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FINITE WIDTH FOR A RANDOM STATIONARY INTERFACE

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Abstract: We study the asymptotic shape of the solution $u(t,x) \in [0,1]$ to a one-dimensional heat equation with a multiplicative white noise term. At time zero the solution is an interface, that is $u(0,x)$ is 0 for all large positive x and $u(0,x)$ is 1 for all large negative x. The special form of the noise term preserves this property at all times $t \geq 0$. The main result is that, in contrast to the deterministic heat equation, the width of the interface remains stochastically bounded.

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1. Introduction

Our goal is to study the shape of the wavefront for the following stochastic partial differential equation (SPDE)

$$
\begin{aligned}
(1.1)\qquad u_t &= \frac{1}{2}u_{xx} + |u(1-u)|^{1/2}\dot{W} \quad \text{for } (x,t) \in \mathbf{R} \times [0,\infty) \\
u(0,x) &= u_0(x).
\end{aligned}
$$

Here we write u_t, u_x, u_{xx} for the partial derivatives of the function $u(t,x)$. We shall also write $u(t)$ as shorthand for the function $u(t,x)$. The noise $\dot{W} = \dot{W}(t,x)$ is 2-parameter white noise. We interpret (1.1) in terms of the integral equation

(1.2)
$$
\begin{aligned}
u(t,x) = &\int_{-\infty}^{\infty} g(t,x-y)u_0(y)dy \\
&+ \int_0^t \int_{-\infty}^{\infty} g(t-s,x-y)|u(s,y)(1-u(s,y))|^{1/2}W(dyds)
\end{aligned}
$$

where $g(t,x) = (2\pi t)^{-1/2}\exp(-x^2/2t)$ is the fundamental solution of the heat equation. See Walsh [Wal86] for basic theory of equations driven by space-time white noise. For future use, let G_t denote the heat semigroup generated by $g(t,x)$.

If the initial function $u_0(x)$ is continuous and satisfies $u_0(x) \in [0,1]$ for all $x \in \mathbf{R}$, then it is possible to construct solutions $u(t,x)$ for which $u(t,x) \in [0,1]$ for all t,x (see section 2 in [Shi94]). Furthermore, the solutions are jointly continuous in (t,x). Throughout the paper we shall consider only such solutions.

The equation (1.1) arises in population biology; see Shiga [Shi88]. Roughly speaking, $u(t,x)$ represents the proportion of the population at position x and at time t which has a certain trait. The term u_{xx} represents the random motion of individuals. The number of matings at site x and at time t between individuals with and without the trait is proportional to $|u(1-u)|$. The trait is neutral, so there is no drift term in (1.1). The term $|u(1-u)|^{1/2}\dot{W}$ represents random fluctions in the frequency of mating.

From our point of view, however, (1.1) is interesting because it may be the simplest SPDE exhibiting a nontrivial interface. We define the interface below, but first we give an intuitive description. One imagines that there is a region in which everyone has the trait, and hence $u = 1$, and a region in which no one has the trait, and hence $u = 0$. The region in between is called the interface. We note that interface problems have a long history, for example in the Ising model and in growth models such as first passage percolation. In the context of first passage percolation, we refer the reader to Kesten [Kes93].

Let $\mathcal{C}$ be the space of continuous functions from $\mathbf{R}$ to $[0,1]$, with the topology of uniform convergence on compacts. Let $\Omega = C([0,\infty) \to \mathcal{C})$ be the space of continuous paths, let U_t be the coordinate process on Ω, let $\mathcal{F}$ be the Kolmogorov σ-field on Ω, and let $\mathcal{F}_t$ be the σ-field on Ω generated by $\{U_s : s \le t\}$.

In [Shi88], Shiga used duality to show that, for each $f \in \mathcal{C}$, a continuous $\mathcal{C}$-valued solution to (1.1) satisfying $u_0 = f$ is unique in law. Let P_f be the law on Ω induced by this solution. Then $(P_f)_{f\in\mathcal{C}}$ forms a strong Markov family. If μ is a probability measure on $\mathcal{C}$, we define $P_\mu = \int_{\mathcal{C}} P_f \, \mu(df)$.

We now define the interface of a solution. For $f \in \mathcal{C}$, let

$$L(f) = \inf\{x \in \mathbf{R} : f(x) < 1\}$$
$$R(f) = \sup\{x \in \mathbf{R} : f(x) > 0\}$$

be the left and right hand edges of the interface, respectively. Let $\mathcal{C}_I$ be the subset of functions $f \in \mathcal{C}$ for which $-\infty < L(f) < R(f) < \infty$. If $u_0 \in \mathcal{C}_I$, then $u(t) \in \mathcal{C}_I$ for all $t \ge 0$. This is a variant of the compact support property, applied both to the process $u(t,x)$ and to the process $1-u(t,-x)$. The compact support property follows from the same line of argument that is used for the super-Brownian motion. We quote the following lemma from section 3 of Tribe [Tri95].

Lemma 1.1. *Let u be a solution to (1.1) such that $R(u_0) \le 0$. Then for all $t \ge 0$, $b \ge 4t^{1/2}$*

$$P(\sup_{s\le t} R(u_s) \ge b) \le C(t^{-1/2} \vee t^{23}) e^{-b^2/16t}.$$

Applying this lemma to the process $1 - u(t,-x)$, which is also a solution to (1.1), gives a similar result for the left hand edge $L(u(t))$.

We now suppose that $u_0 \in \mathcal{C}_I$. For a solution $u(t,x)$ of (1.1), we let $\bar{u}(t,x) = u(t, x + L(u(t)))$, which is the solution viewed from its left hand edge. Note that $L(\bar{u}(t)) = 0$. We also define the translated coordinate process by

$$\bar{U}_t(x) = \begin{cases} U_t(x + L(U_t)) & \text{if } L(U_t) > -\infty \\ (1-x)_+ \wedge 1 & \text{if } L(U_t) = -\infty. \end{cases}$$

Note again that $L(U_t) = 0$. We say that μ, a probability measure on $\mathcal{C}_I$ for which $\mu\{f \in \mathcal{C}_I : L(f) = 0\} = 1$, is the law of a stationary interface if, under P_μ, the law of $\bar{U}_t$ is μ for all $t > 0$.

Theorem 1. *There exists a unique stationary interface law μ on $\mathcal{C}_I$. Furthermore, for each $f \in \mathcal{C}_I$, we have that the measure $P_f(\bar{U}_t \in \cdot)$ converges in total variation to μ as $t \to \infty$. In addition, the moment of the width of the interface $\int_{\mathcal{C}_I}(R(f)-L(f))^p\mu(df)$ is finite if $0 \le p < 1$, and infinite if $p \ge 1$.*

Note that without the noise term, the infinite speed of propagation of the heat equation would result in $L(u(t)) = -\infty$ and $R(u(t)) = \infty$ for all $t > 0$. Furthermore, the solution would spread out so that $u(t,x) \to 1/2$ as $t \to \infty$ for each x.

We now compare this result with some other recent results about stationary solutions for stochastic pde's. Mueller and Sowers ([MS95]) study the equation

$$u_t = u_{xx} + u(1-u) + \varepsilon\sqrt{u(1-u)}\dot{W}. \tag{1.3}$$

It is proved that for small ε in (1.3), the law of $R(u(t)) - L(u(t))$ tends toward a stationary distribution and that the interface travels with linear speed. The tools used in [MS95] are very different, and would apply to a class of equations with coefficients satisfying the same general properties. However, the result relies on taking ε small, so that the equation closely follows the underlying deterministic KPP equation over finite time intervals. Another stationary travelling wave was found in [Tri96] for the equation

$$u_t = u_{xx} + \lambda u - u^2 + \sqrt{u}\dot{W}. \tag{1.4}$$

This result does not rely on small noise but, as in [MS95], relies on the mass creation term $\lambda u - u^2$ that drives the solution through space. We believe that this driving force makes the finite width of the interface more plausible. Thus, the existence of a stationary interface for (1.1), where there is no such driving force, is more interesting. To obtain this more delicate result, however, we rely heavily on the explicit moment formulae given by duality. We do not yet have general techniques that will establish the existence of an interface for a class of stochastic pde's. We note that the interface for (1.2) does not have a linear speed, and indeed has been shown to move in an asymptotically Brownian way (see [Tri95]).

While preparing this paper, we received a preprint from T. Cox and R. Durrett which deals with a related problem in particle systems. They consider the 1-dimensional unbiased voter model $\xi_t(k)$, with long range interactions which are symmetric with respect to reflections in the $k = 0$ axis. The process begins with 1's to the left of 0, and 0's to the right. Using duality, they show that $\sum_{i<j} \mathbf{1}(\xi_t(i) = 0, \xi_t(i) = 1)$ is not likely to be large. This leads to a proof that there is a stationary interface solution in their discrete space situation. They give various conjectures about moments for the length of the interface. The stochastic pde (1.1) can be derived from the long range voter process (see Mueller and Tribe [MT95]) and we believe that theorem 1 sheds light on their conjectures about the length of the interface.

