15) is equivalent to say that for $\mathbb{Q}$–a.a. ω there exists a constant $a(\omega)$ such that $u(\tau_x\omega) = a(\omega)$ for all $x \in \mathscr{C}(\omega)$. Trivially, the function $a(\omega)$ is translation invariant. Hence, due to the ergodicity of $\mathbb{Q}$ we can conclude that $a(\omega)$ is constant $\mathbb{Q}$–a.s. Since $a(\omega) = u(\omega)$ if $0 \in \mathscr{C}(\omega)$, we conclude that $u(\omega)$ is constant for μ–a.a. ω. □

We have now all the tools in order to define the set of regular environments Ω_*. To this aim we first observe that $L^2_{\rm sol}$ is separable since it is a subset of the separable metric space $L^2(M)$. We fix once and for all a sequence $\{\psi_j\}_{j\geqslant 1}$ dense in $L^2_{\rm sol}$. Since elements of $L^2(M)$ are equivalent if, as functions, they differ on a zero measure set, we fix a representative ψ_j and from now on we think of ψ_j as pointwise function $\psi_j : \Omega\times\mathscr{B}\to\mathbb{R}$. Since $\psi_j\in L^2(M)$ it must be

$$\Psi_{j,e}(\omega):=\sqrt{\omega(0,e)}\psi_j(\cdot,e)\in L^2(\mu)\,. \tag{4.16}$$

For each $j\geqslant 1$ and $e\in\mathscr{B}$ we fix a sequence of continuous functions $f^{(k)}_{j,e}\in C(\Omega)$ such that $f^{(k)}_{j,e}$ converges to $\Psi_{j,e}$ in $L^2(\mu)$ as $k\to\infty$. We first make a simple observation:

Lemma 4.4. *Given $j\geqslant 1$, define the Borel set $\Omega_{*,j}$ as the set of configurations $\omega\in\Omega_0$ such that*

$$\nabla^{(\omega)*}\psi_j(\tau_x\omega)=0\ \ \forall x\in\mathscr{C}(\omega)\,, \tag{4.17}$$

$$\lim_{\varepsilon\downarrow 0}\int_{[-n,n]^d}\left(f^{(k)}_{j,e}(\tau_{z/\varepsilon}\omega)-\Psi_{j,e}(\tau_{z/\varepsilon}\omega)\right)^2\mu^\varepsilon_\omega(dz)=(2n)^d\left\|f^{(k)}_{j,e}-\Psi_{j,e}\right\|^2_{L^2(\mu)} \tag{4.18}$$

for each $e\in\mathscr{B}$ and $k,n\geqslant 1$. Then $\mathbb{Q}(\Omega_{*,j})=1$.

Proof. Let us define the set A_x as

$$A_x:=\{\omega\in\Omega_0\,:\,\nabla^{(\omega)*}\psi_j(\tau_x\omega)\neq 0 \text{ and } x\in\mathscr{C}(\omega)\}\,,\qquad x\in\mathbb{Z}^d\,.$$

Due to Lemma 4.1, $\mathbb{Q}(A_0)=\mu(A_0)=0$. Since $\omega\in A_x$ if and only if $\tau_x\omega\in A_0$, by the translation invariance of $\mathbb{Q}$ we obtain that $\mathbb{Q}(A_x)=0$ for all $x\in\mathbb{Z}^d$. Hence, setting $A=\cup_{x\in\mathbb{Z}^d}A_x$, it must be $\mathbb{Q}(A)=0$. Since $\Omega_0\setminus A$ coincides with the set of $\omega\in\Omega_0$ satisfying (4.17), we only need to prove that (4.18) is satisfied $\mathbb{Q}$–a.s. for each $k,n\geqslant 1$ and $e\in\mathscr{B}$. This is a direct consequence of the L^1–ergodic theorem. □

Recall the definition of $\Omega_1\subset\Omega_0$ given before Theorem 2.4. We can finally define the set Ω_*:

Definition 3. *We define the set Ω_* of regular environments as*

$$\Omega_*:=\Omega_1\cap\left(\cap_{j=1}^\infty\Omega_{*,j}\right)\subset\Omega_0\,.$$

We conclude this section by reformulating the variational characterization (2.1) of the diffusion matrix $\mathscr{D}$ in terms of square integrable forms. To this aim, given a vector $\xi\in\mathbb{R}^{\mathscr{B}_*}$, we write w^ξ for the square integrable form

$$w^\xi(\omega,\pm e):=\pm\xi_e\,,\qquad e\in\mathscr{B}_*,\ \ \omega\in\Omega\,. \tag{4.19}$$

Let $\pi: L^2(M)\to L^2_{\rm sol}(M)$ be the orthogonal projection of $L^2(M)$ onto $L^2_{\rm sol}(M)$ and let Φ be the bilinear form on $\mathbb{R}^{\mathscr{B}_*}\times\mathbb{R}^{\mathscr{B}_*}$ defined as

$$\Phi(\zeta,\xi)=(w^\zeta,\pi w^\xi)_{L^2(M)}\,,$$

where $(\cdot,\cdot)_{L^2(M)}$ denotes the inner product in $L^2(M)$.

Since Φ is bilinear and symmetric, there exists a symmetric matrix D indexed on $\mathscr{B}_* \times \mathscr{B}_*$ such that

$$(\zeta, D\xi) = (w^\zeta, \pi w^\xi)_{L^2(M)}. \tag{4.20}$$

We give an integral representation of $D\xi$ which will be useful in what follows. Since

$$(w^\zeta, \pi w^\xi)_{L^2(M)} = \sum_{e\in\mathscr{B}_*} \zeta_e \int_\Omega \mu(d\omega)\left[\omega(0,e)\big(\pi w^\xi\big)(\omega,e) - \omega(0,-e)\big(\pi w^\xi\big)(\omega,-e)\right], \tag{4.21}$$

it must be

$$(D\xi)_e = \int_\Omega \mu(d\omega)\left[\omega(0,e)\big(\pi w^\xi\big)(\omega,e) - \omega(0,-e)\big(\pi w^\xi\big)(\omega,-e)\right]. \tag{4.22}$$

Moreover, due to the definition of orthogonal projection, we get

$$\begin{aligned}(\xi, D\xi) &= (w^\xi, \pi w^\xi)_{L^2(M)} = \|\pi w^\xi\|^2_{L^2(M)} = \inf_{v\in L^2_{\text{pot}}(M)} \|w^\xi - v\|^2_{L^2(M)}\\ &= \inf_{\psi\in B(\Omega)} \|w^\xi - \nabla^{(\omega)}\psi\|^2_{L^2(M)}.\end{aligned} \tag{4.23}$$

By definition,

$$\begin{aligned}\|w^\xi - \nabla^{(\omega)}\psi\|^2_{L^2(M)} &= \sum_{e\in\mathscr{B}_*}\int_\Omega \mu(d\omega)\omega(0,e)\big[\xi_e - \hat\omega(0,e)\big(\psi(\tau_e\omega) - \psi(\omega)\big)\big]^2\\ &+ \sum_{e\in\mathscr{B}_*}\int_\Omega \mu(d\omega)\omega(0,-e)\big[-\xi_e - \hat\omega(0,-e)\big(\psi(\tau_{-e}\omega) - \psi(\omega)\big)\big]^2\\ &= \sum_{e\in\mathscr{B}_*}\int_\Omega \mu(d\omega)\omega(0,e)\big(\xi_e - \psi(\tau_e\omega) + \psi(\omega)\big)^2\\ &+ \sum_{e\in\mathscr{B}_*}\int_\Omega \mu(d\omega)\omega(0,-e)\big(-\xi_e - \psi(\tau_{-e}\omega) + \psi(\omega)\big)^2.\end{aligned} \tag{4.24}$$

We can rewrite the last term in a more useful form. In fact, due to the translation invariance of $\mathbb{Q}$, we get

$$\begin{aligned}&\int_\Omega \mu(d\omega)\omega(0,-e)\big(-\xi_e - \psi(\tau_{-e}\omega) + \psi(\omega)\big)^2 =\\ &\int_\Omega \mathbb{Q}(d\omega)\mathbb{I}_{0,e\in\mathscr{C}(\tau_{-e}\omega)}\tau_{-e}\omega(0,e)\big(-\xi_e - \psi(\tau_{-e}\omega) + \psi\big(\tau_e(\tau_{-e}\omega)\big)\big)^2 =\\ &\int_\Omega \mathbb{Q}(d\omega)\mathbb{I}_{0,e\in\mathscr{C}(\omega)}\omega(0,e)\big(\xi_e + \psi(\omega) - \psi(\tau_e\omega)\big)^2 = \int_\Omega \mu(d\omega)\omega(0,e)\big(\xi_e - \psi(\tau_e\omega) + \psi(\omega)\big)^2.\end{aligned} \tag{4.25}$$

Due to (4.23), (4.24) and (4.25) we conclude that

$$(\xi, D\xi) = \inf_{\psi\in B(\Omega)} 2\sum_{e\in\mathscr{B}_*}\int_\Omega \mu(d\omega)\omega(0,e)\big(\xi_e - \psi(\tau_e\omega) + \psi(\omega)\big)^2. \tag{4.26}$$

In particular, the matrix D is related to the matrix $\mathscr{D}$ via the identity

$$D = 2m\mathscr{D}\,. \tag{4.27}$$

From the above observations and the non–degeneracy of $\mathscr{D}$ given by hypothesis (H3) we get:

Lemma 4.5. *The vectorial space given by the vectors*

$$\left(\int_\Omega \mu(d\omega) \left[\omega(0,e)\psi(\omega,e) - \omega(0,-e)\psi(\omega,-e) \right] \right)_{e\in\mathscr{B}_*}, \qquad \psi \in L^2_{sol},$$

coincides with $\mathbb{R}^{\mathscr{B}_*}$.

