Electronic Journal of Probability

Vol. 4 (1999) Paper no. 18, pages 1–32.

Journal URL
http://www.math.washington.edu/~ejpecp
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol4/paper18.abs.html

ON SEMI-MARTINGALE CHARACTERIZATIONS OF FUNCTIONALS OF SYMMETRIC MARKOV PROCESSES

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Abstract For a quasi-regular (symmetric) Dirichlet space $(\mathcal{E}, \mathcal{F})$ and an associated symmetric standard process (X_t, P_x), we show that, for $u \in \mathcal{F}$, the additive functional $u^*(X_t) - u^*(X_0)$ is a semimartingale if and only if there exists an $\mathcal{E}$-nest $\{F_n\}$ and positive constants C_n such that $|\mathcal{E}(u,v)| \leq C_n \|v\|_\infty, \ v \in \mathcal{F}_{F_n,b}$. In particular, a signed measure resulting from the inequality will be automatically smooth. One of the variants of this assertion is applied to the distorted Brownian motion on a closed subset of R^d, giving stochastic characterizations of BV functions and Caccioppoli sets.

Keywords Quasi-regular Dirichlet form, strongly regular representation, additive functionals, semimartingale, smooth signed measure, BV function

AMS subject classification 60J45, 60J55, 31C25

Submitted to EJP on July 9, 1999. Final version accepted on October 6, 1999

1 Introduction

We consider a Hausdorff topological space X and a σ-finite positive Borel measure m on X. Let $(\mathcal{E},\mathcal{F})$ be a (symmetric) Dirichlet form on $L^2(X;m)$, namely, $\mathcal{E}$ is a Markovian closed symmetric form with domain $\mathcal{F}$ linear dense in $L^2(X;m)$. In this paper, we follow exclusively [MR 92] for the definition of $\mathcal{E}$-quasi notions. For a closed set $F \subset X$, we set

$$\mathcal{F}_F = \{u \in \mathcal{F} : u = 0 \ m\text{−a.e. on } X \setminus F\} \quad \mathcal{F}_{b,F} = \mathcal{F}_F \cap L^\infty(X;m).$$

An increasing family $\{F_n\}$ of closed sets is called an *$\mathcal{E}$-nest* if the space $\cup_{n=1}^\infty \mathcal{F}_{F_n}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$. A set $N \subset X$ is said to be *$\mathcal{E}$-exceptional* if $N \subset \cap_{n=1}^\infty F_n^c$ for some $\mathcal{E}$-nest $\{F_n\}$. '*$\mathcal{E}$-quasi-everywhere*' or '*$\mathcal{E}$-q.e.*' will mean 'except for an $\mathcal{E}$-exceptional set'. A function defined $\mathcal{E}$-q.e. on X is said to be *$\mathcal{E}$-quasicontinuous* if there exists an $\mathcal{E}$-nest $\{F_n\}$ such that the restriction of u to each set F_n is continuous there. When the Dirichlet form is quasi-regular in the sense of [MR 92], any $u \in \mathcal{F}$ admits an $\mathcal{E}$-quasicontinuous version which will be denoted by u^*.

We consider a quasi-regular Dirichlet form $(\mathcal{E},\mathcal{F})$ and an associated standard process $\mathbf{M} = (X_t, P_x)$ on X. A purpose of the present paper is to prove that, for $u \in \mathcal{F}$, the additive functional(AF in abbreviation)

$$A_t^{[u]} = u^*(X_t) - u^*(X_0) \tag{1}$$

of the process $\mathbf{M}$ is a semimartingale ,namely, a sum of a martingale and a process of bounded variation, if and only if there exists an $\mathcal{E}$-nest $\{F_n\}$ and positive constants $C_n,\ n = 1, 2, \cdots$ such that u satisfies

$$|\mathcal{E}(u,v)| \le C_n \, \|v\|_\infty, \quad \forall v \in \mathcal{F}_{b,F_n}, \quad n = 1, 2, \cdots, \tag{2}$$

where $\|v\|_\infty$ denotes the m-essential sup norm of $v \in L^\infty(X;m)$.

The existence of a special standard process associated with a quasi-regular Dirichlet form is well known [MR 92]. In the present paper, we give another construction of an associated tight special standard process as an image by a quasi-homeomorphism of a Hunt modification of a Ray process. The importance of the notion of quasi-regularity of a Dirichlet form is in that it is not only sufficient but also necessary for the existence of an associated right process which is m-special standard and m-tight ([AM 91],[MR 92]). For instance, given simply an m-symmetric right process $\mathbf{M}$ on a Lusin topological space X, $\mathbf{M}$ automatically becomes m-tight and m-special standard, and consequently the associated Dirichlet form $\mathcal{E}$ on $L^2(X;m)$ becomes quasi-regular and $\mathbf{M}$ can be modified outside some $\mathcal{E}$-exceptional set to be a tight special standard process ([MR 92], [Fi 97]).

We note here that, when the Dirichlet space is regular in the sense of [FOT 94], the $\mathcal{E}$-quasi notions of [MR 92] defined above can be identified with those classical quasi notions introduced for instance in [FOT 94] in terms of the ($\mathcal{E}_1$-)capacity. Indeed, an increasing sequence of closed sets is an $\mathcal{E}$-nest iff it is a generalized nest in the sense of [FOT 94], a set is $\mathcal{E}$-exceptional iff it is of zero capacity, and a function is $\mathcal{E}$-quasicontinuous iff it is quasi continuous in the sense of [FOT 94] (see Lemma 2.1). We shall take those identifications for granted for the moment.

When the Dirichlet space is regular and $\mathbf{M}$ is an associated Hunt process, the following facts are already known ([F 80], [FOT 94, Theorem 5.4.2]): the AF $A^{[u]}$ for $u \in \mathcal{F}$ can be uniquely decomposed as

$$A^{[u]} = M^{[u]} + N^{[u]}, \tag{3}$$

where $M^{[u]}$ is a martingale AF of finite energy and $N^{[u]}$ is a continuous AF of zero energy. $N^{[u]}$ needs not be of bounded variation. It is of bounded variation if and only if there exists a set function ν on X and an $\mathcal{E}$-nest $\{F_n\}$ such that $\nu|_{F_n}$ is a finite signed measure for each n and ν charges no $\mathcal{E}$-exceptional set (such a set function ν is called a *smooth signed measure with an attached $\mathcal{E}$-nest $\{F_n\}$*) and further

$$\mathcal{E}(u,v) = -\int_X v^*(x)d\nu(x), \qquad \forall v \in \cup_{n=1}^{\infty} \mathcal{F}_{b,F_n}. \tag{4}$$

In this case moreover, expressing ν as a difference $\nu^1 - \nu^2$ of two positive smooth measures and denoting by A^k the positive continuous AF associated with ν^k by the Revuz correspondence, $k = 1, 2$, it holds that

$$N^{[u]} = A^1 - A^2. \tag{5}$$

The above mentioned facts for a regular Dirichlet form and an associated Hunt process will be systematically extended in §5 to a general quasi-regular Dirichlet space and an associated standard process. Here we will make use of a regular representation and an associated quasi-homeomorphism of the underlying spaces as will be formulated in §2 following [F 71a], [F 71b], [FOT 94], [CMR 94]. Such a method was already adopted in [MR 92] in their specific context of a (local) compactification where a quasi-homeomorphism is realized by an embedding, and they called it a *transfer method.* But we would like to use this term in the present more general context. A Dirichlet space is called strongly regular if the associated resolvent admits a version possessing a Ray property (see §2 for precise definition). In §4, we crucially need a refined transfer method involving a strongly regular representation.

Thus we basically need to show for $u \in \mathcal{F}$ the equivalence between the inequality (2) and the existence of a signed smooth measure ν satisfying equation (4). This will be done in §3 and §4. While the inequality (2) follows readily from the equation (4), the proof of the converse implication, especially the derivation of the smoothness of a signed measure is a very delicate matter. It has been known however that a kind of strong Feller property of the resolvent of the associated Markov process $\mathbf{M}$ yields the desired smoothness([CFW 93], [FOT 94, Theorem 5.4.3], [F 97a]). In §3, we shall work with a strongly regular Dirichlet space to show that this requirement can be weakened to a Ray property of the resolvent of the associated process. We shall then employ in §4 a transfer method involving a strongly regular representation of the Dirichlet space prepared in §2 to complete the proof of the desired equivalence.

The condition (2) is accordingly more easily verifiable than the existence of a signed smooth measure satisfying equation (4). Actually it is enough to require inequality (2) holding for v in a more tractable dense subspace of $\mathcal{F}_{b,F_n}$. For instance, consider the simple case that the nest is trivial: $F_n = X, n = 1, 2, \cdots$. Then (2) is reduced to the condition that, for $u \in \mathcal{F}$, there exists a positive constant C such that the inequality

$$|\mathcal{E}(u,v)| \leq C \, \|v\|_\infty, \tag{6}$$

holds for any v in the space

$$\mathcal{F}_b = \mathcal{F} \cap L^\infty(X; m).$$

As will be seen in §6, the above condition is equivalent to the one obtained by replacing the space $\mathcal{F}_b$ with its subspace $\mathcal{L}$ satisfying

($\mathcal{L}$) $\mathcal{L}$ is an $\mathcal{E}_1$-dense linear subspace of $\mathcal{F}_b$, and, for any $\epsilon > 0$, there exists a real function $\phi_\epsilon(t)$ such that

$$\begin{aligned} |\phi_\epsilon(t)| &\leq 1+\epsilon,\ t \in R; \quad \phi_\epsilon(t) = t,\ t \in [-1,1]; \\ 0 &\leq \phi_\epsilon(t) - \phi_\epsilon(s) \leq t - s,\ s < t,\ s,t \in R, \end{aligned}$$

and $\phi_\epsilon(\mathcal{L}) \subset \mathcal{L}$.

We shall further see in §6 that inequality (6) holding for $v \in \mathcal{L}$ is not only sufficient but also necessary for the AF $N_t^{[u]}$ to be of bounded variation with an additional property that

$$\lim_{t \downarrow 0} \frac{1}{t} E_m \left(\int_0^t |dN_s^{[u]}| \right) < \infty, \tag{7}$$

where the integral inside the braces denotes the total variation of $N^{[u]}$ on the interval $[0,t]$. The property (7) says that this PCAF has a finite Revuz measure.

When X is an infinite dimensional vector space and $\mathcal{E}$ is obtained by closing a pre-Dirichlet form defined for smooth cylindrical functions on X, we may take as $\mathcal{L}$ the set of all smooth cylindrical functions to check inequality (6).

When the Dirichlet space is regular, a natural choice of $\mathcal{L}$ is a dense subspace of $C_0(X)$ satisfying the condition ($\mathcal{L}$). In this case, inequality (6) for $\mathcal{L}$ is evidently equivalent to the existence of a unique finite signed measure ν satisfying the equation (4) for all $v \in \mathcal{L}$. Our general theorem in §4 assures that this ν is automatically smooth, namely, it charges no set of zero $\mathcal{E}_1$-capacity. When the Dirichlet space is not only regular but also strongly local, we shall extend in §6 the last statement to a function $u \in \mathcal{F}_{loc}$ satisfying equation (4) for a signed Radon measure ν and for all v belonging to a more specific subspace $\mathcal{L}$ of $\mathcal{F} \cap C_0(X)$. We will see that this property of u is equivalent to the condition that $N_t^{[u]}$ is of bounded variation and satisfies, for any compact set K,

$$\lim_{t \downarrow 0} \frac{1}{t} E_m \left(\int_0^t I_K(X_s) |dN_s^{[u]}| \right) < \infty. \tag{8}$$

In §7 we will apply the last theorem of §6 to the energy forms $\mathcal{E}$ and the associated distorted Brownian motions $\mathbf{M}$ living on closed subsets of R^d. In particular, we shall improve those results obtained in [F 97a],[F 97b] to complete stochastic characterizations of BV functions and Caccioppoli sets.

Finally we mention a celebrated paper [CJPS 80], in which it was already proved that, given a general Markov process $\mathbf{M} = (X_t, P_x)$ on a general state space X, a function u on X produces a semimartingale $u(X_t)$ under P_x for every $x \in X$ if and only if there exist finely open sets E_n with

$$\cup_{n=1}^{\infty} E_n = X, \qquad P_x(\lim_{n\to\infty} \tau_{E_n} = \infty) = 1 \quad \forall x \in X$$

such that u is a difference of two excessive functions on each set E_n. In the present paper, we restrict ourselves to the case that $\mathbf{M}$ is symmetric and the semimartingale property of $u(X_t)$ is required to hold only for $\mathcal{E}$-q.e. starting point $x \in X$. As is clear from the above explanations, our necessary and sufficient conditions of the type (2) are more easily verifiable in many cases where X are of higher dimensions.