We now discuss the proof, which has two ingredients. Duality gives explicit formulae for the moments, which are used in section 2 to prove the following lemma. This lemma gives the stochastic compactness of the width of the interface.

Lemma 1.2. *Let u be any solution to (1.1) with deterministic initial condition $u_0 = f \in \mathcal{C}_I$. Then*

$$E(|R(u(t)) - L(u(t))|^p) \leq C(f,p) < \infty \quad \textit{for all } t \geq 0, p \in [0,1).$$

This lemma is used to establish the existence of a stationary interface. The second ingredient is the construction of certain coupled solutions to (1.1). We say that two solutions u, v are completely coupled at time t if there exists $y \in \mathbf{R}$ such that

$$u(t,x) = v(t, x+y) \quad \text{for all } x \in \mathbf{R}.$$

In section 3 we construct two coupled solutions with finite interfaces at time zero. We show that these solutions have positive probability of completely coupling. In section 4 we use this coupling to show the uniqueness of the stationary interface, and then finish the proof of theorem 1.

2. Stochastic compactness for the width of the interface

In this section we prove Lemma 1.2. We will use C to denote a quantity whose dependence will be indicated, but whose exact value is unimportant and may vary from line to line.

There is a duality relation for (1.1), and it gives a formula for moments $E(\prod_{i=1}^n u(t,x_i))$ in terms of a system of annihilating Brownian motions. This is used in Tribe [Tri95] Lemma 2.1 to obtain the following estimates for a solution u satisfying $u(0) = f \in \mathcal{C}_I$: if $\varepsilon > 0$, there exists $C(\varepsilon)$ so that whenever $|z_1 - z_4| \vee |z_2 - z_3| \leq 1$

and $d := \min\{z_1, z_4\} - \max\{z_2, z_3\} \geq 0$, then for all $t \geq 0$

$$E\Big(\int_{\mathbf{R}} u(t, z_1 + x)u(t, z_2 + x)(1 - u(t, z_3 + x))(1 - u(t, z_4 + x))dx\Big) \tag{2.1}$$
$$\leq C(\varepsilon)d^{-2(1-\varepsilon)}(1 + R(f) - L(f))$$

and for all $z \in \mathbf{R}$

$$E\left(\int_{\mathbf{R}} u(t, x)(1 - u(t, x + z))dx\right) \leq 1 + (z \vee 0) + R(f) - L(f). \tag{2.2}$$

Define

$$\tilde{u}(t, x) = \int_{x-(1/2)}^{x+(1/2)} u(t, z)dz$$
$$I_p(t) = \int_{\mathbf{R}}\int_{\mathbf{R}} \tilde{u}(t, x)(1 - \tilde{u}(t, x))|x - y|^p \tilde{u}(t, y)(1 - \tilde{u}(t, y))dxdy.$$

The smoothed density $\tilde{u}$ is used to ensure that an interface, where the solution changes from 1 to 0 or from 0 to 1, will give a contribution to the integral $I_p(t)$ no matter how quick the change is. From the estimates in (2.1) and (2.2) above we shall argue that if $p \in [0, 1)$ then

$$E(I_p(t)) \leq C(f, p) < \infty \text{ for all } t \geq 0. \tag{2.3}$$

Indeed,

$$\begin{aligned}
&E(I_p(t)) \\
&\quad = 2E\left(\int_{\mathbf{R}}\int_{\{z>0\}} \tilde{u}(t, x)(1 - \tilde{u}(t, x))|z|^p \tilde{u}(t, x + z)(1 - \tilde{u}(t, x + z))dz\, dx\right) \\
&\quad = 2E\Big(\int_{\mathbf{R}} dx \int_{\{z>0\}} dz \int_{-1/2}^{1/2} dy_2 \int_{-1/2}^{1/2} dy_3 \int_{-1/2}^{1/2} dy_1 \int_{-1/2}^{1/2} dy_4 \\
&\qquad\qquad u(t, y_2 + x)(1 - u(t, y_3 + x))|z|^p u(t, y_1 + z + x)(1 - u(t, y_4 + z))\Big).
\end{aligned}$$

Now one applies the bound in (2.1) over the region $\{z > 1\}$, with the choice $z_2 = y_2, z_3 = y_3, z_1 = y_1 + x, z_4 = y_4 + x$. Then one applies the bound in (2.2) over the region $\{0 < z < 1\}$ (throwing away the terms in z_1 and z_4 in this second case). A little algebra then results in (2.3).

We now need to use this information about the amount of mass in the interface to control its width.

Lemma 2.1. *Let u be a solution of (1.1). There exists a constant C so that for all α, γ with $\gamma \geq \alpha + 4$ we have*

$$P(R(u(2)) \geq \gamma) \leq CE\left(\int_\alpha^\infty u(0,x)dx\right) + C\exp(-(\gamma-\alpha)^2/64).$$

Proof. The basic method is to obtain a lower bound for the Laplace transform $E(\exp(-\lambda \int_\gamma^\infty u(2,x)dx))$. As $\lambda \to \infty$, the transform converges to $P(\int_\gamma^\infty u(2,x)dx = 0) = P(R(u(2)) \leq \gamma)$. This mimics the approach used in Dawson, Iscoe and Perkins [DIP89] to study super-Brownian motion.

Fix $\alpha + 4 \leq \beta + 1 \leq \gamma$. For $\lambda > 0$, let $(\phi^\lambda(t,x) : x \in \mathbf{R}, t \in [0,2])$ be the unique bounded non-negative solution to

$$-\phi_t^\lambda = \frac{1}{2}\phi_{xx}^\lambda - \frac{1}{4}(\phi^\lambda)^2$$
$$\phi^\lambda(2,x) = \lambda((x-\gamma)_+ \wedge 1).$$

Then, arguing as in [DIP89] Lemma 3.2, we find that the functions ϕ^λ converge monotonically to a limit function ϕ as $\lambda \to \infty$. Moreover, the function ϕ takes values in $[0,\infty]$ and one has the two bounds

$$\begin{aligned} \phi(s,x) &\leq C(2-s)^{-1} \quad \text{for } x \in \mathbf{R}, s \in [0,2), \\ \phi(s,x) &\leq C\exp(-(\gamma-x)^2/4) \quad \text{for } x \leq \gamma - 1, s \in [1,2]. \end{aligned} \tag{2.4}$$

Let $\tau_\beta = \inf\{t \geq 1 : \sup_{x\geq\beta} |u(t,x)| \geq 1/2\}$. Then

$$(1/2)(1-u(s,x)) \geq 1/4 \quad \text{for } x \geq \beta \text{ and } s \in [1,\tau_\beta). \tag{2.5}$$

From Ito's formula, the drift part of $X_s := \exp(-\int_{-\infty}^\infty u(s,x)\phi^\lambda(s,x)dx)$ is

$$\begin{aligned} X_s &\int_{-\infty}^\infty u(s,x)\left(-\phi_s^\lambda(s,x) - \frac{1}{2}\phi_{xx}^\lambda(s,x) + \frac{1}{2}(1-u(s,x))(\phi^\lambda(s,x))^2\right)dx \\ &= X_s \int_{-\infty}^\infty u(s,x)\left(\frac{1}{2}(1-u(s,x)) - \frac{1}{4}\right)(\phi^\lambda(s,x))^2 dx \\ &\geq -\frac{1}{4}\int_{-\infty}^\beta (\phi^\lambda(s,x))^2 dx \quad \text{for } s \in [1,\tau_\beta \wedge 2) \text{ using (2.5)} \\ &\geq -C\exp(-(\gamma-\beta)^2/4) \end{aligned}$$

where in the last inequality we used (2.4) and the well known inequality

$$\int_R^\infty \exp(-z^2/2)dz \leq \exp(-R^2/2) \quad \text{for } R \geq 1.$$

We now take expectations of X_t and let $\lambda \to \infty$.

