

$L_p$  ESTIMATES FOR SPDE WITH DISCONTINUOUS  
COEFFICIENTS IN DOMAINS

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ABSTRACT. Stochastic partial differential equations of divergence form with discontinuous and unbounded coefficients are considered in  $C^1$  domains. Existence and uniqueness results are given in weighted  $L_p$  spaces, and Hölder type estimates are presented.

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## 1. INTRODUCTION

Let  $G$  be an open set in  $\mathbb{R}^d$ . We consider parabolic stochastic partial differential equations of the form

$$du = (D_i(a^{ij}u_{xj} + b^i u + f^i) + \bar{b}^i u_{x^i} + cu + \bar{f}) dt + (\nu^k u + g^k) dw_t^k, \quad (1.1)$$

given for  $x \in G, t \geq 0$ . Here  $w_t^k$  are independent one-dimensional Wiener processes,  $i$  and  $j$  go from 1 to  $d$ , and  $k$  runs through  $\{1, 2, \dots\}$ . The coefficients  $a^{ij}, b^i, \bar{b}^i, c, \nu^k$  and the free terms  $f^i, \bar{f}, g^k$  are random functions depending on  $t > 0$  and  $x \in G$ .

This article is a natural continuation of the article [15], where  $L_p$  estimates for the equation

$$du = D_i(a^{ij}u_{xj} + f^i) dt + (\nu^k u + g^k) dw_t^k \quad (1.2)$$

with discontinuous coefficients was constructed on  $\mathbb{R}^d$ .

Our approach is based on Sobolev spaces with or without weights, and we present the unique solvability result of equation (1.1) on  $\mathbb{R}^d, \mathbb{R}_+^d$  (half space) and on bounded  $C^1$  domains. We show that  $L_p$ -norm of  $u_x$  can be controlled by  $L_p$ -norms of  $f^i, \bar{f}$  and  $g$  if  $p$  is sufficiently close to 2.

Pulvirenti [13] showed by example that without the continuity of  $a^{ij}$  in  $x$  one can not fix  $p$  even for deterministic parabolic equations. For an  $L_p$  theory of linear SPDEs with continuous coefficients on domains, we refer to [1], [2] and [7].

Actually  $L_2$  theory for type (1.1) with bounded coefficients was developed long times ago on the basis of monotonicity method, and an account of it can be found in [14]. But our results are new even for  $p = 2$  (and probably even for deterministic equation) since, for instance, we are only assuming the functions

$$\rho b^i, \quad \rho \bar{b}^i, \quad \rho^2 c, \quad \rho \nu^k$$

are bounded, where  $\rho(x) = \text{dist}(x, \partial G)$ . Thus we are allowing our coefficients to blow up near the boundary of  $G$ .

An advantage of  $L_p(p > 2)$  theory can be found, for instance, in [16], where solvability of some nonlinear SPDEs was presented with the help of  $L_p$  estimates for linear SPDEs with discontinuous coefficients. Also we will see that some Hölder type estimates are valid only for  $p > 2$  (Corollary 2.5).

We finish the introduction with some notations. As usual  $\mathbb{R}^d$  stands for the Euclidean space of points  $x = (x^1, \dots, x^d)$ ,  $\mathbb{R}_+^d = \{x \in \mathbb{R}^d : x^1 > 0\}$  and  $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ . For  $i = 1, \dots, d$ , multi-indices

$\alpha = (\alpha_1, \dots, \alpha_d)$ ,  $\alpha_i \in \{0, 1, 2, \dots\}$ , and functions  $u(x)$  we set

$$u_{x^i} = \partial u / \partial x^i = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdot \dots \cdot D_d^{\alpha_d} u, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

## 2. MAIN RESULTS

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space,  $\{\mathcal{F}_t, t \geq 0\}$  be an increasing filtration of  $\sigma$ -fields  $\mathcal{F}_t \subset \mathcal{F}$ , each of which contains all  $(\mathcal{F}, P)$ -null sets. By  $\mathcal{P}$  we denote the predictable  $\sigma$ -field generated by  $\{\mathcal{F}_t, t \geq 0\}$  and we assume that on  $\Omega$  we are given independent one-dimensional Wiener processes  $w_t^1, w_t^2, \dots$ , each of which is a Wiener process relative to  $\{\mathcal{F}_t, t \geq 0\}$ .

Fix an increasing function  $\kappa_0$  defined on  $[0, \infty)$  such that  $\kappa_0(\varepsilon) \rightarrow 0$  as  $\varepsilon \downarrow 0$ .

**Assumption 2.1.** The domain  $G \subset \mathbb{R}^d$  is of class  $C_u^1$ . In other words, there exist constants  $r_0, K_0 > 0$  such that for any  $x_0 \in \partial G$  there exists a one-to-one continuously differentiable mapping  $\Psi$  from  $B_{r_0}(x_0)$  onto a domain  $J \subset \mathbb{R}^d$  such that

- (i)  $J_+ := \Psi(B_{r_0}(x_0) \cap G) \subset \mathbb{R}_+^d$  and  $\Psi(x_0) = 0$ ;
- (ii)  $\Psi(B_{r_0}(x_0) \cap \partial G) = J \cap \{y \in \mathbb{R}^d : y^1 = 0\}$ ;
- (iii)  $\|\Psi\|_{C^1(B_{r_0}(x_0))} \leq K_0$  and  $|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \leq K_0|y_1 - y_2|$  for any  $y_i \in J$ ;
- (iv)  $|\Psi_x(x_1) - \Psi_x(x_2)| \leq \kappa_0(|x_1 - x_2|)$  for any  $x_i \in B_{r_0}(x_0)$ .

**Assumption 2.2.** (i) For each  $x \in G$ , the functions  $a^{ij}(t, x)$ ,  $b^i(t, x)$ ,  $\bar{b}^i(t, x)$ ,  $c(t, x)$  and  $\nu^k(t, x)$  are predictable functions of  $(\omega, t)$ .

(ii) There exist constants  $\lambda, \Lambda \in (0, \infty)$  such that for any  $\omega, t, x$  and  $\xi \in \mathbb{R}^d$ ,

$$\lambda|\xi|^2 \leq a^{ij}\xi^i\xi^j \leq \Lambda|\xi|^2.$$

(iii) For any  $x, t$  and  $\omega$ ,

$$\rho(x)|b^i(t, x)| + \rho(x)|\bar{b}^i(t, x)| + \rho(x)^2|c(t, x)| + \rho(x)|\nu^k(t, x)|_{\ell_2} \leq K.$$

(iv) There is control on the behavior of  $b^i, \bar{b}^i, c, \nu$  near  $\partial G$ , namely,

$$\lim_{\substack{\rho(x) \rightarrow 0 \\ x \in G}} \sup_{t, \omega} \rho(x)(|b^i(t, x)| + |\bar{b}^i(t, x)| + \rho(x)|c(t, x)| + |\nu(t, x)|_{\ell_2}) = 0. \quad (2.1)$$

To describe the assumptions of  $f^i, \bar{f}$  and  $g$  we use Sobolev spaces introduced in [7], [8] and [12]. If  $n$  is a non negative integer, then

$$\begin{aligned} H_p^n &= H_p^n(\mathbb{R}^d) = \{u : u, Du, \dots, D^\alpha u \in L_p : |\alpha| \leq n\}, \\ L_{p, \theta}(G) &:= H_{p, \theta}^0(G) = L_p(G, \rho^{\theta-d} dx), \quad \rho(x) := \text{dist}(x, \partial G), \\ H_{p, \theta}^n(G) &:= \{u : u, \rho u_x, \dots, \rho^{|\alpha|} D^\alpha u \in L_{p, \theta}(G) : |\alpha| \leq n\}. \end{aligned} \quad (2.2)$$

In general, by  $H_p^\gamma = H_p^\gamma(\mathbb{R}^d) = (1 - \Delta)^{-\gamma/2} L_p$  we denote the space of Bessel potential, where

$$\|u\|_{H_p^\gamma} = \|(1 - \Delta)^{\gamma/2} u\|_{L_p},$$

and the weighted Sobolev space  $H_{p,\theta}^\gamma(G)$  is defined as the set of all distributions  $u$  on  $G$  such that

$$\|u\|_{H_{p,\theta}^\gamma(G)}^p := \sum_{n=-\infty}^{\infty} e^{n\theta} \|\zeta_{-n}(e^n \cdot) u(e^n \cdot)\|_{H_p^\gamma}^p < \infty, \quad (2.3)$$

where  $\{\zeta_n : n \in \mathbb{Z}\}$  is a sequence of functions  $\zeta_n \in C_0^\infty(G)$  such that

$$\sum_n \zeta_n \geq c > 0, \quad |D^m \zeta_n(x)| \leq N(m) e^{mn}.$$

If  $G = \mathbb{R}_+^d$  we fix a function  $\zeta \in C_0^\infty(\mathbb{R}_+)$  such that

$$\sum_{n \in \mathbb{Z}} \zeta(e^{n+x}) \geq c > 0, \quad \forall x \in \mathbb{R}, \quad (2.4)$$

and define  $\zeta_n(x) = \zeta(e^n x)$ , then (2.3) becomes

$$\|u\|_{H_{p,\theta}^\gamma}^p := \sum_{n=-\infty}^{\infty} e^{n\theta} \|\zeta(\cdot) u(e^n \cdot)\|_{H_p^\gamma}^p < \infty. \quad (2.5)$$

It is known that up to equivalent norms the space  $H_{p,\theta}^\gamma$  is independent of the choice  $\zeta$ , and  $H_{p,\theta}^\gamma(G)$  and its norm are independent of  $\{\zeta_n\}$  if  $G$  is bounded.