2 Representation and quasi-homeomorphism

We say that a quadruplet $(X, m, \mathcal{E}, \mathcal{F})$ is a *Dirichlet space* if X is a Hausdorff topological space with a countable base, m is a σ-finite positive Borel measure on X and $\mathcal{E}$ with domain $\mathcal{F}$ is a Markovian closed symmetric form on $L^2(X; m)$. The inner product in $L^2(X; m)$ is denoted by $(\cdot, \cdot)_X$ and we let

$$\mathcal{E}_\alpha(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) + \alpha(\cdot, \cdot)_X \quad \alpha > 0.$$

We note that the space $\mathcal{F}_b = \mathcal{F} \cap L^\infty(X; m)$ is an algebra.

Given two Dirichlet spaces

$$(X, m, \mathcal{E}, \mathcal{F}), \qquad (\tilde{X}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}),$$

we call them *equivalent* if there is an algebraic isomorphism Φ from $\mathcal{F}_b$ onto $\tilde{\mathcal{F}}_b$ preserving three kinds of metrics: for $u \in \mathcal{F}_b$

$$\|u\|_\infty = \|\Phi u\|_\infty, \ (u, u)_X = (\Phi u, \Phi u)_{\tilde{X}}, \ \mathcal{E}(u, u) = \tilde{\mathcal{E}}(\Phi u, \Phi u).$$

One of the two equivalent Dirichlet spaces is called a *representation* of the other.

The underlying spaces X, $\tilde{X}$ are said to be *quasi-homeomorphic* if there exist $\mathcal{E}$-nest $\{F_n\}$, $\tilde{\mathcal{E}}$-nest $\{\tilde{F}_n\}$ and a one to one mapping q from $X_0 = \cup_{n=1}^{\infty} F_n$ onto $\tilde{X}_0 = \cup_{n=1}^{\infty} \tilde{F}_n$ such that its restricition to each F_n is homeomorphic to $\tilde{F}_n$.

We say that the equivalence as above is *induced by a quasi-homeomorphism* if there exists a mapping q as above such that

$$\Phi u(\tilde{x}) = u(q^{-1}(\tilde{x})) \qquad \tilde{x} \in X_0.$$

Then $\tilde{m}$ is the image measure of m by q and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the image of $(\mathcal{E}, \mathcal{F})$ by q. Furthermore q is quasi-notion preserving ([CMR 94, Cor.3.6]):

1. Let $\{E_n\}$ be an increasing sequence of closed subsets of X. It is an $\mathcal{E}$-nest if and only if $\{q(F_n \cap E_n)\}$ is an $\tilde{\mathcal{E}}$-nest.

2. $N \subset X$ is $\mathcal{E}$-exceptional if and only if $q(X_0 \cap N)$ is $\tilde{\mathcal{E}}$-exceptional.

3. A function f, $\mathcal{E}$-q.e. defined on X, is $\mathcal{E}$-quasicontinuous if and only if $f \circ q^{-1}$ is $\tilde{\mathcal{E}}$-quasicontinuous.

Let us now recall three kinds of regularity of Dirichlet spaces. We call a Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ *quasi-regular* if there exists an $\mathcal{E}$-nest consisting of compact sets, each element in a certain $\mathcal{E}_1$ dense subspace of $\mathcal{F}$ admits its $\mathcal{E}$-quasicontinuous version and there exists an $\mathcal{E}$-nest $\{F_n\}$ such that the points of $\cup_{n=1}^{\infty} F_n$ are separated by a certain coutable familiy of $\mathcal{E}$-quasicontinuous functions belonging to $\mathcal{F}$. Every element of $\mathcal{F}$ then admits a quasicontinuous version.

When X is locally compact, we denote by $C_0(X)$ (resp. $C_\infty(X)$) the space of continuous functions on X with compact support (resp. vanishing at infinity). We call a Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ *regular* if X is a locally compact separable metric space, m is a positive Radon measure on X with full support and the space $\mathcal{F} \cap C_0(X)$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ and uniformly dense in $C_0(X)$.

A submarkovian resolvent kernel $R_\alpha(x, B)$ is said to be a *Ray resolvent* if

$$R_\alpha(C_\infty(X)) \subset C_\infty(X) \quad \alpha > 0$$

and there is a countable family C_1 of non-negative function in $C_\infty(X)$ separating points of X_Δ such that

$$\alpha R_{\alpha+1} u \leq u \quad u \in C_1 \quad \alpha > 0.$$

Such a family C_1 is said to be *attached* to the Ray resolvent $R_\alpha(x, B)$.

A Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ is called a *strongly regular* if X is a locally compact separable metric space, m is a positive Radon measure on X with full support and the associated L^2-resolvent is generated by a Ray resolvent and a set C_1 attached to this Ray resolvent is contained in the space $\mathcal{F} \cap C_\infty(X)$. Any strongly regular Dirichlet space is regular (cf.[F 71b, Remark 2.2] and [FOT 94, Lemma 1.4.2]).

We shall prove the following two theorems by combining those results in [F 71a], [F 71b], [FOT 94] and [CMR 94].

Theorem 2.1 *Any Dirichlet space admits its strongly regular representation.*

Theorem 2.2 *A Dirichlet space is quasi-regular if and only if some (and equivalently any) of its regular representations is induced by a quasi-homeomorphism.*

We prepare a lemma about identifications of quasi-notions in the regular case. Suppose $(X, m, \mathcal{E}, \mathcal{F})$ is a regular Dirichlet space. The associated capacity Cap is defined for any open set $A \subset X$ by

$$Cap(A) = \inf\{\mathcal{E}_1(u, u) : u \in \mathcal{F},\ u \geq 1_A\} \quad \inf \phi = \infty$$

and for any set $B \subset X$ by

$$Cap(B) = \inf\{Cap(A) : B \subset A \text{ open }\}.$$

In [FOT 94], 'q.e.' means 'except for a set of zero capacity'. A family $\{F_n\}$ of increasing closed subsets of X is then said in [FOT 94] to be a *nest* if

$$\lim_{n\to\infty} Cap(X \setminus F_n) = 0$$

and to be a *generalized nest* if for any compact set $K \subset X$

$$\lim_{n\to\infty} Cap(K \setminus F_n) = 0.$$

A function u defined 'q.e.' on X is said in [FOT 94] to be *quasicontinuous* if for any $\epsilon > 0$ there exists an open set A with $Cap(A) < \epsilon$ such that $u|_{X-A}$ is continuous.

Lemma 2.1 *Suppose* $(X, m, \mathcal{E}, \mathcal{F})$ *is regular, then*

(i) *a family of increasing closed sets is an* $\mathcal{E}$*-nest iff it is a generalized nest in the sense of* [FOT 94],

(ii) $N \subset X$ *is* $\mathcal{E}$*-exceptional iff* $Cap(N) = 0$,

(iii) *a function on* X *is* $\mathcal{E}$*-quasicontinuous iff it is quasicontinuous in the sense of* [FOT 94].

Proof. Let $\mathbf{M} = (X_t, P_x)$ be a Hunt process on X which is *associated with the form* $\mathcal{E}$ in the sense that the transition semigroup $p_t f$ of the process $\mathbf{M}$ is a version of the L^2 semigroup $T_t f$ associated with $\mathcal{E}$ for any non-negative Borel function $f \in L^2(X; m)$.

(i) Denote by σ_E the hitting time of a set E:

$$\sigma_E = \inf\{t > 0 : X_t \in E\} \quad (\inf \phi = \infty).$$

In view of Lemma 5.1.6 of [FOT 94], we know that an increasing sequence $\{F_n\}$ of closed sets is a generalized nest iff

$$P_x(\lim_{n\to\infty} \sigma_{X-F_n} < \zeta) = 0 \quad \text{q.e. } x \in X, \tag{9}$$

where ζ denotes the life time of $\mathbf{M}$.

If (9) is true, then, for any bounded Borel $\varphi \in L^2(X; m)$, the function

$$R_1^{(n)}\varphi(x) = E_x\left(\int_0^{\sigma_{X-F_n}} e^{-t}\varphi(X_t)dt\right), \; x \in X, \tag{10}$$

belongs to the space $\mathcal{F}_{F_n}$ and converges as $n \to \infty$ to the 1-resolvent $R_1\varphi$ of $\mathbf{M}$ m-a.e. and in $\mathcal{E}_1$-metric as well. Here we set the value of φ at the cemetery Δ to be zero. Since the family $R_1\varphi$ is dense in $\mathcal{F}$, $\{F_n\}$ is an $\mathcal{E}$-nest.

If conversely $\{F_n\}$ is an $\mathcal{E}$-nest, then, for $\sigma = \lim_{n\to\infty} \sigma_{X-F_n}$, the function

$$u(x) = E_x\left(\int_{\sigma\wedge\zeta}^{\zeta} e^{-t}\varphi(X_t)dt\right) \tag{11}$$

must vanish m-a.e. because $u \in \mathcal{F}$ is $\mathcal{E}_1$-orthogonal to $\cup_{n=1}^{\infty}\mathcal{F}_{F_n}$. Since u is quasicontinuous, it vanishes q.e. Choosing φ in (11) to be strictly positive on X, we arrive at the property (9).

(ii) If N is $\mathcal{E}$-exceptional, then

$$N \subset \cap_{n=1}^{\infty}(X - F_n) \tag{12}$$

for some generalized nest $\{F_n\}$ by virtue of (i). Then $Cap(K \cap N) = 0$ for any compact set K and hence $Cap(N) = 0$. Conversely, any set N of zero capacity satisfies the inclusion (12) for a certain nest $\{F_n\}$ in the sense of [FOT 94] ,which is an $\mathcal{E}$-nest by (i). Hence N is $\mathcal{E}$-exceptional.

To see the equivalence (iii), let u be $\mathcal{E}$-quasicontinuous with associated $\mathcal{E}$-nest $\{F_n\}$. Since $\{F_n\}$ is a generalized nest by (i), u is quasicontinuous in the sense of [FOT 94] on each relatively compact open subset of X, which in turn readily implies that it is quasicontinuous on X in the sense of [FOT 94]. The converse implication is trivial as in the proof of (ii). □

For a regular Dirichlet space, the notion 'q.e.' now becomes a synomym for '$\mathcal{E}$-q.e.' Further the condition (9) with 'q.e.' being replaced by '$\mathcal{E}$-q.e.' becomes a stochastic characterization of an $\mathcal{E}$-nest.

Proof of Theorem 2.1 Given a general Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$, a subalgebra L of $L^\infty(X; m)$ is said to satisfy condition (L) if
(L.1) L is a countably generated closed subalgebra of $L^\infty(X; m)$,
(L.2) $\mathcal{F} \cap L$ is dense both in $(\mathcal{F}, \mathcal{E}_1)$ and in $(L, \|\cdot\|_\infty)$,
(L.3) $L^1(X; m) \cap L$ is dense in $(L, \|\cdot\|_\infty)$.

Denote by $L_0^\infty(X; m)$ the closure of $L^2 \cap L^\infty$ in L^∞ and by $\bar{G}_\alpha$ the extension of the L^2 resolvent operator G_α associated with $\mathcal{E}$ from $L^2 \cap L^\infty$ to L_0^∞. A closed subalgebra L of $L_0^\infty(X; m)$ is said to satisfy condition (R) if
(R.1) $\bar{G}_\alpha(L) \subset L$ for any $\alpha > 0$,
(R.2) L is generated by a countable subset L_0 of $\mathcal{F} \cap L$ such that each $u \in L_0$ is non-negative and satisfies

$$\alpha \bar{G}_{\alpha+1} u \le u, \quad \alpha > 0.$$

Let L be a closed subalgebra of $L^\infty(X; m)$ satisfying condition (L) and $\tilde{X}$ be its character space. By virtue of [FOT 94, Th.A.4.1], there exists then a regular Dirichlet space with underlying space $\tilde{X}$ which is equivalent to the given Dirichlet space. Such a regular Dirichlet space is called a regular representation with respect to L. [F 71a, Th.3] went further asserting that there exists a subalgebra L satisfying not only (L) but also (R), and the regular representation with respect to this L becomes strongly regular.