Proof. If the statement was not true, then there would exist $\xi \in \mathbb{R}^{\mathscr{B}_*} \setminus \{0\}$ such that

$$\sum_{e\in\mathscr{B}_*} \xi_e \int_\Omega \mu(d\omega) \left[\omega(0,e)\psi(\omega,e) - \omega(0,-e)\psi(\omega,-e) \right] = 0, \qquad \forall \psi \in L^2_{\mathrm{sol}}.$$

In particular, the above identity would hold with $\psi = \pi w^\xi$. Due to (4.20) and (4.21), this would imply that $(\xi, D\xi) = 0$, which is absurd due to hypothesis (H3). □

Finally, we conclude with a simple but crucial observation. Given $\xi \in \mathbb{R}^{\mathscr{B}_*}$, there exists a unique form $v \in L^2_{\mathrm{pot}}$ such that $w^\xi + v \in L^2_{\mathrm{sol}}$. In fact, these requirements imply that $w^\xi + v = \pi w^\xi$.

5 Two–scale convergence

In this section we analyze the weak two–scale convergence for our disordered model. We recall that Ω_* denotes the set of regular environments ω defined in the previous section, and we recall that $(\cdot,\cdot)_{\mu^\varepsilon_\omega}$ and $\|\cdot\|_{\mu^\varepsilon_\omega}$ denote respectively the inner product and the norm in $L^2(\mu^\varepsilon_\omega)$. In our context the two–scale convergence [ZP][Section 5] can be defined as follows:

Definition 4. *Fix $\omega \in \Omega_*$. Let v^ε be a family of functions parameterized by $\varepsilon > 0$ such that $v^\varepsilon \in L^2(\mu^\varepsilon_\omega)$. Then the function $v \in L^2(\mathbb{R}^d \times \Omega, dx \times \mu)$ is the weak two–scale limit of v^ε as $\varepsilon \downarrow 0$ (shortly, $v^\varepsilon \xrightarrow{2} v$) if the following two conditions are fulfilled:*

$$\limsup_{\varepsilon\downarrow 0} \|v^\varepsilon\|_{\mu^\varepsilon_\omega} < \infty\,, \tag{5.1}$$

and

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^d} v^\varepsilon(x)\varphi(x)\psi\big(\tau_{x/\varepsilon}\omega\big)\mu^\varepsilon_\omega(dx) = \int_{\mathbb{R}^d} dx \int_\Omega v(x,\omega')\varphi(x)\psi(\omega')\mu(d\omega')\,, \tag{5.2}$$

for all $\varphi \in C^\infty_c(\mathbb{R}^d)$ and $\psi \in C(\Omega)$.

Let us first collect some technical results concerning the weak two–scale convergence. For the next lemma, recall the definition of the function $\Psi_{j,e} \in L^2(\mu)$ given in (4.16).

Lemma 5.1. *Fix $\omega \in \Omega_*$ and suppose that $L^2(\mu^\varepsilon_\omega) \ni v^\varepsilon \xrightarrow{2} v \in L^2(\mathbb{R}^d \times \Omega, dx \times \mu)$. Then, for each $j \geqslant 1$, $e \in \mathscr{B}$, $\psi \in C(\Omega)$ and $\varphi \in C_c(\mathbb{R}^d)$, it holds*

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} v^\varepsilon(x)\varphi(x)\psi(\tau_{x/\varepsilon}\omega)\Psi_{j,e}(\tau_{x/\varepsilon}\omega)\mu^\varepsilon_\omega(dx) = \int_{\mathbb{R}^d} dx \int_\Omega v(x,\omega')\varphi(x)\psi(\omega')\Psi_{j,e}(\omega')\mu(d\omega'). \tag{5.3}$$

Proof. Suppose that the support of φ is included in $[-n,n]^d$. Recall the definition of the functions $f^{(k)}_{j,e} \in C(\Omega)$ given in Section 4. Then, by Schwarz inequality, we get

$$\left| \int_{\mathbb{R}^d} v^\varepsilon(x)\varphi(x)\psi(\tau_{x/\varepsilon}\omega)\big[\Psi_{j,e}(\tau_{x/\varepsilon}\omega) - f^{(k)}_{j,e}(\tau_{x/\varepsilon}\omega)\big]\mu^\varepsilon_\omega(dx) \right| \leqslant \\ \|\varphi\|_\infty \|\psi\|_\infty \|v^\varepsilon\|_{\mu^\varepsilon_\omega} \Big\{ \int_{[-n,n]^d} \big(\Psi_{j,e}(\tau_{x/\varepsilon}\omega) - f^{(k)}_{j,e}(\tau_{x/\varepsilon}\omega)\big)^2 \mu^\varepsilon_\omega(dx) \Big\}^{1/2}. \tag{5.4}$$

Since $\omega \in \Omega_* \subset \Omega_{*,j}$ and since $f^{(k)}_{j,e} \to \Psi_{j,e}$ in $L^2(\mu)$, we conclude that the upper limit of the r.h.s. as $\varepsilon \downarrow 0$ and then $k \uparrow \infty$ is zero. On the other hand, since $v^\varepsilon \xrightarrow{2} v$ and $f^{(k)}_{j,e} \to \Psi_{j,e}$ in $L^2(\mu)$, we obtain that

$$\lim_{k\uparrow\infty} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} v^\varepsilon(x)\varphi(x)\psi(\tau_{x/\varepsilon}\omega) f^{(k)}_{j,e}(\tau_{x/\varepsilon}\omega)\mu^\varepsilon_\omega(dx) = \\ \lim_{k\uparrow\infty} \int_{\mathbb{R}^d} dx \int_\Omega v(x,\omega')\varphi(x)\psi(\omega') f^{(k)}_{j,e}(\omega')\mu(d\omega') = \\ \int_{\mathbb{R}^d} dx \int_\Omega v(x,\omega')\varphi(x)\psi(\omega')\Psi_{j,e}(\omega')\mu(d\omega'). \tag{5.5}$$

This allows to get (5.3). □

By the same arguments leading to [ZP][Lemma 5.1] and [Z][Prop. 2.2] one can easily prove the following result:

Lemma 5.2. *Fix $\omega \in \Omega_*$. Suppose that the family of functions $v^\varepsilon \in L^2(\mu^\varepsilon_\omega)$ satisfies (5.1). Then from each sequence ε_k converging to zero, one can extract a subsequence ε_{k_n} such that v^ε converges along ε_{k_n} to some $v \in L^2(\mathbb{R}^d \times \Omega, dx \times \mu)$ in the sense of weak two–scale convergence.*

We give the proof for the reader's convenience:

Proof. Given $\varphi \in C^\infty_c(\mathbb{R}^d)$ and $\psi \in C(\Omega)$, we can bound

$$\limsup_{k\to\infty} \left| \int_{\mathbb{R}^d} v^{\varepsilon_k}(x)\varphi(x)\psi(\tau_{x/\varepsilon_k}\omega)\mu^{\varepsilon_k}_\omega(dx) \right| \leqslant \\ \limsup_{k\to\infty} \|v^{\varepsilon_k}\|_{\mu^{\varepsilon_k}_\omega} \left(\int_{\mathbb{R}^d} \varphi^2(x)\psi^2(\tau_{x/\varepsilon_k}\omega)\mu^{\varepsilon_k}_\omega(dx) \right)^{1/2} \leqslant \\ C(\omega) \left(\int_{\mathbb{R}^d} \varphi^2(x)\,dx \int_\Omega \psi^2(\omega')\mu(d\omega') \right)^{1/2} = C(\omega) \big\| \varphi\psi \big\|_{L^2(\mathbb{R}^d \times \Omega, dx \times \mu)}$$

(note that the first estimate follows from Schwarz inequality, while the second one follows from (2.17) and (5.1)).

Using a standard diagonal argument and the separability of the space of test functions φ, ψ, we can conclude that there exists a subsequence $\{\varepsilon_{k_n}\}_{n\geqslant 1}$ along which the limit in the l.h.s. of (5.2) exists and can be extended to a continuous linear functional on $L^2(\mathbb{R}^d\times\Omega, dx\times\mu)$. Therefore, this limit can be written as the inner product in $L^2(\mathbb{R}^d\times\Omega, dx\times\mu)$ with a suitable function v.