We use above notations for  $\ell_2$ -valued functions  $g = (g_1, g_2, \dots)$ . For instance

$$\|g\|_{H_p^\gamma(\ell_2)} = \|(1 - \Delta)^{\gamma/2} g\|_{\ell_2} \|L_p.$$

For any stopping time  $\tau$ , denote  $(0, \tau] = \{(\omega, t) : 0 < t \leq \tau(\omega)\}$ ,

$$\mathbb{H}_p^\gamma(\tau) = L_p((0, \tau], \mathcal{P}, H_p^\gamma), \quad \mathbb{H}_{p,\theta}^\gamma(G, \tau) = L_p((0, \tau], \mathcal{P}, H_{p,\theta}^\gamma(G)),$$

$$\mathbb{H}_{p,\theta}^\gamma(\tau) = L_p((0, \tau], \mathcal{P}, H_{p,\theta}^\gamma), \quad \mathbb{L}_{\dots}(\dots) = \mathbb{H}_{\dots}^0(\dots).$$

Fix (see [5]) a bounded real-valued function  $\psi$  defined in  $\bar{G}$  such that for any multi-index  $\alpha$ ,

$$[\psi]_{|\alpha|}^{(0)} := \sup_G \rho^{|\alpha|}(x) |D^\alpha \psi_x(x)| < \infty$$

and the functions  $\psi$  and  $\rho$  are comparable in a neighborhood of  $\partial G$ . As in [11], by  $M^\alpha$  we denote the operator of multiplying by  $(x^1)^\alpha$  and  $M = M^1$ . Define

$$U_p^\gamma = L_p(\Omega, \mathcal{F}_0, H_p^{\gamma-2/p}), \quad U_{p,\theta}^\gamma = M^{1-2/p} L_p(\Omega, \mathcal{F}_0, H_{p,\theta}^{\gamma-2/p}),$$

$$U_{p,\theta}^\gamma(G) = \psi^{1-2/p} L_p(\Omega, \mathcal{F}_0, H_{p,\theta}^{\gamma-2/p}(G)).$$

By  $\mathfrak{H}_{p,\theta}^\gamma(G, \tau)$  we denote the space of all functions  $u \in \psi \mathbb{H}_{p,\theta}^\gamma(G, \tau)$  such that  $u(0, \cdot) \in U_{p,\theta}^\gamma(G)$  and for some  $f \in \psi^{-1} \mathbb{H}_{p,\theta}^{\gamma-2}(G, \tau)$ ,  $g \in \mathbb{H}_{p,\theta}^{\gamma-1}(G, \tau)$ ,

$$du = f dt + g^k dw_t^k, \quad (2.6)$$

in the sense of distributions. In other words, for any  $\phi \in C_0^\infty(G)$ , the equality

$$(u(t, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^t (f(s, \cdot), \phi) ds + \sum_0^\infty \int_0^t (g^k(s, \cdot), \phi) dw_s^k$$

holds for all  $t \leq \tau$  with probability 1.

The norm in  $\mathfrak{H}_{p,\theta}^\gamma(G, \tau)$  is introduced by

$$\begin{aligned} \|u\|_{\mathfrak{H}_{p,\theta}^\gamma(G, \tau)} &= \|\psi^{-1}u\|_{\mathbb{H}_{p,\theta}^\gamma(G, \tau)} + \|\psi f\|_{\mathbb{H}_{p,\theta}^{\gamma-2}(G, \tau)} \\ &\quad + \|g\|_{\mathbb{H}_{p,\theta}^{\gamma-1}(G, \tau)} + \|u(0, \cdot)\|_{U_{p,\theta}^\gamma(G)}. \end{aligned}$$

It is easy to check that up to equivalent norms the space  $\mathfrak{H}_{p,\theta}^\gamma(G, \tau)$  and its norm are independent of the choice of  $\psi$  if  $G$  is bounded.

We write  $u \in \mathfrak{H}_{p,\theta}^\gamma(\tau)$  if  $u \in M \mathbb{H}_{p,\theta}^\gamma(\tau)$  satisfies (2.6) for some  $f \in M^{-1} \mathbb{H}_{p,\theta}^{\gamma-2}(\tau)$ ,  $g \in \mathbb{H}_{p,\theta}^{\gamma-1}(\tau, \ell_2)$ , and we define

$$\begin{aligned} \|u\|_{\mathfrak{H}_{p,\theta}^\gamma(\tau)} &= \|M^{-1}u\|_{\mathbb{H}_{p,\theta}^\gamma(\tau)} + \|Mf\|_{\mathbb{H}_{p,\theta}^{\gamma-2}(\tau)} \\ &\quad + \|g\|_{\mathbb{H}_{p,\theta}^{\gamma-1}(\tau)} + \|u(0, \cdot)\|_{U_{p,\theta}^\gamma}. \end{aligned}$$

Similarly we define stochastic Banach space  $\mathcal{H}_p^\gamma(\tau)$  on  $\mathbb{R}^d$  (and its norm) by formally taking  $\psi = 1$  and replacing  $H_{p,\theta}^\gamma(G), U_{p,\theta}^\gamma(G)$  by  $H_p^\gamma, U_p^\gamma$ , respectively, in the definition of the space  $\mathfrak{H}_{p,\theta}^\gamma(G, \tau)$ .

We drop  $\tau$  in the notations of appropriate Banach spaces if  $\tau \equiv \infty$ . Note that if  $G = \mathbb{R}_+^d$ , then  $\mathfrak{H}_{p,\theta}^\gamma(G, \tau)$  is slightly different from  $\mathfrak{H}_{p,\theta}^\gamma(\tau)$  since  $\psi(x)$  is bounded. Finally we define

$$\begin{aligned} \mathfrak{H}_{p,\theta,0}^\gamma(\dots) &= \mathfrak{H}_{p,\theta}^\gamma(\dots) \cap \{u : u(0, \cdot) = 0\}, \\ \mathcal{H}_{p,0}^\gamma(\dots) &= \mathcal{H}_p^\gamma(\dots) \cap \{u : u(0, \cdot) = 0\}. \end{aligned}$$

Some properties of the spaces  $H_{p,\theta}^\gamma, \mathfrak{H}_{p,\theta}^\gamma(G, \tau)$  and  $\mathcal{H}_p^\gamma(\tau)$  are collected in the following lemma (see [3],[7], [8] and [12] for detail). From now on we assume that

$$p \geq 2, \quad d-1 < \theta < d-1+p.$$

**Lemma 2.3.** (i) *The following are equivalent:*

- (a)  $u \in H_{p,\theta}^\gamma(G)$ ,
- (b)  $u \in H_{p,\theta}^{\gamma-1}(G)$  and  $\psi Du \in H_{p,\theta}^{\gamma-1}(G)$ ,
- (c)  $u \in H_{p,\theta}^{\gamma-1}(G)$  and  $D(\psi u) \in H_{p,\theta}^{\gamma-1}(G)$ .

In addition, under either of these three conditions

$$\|u\|_{H_{p,\theta}^\gamma(G)} \leq N\|\psi u_x\|_{H_{p,\theta}^{\gamma-1}(G)} \leq N\|u\|_{H_{p,\theta}^\gamma(G)}, \quad (2.7)$$

$$\|u\|_{H_{p,\theta}^\gamma(G)} \leq N\|(\psi u)_x\|_{H_{p,\theta}^{\gamma-1}(G)} \leq N\|u\|_{H_{p,\theta}^\gamma(G)}. \quad (2.8)$$

(ii) For any  $\nu, \gamma \in \mathbb{R}$ ,  $\psi^\nu H_{p,\theta}^\gamma(G) = H_{p,\theta-p\nu}^\gamma(G)$ , and

$$\|u\|_{H_{p,\theta-p\nu}^\gamma(G)} \leq N\|\psi^{-\nu}u\|_{H_{p,\theta}^\gamma(G)} \leq N\|u\|_{H_{p,\theta-p\nu}^\gamma(G)}.$$

(iii) There exists a constant  $N$  depending only on  $d, p, \gamma, T$  (and  $\theta$ ) such that for any  $t \leq T$ ,

$$\|u\|_{H_{p,\theta}^\gamma(G,t)}^p \leq N \int_0^t \|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+1}(G,s)}^p ds \leq Nt\|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+1}(G,t)}^p, \quad (2.9)$$

$$\|u\|_{H_p^\gamma(t)}^p \leq N \int_0^t \|u\|_{\mathcal{H}_p^{\gamma+1}(s)}^p ds \leq Nt\|u\|_{\mathcal{H}_p^{\gamma+1}(t)}^p. \quad (2.10)$$

(iv) Let  $\gamma - d/p = m + \nu$  for some  $m = 0, 1, \dots$  and  $\nu \in (0, 1)$ , then for any  $k \leq m$ ,

$$|\psi^{k+\theta/p} D^k u|_{C^0} + [\psi^{m+\nu+\theta/p} D^m u]_{C^\nu(G)} \leq N\|u\|_{H_{p,\theta}^\gamma(G)}.$$

(v) Let

$$2/p < \alpha < \beta \leq 1.$$

Then for any  $u \in \mathfrak{H}_{p,\theta,0}^\gamma(G, \tau)$  and  $0 \leq s < t \leq \tau$ ,

$$E\|\psi^{\beta-1}(u(t) - u(s))\|_{H_{p,\theta}^{\gamma-\beta}(G)}^p \leq N|t - s|^{p\beta/2-1}\|u\|_{\mathfrak{H}_{p,\theta}^\gamma(G,\tau)}^p, \quad (2.11)$$

$$E\|\psi^{\beta-1}u\|_{C^{\alpha/2-1/p}([0,\tau], H_{p,\theta}^{\gamma-\beta}(G))}^p \leq N\|u\|_{\mathfrak{H}_{p,\theta}^\gamma(G,\tau)}^p. \quad (2.12)$$

Here are our main results.