[FOT 94, Th.A.4.1] is a reformulation of [F 71a, Th.2] just by removing the irrelevent assumption that X is locally compact and m is Radon. In the same way, [F 71a, Th.3] can be reformulated. □

Proof of Theorem 2.2 In view of [F 71b, Th.2.1] (c.f. [FOT 94, Th.A.4.2]), we see that, whenever two regular Dirichlet spaces are equivalent, then the equivalence is induced by a quasi-homeomorphism. Here, the quasi-homeomorphism was formulated by the nest defined by the associated capacities, but it is a quasi-homeomorphism in the present sense because of Lemma 2.1 (i).

On the other hand, [CMR 94, Th.3.7] shows that a Dirichlet space is quasi-regular if and only if it is equivalent to some regular Dirichlet space by means of a quasi-homeomorphism. We arrive at Theorem 2.2 by combining those two facts. □

Corollary 2.1 *If two quasi-regular Dirichlet spaces are equivalent, then the equivalence is induced by a quasi-homeomorphism.*

Corollary 2.2 *Any quasi-regular Dirichlet space is equivalent to a strongly regular Dirichlet space by means of a quasi-homeomorphsim.*

3 Smoothness of signed measures for a strongly regular Dirichlet spaces

In this section, we work with a fixed strongly regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$. For the associated Ray resolvent $R_\alpha(x, B)$, there exists a substochastic kernel $\mu(x, B)$ such that

$$\lim_{\alpha\to\infty} \alpha R_\alpha f(x) = \int_X f(y)\mu(x, dy) \quad \forall f \in C_\infty(X).$$

A point $x \in X$ is called a *branching point* if the measure $\mu(x, \cdot)$ is not concentrated on $\{x\}$. The set X_b of all branching points is called the *branch set.* One can then construct a Markov process $\mathbf{M} = (X_t, P_x)$ on X called a *Ray process* with resolvent $R_\alpha(x, B)$, which is known to enjoy the following specific properties ([R 59], [KW 67]). We denote by X_Δ the one point compactification of X and put $\zeta(\omega) = \inf\{t \geq 0 : X_t(\omega) = \Delta\}$.

($\mathbf{M}$.1) $P_x(X_0 = x) = 1 \quad x \in X - X_b$.

($\mathbf{M}$.2) The sample path $X_t = X_t(\omega)$ is cadlag; $X_t(\omega) \in X_\Delta$ is right continuous for all $t \geq 0$, and have the left limit $X_{t-}(\omega) \in X_\Delta$ for all $t > 0$. $X_t(\omega) = \Delta$ for any $t \geq \zeta(\omega)$.

($\mathbf{M}$.3) $P_x(X_t \in X_\Delta - X_b) = 1 \quad \forall x \in X,\ \forall t \geq 0$.

($\mathbf{M}$.4) $\mathbf{M}$ is strong Markov.

($\mathbf{M}$.5) (quasi-left continuity in a restricted sense)
If stopping times σ_n increase to σ, then, for any $x \in X$,

$$P_x\left(\lim_{n\to\infty} X_{\sigma_n} = X_\sigma | \sigma < \infty, \lim_{n\to\infty} X_{\sigma_n} \in X_\Delta - X_b\right) = 1.$$

A non-negative universally measurable function f on X is said to be 1-*supermedian* if

$$\alpha R_{\alpha+1} f(x) \leq f(x) \quad \alpha > 0,\ x \in X,$$

and to be 1-*excessive* if

$$\alpha R_{\alpha+1} f(x) \uparrow f(x) \quad \alpha \to \infty,\ x \in X.$$

For a 1-supermedian function f, $\hat{f}$ denotes its 1-excessive regularization :

$$\hat{f}(x) = \lim_{\alpha\to\infty} \alpha R_{\alpha+1} f(x).$$

Actually $\hat{f}$ is then 1-excessive and $\hat{f}(x) \leq f(x)$ $x \in X$. We shall use the following description of the branch set(cf. [KW 67]). For the family $C_1 (\subset \mathcal{F} \cap C_\infty(X))$ of 1-supermedian functions attached to the Ray resolvent, we set

$$C_1' = \{f \wedge c : f \in C_1,\ c \text{ positive rational }\}.$$

Then

$$X_b = \bigcup_{g \in C_1'} \{x \in X : g(x) > \hat{g}(x)\}. \tag{13}$$

For a Borel set $B \subset X$, we let

$$\sigma_B(\omega) = \inf\{t > 0 : X_t \in B\} \quad \dot{\sigma}(\omega) = \inf\{t \geq 0 : X_t \in B\},$$

with the convention that $\inf \phi = \infty$ and we further let

$$p_B(x) = E_x\left(e^{-\sigma_B}\right) \quad \dot{p}_B(x) = E_x\left(e^{-\dot{\sigma}_B}\right).$$

When B is open, $p_B = \dot{p}_B$ and it is a 1-excessive Borel measurable function.

The facts stated in the next lemma are taken from [F 71b] but we shall present alternative elementary proof of them.

Lemma 3.1 (i) $Cap(X_b) = 0$.

(ii) *Let A be an open set with $Cap(A) < \infty$ and $e_A \in \mathcal{F}$ be its (1-)equilibrium potential. Then $p_A(x)$ is a Borel 1-excessive version of e_A.*

Proof. (i) Take any $g \in C_1'$. Since $\alpha R_{\alpha+1} g$ is $\mathcal{E}_1$-convergent to $g \in \mathcal{F}$, $\hat{g}$ is an $\mathcal{E}$-quasicontinuous version of g. But g is continuous and hence $g = \hat{g}$ $\mathcal{E}$-q.e., namely, $Cap(g > \hat{g}) = 0$. By (13) and the countable subadditivity of the capacity, we arrive at (i).

(ii) Since $p_A(x) = 1$ for all $x \in A - X_b$ and consequently m-a.e. on A by (**M**.6), the proof of [FOT 94, Lemma 4.2.1] (where $p_A(x) = 1$ $\forall x \in A$) works without any change in proving that p_A is a version of e_A.

□

We now proceed to the proof of a proposition which is an intermidiate but crucial step in establishing the equivalence of the inequality (2) and the exisitence of a smooth signed measure satisfying (4).

Proposition 3.1 *Let $u \in \mathcal{F}$ and $w \in \mathcal{F}_b$. Suppose there exists a finite signed measure $\nu = \nu_{u,w}$ on X such that*

$$\mathcal{E}(u, vw) = -\int_X v\, d\nu \quad \forall v \in \mathcal{F} \cap C_\infty(X). \tag{14}$$

Then ν is smooth, namely, ν charges no set of zero capacity. Moreover it holds that

$$\mathcal{E}(u, vw) = -\int_X v^* d\nu \quad \forall v \in \mathcal{F}_b, \tag{15}$$

where v^* is any $\mathcal{E}$-quasicontinuous version of v.

Lemma 3.2 *Assume the condition of Proposition 3.1.*

(i) *Suppose v_n satisfies the equation* (14), *$\mathcal{E}(v_n, v_n)$ is bounded, $v_n(x)$ is uniformly bounded and v_n converges to a function $v \in \mathcal{F}$ pointwise and in $L^2(X; m)$. Then v satisfies* (14).

(ii) *The equation* (14) *holds for the product $v = v_1 \cdot v_2$ for any $v_1 \in \mathcal{F} \cap C_\infty(X)$ and for any bounded Borel 1-excessive function $v_2 \in \mathcal{F}$.*

Proof. (i) $w \cdot v_n$ then $\mathcal{E}$-weakly convergent to $w \cdot v$(cf.[FOT 94, Th.1.4.2]).

(ii) Fix an arbitrary non-negative $v_1 \in \mathcal{F} \cap C_\infty(X)$ and $\alpha > 0$ and let

$$H = \{f \in L^2_+(X; m) : \text{ bounded Borel, (14) holds for } v = v_1 \cdot R_\alpha f\}.$$

Since $R_\alpha(C_0(X)) \subset \mathcal{F} \cap C_\infty(X)$, $C_0(X) \subset H$. If $f_1, f_2 \in H, c_1 f_1 + c_2 f_2 \geq 0$ for some constants c_1, c_2, then clearly $C_1 f_1 + c_2 f_2 \in H$. If $f_n \in H$ increases to a bounded function $f \in L^2(X; m)$, then $v_1 \cdot R_\alpha f_n$ is $\mathcal{E}$-bounded, uniformly bounded, and convergent to $v_1 \cdot R_\alpha f$ pointwise and in L^2. Hence $f \in H$ by virtue of (i). By the monotone lemma, we see that equation (14) holds for $v = v_1 \cdot R_\alpha f$ for any nonnegative bounded Borel $f \in L^2$.

Next take any bounded Borel 1-excessive function $v_2 \in \mathcal{F}$. Since $\alpha R_{\alpha+1} v_2$ is $\mathcal{E}_1$-convergent to v_2 as $\alpha \to \infty$, $v_1 \cdot \alpha R_{\alpha+1} v_2$ is $\mathcal{E}$-bounded, uniformly bounded and convergent to $v_1 \cdot v_2$ pointwise and in L^2. Hence (14) holds for $v = v_1 \cdot v_2$ by (i) again. □

Lemma 3.3 *Assume the condition of Proposition 3.1 and denote by $\bar{\nu}$ the total variation of the finite signed measure ν.*

(i) $\bar{\nu}(X_b) = 0$.

(ii)

$$P_{\bar{\nu}}(X_{t-} \in X_b \quad \exists t \in (0, \infty)) = 0. \tag{16}$$

Proof. (i) We use the description (13) of the branch set. Take $g \in C'_1$. For any $h \in \mathcal{F} \cap C_\infty(X)$, $hg \in \mathcal{F} \cap C_\infty(X)$ and

$$\mathcal{E}(u, whg) = -\int_X hg d\nu.$$

On the other hand, $\hat{g}$ is a bounded Borel 1-excessive function and defines the same element of $\mathcal{F}$ as g because $\alpha R_{\alpha+1} g$ is $\mathcal{E}_1$-convergent to $g \in \mathcal{F}$. Therefore, by Lemma 3.2 (i),

$$\mathcal{E}(u, whg) = -\int_X h\hat{g} d\nu,$$

and consequently

$$\int_X h(g - \hat{g}) d\nu = 0 \quad \forall h \in \mathcal{F} \cap C_\infty(X),$$

which implies that $\bar{\nu}(\{x \in X : g(x) > \hat{g}(x)\}) = 0$.

(ii) Because of lemma 3.1 (i), there exists an decresing open sets A_n including X_b such that $\lim_{n\to\infty} Cap(A_n) = 0$. Due to Lemma 3.1 (ii) and Lemma 3.2 (ii), we then have

$$\mathcal{E}(u, whp_{A_n}) = -\int_X hp_{A_n} d\nu \quad \forall h \in \mathcal{F} \cap C_\infty(X). \tag{17}$$

In view of

$$Cap(A_n) = \mathcal{E}_1(p_{A_n}, p_{A_n}) \geq (p_{A_n}, p_{A_n})_{L^2},$$

p_{A_n} is $\mathcal{E}$-bounded, uniformly bounded and L^2-convergent to zero. Therefore, from (17) and Lemma 3.2 (i), we have for

$$p(x) = \lim_{n\to\infty} p_{A_n}(x)$$

the identity

$$\int_X h(x)p(x)d\nu(x) = 0 \quad \forall h \in \mathcal{F} \cap C_\infty(X),$$

which implies that

$$0 = \bar{\nu}(\{x \in X : p(x) > 0\}) = P_{\bar{\nu}}(\Lambda),$$

where

$$\Lambda = \{\omega : \lim_{n\to\infty} \sigma_{A_n}(\omega) < \infty\}.$$

Since the ω-set in the braces of the left hand side of (16) is contained in the measurable ω-set Λ, we get to (16).

□

Proof of Porposition 3.1 Take any compact set K with $Cap(K) = 0$. For the first assertion of the proposition, it suffices to show that

$$\bar{\nu}(K) = 0. \tag{18}$$

Choose a sequence $\{A_n\}$ of relatively compact open sets such that

$$A_{n+1} \supset \bar{A}_n \supset K \quad \bigcap_{n=1}^{\infty} A_n = K.$$

Since Cap is a Choquet capacity,

$$\lim_{n\to\infty} Cap(A_n) = \lim_{n\to\infty} Cap(\bar{A}_n) = Cap(K) = 0.$$

On the other hand, $\mathbf{M}$ is quasi-left continuous under $P_{\bar{\nu}}$ by virtue of ($\mathbf{M}$.5) and Lemma 3.3 (ii): for any stopping times σ_n increasing to σ,

$$P_{\bar{\nu}}\left(\lim_{n\to\infty} X_{\sigma_n} = X_\sigma,\ \sigma < \infty\right) = P_{\bar{\nu}}(\sigma < \infty).$$

Hence

$$P_{\bar{\nu}}\left(\lim_{n\to\infty}\sigma_{A_n}\neq\dot{\sigma}_K\right)=0,$$

and accordingly

$$\lim_{n\to\infty}p_{A_n}(x)=\dot{p}_K(x)\quad\text{for } \bar{\nu}-a.e.\ x\in X.$$

We now have the equation (17) for p_{A_n} by Lemma 3.1 (i) and Lemma 3.2 (ii). In the same way as in the proof of the preceding lemma, we see that the left hand side of this equation tends to zero and consequently

$$\int_X h(x)\dot{p}_K(x)d\nu(x)=0\quad\forall h\in\mathcal{F}\cap C_\infty(X),$$

which means that

$$\bar{\nu}(\dot{p}_K>0)=0.$$

But $\dot{p}_K(x)=1$ for $x\in K-X_b$ and hence $\bar{\nu}(K-X_b)=0$, which combined with Lemma 3.1 (i) proves the desired (18).

