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Homogeneous Random Measures and Strongly Supermedian Kernels of a Markov Process

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Abstract. The potential kernel of a positive *left* additive functional (of a strong Markov process X) maps positive functions to strongly supermedian functions and satisfies a variant of the classical *domination principle* of potential theory. Such a kernel V is called a *regular strongly supermedian* kernel in recent work of L. Beznea and N. Boboc. We establish the converse: Every regular strongly supermedian kernel V is the potential kernel of a random measure homogeneous on $[0, \infty[$. Under additional finiteness conditions such random measures give rise to left additive functionals. We investigate such random measures, their potential kernels, and their associated characteristic measures. Given a left additive functional A (not necessarily continuous), we give an explicit construction of a simple Markov process Z whose resolvent has initial kernel equal to the potential kernel U_A. The theory we develop is the probabilistic counterpart of the work of Beznea and Boboc. Our main tool is the Kuznetsov process associated with X and a given excessive measure m.

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1. Introduction.

In a recent series of papers [**BB00, BB01a, BB01b, BB02**], L. Beznea and N. Boboc have singled out an important class of kernels for which they have developed a rich potential theory. These kernels (called *regular strongly supermedian* kernels) are those that map positive Borel functions to strongly supermedian functions of a strong Markov process X and that satisfy a form of the domination principle. If κ is a random measure of X, homogeneous on $[0,\infty[$ as in [**Sh88**], then the potential kernel U_κ is a strongly supermedian kernel. Using entirely potential theoretic arguments, Beznea and Boboc were able to develop a theory of characteristic (Revuz) measures, uniqueness theorems, etc., for regular strongly supermedian kernels that parallels a body of results on homogeneous random measures (under various sets of hypotheses) due to J. Azéma [**A73**], E.B. Dynkin [**Dy65, Dy75**], and others. In fact, the theory developed by Beznea and Boboc goes far beyond that previously developed for homogeneous random measures. The question of the precise relationship between regular strongly supermedian kernels and homogeneous random measures poses itself. One of our goals in this paper is to show that the class of regular strongly supermedian kernels is coextensive with the class of potential kernels U_κ as κ varies over the class of (optional, co-predictable) homogeneous random measures. Our examination of these matters will be from a probabilistic point of view. Before describing our work in more detail we shall attempt to provide some historical background.

Let $X = (X_t, \mathbf{P}^x)$ be a strong Markov process with transition semigroup (P_t) and state space E. To keep things simple, in this introduction we assume that X is transient and has infinite lifetime.

If we are given an excessive function u of X (bounded, for simplicity) such that $P_t u \to 0$ as $t \to \infty$, then the right continuous supermartingale $(u(X_t))_{t\ge 0}$ admits a Doob-Meyer decomposition $u(X_t) = M_t - A_t$ in which M is a uniformly integrable martingale and A is a predictable increasing process, with $\mathbf{P}^x[A_\infty] = u(x)$ for all $x \in E$. Both M and A are right-continuous, and the uniqueness of the Doob-Meyer decomposition implies that $M - M_0$ and A are *additive functionals* of X; thus, for example, $A_{t+s} = A_t + A_s \circ \theta_t$ for all $s, t \ge 0$. The additive functional A is continuous if and only if the excessive function u is *regular*; that is, $T \mapsto \mathbf{P}^x[u(X_T)]$ is left-continuous along increasing sequences of stopping times. In general, A admits a decomposition $A = A^c + A^d$ into continuous and purely discrete additive functional components. The continuous component A^c has been well-understood since the early 60s; see [**V60, Su61**]. Our knowledge of the discontinuous component A^d begins (for standard processes) with the work of P.-A. Meyer [**My62**], but a complete description of A^d in the context of right processes requires Ray-Knight methods;

details can be found in [**Sh88**].

The story simplifies greatly in case X is in weak duality with a second strong Markov process $\hat{X}$, with respect to an excessive measure m. In this case there is a Borel function $a \geq 0$ with $\{a > 0\}$ m-semipolar such that A^d is $\mathbf{P}^m$-indistinguishable from $t \mapsto \sum_{0<s\leq t} a(X_{s-})$, the existence of the left limit X_{s-} being guaranteed $\mathbf{P}^m$-a.s. by weak duality. See [**GS84**; (16.8)(i)]. Associated with A are its *left potential kernel*

$$U_A^- f(x) := \mathbf{P}^x \int_{]0,\infty[} f(X_{t-})\, dA_t, \qquad f \in p\mathcal{E}, \tag{1.1}$$

defined outside an m-polar set, and its *Revuz measure*

$$\mu_A^-(f) := \lim_{q\to\infty} q\mathbf{P}^m \int_{]0,\infty[} e^{-qt} f(X_{t-})\, dA_t, \qquad f \in p\mathcal{E}. \tag{1.2}$$

The kernel U_A^-, which maps positive Borel functions to excessive functions, determines A up to $\mathbf{P}^m$-evanescence, and it is clear from the *Revuz formula*

$$\int_E g(x) U_A^- f(x)\, m(dx) = \int_E \hat{U} g(x) f(x)\, \mu_A^-(dx), \tag{1.3}$$

that μ_A^- determines U_A^-, modulo an m-polar set. See [**Re70a, Re70b**] in the context of standard processes in strong duality and [**GS84**] in the context of right processes in weak duality as above.

Can the duality hypothesis imposed above be relaxed? Given a right-continuous strong Markov process X (more precisely, a Borel right Markov process) and an excessive measure m, there is always a dual process $\hat{X}$ (essentially uniquely determined), but in general it is a *moderate* Markov process: the Markov property holds only at predictable times. If, in the general case, we must use the left-handed dual process, then time reversal dictates that we must trade in A for its right-handed counterpart. Thus, we are mainly concerned with additive functionals (and, more generally, homogeneous random measures) that can be expressed as

$$A_t = A_t^c + \sum_{0\leq s<t} a(X_s), \tag{1.4}$$

where A^c is a continuous additive functional and $\{a > 0\}$ is m-semipolar. Observe that this additive functional is left-continuous and adapted. As we shall see, A is *co-predictable*, meaning (roughly) that it is predictable as a functional of $\hat{X}$. Such additive functionals were introduced and studied by J. Azéma in his pioneering work [**A73**], under the name *d-fonctionelle*.

The potential kernel U_A associated with the additive functional defined in (1.4) is

$$U_A f(x) := \mathbf{P}^x \int_{[0,\infty[} f(X_t)\, dA_t, \qquad f \in p\mathcal{E}. \tag{1.5}$$

In contrast to U_A^-, if $f \in p\mathcal{E}$ then the function $U_A f$ is strongly supermedian and regular, but is excessive only when A^d vanishes. Azéma showed, among other things, that A is uniquely determined by $U_A 1$ provided this function is finite, and that any regular strongly supermedian function u of class (D) is equal to $U_A 1$ for a unique A. It is a crucial observation of Beznea and Boboc, foreshadowed by a remark of Mokobodzki [**Mo84**; p. 463], that the analytic concept of regular strongly supermedian kernel corresponds to the probabilistic conditions (1.4) and (1.5). Beznea and Boboc develop their theory analytically; our approach to these matters is largely probabilistic.

Our setting will be a Borel right Markov process X coupled with a fixed excessive measure m. Our main tool will be the stationary Kuznetsov process $Y = ((Y_t)_{t\in\mathbf{R}}, \mathbf{Q}_m)$ associated with X and m. The theory of HRMs over Y has been developed in [**Fi87, G90**] and applied to the study of (continuous) additive functionals in [**FG96, G99**]. In section 2 we recall the basic definitions and notation concerning Y; other facts about Y will be introduced as the need arises. In section 2 we also present a small but necessary refinement of the strong Markov property of Y proved in [**Fi87**], and in section 3 we record some basic facts about HRMs, drawn mainly from [**Fi87**]. Section 4 contains fundamentals on potential kernels and characteristic measures of optional HRMs that are either co-natural or co-predictable. We show, in particular, that the potential kernel of a suitably perfected optional co-predictable HRM satisfies the domination principle. We also prove a formula for the characteristic measure of such an HRM that provides a first link with the work of Beznea and Boboc. Section 5 (and the accompanying appendix) contains a probabilistic approach to some results in [**BB01b**]. Starting from a regular strongly supermedian kernel, assumed to be proper in a suitable sense, we construct the associated characteristic measure and HRM. The analog of (1.3) in this context appears in section 5 as well. We specialize, in section 6, to the situation of additive functionals. Of particular interest are criteria, based on the characteristic measure μ_κ or the potential kernel U_κ of an optional co-predictable HRM κ, ensuring that $A_t := \kappa([0,t[)$ defines a *finite* additive functional. The probabilistic approach used here extends that found in [**FG96**] in the context of continuous additive functionals. If a regular strongly supermedian kernel V satisfies a suitable "properness" condition (such conditions are discussed in section 6) then V is the initial kernel of a subMarkovian resolvent. Results of this type have a history going back to Hunt [**Hu57**], including work of Taylor [**T72, T75**] and Hirsch [**Hi74**], and culminating in [**BB01a**]. We give a probabilistic treatment of this topic in section 7, making use of the additive functional material from section 6, thereby obtaining an explicit expression for the resolvent associated with V. Finally, the appendix contains technical results on strongly supermedian functions as well as a proof of the main result of section 5.

We close this introduction with a few words on notation. We shall use $\mathcal{B}$ to denote the Borel subsets of the real line $\mathbf{R}$. If $(F, \mathcal{F}, \mu)$ is a measure space, then $b\mathcal{F}$ (resp. $p\mathcal{F}$) denotes the class of bounded real-valued (resp. $[0, \infty]$-valued) $\mathcal{F}$-measurable functions on F. For $f \in p\mathcal{F}$ we use $\mu(f)$ to denote the integral $\int_F f\,d\mu$; similarly, if $D \in \mathcal{F}$ then $\mu(f; D)$ denotes $\int_D f\,d\mu$. We write $\mathcal{F}^*$ for the universal completion of $\mathcal{F}$; that is, $\mathcal{F}^* = \cap_\nu \mathcal{F}^\nu$, where $\mathcal{F}^\nu$ is the ν-completion of $\mathcal{F}$ and the intersection runs over all finite measures on $(F, \mathcal{F})$. If $(E, \mathcal{E})$ is a second measurable space and $K = K(x, dy)$ is a kernel from $(F, \mathcal{F})$ to $(E, \mathcal{E})$ (*i.e.*, $F \ni x \mapsto K(x, A)$ is $\mathcal{F}$-measurable for each $A \in \mathcal{E}$ and $K(x, \cdot)$ is a measure on $(E, \mathcal{E})$ for each $x \in F$), then we write μK for the measure $A \mapsto \int_F \mu(dx) K(x, A)$ and Kf for the function $x \mapsto \int_E K(x, dy) f(y)$.

2. Preliminaries.

Throughout this paper $(P_t : t \geq 0)$ will denote a Borel right semigroup on a Lusin state space $(E, \mathcal{E})$, and $X = (X_t, \mathbf{P}^x)$ will denote a right-continuous strong Markov process realizing (P_t). We shall specify the realization shortly. Recall that a (positive) measure m on $(E, \mathcal{E})$ is *excessive* provided $mP_t \leq m$ for all $t \geq 0$. Since (P_t) is a right semigroup, it follows that $mP_t \uparrow m$ setwise as $t \downarrow 0$. See [**DM87**; XII.36–37]. Let Exc denote the cone of excessive measures. In general, we shall use the standard notation for Markov processes without special mention. See, for example, [**BG68**], [**DM87**], [**Sh88**], and [**G90**]. In particular, $U^q := \int_0^\infty e^{-qt} P_t\,dt$, $q \geq 0$, denotes the resolvent of (P_t).

We are going to need the Kuznetsov process (or measure) $\mathbf{Q}_m$ associated with (P_t) and a given $m \in$ Exc. We refer the reader to Section 6 of [**G90**] for notation and definition. See also [**DMM92**]. For the convenience of the reader we shall review some of the basic notation here. Thus W denotes the space of all paths $w : \mathbf{R} \to E_\Delta := E \cup \{\Delta\}$ that are right continuous and E-valued on an open interval $]\alpha(w), \beta(w)[$ and take the value Δ outside of this interval. Here Δ is a point adjoined to E as an isolated point. The dead path $[\Delta]$, constantly equal to Δ, corresponds to the interval being empty; by convention $\alpha([\Delta]) = +\infty$, $\beta([\Delta]) = -\infty$. The σ-algebra $\mathcal{G}^\circ$ on W is generated by the coordinate maps $Y_t(w) = w(t)$, $t \in \mathbf{R}$, and $\mathcal{G}_t^\circ := \sigma(Y_s : s \leq t)$. Two families of shift operators are defined on W: the simple shifts σ_t, $t \in \mathbf{R}$,

$$\sigma_t w(s) = [\sigma_t w](s) := w(t+s), \qquad s \in \mathbf{R},$$

and the truncated shifts θ_t, $t \in \mathbf{R}$,

$$\theta_t w(s) = [\theta_t w](s) := \begin{cases} w(t+s), & s > 0; \\ \Delta, & s \leq 0. \end{cases}$$

(In [**Fi87**], the truncated shift operator was denoted τ_t; here we follow [**G90**] in using θ_t.) We refer the reader to [**G90**] for additional notation and terminology. Given $m \in \mathrm{Exc}$, the Kuznetsov measure $\mathbf{Q}_m$ is the unique σ-finite measure on $\mathcal{G}^\circ$ not charging $\{[\Delta]\}$ such that, for $-\infty < t_1 < t_2 < \cdots < t_n < +\infty$,

$$\begin{aligned} \mathbf{Q}_m(Y_{t_1} \in dx_1, Y_{t_2} \in dx_2, \ldots, Y_{t_n} \in dx_n) & \\ = m(dx_1)\, P_{t_2 - t_1}(x_1, dx_2) \cdots P_{t_n - t_{n-1}}(x_{n-1}, dx_n). \end{aligned} \tag{2.1}$$

We let $X = (X_t, \mathbf{P}^x)$ be the realization of (P_t) described on page 53 of [**G90**]. In particular, the sample space for X is

$$\Omega := \{\alpha = 0, Y_{\alpha+} \text{ exists in } E\} \cup \{[\Delta]\},$$

X_t is the restriction of Y_t to Ω for $t > 0$, and X_0 is the restriction of Y_{0+}. Moreover, $\mathcal{F}^\circ := \sigma(X_t : t \geq 0)$ is the trace of $\mathcal{G}^\circ$ on Ω.

Because of its crucial role in our development we recall the modified process Y^* of [**G90**; (6.12)]. Let d be a totally bounded metric on E compatible with the topology of E, and let $\mathcal{D}$ be a countable uniformly dense subset of the d-uniformly continuous bounded real-valued functions on E. Given a strictly positive $h \in b\mathcal{E}$ with $m(h) < \infty$ define $W(h) \subset W$ by the conditions:

(2.2) (i) $\alpha \in \mathbf{R}$;

(ii) $Y_{\alpha+} := \lim_{t \downarrow \alpha} Y_t$ exists in E;

(iii) $U^q g(Y_{\alpha+1/n}) \to U^q g(Y_{\alpha+})$ as $n \to \infty$, for all $g \in \mathcal{D}$ and all rationals $q > 0$;

(iv) $Uh(Y_{\alpha+1/n}) \to Uh(Y_{\alpha+})$ as $n \to \infty$.