**Theorem 2.4.** Assume  $G$  is bounded and  $\tau \leq T$ . Under the above assumptions, there exist  $p_0 = p_0(\lambda, \Lambda, d) > 2$  and  $\chi = \chi(p, d, \lambda, \Lambda) > 0$  such that if  $p \in [2, p_0)$  and  $\theta \in (d - \chi, d + \chi)$ , then

(i) for any  $f^i \in \mathbb{L}_{p,\theta}(G, \tau)$ ,  $\bar{f} \in \psi^{-1}\mathbb{H}_{p,\theta}^{-1}(G, \tau)$ ,  $g \in \mathbb{L}_{p,\theta}(G, \tau)$  and  $u_0 \in U_{p,\theta}^1(G)$  equation (1.1) admits a unique solution  $u \in \mathfrak{H}_{p,\theta}^1(G, \tau)$ ,

(ii) for this solution

$$\|\psi^{-1}u\|_{\mathbb{H}_{p,\theta}^1(G,\tau)} \leq N(\|f^i\|_{\mathbb{L}_{p,\theta}(G,\tau)} + \|\psi\bar{f}\|_{\mathbb{H}_{p,\theta}^{-1}(G,\tau)} + \|g\|_{\mathbb{L}_{p,\theta}(G,\tau)} + \|u_0\|_{U_{p,\theta}^1(G)}), \quad (2.13)$$

where the constant  $N$  is independent of  $f^i, \bar{f}, g, u$  and  $u_0$ .

Lemma 2.3 (iv) and (v) yield the following results. It is crucial that  $p$  is bigger than 2.

**Corollary 2.5.** *Let  $u \in \mathfrak{H}_{p,\theta,0}^1(G, \tau)$  be the solution of (1.1) and*

$$2/p < \alpha < \beta \leq 1.$$

(i) *Then for any  $0 \leq s < t \leq \tau$ ,*

$$E \|\psi^{\beta-1}(u(t) - u(s))\|_{H_{p,\theta}^{1-\beta}(G)}^p \leq N |t - s|^{p\beta/2-1} C(f^i, \bar{f}, g, \theta) \quad (2.14)$$

$$E |\psi^{\beta-1}u|_{C^{\alpha/2-1/p}([0,\tau], H_{p,\theta}^{1-\beta}(G))}^p \leq NC(f^i, \bar{f}, g, \theta), \quad (2.15)$$

where  $C(f^i, \bar{f}, g, \theta) := \|f^i\|_{\mathbb{L}_{p,\theta}(G,\tau)} + \|\psi \bar{f}\|_{\mathbb{H}_{p,\theta}^{-1}(G,\tau)} + \|g\|_{\mathbb{L}_{p,\theta}(G,\tau)}$ .

(ii) *If  $d \leq 2, 1 - d/p =: \nu$ , then*

$$E \int_0^\tau (|\psi^{\theta/p-1}u|_{C^0} + [\psi^{(\theta-d)/p}u]_{C^\nu(G)}) dt \leq NC(f^i, \bar{f}, g, \theta), \quad (2.16)$$

thus if  $\theta \leq d$ , then the function  $u$  itself is Hölder continuous in  $x$ .

The following corollary shows that if some extra conditions are assumed, then the solutions are Hölder continuous in  $(t, x)$  (regardless of the dimension  $d$ ).

**Corollary 2.6.** *Let  $u \in \mathfrak{H}_{p,d,0}^1(G, T)$  be the solution of (1.1). Assume that  $b^i, \bar{b}, c$  are bounded,  $\nu = 0$  and*

$$1 - 2/q - d/r > 0, \quad q \geq r > 2,$$

$$f^i, f, g \in L_q(\Omega \times [0, T], \mathcal{P}, L_r(G)).$$

Then there exists  $\alpha = \alpha(q, r, d, G) > 0$  such that

$$E |u|_{C^\alpha(G \times [0, T])}^q < \infty. \quad (2.17)$$

*Proof.* It is shown in [3] that under the conditions of the corollary, there is a solution  $v \in \mathfrak{H}_{2,d,0}^1(G, T)$  satisfying (2.17). By the uniqueness result (Theorem 2.4) in the space  $\mathfrak{H}_{2,d}^1(G, T)$ , we conclude that  $u = v$  and thus  $v \in \mathfrak{H}_{p,d}^1(G, T)$ .  $\square$

We will see that the proof of Theorems 2.4 depends also on the following results on  $\mathbb{R}_+^d$  and  $\mathbb{R}^d$ .

**Theorem 2.7.** *Assume that*

$$x^1 |b^i(t, x)| + x^1 |\bar{b}^i(t, x)| + (x^1)^2 |c(t, x)| + x^1 |\nu(t, x)| \leq \beta, \quad \forall \omega, t, x.$$

Then there exist  $p_0 = p_0(\lambda, \Lambda, d) > 2$ ,  $\beta_0 = \beta_0(p, d, \lambda, \Lambda) \in (0, 1)$  and  $\chi = \chi(p, d, \lambda, \Lambda) > 0$  such that if

$$\beta \leq \beta_0, \quad p \in [2, p_0), \quad d - \chi < \theta < d + \chi, \quad (2.18)$$

then for any  $f^i \in \mathbb{L}_{p,\theta}(\tau)$ ,  $\bar{f} \in M^{-1}\mathbb{H}_{p,\theta}^{-1}(\tau)$ ,  $g \in \mathbb{L}_{p,\theta}(\tau)$  and  $u_0 \in U_{p,\theta}^1$  equation (1.1) with initial data  $u_0$  admits a unique solution  $u$  in the class  $\mathfrak{H}_{p,\theta}^1(\tau)$  and for this solution,

$$\|M^{-1}u\|_{\mathbb{H}_{p,\theta}^1(\tau)} \leq N(\|f^i\|_{\mathbb{L}_{p,\theta}(\tau)} + \|M\bar{f}\|_{\mathbb{H}_{p,\theta}^{-1}(\tau)} + \|g\|_{\mathbb{L}_{p,\theta}(\tau)} + \|u_0\|_{U_{p,\theta}^1}), \quad (2.19)$$

where  $N$  depends only  $d, p, \theta, \lambda$  and  $\Lambda$ .

**Theorem 2.8.** *Assume that*

$$|b^i(t, x)| + |\bar{b}^i(t, x)| + |c(t, x)| + |\nu(t, x)| \leq K, \quad \forall \omega, t, x.$$

Then there exists  $p_0 > 2$  such that if  $p \leq [2, p_0)$ , then for any  $f^i \in \mathbb{L}_p(\tau)$ ,  $\bar{f} \in \mathbb{H}_p^{-1}(\tau)$ ,  $g \in \mathbb{L}_p(\tau)$ ,  $u_0 \in U_p^1$  equation (1.1) with initial data  $u_0$  admits a unique solution  $u$  in the class  $\mathcal{H}_p^1(\tau)$  and for this solution,

$$\|u\|_{\mathbb{H}_p^1(\tau)} \leq N(\|f^i\|_{\mathbb{L}_p(\tau)} + \|\bar{f}\|_{\mathbb{H}_p^{-1}(\tau)} + \|g\|_{\mathbb{L}_p(\tau)} + \|u_0\|_{U_p^1}), \quad (2.20)$$

where  $N$  depends only  $d, p, \lambda, \Lambda, K$  and  $T$ .

### 3. PROOF OF THEOREM 2.7

First we prove the following lemmas.