For the second assertion, take any $v\in\mathcal{F}_b$ and let $\|v\|_\infty=M$. We can then find a sequence of functions $v_n\in\mathcal{F}\cap C_\infty(X)$ $\mathcal{E}_1$-convergent to v such that $\sup_{x\in X}|v_n(x)|\le M$ and further v_n converges $\mathcal{E}$-q.e. to an $\mathcal{E}$-quasicontinuous version v^* of v. As lemma 3.2 (i), the desired identity (15) now follows from (14) for v_n. □

We are now in a position to prove the main theorem of this section.

Theorem 3.1 *For $u\in\mathcal{F}$, the next two conditions are equaivalent:*

(i) *There exists an $\mathcal{E}$-nest $\{F_n\}$ for which the inequality* (2) *is valid for some positive constants C_n.*

(ii) *There exists a signed smooth measure ν with some attached $\mathcal{E}$-nest $\{F_n\}$ for which the equation* (4) *is valid.*

Proof of the implication (ii) $\Rightarrow$ (i). Suppose (ii) is fulfilled. Take any $v\in\mathcal{F}_{b,F_n}$ and let $M=\|v\|_\infty$. Then $v^*\le M\ \mathcal{E}-q.e.$ and further $v^*=0\ \mathcal{E}-q.e.$ on $X\setminus F_n$. Therefore the absolute value of the right hand side of (4) is domonated by $C_n\cdot M$ with C_n being the total variation of ν on the set F_n.

□

The converse implication (i) $\Rightarrow$ (ii) will be proved in the following more specific form.

Proposition 3.2 *Suppose, for $u\in\mathcal{F}$, the inequality* (1) *is valid for some $\mathcal{E}$-nest $\{F_n\}$. Then there exists a smooth signed measure ν with an attached $\mathcal{E}$-nest $\{F'_n\}$ with $F'_n\subset F_n$, $n=1,2,\cdots$, for which the equation* (4) *is valid.*

In the rest of this section, we assume that $u \in \mathcal{F}$ satisfies (1) for an $\mathcal{E}$-nest $\{F_n\}$. We shall construct ν and $\{F_n'\}$ of Proposition 3.2 by a series of lemmas.

First fix an arbitrary w belonging to the space $\mathcal{F}_{F_n,b}$ for some n. Then $v \cdot w \in \mathcal{F}_{F_n,b}$ for any $v \in \mathcal{F}_b$,

$$|\mathcal{E}(u, v \cdot w)| \le C_n \|v \cdot w\|_\infty \le C_n \|w\|_\infty \cdot \|v\|_\infty, \ v \in \mathcal{F}_b.$$

Since this inequality holds for any v in the space $\mathcal{F} \cap C_\infty(X)$ which is uniformly dense in $C_\infty(X)$, there exists a unique finite signed measure $\nu = \nu_w$ for which the equation (14) is valid. In virtue of Proposition 3.1, ν_w charges no set of zero capacity and

$$\mathcal{E}(u, \ v \cdot w) = -\int_X v^*(x) d\nu_w(x) \quad \forall v \in \mathcal{F}_b. \tag{19}$$

Lemma 3.4

$$w_1^* d\nu_{w_2} = w_2^* d\nu_{w_1} \quad w_1, \ w_2 \in \cup_{n=1}^\infty \mathcal{F}_{F_n,b}.$$

Proof. For any $v \in \mathcal{F}_b$,

$$\mathcal{E}(u, \ v w_1 w_2) = -\int_X v^* w_1^* d\nu_{w_2} = -\int_X v^* w_2^* d\nu_{w_1}.$$

□

This lemma says that, roughly speaking, $(w^*)^{-1} d\nu_w$ is independent of w and a candidate of the measure ν we want to construct. To make a rigorous construction, we need to consider a Hunt process associated with the given strongly regular Dirichlet space in order to use a general theory in [FOT 94]. Such a process can be immediately constructed from the Ray process $\mathbf{M} = (X_t, P_x)$ already being considered . In fact, Lemma 3.1 (ii) readily implies the following ([F 71b, Th.3.9]): for any set $B \subset X$ with $Cap(B) = 0$, there exists a Borel set $N \supset B$ with $Cap(N) = 0$ such that $X - N$ is $\mathbf{M}$-*invariant* in the sense that

$$P_x(X_t \in X_\Delta - N, \ X_{-t} \in X_\Delta - N \ \forall t \ge 0) = 1. \ \forall x \in X - N.$$

Since $Cap(X_b) = 0$ by Lemma 3.1 (i), the branch set is included in a set N of the above type. On account of the properties ($\mathbf{M}$.1) $\sim$ ($\mathbf{M}$.5) of the Ray process $\mathbf{M}$, we can get a Hunt process on X still associated with the form $\mathcal{E}$ first by restricting the state space of $\mathbf{M}$ to $X_\Delta - N$ and then by making each point of N to be a trap(see [FOT 94, Th.A.2.8,A.2.9] for those procedures). We may call the resulting Hunt process a *Hunt modification of the Ray process* $\mathbf{M}$.

In what follows in this section, $\mathbf{M} = (X_t, P_x)$ denotes a Hunt process on X associated with the form $\mathcal{E}$. For a strictly positive bounded Borel function φ on X with $\varphi \in L^2(X; m)$, we put

$$\rho_n(x) = R_1^{(n)} \varphi(x), \ x \in X,$$

where the right hand side is defined by (10). Here, we set $\varphi(\Delta) = 0$. By [FOT 94, Th.4.4.1], we know that ρ_n is an $\mathcal{E}$-quasicontinuous Borel function in $\mathcal{F}_{F_n,b}$. We then introduce the sets

$$E_n = \{x \in X : \rho_n(x) \ge \frac{1}{n}\}, \ n = 1, 2, \cdots, \quad N_0 = X - (\cup_{n=1}^\infty E_n). \tag{20}$$

E_n is a quasi-closed Borel set, increasing in n and $E_n \subset F_n,\ n = 1, 2, \cdots$. Since $N_0 = \{x \in X : \lim_{n\to\infty} \rho_n(x) = 0\}$, we see that

$$Cap(N_0) = 0 \tag{21}$$

owing to the stochastic charcterization (9) of the $\mathcal{E}$-nest $\{F_n\}$.

We define ν by

$$\nu(dx) = \frac{1}{\rho_n(x)}\, \nu_{\rho_n}(dx) \text{ on } E_n,\ n = 1, 2, \cdots,\ \nu(N_0) = 0. \tag{22}$$

For $m > n$, we have from Lemma 3.4

$$\frac{1}{\rho_m(x)}\nu_{\rho_m}(dx) = \frac{1}{\rho_n(x)}\nu_{\rho_n}(dx) \text{ on } E_n$$

which means that the above definition makes sense. $\nu|_{E_n}$ is then a finite signed measure for each n and ν charges no set of zero capacity. Moreover

$$\mathcal{E}(u,\ v \cdot w) = -\int_X v^* w^* \nu(dx) \quad v \in \mathcal{F}_b,\ w \in \cup_{n=1}^{\infty} \mathcal{F}_{F_n,b}. \tag{23}$$

In fact, the above definition of ν and Lemma 3.4 imply that

$$\nu_w(dx) = w^* \nu(dx) \quad w \in \cup_{n=1}^{\infty} \mathcal{F}_{F_n,b},$$

and we are led to (23) from (19).

In order to construct an appropriate $\mathcal{E}$-nest from $\{E_n\}$, we prepare a lemma.

Lemma 3.5 *Suppose $\Gamma_n,\ n = 1, 2, \cdots$, are quasi-closed, decreasing in n and*

$$Cap\,(\cap_{n=1}^{\infty} \Gamma_n) = 0.$$

Then

$$\lim_{n\to\infty} Cap(\Gamma_n \cap K) = 0 \quad \text{for any compact set } K.$$

Proof. In view of the definition of the quasi-closed set ([FOT 94, pp68]), we can find, for any $\epsilon > 0$, an open set ω with $Cap(\omega) < \epsilon$ such that $\Gamma_n - \omega$ are closed for all n. For any compact K, $(\Gamma_n - \omega) \cap K$ are decreasing compact sets. Since Cap is a Choquet capacity,

$$\lim_{n\to\infty} Cap((\Gamma_n - \omega) \cap K) = Cap(\cap_{n=1}^{\infty} (\Gamma_n - \omega) \cap K) \le Cap(\cap_{n=1}^{\infty} \Gamma_n) = 0.$$

We then get $\overline{\lim}_{n\to\infty} Cap(\Gamma_n \cap K) \le \epsilon$ from

$$Cap(\Gamma_n \cap K) \le Cap((\Gamma_n - \omega) \cap K) + Cap(\omega).$$

□

Proof of Proposition 3.2 Take a sequence $\epsilon_\ell \downarrow 0$. Since the sets E_n defined by (20) are quasi-closed, we can find decresing open sets ω_ℓ with $Cap(\omega_\ell) < \epsilon_\ell$ and $E_n - \omega_\ell$ are closed for all n and ℓ.

Let us define increasing closed sets F_n' by

$$F_n' = E_n - \omega_n \quad n = 1, 2, \cdots \tag{24}$$

and prove that

$$\lim_{n\to\infty} Cap(K - F_n') = 0 \quad \text{for any compact } K. \tag{25}$$

Since

$$K - F_n' = (K \cap E_n^c) \cup (K \cap \omega_n),$$

we have

$$Cap(K - F_n') \leq Cap(K \cap E_n^c) + \frac{1}{n}.$$

We let

$$\Gamma_n = \{\rho_n \leq \frac{1}{n}\}.$$

Γ_n are then quasi-closed and

$$E_n^c \subset \Gamma_n \quad \cap_{n=1}^\infty E_n^c = \cap_{n=1}^\infty \Gamma_n = N_0$$

because $E_n^c = \{\rho_n < \frac{1}{n}\}$. On account of the preceding Lemma and (21), we conclude that

$$Cap(K \cap E_n^c) \leq Cap(K \cap \Gamma_n) \to 0 \quad n \to \infty.$$

(25) is proven and $\{F_n'\}$ is an $\mathcal{E}$-nest by Lemma 2.1. Moreover

$$F_n' \subset E_n \subset F_n, n = 1, 2 \cdots .$$

For the measure ν defined by (22), $\nu|_{F_n'}$ is therefore a finite signed measure for each n. Since ν charges no set of zero capacity, it becomes a smooth measure for which the present $\mathcal{E}$-nest $\{F_n'\}$ is attached.

Finally take any w belonging to the space $\mathcal{F}_{F_n',b}$ for some n. Let $v = (n\rho_n) \wedge 1$. Then $v = 1$ on $E_n (\supset F_n')$ and $v \cdot w = w$. Thus the equation (23) leads us to

$$\mathcal{E}(u, w) = \int_X w^*(x)\nu(dx) \quad \forall w \in \cup_{n=1}^\infty \mathcal{F}_{F_n',b}, \tag{26}$$

completing the proof of Proposition 3.2. $\square$

Theorem 3.1 is proved. Here we add a theorem corresponding to a special case of Theorem 3.1 where the $\mathcal{E}$-nests are trivial.

Theorem 3.2 *For $u \in \mathcal{F}$, the next two conditions are equivalent:*

(i) *The inequality* (6) *holds for some constant $C > 0$.*

(ii) *There exists a finite signed measure ν charging no $\mathcal{E}$-exceptional set such that the equation* (4) *holds for any $v \in \mathcal{F}_b$.*

Proof. The proof of the implication (i) $\Rightarrow$ (ii) is the same as the corresponding proof of Theorem 3.1. The converse can be viewed as a special case of Proposition 3.1 (the case where $w = 1$). $\square$

4 Transfers of analytic theorems to a quasi-regular Dirichlet space

First, we transfer Theorem 3.1 from a strongly regular Dirichlet space to a quasi-regular Dirichlet space.