$$
\begin{aligned}
&P\left(\int_\gamma^\infty u(2,x)dx = 0\right) + P(\tau_\beta < 2) \\
&\quad \geq P\left(\tau_\beta \geq 2, \int_\gamma^\infty u(2,x)dx = 0\right) \\
&\qquad + E\left(\mathbf{1}_{(\tau_\beta<2)} \exp\left(-\int_{-\infty}^\infty u(\tau_\beta,x)\phi(\tau_\beta,x)dx\right)\right) \\
&\quad \geq E\left(\exp\left(-\int_{-\infty}^\infty u(2\wedge\tau_\beta,x)\phi(2\wedge\tau_\beta,x)dx\right)\right) \\
&\quad = E\left(\exp\left(-\int_{-\infty}^\infty u(1,x)\phi(1,x)dx\right)\right) \\
&\qquad + E\left(\int_1^{2\wedge\tau_\beta} X_s \int_{-\infty}^\infty u(s,x)\left(\frac{1}{2}(1-u(s,x)) - \frac{1}{4}\right)(\phi(s,x))^2 dxds\right) \\
&\quad \geq 1 - E\left(\int_{-\infty}^\infty u(1,x)\phi(1,x)dx\right) - C\exp(-(\gamma-\beta)^2/4) \\
&\quad = 1 - E\left(\int_{-\infty}^\infty u(0,x)G_1\phi(1,x)dx\right) - C\exp(-(\gamma-\beta)^2/4) \\
&\quad \geq 1 - CE\left(\int_\alpha^\infty u(0,x)dx\right) \\
&\qquad - C\exp(-(\gamma-\alpha)^2/4) - C\exp(-(\gamma-\beta)^2/4)
\end{aligned}
$$

where in the last step we use the bounds on $\phi(1)$ from (2.4). Rearranging and choosing $\beta = (3/4)\gamma + (1/4)\alpha$ gives

(2.6)
$$
\begin{aligned}
&P\left(\int_\gamma^\infty u(2,x)dx > 0\right) \\
&\quad \leq P(\tau_\beta < 2) + CE\left(\int_\alpha^\infty u(0,x)dx\right) + C\exp(-(\gamma-\alpha)^2/64).
\end{aligned}
$$

We now estimate $P(\tau_\beta < 2)$. Bounds on the deviation arising from the noise show that the solution $u(t,x)$ lies close to $G_t u(0,x)$ for $x \geq \beta, t \leq 2$, provided that $\int_\alpha^\infty u(0,x)dx$ is small. Indeed, Lemma 3.1 from Tribe

[Tri95] implies that

$$P\left(\sup_{t\le 2}\sup_{x\ge\beta}|u(t,x)-G_t u(0,x)|\ge\frac{1}{8}\right) \le CE\left(\int_{-\infty}^{\infty}u(0,x)G_2\mathbf{1}_{(\beta,\infty)}(x)dx\right).$$

Note that

$$G_t u(0,x)\le G_t\mathbf{1}_{(-\infty,\alpha)}(x)+(2\pi t)^{-1/2}\int_{\alpha}^{\infty}u(0,x)dx\le\frac{1}{4}$$

provided $t\in[1,2]$, $x\ge\alpha+3$ and $\int_{\alpha}^{\infty}u(0,x)dx\le\frac{1}{4}$. So

$$\begin{aligned}
&P(\tau_\beta<2)\\
&\quad\le P\left(\sup_{t\le 2}\sup_{x\ge\beta}|u(t,x)-G_t u(0,x)|\ge\frac{1}{8}\right)\\
&\qquad\quad+P\left(\int_{\alpha}^{\infty}u(0,x)dx\ge\frac{1}{4}\right)\\
&\quad\le CE\left(\int_{\alpha}^{\infty}u(0,x)dx\right)+CE\left(\int_{-\infty}^{\infty}u(0,x)G_2\mathbf{1}_{(\beta,\infty)}(x)dx\right)\\
&\quad\le CE\left(\int_{\alpha}^{\infty}u(0,x)dx\right)+C\int_{-\infty}^{\alpha}G_2\mathbf{1}_{(\beta,\infty)}(x)dx\\
&\quad\le CE\left(\int_{\alpha}^{\infty}u(0,x)dx\right)+C\exp(-(\beta-\alpha)^2/4)
\end{aligned}$$

where the last inequality comes from estimating the double integral. Combined with (2.6) this completes the proof. □

We now complete the proof of lemma 1.2. Define approximate right and left hand edges of the interface by

$$\tilde{L}(t)=\inf\{x:\tilde{u}(t,x)=1/2\}\quad \tilde{R}(t)=\sup\{x:\tilde{u}(t,x)=1/2\}.$$

The smoothed solution $\tilde{u}$ satisfies $|\tilde{u}_x(t,x)|\le 1$, which implies that

$$\int_{-\infty}^{\tilde{L}(t)}\tilde{u}(t,x)(1-\tilde{u}(t,x))dx\ge\frac{1}{16}.$$

Therefore, for $z \geq 0$,

$$
\begin{aligned}
&\int_{\tilde{L}(t)+z}^{\infty} \tilde{u}(t,y)(1-\tilde{u}(t,y))dy \\
&\qquad \leq 16 \int_{-\infty}^{\tilde{L}(t)} \tilde{u}(t,x)(1-\tilde{u}(t,x))dx \int_{\tilde{L}(t)+z}^{\infty} \tilde{u}(t,y)(1-\tilde{u}(t,y))dy \\
&\qquad \leq 16 z^{-p} I_p(t).
\end{aligned}
$$

By symmetry $\int_{\tilde{R}(t)}^{\infty} \tilde{u}(t,x)(1-\tilde{u}(t,x))dx \geq \frac{1}{16}$. So if $16z^{-p}I_p(t) \leq \frac{1}{16}$ then $\tilde{R}(t) \leq \tilde{L}(t)+z$, implying $\tilde{u}(t,y) \leq \frac{1}{2}$ for $y \geq \tilde{L}(t)+z$. So on the set $\{16z^{-p}I_p(t) \leq \frac{1}{16}\}$ we have

$$
\begin{aligned}
&\int_{\tilde{L}(t)+z+1}^{\infty} u(t,y)dy \\
\leq\ & \int_{\tilde{L}(t)+z}^{\infty} \tilde{u}(t,y)dy \\
\leq\ & 2\int_{\tilde{L}(t)+z}^{\infty} \tilde{u}(t,y)(1-\tilde{u}(t,y))dy \\
\leq\ & 32 z^{-p} I_p(t). \qquad (2.7)
\end{aligned}
$$

Let Ω_0 be the set $\{16z^{-p}I_p(t) > \frac{1}{16}\}$. Note that $P(\Omega_0) \leq Cz^{-p}E(I_p(t))$ by Chebychev's inequality. For $z \geq 4$ we have

$$
\begin{aligned}
&P(R(u(t+2)) \geq \tilde{L}(t)+2z+1) \\
&\qquad \leq P(\Omega_0) + P(R(u(t+2)) \geq \tilde{L}(t)+2z+1; \Omega_0^c) \\
&\qquad \leq P(\Omega_0) + CE\left(\int_{\tilde{L}(t)+z+1}^{\infty} u(t,y)dy; \Omega_0^c \right) + Ce^{-z^2/64} \\
&\qquad\qquad \text{(using lemma 2.1 and the Markov property)} \\
&\qquad \leq Cz^{-p}E(I_p(t)) + Ce^{-z^2/64} \quad \text{(using 2.7)}.
\end{aligned}
$$

A similar bound holds for $P(L(u(t+2)) \leq \tilde{R}(t)-2z-1)$ and thence for $P(R(u(t+2)) - L(u(t+2)) \geq 4z+2)$. So, for $0 \leq q < p < 1$,

$$
\begin{aligned}
&E(|R(u(t+2)) - L(u(t+2))|^q) \\
&\quad \leq C(q) + C(q)\int_1^{\infty} z^{q-1} P(R(u(t+2)) - L(u(t+2)) \geq 4z+2)dz \\
&\quad \leq C(q) + C(q)\int_1^{\infty} z^{q-1}\left(z^{-p}E(I_p(t)) + e^{-z^2/64} \right) dz \\
&\quad \leq C(q,p)(1+E(I_p(t))) \\
&\quad \leq C(f,q,p).
\end{aligned}
$$

To bound the expectation for $t \in [0, 2]$ one may use the finite speed of motion of $L(u(t))$ and $R(u(t))$ in Lemma 1.1. Indeed the super exponential decay of the tail probability of $R(u(t))$ implies that bothe the expectations $E(\sup_{t \le 2} |R(u(t)) - R(f)|^q)$ and $E(\sup_{t \le 2} |L(u(t)) - L(f)|^q)$ are finite for any positive q. This completes the proof. $\square$

3. The Coupling Method

In this section we describe our coupling method. It is based on similar ideas in [Mue93] and [MS95]. In this section we prove the following result:

Lemma 3.1. *For $K > 0$ there exists $p_0(K) > 0$ so that for any (possibly random) initial conditions u_0, v_0 whose interfaces have length at most K, there exist solutions u, v to (1.1) with initial conditions u_0, v_0 satisfying*

$$P(u \text{ and } v \text{ completely couple at some time } t \le 1) \ge p_0(K).$$

Proof. We shall include in the proof several lemmas whose proof we delay until after we complete the main argument.