**Lemma 3.1.** *Let  $f = (f^1, f^2, \dots, f^d), g = (g^1, g^2, \dots) \in \mathbb{L}_{2,d}(T)$  and  $u \in \mathfrak{H}_{2,d,0}^1(T)$  be a solution of*

$$du = (\Delta u + f_{x^i}^i)dt + g^k dw_t^k. \quad (3.1)$$

Then

$$\|u_x\|_{\mathbb{L}_{2,d}(T)}^2 \leq \|f\|_{\mathbb{L}_{2,d}(T)}^2 + \|g\|_{\mathbb{L}_{2,d}(T)}^2. \quad (3.2)$$

*Proof.* It is well known (see [11]) that (3.1) has a unique solution  $u \in \mathfrak{H}_{p,d,0}^1(T)$  and

$$\|u_x\|_{\mathbb{L}_{p,d}(T)}^p \leq N(p)(\|f\|_{\mathbb{L}_{p,d}(T)}^p + \|g\|_{\mathbb{L}_{p,d}(T)}^p). \quad (3.3)$$

We will show that one can take  $N(2) = 1$ . Let  $\Theta$  be the collections of the form

$$f(t, x) = \sum_{i=1}^m I_{(\tau_{i-1}, \tau_i]}(t) f_i(x),$$

where  $f_i \in C_0^\infty(\mathbb{R}_+^d)$  and  $\tau_i$  are stopping times,  $\tau_i \leq \tau_{i+1} \leq T$ . It is well known that the set  $\Theta$  is dense in  $\mathbb{H}_{p,\theta}^\gamma(T)$  for any  $\gamma, \theta \in \mathbb{R}$ . Also the collection of sequences  $g = (g^k)$ , such that each  $g_k \in \Theta$  and only finitely many of  $g_k$  are different from zero, is dense in  $\mathbb{H}_{p,\theta}^\gamma(T, \ell_2)$ . Thus by considering approximation argument, we may assume that  $f$  and  $g$  are of this type.



We continue  $f(t, x)$  to be an even function and  $g(t, x)$  to be an odd function of  $x^1$ . Then obviously  $f, g \in \mathbb{H}_p^\gamma(T)$  for any  $\gamma$  and  $p$ . By Theorem 5.1 in [7], equation (3.1) considered in the whole  $\mathbb{R}^d$  has a unique solution  $v \in \mathcal{H}_p^1$  and  $v \in \mathcal{H}_p^\gamma$  for any  $\gamma$ . Also by the uniqueness it follows that  $v$  is an odd function of  $x^1$  and vanishes at  $x^1 = 0$ . Moreover remembering the fact that  $v$  satisfies

$$dv = \Delta v dt$$

outside the support of  $f$  and  $g$ , we conclude (see the proof of Lemma 4.2 in [10] for detail) that  $v \in \mathfrak{H}_{p,d}^\gamma$  for any  $\gamma$ .

Thus, both  $u$  and  $v$  satisfy (3.1) considered in  $\mathbb{R}_+^d$  and belong to  $\mathfrak{H}_{p,d}^1$ . By the uniqueness result (Theorem 3.3 in [11]) on  $\mathbb{R}_+^d$ , we conclude that  $u = v$ .

Finally, we see that (3.2) follows from Itô's formula. Indeed (remember that  $u$  is infinitely differentiable and vanishes at  $x^1 = 0$ ),

$$|u(t, x)|^2 = \int_0^t (2u\Delta u + 2u f_{x^i}^i + |g|_{\ell_2}^2) dt + 2 \int_0^t u g^k dw_t^k,$$

therefore

$$\begin{aligned} 0 &\leq E \int_{\mathbb{R}_+^d} |u(t, x)|^2 dx = -2E \int_0^t \int_{\mathbb{R}_+^d} |Du(s, x)|^2 dx dt \\ &\quad - 2E \int_0^t \int_{\mathbb{R}_+^d} f^i D^i u dx dt + E \int_0^t \int_{\mathbb{R}_+^d} |g|_{\ell_2}^2 dx dt \\ &\leq -E \int_0^t \int_{\mathbb{R}_+^d} |Du(s, x)|^2 dx dt \\ &\quad + E \int_0^t \int_{\mathbb{R}_+^d} |f|^2 dx dt + E \int_0^t \int_{\mathbb{R}_+^d} |g|_{\ell_2}^2 dx dt. \end{aligned}$$

□

**Lemma 3.2.** *There exists  $p_0 = p_0(\lambda, \Lambda, d) > 2$  such that if  $p \in [2, p_0)$  and  $u \in \mathfrak{H}_{p,d,0}^1(T)$  is a solution of*

$$du = D_i(a^{ij}u_{x^j} + f^i)dt + g^k dw_t^k, \quad (3.4)$$

then

$$\|u_x\|_{\mathbb{L}_{p,d}(T)} \leq N(\|f\|_{\mathbb{L}_{p,d}(T)} + \|g\|_{\mathbb{L}_{p,d}(T)}), \quad (3.5)$$

where  $N$  is independent of  $T$ .

*Proof.* We repeat arguments in [15]. Take  $N(p)$  from (3.3). By (real-valued version) Riesz-Thorin theorem we may assume that  $N(p) \searrow 1$  as  $p \searrow 2$ . Indeed, consider the operator

$$\Phi : (f^i, g) \rightarrow Du,$$

where  $u \in \mathfrak{H}_{p,d,0}^1$  is the solution of (3.1). Then for any  $r > 2$  and  $p \in [2, r]$ ,

$$\|\Phi\|_p \leq \|\Phi\|_2^{1-\alpha} \|\Phi\|_r^\alpha, \quad 1/p = (1-\alpha)/2 + \alpha/r,$$

and (as  $p \rightarrow 2$ )

$$\|\Phi\|_p \leq \|\Phi\|_r^\alpha = \|\Phi\|_r^{(1/2-1/p)/(1/2-1/r)} \rightarrow 1.$$

Denote  $A := (a^{ij})$ ,  $\kappa := \frac{\lambda+\Lambda}{2}$  and observe that the eigenvalues of  $A - \kappa I$  satisfy

$$-(\Lambda - \lambda)/2 = \lambda - \kappa \leq \lambda_1 - \kappa \leq \dots \leq \lambda_d - \kappa \leq \Lambda - \kappa = (\Lambda - \lambda)/2,$$

and therefore for any  $\xi \in \mathbb{R}^d$ ,

$$|(a^{ij} - \kappa I)\xi| \leq \frac{\Lambda - \lambda}{2} |\xi|. \quad (3.6)$$

Assume that  $v \in \mathfrak{H}_{p,d,0}^1(T)$  satisfies

$$dv = (\kappa \Delta v + f_{x^i}^i) dt + g^k dw_t^k.$$

Then  $\bar{v}(t, x) := v(t, \sqrt{\kappa}x)$  satisfies

$$d\bar{v} = (\Delta \bar{v} + \bar{f}_{x^i}^i) dt + \bar{g}^k dw_t^k,$$

where  $\bar{f}^i(t, x) = \frac{1}{\sqrt{\kappa}} f^i(t, \sqrt{\kappa}x)$  and  $\bar{g}^k(t, x) = g^k(t, \sqrt{\kappa}x)$ . Thus by (3.3),

$$\|v_x\|_{\mathbb{L}_{p,d}(T)}^p \leq \frac{N(p)}{\kappa^p} \|f\|_{\mathbb{L}_{p,d}(T)}^p + \frac{N(p)}{\kappa^{p/2}} \|g\|_{\mathbb{L}_{p,d}(T)}^p. \quad (3.7)$$

Therefore we conclude that if  $u \in \mathfrak{H}_{p,d,0}^1(T)$  is a solution of (3.4), then  $u$  satisfies

$$du = (\kappa \Delta u + (f^i + (A - \kappa I)u_{x^j})_{x^i}) dt + g^k dw_t^k,$$

and

$$\|u_x\|_{\mathbb{L}_p(T)}^p \leq \frac{N(p)}{\kappa^p} \|F\|_{\mathbb{L}_{p,d}(T)}^p + \frac{N(p)}{\kappa^{p/2}} \|g\|_{\mathbb{L}_{p,d}(T)}^p,$$

where  $F^i = (A - \kappa I)u_{x^j} + f^i$ . By (3.6)

$$|F|^p \leq (1 + \epsilon) \frac{(\Lambda - \lambda)^p}{2^p} |u_x|^p + N(\epsilon) |f|^p.$$

Thus, for sufficiently small  $\epsilon$ , (since  $N(p) \searrow 1$  as  $p \searrow 2$ )

$$\frac{N(p)}{\kappa^p} (1 + \epsilon) \frac{(\lambda - \lambda)^p}{2^p} = N(p) (1 + \epsilon) \frac{(\Lambda - \lambda)^p}{(\Lambda + \lambda)^p} < 1. \quad (3.8)$$

Obviously the claims of the lemma follow from this.  $\square$

**Lemma 3.3.** *Assume that for any solution  $u \in \mathfrak{H}_{p,\theta_0}^1(\tau)$  of (1.1), we have estimate (2.19) for  $\theta = \theta_0$ , then there exists  $\chi = \chi(d, p, \theta_0, \lambda, \Lambda) > 0$  such that for any  $\theta \in (\theta_0 - \chi, \theta_0 + \chi)$ , estimate (2.19) holds whenever  $u \in \mathfrak{H}_{p,\theta}^1(\tau)$  is a solution of (1.1).*