Evidently $\sigma_t^{-1}(W(h)) = W(h)$ for all $t \in \mathbf{R}$, and $W(h) \in \mathcal{G}^\circ_{\alpha+}$ since E is a Lusin space. We now define

$$Y_t^*(w) = \begin{cases} Y_{\alpha+}(w), & \text{if } t = \alpha(w) \text{ and } w \in W(h), \\ Y_t(w), & \text{otherwise.} \end{cases} \tag{2.3}$$

Fix $m \in \mathrm{Exc}$ and h as above. If $m = \eta + \pi = \eta + \rho U$ is the Riesz decomposition [**G90**; (5.33), (6.19)] of m into harmonic and potential parts, then $\mathbf{Q}_m = \mathbf{Q}_\eta + \mathbf{Q}_\pi$, $\mathbf{Q}_\eta(W(h)) = 0$, and $\mathbf{Q}_m(\,\cdot\,; W(h)) = \mathbf{Q}_\pi$. See [**G90**; (6.19)]. (In particular, if h' is another function with the properties of h then $\mathbf{Q}_m(W(h) \triangle W(h')) = 0$.) Moreover it follows readily from (6.20) and (8.23) of [**G90**] that

$$\mathbf{Q}_m(f(\alpha, Y^*_\alpha);\ W(h)) = \int_E \int_{\mathbf{R}} f(t,x)\, dt\, \rho(dx), \tag{2.4}$$

for all $f \in p(\mathcal{B} \otimes \mathcal{E})$. In particular, $\mathbf{Q}_m$ is σ-finite on $\mathcal{G}^\circ_{\alpha+} \cap W(h)$. The other important feature of Y^* is the strong Markov property recorded in (2.5) below. For a proof of the following result see [**G90**; (6.15)]. The filtration $(\mathcal{G}^m_t)_{t \in \mathbf{R}}$ is obtained by augmenting $(\mathcal{G}^\circ_t)_{t \in \mathcal{R}}$ with the $\mathbf{Q}_m$ null sets in the usual way.

(2.5) Proposition. *Let T be a $(\mathcal{G}^m_t)$-stopping time. Then $\mathbf{Q}_m$ restricted to $\mathcal{G}^m_T \cap \{Y^*_T \in E\}$ is a σ-finite measure and*

$$\mathbf{Q}_m(F \circ \theta_T \mid \mathcal{G}^m_T) = P^{Y^*_T}(F), \quad \mathbf{Q}_m\text{-a.e.} \quad \text{on} \quad \{Y^*_T \in E\}$$

for all $F \in p\mathcal{F}^\circ$.

We shall also require the following form of the section theorem. Define

$$\Lambda^*\colon = \{(t,w) \in \mathbf{R} \times W\colon\ Y^*_t(w) \in E\}\,;$$

evidently Λ^* is $(\mathcal{G}^\circ_{t+})$-optional.

(2.6) Proposition. *Let $(H_t)_{t \in \mathbf{R}}$ and $(K_t)_{t \in \mathbf{R}}$ be positive $(\mathcal{G}^m_t)$-optional processes. If*

$$\mathbf{Q}_m(H_T;\ Y^*_T \in E) = \mathbf{Q}_m(K_T;\ Y^*_T \in E)$$

for all $(\mathcal{G}^m_t)$-stopping times T, then $H1_{\Lambda^*}$ and $K1_{\Lambda^*}$ are $\mathbf{Q}_m$-indistinguishable.

See [**FG91**] for a proof of (2.6).

Certain results from [**Fi87**] will be crucial for our development. We shall recall some definitions from [**Fi87**] and give precise references to the results we shall need. Fitzsimmons defines $\ell := {}^{\hat p}(1_{]\!]\alpha,\beta[\![})$, the co-predictable projection of $1_{]\!]\alpha,\beta[\![}$, in [**Fi87**; (3.3)], and he then defines $\Lambda := \{\ell > 0\}$. One readily checks, using the argument in [**Fi87**; (3.6)], that $\Lambda \subset \Lambda^*$, modulo $\mathcal{I}^m$, the class of $\mathbf{Q}_m$-evanescent processes. See page 436 of [**Fi87**]. It follows that the process $\overline{Y}$ defined in [**Fi87**; (3.8)] is related to Y^* defined above in (2.3) as follows:

$$\overline{Y}_t(w) = \begin{cases} Y^*_t(w), & \text{if } (t,w) \in \Lambda, \\ \Delta, & \text{if } (t,w) \notin \Lambda. \end{cases}$$

In particular, $\Lambda = \{(t,w) : \overline{Y}_t(w) \in E\}$. Many of the definitions and results in [**Fi87**] involve $\overline{Y}$ and Λ. We shall need the extensions of the results in which $\overline{Y}$ and Λ are replaced by Y^* and Λ^*. The keys to these extensions are the strong Markov property (2.5) and the section theorem (2.6). Using them in place of (3.10) and (3.16)(b) in [**Fi87**], the results we require are proved with only minor modifications of the arguments given in [**Fi87**]. For example, using (2.6) the next result is proved exactly as (3.20) is proved in [**Fi87**]. The optional (resp. co-predictable) σ-algebra $\mathcal{O}^m$ (resp. $\hat{\mathcal{P}}^m$) is defined on page 436 in [**Fi87**]. A process Z defined over $(W, \mathcal{G}^m, \mathbf{Q}_m)$ is *homogeneous* provided $t \mapsto Z_{t+s}$ is $\mathbf{Q}_m$ indistinguishable from $t \mapsto Z_t \circ \sigma_s$ for each $s \in \mathbf{R}$.

(2.7) Proposition. *Let $Z \geq 0$ be $\mathcal{O}^m \cap \hat{\mathcal{P}}^m$-measurable. Then $Z 1_{\Lambda^*}$ is $\mathbf{Q}_m$-indistinguishable from a process of the form $t \mapsto F(t, Y_t^*)$, where $F \in p(\mathcal{B} \otimes \mathcal{E})$ and $F(t, \Delta) = 0$. If, in addition, Z is homogeneous, then $Z 1_{\Lambda^*} = f \circ Y^*$, modulo $\mathcal{I}^m$, where $f \in p\mathcal{E}$ with $f(\Delta) = 0$.*

We adhere to the convention that a function defined on E takes the value 0 at Δ.

In general, we shall just use the corresponding result with Y^* and Λ^* without special mention.

3. Homogeneous Random Measures.

In this section we shall expand on some of the results in section 5 of [**Fi87**]. Our definition of random measure $W \times \mathcal{B} \ni (w, B) \mapsto \kappa(w, B)$ is exactly that of [**Fi87**; (5.1)]. A random measure κ is *homogeneous* (in abbreviation, an HRM) provided the measures $B \mapsto \kappa(\sigma_t w, B)$ and $B \mapsto \kappa(w, B + t)$ coincide for $\mathbf{Q}_m$-a.e. $w \in W$, for each $t \in \mathbf{R}$. This notion of HRM, which clearly depends on the choice of m, differs from Definition 8.19 in [**G90**]—what is defined there is essentially a perfect HRM, to be discussed later. We emphasize that for each w, $\kappa(w, \cdot)$ is a measure on $(\mathbf{R}, \mathcal{B})$ that is carried by $\mathbf{R} \cap]\alpha(w), \beta(w)[$. As in [**Fi87**], a random measure κ is *σ-integrable* over a class $\mathcal{H} \subset p\mathcal{M}^m := p[(\mathcal{B} \otimes \mathcal{G}^\circ) \vee \mathcal{I}^m]$ provided there exists $Z \in \mathcal{H}$ with $\int_{\mathbf{R}} 1_{\{Z=0\}}(t, w)\, \kappa(w, dt) = 0$ for $\mathbf{Q}_m$-a.e. $w \in W$ and $\mathbf{Q}_m \int_{\mathbf{R}} Z_t\, \kappa(dt) < \infty$. The class of such random measures is denoted $\sigma\mathcal{I}(\mathcal{H})$. In keeping with our program of using Y^* and Λ^* systematically, we alter the definition [**Fi87**; (5.6)] of optional random measure by substituting Λ^* for Λ there. We say that an HRM κ is *carried* by a set $\Gamma \in (\mathcal{B} \otimes \mathcal{G})^*$ provided $\mathbf{Q}_m \int_{\mathbf{R}} 1_{\Gamma^c}(t)\, \kappa(dt) = 0$. If $\kappa \in \sigma\mathcal{I}(\mathcal{O}^m)$ is carried by Λ^*, then the dual optional projection κ^o of κ is defined as in [**Fi87**]: κ^o is the unique optional HRM carried by Λ^* satisfying

$$\mathbf{Q}_m \int_{\mathbf{R}} Z_t\, \kappa^o(dt) = \mathbf{Q}_m \int_{\mathbf{R}} {}^o Z_t\, \kappa(dt),$$

for all $Z \in p\mathcal{M}^m$, where ${}^oZ$ denotes the $\mathbf{Q}_m$ optional projection of Z. The properties of $\kappa \mapsto \kappa^o$ elucidated in [**Fi87**] remain valid with the obvious modifications.

If κ is an optional HRM, its *Palm measure* $\mathbf{P}_\kappa$ is the measure on $(W, \mathcal{G}^\circ)$ defined by

$$\mathbf{P}_\kappa(G) := \mathbf{Q}_m \int_{]0,1]} G\circ\theta_t \, \kappa(dt), \qquad G \in p\mathcal{G}^\circ. \tag{3.1}$$

Note that $\mathbf{P}_\kappa$ is carried by $\{\alpha = 0\}$. Moreover, [**Fi87**; (5.11)] states that if $F \in p(\mathcal{B}\otimes\mathcal{G}^\circ)$, then

$$\mathbf{Q}_m \int_{\mathbf{R}} F(t, \theta_t)\, \kappa(dt) = \int_{\mathbf{R}} dt \int_W F(t, w)\, \mathbf{P}_\kappa(dw). \tag{3.2}$$

In particular, taking $F(t, w) = \varphi(t)G(w)$ with $\varphi \geq 0$, $\int_{\mathbf{R}} \varphi(t)\, dt = 1$, and $G \in p\mathcal{G}^\circ$, we see that

$$\mathbf{P}_\kappa(G) = \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) G\circ\theta_t \, \kappa(dt). \tag{3.3}$$

Suppose that $\kappa \in \sigma\mathcal{I}(\hat{\mathcal{P}}^m)$. Because $t \mapsto G\circ\theta_t$ is co-predictable, $\mathbf{P}_\kappa = \mathbf{P}_{\kappa^{\hat{p}}}$, where $\kappa^{\hat{p}}$, the dual co-predictable projection of κ, is uniquely determined by

$$\mathbf{Q}_m \int_{\mathbf{R}} Z_t \, \kappa^{\hat{p}}(dt) = \mathbf{Q}_m \int_{\mathbf{R}} {}^{\hat{p}}Z_t \, \kappa(dt), \qquad Z \in p\mathcal{M}^m.$$

We say that κ is co-predictable provided $\kappa^{\hat{p}} = \kappa$, up to $\mathbf{Q}_m$-indistinguishability.

Observe that $B \in \mathcal{E}$ is m-polar if and only if $\{Y \in B\} := \{(t, w) : Y_t(w) \in B\}$ is $\mathbf{Q}_m$-evanescent. As before, let $m = \eta + \rho U$ be the decomposition of m into harmonic and potential components. Then [**FG91**; (2.3)] implies that for $B \in \mathcal{E}$, the set $\{Y^* \in B\}$ is $\mathbf{Q}_m$-evanescent if and only if B is both m-polar and ρ-null. It will be convenient to name this class of sets:

$$\mathcal{N}(m) := \{B \in \mathcal{E} : B \text{ is } m\text{-polar and } \rho(B) = 0\}, \tag{3.4}$$

and to refer to the elements of $\mathcal{N}(m)$ as m-exceptional sets. It follows immediately from [**Fi89**; (3.9)] that m-polar sets are m-exceptional when X is *nearly symmetric.*

Here is the basic existence theorem for HRMs, taken from [**Fi87**; (5.20)].

(3.5) Theorem. *Let μ be a σ-finite measure on $(E, \mathcal{E})$. Then $\mathbf{P}^\mu := \int_E \mu(dx)\mathbf{P}^x$ is the Palm measure of a (necessarily unique) optional co-predictable HRM if and only if μ charges no element of $\mathcal{N}(m)$.*

(3.6) Remarks. (a) Of course, $\mathbf{P}^\mu$ is carried by $\Omega \subset \{\alpha = 0\}$, so it makes sense to think of $\mathbf{P}^\mu$ as a Palm measure. Actually, since X is a Borel right process, $\mathbf{P}^\mu$ is carried by $\{\alpha = 0\} \cap W(h)$.

(b) The uniqueness assertion in (3.5) is to be understood as modulo $\mathbf{Q}_m$-indistinguishability.

We next define the *characteristic measure* μ_κ (sometimes called the Revuz measure) of an optional HRM κ by

$$\mu_\kappa(f) := \mathbf{P}_\kappa[f(Y_0^*)] = \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) f(Y_t^*)\, \kappa(dt), \qquad f \in p\mathcal{E}, \tag{3.7}$$

where the second equality follows from (3.3), and φ is any positive Borel function on the real line with $\int_{\mathbf{R}} \varphi(t)\, dt = 1$. This definition differs from [**Fi87**; (5.21)] where $\overline{Y}$ is used in place of Y^*. If κ is carried by Λ, then one may replace Y^* by $\overline{Y}$ in (3.7). In view of the remarks made above (3.4), the measure μ_κ charges no element of $\mathcal{N}(m)$. Also, if $f \in p\mathcal{E}$ then $f \circ Y^* \in p(\hat{\mathcal{P}}^\circ \cap \mathcal{O}^\circ)$, so when κ is carried by Λ^*, (3.7) implies that $\kappa \in \sigma\mathcal{I}(\hat{\mathcal{P}}^\circ \cap \mathcal{O}^\circ)$ whenever μ_κ is σ-finite.

The following result is drawn from [**Fi87**; §5]; we omit the proof.

(3.8) Proposition. *(i) Let κ be an optional HRM. Then $\kappa \in \sigma\mathcal{I}(\hat{\mathcal{P}}^m)$ if and only if $\mathbf{P}_\kappa$ is σ-finite. If κ is carried by Λ^* then $\mathbf{P}_\kappa = \mathbf{P}^{\mu_\kappa}$, and $\mathbf{P}_\kappa$ is σ-finite if and only if μ_κ is σ-finite.*

(ii) Let κ_1 and κ_2 be optional HRMs. If κ_1 and κ_2 are $\mathbf{Q}_m$-indistinguishable, then $\mu_{\kappa_1} = \mu_{\kappa_2}$. Conversely, if, in addition, κ_1 and κ_2 are co-predictable and carried by Λ^*, and if $\mu_{\kappa_1} = \mu_{\kappa_2}$ is a σ-finite measure, then κ_1 and κ_2 are $\mathbf{Q}_m$-indistinguishable.

(3.9) Definition. We use $\mathcal{S}_0^{\#}(m)$ to denote the class of σ-finite measures on $(E, \mathcal{E})$ charging no element of $\mathcal{N}(m)$. [The symbol $\mathcal{S}_0(m)$ was used in [**G95**] to denote the class of σ-finite measures on $(E, \mathcal{E})$ charging no m-semipolar set.]

The next result is the fundamental existence theorem for homogeneous random measures. A proof can be fashioned from [**Fi87**; §5], especially (5.22). As before, the stated uniqueness is modulo $\mathbf{Q}_m$-indistinguishability.

(3.10) Theorem. *Fix $\mu \in \mathcal{S}_0^{\#}(m)$. Then there exists a unique optional co-predictable HRM κ carried by Λ^* with $\mu_\kappa = \mu$. The HRM κ is carried by Λ if and only if μ charges no m-polar set. Moreover, κ is diffuse if and only if μ charges no m-semipolar set.*

Next we have an important improvement of (3.10). For the statement of this theorem we recall that the birthing and killing operators are defined on W by $b_t w(s) := w(s)$ if $t < s$, $:= \Delta$ if $t \ge s$, and $k_t w(s) := w(s)$ if $s < t$, $:= \Delta$ if $s \ge t$.