Theorem 4.1 *Let $(X, m, \mathcal{E}, \mathcal{F})$ be a quasi-regular Dirichlet space. For $u \in \mathcal{F}$, the next two conditions are equivalent:*

(i) *There exists an $\mathcal{E}$-nest $\{E_n\}$ for which the inequality* (2) *is valid for some positive constants C_n.*

(ii) *There exists a signed smooth measure ν with some attached $\mathcal{E}$-nest $\{E_n\}$ for which the equation* (4) *is valid.*

Proof. By Corollary 2.2, the quasi-regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ is equaivalent to a certain strongly regular Dirichlet space $(\tilde{X}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ by a quasi-homeomprphism q; there exist $\mathcal{E}$-nest $\{F_n\}$, $\tilde{\mathcal{E}}$-nest $\{\tilde{F_n}\}$, q is a one to one mapping from $X_0 = \cup_{n=1}^{\infty} F_n$ onto $\tilde{X}_0 = \cup_{n=1}^{\infty} \tilde{F}_n$, its restriction to each F_n is homeomprphic to $\tilde{F}_n$, and the map from $\mathcal{F}_b$ to $\tilde{\mathcal{F}}_b$ defined by

$$(\Phi u)(\tilde{x}) = u(q^{-1}(\tilde{x})) \quad \tilde{x} \in \tilde{X}_0$$

satisfies

$$\|u\|_\infty = \|\Phi u\|_\infty, \ (u, u)_X = (\Phi u, \Phi u)_{\tilde{X}}, \ \mathcal{E}(u, u) = \tilde{\mathcal{E}}(\Phi u, \Phi u).$$

q is $\mathcal{E}$-quasi notions preserving as is explained in §2.

Suppose that the condition (i) is fulfilled. Let

$$\tilde{u} = \Phi(u)(\in \tilde{\mathcal{F}}), \quad \tilde{E}_n = q(E_n \cap F_n) \ n = 1, 2, \cdots$$

Then, $\{\tilde{E}_n\}$ is an $\tilde{\mathcal{E}}$-nest and

$$|\tilde{\mathcal{E}}(\tilde{u}, \tilde{v})| \leq C_n \|\tilde{v}\|_\infty, \quad \forall \tilde{v} \in \tilde{\mathcal{F}}_{b, \tilde{E}_n}, \ n = 1, 2, \cdots$$

for the same constant C_n as in (i).

By virtue of Theorem 3.1, there exists a smooth signed measure $\tilde{\nu}$ on $\tilde{X}$ with an attached $\tilde{\mathcal{E}}$-nest $\{\tilde{E}'_n\}$ for which the equation

$$\tilde{\mathcal{E}}(\tilde{u},\tilde{v}) = -\int_{\tilde{X}} \tilde{v}^*(\tilde{x})d\tilde{\nu}(\tilde{x}), \quad \forall \tilde{v} \in \cup_{n=1}^{\infty} \tilde{\mathcal{F}}_{b,\tilde{E}_n}$$

holds. Let

$$\nu(B) = \tilde{\nu}(q(B \cap X_0)), \quad E'_n = q^{-1}(\tilde{E}'_n \cap \tilde{F}_n)\ n = 1, 2, \cdots$$

Then we can easily see that $\{E'_n\}$ is an $\mathcal{E}$-nest, ν is a smooth signed measure on X with the attached $\mathcal{E}$-nest $\{E'_n\}$. By rewiting the above equation and noting that $v^*(x) = \tilde{v}^*(qx)$ gives an $\mathcal{E}$-quasicontinuous version of $v = \Phi^{-1}(\tilde{v})$, we arrive at the equation (4) holding for $u, \nu, \{\tilde{E}'_n\}$, getting the conclusion (ii).

The converse implication (ii) $\Rightarrow$ (i) can be directly shown as the corresponding proof of Theorem 3.1. □

In exactly the same way, Theorem 3.2 can be transfered from a strongly regular Dirichlet space to a quasi-regular Dirichlet space.

Theorem 4.2 *Let $(X, m, \mathcal{E}, \mathcal{F})$ be a quasi-regular Dirichlet space. For $u \in \mathcal{F}$, the next two conditions are equivalent:*

(i) *The inequality* (6) *holds for some constant $C > 0$.*

(ii) *There exists a finite signed measure ν charging no $\mathcal{E}$-exceptional set such that the equation* (4) *holds for any $v \in \mathcal{F}_b$.*

For later convenience, we also transfer Proposition 3.1 in the following manner:

Proposition 4.1 *Let $(X, m, \mathcal{E}, \mathcal{F})$ be a quasi-regular Dirichlet space. Let $u \in \mathcal{F}$ and $w \in \mathcal{F}_b$. Suppose there exists a positive constant $C = C_{u,w}$ such that*

$$|\mathcal{E}(u, vw)| \leq C\|v\|_\infty \quad \forall v \in \mathcal{F}_b.$$

Then there exists uniquely a finite signed smooth measure ν for which the equation (15) *is valid.*

Proof. In the same way as in the proof of Theorem 4.1, the above inequality is honestly inherited to a strongly regular representation, where we can use Proposition 3.1 to obtain the conclusion as above which can be transfer back to the quasi-regular Dirichlet space also in the same way as in the proof of Theorem 4.1. □

5 Transfers of stochastic contents to a quasi-regular Dirichlet space

This section is devoted to transfers of probabilistic notions and theorems from a regular Dirichlet space to a quasi-regular Dirichlet space. In particular, we formulate the stochastic significance of the condition (ii) in Theorem 4.1 more precisely than what was mentioned in §1.

To this end, let us first consider a regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ and an associated Hunt process $\mathbf{M} = (X_t, P_x)$ on X. In this context, we start with mentioning some probabilistic characterizations of $\mathcal{E}$-exceptional sets in a convenient way for later use.

A set $N \subset X$ is called $\mathbf{M}$-*exceptional* if N is contained in a nearly Borel set $\hat{N}$ such that $P_m(\sigma_{\hat{N}} < \infty) = 0$. A set N is said to be ($\mathbf{M}$-)*properly exceptional* if it is a nearly Borel measurable m-negligible set and its complement $X - N$ is $\mathbf{M}$-invariant. We call N ($\mathbf{M}$-)*properly exceptional in the standard sense* if, in the above definition of the proper exceptionality, the $\mathbf{M}$-invarance of $X - N$ is weakened to the $\mathbf{M}$-invariance up to the life time in the following sense:

$$P_x(X_t,\ X_{t-} \in X - N\ \forall t \in [0, \zeta)) = 1,\ \forall x \in X - N.$$

Remark 5.1 The last notion of the exceptionality makes sense not only for the present Hunt process but also for a standard process, and we shall later utilize it for a standard process associated with a quasi-regular Dirichlet space.

Lemma 5.1 *The following conditions for a set $N \subset X$ are equivalent each other:*

(i) *N is $\mathcal{E}$-exceptional.*

(ii) *N is $\mathbf{M}$-exceptional.*

(iii) *N is contained in a properly exceptional set.*

(iv) *N is contained in a properly exceptional set in the standard sense.*

Proof. The equivalence of the first three conditions are proven in [FOT 94]. The implication (iii)$\Rightarrow$(iv)$\Rightarrow$(ii) is also obvious. □

Remark 5.2 (i) In studying a Hunt process $\mathbf{M}$ associated with a regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$, the state space of $\mathbf{M}$ needs not to be the entire space X. Indeed, we may well consider a Borel subset N_0 of X with $m(N_0) = 0$ and a Hunt process $\mathbf{M}$ with state space $X - N_0$ such that it is associated with a Dirichlet form $\mathcal{E}$ on $L^2(X; m)$ in the sense that its transition function on $X - N_0$ generates the L^2-semigroup corresponding to $\mathcal{E}$. Then the set N_0 becomes automatically $\mathcal{E}$-exceptional, because it is properly exceptional with respect to the trivial extension $\mathbf{M}'$ of $\mathbf{M}$ to X ($\mathbf{M}'$ is obtained from $\mathbf{M}$ by joining every point of N_0 as a trap, see [FOT 94, Th.A.2.9]). $\mathbf{M}'$ is still associated with $\mathcal{E}$ and hence Lemma 5.1 applies.

(ii) Given a Hunt process $\mathbf{M}$ on $X - N_0$ as above, we call a set N $\mathbf{M}$-properly exceptional (resp. $\mathbf{M}$-properly exceptional in the standard sense) if $N \supset N_0$, N is nearly Borel, $m(N) = 0$ and $X - N (\subset X - N_0)$ is $\mathbf{M}$-invariant (resp. $\mathbf{M}$-invariant up to the life time). With this slight modification of the notion of proper exceptionality, not only Lemma 5.1 but also what will be stated below about additive functionals remain valid for a Hunt process $\mathbf{M}$ on $X - N_0$ as above.

(iii) We can and we shall allow an analogous freedom of choice of the state space about a standard process associated with a quasi-regular Dirichlet space, accompanyed by an analogous modification of the proper exceptionality in the standard sense.

We quote from [FOT 94] those basic notions and theorems concerning additive functionals of the Hunt process $\mathbf{M}$ on X. By convention, any numerical function f on X is extended to the one-point compactification X_Δ by setting $f(\Delta)=0$. Let $\Omega, \{\mathcal{F}_t\}_{t\in[0,\infty]}, \zeta, \theta_t$ be the sample space, the minimum completed admissible filtration, the life time and the shift operator respectively attached to the Hunt process $\mathbf{M}$. An extended real valued function $A_t(\omega)$ of $t \geq 0$ and $\omega \in \Omega$ is called an *additive functional* (AF in abbreviation) if it is $\{\mathcal{F}_t\}$-adapted and there exist $\Lambda \in \mathcal{F}_\infty$ with $\theta_t\Lambda \subset \Lambda$, $\forall t>0$ and a properly exceptional set $N \subset X$ with $P_x(\Lambda)=1$, $\forall x \in X-N$, such that, for each $\omega \in \Omega$, $A_0(\omega)=0$, $A_t(\omega)$ is cadlag and finite on $[0,\zeta(\omega))$, $A_t(\omega)=A_{\zeta(\omega)}(\omega)$, $t \geq \zeta(\omega)$, and

$$A_{t+s}(\omega)=A_s(\omega)+A_t(\theta_s\omega), \quad s,t\geq 0.$$

Λ (resp. N) in the above definition is called a *defining* (resp. *exceptional*) *set* for the AF A. We regard two AF's to be *equivalent* if

$$P_x\left(A_t^{(1)}=A_t^{(2)}\right)=1, \quad t\geq 0, \quad \mathcal{E}-\text{q.e. } x\in X.$$

Then we can find a common defining set Λ and a common properly exceptional set N of $A^{(1)}$ and $A^{(2)}$ such that $A_t^{(1)}(\omega)=A_t^{(2)}(\omega)$, $\forall t\geq 0$, $\forall \omega\in\Lambda$. Here we have to use Lemma 4.1 together with the fact that the ω-set

$$\Gamma=\{\omega\in\Omega : X_t(\omega),\ X_{t-}(\omega)\in X_\Delta - N\ \forall t\geq 0\}$$

is $\mathcal{F}_\infty$-measurable.

An AF $A_t(\omega)$ is said to be *finite, cadlag* and *continuous* respectively if it satisfies the respective property at every $t\in[0,\infty)$ for each ω in its defining set. A $[0,\infty]$-valued continuous AF is called a *posiitve continuous* AF (PCAF in abbreviation). We denote by $\mathbf{A}_c^+$ the set of all PCAF's. We shall call an AF $A_t(\omega)$ *of bounded variation* if it is of bounded variation in t on each compact subinterval of $[0,\zeta(\omega))$ for every fixed ω in a defining set of A.

A positive Borel measure μ is called a *smooth measure* if μ charges no $\mathcal{E}$-exceptional set and there is an $\mathcal{E}$-nest $\{F_n\}$ such that $\mu(F_n)$ is finite for each n. The totality of smooth measures is denoted by S. There is a one to one correspondence between (the equivalence classes of) $\mathbf{A}_c^+$ and S by the *Revuz correspondence*:

$$\lim_{t\downarrow 0}\frac{1}{t}E_{h\cdot m}\left(\int_0^t f(X_t)dA_t\right)=\int_X h\cdot f d\mu, \quad A\in\mathbf{A}_c^+,\ \mu\in S, \tag{27}$$

holding for any non-negative Borel function f and γ-excessive function h, $\gamma\geq 0$. The smooth measure corresponding to a PCAF A in the above way is said to be the *Revuz measure of* A.