*Proof.* The lemma is essentially proved in [6] for SPDEs with constant coefficients. By Lemma 2.3,  $u \in \mathfrak{H}_{p,\theta}^1(\tau)$  if and only if  $v := M^{(\theta-\theta_0)/p}u \in \mathfrak{H}_{p,\theta_0}^1(\tau)$  and the norms  $\|u\|_{\mathfrak{H}_{p,\theta}^1(\tau)}$  and  $\|v\|_{\mathfrak{H}_{p,\theta_0}^1(\tau)}$  are equivalent. Denote  $\varepsilon = (\theta - \theta_0)/p$  and observe that  $v$  satisfies

$$dv = (D_i(a^{ij}v_{xj} + b^i v + \tilde{f}^i) + \bar{b}^i v_{x_i} + cv + \tilde{f})dt + (\nu^k v + M^\varepsilon g^k)dw_t^k,$$

where

$$\begin{aligned} \tilde{f}^i &= M^\varepsilon f^i - \varepsilon a^{i1} M^{-1} v, \\ \tilde{f} &= M^\varepsilon \bar{f} - M^{-1} \varepsilon (\bar{b}^1 v + a^{1j} v_{xj} - a^{11} \varepsilon M^{-1} v + b^1 v + M^\varepsilon f^i). \end{aligned}$$

By assumption (remember that  $Mb^i$  and  $M\bar{b}$  are bounded),

$$\begin{aligned} \|v\|_{\mathfrak{H}_{p,\theta_0}^1(\tau)} &\leq N(\|\tilde{f}^i\|_{\mathbb{L}_{p,\theta_0}(\tau)} + \|M\tilde{f}\|_{\mathbb{H}_{p,\theta_0}^{-1}(\tau)} + \|M^\varepsilon u_0\|_{U_{p,\theta_0}}) \\ &\leq N(\|f^i\|_{\mathbb{L}_{p,\theta}(\tau)} + \|M\bar{f}\|_{\mathbb{H}_{p,\theta}^{-1}(\tau)} + \|u_0\|_{U_{p,\theta}}) \\ &\quad + N\varepsilon(\|M^{-1}v\|_{\mathbb{L}_{p,\theta_0}(\tau)} + \|v_x\|_{\mathbb{L}_{p,\theta_0}(\tau)}). \end{aligned}$$

Thus it is enough to take  $\varepsilon$  sufficiently small (see (2.8)). The lemma is proved.  $\square$

Now we come back to our proof. As usual we may assume  $\tau \equiv T$  (see [7]), and due to Lemma 3.3, without loss of generality we assume that  $\theta = d$ .

Take  $p_0$  from Lemma 3.2. The method of continuity shows that to prove the theorem it suffices to prove that if  $p \leq p_0$ , then (2.19) holds true given that a solution  $u \in \mathfrak{H}_{p,d}^1(T)$  already exists.

**Step 1.** We assume that  $b^i = \bar{b}^i = c = \nu^k = 0$ . By (2.8) (or see Lemma 1.3 (i) in [11])

$$\|u_x\|_{H_{p,\theta}^\gamma} \sim \|M^{-1}u\|_{H_{p,\theta}^{\gamma+1}}.$$

Thus we estimate  $\|u_x\|_{\mathbb{L}_{p,d}(T)}$  instead of  $\|M^{-1}u\|_{\mathbb{H}_{p,d}^1(T)}$ . By Theorem 3.3 in [11] there exists a solution  $v \in \mathfrak{H}_{p,d}^1(T)$  of

$$dv = (\Delta v + \bar{f}) dt, \quad v(0, \cdot) = u_0,$$

and furthermore

$$\|v_x\|_{\mathbb{L}_{p,d}(T)} \leq N\|M\bar{f}\|_{\mathbb{H}_{p,d}^{-1}(T)} + N\|u_0\|_{U_{p,d}^1}. \quad (3.9)$$

Observe that  $u - v$  satisfies

$$d(u - v) = D_i(a^{ij}(u - v)_{x^j} + \tilde{f}^i) dt + g^k dw_t^k, \quad (u - v)(0, \cdot) = 0,$$

where  $\tilde{f}^i = f^i + (a^{ij} - \delta^{ij})v_{x^j}$ . Therefore (2.19) follows from Lemma 3.2 and (3.9).

**Step 2**(general case). By the result of step 1,

$$\begin{aligned} \|M^{-1}u\|_{\mathbb{H}_{p,d}^1(T)} &\leq N\|Mb^i M^{-1}u + f^i\|_{\mathbb{L}_{p,d}(T)} + N\|u_0\|_{U_{p,d}^1} \\ &+ N\|M\bar{b}^i u_{x^i} + M^2 c M^{-1}u + M\bar{f}\|_{\mathbb{H}_{p,d}^{-1}(T)} + N\|M\nu M^{-1}u + g\|_{\mathbb{L}_{p,d}(T)} \\ &\leq N\beta(\|M^{-1}u\|_{\mathbb{L}_{p,d}(T)} + \|u_x\|_{\mathbb{L}_{p,d}(T)}) \\ &+ N\|u_0\|_{U_{p,d}^1} + N\|f^i\|_{\mathbb{L}_{p,d}(T)} + N\|M\bar{f}\|_{\mathbb{H}_{p,d}^{-1}(T)} + N\|g\|_{\mathbb{L}_{p,d}(T)}. \end{aligned}$$

Now it is enough to choose  $\beta_0$  such that for any  $\beta \leq \beta_0$ ,

$$N\beta(\|M^{-1}u\|_{\mathbb{L}_{p,d}(T)} + \|u_x\|_{\mathbb{L}_{p,d}(T)}) \leq 1/2\|M^{-1}u\|_{\mathbb{H}_{p,d}^1(T)}.$$

The theorem is proved.

#### 4. PROOF OF THEOREM 2.8

First we need the following result on  $\mathbb{R}^d$  proved in [15].

**Lemma 4.1.** *There exists  $p_0 = p_0(\lambda, \Lambda, d) > 2$  such that if  $p \in [2, p_0)$  and  $u \in \mathcal{H}_{p,0}^1(T)$  is a solution of*

$$du = D_i(a^{ij}u_{x^j} + f^i)dt + g^k dw_t^k, \quad (4.1)$$

then

$$\|u_x\|_{\mathbb{L}_p(T)} \leq N(\|f\|_{\mathbb{L}_p(T)} + \|g\|_{\mathbb{L}_p(T)}).$$

Again, to prove the theorem, we only show that the apriori estimate (2.20) holds for  $p < p_0$  (also see step 1 below).

As in theorem 5.1 in [7], considering  $u - v$ , where  $v \in \mathcal{H}_p^1(T)$  is the solution of

$$dv = \Delta v dt, \quad v(0, \cdot) = u_0,$$

without loss of generality we assume that  $u(0, \cdot) = 0$ .

**Step 1.** Assume that  $b^i = \bar{b}^i = c = \nu^k = 0$ . By Theorem 5.1 in [7], there exists a solution  $v \in \mathcal{H}_{p,0}^1(T)$  of

$$dv = (\Delta v + \bar{f})dt,$$

and it satisfies

$$\|v_x\|_{\mathbb{L}_p(T)} \leq N\|\bar{f}\|_{\mathbb{H}_p^{-1}(T)}. \quad (4.2)$$

Observe that  $\bar{u} := u - v$  satisfies

$$d\bar{u} = D_i(a^{ij}\bar{u}_{x^j} + \tilde{f}^i) dt + g^k dw_t^k,$$

where  $\tilde{f}^i = f^i + (A - I)v_{x^j}$ . Thus the estimate (2.20) follows from Lemma 4.1 and (4.2).

**Step 2.** We show that there exists  $\epsilon_1 > 0$  such that if  $T \leq \epsilon_1$ , then all the assertions of the theorem hold true. Thus without loss of generality we assume that  $T \leq 1$ .

Note that  $\bar{b}^i u_{x^i} \in \mathbb{L}_p(T)$  since  $u \in \mathbb{H}_p^1(T)$ , so by Theorem 5.1 in [7], there exists a unique solution  $v \in \mathcal{H}_{p,0}^2(T)$  of

$$dv = (\Delta v + \bar{b}^i u_{x^i}) dt,$$

and  $v$  satisfies

$$\|v\|_{\mathcal{H}_{p,0}^2(T)}^p \leq N \|u_x\|_{\mathbb{L}_p(T)}^p.$$

By (2.10),

$$\|v_x\|_{\mathbb{L}_p(T)}^p \leq N \|v\|_{\mathbb{H}_p^1(T)}^p \leq N(T) \|u_x\|_{\mathbb{L}_p(T)}, \quad (4.3)$$

where  $N(T) \rightarrow 0$  as  $T \rightarrow 0$ . Observe that  $u - v$  satisfies

$$\begin{aligned} d(u - v) &= (D_i(a^{ij}(u - v)_{x^j} + (a^{ij} - \delta^{ij})v_{x^i} + b^i u + f^i) + cu + \bar{f}) dt \\ &\quad + (\nu^k u + g^k) dw_t^k. \end{aligned}$$