(3.11) Theorem. *Fix $\mu \in \mathcal{S}_0^{\#}(m)$. Then the HRM κ in (3.10) may be taken to be perfect; that is, to have the following additional properties:*

(i) κ is a kernel from $(W, \mathcal{G}^*)$ to $(\mathbf{R}, \mathcal{B})$, where $\mathcal{G}^*$ is the universal completion of $\mathcal{G}^\circ$;

(ii) $\kappa(\sigma_t w, B) = \kappa(w, B + t)$ for all $w \in W$, $t \in \mathbf{R}$, and $B \in \mathcal{B}$;

(iii) $\kappa(b_t w, B) = \kappa(w, B \cap [t, \infty[)$ for all $w \in W$, $t \in \mathbf{R}$, and $B \in \mathcal{B}$;

(iv) $\kappa(k_t w, B) = \kappa(w, B \cap]-\infty, t[)$ for all $w \in W$, $t \in \mathbf{R}$, and $B \in \mathcal{B}$.

Moreover, κ may be chosen so that

$$\kappa = \sum_{t \in \mathbf{R}} j(Y_t^*)\epsilon_t + \kappa^c, \tag{3.12}$$

where $j \in p\mathcal{E}$ with $\{j > 0\}$ semipolar, and κ^c is diffuse.

Proof. With the exception of the last sentence, everything stated here follows immediately from Theorem (5.27) in [**Fi87**]. Using (2.7) and the fact that κ is carried by Λ^*, and arguing as in the proof of [**Fi87**; (5.27)], one obtains a weak form of (3.12) with $\{j > 0\}$ m-semipolar. To complete the proof of (3.11) we require the following lemma, which is of interest in its own right.

(3.13) Lemma. *A Borel m-polar set is the union of a Borel semipolar set and a Borel set in $\mathcal{N}(m)$. In particular, a Borel m-semipolar set is the union of a Borel semipolar set and a Borel set in $\mathcal{N}(m)$.*

We shall use (3.13) to complete the proof of (3.11), after which we shall prove (3.13). Thus, let (3.12) hold with $\{j > 0\}$ m-semipolar. Then by (3.13) we can write $\{j > 0\} = A \cup B$ where $A \in \mathcal{E} \cap \mathcal{N}(m)$ and $B \in \mathcal{E}$ is semipolar. Define $j' := j1_B$, so that $j' \in p\mathcal{E}$ and $\{j' > 0\} = B$ is semipolar. Let $\kappa' := \sum_{t \in \mathbf{R}} j'(Y_t^*)\epsilon_t + \kappa^c$. If $D \in \mathcal{E}$ and $\varphi \geq 0$ with $\int_{\mathbf{R}} \varphi(t)\, dt = 1$, then

$$\mu_\kappa(D) = \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) 1_D(Y_t^*)\, \kappa'(dt) + \mathbf{Q}_m \sum_{t \in \mathbf{R}} \varphi(t) 1_D(Y_t^*) 1_A(Y_t^*) j(Y_t^*).$$

But $A \in \mathcal{N}(m)$, hence $\{Y^* \in A\}$ is $\mathbf{Q}_m$-evanescent. Therefore $\mu_\kappa = \mu_{\kappa'}$, and so (3.12) obtains with κ' replacing κ. Since κ' is carried by Λ^*, κ' and κ are $\mathbf{Q}_m$-indistinguishable, and we are done. ∎

Proof of (3.13). According to a theorem of Dellacherie [**De88**; p. 70], an m-semipolar set B is of the form $B_1 \cup B_2$, with B_1 m-polar and B_2 semipolar. (In fact, this is the definition of "m-semipolar"; its equivalence with the $\mathbf{P}^m$-a.e. countability of $\{t : X_t \in B\}$ is part of the result referred to in the preceding sentence.) Moreover, using the fact that an m-polar set is contained in a Borel m-polar set it is easy to see that if B is Borel then both B_1 and B_2 may be chosen Borel. Suppose now that $B \in \mathcal{E}$ is m-polar. Let $m = \eta + \rho U$ be the Riesz decomposition of m into harmonic and potential parts. Then B is ρU-polar. Let $D = D_B := \inf\{t \ge 0 : X_t \in B\}$ denote the début of B. Recall that for a σ-finite measure μ, the phrase "B is μ-polar" means that $\mathbf{P}^\mu[D < \infty] = 0$. Thus,

$$0 = \mathbf{P}^{\rho U}[D < \infty] = \int_0^\infty \mathbf{P}^{\rho P_t}[D < \infty]\, dt = \int_0^\infty \mathbf{P}^{\rho}[D \circ \theta_t < \infty]\, dt,$$

and so $\mathbf{P}^\rho[D \circ \theta_t < \infty] = 0$ for (Lebesgue) a.e. $t > 0$. It follows that $\{t \ge 0 : X_t \in B\} \subset \{0\}$, $\mathbf{P}^\rho$-a.s. Using Dellacherie's theorem again, B is ρ-semipolar, so $B = B_1 \cup B_2$ with B_1 ρ-polar and B_2 semipolar. Hence $\rho(B_1) = 0$ and $B_1 \subset B$, so B_1 is m-polar; that is, $B_1 \in \mathcal{N}(m)$. ☐

We find it convenient to make the following definition.

(3.14) Definition. An HRM κ is *perfect* provided it is carried by Λ^* and satisfies conditions (i)-(iv) of (3.11), and (3.12).

Thus (3.11) may be rephrased as follows: Each $\mu \in \mathcal{S}_0^{\#}(m)$ is the characteristic measure of a unique perfect HRM κ. It follows from (5.24) and (5.25) of [**Fi87**] that a perfect HRM is optional and co-predictable provided it is in $\sigma\mathcal{I}(\hat{\mathcal{P}}^m \cap \mathcal{O}^m)$—in particular, if μ_κ is σ-finite.

When μ charges no m-polar set there is second HRM γ, carried by $]\alpha, \beta[$, with characteristic measure μ. The discussion of "co-natural" that follows is dual to that found in [**GS84**] concerning natural HRMs.

Suppose μ is a σ-finite measure charging no m-polar set. Then Theorems (3.10) and (3.11) apply, so there is a perfect HRM κ, carried by Λ, with characteristic measure μ. Fix $\varphi \ge 0$ with $\int_{\mathbf{R}} \varphi(t)\, dt = 1$. If $f \in p\mathcal{E}$, then since $\Lambda = \{\ell > 0\}$,

$$\begin{aligned} \mu(f) &= \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) f(Y_t^*) 1_\Lambda(t)\, \kappa(dt) \\ &= \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) f(Y_t^*) \ell_t \ell_t^{-1} 1_\Lambda(t)\, \kappa(dt) \\ &= \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t)\, {}^{\hat{p}}[f \circ Y]_t \ell_t^{-1} 1_\Lambda(t)\, \kappa(dt), \end{aligned}$$

since ${}^{\hat{p}}[f \circ Y] = \ell \cdot f \circ Y^*$ on Λ according to [**Fi87**; (3.9)]. But the process $\ell^{-1} 1_\Lambda$ is co-predictable, so

$$\mu(f) = \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) f(Y_t) \ell_t^{-1} \kappa(dt) = \mathbf{Q}_m \int_{]\alpha,\beta[} \varphi(t) f(Y_t) \ell_t^{-1} \kappa(dt). \tag{3.15}$$

The last equality holds because $f(Y_t) = 0$ if $t \notin]\alpha, \beta[$.

(3.16) Definition. An HRM γ is *co-natural* provided there exists a perfect HRM κ, carried by Λ, such that $\gamma = 1_{]\!]\alpha,\infty[\![}\,\kappa$.

A co-natural HRM is carried by $]\!]\alpha, \beta[\![$ and, if it lies in $\sigma\mathcal{I}(\mathcal{O}^m)$, then it is optional in view of [**Fi87**; (5.25)]. It has all of the properties listed in (3.11), except that (iii) must be replaced by

(iii') $\kappa(b_t w, B) = \kappa(w, B \cap]t, \infty[)$ for all $w \in W$, $t \in \mathbf{R}$, and $B \in \mathcal{B}$.

(3.17) Theorem. *Let μ be a σ-finite measure charging no m-polar set. Then there exists a unique co-natural HRM γ with characteristic measure μ.*

Proof. Clearly $\mu \in \mathcal{S}_0^{\#}(m)$, so by Theorems (3.10) and (3.11) there is a perfect HRM κ, carried by Λ, with characteristic measure μ. From (3.15), $\mu(f) = \mathbf{Q}_m \int \varphi(t) f(Y_t)\, \gamma(dt)$, where $\gamma := 1_{]\!]\alpha,\infty[\![}\,\ell^{-1}\,\kappa$ is obviously co-natural since $\ell^{-1}\kappa$ has the properties required by Definition (3.16). For the uniqueness, let γ be as above and define $\bar{\kappa}(dt) := \ell_t \gamma(dt)$. The computation leading to (3.15) gives

$$\mu_{\bar{\kappa}}(f) = \mathbf{Q}_m \int_{]\alpha,\beta[} \varphi(t) f(Y_t) \ell_t^{-1}\, \bar{\kappa}(dt) = \mathbf{Q}_m \int_{]\alpha,\beta[} \varphi(t) f(Y_t)\, \gamma(dt) = \mu(f).$$

Consequently $\bar{\kappa}$ is uniquely determined by μ; κ is likewise determined since it is carried by $\Lambda = \{\ell > 0\}$. It follows that γ is uniquely determined by μ. $\square$

(3.18) Remark. Suppose μ is σ-finite and charges no m-polar set. Let κ correspond to μ as in (3.11). Then from (3.15) the unique co-natural HRM with characteristic measure μ is $\gamma = 1_{]\!]\alpha,\infty[\![}\,\ell^{-1}\kappa$. It follows from [**Fi87**; (3.5)] and (2.7) that there exists $p \in p\mathcal{E}$ with $\ell = p \circ Y^*$ on Λ^*. Therefore, using the representation (3.12) for κ and the fact that κ is carried by Λ, we find

$$\gamma = \sum_{t \in]\alpha,\beta[} j(Y_t) p^{-1}(Y_t) \epsilon_t + p^{-1} * \kappa^c,$$

where, for an HRM λ, $(f * \lambda)(dt) := f(Y_t^*)\lambda(dt)$. Of course, κ^c doesn't charge $\{\alpha\}$.

We close this section with a useful characterization of co-natural HRMs.

(3.19) Theorem. *An HRM γ is co-natural if and only if $\gamma = \sum_{t \in]\alpha,\beta[} g(Y_t)\epsilon_t + \gamma^c$, where γ^c is a diffuse (optional, co-predictable) perfect HRM and $g \in p\mathcal{E}$ with $\{g > 0\}$ semipolar.*

Proof. If γ is co-natural then it is immediate from (3.11) that γ is of the stated form. Conversely, a γ of the stated form is clearly carried by $]\!]\alpha, \beta[\![$. Define $\kappa := \sum_{t \in [\alpha,\beta[} g(\overline{Y}_t)\epsilon_t + \gamma^c$. Then κ is carried by Λ and is a perfect HRM. Since $\gamma = 1_{]\!]\alpha,\infty[\![}\,\kappa$, γ is co-natural. $\square$

4. Homogeneous Random Measures and Potential Kernels.

In this section we fix a perfect HRM κ and we let $\mu = \mu_\kappa$ denote the associated characteristic measure. Recall the "jump function" j as in the representation (3.12). In what follows, named subsets of $\mathbf{R}$ are taken to be Borel sets, unless mention is made to the contrary.

If $B \subset [t, s[$ where $t < s$, then (3.11) implies that $\kappa(B) \in b_t^{-1}\mathcal{G}^* \cap k_s^{-1}\mathcal{G}^*$. For the moment let $\kappa_\Omega(B)$ denote the restriction of $\kappa(B)$ to Ω, for $B \subset [0, \infty[$. In view of the above comments we have $\kappa_\Omega(B) \in \mathcal{F}^*_{[t,s[}$ (the universal completion of $\sigma(X_u : t \le u < s)$) provided $0 \le t < s$ and $B \subset [t, s[$. Moreover, if $B \subset [0, \infty[$, $t \ge 0$, and $\omega \in \Omega$, then

$$\kappa_\Omega(\theta_t\omega, B) = \kappa(\sigma_t b_t\omega, B) = \kappa(b_t\omega, B+t) = \kappa(\omega, (B+t) \cap [t, \infty[) = \kappa_\Omega(\omega, B+t).$$

Thus κ_Ω is an optional random measure over X, homogeneous on $[0, \infty[$ in the sense of [**Sh88**], and κ_Ω is perfect. For notational simplicity we now drop the subscript Ω from our notation, but it should be clear from context when we are restricting κ to Ω.

We define the *potential kernel* U_κ of κ by setting, for $f \in p\mathcal{E}^*$,

$$U_\kappa f(x) := \mathbf{P}^x \int_{[0,\infty[} f(X_t)\,\kappa(dt) = f(x)j(x) + \mathbf{P}^x \int_{]0,\infty[} f(X_t)\,\kappa(dt), \tag{4.1}$$

where $j \in p\mathcal{E}$ comes from (3.12). Define $\bar\kappa := 1_{]\!]\alpha,\infty[\![}\,\kappa$ on Ω. Then

$$U_{\bar\kappa} f(x) = \mathbf{P}^x \int_{]0,\infty[} f(X_t)\,\kappa(dt),$$

and using the Markov property one sees that $U_{\bar\kappa} f$ is an excessive function of X; indeed, $U_{\bar\kappa} f$ is the excessive regularization of $U_\kappa f$.

If T is a stopping time then the associated hitting operator P_T is defined by $P_T f(x) := \mathbf{P}^x[f(X_T); T < \zeta]$, $x \in E$. The σ-algebra of *nearly Borel* subsets of E is denoted $\mathcal{E}^n$. If $B \in \mathcal{E}^n$ then $T_B := \inf\{t > 0 : X_t \in B\}$ is the *hitting time* of B and $D_B := \inf\{t \ge 0 : X_t \in B\}$ is the *début* of B, and these are both stopping times. We shall write P_B and H_B as abbreviations of P_{T_B} and P_{D_B}.