□

In what follows, we will apply the concept of weak two–scale convergence to the solution $u^\varepsilon_\omega \in L^2(\mu^\varepsilon_\omega)$ of (2.18) and to its gradients, for a fixed sequence $f^\varepsilon_\omega \rightharpoonup f$. To this aim we start with some simple observations.

We note that, given $u, v \in L^2(\mu^\varepsilon_\omega)$, it holds

$$(u, -\mathscr{L}^\varepsilon_\omega v)_{\mu^\varepsilon_\omega} = \frac{\varepsilon^{d-2}}{2}\sum_{z\in\mathscr{C}(\omega)}\sum_{e\in\mathscr{B}}\omega(z,z+e)\,[u(\varepsilon z+\varepsilon e)-u(\varepsilon z)]\,[v(\varepsilon z+\varepsilon e)-v(\varepsilon z)]$$
$$= \frac{\varepsilon^d}{2}\sum_{z\in\mathscr{C}(\omega)}\sum_{e\in\mathscr{B}}\omega(z,z+e)\nabla^\varepsilon_e u(\varepsilon z)\nabla^\varepsilon_e v(\varepsilon z) = \frac{1}{2}\sum_{e\in\mathscr{B}}\int_{\mathbb{R}^d}\mu^\varepsilon_\omega(dx)\tau_{x/\varepsilon}\omega(0,e)\nabla^\varepsilon_e u(x)\nabla^\varepsilon_e v(x)\,. \tag{5.6}$$

In particular, we can write

$$(u^\varepsilon_\omega, -\mathscr{L}^\varepsilon_\omega u^\varepsilon_\omega)_{\mu^\varepsilon_\omega} = \frac{1}{2}\sum_{e\in\mathscr{B}}\int_{\mathbb{R}^d}\mu^\varepsilon_\omega(dx)\tau_{x/\varepsilon}\omega(0,e)\big(\nabla^\varepsilon_e u^\varepsilon_\omega(x)\big)^2 = \frac{1}{2}\sum_{e\in\mathscr{B}}\Big\|\sqrt{\tau_{x/\varepsilon}\omega(0,e)}\nabla^\varepsilon_e u^\varepsilon_\omega(x)\Big\|^2_{\mu^\varepsilon_\omega}. \tag{5.7}$$

Moreover, taking the inner product of (2.18) with u^ε_ω, we obtain

$$\lambda\|u^\varepsilon_\omega\|^2_{\mu^\varepsilon_\omega} \leqslant \lambda(u^\varepsilon_\omega, u^\varepsilon_\omega)_{\mu^\varepsilon_\omega} + (u^\varepsilon_\omega, -\mathscr{L}^\varepsilon_\omega u^\varepsilon_\omega)_{\mu^\varepsilon_\omega} = (u^\varepsilon_\omega, f^\varepsilon_\omega)_{\mu^\varepsilon_\omega} \leqslant \|u^\varepsilon_\omega\|_{\mu^\varepsilon_\omega}\|f^\varepsilon_\omega\|_{\mu^\varepsilon_\omega}\,.$$

Hence, since $f^\varepsilon_\omega \rightharpoonup f$, for any $\lambda > 0$ it holds that

$$\sup_{\varepsilon>0}\|u^\varepsilon_\omega\|^2_{\mu^\varepsilon_\omega} < \infty\,,\ \sup_{\varepsilon>0}(u^\varepsilon_\omega, -\mathscr{L}^\varepsilon_\omega u^\varepsilon_\omega)_{\mu^\varepsilon_\omega} < \infty\,,\ \sup_{\varepsilon>0, e\in\mathscr{B}}\Big\|\sqrt{\tau_{x/\varepsilon}\omega(0,e)}\nabla^\varepsilon_e u^\varepsilon_\omega(x)\Big\|_{\mu^\varepsilon_\omega} < \infty\,. \tag{5.8}$$

Lemma 5.3. *Fix $\tilde\omega \in \Omega_*$. The family $u^\varepsilon_{\tilde\omega}$ converges along a subsequence to a function $u^0 \in L^2(\mathbb{R}^d\times\Omega, dx\times\mu)$ in the sense of weak two–scale convergence and u^0 does not depend on ω, i.e. $u^0 \in L^2(\mathbb{R}^d, dx)$.*

Proof. Due to Lemma 5.2, the sequence $u^\varepsilon_{\tilde\omega}$ converges along a subsequence $\varepsilon_k \downarrow 0$ to a function $u^0 \in L^2(\mathbb{R}^d\times\Omega, dx\times\mu)$ in the sense of weak two–scale convergence. In order to simplify the notation, we suppose that this convergence holds for $\varepsilon\downarrow 0$. We need to prove that u^0 does not depend on ω. To this aim, fix $e\in\mathscr{B}$, $\varphi\in C^\infty_c(\mathbb{R}^d)$ and a function $\psi\in C(\Omega)$. We define $v(\omega, e') = \psi(\omega)\delta_{e,e'}$. Due to the definition of weak two–scale convergence, it holds

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d} u^\varepsilon_{\tilde\omega}(x)\varphi(x)\nabla^{(\omega)*}v(\tau_{x/\varepsilon}\tilde\omega)\mu^\varepsilon_{\tilde\omega}(dx) = \int_{\mathbb{R}^d}dx\int_\Omega \mu(d\omega)u^0(x,\omega)\varphi(x)\nabla^{(\omega)*}v(\omega)\,, \tag{5.9}$$

while due to Lemma 4.2

$$\int_{\mathbb{R}^d} u^\varepsilon_{\tilde\omega}(x)\varphi(x)\nabla^{(\omega)*}v(\tau_{x/\varepsilon}\tilde\omega)\mu^\varepsilon_{\tilde\omega}(dx) = -\varepsilon\int_{\mathbb{R}^d}\nabla^\varepsilon_{\omega,e}\left(u^\varepsilon_{\tilde\omega}(x)\varphi(x)\right)\psi(\tau_{x/\varepsilon}\tilde\omega)\mu^\varepsilon_{\tilde\omega}(dx)\,. \tag{5.10}$$

The r.h.s. in (5.10) is bounded by

$$\varepsilon\|\psi\|_\infty\int_{\mathbb{R}^d}\left|\nabla^\varepsilon_{\omega,e}\left(u^\varepsilon_{\tilde\omega}(x)\varphi(x)\right)\right|\mu^\varepsilon_{\tilde\omega}(dx)\leqslant I_1+I_2\,, \tag{5.11}$$

where

$$I_1=\varepsilon\|\psi\|_\infty\int_{\mathbb{R}^d}\left|\nabla^\varepsilon_{\omega,e}u^\varepsilon_{\tilde\omega}(x)\cdot\varphi(x+\varepsilon e)\right|\mu^\varepsilon_{\tilde\omega}(dx)\,,$$

$$I_2=\varepsilon\|\psi\|_\infty\int_{\mathbb{R}^d}\left|u^\varepsilon_{\tilde\omega}(x)\cdot\nabla^\varepsilon_{\omega,e}\varphi(x)\right|\mu^\varepsilon_{\tilde\omega}(dx)\,.$$

By Schwarz inequality and (5.7) we can bound

$$\begin{aligned} I_1 &\leqslant \varepsilon\|\psi\|_\infty\left[\int_{\mathbb{R}^d}\tau_{x/\varepsilon}\tilde\omega(0,e)\left(\nabla^\varepsilon_e u^\varepsilon_{\tilde\omega}(x)\right)^2\mu^\varepsilon_{\tilde\omega}(dx)\right]^{1/2}\left[\int_{\mathbb{R}^d}\tau_{x/\varepsilon}\tilde\omega(0,e)\,\varphi(x+\varepsilon e)^2\mu^\varepsilon_{\tilde\omega}(dx)\right]^{1/2}\\ &\leqslant \varepsilon\, c(\psi,\varphi)\big(u^\varepsilon_{\tilde\omega},-\mathscr{L}^\varepsilon_{\tilde\omega}u^\varepsilon_{\tilde\omega}\big)^{1/2}_{\mu^\varepsilon_{\tilde\omega}}\,, \end{aligned} \tag{5.12}$$

for a suitable positive constant $c(\psi,\varphi)$ depending on ψ and φ. Due to (5.8), we obtain that $I_1\leqslant c(\psi,\varphi,\tilde\omega)\varepsilon$.