By the result of step 1,

$$\begin{aligned} \|(u - v)_x\|_{\mathbb{L}_p(T)} &\leq N(\|(a^{ij} - \delta^{ij})v_{x^i} + b^i u + f^i\|_{\mathbb{L}_p(T)} \\ &\quad + \|cu + \bar{f}\|_{\mathbb{H}_p^{-1}(T)} + \|\nu^k u + g\|_{\mathbb{L}_p(T)}) \\ &\leq N(\|v_x\|_{\mathbb{L}_p(T)} + \|f^i\|_{\mathbb{L}_p(T)} + \|\bar{f}\|_{\mathbb{H}_p^{-1}(T)} + \|g\|_{\mathbb{L}_p(T)} + \|u\|_{\mathbb{L}_p(T)}), \end{aligned}$$

where constants  $N$  are independent of  $T$  ( $T \leq 1$ ). This and (4.3) yield

$$\begin{aligned} \|u_x\|_{\mathbb{L}_p(T)} &\leq NN(T) \|u_x\|_{\mathbb{L}_p(T)} + N \|f^i\|_{\mathbb{L}_p(T)} + N \|\bar{f}\|_{\mathbb{H}_p^{-1}(T)} \\ &\quad + N \|g\|_{\mathbb{L}_p(T)} + N \|u\|_{\mathbb{L}_p(T)}. \end{aligned}$$

Note that the above inequality holds for all  $t \leq T$ . Choose  $\epsilon_1$  so that  $NN(T) \leq 1/2$  for all  $T \leq \epsilon_1$ , then for any  $t \leq T \leq \epsilon_1$  (see Lemma 2.3),

$$\begin{aligned} \|u\|_{\mathcal{H}_p^1(t)}^p &\leq N \|u\|_{\mathbb{L}_p(t)}^p + N(\|f^i\|_{\mathbb{H}_p^{-1}(T)}^p + \|\bar{f}\|_{\mathbb{L}_p(T)}^p + \|g\|_{\mathbb{L}_p(T)}^p) \\ &\leq N \int_0^t \|u\|_{\mathcal{H}_p^1(t)}^p dt + N(\|f^i\|_{\mathbb{H}_p^{-1}(T)}^p + \|\bar{f}\|_{\mathbb{L}_p(T)}^p + \|g\|_{\mathbb{L}_p(T)}^p). \end{aligned}$$

Gronwall's inequality leads to (2.20).

**Step 3.** Consider the case  $T > \epsilon_1$ . To proceed further, we need the following lemma.

**Lemma 4.2.** *Let  $\tau \leq T$  be a stopping and  $du(t) = f(t)dt + g^k(t)dw_t^k$ .*

(i) *Let  $u \in \mathcal{H}_{p,0}^{\gamma+2}(\tau)$ . Then there exists a unique  $\tilde{u} \in \mathcal{H}_{p,0}^{\gamma+2}(T)$  such that  $\tilde{u}(t) = u(t)$  for  $t \leq \tau$  (a.s) and, on  $(0, T)$ ,*

$$d\tilde{u} = (\Delta\tilde{u}(t) + \tilde{f}(t))dt + g^k I_{t \leq \tau} dw_t^k, \quad (4.4)$$

where  $\tilde{f} = (f(t) - \Delta u(t))I_{t \leq \tau}$ . Furthermore,

$$\|\tilde{u}\|_{\mathcal{H}_p^{\gamma+2}(T)} \leq N \|u\|_{\mathcal{H}_p^{\gamma+2}(\tau)}, \quad (4.5)$$

where  $N$  is independent of  $u$  and  $\tau$ .

(ii) *all the claims in (i) hold true if  $u \in \mathfrak{H}_{p,\theta,0}^{\gamma+2}(G, \tau)$  and if one replace the space  $\mathcal{H}_p^{\gamma+2}(\tau)$  and  $\mathcal{H}_p^{\gamma+2}(T)$  with  $\mathfrak{H}_{p,\theta}^{\gamma+2}(G, \tau)$  and  $\mathfrak{H}_{p,\theta}^{\gamma+2}(G, T)$ , respectively.*

*Proof.* (i) Note  $\tilde{f} \in \mathbb{H}_p^\gamma(T)$ ,  $g I_{t \leq \tau} \in \mathbb{H}_p^{\gamma+1}(T)$ , so that, by Theorem 5.1 in [7], equation (4.4) has a unique solution  $\tilde{u} \in \mathcal{H}_{p,0}^{\gamma+2}(T)$  and (4.5) holds. To show that  $\tilde{u}(t) = u(t)$  for  $t \leq \tau$ , notice that, for  $t \leq \tau$ , the function  $v(t) = \tilde{u}(t) - u(t)$  satisfies the equation

$$v(t) = \int_0^t \Delta v(s) ds, \quad v(0, \cdot) = 0.$$

Theorem 5.1 in [7] shows that  $v(t) = 0$  for  $t \leq \tau$  (a.e).

(ii) It is enough to repeat the arguments in (i) using Theorem 2.9 in [1] (instead of Theorem 5.1 in [7]).  $\square$

Now, to complete the proof, we repeat the arguments in [4]. Take an integer  $M \geq 2$  such that  $T/M \leq \varepsilon_1$ , and denote  $t_m = Tm/M$ . Assume that, for  $m = 1, 2, \dots, M-1$ , we have the estimate (2.20) with  $t_m$  in place of  $\tau$  (and  $N$  depending only on  $d, p, \lambda, \Lambda, K$  and  $T$ ). We are going to use the induction on  $m$ . Let  $u_m \in \mathcal{H}_{p,0}^1$  be the continuation of  $u$  on  $[t_m, T]$ , which exists by Lemma 4.2(i) with  $\gamma = -1$  and  $\tau = t_m$ . Denote  $v_m := u - u_m$ , then (a.s) for any  $t \in [t_m, T]$ ,  $\phi \in C_0^\infty(G)$  (since  $du_m = \Delta u_m dt$  on  $[t_m, T]$ )

$$\begin{aligned} (v_m(t), \phi) &= - \int_{t_m}^t (a^{ij} v_{mx^j} + b^i v_m + f_m^i, \phi_{x^i})(s) ds \\ &+ \int_{t_m}^t (\bar{b}^i v_{mx^i} + c v_m + \bar{f}_m, \phi)(s) ds + \int_{t_m}^t (\nu^k v_m + g_m^k, \phi)(s) dw_s^k, \end{aligned}$$

where

$$\begin{aligned} f_m^i &= (a^{ij} - \delta^{ij}) u_{mx^j} + b^i u_m + f^i, & \bar{f}_m &= \bar{b}^i u_{mx^i} + c u_m + \bar{f}, \\ g_m^k &= \nu^k u_m + g^k. \end{aligned}$$

Next instead of random processes on  $[0, T]$  one considers processes given on  $[t_m, T]$  and, in a natural way, introduce spaces  $\mathcal{H}_p^\gamma([t_m, T])$ ,  $\mathbb{L}_p([t_m, t])$ ,  $\mathbb{H}_p^\gamma([t_m, T])$ . Then one gets a counterpart of the result of step 2 and concludes that

$$\begin{aligned} & E \int_{t_m}^{t_{m+1}} \|(u - u_m)(s)\|_{H_p^1}^p ds \\ & \leq NE \int_{t_m}^{t_{m+1}} (\|f_m^i(s)\|_{L_p}^p + \|\bar{f}_m(s)\|_{H_p^{-1}}^p + \|g_m(s)\|_{L_p}^p) ds. \end{aligned}$$

Thus by the induction hypothesis we conclude

$$\begin{aligned} & E \int_0^{t_{m+1}} \|u(s)\|_{H_p^1}^p ds \leq NE \int_0^T \|u_m(s)\|_{H_p^1}^p ds \\ & \quad + NE \int_{t_m}^{t_{m+1}} \|(u - u_m)(s)\|_{H_p^1}^p ds \\ & \leq N(\|f^i\|_{\mathbb{L}_p(t_{m+1})}^p + \|\bar{f}\|_{\mathbb{H}_p^{-1}(t_{m+1})}^p + \|g\|_{\mathbb{L}_p(t_{m+1})}^p). \end{aligned}$$

We see that the induction goes through and thus the theorem is proved.

## 5. PROOF OF THEOREM 2.8

As usual we may assume  $\tau \equiv T$ . It is known (see [1]) that for any  $u_0 \in U_{p,\theta}^1(G)$  and  $(f, g) \in \psi^{-1}\mathbb{H}_{p,\theta}^{-1}(G, T) \times \mathbb{L}_{p,\theta}(G, T)$ , there exists  $u \in \mathfrak{H}_{p,\theta}^1(G, T)$  such that  $u(0, \cdot) = u_0$  and

$$du = (\Delta u + f) dt + g^k dw_t^k. \quad (5.1)$$

Thus as before, to finish the proof of the theorem, we only need to establish the apriori estimate (2.13) assuming that  $u \in \mathfrak{H}_{p,\theta}^1(G, T)$  satisfies (1.1) with initial data  $u_0 = 0$ , where  $p \in [2, p_0)$  and  $\theta \in (d - \chi, d + \chi)$ .

To proceed we need the following results.