A function f is *strongly supermedian* provided $f \in p\mathcal{E}^n$ and $P_T f \le f$ for all stopping times T. This definition differs slightly from the definition in [**FG96**] where f was required to be measurable over the σ-algebra $\mathcal{E}^e$ generated by the 1-excessive functions. In this paper, because we are assuming that X is a Borel right process, it is more natural to use $\mathcal{E}^n$. The critical point is that $t \mapsto f \circ X_t$ is nearly optional over $(\mathcal{F}_t)$; see [**Sh88**; (5.2)]. In the present case $\mathcal{E}^e \subset \mathcal{E}^n$, so it follows from (4.1) that $U_\kappa f$ is nearly Borel measurable whenever

f is so. Therefore the strong Markov property implies that $U_\kappa f$ is strongly supermedian if $f \in p\mathcal{E}^n$. Consequently, the kernel U_κ is a strongly supermedian kernel as defined in [**BB01b**]. The content of the next proposition is that U_κ is a *regular* strongly supermedian kernel, provided it is proper; see (5.1) and [**BB01b**; §2]. The property asserted is a form of the familiar *domination principle.*

(4.2) Proposition. *Fix a strongly supermedian function u and $f \in p\mathcal{E}^n$. If $U_\kappa f \le u$ on $\{f > 0\}$, then $U_\kappa f \le u$ on all of E.*

Proof. Let $B := \{j > 0\} \cap \{f > 0\}$ and $C := \{j = 0\} \cap \{f > 0\}$. Since B is semipolar, there is an increasing sequence (B_n) of thin sets with union B. Define $g_n := 1_{B_n \cup C} f$, so that $\uparrow \lim_n g_n = f$. Notice that $\{g_n > 0\} \subset \{f > 0\}$. Fix $x \in E$ and let K_p be an increasing sequence of compact subsets of C such that $T_{K_p} \downarrow T_C$, $\mathbf{P}^x$-a.s. Finally, define $T_{n,p} := T_{B_n} \wedge T_{K_p}$. Recalling from (3.12) that κ^c denotes the diffuse part of κ, we see that

$$U_\kappa g_n(x) = U_\kappa(1_{B_n} f)(x) + U_{\kappa^c}(1_C f)(x)$$

since $U_{\kappa^c}(1_{B_n}) = 0$. But

$$U_\kappa(1_{B_n} f)(x) = \mathbf{P}^x \int_{[T_{B_n},\infty[} (1_{B_n} f)(X_t)\,\kappa(dt),$$

and, since κ^c is diffuse,

$$U_\kappa(1_C f)(x) = \mathbf{P}^x \int_{]T_C,\infty[} (1_C f)(X_t)\,\kappa^c(dt).$$

On the other hand, using the strong Markov property and the fact that $1_{B_n} \circ X$ vanishes on $[\![T_{n,p}, T_{B_n}[\![$, we find

$$P_{T_{n,p}} U_\kappa g_n(x) = \mathbf{P}^x \int_{[T_{B_n},\infty[} (1_{B_n} f)(X_t)\,\kappa(dt) + \mathbf{P}^x \int_{[T_{n,p},\infty[} (1_C f)(X_t)\,\kappa(dt).$$

As $p \to \infty$ we have $T_{n,p} \downarrow T_{B_n} \wedge T_C$, $\mathbf{P}^x$-a.s. But $1_C \circ X$ vanishes on $]\!]T_{B_n} \wedge T_C, T_C[\![$ and $1_C * \kappa$ is diffuse. Combining these observations with the expression for $U_\kappa g_n(x)$ obtained above, we find that $\lim_{p\to\infty} P_{T_{n,p}} U_\kappa g_n(x) = U_\kappa g_n(x)$. Now B_n, being thin, is finely closed, hence so is $B_n \cup K_p$. Therefore $P_{T_{n,p}}(x,\cdot)$ is carried by $B_n \cup K_p \subset \{g_n > 0\}$. Finally, note that $U_\kappa g_n \le U_\kappa f \le u$ on $\{f > 0\}$, so that $U_\kappa g_n \le u$ on $\{g_n > 0\}$. Putting these facts together gives $P_{T_{n,p}} U_\kappa g_n(x) \le P_{T_{n,p}} u(x) \le u(x)$, the last inequality following because u is strongly supermedian. Therefore $U_\kappa g_n(x) \le u(x)$. Sending $n \to \infty$ we obtain $U_\kappa f(x) \le u(x)$ by monotone convergence. Since $x \in E$ was arbitrary, (4.2) is established. $\square$

Our next task is to express μ directly in terms of the kernel U_κ. To this end we begin by extending [**G90**; (7.2)] to a class of strongly supermedian functions. Let $H \in p\mathcal{F}$, so that H is defined on Ω. Suppose that $H\circ\theta_t \le H$ for all $t > 0$. Let $u(x) := \mathbf{P}^x[H]$. If T is an $(\mathcal{F}_t)$-stopping time, then

$$P_T u(x) = \mathbf{P}^x[H\circ\theta_T] \le \mathbf{P}^x[H] = u(x). \tag{4.3}$$

In particular, if u is $\mathcal{E}^n$ measurable, then u is strongly supermedian. If $\alpha(w) < s < t$ then one can check as in section 7 of [**G90**] that $H(\theta_t w) \le H(\theta_s w)$. Define $H^* : W \to [0,\infty]$ by

$$H^* :=\uparrow \lim_{t\downarrow\alpha} H\circ\theta_t = \sup_{r\in\mathbf{Q}, r>\alpha} H\circ\theta_r. \tag{4.4}$$

It is evident that $H^* \in p\mathcal{G}$, and if $H \in p\mathcal{F}^*$ then $H^* \in p\mathcal{G}^*$. Also, H^* is (σ_t)-invariant: $H^*\circ\sigma_t = H^*$ for all $t \in \mathbf{R}$. The next result should be compared with [**G90**; (7.2)]. Once again we use the Riesz decomposition $m = \eta + \rho U$ of m into harmonic and potential components. Also, L is the energy functional defined for $\xi \in$ Exc and excessive h by $L(\xi, h) := \sup\{\nu(h) : \nu U \le \xi\}$; see [**G90**; (3.1)]. Recall that if $\xi \in$ Exc is conservative then ξ dominates no non-zero potential, so $L(\xi, h) = 0$ for all excessive h.

(4.5) Lemma. *Let H, H^*, and u be as above, and let $\bar{u} :=\uparrow \lim_{t\downarrow 0} P_t u$ denote the excessive regularization of u. Then*

$$\sup\{\nu(u) : \nu U \le m\} = \rho(u) + L(\eta, \bar{u}),$$

where ν represents a generic σ-finite measure on $(E, \mathcal{E}^n)$.

Proof. If $\nu U \le \eta + \rho U$, then by [**G90**; (5.9),(5.23)] we can decompose $\nu U = \nu_1 U + \nu_2 U$ with $\nu_1 U \le \eta$ and $\nu_2 U \le \rho U$. Moreover, (4.3) and [**G90**; (5.23)] imply that $\nu_2(u) \le \rho(u)$, so $\nu(u) \le \nu_1(u) + \rho(u)$. Therefore, it suffices to prove (4.5) when $m \in$ Har (the class of *harmonic* elements of Exc). We may also assume that $m \in$ Dis, the class of *dissipative* elements of Exc. Indeed, let $m = m_c + m_d$ be the decomposition of m into conservative and dissipative components. If $\nu U \le m$ then $\nu U = \nu_c U + \nu_d U$ with $\nu_c U \le m_c$ and $\nu_d U \le m_d$. Hence (4.5) holds if and only if it holds for m_d.

In the remainder of the proof we suppose that $\nu U \le m$ and that $m \in \text{Har} \cap \text{Dis}$. Combining (7.10), (7.5i), and (6.20) in [**G90**] we find that

$$\nu(f) = \int_0^1 \mathbf{Q}_m[f(Y^*_{T(t)}); 0 < T(t) \le 1, b^{-1}_{T(t)} W(h)]\, dt, \tag{4.6}$$

where $(T(t) : 0 \le t \le 1)$ is an increasing family of stationary terminal times with $\alpha \le T(t)$, and $T(t) < \beta$ on $\{T(t) < \infty\}$. But $m \in \mathrm{Har}$, so for $f \in p\mathcal{E}$

$$\mathbf{Q}_m[f(Y^*_{T(t)}); 0 < \alpha = T(t) \le 1, b^{-1}_{T(t)} W(h)] \le \mathbf{Q}_m[f(Y^*_{T(t)}); W(h)] = 0,$$

in view of [**G90**; (6.19)] and the fact that $b^{-1}_{T(t)} W(h) \subset W(h)$ on $\{\alpha \le T(t)\}$. Therefore,

$$\nu(f) \le \int_0^1 \mathbf{Q}_m[f(Y_{T(t)}); \alpha < T(t), 0 < T(t) \le 1]\, dt,$$

hence

$$\begin{aligned}\nu(u) &\le \int_0^1 \mathbf{Q}_m[\mathbf{P}^{Y(T(t))}(H); \alpha < T(t), 0 < T(t) \le 1]\, dt \\ &= \int_0^1 \mathbf{Q}_m[H \circ \theta_{T(t)}; \alpha < T(t), 0 < T(t) \le 1]\, dt \\ &\le \int_0^1 \mathbf{Q}_m[H^*; 0 < T(t) \le 1]\, dt.\end{aligned}$$

Let S be a stationary time with $\alpha < S < \beta$, $\mathbf{Q}_m$-a.e. Since $m \in \mathrm{Dis}$, such an S exists according to [**G90**; (6.24)(iv)]. Because H^* is (σ_t)-invariant, [**G90**; (6.27)] gives

$$\mathbf{Q}_m[H^*; 0 < T(t) \le 1] = \mathbf{Q}_m[H^*; 0 < S \le 1, T(t) \in \mathbf{R}],$$

and so $\nu(u) \le \mathbf{Q}_m[H^*; 0 < S \le 1]$. Note that

$$\bar{u}(x) = \lim_{t \downarrow 0} P_t u(x) = \lim_{t \downarrow 0} \mathbf{P}^x[H \circ \theta_t] = \mathbf{P}^x[\bar{H}],$$

by monotone convergence, where

$$\bar{H} := \uparrow \lim_{t \downarrow 0} H \circ \theta_t = \sup_{r \in \mathbf{Q}, r > 0} H \circ \theta_r.$$

Moreover,

$$\lim_{t \downarrow 0} \bar{H} \circ \theta_t = \lim_{t \downarrow 0} \lim_{s \downarrow 0} H \circ \theta_{t+s} = \sup_{r > 0} H \circ \theta_r = \bar{H},$$

and it is now evident that $\bar{H}$ is excessive as defined in section 7 of [**G90**]. Finally, observe that

$$(\bar{H})^* = \sup_{r > \alpha} \bar{H} \circ \theta_r = \sup_{r > \alpha} \sup_{q > 0} H \circ \theta_{r+q} = H^*.$$

Therefore we may apply [**G90**; (7.2)] to obtain

$$\nu(u) \le \mathbf{Q}_m[(\bar{H})^*; 0 < S \le 1] = L(m, \bar{u}).$$

On the other hand, if $\nu_n U \uparrow m$, then

$$L(m,\bar{u}) = \lim_n \nu_n(\bar{u}) \le \limsup_n \nu_n(u) \le \sup\{\nu(u) : \nu U \le m\}.$$

This establishes (4.5) when $m \in \mathrm{Har} \cap \mathrm{Dis}$, hence in general by the earlier discussion. □

(4.7) Theorem. *If u is a strongly supermedian function, then*

$$\sup\{\nu(u) : \nu U \le m\} = \rho(u) + L(\eta,\bar{u}). \tag{4.8}$$

Proof. First observe that it suffices to prove (4.8) for bounded u. Indeed, given an arbitrary strongly supermedian function u, define $u_k := u \wedge k$ for $k \in \mathbf{N}$. By monotonicity

$$\lim_k \sup\{\nu(u_k) : \nu U \le m\} = \sup\{\nu(u) : \nu U \le m\}$$

and $\lim_k \rho(u_k) = \rho(u)$. By the same token, since the limits defining the excessive regularizations $\bar{u}_k$ and $\bar{u}$ are monotone increasing, we have $\lim_k L(\eta,\bar{u}_k) = L(\eta,\bar{u})$.

Next, arguing as at the beginning of the proof of Lemma (4.5) we can assume that m is dissipative, and then, without loss of generality, that X is transient.

Now let u be a bounded strongly supermedian function. By Corollary (A.7) in the appendix, there is an increasing sequence $\{v_n\}$ of regular strongly supermedian functions with pointwise limit u. By [**Sh88**; (38.2)], for each n there is a PLAF $A^{(n)}$ with $v_n(x) = \mathbf{P}^x[A^{(n)}_\infty]$ for all $x \in E$. Evidently $t \mapsto A^{(n)}_\infty \circ \theta_t = A^{(n)}_\infty - A^{(n)}_t$ is decreasing, so Lemma (4.5) applies and we find that

$$\sup\{\nu(v_n) : \nu U \le m\} = \rho(v_n) + L(\eta,\bar{v}_n), \qquad n \in \mathbf{N}. \tag{4.9}$$

As in the first paragraph of the proof, monotonicity allows us to let $n \to \infty$ in (4.9) to arrive at (4.8). □

We are now in a position to express μ in terms of the kernel U_κ. As before, κ is a perfect HRM and $\mu = \mu_\kappa$. Also, $m = \eta + \rho U$.

(4.10) Theorem. *Suppose $m \in \mathrm{Dis}$. If $f \in p\mathcal{E}$, then*

$$\mu(f) = \sup\{\nu U_\kappa f : \nu U \le m\} = \rho U_\kappa f + L(\eta,\bar{U}_\kappa f), \tag{4.11}$$

where $\bar{U}_\kappa f := \overline{U_\kappa f}$ denotes the excessive regularization of $U_\kappa f$. [Recall that $\bar{U}_\kappa f = U_{\bar{\kappa}} f$, where $\bar{\kappa} := 1_{]\!]\alpha,\infty[\![}\,\kappa$.]

Proof. Fix a positive Borel function φ on $\mathbf{R}$ with $\int_{\mathbf{R}} \varphi(t)\,dt = 1$. Define, for a given $f \in p\mathcal{E}$, $G := \int_{\mathbf{R}} \varphi(t) f(Y_t^*)\,\kappa(dt)$. From (3.7), $\mu(f) = \mathbf{Q}_m[G]$. Moreover,

$$\begin{aligned}\bar{G} := \int_{\mathbf{R}} G\circ\theta_t\,dt &= \int_{\mathbf{R}} \left[\int_{\mathbf{R}} \varphi(s) f(Y_{t+s}^*)\,\kappa(ds+t)\right] dt \\ &= \int_{\mathbf{R}} \left[\int_{\mathbf{R}} \varphi(s-t) f(Y_s^*)\,\kappa(ds)\right] dt = \int_{\mathbf{R}} f(Y_s^*)\,\kappa(ds).\end{aligned}$$

But $m \in \mathrm{Dis}$ and so there is a stationary time S with $\alpha < S < \beta$, $\mathbf{Q}_m$-a.s. Thus, from [**G90**; (6.27)] we obtain

$$\mu(f) = \mathbf{Q}_m\left[\int_{\mathbf{R}} f(Y_t^*)\,\kappa(dt); 0 < S \le 1\right]. \tag{4.12}$$

Define $H := \int_{[0,\zeta[} f(X_t)\,\kappa(dt)$ on Ω, so that $\mathbf{P}^x[H] = U_\kappa f(x)$ for all $x \in E$. From (4.12),

$$\begin{aligned}\mu(f) = \mathbf{Q}_m &\left[\int_{\mathbf{R}} f(Y_t^*)\,\kappa(dt); 0 < S \le 1, W(h)\right] \\ &+ \mathbf{Q}_m\left[\int_{\mathbf{R}} f(Y_t^*)\,\kappa(dt); 0 < S \le 1, W(h)^c\right].\end{aligned}$$

The first term on the right equals $\mathbf{Q}_m[H\circ\theta_\alpha; 0 < S \le 1, W(h)]$. Now $\alpha \in \mathbf{R}$ on $W(h)$, and so using [**G90**; (6.27)] this may be written, since $H\circ\theta_\alpha$ and $W(h)$ are invariant,

$$\begin{aligned}\mathbf{Q}_m[H\circ\theta_\alpha; 0 < \alpha \le 1, W(h)] &= \mathbf{Q}_m[\mathbf{P}^{Y^*(\alpha)}[H]; 0 < \alpha \le 1, W(h)] \\ &= \mathbf{Q}_m[U_\kappa f(Y_\alpha^*); 0 < \alpha \le 1, W(h)] = \rho U_\kappa f,\end{aligned}$$

where the first equality just above comes from (2.5) and the last from (2.4). Because $\mathbf{Q}_m[\cdot\,; W(h)^c] = \mathbf{Q}_\eta$ and $Y^* = Y$ on $W(h)^c$, the second piece reduces to

$$\mathbf{Q}_\eta\left[\int_{]\alpha,\infty[} f(Y_t)\,\kappa(dt); 0 < S \le 1\right].$$