For any $u\in\mathcal{F}$, there exists a unique finite smooth measure $\mu_{\langle u\rangle}$ on X, which satisfies the following equation in the case that $u\in\mathcal{F}_b$:

$$\int_X f^*(x)\mu_{\langle u\rangle}(dx)=2\mathcal{E}(u\cdot f,u)-\mathcal{E}(u^2.f) \quad \forall f\in\mathcal{F}_b. \tag{28}$$

$\mu_{\langle u\rangle}$ is called the *energy measure of* $u\in\mathcal{F}$.

The *energy* $e(A)$ of an AF A is defined by

$$e(A)=\lim_{t\downarrow 0}\frac{1}{2t}E_m(A_t^2).$$

For $u \in \mathcal{F}$, $A_t^{[u]}$ defined by (1) is (up to the equivalence specified above) a finite cadlag AF of finite energy. The families of *martingale AF's of finite energy* and *CAF's of zero energy* are defined respectively by

$$\overset{\circ}{\mathcal{M}} = \{M : \text{ finite cadlag AF } E_x(M_t^2) < \infty, E_x(M_t) = 0 \text{ q.e. and } e(M) < \infty\},$$

$$\mathcal{N}_c = \{N : CAF \ E_x(|N_t|) < \infty \text{ q.e. and } e(N) = 0\}.$$

For any $u \in \mathcal{F}$, the AF $A^{[u]}$ admits a decomposition (3) uniquely up to the equivalence specified above, where $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and $N^{[u]} \in \mathcal{N}_c$ ([FOT 94, Th.5.2.2]). The energy measure $\mu_{\langle u \rangle}$ of u coincides with the Revuz measure of the quadratic variation $\langle M^{[u]} \rangle \in \mathbf{A}_c^+$ of the AF $M^{[u]}$ ([FOT 94, Th. 5.2]). Furthermore [FOT 94, Th.5.4.2] asserts the following: the CAF $N^{[u]}$ is of bounded variation if and only if the condition (ii) of Theorem 4.1 is valid for some signed smooth measure ν. In this case moreover, $N^{[u]}$ admits an expression (5) for some PCAF's A^k corresponding to smooth measures ν^k, $k = 1, 2$, by the Revuz correspondence and it holds that $\nu = \nu^1 - \nu^2$.

Now we turn to a general quasi-regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$. Let $(\tilde{X}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ be its regular representation and

$$\tilde{\mathbf{M}} = (\tilde{\Omega}, \{\tilde{\mathcal{F}}_t\}_{t \in [0,\infty]}, \tilde{X}_t, \tilde{\zeta}, \tilde{P}_{\tilde{x}})$$

be a Hunt process on $\tilde{X}$ associated with the latter (one may take for instance a strongly regular representation and a Hunt modification of the associated Ray process). By Theorem 2.2, the two Dirichlet spaces are related each other just as in the first paragraph of the proof of Theorem 4.1. We shall use the notations $\{F_n\}, X_0, \{\tilde{F}_n\}, \tilde{X}_0, q, \Phi$, appearing in that paragraph without repeating the explanation. We are ready to construct a standard process on X associated with $\mathcal{E}$ as an image of $\tilde{\mathbf{M}}$ by q^{-1} in a similar way to [F 71b].

Applying the stochastic characterization (9) to the $\tilde{\mathcal{E}}$-nest $\{\tilde{F}_n\}$, we can find an $\tilde{\mathbf{M}}$-properly exceptional Borel set $\tilde{N}_1$ including $\tilde{X} - \tilde{X}_0$ such that

$$\tilde{P}_x(\lim_{n \to \infty} \sigma_{\tilde{X} - \tilde{F}_n} < \tilde{\zeta}) = 0 \quad \forall \tilde{x} \in \tilde{X} - \tilde{N}_1. \tag{29}$$

In other words, we have

$$\tilde{P}_x(\tilde{\Lambda}_1) = 1 \quad \forall \tilde{x} \in \tilde{X} - \tilde{N}_1$$

for the set $\tilde{\Lambda}_1 = \tilde{\Lambda}_{11} \cap \tilde{\Lambda}_{12}$ where

$$\tilde{\Lambda}_{11} = \{\tilde{\omega} \in \tilde{\Omega} : \tilde{X}_t, \ \tilde{X}_{t-} \in \tilde{X}_{\tilde{\Delta}} - \tilde{N}_1 \ \forall t \geq 0\}$$

$$\tilde{\Lambda}_{12} = \{\tilde{\omega} \in \tilde{\Omega} : \lim_{t \to \infty} \sigma_{\tilde{X} - \tilde{F}_n} \geq \tilde{\zeta}(\tilde{\omega})\}.$$

Adjoin a point Δ to X as an extra point (as a point at infinity if X is locally compact) and extend q to a one to one mapping from $X_0 \cup \Delta$ onto $\tilde{X}_0 \cup \tilde{\Delta}$ by setting $q(\Delta) = \tilde{\Delta}$. We define an $\mathcal{E}$-exceptional Borel set $N_1 \subset X$ by

$$X - N_1 = q^{-1}(\tilde{X} - \tilde{N}_1). \tag{30}$$

We let

$$\Omega = \tilde{\Lambda}_1 \qquad \mathcal{F}_t = \tilde{\mathcal{F}}_t \cap \tilde{\Lambda}_1 \ t \in [0, \infty]. \tag{31}$$

The element of Ω (resp. $\{\mathcal{F}_t\}$) is denoted by ω (resp. Λ) instead of $\tilde{\omega}$ (resp. $\tilde{\Lambda}$). Finally let us define X_t, ζ, P_x by

$$X_t(\omega) = q^{-1}(\tilde{X}_t(\omega)) \quad \omega \in \Omega,\ t \geq 0, \quad \zeta(\omega) = \tilde{\zeta}(\omega),\ \omega \in \Omega, \tag{32}$$

$$P_x(\Lambda) = \tilde{P}_{qx}(\Lambda) \quad x \in X \cup \Delta - N_1,\ \Lambda \in \mathcal{F}_\infty. \tag{33}$$

With these definitions of elements, we put

$$\tilde{\mathbf{M}}_1 = (\Omega, \{\mathcal{F}_t\}_{t \in [0,\infty]}, \tilde{X}_t, \tilde{\zeta}, \tilde{P}_{\tilde{x}}),$$

$$\mathbf{M}_1 = (\Omega, \{\mathcal{F}_t\}_{t \in [0,\infty]}, X_t, \zeta, P_x).$$

$\tilde{\mathbf{M}}_1$ is a Hunt process on $\tilde{X} - \tilde{N}_1$ which is associated with the regular Dirichlet form $\tilde{\mathcal{E}}$. We shall call $\mathbf{M}_1$ *the image of the Hunt process* $\tilde{\mathbf{M}}_1$ *by the quasi-homeomorphism* q^{-1}.

Theorem 5.1 $\mathbf{M}_1$ *defined by* (30)$\sim$(33) *is a standard process on* $X - N_1$ *associated with the quasi-regular Dirichlet form* $\mathcal{E}$. *Further* $\mathbf{M}_1$ *is special and tight.*

Proof. The first assertion can be proved in the same way as in [F 71b, §4] where $(X, m, \mathcal{E}, \mathcal{F})$, $(\tilde{X}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ and $\tilde{\mathbf{M}}$ were a regular Dirichlet space, its strongly regular representation, and a Hunt modification of the Ray process associated with the latter respectively, and the process $\mathbf{M}_1$ defined by (30)$\sim$(33) was shown to be a Hunt process on $X - N_1$. The only defference from the present situation was in that $\{F_n\}$, $\{\tilde{F}_n\}$ were nests in the sense of [FOT 94] rather than $\mathcal{E}$-nest and $\tilde{\mathcal{E}}$-nest, and accordingly we had the following stronger property than (29):

$$\tilde{P}_x(\lim_{n\to\infty} \sigma_{\tilde{X} - \tilde{F}_n} < \infty) = 0 \quad \forall \tilde{x} \in \tilde{X} - \tilde{N}_1. \tag{34}$$

In the present case, we can also see as in [F 71b] that $\{\mathcal{F}_t\}_{t\in[0,\infty]}$ defined by (31) is the minimum completed admissible filtration for X_t defined by (32). Thus $\mathbf{M}_1$ is special, namely, $\{\mathcal{F}_t\}$ is quasi-left continuous because so is $\{\tilde{\mathcal{F}}_t\}$. Let $\tilde{K}_n$ be compact sets increasing to $\tilde{X}$ such that $\tilde{K}_n$ is included in the interior of $\tilde{K}_{n+1}$ and put $K_n = q^{-1}(\tilde{K}_n \cap \tilde{F}_n)$. Then $\{K_n\}$ are increasing compact subsets of $X - N_1$ and we get obviously the tightness of $\mathbf{M}_1$:

$$P_x(\lim_{n\to\infty} \sigma_{X - K_n} < \zeta) = 0 \quad \forall x \in X - N_1.$$

□

The state space of $\mathbf{M}_1$ can be enlarged to X if necessary by an trivial extension (namely, by making each point of N_1 to be a trap.) [MR 92, Th. 3.5] has given another construction of an m-tight special standard process associated with a quasi-regular Dirichlet space.

Remark 5.3 According to [MR 92, Th. 5.1, Th.6.4] however, two standard processes associated with the same quasi-regular Dirichlet space admit a common properly exceptional set in the standard sense such that their restrictions to the complement of this set share a common transition function. Therefore, in dealing with a standard process associated with a quasi-regular Dirichlet space, we may assume without loss of generality that it is an image of a Hunt process by a quasi-homeomorphism. This viewpoint is very convenient in utilizing the transfer method.

Thus, in the rest of this section, we continue to work with a quasi-regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ and an associated standard process $\mathbf{M}_1 = (\Omega, \{\mathcal{F}_t\}, X_t, \zeta, P_x)$ on $X - N_1$ defined by (30)$\sim$(33). We can transfer Lemma 5.1 as follows:

Lemma 5.2 *The following conditions for a set $N \subset X$ are equivalent each other:*

(i) *N is $\mathcal{E}_1$-exceptional.*

(ii) *N is $\mathbf{M}_1$-exceptional.*

(iii) *N is contained in a properly exceptional set in the standard sense.*

Remark 5.4 Since the state space of $\mathbf{M}_1$ is $X - N_1$, any $\mathbf{M}_1$-properly exceptional set in the standard sense is required to contain the set N_1 (see Remark 5.2).

Proof. For simplicity, let N be a Borel subset of $X - N_1$ and put $\tilde{N} = q(N)(\subset \tilde{X} - \tilde{N}_1)$. Of course, N is $\mathcal{E}$-exceptional iff $\tilde{N}$ is $\tilde{\mathcal{E}}$-exceptional. By our construction of $\mathbf{M}_1$, we see that the $\mathbf{M}_1$-exceptionality of N (resp. $\mathbf{M}_1$-proper exceptionality in the standard sense of $N \cup N_1$) is equivalent to the $\tilde{\mathbf{M}}_1$-exceptionality of $\tilde{N}$ (resp.$\tilde{\mathbf{M}}_1$-proper exceptionality of $\tilde{N} \cup \tilde{N}_1$). But, in view of Lemma 5.1 and Remark 5.1, three conditions of the present lemma are equivalent for $\tilde{N}$, the regular Dirichlet form $\tilde{\mathcal{E}}$ and the Hunt process $\tilde{\mathbf{M}}_1$. □

Remark 5.5 In relation to the ω-set involved in a properly exceptional set in the standard sense, we make the following remark : for a Borel set $A \subset X$, the ω-set

$$\Gamma = \{\omega \in \Omega : X_t(\omega),\ X_{t-}(\omega) \in A\ \forall t \in [0, \zeta(\omega))\}$$

is $\mathcal{F}_\infty$-measurable, because, for $B = X - A$, and each $T > 0$, the ω-set

$$\{\omega : X_{t-} \in B \cup \{\Delta\}\ \exists t \in [0, T \wedge \zeta(\omega))\}$$

is contained in

$$\{\omega : X_{t-} \text{ exists and is in } B \cup \{\Delta\}\ \exists t \in [0, T]\},$$

and by [BG 68, Prop.10.20] we can see that $\Omega \setminus \Gamma \in \mathcal{F}_\infty$.