We may assume, by applying a possibly random translation at time zero, that the interface of u_0 is contained in $[0, K]$ and the interface of v_0 is contained in $[-K, 0]$. Thus $v_0(x) \le u_0(x)$. We shall take a coupling of solutions u, v so that the difference $D(t, x) = v(t, x) - u(t, x)$ remains non-negative and of compact support for all time, and which will be an approximate solution to (1.1). We shall then compare the total mass $\int D(t, x)dx$ with a one dimensional diffusion to show that D may die out by time one.

Take two independent white noises W_1, W_2. Let $g(z) = |z(1-z)|^{1/2}$. We take solutions u, v satisfying for $t \ge 0$ and $x \in \mathbf{R}$

$$\begin{aligned} v_t &= \frac{1}{2} v_{xx} + g(v)\dot{W}_v \\ u_t &= \frac{1}{2} u_{xx} + g(u)\dot{W}_u \end{aligned} \tag{3.1}$$

where the white noises W_u, W_v satisfy

$$\begin{aligned} \dot{W}_v &= \dot{W}_1 \\ \dot{W}_u &= \left(1 - f^2\right)^{\frac{1}{2}} \dot{W}_1 + f\dot{W}_2 \\ f = f(u, v; t, x) &= \min\left\{ \frac{g(D)}{(g(u)g(v))^{\frac{1}{2}}}, 1 \right\}. \end{aligned} \tag{3.2}$$

Lemma 3.2. *On a probability space $(\Omega^0, (\mathcal{F}_t^0), P)$, there exists a solution (u, v) to (3.1) and (3.2) such that with probability 1,*

$$0 \le v(t,x) \le u(t,x)$$

for all $t \ge 0$ and $x \in \mathbf{R}$.

Next, a short calculation shows that D satisfies

$$D_t = \frac{1}{2} D_{xx} + h(u,v)\dot{W} \tag{3.3}$$

where $\dot{W}$ is another white noise and

$$h(u,v) = h(u,v;t,x) = \left((g(u)-g(v))^2 + 2\frac{g(u)g(v)f^2}{1+(1-f^2)^{\frac{1}{2}}} \right)^{\frac{1}{2}}.$$

We claim that

$$g(D) \le h(u,v) \le 2g(D). \tag{3.4}$$

The lower bound in (3.4) is immediate if $g^2(D) \le g(u)g(v)$, since in that case

$$h(u,v) = \left((g(u)-g(v))^2 + 2\frac{g^2(D)}{(1+(1-f^2)^{1/2})} \right)^{\frac{1}{2}} \ge g(D).$$

However, if $g^2(D) \ge g(u)g(v)$, then we have

$$\begin{aligned} h(u,v) &= \left((g(u)-g(v))^2 + 2g(u)g(v) \right)^{\frac{1}{2}} \\ &= \left(g^2(u) + g^2(v) \right)^{\frac{1}{2}} \\ &= \left(u(1-u) + v(1-v) \right)^{\frac{1}{2}} \\ &= \left(D(1-D) + 2u(1-v) \right)^{\frac{1}{2}} \ge g(D). \end{aligned}$$

Using the bound $|g(u)-g(v)| \le g(D)$, the upper bound can be checked in a similar manner.

To complete the proof we need to show that the process D has some chance of dying out by time one. Since D remains non-negative, this is equivalent to the integral $\int D(t,x)dx$ reaching zero. Note that this integral is a non-negative martingale and hence converges. We have been unable to exploit this fact to give a quick proof, however. If D were an exact solution to (1.1) then one could give a relatively short proof (see [Tri95]). However, for our equation we seem to need a longer argument, which is the content of the rest of this section.

Since we need only show a positive chance of dying out, we consider an absolutely continuous change of measure under which D has a large negative drift. For $\zeta \geq 0$, define on $\mathcal{F}_1^0$ a new measure P_ζ by

(3.5)
$$\frac{dP_\zeta}{dP} = \exp\bigg(-\zeta \int_0^1 \int_{-\infty}^{\infty} D(s,x)(1-D(s,x))h^{-1}(u,v;s,x)W(dx,ds) - \frac{\zeta^2}{2}\int_0^1 \int_{-\infty}^{\infty} (D(s,x)(1-D(s,x)))^2 h^{-2}(u,v;s,x)dxds \bigg).$$

Lemma 3.3. *The exponential in (3.5) has mean one under P, so that P_ζ is a true probability for any $\zeta \geq 0$. Moreover, under P_ζ, the process D satisfies*

$$D_t = \frac{1}{2}D_{xx} - \zeta D(1-D) + h(u,v)\dot{W}^\zeta \tag{3.6}$$

with respect to some new white noise W^ζ.

We shall show shortly (in Lemma 3.4) that for suitable Δ and for all large enough ζ, we have $D(\Delta/2, x) \leq 1/2$ for all x with high probability. Keeping this in mind, we shall now give the main argument showing that D may die out. Define

$$M(t) = \int_{-\infty}^{\infty} D(t,x)dx$$

and stopping times

$$\begin{aligned} \sigma &= \inf\{t \geq 0 : M(t) = 0\} \\ \tau &= \inf\{t \geq \Delta/2 : D(t,x) \geq 3/4 \text{ for some } x\}. \end{aligned}$$

Note that u and v are completely coupled at time σ. The process $M(t)$ under P_ζ is a supermartingale. On $[\Delta/2, \tau]$, it has drift less than $-\frac{\zeta}{4}M(t)$ and square variation satisfying

$$\begin{aligned} d[M](t) &\geq \int_{-\infty}^{\infty} D(t,x)(1-D(t,x))dxdt \\ &\geq \frac{1}{4}\int_{-\infty}^{\infty} D(t,x)dxdt = \frac{1}{4}M(t)dt. \end{aligned}$$

If $\tau > \Delta$, then for large ζ the process M should be likely to die out before time Δ. Indeed, Ito's formula shows that, provided $2\zeta\exp(-\zeta\Delta/4) \leq 1$, the process $X_s = \exp(-M(s)(e^{-\zeta s/4} - e^{-\zeta\Delta/4})^{-1})$ has non-negative drift on $[\Delta/2, \tau \wedge \Delta)$. Note that on the set $\{\sigma > \Delta, \tau \geq \Delta\}$ the process X_t converges to zero as $t \uparrow \tau \wedge \Delta$. On the complement we bound X by one. So taking expectations of $X_{t\wedge\Delta}$ and letting $t \uparrow \tau$ we have that

$$\begin{aligned}
&P_\zeta(\sigma \leq \Delta \leq \tau) + P_\zeta(\tau < \Delta) \\
&\quad \geq E_\zeta(\exp(-M(\tau \wedge \Delta)(e^{-\zeta(\tau\wedge\Delta)/4} - e^{-\zeta\Delta/4})^{-1})) \\
&\quad \geq E_\zeta(\exp(-M(\Delta/2)(e^{-\zeta\Delta/8} - e^{-\zeta\Delta/4})^{-1})) \\
&\quad \geq E_\zeta(\exp(-M(0)(e^{-\zeta\Delta/8} - e^{-\zeta\Delta/4})^{-1})) \\
&\quad \geq \exp(-2K(e^{-\zeta\Delta/8} - e^{-\zeta\Delta/4})^{-1})
\end{aligned}$$

where in the penultimate inequality we used the fact that $\exp(-aM(s))$ is a submartingale for $a > 0$. Rearranging terms, we have for $\varepsilon > 0$ that there exists $\zeta_0(\Delta, K, \varepsilon)$ so that for $\zeta \geq \zeta_0$

$$P_\zeta(\sigma \leq \Delta) \geq 1 - \varepsilon - P_\zeta(\tau < \Delta). \tag{3.7}$$

To finish the proof, we must now show that $P_\zeta(\tau < \Delta)$ is small. To control τ, we shall compare the solution D with a process $\bar{D}$ which evolves deterministically according to the equation

$$\bar{D}_t = \frac{1}{2}\bar{D}_{xx} - \zeta\bar{D}(1 - \bar{D}). \tag{3.8}$$

Over short time periods, D and $\bar{D}$ have high probability of remaining close. However, the process $\bar{D}$ has been well studied and is known to have wavefronts that will travel at least at speed $(2\zeta)^{1/2}$. From an initial condition supported inside $[-K, K]$, the process $\bar{D}$ would satisfy $\bar{D}(t, x) \leq 1/2$ before time $K\zeta^{-1/2}$. Using this idea, we will be able to show the following key lemma.