But H^* defined in (4.4) is given by $\int_{]\alpha,\infty[} f(Y_t)\,\kappa(dt)$ in the present situation. As in the proof of (4.5), with m replaced by η, this last expression equals $L(\eta, \bar{U}_\kappa f)$. Clearly $\bar{U}_\kappa f = U_{\bar{\kappa}} f$ with $\bar{\kappa}$ as in (4.10). Consequently, $\mu(f) = \rho U_\kappa f + L(\eta, U_{\bar{\kappa}} f)$ and the fact that this in turn equals $\sup\{\nu U_\kappa f : \nu U \le m\}$ results from one final appeal to (4.5). ∎

The most interesting case ($m \in \mathrm{Dis}$, which occurs if X is transient) is covered by (4.10). However if m is invariant, in particular if m is conservative, then one has

$$\mu(f) = \mathbf{P}^m \int_{]0,1]} f(X_t)\,\kappa(dt). \tag{4.13}$$

Indeed, since $\mathbf{Q}_m[\alpha > -\infty] = 0$ when m is invariant,

$$\begin{aligned}\mu(f) = \mathbf{Q}_m \int_{]0,1]} f(Y_t)\,\kappa(dt) &= \mathbf{Q}_m\left[\left(\left(\int_{]0,1]} f(Y_t)\,\kappa(dt)\right)\circ\theta_0\right)\right]\\ &= \mathbf{Q}_m\left[\mathbf{P}^{X(0)}\int_{]0,1]} f(Y_t)\,\kappa(dt)\right] = \mathbf{P}^m \int_{]0,1]} f(X_t)\,\kappa(dt).\end{aligned}$$

(4.14) Remark. As observed in Theorem (4.10), the co-natural HRM $\bar\kappa := 1_{]\!]\alpha,\infty[\![}\kappa$ has potential kernel $U_{\bar\kappa} = \bar U_\kappa$, which is a semi-regular excessive kernel in the sense of Definition (5.1) below. By an argument used in the proof of Theorem (3.17), the characteristic measure $\mu_{\bar\kappa}$ of $\bar\kappa$ is $p\cdot\mu_\kappa$, where $p\circ Y^* = \ell$ as in Remark (3.18); *cf.* (5.11). Observe that (4.5) and the proof of (4.10) together imply that $\mu_{\bar\kappa}(f) = L(m, \bar U_\kappa f)$. Combining (8.21) and (8.9) of [**G90**] we obtain the classical expressions for the characteristic measure of $\bar\kappa$:

$$\begin{aligned}\mu_{\bar\kappa}(f) &=\uparrow \lim_{t\downarrow 0} t^{-1}\mathbf{P}^m \int_{]0,t[} f(X_s)\,\bar\kappa(ds)\\ &=\uparrow \lim_{q\uparrow\infty} q\mathbf{P}^m \int_{]0,\infty[} e^{-qs} f(X_s)\,\bar\kappa(ds).\end{aligned}\tag{4.15}$$

5. Strongly Supermedian Kernels.

We shall now examine some of the relationships between our results on HRMs and the material presented in [**BB01b**]. Beznea and Boboc assume that the potential kernel U is proper (equivalently, that X is transient), so throughout this section and the next we shall assume that U is proper. More precisely, we assume that there is a function $b \in \mathcal{E}$ with $0 < b \le 1$ and $Ub \le 1$. Reducing b if necessary, we can (and do) assume that $m(b) < \infty$. In particular, each excessive measure of X is dissipative.

It is well known that if u is a strongly supermedian function and $0 \le S \le T$ are stopping times, then $P_T u \le P_S u$ everywhere on E. More generally, let us say that a σ-finite measure μ is dominated by another σ-finite measure ν in the *balayage order* (and write $\mu \dashv \nu$) provided $\mu U \le \nu U$. Then for strongly supermedian u we have $\mu(u) \le \nu(u)$ whenever $\mu \dashv \nu$. (These assertions follow, for example, from Rost's theorem; see [**G90**; (5.23)].) It follows in turn that if T is a terminal time (namely, $T = t + T\circ\theta_t$ on $\{t < T\}$, for each $t > 0$) then $P_T u$ is strongly supermedian.

We modify slightly the definition of *regular strongly supermedian kernel* found in [**BB01b**] by dropping the assumption that such a kernel is proper. The connection between regular strongly supermedian kernels and the notion of regularity for strongly supermedian functions (as in [**Sh88**; (36.7)] or (A.1) in the appendix to this paper) is made in Proposition (A.8) and Remark (A.11).

(5.1) Definition. (a) A *strongly supermedian* (resp. *excessive*) *kernel* V is a kernel on $(E, \mathcal{E}^n)$ such that Vf is a strongly supermedian (resp. excessive) function for each $f \in p\mathcal{E}^n$.

(b) A strongly supermedian kernel V is *regular* provided, for each $f \in p\mathcal{E}^n$ and each strongly supermedian function u, if $Vf \le u$ on $\{f > 0\}$ then $Vf \le u$ on all of E.

(c) An excessive kernel W is *semi-regular* provided there is a regular strongly supermedian kernel V such that $W = \bar{V}$; that is, $Wf = \overline{Vf}$ (the excessive regularization of Vf) for each $f \in p\mathcal{E}^n$.

Proposition (4.2) may now be restated as follows: If κ is a perfect HRM, then U_κ is a regular strongly supermedian kernel.

The following result records some facts that have familiar analogs in the context of excessive functions. Recall that $B \in \mathcal{E}^n$ is *absorbing* provided $\mathbf{P}^x[T_{E\setminus B} < \infty] = 0$ for all $x \in B$. For instance, if h is an excessive function of X then $\{h = 0\}$ is finely closed and absorbing (hence also finely open); in particular, if $\{h > 0\}$ is m-null, then $\{h > 0\}$ is m-exceptional.

(5.2) Lemma. *(i) If u is a strongly supermedian function, then $\{u < \infty\}$ is an absorbing set, hence finely open. In particular, if $u < \infty$, m-a.e. then $u < \infty$ off an m-polar set.*

(ii) Let V be a regular strongly supermedian kernel. If B is m-exceptional (resp. m-polar) then $\{V1_B > 0\}$ is m-exceptional (resp. m-polar).

Proof. (i) Define $B := \{u = \infty\}$ and fix $x \in E \setminus B$. If $\mathbf{P}^x[T_B < \infty] > 0$ then there exists a compact set $K \subset B$ with $\mathbf{P}^x[T_K < \infty] > 0$. It then follows that $\infty = P_K u(x) \le u(x) < \infty$, which is absurd. Therefore $\mathbf{P}^x[T_B < \infty] = 0$ for all $x \in E \setminus B$. Consequently $\{u < \infty\} = E \setminus B$ is absorbing and finely open. If, in addition, $m(u = \infty) = 0$ then $\mathbf{P}^x[T_B < \infty] = 0$, m-a.e.; that is, B is m-polar.

(ii) Now define $D_B := \inf\{t \ge 0 : X_t \in B\}$, the début of B. Clearly $D_B = T_B$, $\mathbf{P}^x$-a.s. for each $x \notin B$. If B is m-polar then $\mathbf{P}^x[T_B < \infty] = 0$, m-a.e., hence off an m-polar set. It follows that if B is m-polar (resp. m-exceptional) then $\{x \in E : \mathbf{P}^x[D_B < \infty] > 0\}$ is m-polar (resp. m-exceptional). Now observe that $V1_B$ and $P_{D_B} V1_B$ are strongly supermedian functions with $P_{D_B} V1_B \le V1_B$. But $P_{D_B} V1_B = V1_B$ on B, so the regularity of V implies that $P_{D_B} V1_B \ge V1_B$ everywhere, so $P_{D_B} V1_B = V1_B$ everywhere. It follows that if B is m-exceptional (resp. m-polar) then $\{V1_B > 0\}$ is m-exceptional (resp. m-polar). ∎

(5.3) Definition. Let ν be a measure on $(E, \mathcal{E})$. A strongly supermedian kernel V is ν-proper provided there exists a strictly positive function $g \in \mathcal{E}^n$ with $Vg < \infty$, ν-a.e.

(5.4) Remark. We shall use this definition in two cases: $\nu = m$ and $\nu = m+\rho$, where, as usual, ρU is the potential part of m. It follows immediately from (5.2)(i) that if $Vg < \infty$, m-a.e. (resp. $(m+\rho)$-a.e.), then $\{Vg = \infty\}$ is m-polar (resp. m-exceptional).

(5.5) Proposition. *(a) Let V be a regular strongly supermedian kernel. If V is m-proper (resp. $(m+\rho)$-proper) then there exists $g \in \mathcal{E}^n$ with $0 < g \le 1$ such that $Vg \le Ub$ off an m-polar (resp. m-exceptional) set.*

(b) Let W be a semi-regular excessive kernel. If W is m-proper, then there exists $g \in \mathcal{E}^n$ with $0 < g \le 1$ such that $Wg \le Ub$ off an m-polar set.

Proof. (a) Choose $g_0 \in \mathcal{E}^n$ with $0 < g_0 \le 1$ such that $\{Vg_0 = \infty\}$ is m-polar (resp. m-exceptional). Define $u := Ub$, so that $0 < u \le 1$. For $n \in \mathbf{N}$ define $B_n := \{Vg_0 \le n \cdot u\}$. Clearly $B_n \uparrow \{Vg_0 < \infty\}$ as $n \to \infty$. Also, $V(1_{B_n} g_0) \le Vg_0 \le n \cdot u$ on B_n, so $V(1_{B_n} g_0) \le n \cdot u$ on all of E by the regularity of V. We now define $g := 1_B + g_0 \sum_{n=1}^{\infty} (n 2^n)^{-1} 1_{B_n}$, where $B := \{Vg_0 = \infty\}$. Evidently $0 < g \le 1$ and $Vg \le u + V1_B$. But Lemma (5.2)(ii) implies that $V1_B = 0$ off an m-polar (resp. m-exceptional) set.

(b) We can write $W = \bar{V}$, where V is a regular strongly supermedian kernel. Let $g_0 > 0$ be such that $Wg_0 < \infty$, m-a.e. The set $\{Wg_0 \ne Vg_0\}$ is semipolar, hence m-null. It follows that V is m-proper, so assertion (b) follows from (a) because $Wf \le Vf$ for all $f \in p\mathcal{E}^n$. □

Given a strongly supermedian kernel V, we follow Beznea and Boboc [**BB01b**] in defining

$$\mu_V(f) := \sup\{\nu V f : \nu U \le m\}, \qquad f \in p\mathcal{E}^n. \tag{5.6}$$

[As before, ν on the right side of (5.6) represents a generic σ-finite measure on $(E, \mathcal{E}^n)$.]

(5.7) Proposition. *Let V be a regular strongly supermedian kernel. Then μ_V (resp. $\mu_{\bar{V}}$) is a measure on $(E, \mathcal{E}^n)$ charging no m-exceptional (resp. m-polar) set. Moreover, μ_V is σ-finite if and only if V is $(m+\rho)$-proper. If $\bar{V}$ is m-proper then $\mu_{\bar{V}}$ is σ-finite.*

Proof. (The first paragraph of the proof is stated for V, but the arguments work just as well for $\bar{V}$.) Clearly μ_V is positive homogeneous, monotone increasing, and subadditive on $p\mathcal{E}$. If $0 \le f_n \uparrow f$ then

$$\sup_n \mu_V(f_n) = \sup_n \sup_\nu \{\nu V f_n : \nu U \le m\} = \sup_\nu \{\sup_n \nu V f_n : \nu U \le m\} = \mu_V(f).$$

Thus, to show that μ_V is a measure, it suffices to show that $\mu_V(f_1+f_2) \ge \mu_V(f_1)+\mu_V(f_2)$. This inequality is evident if either term on the right is infinite, so we assume that $\mu_V(f_1)+$

$\mu_V(f_2)$ is finite. Given $\epsilon > 0$, there exist ν_1 and ν_2 with $\nu_i U \le m$ and $\nu_i V(f_i) \ge \mu_V(f_i) - \epsilon$ for $i = 1, 2$. Define $\xi := \inf\{\eta \in \text{Exc} : \eta \ge \nu_1 U \vee \nu_2 U\}$. Then $\xi \in \text{Exc}$, $\xi \le m$, and (because ξ is dominated by the potential $(\nu_1 + \nu_2)U$) ξ is a potential, say $\xi = \nu U$. Also, $\nu_i(Vf_i) \le \nu(Vf_i)$ since $\nu_i \dashv \nu$ for $i = 1, 2$. Therefore

$$\mu_V(f_1 + f_2) \ge \nu V(f_1 + f_2) \ge \nu_1 V f_1 + \nu_2 V f_2 \ge \mu_V(f_1) + \mu_V(f_2) - 2\epsilon.$$

As $\epsilon > 0$ was arbitrary, it follows that μ_V is a measure.

Now suppose that B is an element of $\mathcal{N}(m)$. Then $\{V1_B > 0\} \in \mathcal{N}(m)$ by (5.2)(ii). If $\nu U \le m$ then ν doesn't charge sets in $\mathcal{N}(m)$ by (4.6). Consequently $\mu_V(B) = 0$. Suppose that V is $(m + \rho)$-proper. Using (5.5) we see that there exists $0 < g \le 1$ with $\{Vg > Ub\} \in \mathcal{N}(m)$. Thus, if $\nu U \le m$ then $\nu V g \le \nu U b \le m(b) < \infty$, and so $\mu_V(g) < \infty$, proving the σ-finiteness of μ_V. Conversely, suppose there exists $0 < f \le 1$ with $\mu_V(f) < \infty$. If $\nu U \le m$ then $\nu P_t V f \le \nu V f \le \mu_V(f) < \infty$. In particular, $\nu P_t\{Vf = \infty\} = 0$ for each $t > 0$. Integrating with respect to t gives $\nu U\{Vf = \infty\} = 0$. Choosing a sequence $\{\nu_n\}$ with $\nu_n U \uparrow m$ (possible because X is transient), we see that $Vf < \infty$, m-a.e. But $\rho U \le m$, so $\rho\{Vf = \infty\} = 0$. Consequently, $Vf < \infty$, $(m + \rho)$-a.e.

Finally, suppose that $\bar V$ is m-proper. By (5.5)(b) there exists $g \in \mathcal{E}^n$ with $0 < g \le 1$ such that $Vg \le Ub$ off an m-polar set. By [**G90**; (2.17)] there is a sequence (ν_n) of measures, each absolutely continuous with respect to m, such that $\nu_n U$ increases setwise to m, and then $\nu_n \bar V g$ increases to $L(m, \bar V g) = \mu_{\bar V}(g)$. Because the set $\{Vg > Ub\}$ is m-polar, hence m-null, it follows that $\mu_{\bar V}(g) \le \lim_n \nu_n U b = m(b) < \infty$, proving that $\mu_{\bar V}$ is σ-finite. ∎

The measure μ_V defined in (5.6) is called the *characteristic measure* of V. Note that if κ is a perfect HRM then, in light of Theorem (4.10), the characteristic measure μ_κ of κ (defined in (3.7)) is the characteristic measure of the regular strongly supermedian kernel U_κ. Writing $\bar\kappa := 1_{]\!]\alpha,\infty[\![}\,\kappa$ as before, the potential kernel of $\bar\kappa$ is the semi-regular excessive kernel $\bar U_\kappa := \overline{U_\kappa}$, and the characteristic measure of $\bar\kappa$ is $p \cdot \mu_\kappa$; see Remark (4.14).

We come now to the main result of this development.