Moreover, by Schwarz inequality we have

$$I_2\leqslant\varepsilon\|\psi\|_\infty\|u^\varepsilon_{\tilde\omega}\|_{\mu^\varepsilon_{\tilde\omega}}\left\|\nabla^\varepsilon_{\omega,e}\varphi\right\|_{\mu^\varepsilon_{\tilde\omega}}\,,$$

and again from (5.8) we deduce that $I_2\leqslant c(\psi,\varphi,\tilde\omega)\varepsilon$. Hence the r.h.s. of (5.10) is bounded by $c\varepsilon$ and due to (5.9) we get that

$$\int_{\mathbb{R}^d}dx\int_\Omega\mu(d\omega)u^0(x,\omega)\varphi(x)\nabla^{(\omega)*}v(\omega)=0\,.$$

Since this holds for all $\varphi\in C^\infty_c(\mathbb{R}^d)$ we get that

$$\int_\Omega\mu(d\omega)u^0(x,\omega)\nabla^{(\omega)*}v(\omega)=0$$

for Lebesgue a.a. $x\in\mathbb{R}^d$. Due to separability, we conclude that for Lebesgue a.a. $x\in\mathbb{R}^d$ the above identity is valid for all v of the form $v(\omega,e')=\psi(\omega)\delta_{e,e'}$, for some function $\psi\in C(\Omega)$ and some $e\in\mathscr{B}$. By Lemma 4.3 we conclude that for these points x, the function $u^0(x,\cdot)$ is constant μ–almost everywhere. This concludes the proof. □

In what follows, u^0 will be as in Lemma 5.3 for a fixed $\tilde\omega\in\Omega_*$. We will prove at the end that u^0 coincides with the solution of (2.19) and in particular that u^0 does not depend on $\tilde\omega$.

Lemma 5.4. *Fix $\tilde{\omega} \in \Omega_*$. The function u^0 belongs to the Sobolev space $H^1(\mathbb{R}^d, dx)$. Moreover, along a suitable subsequence and for all $e \in \mathscr{B}$ it holds*

$$u^\varepsilon_{\tilde{\omega}}(x) \xrightarrow{2} u^0(x)\,, \tag{5.13}$$

$$\sqrt{\tau_{x/\varepsilon}\tilde{\omega}(0,e)}\nabla^\varepsilon_e u^\varepsilon_{\tilde{\omega}}(x) \xrightarrow{2} v^0_e(x,\omega) \tag{5.14}$$

for some $v^0_e \in L^2(\mathbb{R}^d \times \Omega, dx \times \mu)$.
Given $x \in \mathbb{R}^d$, consider the forms θ_x, Γ_x in $L^2(M)$ defined as

$$\theta_x : \Omega \times \mathscr{B} \ni (\omega, e) \to v^0_e(x,\omega)/\sqrt{\omega(0,e)} \in \mathbb{R}\,, \tag{5.15}$$

$$\Gamma_x : \Omega \times \mathscr{B} \ni (\omega, e) \to \partial_e u^0(x) \in \mathbb{R}^d\,, \tag{5.16}$$

where $\partial_e u^0(x)$ denotes a representative of the weak derivative in $L^2(dx)$ of u^0, along the direction e. Then, for Lebesgue a.a. $x \in \mathbb{R}^d$, it holds

$$\theta_x \in L^2_{sol}(M)\,, \qquad \theta_x = \pi\Gamma_x\,, \tag{5.17}$$

where $\pi : L^2(M) \to L^2_{sol}(M)$ is the orthogonal projection onto $L^2_{sol}(M)$.

Note that the form θ_x is well defined, since for M–a.a. (ω, e) it holds $\omega(0,e) > 0$. Moreover, $\theta_x \in L^2(M)$ for Lebesgue a.a. $x \in \mathbb{R}^d$. In fact,

$$\|\theta_x\|^2_{L^2(M)} = \sum_{e\in\mathscr{B}} \int_\Omega \mu(d\omega)\omega(0,e)\theta_x(\omega,e)^2 = \sum_{e\in\mathscr{B}} \int_\Omega \mu(d\omega) v^0_e(x,\omega)^2$$

and $v^0_e \in L^2(\mathbb{R}^d \times \Omega, dx \times \mu)$.

Proof. (5.13) follows from Lemma 5.3. At cost to take a sub–subsequence, due to Lemma 5.2 and (5.8), (5.14) holds for all $e \in \mathscr{B}$.

Let us prove that $u^0 \in H^1(\mathbb{R}^d, dx)$. To this aim for each $j \geqslant 1$ we consider the function $\psi_j : \Omega \times \mathscr{B} \to \mathbb{R}$ introduced before Lemma 4.4 (we recall that $\psi_j \in L^2_{\mathrm{sol}}(M)$) and we take a function $\varphi \in C^\infty_c(\mathbb{R}^d)$. Then by (4.10) we can write

$$\begin{aligned} &\sum_{e\in\mathscr{B}} \int_{\mathbb{R}^d} \big(\nabla^\varepsilon_{\omega,e} u^\varepsilon_{\tilde{\omega}}(x)\big)\, \varphi(x)\, \psi_j(\tau_{x/\varepsilon}\tilde{\omega}, e)\mu^\varepsilon_{\tilde{\omega}}(dx) = \\ &\qquad \sum_{e\in\mathscr{B}} \int_{\mathbb{R}^d} \big(\nabla^\varepsilon_{\omega,e} u^\varepsilon_{\tilde{\omega}}(x)\big)\, \varphi(x+\varepsilon e)\, \psi_j(\tau_{x/\varepsilon}\tilde{\omega}, e)\mu^\varepsilon_{\tilde{\omega}}(dx) + o(1) = \\ &\qquad \sum_{e\in\mathscr{B}} \int_{\mathbb{R}^d} \nabla^\varepsilon_{\omega,e}\big(u^\varepsilon_{\tilde{\omega}}(x)\,\varphi(x)\big)\, \psi_j(\tau_{x/\varepsilon}\tilde{\omega}, e)\mu^\varepsilon_{\tilde{\omega}}(dx) - \\ &\qquad \sum_{e\in\mathscr{B}} \int_{\mathbb{R}^d} u^\varepsilon_{\tilde{\omega}}(x)\big(\nabla^\varepsilon_{\omega,e}\varphi(x)\big)\, \psi_j(\tau_{x/\varepsilon}\tilde{\omega}, e)\mu^\varepsilon_{\tilde{\omega}}(dx) + o(1)\,. \end{aligned} \tag{5.18}$$

Due to Lemma 4.2, we can rewrite the first addendum in the r.h.s. as

$$\sum_{e\in\mathcal{B}}\int_{\mathbb{R}^d}\nabla^\varepsilon_{\omega,e}\big(u^\varepsilon_{\tilde\omega}(x)\,\varphi(x)\big)\,\psi_j(\tau_{x/\varepsilon}\tilde\omega,e)\mu^\varepsilon_{\tilde\omega}(dx)= \\ -\varepsilon^{-1}\sum_{e\in\mathcal{B}}\int_{\mathbb{R}^d}u^\varepsilon_{\tilde\omega}(x)\varphi(x)\big(\nabla^{(\omega)*}\psi_j\big)(\tau_{x/\varepsilon}\tilde\omega,e)\mu^\varepsilon_{\tilde\omega}(dx)\,. \quad (5.19)$$

Since $\tilde\omega\in\Omega_*\subset\Omega_{*,j}$, the r.h.s. is zero (see (4.17)). We conclude that

$$\sum_{e\in\mathcal{B}}\int_{\mathbb{R}^d}\big(\nabla^\varepsilon_{\omega,e}u^\varepsilon_{\tilde\omega}(x)\big)\,\varphi(x)\,\psi_j(\tau_{x/\varepsilon}\tilde\omega,e)\mu^\varepsilon_{\tilde\omega}(dx)= \\ -\sum_{e\in\mathcal{B}}\int_{\mathbb{R}^d}u^\varepsilon_{\tilde\omega}(x)\partial_e\varphi(x)\,\psi_j(\tau_{x/\varepsilon}\tilde\omega,e)\tau_{x/\varepsilon}\tilde\omega(0,e)\mu^\varepsilon_{\tilde\omega}(dx)+o(1)\,. \quad (5.20)$$

We now take the limit $\varepsilon\downarrow 0$ along the suitable subsequence of the above indentity. Since

$$\nabla^\varepsilon_{\omega,e}u^\varepsilon_{\tilde\omega}(x)=\sqrt{\tau_{x/\varepsilon}\tilde\omega(0,e)}\left[\sqrt{\tau_{x/\varepsilon}\tilde\omega(0,e)}\nabla^\varepsilon_e u^\varepsilon_{\tilde\omega}(x)\right]\,, \quad (5.21)$$

the above identity and Lemma 5.1 imply that

$$\begin{aligned}&\sum_{e\in\mathcal{B}}\int_{\mathbb{R}^d}dx\int_\Omega\mu(d\omega)\sqrt{\omega(0,e)}v^0_e(x,\omega)\varphi(x)\psi_j(\omega,e)=\\ &-\sum_{e\in\mathcal{B}}\int_{\mathbb{R}^d}dx\,u^0(x)\partial_e\varphi(x)\int_\Omega\mu(d\omega)\psi_j(\omega,e)\omega(0,e)=\\ &-\sum_{e\in\mathcal{B}_*}\int_{\mathbb{R}^d}dx\,u^0(x)\partial_e\varphi(x)\int_\Omega\mu(d\omega)\left[\psi_j(\omega,e)\omega(0,e)-\psi_j(\omega,-e)\omega(0,-e)\right]\,.\end{aligned} \quad (5.22)$$

Given $\psi\in L^2_{\rm sol}$, we define $a(\psi)\in\mathbb{R}^d$ as

$$a(\psi)_e=\int_\Omega\mu(d\omega)\left[\psi(\omega,e)\omega(0,e)-\psi(\omega,-e)\omega(0,-e)\right]\,,\qquad e\in\mathcal{B}_*\,.$$