Lemma 3.4. *Given $\varepsilon > 0$, there exists $\zeta_1(K, \varepsilon)$ and $\Delta(K, \varepsilon) \in (0, 1]$ so that for all $\zeta \geq \zeta_1$ satisfying $K\zeta^{-1/2} \leq \Delta$,*

$$P_\zeta(D(t, x) < 3/4 \text{ for all } x \in \mathbf{R} \text{ and } t \in [K\zeta^{-1/2}, \Delta]) \geq 1 - \varepsilon$$

Finally, we need to control the support of D.

Lemma 3.5. *Define*

$$\Omega(K_1) = \{D \text{ is supported inside } [-K_1, K_1] \text{ for all time } t \leq 1\}.$$

Given $\varepsilon > 0$, we may pick $K_1(K, \varepsilon)$ so that for all ζ we have

$$P_\zeta(\Omega(K_1)) \geq 1 - \varepsilon.$$

Now we can finish the proof of Lemma 3.1. Given K and taking $\varepsilon = 1/4$, we pick $K_1(K, \frac{1}{4})$ as in Lemma 3.5, and then choose $\Delta = \Delta(K, \frac{1}{4})$ as in Lemma 3.4. Fix $\zeta \geq \zeta_0(\Delta, K, \frac{1}{4}) \vee \zeta_1(K, \frac{1}{4})$ satisfying $K\zeta^{-1/2} \leq \Delta/2$. Then from (3.7) and Lemmas 3.4 and 3.5, we have

$$P_\zeta(\sigma \leq \Delta; \Omega(K_1)) \geq \frac{1}{4}.$$

Note that ζ, Δ and K_1 depend only on K. Finally we use the change of measure as follows.

$$\begin{aligned} P_\zeta(\sigma \leq \Delta; \Omega(K_1)) &= E(\mathbf{1}_{(\sigma \leq \Delta, \Omega(K_1))}(dP_\zeta/dP)) \\ &\leq \big(P(\sigma \leq \Delta)\big)^{\frac{1}{2}} \big(E((dP_\zeta/dP)^2; \Omega(K_1))\big)^{\frac{1}{2}} \end{aligned} \tag{3.9}$$

Also, on the set $\Omega(K_1)$ we have that

$$\int_{-\infty}^{\infty} D(s,x)(1-D(s,x))h^{-2}(u,v;s,x)dx \leq 2K_1$$

so that

(3.10)

$$\begin{aligned}
&E((dP_\zeta/dP)^2; \Omega(K_1)) \\
&= E\Big[\exp\Big(-2\zeta \int_0^1 \int_{-\infty}^{\infty} D(s,x)(1-D(s,x))h^{-1}(u,v;s,x)W(dx,ds) \\
&\qquad - \zeta^2 \int_0^1 \int_{-\infty}^{\infty} (D(s,x)(1-D(s,x)))^2 h^{-2}(u,v;s,x)dxds\Big); \Omega(K_1)\Big] \\
&\leq E\Big[\exp\Big(-2\zeta \int_0^1 \int_{-\infty}^{\infty} D(s,x)(1-D(s,x))h^{-1}(u,v;s,x)W(dx,ds) \\
&\qquad - 2\zeta^2 \int_0^1 \int_{-\infty}^{\infty} (D(s,x)(1-D(s,x)))^2 h^{-2}(u,v;s,x)dxds\Big); \Omega(K_1)\Big] \\
&\qquad\quad \exp(2\zeta^2 K_1) \\
&\leq \exp(2\zeta^2 K_1).
\end{aligned}$$

In the last inequality we used the fact that the exponential is a non-negative local martingale and therefore has expectation bounded by one. Substituting (3.10) in (3.9) shows that $P(\sigma \leq \Delta) \geq \frac{1}{16}\exp(-2\zeta^2 K_1)$. This completes the proof of Lemma 3.1. □

In the rest of this section we complete the proofs of lemmas 3.2 - 3.5.

Proof of Lemma 3.2. One may use methods similar to those used in Theorem 2.2 of Shiga [Shi94]. We summarize the argument used there. The functions g and f are approximated by Lipschitz functions, the Laplacian by a bounded operator, and the white noise by a smoothed white noise. The resulting equations have unique solutions for which the functions $u(t,x)$ are semimartingales for each x, and one may use Ito calculus to verify that the required inequalities are satisfied. The approximations may be checked to be relatively compact, and any limit point will be a solution which still satisfies the required inequalities. □

Proof of Lemma 3.3. The martingale

$$Z_t = \zeta \int_0^t \int D(s,x)(1-D(s,x))h^{-1}(u,v;s,x)W(dxds)$$

has brackets process bounded (using $h \geq (D(1-D))^{1/2}$) by

$$\begin{aligned}[Z]_t &\leq \zeta^2 \int_0^t \int D(s,x)(1-D(s,x))dxds \\ &\leq \zeta^2 \int_0^t \left[R(u(s)) - L(u(s))\right] ds.\end{aligned}$$

Then Lemma 1.1 shows that $E(\exp([Z]_t)) < \infty$, and Novikov's criterion ([RY91] VIII.1.15) implies that the exponential martingale $\exp(Z_t - \frac{1}{2}[Z]_t)$ is a true martingale. Equation (3.6) then follows by applying Girsanov's theorem. See [Daw78] for the use of Girsanov's theorem for stochastic PDE's. □

Proof of Lemma 3.5. When $\zeta = 0$ we may deduce this lemma from Lemma 1.1. In fact, Lemma 1.1 controls the left and right hand edges of u and v and hence of their difference D. Note also that the estimate does not depend on the exact shape of the initial condition. When $\zeta > 0$ we have to argue anew. The difference $D(x)$ is zero for large x, so we may define its right hand edge R_D. The estimate on the compact support of R_D still holds exactly as stated in Lemma 1.1. To see this one must work through the proof of Lemma 1.1, making only two small changes. Firstly, the equation (3.6) for the evolution of D has a negative drift term. This actually helps the proof of the compact support, in that the proof involves a series of inequalities which remain true with this extra negative term. The second change is that the coefficient of the noise is not exactly $|D(1-D)|^{1/2}$. However, there is in fact more noise, and again terms involving the noise may be replaced by terms with the coefficient $|D(1-D)|^{1/2}$ with no cost. The same bound also holds for the left hand edge, and together these bounds imply the lemma. □

To complete the last proof in this section we need a large deviations lemma. Let W be an adapted white noise on a filtered probability space $(E, \mathcal{E}_t)$, and $H(s,y)$ a predictable integrand with $|H| \leq 1$. Define

$$N(t,x) = \int_0^t \int g(t-r, x-y) H(r,y) W(dydr). \tag{3.11}$$

Lemma 3.6. *For any $p \geq 10$ there exist constants $C_1(p), C_2(p)$ so that for any $\lambda \geq 1$*

$$\begin{aligned} &P\big(|N(t,x) - N(s,y)| \geq C_1(p)\lambda(|x-y|^{1/10} + |s-t|^{1/10}) \\ &\qquad \text{for some } |x-y| \leq 1,\ s,t \in [0,1], |\mathcal{E}_0\big) \\ &\leq C_2(p)\lambda^{-2p} \int_0^1 \int_{\mathbf{R}} E(H^2(r,z)|\mathcal{E}_0) dz\, dr. \end{aligned}$$

Proof of Lemma 3.6. Such estimates for white noise integrals have been proved in several papers ([Sow92],[Mue91],[Tri95]). Alas, none of them quite apply here. We sketch the argument and leave the calculations to the reader. Arguing exactly as in Lemma 3.1 in [Tri95], one obtains the following bounds. For $0 \leq s \leq t \leq 1$,

$$\begin{aligned} &E(|N(t,x) - N(t,y)|^{2p}|\mathcal{E}_0) \\ &\quad \leq C(p)|x-y|^{p-1} \int_0^t (t-r)^{-1/2} \\ &\qquad \int_{\mathbf{R}} (g(t-r, x-z) + g(t-r, y-z)) E(H^2(r,z)|\mathcal{E}_0) dz\, dr \end{aligned}$$

and

$$\begin{aligned} &E(|N(t,x) - N(s,x)|^{2p}|\mathcal{E}_0) \\ &\quad \leq C(p)|t-s|^{(p-1)/2} \\ &\qquad \bigg(\int_0^t (t-r)^{-1/2} \int_{\mathbf{R}} g(t-r, x-z) E(H^2(r,z)|\mathcal{E}_0) dz\, dr \\ &\qquad + \int_0^s (s-r)^{-1/2} \int_{\mathbf{R}} g(s-r, x-z) E(H^2(r,z)|\mathcal{E}_0) dz\, dr\bigg). \end{aligned}$$

These moments can be used in Chebychev's inequality to estimate the probability of the event

$$A(\lambda) = \bigcup_{n\geq 1}\bigcup_{j=1}^{2^n}\bigcup_{k=-\infty}^{\infty} \Big[\{|N(j2^{-n},k2^{-n}) - N((j-1)2^{-n},k2^{-n})| \geq \lambda 2^{-n/10}\} \cup \{|N(j2^{-n},(k+1)2^{-n}) - N(j2^{-n},k2^{-n})| \geq \lambda 2^{-n/10}\}\Big].$$