(5.8) Theorem. *Let V be a regular strongly supermedian kernel.*

(a) If V is $(m + \rho)$-proper, then there exists a unique perfect HRM κ with $\{x \in E : V(x, \cdot) \ne U_\kappa(x, \cdot)\} \in \mathcal{N}(m)$.

(b) If $\bar V$ is m-proper, then there exists a unique co-natural HRM γ such that $\{x \in E : \bar V(x, \cdot) \ne U_\gamma(x, \cdot)\}$ is m-polar.

Proof. (a) Let μ_V be the characteristic measure of V. Then (5.7) states that $\mu_V \in \mathcal{S}_0^{\#}(m)$. Consequently, according to (3.11), there exists a unique perfect HRM κ with $\mu_\kappa = \mu_V$. Now

Theorem 3.2(i) in [**BB01b**] implies $V(x,\cdot) = U_\kappa(x,\cdot)$ for all x outside an m-exceptional set. This yields the desired conclusion. Since the proof of this uniqueness theorem in [**BB01b**] depends on some rather deep results in potential theory, we shall give an alternate proof of Theorem (5.8) in the Appendix; this proof may be more palatable to probabilists.

(b) By (5.7) the characteristic measure $\mu_{\bar V}$ is σ-finite and charges no m-polar set. Theorem (3.17) guarantees the existence of a co-natural HRM with characteristic measure $\mu_{\bar V}$. Theorem 3.2(ii) in [**BB01b**] now implies that $\bar V(x,\cdot) = U_\gamma(x,\cdot)$ for all x outside an m-polar set. $\square$

The following variant of Theorem (5.8) can be proved by similar methods; see the appendix. This result for everywhere finite v is [**A73**; Thm. 3.3, p. 489]; see also [**Sh88**; (38.2)]. If v is a strongly supermedian function and $E' \in \mathcal{E}^n$ is an absorbing set, then we say that v is *regular on* E' provided $T \mapsto P_T v(x)$ is left-continuous along increasing sequences of stopping times, for each $x \in E'$; see (A.1) for the precise definition.

(5.9) Theorem. *Let v be a strongly supermedian function such that $v < \infty$ off an m-exceptional set. If v is regular on $\{v < \infty\}$ then there exists a unique perfect HRM κ such that $v = U_\kappa 1$ off an m-exceptional set.*

We now record some corollaries of Theorem (5.8) that are of interest. The first of these is an immediate consequence of (4.8) and (5.6).

(5.10) Corollary. *Let V be a regular strongly supermedian kernel. Then*

$$\mu_{\bar V}(f) = L(m, \bar V f)$$

and

$$\mu_V(f) = \rho V f + L(\eta, \bar V f)$$

for all $f \in \mathcal{E}^n$, where $\bar V f$ denote the excessive regularization of Vf.

The second corollary sharpens Theorem 3.6 in [**BB01b**], by showing that the Radon-Nikodym derivative $d(\mu_{\bar V})/d\mu_V$ does not depend on V. Recall from Remark (3.18) that there is a Borel function p with values in $[0,1]$ such that $\ell = {}^{\hat p}1_{]\!]\alpha,\beta[\![}$ is $\mathbf{Q}_m$-indistinguishable from $p \circ Y^*$. In fact, $p(x) := \hat{\mathbf{P}}^x(\hat\zeta > 0)$ does the job, where the "hats" indicate the moderate Markov dual process $\hat X$ mentioned in Theorem (5.12) below. This representation leads easily to the conclusion that the set $\{x \in E : p(x) < 1\}$ is m-semipolar.

(5.11) Corollary. *If V is an $(m+\rho)$-proper regular strongly supermedian kernel, then the characteristic measures μ_V and $\mu_{\bar V}$ are related by*

$$\mu_{\bar V} = p \cdot \mu_V.$$

Proof. Let κ and γ be the HRMs associated with V and $\bar{V}$ by Theorem (5.8). Then $U_\gamma = \bar{V} = \overline{U_\kappa} = U_{\bar{\kappa}}$. Therefore $\mu_{\bar{V}} = \mu_\gamma = \mu_{\bar{\kappa}}$, which in turn is equal to $p \cdot \mu_\kappa = p \cdot \mu_V$ by Remark (4.14). $\square$

The third corollary will follow immediately from the next result about HRMs. This result—more precisely its dual—appears for diffuse κ in [**G99**].

(5.12) Theorem. *Let κ be a perfect HRM. Then*

$$\int_E f \cdot U_\kappa g \, dm = \int_E g \cdot \hat{U} f \, d\mu_\kappa, \qquad f, g \in p\mathcal{E}, \tag{5.13}$$

where $\hat{U}$ is the potential kernel of the moderate Markov dual process $(\hat{X}, \hat{\mathbf{P}}^x)$ of X relative to m; see [**Fi87**; §4].

(5.14) Remark. Theorem 4.6 in [**Fi87**] establishes the existence of $(\hat{X}, \hat{\mathbf{P}}^x)$, and it is only stated there that the probability measures $\hat{\mathbf{P}}^x$, $x \in E$, are uniquely determined modulo m-polars. However, using Λ^* in place of Λ and letting μ be a probability measure equivalent to $m + \rho$ (rather than m), the proof given in the appendix of [**Fi87**] is readily modified to show that the family $\{\hat{\mathbf{P}}^x\}$ is unique up to an m-exceptional set. Hence $\hat{U} f$ is uniquely determined up to an m-exceptional set; since μ_κ charges no element of $\mathcal{N}(m)$, the integral on the right side of (5.13) is well defined.

Proof. It suffices to prove (5.13) for $f, g \in bp\mathcal{E}$ and then, replacing κ by $g * \kappa$, for $g \equiv 1$. Fix $\varphi \ge 0$ with $\int_{\mathbf{R}} \varphi(t)\, dt = 1$. In the following computation, the first equality is (3.7) while the third holds because κ is co-predictable:

$$\begin{aligned}
\mu_\kappa \hat{U} f &= \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) \hat{U} f(Y_t^*) \, \kappa(dt) = \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) \hat{\mathbf{P}}^{Y_t^*} \int_0^\infty f(\hat{X}_s) \, ds \, \kappa(dt) \\
&= \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) \left(\int_0^\infty f(\hat{X}_s) \, ds \right) \circ \hat{\theta}_t \, \kappa(dt) = \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) \int_0^\infty f(Y_{t-s}^*) \, ds \, \kappa(dt) \\
&= \mathbf{Q}_m \int_{\mathbf{R}} \varphi(t) \int_{-\infty}^t f(Y_s^*) \, ds \, \kappa(dt) = \mathbf{Q}_m \int_{\mathbf{R}} f(Y_s^*) \int_{[s,\infty[} \varphi(t) \, \kappa(dt) \, ds \\
&= \int_{\mathbf{R}} ds \, \mathbf{Q}_m \left[f(Y_s^*) \int_{[s,\infty[} \varphi(t) \, \kappa(dt) \right] \\
&= \int_{\mathbf{R}} ds \, \mathbf{Q}_m \left[f(Y_s^*) \int_{[0,\infty[} \varphi(t+s) \, \kappa(dt+s) \right] \\
&= \int_{\mathbf{R}} ds \, \mathbf{Q}_m \left[f(Y_0^*) \int_{[0,\infty[} \varphi(t+s) \, \kappa(dt) \right] = \mathbf{Q}_m \left[f(Y_0^*) \int_{[0,\infty[} \kappa(dt) \right] \\
&= \mathbf{Q}_m [f(Y_0^*) U_\kappa 1(Y_0^*)] = m(f \cdot U_\kappa 1).
\end{aligned}$$

The next-to-last equality depends on (3.11)(iii); the homogeneity of κ is used for the fourth-from-the-last equality; the use of Fubini's theorem in the fourth- and sixth-from-the-last equalities is justified since κ is an HRM. □

The following corollary is now evident.

(5.15) Corollary. *Let V be an $(m+\rho)$-proper regular strongly supermedian kernel. Then*

$$\int_E f \cdot Vg\, dm = \int_E g \cdot \hat{U}f\, d\mu_V, \qquad f, g \in p\mathcal{E}, \tag{5.16}$$

where $\hat{U}$ is as in (5.12).

(5.17) Remarks. (i) Taking $g \equiv 1$ in (5.16) we find that $\mu_V \hat{U} = V1 \cdot m$, which is the *Revuz formula* in the present context; *cf.* [**GS84**; (9.5)].

(ii) Clearly, both sides of (5.16) are σ-finite measures as functionals of f and g separately. Hence, if $F \in p(\mathcal{E} \otimes \mathcal{E})$, then

$$\int_E \int_E F(x,y)\, V(x,dy)\, m(dx) = \int_E \int_E F(x,y)\, \hat{U}(y,dx)\, \mu_V(dy). \tag{5.18}$$

More generally, if $F \in \mathcal{E} \otimes \mathcal{E}$ and if either side of (5.18) is finite when F is replaced by $|F|$, then (5.18) holds.

(iii) Formula (5.16) with $f \equiv 1$ implies that $mV \ll \mu_V$, and that a version of the Radon-Nikodym derivative is $\hat{U}1$. Since $\hat{U}1 > 0$ off an m-exceptional set, we also have $\mu_V \ll mV$. It is not too difficult to check this measure equivalence directly, but the fact that the Radon-Nikodym derivative does not depend on V is somewhat surprising.

(iv) A more general version of (5.16) appears as Theorem 5.2 in [**BB02**]. In that result the co-potential $\hat{U}f$ is replaced by (an m-fine version of) a general m-a.e. finite co-excessive function $\hat{u}$. Notice that the left side of (5.16) can be interpreted as $L(\pi U, Vg)$, where $\pi := f \cdot m$; the measure potential πU has Radon-Nikodym derivative $\hat{U}f$ with respect to m. (For an excessive measure ξ and a strongly supermedian function u, we follow [**BB01a**] in defining the energy $L(\xi, u)$ as $\sup\{\nu(u) : \nu U \le \xi\}$.) Thus, in the general case, πU is replaced by the excessive measure $\hat{u} \cdot m$.

The final result of this section is a uniqueness theorem for perfect HRMs.

(5.19) Theorem. *Let κ_1 and κ_2 be perfect HRMs with σ-finite characteristic measures μ_{κ_1} and μ_{κ_2}, and potential kernels U_{κ_1} and U_{κ_2}. The following statements are equivalent:*

(i) κ_1 and κ_2 are $\mathbf{Q}_m$-indistinguishable;

(ii) $\mu_{\kappa_1} = \mu_{\kappa_2}$;

(iii) $\{x \in E : U_{\kappa_1}(x,\cdot) \neq U_{\kappa_2}(x,\cdot)\}$ *is m-exceptional;*

(iv) $\{x \in E : U_{\kappa_1}(x,\cdot) \neq U_{\kappa_2}(x,\cdot)\}$ *is $(m+\rho)$-null;*

(v) There exists a strictly positive function $g \in p\mathcal{E}^n$ such that $U_{\kappa_1} g = U_{\kappa_2} g < \infty$ off an m-exceptional set.

(5.20) Remark. Some of the implications in Theorem (5.19) follow from results in [**BB01b**], but we shall give direct proofs that are more probabilistic in character than those found in [**BB01b**].

Proof. (i)$\Longleftrightarrow$(ii). This follows immediately from (3.8)(ii).

(i)$\Longrightarrow$(iii). Fix $f \in p\mathcal{E}$ and note that for $j = 1,2$ the $\mathbf{Q}_m$-optional projection of the process

$$t \mapsto \int_{[t,\infty[} f(Y_s^*)\,\kappa_j(ds) = \left[\int_{[0,\infty[} f(X_s)\,\kappa_j(ds)\right] \circ \theta_t$$

is $U_{\kappa_j} f(Y_t^*)$. Thus, if (i) holds then the $\mathbf{Q}_m$-optional processes $U_{\kappa_1} f(Y^*)$ and $U_{\kappa_2} f(Y^*)$ are $\mathbf{Q}_m$-indistinguishable, so (iii) follows by a monotone class argument because $\mathcal{E}$ is countably generated; see the sentence preceding (3.4).

(iii)$\Longrightarrow$(iv). This is trivial.

(iv)$\Longrightarrow$(ii). If $f \in p\mathcal{E}^n$ then $U_{\kappa_1} f = U_{\kappa_2} f$, $(m+\rho)$-a.e. But m is excessive, so this implies that $\bar{U}_{\kappa_1} f = \bar{U}_{\kappa_2} f$, m-a.e., hence η-a.e. In view of (4.11), this yields $\mu_{\kappa_1}(f) = \mu_{\kappa_2}(f)$.

(iii)$\Longrightarrow$(v). This is an immediate consequence of (4.2) and (5.5).

(v)$\Longrightarrow$(i). Let $E' \subset \{U_{\kappa_1} g = U_{\kappa_2} g < \infty\}$ be a Borel absorbing set with m-exceptional complement, and let X' denote the restriction of X to E'. Then, for $j = 1,2$, $g * \kappa_j$ (restricted to Ω) may be viewed as a RM of X', perfectly homogeneous on $[0,\infty[$, with left potential function $U_{\kappa_j} g$ as in [**Sh88**]. Clearly $A_t^j := \int_{[0,t[} g(X_s')\,\kappa_j(ds)$ defines an optional LAF of X' with left potential function $v_j = U_{\kappa_j} g$. Since $v_1 = v_2 < \infty$ everywhere on E', [**Sh88**; (37.8)] implies that A^1 and A^2 are $\mathbf{P}^x$-indistinguishable for each starting point $x \in E'$. Since $g > 0$, we deduce that the restrictions to Ω of κ_1 and κ_2 are $\mathbf{P}^x$-indistinguishable for each $x \in E'$. Now define

$$\Omega_0 := \{w \in \Omega : \alpha(w) = 0, \kappa_1(w,B) \neq \kappa_2(w,B) \text{ for some } B \in \mathcal{B}\},$$

and (recalling that κ_1 and κ_2 are carried by Λ^*) observe that

$$\{w \in W : \kappa_1(w) \neq \kappa_2(w)\} \subset \left[\cup_{r \in \mathbf{Q}} \{\alpha < r < \beta\} \cap \theta_r^{-1}\Omega_0\right] \cup \left[\{Y_\alpha^* \in E\} \cap \theta_\alpha^{-1}\Omega_0\right].$$

We have, because $E \setminus E'$ is $(m+\rho)$-null,

$$\mathbf{Q}_m[\theta_r^{-1}\Omega_0; \alpha < r < \beta] = \mathbf{P}^m[\Omega_0] = 0,$$

and

$$\mathbf{Q}_m[\theta_\alpha^{-1}\Omega_0; Y_\alpha^* \in E, u < \alpha < v] = (v-u)\mathbf{P}^\rho[\Omega_0] = 0,$$

the second display following from (2.4). This proves (i). $\square$

(5.21) Remark. (a) There is an analogous uniqueness theorem for co-natural HRMs with σ-finite characteristic measures. It reads exactly like Theorem (5.19) except that "m-exceptional" is replaced by "m-polar" in (iii) and (v), and $(m+\rho)$-null is replaced by m-null in (iv). We leave the details to the reader.

(b) One consequence of Theorem (5.19) is this: A perfect HRM κ with σ-finite characteristic measure is $\mathbf{Q}_m$-indistinguishable from a perfect HRM κ' whose potential kernel is proper. Indeed, if μ_κ is σ-finite then by (4.10) and (5.7) the potential kernel U_κ is $(m+\rho)$-proper. Thus, by (5.5)(a) there is a strictly positive Borel function g and an m-exceptional set N such that $U_\kappa g \le 1$ off N. We can (and do) assume that $N \in \mathcal{E}$ and that $E \setminus N$ is absorbing. The desired modification of κ is then $\kappa'(dt) := 1_{E\setminus N}(Y_t^*)\,\kappa(dt)$. To see this observe that $U_{\kappa'}g(x) = U_\kappa(1_{E\setminus N}g)(x) = U_\kappa g(x) \le 1$ if $x \in E \setminus N$, hence $U_{\kappa'}g \le 1$ on all of E because U_κ is a regular strongly supermedian kernel.