**Lemma 5.1.** *Let  $u \in \mathfrak{H}_{p,\theta,0}^1(G, T)$  be a solution of (1.1). Then*

(i) *there exists  $\varepsilon_0 \in (0, 1)$  (independent of  $u$ ) such that if  $u$  has support in  $B_{\varepsilon_0}(x_0)$ ,  $x_0 \in \partial G$  then (2.13) holds.*

(ii) *if  $u$  has support on  $G_\varepsilon$  for some  $\varepsilon > 0$ , where  $G_\varepsilon := \{x \in G : \text{dist}(x, \partial G) > \varepsilon\}$ , then (2.13) holds.*

*Proof.* The second assertion of the lemma follows from Theorem 2.8 since in this case (see [12])  $u \in \mathcal{H}_p^1(T)$  and

$$\|u\|_{\mathfrak{H}_{p,\theta}^1(G,T)} \sim \|u\|_{\mathcal{H}_p^1(T)}.$$

To prove the first assertion, we use Theorem 2.7. Let  $x_0 \in \partial G$  and  $\Psi$  be a function from Assumption 2.1. It is shown in [5] (or see [1]) that

$\Psi$  can be chosen such that  $\Psi$  is infinitely differentiable in  $G \cap B_{r_0}(x_0)$  and satisfies

$$[\Psi_x]_{n, B_{r_0}(x_0) \cap G}^{(0)} + [\Psi_x^{-1}]_{n, J_+}^{(0)} < N(n) < \infty \quad (5.2)$$

and

$$\rho(x)\Psi_{xx}(x) \rightarrow 0 \quad \text{as } x \in B_{r_0}(x_0) \cap G, \text{ and } \rho(x) \rightarrow 0, \quad (5.3)$$

where the constants  $N(n)$  and the convergence in (5.3) are independent of  $x_0$ .

Define  $r = r_0/K_0$  and fix smooth functions  $\eta \in C_0^\infty(B_r)$ ,  $\varphi \in C^\infty(\mathbb{R})$  such that  $0 \leq \eta, \varphi \leq 1$ , and  $\eta = 1$  in  $B_{r/2}$ ,  $\varphi(t) = 1$  for  $t \leq -3$ , and  $\varphi(t) = 0$  for  $t \geq -1$  and  $0 \geq \varphi' \geq -1$ . Observe that  $\Psi(B_{r_0}(x_0))$  contains  $B_r$ . For  $m = 1, 2, \dots$ ,  $t > 0$ ,  $x \in \mathbb{R}_+^d$  define  $\varphi_m(x) = \varphi(m^{-1} \ln x^1)$ . Also we denote  $\Psi_r^i := D_r \Psi^i$ ,  $\Psi_{rs}^i := D_r D_s \Psi^i$ ,  $\Phi_r^i := D_i(\Psi_{x^r}^i(\Psi^{-1}))(\Psi)$ ,

$$\begin{aligned} \hat{a}_m &:= \tilde{a}\eta(x)\varphi_m + (1 - \eta\varphi_m)I, & \hat{b}_m &:= \tilde{b}\eta\varphi_m, & \hat{\bar{b}}_m &:= \tilde{\bar{b}}\eta\varphi_m, \\ \hat{c}_m &:= \tilde{c}\eta\varphi_m, & \hat{\nu}_m &:= \tilde{\nu}\eta\varphi_m, \end{aligned}$$

where

$$\begin{aligned} \tilde{a}^{ij}(t, x) &= \check{a}^{ij}(t, \Psi^{-1}(x)), & \tilde{b}^i(t, x) &= \check{b}^i(t, \Psi^{-1}(x)), \\ \tilde{\bar{b}}^i(t, x) &= \check{\bar{b}}^i(t, \Psi^{-1}(x)), & \tilde{c}(t, x) &= c(t, \Psi^{-1}(x)) \\ \tilde{\nu}(t, x) &= \nu(t, \Psi^{-1}(x)), \\ \check{a}^{ij} &= a^{rs}\Psi_{x^r}^i\Psi_{x^s}^j, & \check{b}^i &= b^r\Psi_r^i, \\ \check{\bar{b}}^i &= \bar{b}^r\Psi_r^i + a^{rs}\Psi_s^j\Phi_r^i, & \check{c} &= c + b^r\Phi_r^i. \end{aligned}$$

Take  $\beta_0$  from Theorem 2.7. Observe that  $\varphi(m^{-1} \ln x^1) = 0$  for  $x^1 \geq e^{-m}$ . Also we easily see that (5.3) implies  $x^1\Psi_{xx}(\Psi^{-1}(x)) \rightarrow 0$  as  $x^1 \rightarrow 0$ . Using these facts and Assumption 2.2(iv), one can find  $m > 0$  independent of  $x_0$  such that

$$x^1|\hat{b}_m(t, x)| + x^1|\hat{\bar{b}}_m(t, x)| + (x^1)^2|\hat{c}_m(t, x)| + x^1|\hat{\nu}_m(t, x)| \leq \beta_0,$$

whenever  $t > 0$ ,  $x \in \mathbb{R}_+^d$ .

Now we fix a  $\varepsilon_0 < r_0$  such that

$$\Psi(B_{\varepsilon_0}(x_0)) \subset B_{r/2} \cap \{x : x^1 \leq e^{-3m}\}.$$

Let's denote  $v := u(\Psi^{-1})$  and continue  $v$  as zero in  $\mathbb{R}_+^d \setminus \Psi(B_{\varepsilon_0}(x_0))$ . Since  $\eta\varphi_m = 1$  on  $\Psi(B_{\varepsilon_0}(x_0))$ , the function  $v$  satisfies

$$dv = ((\hat{a}_m^{ij}v_{x^i x^j} + \hat{b}_m^i v + \hat{f}^i)_{x^i} + \hat{\bar{b}}_m^i v_{x^i} + \hat{c}_m v + \hat{f}) dt + (\hat{\nu}_m^k v + \hat{g}^k) dw_t^k,$$

where

$$\hat{f}^i = f^i(\Psi^{-1}), \quad \hat{\bar{f}} = \bar{f}(\Psi^{-1}), \quad \hat{g}^k = g^k(\Psi^{-1}).$$



Next we observe that by (5.2) and Theorem 3.2 in [12] (or see [5]) for any  $\nu, \alpha \in \mathbb{R}$  and  $h \in \psi^{-\alpha} H_{p,\theta}^\nu(G)$  with support in  $B_{\varepsilon_0}(x_0)$

$$\|\psi^\alpha h\|_{H_{p,\theta}^\nu(G)} \sim \|M^\alpha h(\Psi^{-1})\|_{H_{p,\theta}^\nu}. \quad (5.4)$$

Therefore we conclude that  $v \in \mathfrak{H}_{p,\theta}^1(T)$ . Also by Theorem 2.7 we have

$$\|M^{-1}v\|_{\mathbb{H}_{p,\theta}^1(T)} \leq N\|\hat{f}\|_{\mathbb{L}_{p,\theta}(T)} + N\|M\hat{f}\|_{\mathbb{H}_{p,\theta}^{-1}(T)} + N\|\hat{g}\|_{\mathbb{L}_{p,\theta}(T)}.$$

Finally (5.4) leads to (2.13). The lemma is proved.  $\square$

Coming back to our proof, we choose a partition of unity  $\zeta^m, m = 0, 1, 2, \dots, N_0$  such that  $\zeta^0 \in C_0^\infty(G)$ ,  $\zeta^{(m)} = \zeta(\frac{2(x-x_m)}{\varepsilon_0}), \zeta \in C_0^\infty(B_1(0))$ ,  $x_m \in \partial G$ ,  $m \geq 1$ , and for any multi-indices  $\alpha$

$$\sup_x \sum |\psi^{|\alpha|} |D^\alpha \zeta^{(m)}| < N(\alpha) < \infty, \quad (5.5)$$

where the constant  $N(\alpha)$  is independent of  $\varepsilon_0$  (see section 6.3 in [9]). Thus it follows (see [12]) that for any  $\nu \in \mathbb{R}$  and  $h \in H_{p,\theta}^\nu(G)$  there exist constants  $N$  depending only  $p, \theta, \nu$  and  $N(\alpha)$  (independent of  $\varepsilon_0$ ) such that

$$\|h\|_{H_{p,\theta}^\nu(G)}^p \leq N \sum \|\zeta^m h\|_{H_{p,d}^\nu(G)}^p \leq N \|h\|_{H_{p,\theta}^\nu(G)}^p, \quad (5.6)$$

$$\sum \|\psi \zeta_x^m h\|_{H_{p,\theta}^\nu(G)}^p \leq N \|h\|_{H_{p,\theta}^\nu(G)}^p. \quad (5.7)$$

Also,

$$\sum \|\zeta_x^{(m)} h\|_{H_{p,\theta}^\nu(G)}^p \leq N(\varepsilon_0) \|h\|_{H_{p,\theta}^\nu(G)}^p, \quad (5.8)$$

where the constant  $N(\varepsilon_0)$  depends also on  $\varepsilon_0$ .

Using the above inequalities and Lemma 5.1 we will show

$$\|u_x\|_{\mathbb{L}_{p,\theta}(G,t)}^p \leq N \|u\|_{\mathbb{L}_{p,\theta}(G,t)}^p + \text{appropriate norms of } f^i, \bar{f}, g \quad (5.9)$$

and we will drop the term  $\|u\|_{\mathbb{L}_{p,\theta}(G,t)}^p$  using (2.9). But as one can see in (5.10) below, one has to handle the term  $a^{ij}u_{x^j}\zeta_{x^i}^m$ . Obviously if the right side of inequality (5.9) contains the norm  $\|u_x\|_{\mathbb{L}_{p,\theta}(G,T)}^p$ , then this is useless. The following arguments below are used just to avoid estimating  $\|a^{ij}u_{x^j}\zeta_{x^i}^m\|_{\mathbb{L}_{p,\theta}(G,T)}^p$ .