Again arguing as in [Tri95], one finds that for $p \geq 10$

$$P(A(\lambda)|\mathcal{E}_0) \leq C(p)\lambda^{-2p}\int_0^1\int_{\mathbf{R}} E(H^2(r,z)|\mathcal{E}_0)dz\,dr.$$

By dividing an increment into dyadic subincrements (as in the proof of the modulus of continuity of Brownian paths), it can be shown that on the set $A(\lambda)$ the desired Hölder continuity holds. □

Proof of Lemma 3.4. The solution $\bar{D}$ to equation (3.8) may be rescaled to satisfy the usual Kolmogorov equation. Indeed, if $z(t,x) = 1 - \bar{D}(\zeta t, \zeta^{1/2}x)$, then z solves the equation

$$z_t = \frac{1}{2}\Delta z + z(1-z).$$

We use one property of the solutions to this equation, whose proof follows from Proposition 3.4 in [Bra83], and the fact that from a decreasing initial condition the solution remains decreasing. If $z(0,x) \geq \frac{1}{2}$ for $x \leq 0$, then there exists an absolute constant time T so that $z(T,x) \geq \frac{3}{4}$ for all $x \leq T$. Undoing the scaling this says that

(3.12) If $\bar{D}(0,x) \leq \frac{1}{2}$ for $x \leq 0$, then $\bar{D}(T\zeta^{-1},x) \leq \frac{1}{4}$ for $x \leq T\zeta^{-1/2}$.

Define

$$\begin{aligned} s_n &= nT\zeta^{-1} \\ I_n &= [-K + nT\zeta^{-1/2}, K - nT\zeta^{-1/2}] \\ J_n &= [-K - nT^{1/2}\zeta^{-1/3}, K + nT^{1/2}\zeta^{1/3}]. \end{aligned}$$

We shall show inductively that with high probability, the process D at time s_n will take values less than $\frac{1}{2}$ outside the interval I_n, whilst being supported inside the interval J_n. Note that the speed of the deterministic equation (3.8) is such that the original interval $I_0 = [-K,K]$ would be reduced by time $s_{KT^{-1}\zeta^{1/2}} = K\zeta^{-1/2}$. First pick

$\zeta_2(K) \geq 1$ large enough that for $\zeta \geq \zeta_2$, we have $J_n \subseteq [-\zeta, \zeta]$ for all $n = 0, 1, \ldots, KT^{-1}\zeta^{1/2}$. We assume inductively that $D(s_n, x) \leq \frac{1}{2}$ for $x \notin I_n$ and that $D(s_n)$ is supported inside J_n. Let $\bar{D}$ solve (3.8), starting at time s_n and with initial condition $D(s_n)$. Then using (3.12), we have that $\bar{D}(s_{n+1}, x) \leq \frac{1}{4}$ for all $x \notin I_{n+1}$. The difference $D - \bar{D}$ starts identically zero at time s_n, and satisfies under P_ζ

$$(D - \bar{D})_t = \frac{1}{2}(D - \bar{D})_{xx} + \zeta(\bar{D}(1 - \bar{D}) - D(1 - D)) + h(u, v)\dot{W}^\zeta.$$

Note that

$$|\bar{D}(1 - \bar{D}) - D(1 - D)| = |\bar{D} - D|.|1 - D - \bar{D}| \leq |D - \bar{D}|.$$

Define

$$S(t) = \sup\{|D(t, x) - \bar{D}(t, x)| : x \in \mathbf{R}\},$$
$$N_1(t, x) = \int_{s_n}^t \int g(t - s, x - y)h(u, v; s, y)W^\zeta(dyds).$$

Then, using the integral form of the equation, we have for $t \geq s_n$

$$\begin{aligned} &D(t, x) - \bar{D}(t, x) \\ &= \zeta \int_{s_n}^t \int g(t - s, x - y)(\bar{D}(1 - \bar{D}) - D(1 - D))(y, s)dyds + N_1(t, x) \\ &\quad \leq \zeta \int_{s_n}^t S(s)ds + |N_1(t, x)|. \end{aligned}$$

The same bound holds for $(\bar{D}(t, x) - D(t, x))$. Taking suprema over x and using Gronwall's Lemma, we see that for $t \geq s_n$

$$S(t) \leq e^{\zeta t} \sup\{|N_1(s, x)| : s_n \leq s \leq t, x \in \mathbf{R}\}.$$

Now we apply Lemma 3.6. In the lemma we take $H(s, y) = \frac{1}{2}h(s_n + s, y) \leq 1$ and $\mathcal{E}_t = \mathcal{F}^0_{s_n+t}$. Note that

$$\begin{aligned} &\int_0^1 \int_{\mathbf{R}} E_\zeta(H^2(r, z)|\mathcal{E}_0)dz\, dr \\ &\qquad \leq \int_0^1 \int_{\mathbf{R}} E_\zeta(D(s_n + r, z)|\mathcal{F}^0_{s_n})dz\, dr \\ &\qquad \leq \int_0^1 \int_{\mathbf{R}} G_r D(s_n, z)dz\, dr \\ &\qquad \leq 2\zeta \end{aligned}$$

since by assumption, $D(s_n)$ is supported inside $[-\zeta, \zeta]$. Applying Lemma 3.6 with $\lambda = \zeta^{1/20}$ and $p = 30$, we may choose $\zeta_3 \geq \zeta_2$ so that for $\zeta \geq \zeta_3$,

$$P_\zeta \left(\sup\{|N_1(s,x)| : x \in \mathbf{R}, s \in [s_n, s_{n+1}]\} \geq \frac{1}{4} e^{-T} \right) \leq \zeta^{-1}.$$

So on this set we have for $x \notin I_{n+1}$

$$D(s_{n+1}, x) \leq \bar{D}(s_{n+1}, x) + S(s_{n+1}) \leq \frac{1}{2}.$$

Finally, we may apply the estimate from Lemma 1.1 to control the left and right hand edges of the support of D. It is explained in the proof of Lemma 3.5 why this estimate still applies. Taking $b = T^{1/2}\zeta^{-1/3}$ in Lemma 1.1, we see that $D(s_{n+1})$ fails to be supported in J_{n+1} only with probability $C\zeta^{-1}$. This completes the inductive step.

We apply the above argument over the time steps $s_0, s_1, \ldots, s_{KT^{-1}\zeta^{1/2}}$. By conditioning inductively, we have for $\zeta \geq \zeta_3$ that

$$\begin{aligned} P_\zeta \left(\sup_x D(K\zeta^{-1/2}, x) \leq \frac{1}{2} \right) &\geq (1 - C\zeta^{-1})^{KT^{-1}\zeta^{1/2}} \\ &\geq 1 - C(K,T)\zeta^{-1/2}. \end{aligned} \tag{3.13}$$

Now set $t_0 = K\zeta^{-1/2}$. To control D over an interval $[t_0 \wedge \Delta, \Delta]$, we write

$$D(t,x) = G_{t-t_0} D(t_0, x) + N_2(t,x) \tag{3.14}$$

where

$$N_2(t,x) = \int_{t_0}^{t} \int G(t-s, x-y) h(s,y) W^\zeta(dyds).$$

Another application of Lemma 3.6 allows us to choose $\Delta(K, \varepsilon)$ so that for all ζ,

$$P_\zeta \left(\sup\{|N_2(t,x)| : x \in \mathbf{R}, t \in [t_0, \Delta]\} \geq \frac{1}{4} \right) \leq \varepsilon/2. \tag{3.15}$$

On the set in (3.13), the deterministic part $G_{t-t_0} D(t_0, x)$ is bounded by $\frac{1}{2}$. Thus from (3.13 - 3.15), if we choose $\zeta \geq \zeta_3$ satisfying $C(K,T)\zeta^{-1} \leq \varepsilon/2$ and $K\zeta^{-1/2} \leq \Delta$, then

$$P_\zeta \left(D(t,x) \leq \frac{3}{4} \text{ for all } x \in \mathbf{R}, t \in [K\zeta^{-1/2}, \Delta] \right) \geq 1 - \varepsilon,$$

which completes the proof. $\square$

4. The stationary distribution for the interface

In this section, we give the proof of theorem 1. We first show the uniqueness of the law of a stationary interface. This follows immediately from the next lemma.