The notion of the additive functional $A_t(\omega)$ of the present standard process $\mathbf{M}_1$ on $X - N_1$ is defined exactly in the same way as for an Hunt process except that we now adopt a properly exceptional set in the standard sense (instead of a properly exceptional set) as an exceptional set N $(N_1 \subset N \subset X)$ of the additive functional A. The equivalence of two AF's $A^{(1)}$, $A^{(2)}$ of

$\mathbf{M}_1$ is defined in the same way as for an Hunt process. On account of Lemma 5.2 and Remark 5.4, we can then find a common defining set $\Lambda \in \mathcal{F}_\infty$ and a common exceptional set $N(\supset N_1)$ such that

$$A_t^{(1)}(\omega) = A_t^{(2)}(\omega), \ \forall t \geq 0, \ \forall \omega \in \Lambda.$$

We can easily observe that, if $A_t(\omega)$ is an AF of $\mathbf{M}_1$ with a defining set Λ and an exceptional set $N(\supset N_1)$, then $A_t(\omega)$ is an AF of $\tilde{\mathbf{M}}_1$ with some defining set contained in Λ and some exceptional set containing $q(N)(\supset \tilde{N}_1)$. Conversely any AF of $\tilde{\mathbf{M}}_1$ with a defining set Λ and an exceptional set $\tilde{N}(\supset \tilde{N}_1)$ can be viewed as an AF of $\mathbf{M}_1$ with the same defining set and the exceptional set $q^{-1}((\tilde{N}))$. Two AF's are equivalent with respect to $\mathbf{M}_1$ iff so they are with respect to $\tilde{\mathbf{M}}_1$.

Various classes of AF's of $\mathbf{M}_1$ are defined in the same way as for an Hunt process. In particlur, we have the classes $\mathbf{A}_c^+$ of PCAF's, $\overset{\circ}{\mathcal{M}}$ of margingale AF's of finite energy and $\mathcal{N}_c$ of continuous AF's of zero energy for the process $\mathbf{M}_1$. For $u \in \mathcal{F}$ and its $\mathcal{E}$-quasicontinuous version u^*, we put

$$\tilde{u}^*(\tilde{x}) = u^*(q^{-1}(\tilde{x})) \quad \tilde{x} \in \tilde{X} - \tilde{N}_1.$$

Then $\tilde{u}^*$ is an $\tilde{\mathcal{E}}$-quasicontinuous version of $\tilde{u} = \Phi u$ and

$$u^*(X_t(\omega)) - u^*(X_0(\omega)) = \tilde{u}^*(\tilde{X}_t(\omega)) - \tilde{u}^*(\tilde{X}_0(\omega)) \quad \omega \in \Omega.$$

Hence $A^{[u]}$ (the left hand side) is (up to the equivalence) a finite cadlag AF and uniquely expressible as (3) for $\mathbf{M}_1$ because so is $A^{[\tilde{u}]}$ (the right hand side) for $\tilde{\mathbf{M}}_1$.

For the present quasi-regular Dirichlet form $\mathcal{E}$, the class S of smooth measures and the notion of the energy measure $\mu_{[u]}$ of $u \in \mathcal{F}_b$ are defined also in the same way as for a regular Dirichlet form. q preserves the notion of the smoothness of positive measures. μ is the Revuz measure of a PCAF A of $\mathbf{M}_1$ if $q\mu$ is the Revuz measure of A as a PCAF of $\tilde{\mathbf{M}}_1$. The energy measure of $u \in \mathcal{F}_b$ characterized by the equation (28) can be constructed as the image by q^{-1} of the energy measure of $\Phi u \in \tilde{\mathcal{F}}_b$ with respect to the regular Dirichlet form $\tilde{\mathcal{E}}$.

Summing up what has been mentioned, we get

Theorem 5.2 (i) *The equivalence classes of PCAF's $\mathbf{A}_c^+$ of $\mathbf{M}_1$ and the smooth measures S of $\mathcal{E}$ are in one to one (Revuz) correspondence by* (27).

(ii) *Any $u \in \mathcal{F}$ admits a unique finite smooth measure $\mu_{\langle u \rangle}$ satisfying the equation* (28) *in the case that $u \in \mathcal{F}_b$.*

(iii) *For any $u \in \mathcal{F}$, the AF* (1) *admits the decomposition* (3) *uniquely up to the equivalence where $M^{[u]} \in \overset{\circ}{\mathcal{M}}$, $N^{[u]} \in \mathcal{N}$. $M^{[u]}$ admits its quadratic variation in $\mathbf{A}_c^+$ whose Revuz measure is the energy measure of u.*

Finally we transfer Theorem 5.4.2 of [FOT 94]. Recall that an AF $A_t(\omega)$ is said to be of bounded variation if it is of bounded variation in t on each compact interval of $[0, \zeta(\omega))$ for every fixed ω in its defining set.

Theorem 5.3 *The following conditions are equivalent for $u \in \mathcal{F}$:*

(i) *The AF $A^{[u]}$ defined by* (1) *is a semimartingale in the sense that the CAF $N^{[u]}$ in its decomposition* (3) *is of bounded variation.*

(ii) *There exists a signed smooth measure ν with some attached $\mathcal{E}$-nest $\{E_n\}$ for which the equation* (4) *holds.*

If condition (ii) *holds, then $N^{[u]}$ admits an expression* (5) *for $A^k \in \mathbf{A}_c^+$ with Revuz measure $\nu^k,\ k = 1, 2,$ where*

$$\nu = \nu^1 - \nu^2 \tag{35}$$

is the Jordan decomposition of the smooth signed measure ν (hence $\nu^k,\ k = 1, 2,$ are automatically smooth). If condition (i) *holds, then $N^{[u]}$ admits an expression* (5) *by some $A^k \in \mathbf{A}_c^+, k = 1, 2,$ and condition* (ii) *is fulfilled for the signed smooth measure ν of* (35) *defined by the Revuz measure ν^k of $A^k,\ k = 1, 2.$*

6 Semimartigale characterizations of AF's

Let $(X, m, \mathcal{E}, \mathcal{F})$ be a quasi-regular Dirichlet space and $\mathbf{M}_1 = (X_t, P_x)$ be an associated standard process specified in Theorem 5.1 as an image of a Hunt process by a quasi-homeomorphism. By Theorem 4.1 and Theorem 5.3, we have

Theorem 6.1 *The following two conditions are equivalent for $u \in \mathcal{F}$:*

(i) *There exists an $\mathcal{E}$-nest $\{E_n\}$ for which the inequality* (2) *is valid for some positive constant C_n.*

(ii) *The AF $A^{[u]}$ defined by* (1) *is a semimartingale in the sense that the CAF $N^{[u]}$ in its decomposition* (3) *is of bounded variation.*

When one of these conditions is satisfied, there exists a signed smooth measure ν with some attached $\mathcal{E}$-nest $\{E_n\}$ for which the equation (4) *holds. Further $N^{[u]}$ admits an expression* (5) *for $A^k \in \mathbf{A}_c^+$ with Revuz measure $\nu^k,\ k = 1, 2,$ which are related to ν by* (35).

We now study a simple case that the $\mathcal{E}$-nest appearing in the inequality (2) is trivial. Then (2) is simplified to the condition that, for $u \in \mathcal{F}$, there exists a positive constant C for which the inequality (6) holds for any v in the space $\mathcal{F}_b$. Let $\mathcal{L}$ be a subspace of $\mathcal{F}_b$ satisfying condition ($\mathcal{L}$) described in §1.

Lemma 6.1 *If, for $u \in \mathcal{F}$, the inequality* (6) *holds for any v in the space $\mathcal{L}$, then* (6) *holds for any v in $\mathcal{F}_b$.*

Proof. Take any $v \in \mathcal{F}_b$ and set $M = \|v\|_\infty$. For any $\epsilon > 0$, we can find a real function $\psi(t)$ such that

$$|\psi(t)| \le M + \epsilon;\ \psi(t) = t,\ -M \le t \le M;\ 0 \le \psi(t) - \psi(s) \le t - s,$$

and $\psi(\mathcal{L}) \subset \mathcal{L}$. Choose $v_n \in \mathcal{L}$ which are $\mathcal{E}$-convergent to v. Then $\psi(v_n) \in \mathcal{L}$, $\psi(v_n)$ is $\mathcal{E}$-convergent to $\psi(v) = v$. From the validity of (6) for functions in $\mathcal{L}$, we get

$$\mathcal{E}(u, \psi(v_n)) \leq \|\psi(v_n)\|_\infty \leq M + \epsilon.$$

It suffices to let $n \to \infty$ and then $\epsilon \downarrow 0$. □

Theorem 6.2 *For $u \in \mathcal{F}$, the following conditions are equivalent:*

(i) *The inequality* (6) *holds for any v in the space $\mathcal{L}$ satisfying condition ($\mathcal{L}$).*

(ii) *There exists a unique finite signed smooth measure ν for which the equation* (4) *is valid for any $v \in \mathcal{F}_b$.*

(iii) *The continuous AF $N_t^{[u]}$ in the decomposition* (3) *of the AF* (1) *is of bounded variation and satisfies the property* (7).

In this case, $N^{[u]}$ admits an expression (5) *by PCAF's A^k, $k = 1, 2$, whose Revuz measures ν^k, $k = 1, 2$, are finite smooth measures related to the signed measure ν of* (ii) *by* (35).

Proof. The first two conditions are equivalent by virtue of Theorem 4.2 and Lemma 6.1. Assume (ii) and let (35) be the Jordan decomposition of ν. Then ν^k, $k = 1, 2$, are finite smooth measures and, on account of Thorem 5.3, $N^{[u]}$ is expressible by the PCAF's A^k with Revuz measure ν^k, $k = 1, 2$. Denote by $\{N^{[u]}\}_t$ the total variation of $N^{[u]}$ on $[0, t]$. It is known to be an element of $\mathbf{A}_c^+$. Since it is dominated by $A_t^1 + A_t^2$, we get from the Revuz correspondence (27)

$$\lim_{t\downarrow 0} \frac{1}{t} E_m(\{N^{[u]}\}_t) \leq \lim_{t\downarrow 0} \frac{1}{t} E_m(A_t^1 + A_t^2) = \nu^1(X) + \nu^2(X) < \infty,$$

arriving at (iii).

Conversely, assume (iii) and let

$$A_t^1 = \frac{1}{2}(\{N^{[u]}\}_t + N_t^{[u]}), \ A_t^2 = \frac{1}{2}(\{N^{[u]}\}_t - N_t^{[u]}).$$

On account of Theorem 5.3, condition (ii) holds for the signed smooth measure (35) where ν^k is defined to be the Revuz measure of A^k, $k = 1, 2$. Then ν is finite, because by condition (7)

$$\nu^1(X) + \nu^2(X) = \lim_{t\downarrow 0} \frac{1}{t} E_m(A_t^1 + A_t^2) = \lim_{t\downarrow 0} \frac{1}{t} E_m(\{N^{[u]}\}_t) < \infty.$$

□

Remark 6.1 Property (7) says that the Revuz measure of the PCAF $\{N^{[u]}\}_t$ has a finite total mass. By [FOT 94, Th.5.3.1], any PCAF A and its Revuz measure μ are related by

$$E_m\left(\int_0^t f(X_s) dA_s\right) = \int_o^t \langle f \cdot p_s 1, \mu \rangle ds, \quad f \in \mathcal{B}.$$

Hence, property (7) implies the P_m-integrability

$$E_m(\{N^{[u]}\}_t) < \infty, \quad t > 0. \tag{36}$$

If the process is conservative in the sense that $p_s 1 = 1$, $s > 0$, then the integrability (36) implies property (7).

In the rest of this section, we assume that $(X, m, \mathcal{E}, \mathcal{F})$ is a regular Dirichlet space and $\mathbf{M} = (X_t, P_x)$ is any associated Hunt process. Of course, the preceding two theorems remain valid under the present assumption. Let us further assume that $\mathcal{E}$ is strongly local: if $u, v \in \mathcal{F}$ are of compact support and v is constant on a neighbourhood of the support of u, then $\mathcal{E}(u, v) = 0$.

Then the associated Hunt process $\mathbf{M}$ can be taken to be a diffusion (namely, of continuous sample paths on $[0, \zeta)$) with no killing inside. A function u is said to be *locally in* $\mathcal{F}$ ($u \in \mathcal{F}_{loc}$ in notation) if for any relatively compact open set G there is a function $w \in \mathcal{F}$ such that $u = w$ $m - a.e.$ on G. Let $u \in \mathcal{F}_{loc}$. u admits an $\mathcal{E}$-quasicontinuous version u^*. The energy measure $\mu_{\langle u \rangle}$ of u is still well defined. The AF $A^{[u]}$ defined by (1) admits a decomposition (3) where $M^{[u]} \in \overset{\circ}{\mathcal{M}}_{loc}$ and $N^{[u]} \in \mathcal{N}_{c,loc}$. The decomposition is unique up to the equivalence of local AF's. The quadratic variation $\langle M^{[u]} \rangle \in \mathbf{A}_c^+$ of $M^{[u]}$ has as its Revuz measure the energy measure $\mu_{\langle u \rangle}$ of u. See [FOT 94, §5.5] for details of the above notions, notations and statements.