Denote  $u^m = u\zeta^m$ ,  $m = 0, 1, \dots, N_0$ . Then  $u^m$  satisfies

$$\begin{aligned} du^m &= (D_i(a^{ij}u_{x^j}^m + b^i u^m + f^{m,i}) + \bar{b}^i u_{x^i}^m + cu^m + \bar{f}^m - a^{ij}u_{x^j}\zeta_{x^i}^m) dt \\ &\quad + (\nu^k u^m + \zeta^m g^k) dw_t^k, \end{aligned} \quad (5.10)$$

where

$$\begin{aligned} f^{m,i} &= f^i \zeta - a^{ij}u\zeta_{x^j}^m, \\ \bar{f}^m &= -b^i u\zeta_{x^i}^m - f^i \zeta_{x^i}^m - \bar{b}^i u\zeta_{x^i}^m + \bar{f}\zeta^m. \end{aligned}$$

Since  $\psi^{-1}a^{ij}u_{x^j}\zeta_{x^i}^m \in \psi^{-1}\mathbb{L}_{p,\theta}(G, T)$ , by Theorem 2.9 in [1] (or Theorem 2.10 in [5]), there exists unique solution  $v^m \in \mathfrak{H}_{p,\theta,0}^2(G, T)$  of

$$dv = (\Delta v - \psi^{-1}a^{ij}u_{x^j}\zeta_{x^i}^m)dt,$$

and furthermore

$$\|v^m\|_{\mathfrak{H}_{p,\theta}^2(G,T)} \leq N\|a^{ij}u_{x^j}\zeta_{x^i}^m\|_{\mathbb{L}_{p,\theta}(G,T)}. \quad (5.11)$$

By (2.2) and Lemma 2.3,

$$\|v^m\|_{\mathbb{L}_{p,\theta}(G,T)} + \|\psi v_x^m\|_{\mathbb{L}_{p,\theta}(G,T)} \leq N(T)\|a^{ij}u_{x^j}\zeta_{x^i}^m\|_{\mathbb{L}_{p,\theta}(G,T)}, \quad (5.12)$$

where  $N(T) \rightarrow 0$  as  $T \rightarrow 0$ .

For  $m \geq 1$ , define  $\eta^m(x) = \zeta(\frac{x-x_m}{\varepsilon_0})$  and fix a smooth function  $\eta^0 \in C_0^\infty(G)$  such that  $\eta^0 = 1$  on the support of  $\zeta^0$ . Now we denote  $\bar{u}^m := \psi v^m \eta^m$ , then  $\bar{u}^m \in \mathfrak{H}_{p,\theta}^2(G, T)$  satisfies

$$d\bar{u}^m = (\Delta \bar{u}^m + \tilde{f}^m - a^{ij}u_{x^j}\zeta_{x^i}^m) dt, \quad (5.13)$$

where  $\tilde{f}^m = -2v_{x^i}^m(\eta^m\psi)_{x^i} - v^m\Delta(\eta^m\psi)$ . Finally by considering  $\tilde{u}^m := u^m - \bar{u}^m$  we can drop the term  $a^{ij}u_{x^j}\zeta_{x^i}^m$  in (5.10) because  $\tilde{u}^m$  satisfies

$$\begin{aligned} d\tilde{u}^m &= (D_i(a^{ij}\tilde{u}_{x^j}^m + b^i\tilde{u}^m + F^{m,i}) + \bar{b}^i\tilde{u}_{x^i}^m + c\tilde{u}^m + \bar{F}_m) dt \\ &\quad + (\nu^k\tilde{u}^m + G^{m,k}) dw_t^k, \end{aligned} \quad (5.14)$$

where

$$\begin{aligned} F^{m,i} &= f^i\zeta^m - a^{ij}u_{x^j}\zeta_{x^i}^m + b^i\bar{u}^m + (a^{ij} - \delta^{ij})\bar{u}_{x^j}^m, \\ \bar{F}^m &= \bar{b}^i\bar{u}_{x^i}^m + c\bar{u}^m - b^i u_{x^i}^m - f^i\zeta_{x^i}^m - \bar{b}^i u_{x^i}^m + \bar{f}\zeta^m + 2v_{x^i}^m(\eta^m\psi)_{x^i} + v^m\Delta(\eta^m\psi), \\ G^{m,k} &= \zeta^m g^k + \nu^k\bar{u}^m. \end{aligned}$$

By Lemma 5.1, for any  $t \leq T$ ,

$$\|\psi^{-1}\tilde{u}^m\|_{\mathbb{H}_{p,\theta}^1(G,t)}^p \leq N\|F^{m,i}\|_{\mathbb{L}_{p,\theta}(G,t)}^p + N\|\psi\bar{F}^m\|_{\mathbb{H}_{p,\theta}^{-1}(G,t)} + N\|G^m\|_{\mathbb{L}_{p,\theta}(G,t)}^p.$$

Remember that  $\psi b^i, \psi \bar{b}^i, \psi^2 c, \psi_x$  and  $\psi\psi_{xx}$  are bounded and  $\|\cdot\|_{H_{p,\theta}^{-1}} \leq \|\cdot\|_{L_{p,\theta}}$ . By (5.6), (5.7) and (5.8),

$$\begin{aligned} \sum \|\psi\bar{F}^m\|_{\mathbb{H}_{p,\theta}^{-1}(G,t)}^p &\leq N(\|\psi\bar{f}\|_{\mathbb{H}_{p,\theta}^{-1}(G,t)}^p + \|f^i\|_{\mathbb{L}_{p,\theta}(G,t)} + \|u\|_{\mathbb{L}_{p,\theta}(G,t)}^p) \\ + N \sum (\|\bar{u}_x^m\|_{\mathbb{L}_{p,\theta}(G,t)}^p &+ \|\psi^{-1}\bar{u}^m\|_{\mathbb{L}_{p,\theta}(G,t)}^p + \|\psi v_x^m\|_{\mathbb{L}_{p,\theta}(G,t)}^p + \|v^m\|_{\mathbb{L}_{p,\theta}(G,t)}^p) \\ &\leq N(\|\psi\bar{f}\|_{\mathbb{H}_{p,\theta}^{-1}(G,t)}^p + \|f^i\|_{\mathbb{L}_{p,\theta}(G,t)} + \|u\|_{\mathbb{L}_{p,\theta}(G,t)}^p) + \sum \|v^m\|_{\mathbb{H}_{p,\theta}^1(G,t)}^p. \end{aligned}$$

Similarly (actually much easily) the sums

$$\sum \|F^{m,i}\|_{\mathbb{L}_{p,\theta}(G,t)}^p, \quad \sum \|G^m\|_{\mathbb{L}_{p,\theta}(G,t)}^p$$

can be handled. Then one gets for each  $t \leq T$  (see (5.12) and note that  $\psi^{-1}\bar{u}^m = v^m\eta^m$ ),

$$\begin{aligned} & \|\psi^{-1}u\|_{\mathbb{H}_{p,\theta}^1(G,t)}^p \leq N \sum \|\psi^{-1}u_m\|_{\mathbb{H}_{p,\theta}^1(G,t)}^p \\ & \leq N \sum \|\psi^{-1}\tilde{u}^m\|_{\mathbb{H}_{p,\theta}^1(G,t)}^p + N \sum \|v^m\eta^m\|_{\mathbb{H}_{p,\theta}^1(G,t)}^p \\ & \leq N\|f^i\|_{\mathbb{L}_{p,\theta}(G,T)} + N\|\psi\bar{f}\|_{\mathbb{H}_{p,\theta}^{-1}(G,T)}^p + N\|g\|_{\mathbb{L}_{p,\theta}(G,T)} \\ & \quad + N\|u\|_{\mathbb{L}_{p,\theta}(G,t)} + NN(t)\|u_x\|_{\mathbb{L}_{p,\theta}(G,t)}^p. \end{aligned}$$

Since  $\|u_x\|_{\mathbb{L}_{p,\theta}} \leq N\|\psi^{-1}u\|_{\mathbb{H}_{p,\theta}^1}$ , we can choose  $\varepsilon_2 \in (0, 1]$  such that

$$NN(t)\|u_x\|_{\mathbb{L}_{p,\theta}(G,t)}^p \leq 1/2\|\psi^{-1}u\|_{\mathbb{H}_{p,\theta}^1(G,t)}^p, \quad \text{if } t \leq T \leq \varepsilon_2,$$

and therefore

$$\begin{aligned} \|u\|_{\mathfrak{H}_{p,\theta}^1(G,t)}^p & \leq N \int_0^t \|u\|_{\mathfrak{H}_{p,\theta}^1(G,s)}^p ds + N\|f^i\|_{\mathbb{L}_{p,\theta}(G,T)} \\ & \quad + N\|\psi\bar{f}\|_{\mathbb{H}_{p,\theta}^{-1}(G,T)}^p + N\|g\|_{\mathbb{L}_{p,\theta}(G,T)}. \end{aligned}$$

This and Gronwall's inequality lead to (2.13) if  $T \leq \varepsilon_2$ . For the general case, one repeats step 3 in the proof of Theorem 2.8 using Lemma 4.2 (ii) instead of Lemma 4.2 (i). The theorem is proved.

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