Lemma 4.1. *Given two probabilities μ^1, μ^2 on $\mathcal{C}_I$, there exist coupled processes (u^1_t, u^2_t), for which u^i has law P_{μ^i} for $i = 1, 2$, and such that, with probability one, u^1 and u^2 are completely coupled for all large times.*

Proof. The basic idea is simple. Lemma 3.1 gives a coupling which has a positive chance of successfully leading to a complete coupling by time one. If it fails we can repeat the attempt. Lemma 1.2 shows that the width of the interfaces will not grow and this leads to repeated attempts at complete coupling with the same chance of success. We now give the details.

It suffices to prove Lemma 4.1 in the case where μ^i gives probability 1 to a single function $f_i \in \mathcal{C}_I$ for $i = 1, 2$. The solutions will be constructed by using the coordinate mappings $u^i_t = U^i_t$, $i = 1, 2$ on the product space $(\Omega \times \Omega, \mathcal{F}_t \times \mathcal{F}_t)$, under a suitable law P. We define the law P inductively over the intervals $[k, k+1], k = 0, 1, \ldots$ Lemma 3.1 constructs a coupling of solutions whose initial conditions u_0, v_0 have interfaces of length at most K. This coupling has probability at least $p_0(K)$ of completely coupling by time 1. Let the law of this coupling over the time interval $[0, 1]$ be $Q(K, u_0, v_0)$. Set $K(k)$ to be the smallest (random) integer greater than $\max\{R(U^1_k) - L(U^1_k), R(U^2_k) - L(U^2_k)\}$. Then the law of $(U^i_t : i = 1, 2,\ t \in [k, k+1])$ conditional on $\mathcal{F}_k \times \mathcal{F}_k$ is defined to be $Q(K(k), U^1_k, U^2_k)$. Define events

$$A_k(l) = \{K(k) \le l\}$$

$$B_k = \{u^1 \text{ and } u^2 \text{ completely couple in the interval } [k, k+1]\}.$$

Lemma 3.1 implies that $P(B_k | \mathcal{F}_k \times \mathcal{F}_k) \ge p_0(K(k))$. Note that we may take $p_0(K)$ to be decreasing in K. To prove that complete coupling occurs at a finite time with probability one, it is enough to show that for any $\varepsilon > 0$ there exists n so that

$$P\left(\bigcap_{k=0}^{n} B^c_k\right) < \varepsilon. \tag{4.1}$$

First use Lemma 1.2 and Chebychev's inequality to pick $l = l(\varepsilon, f_1, f_2)$ so that

$$P(A^c_k(l)) \le \varepsilon/2 \qquad \text{for all } k = 0, 1, \ldots$$

Now suppose that (4.1) fails for all n. Then, for all n,

$$
\begin{aligned}
P\big(B_{n+1}\big| \bigcap_{k=0}^{n} B_k^c\big)& \\
&\geq P\left(B_{n+1}\cap \bigcap_{k=0}^{n} B_k^c\right) \\
&\geq P\left(B_{n+1}\cap A_n(l)\cap \bigcap_{k=0}^{n} B_k^c\right) \\
&= P\left(B_{n+1}\Big| A_n(l)\cap \bigcap_{k=0}^{n} B_k^c\right) P\left(A_n(l)\cap \bigcap_{k=0}^{n} B_k^c\right) \\
&\geq p_0(l)\left(P(A_n(l)) + P(\bigcap_{k=0}^{n} B_k^c) - 1\right) \\
&\geq \frac{p_0(l)\varepsilon}{2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P\left(\bigcap_{k=0}^{n} B_k^c\right) &= P(B_0^c)\prod_{j=0}^{n-1} P\big(B_{j+1}^c\big| \bigcap_{k=0}^{j} B_k^c\big) \\
&\leq \left(1-\frac{p_0(l)\varepsilon}{2}\right)^n.
\end{aligned}
$$

Taking n large enough achieves a contradiction, which proves (4.1) must hold for some n. Now we modify the construction so that after the first time of complete coupling, we let v follow a translated copy of u. This allows the complete coupling to occur for all large times. □

Now we establish the existence of a stationary interface. Fix $f \in \mathcal{C}_I$. Let $\bar{\mu}^t$ be the law of the translated process $\bar{U}_t$ under P_f. The basic idea is to find a limit point of $\{\bar{\mu}_t\}$ and to show that it must be the law of a stationary interface. Our first goal, therefore, is to show that $\{\bar{\mu}^t\}_{t\in[1,\infty)}$ is a tight family of measures on $\mathcal{C}_I$. Fix $\varepsilon > 0$. By Lemma 1.2, there exists $l = l(f,\varepsilon)$ such that for all times $t > 0$, we have

$$
P_f(R(U_t) - L(U_t) \geq l) \leq \varepsilon. \tag{4.2}
$$

By Lemma 1.1 and Chebychev's inequality, there exists $s_0(l,\varepsilon) \leq 1$ so that for any $t \geq 0$,

$$
P_f(L(U_{t+s_0}) \leq L(U_t) - l \text{ or } R(U_{t+s_0}) \geq R(U_t) + l) \leq \varepsilon. \tag{4.3}
$$

Define

$$N_t(s,x) = (U_{t+s}(L(U_t)+x) - G_s U_t(L(U_t)+x))$$

where G_t is the heat semigroup as defined on the first page of the paper. By extending the probability space if necessary, we may construct a white noise so that with respect to this white noise, U_t is a solution to (1.1) under P_ζ. Then, using the integral representation (1.2), we may express $N_t(s,x)$ in the form (3.11). We now apply Lemma 3.6 with $H(s,y) = |U_{t+s}(L(U_t)+y)(1-U_{t+s}(L(U_t)+y))|^{1/2}$. Note that

$$\begin{aligned}
&\int_0^1 \int_{\mathbf{R}} E(U_{t+r}(L(U_t)+z)(1-U_{t+r}(L(U_t)+z))|\mathcal{F}_t)dz\,dr \\
&\qquad \le \int_0^1 \int_{\mathbf{R}} \min\{E(G_r U_t(L(U_t)+z)|\mathcal{F}_t), \\
&\qquad\qquad E(1-G_r U_t(L(U_t)+z)|\mathcal{F}_t)\}dz\,dr \\
&\qquad \le Cl
\end{aligned}$$

on the set $\{R(U_t) - L(U_t) \le l\}$. Then by Lemma 3.6, there exists $\lambda_0(l,\varepsilon)$ such that

(4.4)
$$\begin{aligned}
P_f(|N_t(s_0,x) - N_t(s_0,y)| \le C\lambda_0|x-y|^{1/10} \text{ for all } |x-y| \le 1|\mathcal{F}_t) \\
\ge 1-\varepsilon
\end{aligned}$$

on the set $\{R(U_t) - L(U_t) \le l\}$. Define

$$\begin{aligned}
S(l) &= \{f \in \mathcal{C} : 0 \le L(f) \le R(f) \le l\} \\
H(\lambda) &= \{f \in \mathcal{C} : |f(x)-f(y)| \le \lambda|x-y|^{1/8} \text{ whenever } |x-y| \le 1\}.
\end{aligned}$$

There exists a constant $\lambda_1(s) < \infty$ such that $G_s(f) \in H(\lambda_1(s))$ for all $f \in \mathcal{C}$. By the Arzela-Ascoli theorem, $S(l) \cap H(\lambda)$ is a compact subset of $\mathcal{C}$. Combining (4.2), (4.3), and (4.4) we find that for any $t \ge 0$,

$$P_f(\bar{U}_{t+s_0} \notin S(2l) \cap H(\lambda_1(s_0) + \lambda_0(l,\varepsilon))) \le 3\varepsilon.$$

This proves the desired tightness.

To construct a stationary distribution, we shall need a Feller-like property of the process $\bar{U}_t$. Let Φ be the set of bounded, continuous, non-decreasing functions $\phi : \mathbf{R} \to [0,\infty)$ such that $\phi(x) = 0$ for $x \le 0$.

Lemma 4.2. *For $\phi \in \Phi, l, T > 0$ the map*

$$f \to E_f(\exp(-\int \bar{U}_T(x)\phi(x)dx))$$

is continuous on $S(l) = \{f : 0 \le L(f) \le R(f) \le l\}$.