Due to Lemma 4.5, $\left\{a(\psi)\,:\,\psi\in L^2_{\rm sol}(M)\right\}=\mathbb{R}^{\mathcal{B}_*}$. On the other hand, since the map $L^2_{\rm sol}\ni\psi\to a(\psi)\in\mathbb{R}^{\mathcal{B}_*}$ is continuous, we conclude that $\{a(\psi_j)\}_{j\geqslant 1}$ is dense in $\mathbb{R}^{\mathcal{B}_*}$. Due to (5.22),

$$-\sum_{e\in\mathcal{B}_*}a(\psi_j)_e\int_{\mathbb{R}^d}dx\,u^0(x)\partial_e\varphi(x)=\sum_{e\in\mathcal{B}_*}\int_{\mathbb{R}^d}dx\,\varphi(x)f_e(\psi_j,x) \quad (5.23)$$

where

$$f_e(\psi_j,\cdot)=\int_\Omega\mu(d\omega)\Big(\sqrt{\omega(0,e)}v^0_e(\cdot,\omega)\psi_j(\omega,e)+\sqrt{\omega(0,-e)}v^0_{-e}(\cdot,\omega)\psi_j(\omega,-e)\Big)\in L^2(\mathbb{R}^d)\,. \quad (5.24)$$

As consequence of the density of $\{a(\psi_j)\}_{j\geqslant 1}$ in $\mathbb{R}^{\mathcal{B}_*}$, (5.23) and (5.24), it must be $u^0\in H^1(\mathbb{R}^d,dx)$.

Let us now prove (5.17). Since $u^0 \in H^1(\mathbb{R}^d, dx)$, we are allowed to rewrite the first identity in (5.22) as

$$\sum_{e\in\mathcal{B}} \int_{\mathbb{R}^d} dx\, \varphi(x) \int_\Omega \mu(d\omega) \left(\sqrt{\omega(0,e)} v_e^0(x,\omega) - \omega(0,e)\partial_e u^0(x) \right) \psi_j(\omega,e) = 0\,. \tag{5.25}$$

Then, by means of the arbitrariness of φ and separability arguments, we get that for Lebesgue a.a. $x \in \mathbb{R}^d$

$$\sum_{e\in\mathcal{B}} \int_\Omega \mu(d\omega) \left(\sqrt{\omega(0,e)} v_e^0(x,\omega) - \omega(0,e)\partial_e u^0(x) \right) \psi_j(\omega,e) =$$
$$\sum_{e\in\mathcal{B}} \int_\Omega \mu(d\omega)\omega(0,e) \left(\frac{v_e^0(x,\omega)}{\sqrt{\omega(0,e)}} - \partial_e u^0(x) \right) \psi_j(\omega,e) = 0\,, \qquad \forall j \geqslant 1$$

(with the notational convention, followed also below, that $v_e^0(x,\omega)/\sqrt{\omega(0,e)} := 0$ if $\omega(0,e) = 0$). Hence for Lebesgue a.a. $x \in \mathbb{R}^d$ the form

$$(\omega, e) \to \frac{v_e^0(x,\omega)}{\sqrt{\omega(0,e)}} - \partial_e u^0(x) \tag{5.26}$$

belongs to $L^2_{\rm pot}$. Now fix $\varphi \in C_c^\infty$ and a function $\psi \in C(\Omega)$. Note that, if $\tau_{x/\varepsilon}\tilde\omega(0,e) > 0$, then

$$\varepsilon \nabla_e^\varepsilon \psi(\tau_{x/\varepsilon}\tilde\omega) = \left[\nabla_e^{(\omega)}\psi\right](\tau_{x/\varepsilon}\tilde\omega)\,, \qquad x \in \varepsilon\mathcal{C}(\tilde\omega)\,.$$

Therefore we can write

$$\varepsilon \nabla_e^\varepsilon \left(\varphi(x)\psi(\tau_{x/\varepsilon}\tilde\omega)\right) = \varepsilon \left(\nabla_e^\varepsilon \varphi(x)\right) \psi(\tau_{x/\varepsilon+e}\tilde\omega) + \varphi(x)\left[\nabla_e^{(\omega)}\psi\right](\tau_{x/\varepsilon}\tilde\omega)\,.$$

Due to the above identity and (5.6), taking the inner product of (2.18) with $\varepsilon\varphi(x)\psi(\tau_{x/\varepsilon}\tilde\omega)$, we obtain

$$\varepsilon\lambda \int_{\mathbb{R}^d} u^\varepsilon_{\tilde\omega}(x)\varphi(x)\psi(\tau_{x/\varepsilon}\tilde\omega)\mu^\varepsilon_{\tilde\omega}(dx) +$$
$$\frac{1}{2}\sum_{e\in\mathcal{B}} \varepsilon \int_{\mathbb{R}^d} \left(\tau_{x/\varepsilon}\tilde\omega\right)(0,e)\nabla_e^\varepsilon u^\varepsilon_{\tilde\omega}(x)\, \nabla_e^\varepsilon\varphi(x)\, \psi(\tau_{x/\varepsilon+e}\tilde\omega)\mu^\varepsilon_{\tilde\omega}(dx) +$$
$$\frac{1}{2}\sum_{e\in\mathcal{B}} \int_{\mathbb{R}^d} \left(\tau_{x/\varepsilon}\tilde\omega\right)(0,e)\nabla_e^\varepsilon u^\varepsilon_{\tilde\omega}(x)\, \varphi(x)\left[\nabla_e^{(\omega)}\psi\right](\tau_{x/\varepsilon}\tilde\omega)\mu^\varepsilon_{\tilde\omega}(dx) =$$
$$\varepsilon \int_{\mathbb{R}^d} f^\varepsilon_{\tilde\omega}(x)\varphi(x)\psi(\tau_{x/\varepsilon}\tilde\omega)\mu^\varepsilon_{\tilde\omega}(dx)\,.$$

We note that due to (5.8) all terms but the third one in the l.h.s. are negligible as $\varepsilon \downarrow 0$ along the subsequence satisfying (5.13) and (5.14). Hence, by definition of weak two–scale limit and the trivial identity (5.21), we conclude that

$$\sum_{e\in\mathcal{B}} \int_{\mathbb{R}^d} dx \int_\Omega \mu(d\omega)\sqrt{\omega(0,e)}v_e^0(x,\omega)\varphi(x)\nabla_e^{(\omega)}\psi(\omega) = 0\,.$$

The above identity can be rewritten as

$$\sum_{e\in\mathscr{B}}\int_{\mathbb{R}^d} dx\,\varphi(x)\int_\Omega \mu(d\omega)\omega(0,e)\frac{v_e^0(x,\omega)}{\sqrt{\omega(0,e)}}\nabla_e^{(\omega)}\psi(\omega)=0\,.$$

Due to the arbitrariness of the test functions φ, we get that for Lebesgue a.a. $x\in\mathbb{R}^d$ it holds

$$\sum_{e\in\mathscr{B}}\int_\Omega \mu(d\omega)\omega(0,e)\frac{v_e^0(x,\omega)}{\sqrt{\omega(0,e)}}\nabla_e^{(\omega)}\psi(\omega)=0\,.$$

By a separability argument, this implies that $\theta_x\in L^2_{\rm sol}(M)$ for Lebesgue a.a. $x\in\mathbb{R}^d$. Since we know that the form (5.26) belongs to $L^2_{\rm pot}(M)$, this concludes the proof of (5.17).

□

6 Proof of Theorem 2.4

We start with a technical result, which could be proven in much more generality:

Lemma 6.1. *Fix $\omega\in\Omega_*$. Let $h\in C(\mathbb{R}^d)$ satisfy*

$$|h(x)|\leqslant\frac{c}{1+|x|^{d+1}}\,,\qquad \forall x\in\mathbb{R}^d\,,\tag{6.1}$$

and suppose that

$$L^2(\mu_\omega^\varepsilon)\ni h_\omega^\varepsilon\to h\in L^2(m dx)\,.\tag{6.2}$$

Then

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}\big|h_\omega^\varepsilon(x)-h(x)\big|^2\mu_\omega^\varepsilon(dx)=0\,.\tag{6.3}$$

Proof. Trivially, it is enough to prove the following limits

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}h(x)^2\mu_\omega^\varepsilon(dx)=m\int h(x)^2dx\,,\tag{6.4}$$

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}h_\omega^\varepsilon(x)h(x)\mu_\omega^\varepsilon(dx)=m\int h(x)^2dx\,,\tag{6.5}$$

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}h_\omega^\varepsilon(x)^2\mu_\omega^\varepsilon(dx)=m\int h(x)^2dx\,.\tag{6.6}$$

Since $h\in L^2(\mu_\omega^\varepsilon)$, the integrals in the l.h.s. of (6.4) and (6.5) are meaningful. Moreover, observe that for each $\ell>0$ one can find a function $g_\ell\in C_c(\mathbb{R}^d)$ such that $h(x)=g_\ell(x)$ for any $x\in\mathbb{R}^d$ with $|x|\leqslant\ell$, and $|g_\ell(x)|\leqslant c/(1+|x|^{d+1})$.