As for the semimartingale characterization of $N^{[u]} \in \mathcal{N}_{c,loc}$ for $u \in \mathcal{F}_{loc}$, all statements in Theorem 5.3 remain true except that the $\mathcal{E}$-nest $\{E_n\}$ appearing in the condition (ii) there is now required to consist of compact subsets of X ([FOT 94, th.5.5.4]).

The next condition for a subset $\mathcal{C}$ of $C_0(X)$ is taken from [FOT 94] (condition ($\mathcal{C}$.2) of [FOT 94]).

($\mathcal{C}$) $\mathcal{C}$ is a dense subalgebra of $C_0(X)$. For any compact set K and relatively compact open set G with $K \subset G$, $\mathcal{C}$ admits an element u such that $u \geq 0$, $u = 1$ on K and $u = 0$ on $X - G$.

Let $\mathcal{C}$ be a subset of $\mathcal{F} \cap C_0(X)$ satisfying both conditions $\mathcal{L}$ and $\mathcal{C}$. Such a subset is similar to a special standard core in the sense of [FOT 94], and the only difference lies in the definition of a real function $\phi_\epsilon(t)$ appearing in condition $\mathcal{L}$. For any open set $G \subset X$, we put

$$\mathcal{F}_G = \{u \in \mathcal{F} : u^* = 0 \ \mathcal{E} - q.e. \text{ on } X - G\},$$

$$\mathcal{C}_G = \{u \in \mathcal{C} : Supp[u] \subset G\}.$$

$\mathcal{C}_G$ is uniformly dense in $C_0(G)$ and $\mathcal{E}_1$-dense in $\mathcal{F}_G$ ([FOT 94, Lem.2.3.4]).

Theorem 6.3 *The next conditions are equivalent for $u \in \mathcal{F}_{loc}$:*

(i) *For any relatively compact open set $G \subset X$, there is a positive constant C_G such that*

$$|\mathcal{E}(u, v)| \leq C_G \|v\|_\infty \quad \forall v \in \mathcal{C}_G. \tag{37}$$

(ii) *There exists a signed Radon measure ν on X charging no set of zero capacity such that*

$$\mathcal{E}(u, v) = -\int_X v(x)\nu(dx) \quad \forall v \in \mathcal{C}. \tag{38}$$

(iii) *$N_t^{[u]}$ is of bounded variation and satisfies property* (8).

In this case, $N^{[u]}$ admits an expression (5) *by PCAF's A^k, $k = 1, 2$, whose Revus measures ν^k, $k = 1, 2$, are smooth Radon measures related to the signed measure ν of* (ii) *by* (35).

Proof. The implication (ii) $\Rightarrow$ (i) is trivial. Conversely, suppose condition (i) is fulfilled. Then there exists a unique signed Radon measure on X for which the equation (38) holds. We have to show that ν charges no set of zero capacity.

To this end, fix $w \in \mathcal{C}$ and choose a relatively compact open set G containing the support of w. Take $u_1 \in \mathcal{F}$ satisfying $u = u_1$ on G, then

$$\mathcal{E}(u, wv) = \mathcal{E}(u_1, wv), \;\; v \in \mathcal{C}$$

and we have from (i),

$$|\mathcal{E}(u_1, w \cdot v)| \leq C' \|v\|_\infty \quad \forall v \in \mathcal{C}, \tag{39}$$

where $C' = C_G \cdot \|w\|_\infty$. Since $\mathcal{C}$ is an algebra satisfying condition $(\mathcal{L})$, we can extend inequality (39) from $\mathcal{C}$ to $\mathcal{F}_b$. In fact, keeping the notations in the proof of Lemma 6.1, we can show that $w \cdot \psi(v_n)$ is $\mathcal{E}$-weakly convergent to $w \cdot v$.

By virtue of Proposition 4.1, the inequality (39) holding for u_1 and for any $v \in \mathcal{F}_b$ implies that the equation (15) is valid for u_1 and for a finite signed measure ν_w charging no set of zero capacity. A comparison with (38) yields

$$\nu_w = w \cdot \nu$$

and we conclude that ν charges no set of zero capacity because w is an arbitrary element of $\mathcal{C}$. The proof of the equivalece of (i) and (ii) is complete.

Consider relatively compact open sets $\{G_k\}$ such that

$$\bar{G}_k \subset G_{k+1}, \quad \cup_{k=1}^{\infty} G_k = X.$$

Just as in the proof of [FOT 94, Cor.5.4.1], we can see that condition (ii) is equivalent to the validity of Theorem 5.3 (ii) for $u \in \mathcal{F}_{loc}$, a signed Radon measure ν charging no set of zero capacity and the $\mathcal{E}$-nest $\{\bar{G}_n\}$, which in turn can been seen to be equivalent to the probabilistic condition (iii) in the same way as the corresponding proof of Theorem 6.2, because Theorem 5.3 is applicable to the present situation in view of the remark made in the parpagraph preceding the introducition of the space $\mathcal{C}$.

□

Remark 6.2 Property (8) says that the Revuz measure of the PCAF $\{N^{[u]}\}_t$ is a Radon measure. (8) implies the integrability

$$E_m \left(\int_0^t I_K(X_s) d\{N^{[u]}\}_s \right) < \infty, \quad \forall K \text{ compact}, \; \forall t > 0. \tag{40}$$

Conversely (40) implies (8) if the process is conservative.

7 Stochastic characterizations of BV functions and Caccioppoli sets

Let R^d be the d-dimensional Euclidean space and m_0 be the Lebesgue measure on it. We consider a non-negative locally integrable function ρ on R^d and the associated *energy form* defined by

$$\mathcal{E}^\rho(u,v) = \frac{1}{2}\int_{R^d} \nabla u(x)\cdot\nabla v(x)\ \rho(x)\ m_0(dx), \quad u,v \in C_0^1(R^d). \tag{41}$$

Throughout this section, let us assume the Hamza type condition on ρ :

(H) $\rho = 0 \quad m-$a.e. on $S(\rho)$,
where

$$S(\rho) = \{x \in R^d : \int_{U(x)} \rho(y)^{-1} m_0(dy) = \infty\ \forall U(x)\}$$

the *singular set* of ρ.

The complement $R(\rho)$ is called the *regular set* of ρ. Under (H), the support of the measure ρdm_0 equals $\overline{R(\rho)}$. Further the form $\mathcal{E}^\rho$ is closable on $L^2(R^d;\rho\cdot m_0)$ and the closure $(\mathcal{E}^\rho,\mathcal{F}^\rho)$ is a strongly local regular Dirichlet form on $L^2(\overline{R(\rho)};\rho\cdot m_0)$ ([RW 85], [F 97b]).
The associated diffusion $\mathbf{M}^\rho = (X_t^\rho, P_x^\rho)$ on $\overline{R(\rho)}$ is called a *distorted Brownian motion.* The reason of this naming is in that, if we apply the decomposition (3) to coordinate functions

$$\psi_i(x) = x_i \in \mathcal{F}^\rho_{loc},\ 1 \le i \le d,$$

then we get the expression of the sample path

$$X_t^\rho - X_0^\rho = B_t + N_t^\rho \tag{42}$$

where B_t is a d-dimensional Brownian motion and

$$N_t^\rho = (N_t^1,\cdots,N_t^d), \quad N_t^i = N_t^{[\psi_i]} \in \mathcal{N}_{c,loc},\ 1 \le i \le d.$$

Condition (H) is fulfilled in the following two important cases:
(I) ρ is non-negative continuous m_0-a.e. on R^d.
(II) $\rho(x) = I_D(x),\ x \in R^d$, for an open set $D \subset R^d$.
In the second case, $\overline{R(\rho)} = \overline{D}$ and the distorted Brownian motion $\mathbf{M}^{I_D}$ reduces to the *modified reflecting Brownian motion* on $\overline{D}$ associated with the strongly local regular Dirichlet form

$$\mathcal{E}^{I_D}(u,v) = \frac{1}{2}\int_D \nabla u(x)\cdot\nabla v(x)\ m_0(dx),\ \mathcal{F}^{I_D} = \hat{H}^1(D)$$

on $L^2(\overline{D};I_D\cdot m_0)(= L^2(D;m_0))$ studied in [F 97a]. Here $\hat{H}^1(D)$ denotes the closure of the space $C_0^1(R^d)|_D$ in the Sobolev space $H^1(D)$. The term 'modified' is added because $\hat{H}^1(D)$ could be a proper subset of $H^1(D)$.

A function $\rho \in L^1_{loc}(R^d)$ is called BV (denoted by $\rho \in BV_{loc}$) if for any bounded open set $V \subset R^d$, there exists a positive constant C_V such that

$$|\int_V (div\ v)\rho m_0(dx)| \le C_V \|v\|_\infty \quad \forall v \in C_0^1(V; R^d). \tag{43}$$

Theorem 7.1 *Suppose a non-negative function $\rho \in L^1_{loc}(R^d)$ satisfies the condition* (H). *Then the following conditions are equivalent:*

(i) $\rho \in BV_{loc}$.

(ii) *The distorted Brownian path X_t^ρ is a semimartingale in the sense that each component N_t^i, $(1 \le i \le d)$, in the decomposition* (42) *is of bounded variation and additionally it satisfies that*

$$\lim_{t \downarrow 0} \frac{1}{t} E^\rho_{\rho \cdot m_0}\left(\int_0^t I_K(X_s^\rho)|dN_s^i|\right) < \infty \quad 1 \le i \le d, \tag{44}$$

for every compact set $K \subset \overline{R(\rho)}$.

Proof. We apply Theorem 6.3 to the strongly local, regular Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ on $L^2(X; m)$ and an associated diffusion $\mathbf{M}^\rho = (X_t^\rho, P_x^\rho)$ on X, where

$$X = \overline{R(\rho)} \quad m = \rho \cdot m_0.$$

We take

$$\mathcal{C} = C_0^1(R^d)|_X,$$

which obviously has the properties $(\mathcal{L})$ and $(\mathcal{C})$. Taking as u the coodinate function $\psi_i \in \mathcal{F}_{loc}$, the condition (i) of Theorem 6.3 reads as follows: for any relatively compact open set $G \subset R^d$, there is a positive constant C_G such that

$$|\int_{R(\rho)} \partial_i v(x)\rho(x)m_0(dx)| \le C_G \sup_{x \in R(\rho)} |v(x)|, \quad \forall v \in C_0^1(G).$$

It is easy to see that $\rho \in BV_{loc}$ if and only if the above condition is satisfied for each $i = 1, 2, \cdots, d$. Thus, we get Theorem 7.1 from Theorem 6.3. □

A measurable set $E \subset R^d$ is called *Caccioppoli* if $I_E \in BV_{loc}$. We know ([EG 92]) that E is Caccioppoli if and only if there exists a positive Radon measure σ on ∂E and a σ-measurable vector

$$\mathbf{n}_E : \partial E \to R^d$$

with $|\mathbf{n}_E| = 1$ σ-a.e. such that

$$\int_E div\ v\ m_0(dx) = -\int_{\partial E} v \cdot \mathbf{n}_E\ d\sigma \quad \forall v \in C_0^1(R^d, R^d). \tag{45}$$

By specifying Theorem 7.1 to $\rho = I_D$, we get

Theorem 7.2 *The following conditions are equivalent for an open set $D \subset R^d$:*

(i) D is Caccioppoli.

(ii) The modified reflecting Brownian path $(X_t, P_x) = (X_t^{I_D}, P_x^{I_D})$ on $\overline{D}$ is a semimartingale in the sense that each component N_t^i of the second term $N_t^{I_D}$ in its decomposition (42) *is of bounded variation and satisfies the additional property that*

$$\lim_{t\downarrow 0} \frac{1}{t} E_{I_D m_0} \left(\int_0^t I_K(X_s) |dN_s^i| \right) < \infty$$

In this case, the modified reflecting Brownian motion admits an expession of Skorohod type:

$$X_t - X_0 = B_t + \int_0^t \mathbf{n}(X_s) dL_s \tag{46}$$

for a PCAF L_t with Revuz measure σ.

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