6. Additive Functionals.

It is often important to describe an HRM in terms of the associated distribution function, at least under appropriate finiteness conditions. Thus we are led to the concept of *additive functional* (AF). Our definitions are essentially those given in [**Sh88**], except that we allow an exceptional set of starting points; *cf.* [**Sh71**] and [**FOT94**; Chap. 5]. Throughout this section we work with a fixed $m \in \mathrm{Exc}$ with Riesz decomposition $m = \eta + \rho U$ into harmonic and potential components. As in the last section, we assume in this section that X is transient, meaning that U is a proper kernel: There exists $b \in p\mathcal{E}$ with $0 < b \le 1$, $m(b) < \infty$, and $Ub \le 1$.

Recall that a nearly Borel set N is *m-inessential* provided it is m-polar and $E \setminus N$ is absorbing for X. By [**GS84**; (6.12)], any m-polar set is contained in a Borel m-inessential set. We shall say that a nearly Borel set N is *strongly m-inessential* provided $N \in \mathcal{N}(m)$ and $E \setminus N$ is absorbing. As noted just before (5.2), if G is a finely open m-null set then $G \in \mathcal{N}(m)$. Using this observation, the proof of [**GS84**; (6.12)] is easily modified to show that any set in $\mathcal{N}(m)$ is contained in a Borel strongly m-inessential set.

(6.1) Definition. A *positive left additive functional* (PLAF) is an $(\mathcal{F}_t)$-adapted increasing process $A = (A_t)_{t\ge 0}$ with values in $[0,\infty]$, for which there exist a *defining set* $\Omega_A \in \mathcal{F}$ and a strongly m-inessential Borel set N_A (called an *exceptional set* for A) such that

(i) $\mathbf{P}^x[\Omega_A] = 1$ for all $x \notin N_A$;
(ii) $\theta_t \Omega_A \subset \Omega_A$ for all $t \geq 0$;
(iii) For all $\omega \in \Omega_A$ the mapping $t \mapsto A_t(\omega)$ is left-continuous on $]0, \infty[$, finite valued on $[0, \zeta(\omega)[$, with $A_0(\omega) = 0$;
(iv) For all $\omega \in \Omega_A$, $A_{0+}(\omega) = a(X_0(\omega))$, where $a \in p\mathcal{E}^n$;
(v) For all $\omega \in \Omega_A$ and $s, t \geq 0$: $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$;
(vi) For all $t \geq 0$, $A_t([\Delta]) = 0$.

(6.2) Definition. Two PLAFs A and B are *m-equivalent* provided $\mathbf{P}^{m+\rho}[A_t \neq B_t] = 0$ for all $t \geq 0$.

(6.3) Remarks. (a) Observe that if A is a PLAF then on Ω_A we have $A_t = A_\zeta$ for all $t > \zeta$, and that $A_\zeta = A_{\zeta-}$.

(b) Let κ be a perfect HRM, and define $A_t := \kappa[0, t[$, $t \geq 0$, and $S := \inf\{t : A_t = \infty\}$. Suppose that $\mathbf{P}^{m+\rho}[S < \zeta] = 0$. Then $A = (A_t)_{t \geq 0}$ is a PLAF with defining set $\Omega_A := \{S \geq \zeta\}$ and exceptional set $N_A := \{x : \mathbf{P}^x[S < \zeta] > 0\}$. See Theorem (6.21), where it is proved that N_A just defined is strongly m-inessential and that the construction just given yields the most general PLAF.

The following improvement of property (iv) above is useful.

(6.4) Lemma. *Let A be a PLAF. Then the defining set Ω_A and exceptional set N_A for A can be modified so that the function a in (iv) satisfies: $\{a > 0\}$ is semipolar.*

Proof. From the definition, we have $\Delta A_0 = A_{0+} = a(X_0)$ on Ω_A, where $a \in p\mathcal{E}^n$. Therefore $\Delta A_t = (\Delta A_0) \circ \theta_t = a(X_t)$ on Ω_A, for all $t \geq 0$. Writing $A = A^c + A^d$, where A^c is the continuous and A^d the discontinuous part of A, we have, on Ω_A,

$$A_t^d = \sum_{0 \leq s < t} a(X_s) < \infty, \qquad t \in [0, \zeta[.$$

Thus, if $x \notin N_A$, then $\{t : a(X_t) > 0\}$ is countable, $\mathbf{P}^x$-a.s. Hence $\{a > 0\}$ is m-semipolar, and by (3.13) we can write $\{a > 0\} = S_a \cup N_a$, where S_a is semipolar and $N_a \in \mathcal{N}(m)$. Because $\{a > 0\} \in \mathcal{E}^n$, the argument used in proving (3.13) shows that S_a and N_a can be chosen nearly Borel as well; this we assume done. Now let M be a Borel strongly m-inessential set containing N_a. Define $N^* = N_A \cap M$. Let $D := \inf\{t \geq 0 : X_t \in N^*\}$ be the début of N^*. Then $\mathbf{P}^x[D < \infty] = 0$ for $x \notin N^*$. Define $\Omega_A^* = \Omega_A \cap \{D = \infty\}$ and $a^* = a \cdot 1_{E \setminus N^*}$. It is evident that N^* and Ω_A^* serve as exceptional and defining sets for A, that $\{a^* > 0\}$ is semipolar, and that $A_0 = a^*(X_0)$ on Ω_A^*. □

Theorem (3.18) motivates the following

(6.5) Definition. A *positive co-natural additive functional* (PcNAF) is an $(\mathcal{F}_t)$-adapted increasing process $A = (A_t)_{t\geq 0}$ with values in $[0,\infty]$, for which there exist a *defining set* $\Omega_A \in \mathcal{F}$ and an m-inessential Borel set N_A (called an *exceptional set* for A) such that, in addition to conditions (6.1)(i)(ii)(v)(vi), the following modifications of (6.1)(iii)(iv) hold:

(iii) For all $\omega \in \Omega_A$ the mapping $t \mapsto A_t(\omega)$ is right-continuous on $[0,\infty[$, finite valued on $[0,\zeta(\omega)[$, with $A_0(\omega) = 0$;

(iv) For all $\omega \in \Omega_A$ and all $t > 0$, $\Delta A_t(\omega) := A_t(\omega) - A_{t-}(\omega) = a(X_t(\omega))$, where $a \in p\mathcal{E}^n$;

(6.6) Definition. Two PcNAFs A and B are *m-equivalent* provided $\mathbf{P}^m[A_t \neq B_t] = 0$ for all $t \geq 0$.

(6.7) Remarks. (a) If A is a PcNAF then $A_\zeta = A_{\zeta-}$.

(b) A PcNAF A can be represented as

$$A_t(\omega) = A_t^c(\omega) + \sum_{0<s\leq t} a(X_s(\omega)), \qquad t \in [0,\zeta(\omega)[,\ \omega \in \Omega_A,$$

where A^c is a PCAF (as in [**FG96**]) and $a \in p\mathcal{E}^n$ with $\{a > 0\}$ semipolar.

(c) If A is a PLAF then $B_t := A_{t+} - A_{0+}$ defines a PcNAF (with the same defining and exceptional sets); the potential kernels of A and B, defined in (6.8) below, are related by $U_B f = \overline{U_A f}$.

(d) Let γ be a co-natural HRM, and define $A_t := \lim_n \gamma]0, t + 1/n]$, $t \geq 0$, and $S := \inf\{t : A_t = \infty\}$. Suppose that $\mathbf{P}^m[S < \zeta] = 0$. Then $A = (A_t)_{t\geq 0}$ is a PcNAF with $\Omega_A = \{S \geq \zeta\}$ and $N_A = \{x : \mathbf{P}^x[S < \zeta] > 0\}$. As with PLAFs, it will be clear from the the proof of Theorem (6.21) that N_A is m-inessential and that this example is the most general PcNAF.

We now define the potential kernel of an AF; this notion will play an important role in the sequel.

(6.8) Definition. Let A be a PLAF or a PcNAF. The *potential kernel* U_A of A is defined for $f \in \mathcal{E}^n$ by

$$U_A f(x) := \mathbf{P}^x \int_{[0,\zeta[} f(X_t)\, dA_t, \qquad x \notin N_A, \tag{6.9}$$

where N_A is an exceptional set for A. In particular, $u_A := U_A 1$ is the *potential function* of A.

(6.10) Remarks. Note that $U_A f$ is undefined on N_A. If A is a PcNAF, then dA_t does not charge $\{0\}$, so the integral in (6.9) is really over the open interval $]0,\zeta[$. Since $f(\Delta) = 0$ and $\Delta A_\zeta = 0$, one could just as well integrate over $[0,\infty[$ (or $]0,\infty[$ in case of a PcNAF).

If $f \geq 0$ we define $(f * A)_t := \int_{[0,t[} f(X_s)\, dA_s$ (resp. $(f * A)_t := \int_{]0,t]} f(X_s)\, dA_s$) when A is a PLAF (resp. PcNAF). If, for all $\omega \in \Omega_A$, $(f * A)_t(\omega) < \infty$ for all $t \in [0, \zeta(\omega)[$, then $f * A$ is a PLAF (resp. PcNAF) provided A is a PLAF (resp. PcNAF). Of course the finiteness condition just mentioned is satisfied if f is bounded. Finally, notice that $U_{f*A}(g) = U_A(fg)$.

When A is a PLAF or a PcNAF with exceptional set N_A, it will be convenient to let X^A denote the restriction of X to the absorbing set $E \setminus N_A$. In fact, using this device one can often reduce matters to the situation in which N_A is empty. If A is a PcNAF, one checks easily that $U_A f$ is excessive for X^A, and hence nearly Borel on $E \setminus N_A$. If A is a PLAF, then $U_A f = af + U_{\bar{A}} f$, where $\bar{A}_t := A_{t+} - A_{0+}$ is a PcNAF with the same exceptional set as A; see Remark (6.7)(d). Because $U_{\bar{A}} f$ is excessive for X^A, if $f \geq 0$ is nearly Borel on $E \setminus N_A$ then so is $U_A f$. It is now clear that $U_A f$ is strongly supermedian and that $U_{\bar{A}} f$ is its excessive regularization. Of course, both of these statements must be understood relative to X^A. Henceforth we shall omit such qualification when it is clear from the context that we are referring to X^A.

For the following definition, which is motivated by Theorem (4.10), recall the Riesz decomposition $m = \eta + \rho U$.

(6.11) Definition. Let A be a PLAF or a PcNAF. The characteristic measure μ_A of A (relative to $m \in \mathrm{Exc}$) is defined by

$$\mu_A(f) := \rho U_A f + L(\eta, \bar{U}_A f), \qquad f \in p\mathcal{E}, \tag{6.12}$$

where $\bar{U}_A f$ is the excessive regularization of $U_A f$. (If A is a PcNAF then $\bar{U}_A f = U_A f$.)

Observe that if f is such that $f * A$ is a PLAF or a PcNAF, then $\mu_{f*A} = f \cdot \mu_A$; in particular, $\mu_{f*A}(1) = \mu_A(f)$.

Definition (6.11) requires some justification since $U_A f$ and $\bar{U}_A f$ are only defined on $E \setminus N_A$. Thus, the energy functional L appearing in (6.12) should be regarded as the energy functional L^A of X^A, at least formally. But if $\nu U \leq \eta$ then ν doesn't charge N_A, so $\nu U = \nu U^A$, where U^A is the potential kernel for X^A. Hence $L^A(\eta, \bar{U}_A f) = \sup\{\nu \bar{U}_A f : \nu U \leq \eta\}$. Also, because $\rho U \leq m$, ρ charges no set in $\mathcal{N}(m)$. Since $N_A \in \mathcal{N}(m)$, $\rho U_A f$ is well defined. Thus (6.12) is justified. If A is a PcNAF then (6.12) reduces to $\mu_A(f) = L(m, U_A f)$.

If A is a PcNAF and F is m-polar, then $U_A 1_F = 0$, m-a.e., and so $\mu_A(F) = L(m, U_A 1_F) = 0$. Thus μ_A charges no m-polar set. If A is a PLAF, and $F \in \mathcal{N}(m)$, then $\bar{U}_A 1_F = 0$, m-a.e., and hence η-a.e. Therefore $L(\eta, \bar{U}_A 1_F) = 0$. Moreover, combining the formula [**G90**; (6.20)] for ρ with the strong Markov property we obtain

$$\rho U_A 1_F = \mathbf{Q}_m \left[\left(\int_{[0,\zeta[} 1_F(X_t)\, dA_t \right) \circ \theta_\alpha ; 0 < \alpha \leq 1, W(h) \right] = 0,$$

since $X_t \circ \theta_\alpha = Y^*_{\alpha+t}$ on $W(h)$ for $t \geq 0$, and $\{Y^* \in F\}$ is $\mathbf{Q}_m$-evanescent (because $F \in \mathcal{N}(m)$). Thus μ_A charges no m-exceptional set when A is a PLAF.

(6.13) Definition. A *nest* is an increasing sequence (B_n) of nearly Borel sets such that $\mathbf{P}^m[\lim_n \tau(B_n) < \zeta] = 0$, where $\tau(B) := \inf\{t > 0 : X_t \notin B\}$ denotes the exit time from $B \in \mathcal{E}^n$. A nest (B_n) is a *strong nest* if $\mathbf{P}^{m+\rho}[\lim_n \tau(B_n) < \zeta] = 0$.

(6.14) Definition. A measure ν on $(E, \mathcal{E})$ is smooth (resp. strongly smooth) provided it charges no m-polar set (resp. no element of $\mathcal{N}(m)$) and there is a nest (resp. strong nest) (G_n) of finely open nearly Borel sets such that $\mu(G_n) < \infty$ for all n.

If (G_n) is a nest (resp. strong nest), then $E \setminus \cup_n G_n$ is m-polar (resp. m-exceptional). Thus, a smooth or strongly smooth measure is necessarily σ-finite. If a smooth measure μ charges no m-semipolar set then it is smooth as the term was defined in [**FG96**], where it was proved that such a μ is the characteristic measure of a uniquely determined PCAF A. Below we develop similar results for PLAFs and PcNAFs. We have already seen that μ_A charges no m-polar set (resp. no element of $\mathcal{N}(m)$) if A is a PcNAF (resp. PLAF).

We need to extend the notion of regularity, introduced in section 5 for strongly supermedian functions, as follows: If $E' \in \mathcal{E}^n$ is an absorbing set, then we say that $f \in b\mathcal{E}^n$ is *regular on* E' provided $\lim_n \mathbf{P}^x[f(X_{T_n})] = \mathbf{P}^x[f(X_T)]$ for all $x \in E'$ and every increasing sequence (T_n) of stopping times with limit T.