In order to prove (6.4) we observe that

$$\left|\int_{\mathbb{R}^d}h(x)^2\mu_\omega^\varepsilon(dx)-\int_{\mathbb{R}^d}g_\ell(x)^2\mu_\omega^\varepsilon(dx)\right|\leqslant 2\varepsilon^d\sum_{x\in\varepsilon\mathbb{Z}^d\,:\,|x|>\ell}\frac{c}{1+|x|^{d+1}}\leqslant c(\ell)\,,\tag{6.7}$$

$$\left|\int_{\mathbb{R}^d}h(x)^2 m dx-\int_{\mathbb{R}^d}g_\ell(x)^2 m dx\right|\leqslant 2\int_{\{x\in\mathbb{R}^d\,:\,|x|>\ell\}}\frac{c}{1+|x|^{d+1}}m dx\leqslant c(\ell)\,,\tag{6.8}$$

for a positive constant $c(\ell)$ going to 0 as $\ell \uparrow \infty$. The above estimates (6.7) and (6.8), and the limit

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d} g_\ell(x)^2\mu^\varepsilon_\omega(dx) = m\int g_\ell(x)^2 dx$$

(due to the definition of Ω_*) allow to derive (6.4) by taking the limit $\ell \uparrow \infty$.

In order to prove (6.5) we observe that

$$\left|\int_{\mathbb{R}^d} h^\varepsilon_\omega(x)h(x)\mu^\varepsilon_\omega(dx) - \int_{\mathbb{R}^d} h^\varepsilon_\omega(x)g_\ell(x)\mu^\varepsilon_\omega(dx)\right| \leqslant$$
$$\|h^\varepsilon_\omega\|_{\mu^\varepsilon_\omega}\|h-g_\ell\|_{\mu^\varepsilon_\omega} \leqslant c(\omega)\left(2\varepsilon^d\sum_{x\in\varepsilon\mathbb{Z}^d\,:\,|x|>\ell}\frac{c^2}{(1+|x|^{d+1})^2}\right)^{\frac{1}{2}} \leqslant c(\omega)c(\ell)\,, \quad (6.9)$$

and

$$\left|\int_{\mathbb{R}^d} h^2(x)m dx - \int_{\mathbb{R}^d} h(x)g_\ell(x)m dx\right| \;\leqslant\; 2\|h\|_\infty\int_{\{x\in\mathbb{R}^d\,:\,|x|>\ell\}}\frac{c}{1+|x|^{d+1}}m dx \;\leqslant\; c(\ell)\,, \quad (6.10)$$

for a positive constant $c(\ell)$ going to 0 as $\ell \uparrow \infty$. Since $h^\varepsilon_\omega \to h$ and $g_\ell \in C_c(\mathbb{R}^d)$ we can conclude that

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d} h^\varepsilon_\omega(x)g_\ell(x)\mu^\varepsilon_\omega(dx) = m\int_{\mathbb{R}^d} h(x)g_\ell(x)dx\,.$$

The above limit together with (6.9) and (6.10) implies (6.5).

Finally we observe that (6.6) follows by applying (2.14) in the definition of strong convergence with test functions $\varphi^\varepsilon := h^\varepsilon_\omega$, $\varphi := h$.

□

We have now all the main tools in order to prove Theorem 2.4. We take $\omega \in \Omega_*$, define u^0, v^0_e as in Lemma 5.4 and assume that $f^\varepsilon_\omega \rightharpoonup f_\omega$. We want to prove that u^0 solves equation (2.19) and that (2.20) holds.

First we observe that the weak two–scale convergence (5.13) implies the weak convergence

$$L^2(\mu^\varepsilon_\omega) \ni u^\varepsilon_\omega \rightharpoonup u^0 \in L^2(m dx) \quad (6.11)$$

as $\varepsilon \downarrow 0$ along the subsequence of Lemma 5.4. Taking the inner product of (2.18) with a test function $\varphi \in C^\infty_c(\mathbb{R}^d)$ and using (5.6), we get the identity

$$\lambda\int_{\mathbb{R}^d} u^\varepsilon_\omega(x)\varphi(x)\mu^\varepsilon_\omega(dx) + \frac{1}{2}\sum_{e\in\mathscr{B}}\int_{\mathbb{R}^d}\tau_{x/\varepsilon}\omega(0,e)\nabla^\varepsilon_e u(x)\nabla^\varepsilon_e\varphi(x)\mu^\varepsilon_\omega(dx) =$$
$$\int_{\mathbb{R}^d} f^\varepsilon_\omega(x)\varphi(x)\mu^\varepsilon_\omega(dx)\,. \quad (6.12)$$

By taking the limit $\varepsilon \downarrow 0$ (along the subsequence of Lemma 5.4) and then dividing by m, from the trivial identity (5.21), the limit (5.14) in Lemma 5.4, the limit (6.11) and the hypothesis $L^2(\mu^\varepsilon_\omega) \ni f^\varepsilon_\omega \rightharpoonup f \in L^2(mdx)$ we get

$$\lambda \int_{\mathbb{R}^d} u^0(x)\varphi(x)dx + \frac{1}{2m}\sum_{e\in\mathscr{B}}\int_{\mathbb{R}^d} dx\,\partial_e\varphi(x)\int_\Omega \mu(d\omega')\sqrt{\omega'(0,e)}v^0_e(x,\omega') = \int_{\mathbb{R}^d} f(x)\varphi(x)dx\,. \quad (6.13)$$

The second member in (6.13) can be rewritten as

$$\frac{1}{2m}\sum_{e\in\mathscr{B}_*}\int_{\mathbb{R}^d} dx\,\partial_e\varphi(x)\int_\Omega \mu(d\omega')\left[\omega'(0,e)\theta_x(\omega',e) - \omega'(0,-e)\theta_x(\omega',-e)\right]\,. \quad (6.14)$$

Due to Lemma 5.4, $u^0 \in H^1(\mathbb{R}^d, dx)$. Given $x \in \mathbb{R}^d$ we consider the gradients

$$\zeta(x) := \nabla\varphi(x) = \big(\partial_e\varphi(x)\big)_{e\in\mathscr{B}_*}\,, \qquad \xi(x) := \nabla u^0(x) = \big(\partial_e u^0(x)\big)_{e\in\mathscr{B}_*}$$

(the definition is well posed for Lebesgue a.a. $x \in \mathbb{R}^d$, since $u^0 \in H^1(\mathbb{R}^d, dx)$). Due to Lemma 5.4, we know that for Lebesgue a.a. $x \in \mathbb{R}^d$ the form θ_x defined in (5.15) coincides with the form $\pi w^{\xi(x)}$ (recall definition (4.19)). Therefore, due to (4.20), (4.21) and (4.27), we can rewrite (6.14) as

$$\frac{1}{2m}(w^{\zeta(x)}, \pi w^{\xi(x)})_{L^2(M)} = \frac{1}{2m}(\zeta(x), D\xi(x)) = (\zeta(x), \mathscr{D}\xi(x)) = \big(\nabla\varphi(x), \mathscr{D}\nabla u^0(x)\big)\,. \quad (6.15)$$

In conclusion, (6.13) reads

$$\lambda\int_{\mathbb{R}^d} u^0(x)\varphi(x)dx + \int_{\mathbb{R}^d}\big(\nabla u^0(x), \mathscr{D}\nabla\varphi(x)\big)dx = \int_{\mathbb{R}^d} f(x)\varphi(x)dx\,. \quad (6.16)$$

Hence, the function u^0 of Lemma 5.4 is the solution of equation (2.19), which is unique (in particular u^0 does not depend from $\omega \in \Omega_*$). Due to Lemma 5.2 it is simple to verify that for each sequence $\varepsilon_k \downarrow 0$ one can extract a sub–subsequence ε_{k_n} satisfying Lemma 5.4. Hence, by the previous results, we conclude that for each sequence $\varepsilon_k \downarrow 0$ one can extract a sub–subsequence ε_{k_n} such that

$$L^2(\mu^{\varepsilon_{k_n}}_\omega) \ni u^{\varepsilon_{k_n}}_\omega \rightharpoonup u^0 \in L^2(mdx)\,,$$

thus implying that the functions $u^\varepsilon_\omega \in L^2(\mu^\varepsilon_\omega)$ weakly converge to $u^0 \in L^2(mdx)$. This concludes the proof of point (i).