Proof. Fix $T, l > 0, \phi \in \Phi$ and $f, g \in S(l)$. Take a coupling of four solutions $u^{(f)}, u^{(g)}, u^{(f\wedge g)}, u^{(f\vee g)}$. Here, the superscript denotes the initial value, and we require that $u^{(f\wedge g)} \leq \min(u^{(f)}, u^{(g)})$ and $\max(u^{(f)}, u^{(g)} \leq u^{(f\vee g)}$ for all time. We can construct such a coupling for which all of the solutions are driven by the same white noise (see [Shi94] section 2). In the following we use the simple inequality that $e^{-x} - e^{-y} \leq y - x$ whenever $y \geq x$.

$$\begin{aligned}
&E\left(\exp\left(-\int \bar{u}^{(f)}(T,x)\phi(x)dx)\right) - E(\exp(-\int \bar{u}^{(g)}(T,x)\phi(x)dx\right)\right) \\
&\quad = E\left(\exp\left(-\int u^{(f)}(T,x)\phi(x - L(u^{(f)}(T)))dx\right)\right) \\
&\qquad\quad - E\left(\exp\left(-\int u^{(g)}(T,x)\phi(x - L(u^{(g)}(T)))dx\right)\right) \\
&\quad \leq E\left(\exp\left(-\int u^{(f\wedge g)}(T,x)\phi(x - L(u^{(f\vee g)}(T)))dx\right)\right) \\
&\qquad\quad - E\left(\exp\left(-\int u^{(f\vee g)}(T,x)\phi(x - L(u^{(f\wedge g)}(T)))dx\right)\right) \\
&\quad \leq E\left(\int u^{(f\vee g)}(T,x)\phi(x - L(u^{(f\wedge g)}(T)))dx\right) \\
&\qquad\quad - E\left(\int u^{(f\wedge g)}(T,x)\phi(x - L(u^{(f\vee g)}(T)))dx\right) \\
&\quad \leq \phi(\infty)E\left(\int u^{(f\vee g)}(T,x) - u^{(f\wedge g)}(T,x)dx\right) \\
&\qquad\quad + E\left(\int \phi(x - L(u^{(f\wedge g)}(T))) - \phi(x - L(u^{(f\vee g)}(T)))dx\right) \\
&\quad = \phi(\infty)\int E(u^{(f\vee g)}(T,x)) - E(u^{(f\wedge g)}(T,x))dx \\
&\qquad\quad + \phi(\infty)(E(L(u^{(f\vee g)}(T)) - L(u^{(f\wedge g)}(T))).
\end{aligned}$$

The same bound holds when f and g are interchanged, so that

$$\begin{aligned}
&\left|E_f\left(\exp\left(-\int \bar{U}_T(x)\phi(x)dx\right)\right) - E_g(\exp(-\int \bar{U}_T(x)\phi(x)dx))\right| \\
&\qquad \leq \phi(\infty)\int \left(E_{f\vee g}(U_T(x)) - E_{f\wedge g}(U_T(x))\right)dx \\
&\qquad\quad + \phi(\infty)\left(E_{f\vee g}(L(U_T)) - E_{f\wedge g}(L(U_T))\right).
\end{aligned} \tag{4.5}$$

Suppose that $f, g \in S(l)$ now satisfy $\sup_x |f(x) - g(x)| \leq \delta$. Using the coupling construction of section 3, we may construct another coupling

u, v of solutions to (1.1) so that $v(0) = f \wedge g$, $u(0) = f \vee g$, and the difference process $D = u - v$ remains non-negative and satisfies (3.3). The process $M(t) = \int D(t,x)dx$ is a martingale, and until the stopping time $\tau = \inf\{t : \sup_x D(t,x) \geq 3/4\}$ it satisfies $[M](t) \geq \frac{1}{4}M(t)$. At time zero, we have $\sup_x D(t,x) \leq \delta$ and $M(0) \leq l\delta$. Arguing as in section 3, we may take δ small enough to ensure that $u(T,x) = v(T,x)$ for all x, with probability as close to one as desired. This, and the control on the left and right hand edges of the interface given by Lemma 1.1, are enough to show that by taking δ small, the right hand side of (4.5) can be made arbitrarily small. $\square$

We now complete the proof of existence of a stationary interface. By tightness, there exists a sequence $t_n \to \infty$ and a probability measure ν on $\mathcal{C}$ such that $\bar{\mu}^{t_n} \to \nu$ weakly. We will now show that ν is concentrated on $\{f : L(f) = 0\}$, and is a stationary measure. Fix $T > 0$, and let ν^T be the law of $\bar{U}_T$ under P_ν. For $\phi \in \Phi$,

$$\begin{aligned}
&\int_{\mathcal{C}} \exp\left(-\int f(x)\phi(x)dx\right) \bar{\mu}^{t_n+T}(df) \\
&= \int_{\mathcal{C}} E_f\left(\exp\left(-\int \bar{U}_T(x)\phi(x)dx\right)\right) \bar{\mu}^{t_n}(df). \qquad (4.6)\\
&\to \int_{\mathcal{C}} E_f\left(\exp\left(-\int \bar{U}_T(x)\phi(x)dx\right)\right) \nu(df) \\
&= \int_{\mathcal{C}} \exp\left(-\int f(x)\phi(x)dx\right) \nu^T(df)
\end{aligned}$$

To justify the above convergence, first approximate by reducing the integral to the closed subset $S(l)$. Then apply the weak convergence of $\bar{\mu}^{t_n}$, using Lemma 4.2 to see that the integrand is continuous.

By the coupling Lemma 4.1, there is a coupling of processes u, v where u has law P_f and v has law $P_{\bar{\mu}_T}$, and the processes completely couple with probability 1. Therefore, the total variation distance between the law $\bar{\mu}^{t_n}$ of $\bar{u}(t_n)$ and $\bar{\mu}^{t_n+T}$ of $\bar{v}(t_n)$ tends to 0 as $n \to \infty$. The map $f \to \exp(-\int f(x)\phi(x)dx)$ is continuous on each $S(l)$, so that

$$\begin{aligned}
(4.7) \qquad &\lim_{n\to\infty} \int_{\mathcal{C}} \exp\left(-\int f(x)\phi(x)dx\right) \bar{\mu}^{t_n+T}(df) \\
&= \lim_{n\to\infty} \int_{\mathcal{C}} \exp\left(-\int f(x)\phi(x)dx\right) \bar{\mu}^{t_n}(df) \\
&= \int_{\mathcal{C}} \exp\left(-\int f(x)\phi(x)dx\right) \nu(df).
\end{aligned}$$

Combining (4.7) with (4.6) shows that

$$\int_{\mathcal{C}} \exp\left(-\int f(x)\phi(x)\right)\nu^T(df) = \int_{\mathcal{C}} \exp\left(-\int f(x)\phi(x)\right)\nu(df).$$

Since this is true for all $\phi \in \Phi$, we have that $\nu^T = \nu$. Since ν_T is concentrated on $\{f : L(f) = 0\}$, so is ν, and since $\nu^T = \nu$ for all $T > 0$, we have proved the required stationarity.

Applying Lemma 4.1 with μ^1 a point mass at $f \in \mathcal{C}_I$ and μ^2 the law of the stationary interface, we see that the law $P_f(\bar{U}_t \in \cdot)$ converges in total variation to that of the stationary interface.

Finally, we prove the finiteness of the moments stated in theorem 1. The finite moments of the width follow immediately from Lemma 1.2 and Fatou's lemma. For the blow up of the higher moments, we need another moment result (see [Tri95] Lemma 2.1). If u is a solution to (1.1) with deterministic initial condition $f \in \mathcal{C}_I$, then for any $x \geq 0$,

$$E\left(\int u(t,z+x)(1-u(t,z))dz\right) \to 1 \quad \text{as } t \to \infty. \tag{4.8}$$

Let μ be the law of the stationary interface. Note that $U(t,x+z)(1-U(t,z)) = 0$ if $z \leq L(U_t)$ or if $z + x \geq R(U_t)$. So

$$\begin{aligned}
&\int U_t(x+z)(1-U_t(z))dz \\
&\qquad\qquad \leq \int \mathbf{1}_{(z \leq L(U_t), x+z \geq R(U_t))} dz \\
&\qquad\qquad = (R(U_t) - x - L(U_t))_+ \\
&\qquad\qquad \leq (R(U_t) - L(U_t))\mathbf{1}_{(R(U_t)-L(U_t) \geq x)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\mathcal{C}_I} (R(f) - L(f))\mathbf{1}_{(R(f)-L(f)\geq x)}\mu(df) \\
&\qquad = \int_{\mathcal{C}_I}\int_{\Omega} (R(U_t) - L(U_t))\mathbf{1}_{(R(U_t)-L(U_t)\geq x)} dP_f \mu(df) \\
&\qquad \geq \int_{\mathcal{C}_I}\int_{\Omega}\int_{\mathbf{R}} U_t(x+z)(1-U_t(z))dz dP_f \mu(df).
\end{aligned}$$

Letting $t \to \infty$, and using Fatou's lemma and (4.8), we see that for all $x \geq 0$,

$$\int_{\bar{\mathcal{C}}_I} (R(f) - L(f))\mathbf{1}_{(R(f)-L(f)\geq x)}\mu(df) \geq 1.$$

This is only possible if $\int_{\bar{\mathcal{C}}_I}(R(f) - L(f))\mu(df) = \infty$. This shows the first and hence all higher moments are infinite, and completes the proof of the theorem.

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