In order to prove the strong convergence of $u^\varepsilon_\omega \in L^2(\mu^\varepsilon_\omega)$ to $u^0 \in L^2(mdx)$ in point (ii) one can proceed as in [ZP][Proof of Theorem 6.1]. We give the proof for the reader's convenience. Due to Lemma 2.3 we only need to prove that

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d} u^\varepsilon_\omega(x)^2\mu^\varepsilon_\omega(dx) = m\int_{\mathbb{R}^d} u^0(x)^2 dx\,. \quad (6.17)$$

To this aim, we define v^ε_ω as the solution in $L^2(\mu^\varepsilon_\omega)$ of the equation

$$\lambda v^\varepsilon_\omega - \mathscr{L}^\varepsilon_\omega v^\varepsilon_\omega = u^\varepsilon_\omega\,. \quad (6.18)$$

As already proven, $L^2(\mu^\varepsilon_\omega) \ni u^\varepsilon_\omega \rightharpoonup u^0 \in L^2(mdx)$. Hence, by applying point (i) of Theorem 2.4, we can conclude that

$$L^2(\mu^\varepsilon_\omega) \ni v^\varepsilon_\omega \rightharpoonup v \in L^2(mdx), \tag{6.19}$$

where $v \in L^2(mdx)$ solves the equation

$$\lambda v - \nabla \cdot (\mathscr{D} \nabla v) = u^0 . \tag{6.20}$$

By taking the inner product of (2.18) with v^ε_ω and then subtracting the identity obtained by taking the inner product of (6.18) with u^ε_ω, one obtains that

$$(v^\varepsilon_\omega, f^\varepsilon_\omega)_{\mu^\varepsilon_\omega} = (u^\varepsilon_\omega, u^\varepsilon_\omega)_{\mu^\varepsilon_\omega} . \tag{6.21}$$

Similarly, by taking the inner product of (2.19) with v and then subtracting the identity obtained by taking the inner product of (6.20) with u^0 one obtains that

$$\int_{\mathbb{R}^d} v(x) f(x) dx = \int_{\mathbb{R}^d} u^0(x)^2 dx . \tag{6.22}$$

Since by assumption $L^2(\mu^\varepsilon_\omega) \ni f^\varepsilon_\omega \to f \in L^2(mdx)$, from (6.19) and the definition of strong convergence we derive that

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} v^\varepsilon_\omega(x) f^\varepsilon_\omega(x) \mu^\varepsilon_\omega(dx) = m \int_{\mathbb{R}^d} v(x) f(x) dx . \tag{6.23}$$

Due to (6.21) and (6.22), the above limit is equivalent to (6.17). As already mentioned, this limit and Lemma 2.3 imply (2.21).

We finally prove point (iii). Let $f \in C_c(\mathbb{R}^d)$ and define f^ε_ω as the function f restricted on $\varepsilon \mathscr{C}(\omega)$. Then, due to (2.17), $L^2(\mu^\varepsilon_\omega) \ni f^\varepsilon_\omega \to f \in L^2(mdx)$. Due to point (ii) proven above, we know that $L^2(\mu^\varepsilon_\omega) \ni u^\varepsilon_\omega \to u^0 \in L^2(mdx)$. Since the function u^0 solves (2.19) with $f \in C_c(\mathbb{R}^d)$, u^0 is continuous and decays fast at infinity (see Exercise 3.13 in Section III.3 of [RW]). In order to conclude it is enough to apply Lemma 6.1.

7 Proof of Corollary 2.5

Due to a generalization of the Trotter–Kato Theorem [ZP][Theorem 9.2], [P][Theorem 1.4], Theorem 2.4 (ii) implies for each $\omega \in \Omega_*$ that

$$L^2(\mu^\varepsilon_\omega) \ni P^\varepsilon_{t,\omega} f^\varepsilon_\omega \to P_t f \in L^2(mdx), \tag{7.1}$$

whenever $L^2(\mu^\varepsilon_\omega) \ni f^\varepsilon_\omega \to f \in L^2(mdx)$. Since, it holds $L^2(\mu^\varepsilon_\omega) \ni f \to f \in L^2(mdx)$ for each $f \in C_c(\mathbb{R}^d)$ and each $\omega \in \Omega_*$, (7.1) is verified by setting $f^\varepsilon_\omega := f$. This fact and Lemma 6.1 allow to derive (2.24).

In order to conclude we only need to derive (2.25) from (2.24). To this aim, let $\Lambda_\ell := [-\ell, \ell]^d$, $\ell > 0$. We claim that, for any $\omega \in \Omega_*$, given any $f \in C_c(\mathbb{R}^d)$ it holds

$$\lim_{\varepsilon \downarrow 0} \int_{\Lambda_\ell^c} P^\varepsilon_{t,\omega} f(x) \mu^\varepsilon_\omega(dx) = \int_{\Lambda_\ell^c} P_t f(x) m dx . \tag{7.2}$$

Without loss of generality we can assume that $f \geqslant 0$.
Since

$$P\left[X(t\varepsilon^{-2}|x)=z\right]=P\left[X(t\varepsilon^{-2}|z)=x\right] \qquad \forall t \geqslant 0,\ \forall x,z \in \mathscr{C}(\omega),$$

we can write

$$\int_{\mathbb{R}^d} P^{\varepsilon}_{t,\omega} f(x)\mu^{\varepsilon}_{\omega}(dx)=\varepsilon^d \sum_{x\in\mathscr{C}(\omega)}\sum_{z\in\mathscr{C}(\omega)} f(\varepsilon z)P(X(t\varepsilon^{-2}|x)=z)=$$

$$\varepsilon^d \sum_{x\in\mathscr{C}(\omega)}\sum_{z\in\mathscr{C}(\omega)} f(\varepsilon z)P(X(t\varepsilon^{-2}|z)=x)=\varepsilon^d \sum_{z\in\mathscr{C}(\omega)} f(\varepsilon z)\to m\int_{\mathbb{R}^d} f(z)dz\,. \quad (7.3)$$

The above limit and the identity $\int_{\mathbb{R}^d} P_t f(z)dz = \int_{\mathbb{R}^d} f(z)dz$, following from the symmetry of P_t, implies (7.2) with Λ^c_ℓ replaced by $\mathbb{R}^d$. Therefore, in order to prove (7.2) it is enough to show that

$$\lim_{\varepsilon\downarrow 0}\int_{\Lambda_\ell} P^{\varepsilon}_{t,\omega} f(x)\mu^{\varepsilon}_{\omega}(dx)=\int_{\Lambda_\ell} P_t f(x) m dx\,. \quad (7.4)$$

To this aim we apply Schwarz inequality and obtain the bounds

$$\left|\int_{\Lambda_\ell} P^{\varepsilon}_{t,\omega} f(x)\mu^{\varepsilon}_{\omega}(dx)-\int_{\Lambda_\ell} P_t f(x)\mu^{\varepsilon}_{\omega}(dx)\right| \leqslant \int_{\Lambda_\ell}\left|P^{\varepsilon}_{t,\omega} f(x)-P_t f(x)\right|\mu^{\varepsilon}_{\omega}(dx)$$

$$\leqslant \mu^{\varepsilon}_{\omega}(\Lambda_\ell)^{1/2}\left(\int_{\Lambda_\ell}\left|P^{\varepsilon}_{t,\omega} f(x)-P_t f(x)\right|^2\mu^{\varepsilon}_{\omega}(dx)\right)^{1/2}. \quad (7.5)$$

Since by (2.16) $\mu^{\varepsilon}_{\omega}(\Lambda_\ell)\to m(2\ell)^d$ for each $\omega\in\Omega_*$, the above upper bound and (2.24) imply that the first member in (7.5) goes to 0 as $\varepsilon\downarrow 0$ for each $\omega\in\Omega_*$. To conclude the proof of (7.4) it is enough to observe that for each $\omega\in\Omega_*$ the integral $\int_{\Lambda_\ell} P_t f(x)\mu^{\varepsilon}_{\omega}(dx)$ converges to $m\int_{\Lambda_\ell} P_t f(x)dx$ since $P_t f$ is a regular function fast decaying to infinity (the proof follows the same arguments used in order to check (6.4)). This concludes the proof of (7.2).

Let us come back to (2.25). For each $\ell>0$ we can bound

$$\int_{\mathbb{R}^d}\left|P^{\varepsilon}_{t,\omega} f(x)-P_t f(x)\right|\mu^{\varepsilon}_{\omega}(dx)\leqslant$$

$$\int_{\Lambda_\ell}\left|P^{\varepsilon}_{t,\omega} f(x)-P_t f(x)\right|\mu^{\varepsilon}_{\omega}(dx)+\int_{\Lambda^c_\ell} P^{\varepsilon}_{t,\omega} f(x)\mu^{\varepsilon}_{\omega}(dx)+\int_{\Lambda^c_\ell} P_t f(x)\mu^{\varepsilon}_{\omega}(dx)\leqslant$$

$$\mu^{\varepsilon}_{\omega}(\Lambda_\ell)^{1/2}\left(\int_{\Lambda_\ell}\left|P^{\varepsilon}_{t,\omega} f(x)-P_t f(x)\right|^2\mu^{\varepsilon}_{\omega}(dx)\right)^{1/2}+\int_{\Lambda^c_\ell} P^{\varepsilon}_{t,\omega} f(x)\mu^{\varepsilon}_{\omega}(dx)+\int_{\Lambda^c_\ell} P_t f(x)\mu^{\varepsilon}_{\omega}(dx)\,. \quad (7.6)$$

Due (2.24) and (7.2), by taking $\varepsilon\downarrow 0$ we get that for each $\omega\in\Omega_*$

$$\limsup_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}\left|P^{\varepsilon}_{t,\omega} f(x)-P_t f(x)\right|\mu^{\varepsilon}_{\omega}(dx)\leqslant 2m\int_{\Lambda^c_\ell} P_t f(x)dx\,.$$

By the arbitrariness of ℓ in the above estimate one derives (2.25).

A Proof of Lemma 2.1

Let us suppose that $\hat{\omega}_c$, $c > 0$, stochastically dominates a supercritical Bernoulli bond percolation and prove that hypotheses (H2) and (H3) are satisfied. We call $\mathbb{P}$ the law of $\hat{\omega}_c$ on $\{0,1\}^{\mathbb{E}_d}$. Due to Strassen Theorem, there exists a probability measure $\mathscr{P}$ on the product space $\mathbb{X} := \{0,1\}^{\mathbb{E}_d} \times \{0,1\}^{\mathbb{E}_d}$ such that (i) $\omega_1(b) \geqslant \omega_2(b)$ for each $b \in \mathbb{E}_d$, for $\mathscr{P}$ almost all $(\omega_1, \omega_2) \in \mathbb{X}$, (ii) the marginal law of ω_1 is $\mathbb{P}$ and (iii) the marginal law of ω_2 is a Bernoulli bond percolation with supercritical parameter $p > p_c$. It is well known (see [G]) that ω_2 has a.s. a unique infinite cluster whose complement has only connected components of finite cardinality. This implies the same property for the random field ω_1, thus assuring that the random field ω fulfills hypothesis (H2).

Let us now consider hypothesis (H3). We recall the variational characterization of $\mathscr{D}$:

$$(a, \mathscr{D}a) = \frac{1}{m} \inf_{\psi \in B(\Omega)} \left\{ \sum_{e \in \mathscr{B}_*} \int_\Omega \omega(0,e)(a_e + \psi(\tau_e \omega) - \psi(\omega))^2 \mathbb{I}_{0,e \in \mathscr{C}(\omega)} \mathbb{Q}(d\omega) \right\} . \tag{A.1}$$

Since $\omega(0,e) \geqslant c\, \hat{\omega}_c(0,e)$, we obtain that

$$(a, \mathscr{D}a) \geqslant \frac{c}{m} \inf_{\psi \in B(\Omega)} \left\{ \sum_{e \in \mathscr{B}_*} \int_\Omega \hat{\omega}_c(0,e)(a_e + \psi(\tau_e \omega) - \psi(\omega))^2 \mathbb{I}_{0,e \in \mathscr{C}(\hat{\omega}_c)} \mathbb{Q}(d\omega) \right\} , \tag{A.2}$$

where now $\mathscr{C}(\hat{\omega}_c)$ denotes the unique infinite cluster of $\hat{\omega}_c$ (due to the previous observations, the definition is well posed a.s.). Given $\psi \in B(\Omega)$ we can write $\psi = f + g$ where $f = \mathbb{E}(\psi|\mathscr{F})$ and $g = \psi - \mathbb{E}(\psi|\mathscr{F})$, $\mathbb{E}$ being the expectation w.r.t. $\mathbb{Q}$ and $\mathscr{F}$ being the σ–algebra generated by the random variables $\hat{\omega}_c(b)$, $b \in \mathbb{E}^d$. Since $\mathbb{E}(g|\mathscr{F}) = 0$, it is simple to check that

$$\begin{aligned} &\int_\Omega \hat{\omega}_c(0,e)(a_e + \psi(\tau_e \omega) - \psi(\omega))^2 \mathbb{I}_{0,e \in \mathscr{C}(\hat{\omega}_c)} \mathbb{Q}(d\omega) = \\ &\qquad \int_\Omega \hat{\omega}_c(0,e) \{ [a_e + f(\tau_e \omega) - f(\omega)]^2 + [g(\tau_e \omega) - g(\omega)]^2 \} \mathbb{I}_{0,e \in \mathscr{C}(\hat{\omega}_c)} \mathbb{Q}(d\omega) . \end{aligned} \tag{A.3}$$

This shows that the infimum in the r.h.s. of (A.2) is realized by $\mathscr{F}$–measurable functions. Therefore,

$$\begin{aligned} &\text{r.h.s. of (A.2)} = \frac{c}{m} \inf_{\substack{\psi \in B(\Omega): \\ \psi = \psi(\hat{\omega}_c)}} \left\{ \sum_{e \in \mathscr{B}_*} \int_\Omega \hat{\omega}_c(0,e)(a_e + \psi(\tau_e \omega) - \psi(\omega))^2 \mathbb{I}_{0,e \in \mathscr{C}(\hat{\omega}_c)} \mathbb{Q}(d\omega) \right\} = \\ &\frac{c}{m} \inf_{\substack{\psi \in B(\mathbb{X}) \\ \psi = \psi(\omega_1)}} \left\{ \sum_{e \in \mathscr{B}_*} \int_\mathbb{X} \omega_1(0,e)(a_e + \psi(\tau_e \omega_1) - \psi(\omega_1))^2 \mathbb{I}_{0,e \in \mathscr{C}(\omega_1)} \mathscr{P}(d\omega_1, d\omega_2) \right\} \geqslant \\ &\frac{c}{m} \inf_{\psi \in (\mathbb{X})} \left\{ \sum_{e \in \mathscr{B}_*} \int_\mathbb{X} \omega_1(0,e)(a_e + \psi(\tau_e(\omega_1, \omega_2)) - \psi(\omega_1, \omega_2))^2 \mathbb{I}_{0,e \in \mathscr{C}(\omega_1)} \mathscr{P}(d\omega_1, d\omega_2) \right\} . \end{aligned} \tag{A.4}$$

Above we have used that the law $\hat{\omega}_c$ and the marginal law of ω_1 coincide. Moreover, we recall that $B(\cdot)$ denotes the space of bounded Borel functions on the given topological space.

A this point, since $\omega_1(0,e) \geqslant \omega_2(0,e)$ and $\mathbb{I}_{0,e\in\mathscr{C}(\omega_1)} \geqslant \mathbb{I}_{0,e\in\mathscr{C}(\omega_2)}$ $\mathscr{P}$–a.s., we can obtain another lower bound by substituting in the last expectation $\omega_1(0,e)$ and $\mathscr{C}(\omega_1)$ with $\omega_2(0,e)$ and $\mathscr{C}(\omega_2)$ respectively. By taking the conditional expectation w.r.t. to the σ–algebra generated by ω_2 and using the same arguments as above, we derive that the last expression in (A.4) is bounded from below by

$$\frac{c}{m} \inf_{\substack{\psi\in B(\mathbb{X}):\\ \psi=\psi(\omega_2)}} \left\{ \sum_{e\in\mathscr{B}_*} \int_{\mathbb{X}} \omega_2(0,e)(a_e+\psi(\tau_e\omega_2)-\psi(\omega_2))^2 \mathbb{I}_{0,e\in\mathscr{C}(\omega_2)} \mathscr{P}(d\omega_1, d\omega_2) \right\} =$$

$$\frac{c}{m} \inf_{\psi\in B(\{0,1\}^{\mathbb{E}_d})} \left\{ \sum_{e\in\mathscr{B}_*} \int_{\{0,1\}^{\mathbb{E}_d}} \omega(0,e)(a_e+\psi(\tau_e\omega)-\psi(\omega))^2 \mathbb{I}_{0,e\in\mathscr{C}(\omega)} \mathbb{P}_p(d\omega) \right\}, \quad \text{(A.5)}$$

where $\mathbb{P}_p$ is the Bernoulli bond percolation with parameter $p > p_c$. In order to prove that the diffusion matrix $\mathscr{D}$ is positive defined, we only need to show that the r.h.s. of (A.5) is positive for $a \neq 0$. We point out that, apart multiplicative factors, the last infimum in (A.5) equals $(a, D_p a)$, D_p being the diffusion matrix of the simple random walk on the supercritical infinite cluster. One only needs to prove the positivity of D_p. This result has been proven in [DFGW][pages 828–838] in any dimension for $p > 1/2$ (see in particular Remark 4.16 in [DFGW][page 837]). There the authors are able to bound from below the diffusion matrix by means of the effective conductivities of suitable resistor networks (this reduction works without any restriction on p). The positivity of the effective conductivities is then derived by applying percolation results originally proven for $p > 1/2$. These results have been improved (see[GM][page 454] and references therein) and the improvement allows to extend the positive bound on the effective conductivities to all $p > p_c$. Other derivations of the positivity of D_p can be found in [SS], [BB] and [MP].

If $\hat{\omega}$ is a Bernoulli bond percolation with parameter $p > p_c$, then for each $c > 0$ the random field $\hat{\omega}_c$ is a Bernoulli bond percolation with parameter $p(c)$ such that $\lim_{c\downarrow 0} p(c) = p$. Hence, taking $c > 0$ small enough, we obtain that hypotheses (H2) and (H3) are satisfied.

The last statement regarding the cases of $\mathscr{D}$ diagonal or multiple of the identity can be proved by the same arguments used in the proof of Theorem 4.6 (iii) in [DFGW].

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