(6.15) Theorem. *If A is a PcNAF (resp. PLAF) then μ_A is smooth (resp. strongly smooth). Moreover, there exists a finely lower-semicontinuous nearly Borel function g defined on $E \setminus N_A$ such that (i) $0 < g \leq 1$ on $E \setminus N_A$, (ii) $U_A g \leq 1$ on $E \setminus N_A$, and (iii) g is regular on $E \setminus N_A$.*

Proof. Suppose first that A is a PLAF, with exceptional set $N = N_A$ and defining set Ω_A. On Ω_A let M be the Stieltjes exponential of A; that is,

$$M_t(\omega) = e^{A^c_t(\omega)} \prod_{0 \leq s < t} (1 + \Delta A_s(\omega)), \qquad t < \zeta(\omega), \omega \in \Omega_A, \tag{6.16}$$

where A^c is the continuous part of A and $\Delta A_s := A_{s+} - A_s$ for $s \geq 0$. On Ω_A we have $\sum_{0 \leq s < t} \Delta A_s \leq A_t < \infty$ if $t \in [0, \zeta[$, so M is well defined, increasing, and finite on $[0, \zeta[$, left continuous on $]0, \zeta[$, with $M_0 = 1$, $M_{0+} = 1 + \Delta A_0 = 1 + A_{0+}$. Clearly $M_{t+s} = M_s M_t \circ \theta_s$

for $s, t \geq 0$. Moreover, $\{t : M_{t+} \neq M_t\}$ is countable $\mathbf{P}^x$-a.s. for $x \in E \setminus N$. The reader should keep in mind that M is defined only on Ω_A. Evidently,

$$dM_t = M_t\, dA_t, \qquad t \in [0, \zeta[. \tag{6.17}$$

In addition, $M_t \cdot 1/M_t = 1$, so $M_{t+}d(1/M_t) + (1/M_t)dM_t = 0$, or

$$d\left(\frac{1}{M_t}\right) = -\frac{dM_t}{M_{t+}M_t} = -\frac{1}{M_{t+}}dA_t. \tag{6.18}$$

Recall that $b \in bp\mathcal{E}$ with $0 < b \leq 1$, $m(b) < \infty$, and $Ub \leq 1$. Define, for $x \notin N$,

$$g(x) := \mathbf{P}^x \int_{[0,\zeta[} M_t^{-1} b(X_t)\, dt = \mathbf{P}^x \int_{[0,\zeta[} M_{t+}^{-1} b(X_t)\, dt. \tag{6.19}$$

Clearly $g > 0$ on $E \setminus N$, and, for $x \notin N$,

$$\begin{aligned}
U_A g(x) &= \mathbf{P}^x \int_{[0,\zeta[} g(X_t)\, dA_t = \mathbf{P}^x \int_{[0,\zeta[} dA_t \int_{[0,\zeta\circ\theta_t[} (M_s\circ\theta_t)^{-1} b(X_s\circ\theta_t)\, ds \\
&= \mathbf{P}^x \int_{[0,\zeta[} dA_t \int_{[t,\zeta[} M_t (M_s)^{-1} b(X_s)\, ds = \mathbf{P}^x \int_{[0,\zeta[} ds M_s^{-1} b(X_s) \int_{[0,s]} M_t\, dA_t \\
&= \mathbf{P}^x \int_{[0,\zeta[} M_s^{-1}(M_s - 1) b(X_s)\, ds = \mathbf{P}^x \int_{[0,\zeta[} (1 - M_s^{-1}) b(X_s)\, ds \\
&= Ub(x) - g(x),
\end{aligned}$$

where the fifth equality comes from (6.17). Thus $g + U_A g = Ub$ on $E \setminus N$; because $Ub \leq 1$ and $g \geq 0$, we have $U_A g \leq 1$ on $E \setminus N$. Since $M_{0+} = 1 + \Delta A_0 = 1 + a(X_0)$, where a comes from Definition (6.1)(iv),

$$g(x) = \frac{1}{1 + a(x)} \mathbf{P}^x \int_{[0,\zeta[} M_{0+} M_{t+}^{-1} b(X_t)\, dt. \tag{6.20}$$

Let $L_t := M_{0+}M_{t+}^{-1}$. Then $L = (L_t)_{t\geq 0}$ is a right-continuous, decreasing multiplicative functional of X^A with $L_0 = 1$. Hence L is exact and the expectation in (6.20) is excessive with respect to (X^A, L)—the L subprocess of X^A. In particular, g is nearly Borel measurable. If T is a stopping time then

$$\mathbf{P}^x[g(X_T)] = \mathbf{P}^x \left[M_T \int_{[T,\zeta[} M_t^{-1} b(X_t)\, dt \right],$$

from which it follows that g is regular and finely lower-semicontinuous on $E \setminus N$ (by Theorem 4.9 in [**Dy65**], since $M_{0+} \geq 0$). Thus the sets $G_n := \{g > 1/n\}$, $n \geq 1$, form an increasing sequence of finely open, nearly Borel, subsets of $E \setminus N$. Let τ_n be the exit time of G_n. Since G_n^c is finely closed, $g(X_{\tau_n}) \le 1/n$, $\mathbf{P}^x$-a.s. for $x \in E \setminus N$. Thus, for such x,

$$1/n \ge \mathbf{P}^x[g(X_{\tau_n})] = \mathbf{P}^x \left[M_{\tau_n} \int_{]\tau_n, \zeta[} M_t^{-1} b(X_t)\, dt \right] \ge \mathbf{P}^x \int_{]\tau_n, \zeta[} M_t^{-1} b(X_t)\, dt.$$

But $b > 0$, and $M_t < \infty$ on $[0, \zeta[$ ($\mathbf{P}^x$-a.s.), so we must have $\lim_n \tau_n = \zeta$, $\mathbf{P}^x$-a.s. for all $x \in E \setminus N$. Since N is strongly m-inessential, (G_n) is a strong nest. Finally, $U_A 1_{G_n} \le nU_A g \le nUb$ on $E \setminus N$, and so

$$\begin{aligned} \mu_A(G_n) &= \rho U_A 1_{G_n} + L(\eta, \bar{U}_A 1_{G_n}) \le n\, [\rho Ub + L(\eta, Ub)] \\ &= nL(m, Ub) = n \cdot m(b) < \infty. \end{aligned}$$

This establishes (6.15) when A is a PLAF.

Now consider the case in which A is a PcNAF. Define a PLAF A^* by $A_t^* := A_t^c + \sum_{0 \le s < t} a(X_s)$, where a comes from Definition (6.5)(iv). Then $A_t = A_{t+}^* - A_0^*$. Define M as in (6.16) with A replaced by A^*, and define g as in (6.19). As before, g is nearly Borel measurable, finely lower-semicontinuous, and regular. In the present case the computation just below (6.19) yields

$$\begin{aligned} U_A g(x) &= \mathbf{P}^x \int_{[0,\zeta[} \int_{]0,s]} M_t\, dA_t^*\, M_s^{-1} b(X_s)\, ds \\ &\le \mathbf{P}^x \int_{[0,\zeta[} \int_{[0,s]} M_t\, dA_t^*\, M_s^{-1} b(X_s)\, ds \\ &= Ub(x) - g(x) \end{aligned}$$

for $x \in E \setminus N$. Hence $g \le Ub$ and $U_A g \le Ub \le 1$ on $E \setminus N$. Just as for PLAFs, the sequence defined by $G_n := \{g > 1/n\}$ is a nest (recall that N is m-inessential) with $\mu_A(G_n) < \infty$ for each n. This completes the proof of Theorem (6.15). ∎

The next result is the fundamental existence theorem for AFs. It is essentially the converse of Theorem (6.15). The accompanying uniqueness result is (6.29).

(6.21) Theorem. *Let μ be a strongly smooth (resp. smooth) measure. Then there exists a PLAF (resp. PcNAF) A with characteristic measure μ. Moreover, in either case, there exists a Borel function $j \ge 0$ with $\{j > 0\}$ semipolar such that $\Delta A \equiv j \circ X$.*

Proof. Suppose first that μ is strongly smooth. Clearly $\mu \in \mathcal{S}_0^{\#}(m)$, so there is an HRM κ associated with μ as in (3.11). Let κ_Ω denote the restriction of κ to Ω. If $f \in p\mathcal{E}$ and

$t \geq 0$ then

$$\begin{aligned}
t \cdot \mu(f) &= \mathbf{Q}_m \int_{[0,t[} f(Y_s^*)\, \kappa(ds) \\
&\geq \mathbf{Q}_m \left[\int_{[0,t[} f(Y_s^*)\, \kappa(ds); \alpha < 0 < \beta \right] \\
&\geq \mathbf{Q}_m \left[\mathbf{P}^{Y(0)} \left[\int_{[0,t[} f(X_s)\, \kappa_\Omega(ds) \right]; \alpha < 0 < \beta \right] \\
&= \mathbf{P}^m \int_{[0,t[} f(X_s)\, \kappa_\Omega(ds).
\end{aligned}$$

In addition, from (4.10), we have

$$\mu(f) \geq \rho U_{\kappa_\Omega}(f) = \mathbf{P}^\rho \int_{[0,\zeta[} f(X_s)\, \kappa_\Omega(ds) \geq \mathbf{P}^\rho \int_{[0,t[} f(X_s)\, \kappa_\Omega(ds).$$

Combining these estimates we have, for $G \in \mathcal{E}$,

$$\mathbf{P}^{m+\rho} \int_{[0,t[} 1_G(X_s)\, \kappa_\Omega(ds) \leq (t+1)\mu(G). \tag{6.22}$$

Now let (G_n) be a strong nest with $\mu(G_n) < \infty$ for all n. Then (6.22) implies

$$\mathbf{P}^{m+\rho} \left[\kappa_\Omega[0,t[; t < \tau(G_n) \right] \leq (t+1)\mu(G_n) < \infty. \tag{6.23}$$

Define $A_t := \kappa_\Omega[0,t[$, $t \geq 0$. In view of the properties of κ in (3.11) one has, on all of Ω, (i) $A_0 = 0$, (ii) $t \mapsto A_t$ is left continuous on $]0,\infty[$ and increasing on $[0,\infty[$, and (iii) $A_{t+s} = A_t + A_s \circ \theta_t$ for all $s,t \geq 0$. Also, A_t is measurable over $\mathcal{F}^*_{[0,t[} \subset \mathcal{F}^*_t = \mathcal{F}^*_{[0,t]}$. (See the second paragraph of section 4 for notation.) Let $S := \inf\{t : A_t = \infty\}$. It is evident that S is an $(\mathcal{F}^*_t)$-stopping time and that $S = t + S \circ \theta_t$ on $\{S > t\}$. Now (6.23) implies that $A_t < \infty$ on $[0,\tau(G_n)[$, $\mathbf{P}^{m+\rho}$-a.s. But $P^{m+\rho}[\lim_n \tau(G_n) < \zeta] = 0$ since (G_n) is a strong nest, hence $\mathbf{P}^{m+\rho}[S < \zeta] = 0$. Therefore the set $\tilde{N} := \{x : \mathbf{P}^x[S < \zeta] > 0\}$ is $(m+\rho)$-null. We need the following lemma, which we shall prove after using it to complete the proof of the theorem.

(6.24) Lemma. *The function h defined on E by $h(x) := \mathbf{P}^x[S < \zeta]$ is strongly supermedian.*

Since $\tilde{N} = \{h > 0\}$ and $m(\tilde{N}) = 0$, it follows from [**FG96**; Lem. 2.1] that $\tilde{N}$ is m-polar. Hence $\tilde{N} \in \mathcal{N}(m)$. Let N be a strongly m-inessential Borel set containing $\tilde{N}$. It is now easy to see that A is a PLAF with defining set $\{S \geq \zeta\}$ and exceptional set N. The form of ΔA comes from (3.12).

Suppose next that μ is smooth. Because μ is a σ-finite measure charging no m-polar set, Theorem (3.16) guarantees the existence of a co-natural HRM κ with characteristic measure μ. In this case $\kappa_\Omega(\{0\}) = 0$. Let (G_n) be a nest of finely open sets with $\mu(G_n) < \infty$. Dropping the subscript Ω from κ, it follows just as before that

$$\mathbf{P}^m[\kappa]0,t]; t < \tau(G_n)] \leq t\mu(G_n) < \infty.$$

This time we define $A_t := \lim_{n\to\infty} \kappa]0, t+1/n]$, and $S := \inf\{t : A_t = \infty\} = \inf\{t : \kappa]0,t] = \infty\}$. Arguing exactly as in the proof of [**G95**; Prop. 4.3], one see that A is an adapted, right-continuous increasing process with $A_{t+s} = A_t + A_s \circ \theta_t$ for $s, t \geq 0$, and that A is exact in the sense that

$$\lim_{s\downarrow 0} A_{t-s} \circ \theta_s = A_t, \qquad \forall t > 0. \tag{6.25}$$

All of these statements hold identically on Ω. (The diffuseness hypothesis imposed in [**G95**; (4.3)] is not used in the proof of the above statements. The sentence "If ... cases" on line -13, page 86 of [**G95**] should be deleted.) Clearly $A_t = \kappa]0,t]$ if $t < S$, and $A_0 = 0$ on $\{S > 0\}$. Just as in the PLAF case, we find that $\mathbf{P}^m[S < \zeta] = 0$ and that $\tilde{N} := \{x \in E : \mathbf{P}^x[S < \zeta] > 0\}$ is m-polar. Thus, A is a PcNAF with defining set $\{S \geq \zeta\}$ and exceptional set a Borel m-inessential set N containing $\tilde{N}$. Invoking (3.18) one sees that $A_t^d = \sum_{0<s\leq t} j(X_s)$, where $j \in p\mathcal{E}$ and $\{j > 0\}$ is semipolar. Hence A is a PcNAF with $\mu_A = \mu$. ☐

It remains to prove Lemma (6.24). So let h and S be as there. Since $\kappa(\{0\}) = j(X_0) < \infty$,

$$S = \inf\{t : \kappa]0,t[= \infty\} = \inf\{t : \kappa]0,t] = \infty\},$$

the second equality being easily verified. Define $B_t := \lim_{n\to\infty} \kappa]0,t+1/n]$. Then, as in [**G95**; (4.3)], B is exact; *i.e.*, it satisfies (6.25). Clearly $S = \inf\{t : B_t = \infty\}$. Now the argument at the top of page 91 of [**G95**] shows that $h(x) := \mathbf{P}^x[S < \zeta]$ is $\mathcal{E}^e$-measurable, where $\mathcal{E}^e$ is the σ-algebra generated by the 1-excessive functions of X. Since X is a Borel right process, we have $\mathcal{E}^e \subset \mathcal{E}^n$. It is evident that $P_T h \leq h$ for all stopping times T, and so h is strongly supermedian. ☐

(6.26) Remarks. The AFs constructed in the proof of (6.21) have better properties than required by the definitions. The shift property (6.1)(v) holds for all $\omega \in \Omega$ and the "jump" function j is Borel measurable with $\{j > 0\}$ semipolar. Also, the PcNAF produced is exact. The PLAF constructed is $(\mathcal{F}^*_{t-})$-adapted and the PcNAF is $(\mathcal{F}^*_{t+})$-adapted.

Here is the analog of (6.21) for regular strongly supermedian kernels.

(6.27) Theorem. *(a) Let V be a regular strongly supermedian kernel and suppose there are a strongly m-inessential set N and a finely lower-semicontinuous function $g : E \setminus N \to]0,1]$ that is regular on $E \setminus N$, such that $Vg \le 1$ on $E \setminus N$. Then there is a unique PLAF A with exceptional set $N_A \supset N$ such that $U_A(x,\cdot) = V(x,\cdot)$ for all $x \notin N_A$.*

(b) Let V be a regular strongly supermedian kernel and define a semi-regular excessive kernel W by $Wf := \overline{Vf}$ (excessive regularization). Suppose there are an m-inessential set N and a finely lower-semicontinuous function $g : E \setminus N \to]0,1]$ that is regular on $E \setminus N$, such that $Wg \le 1$ on $E \setminus N$. Then there is a unique PcNAF A with exceptional set $N_A \supset N$ such that $U_A(x,\cdot) = W(x,\cdot)$ for all $x \notin N_A$.

Proof. (a) It is clear that V is $(m+\rho)$-proper, so by Theorem (5.8) there is a unique perfect HRM κ such that $\{x \in E \setminus N : U_\kappa(x,\cdot) \ne V(x,\cdot)\}$ is contained in a strongly m-inessential set $N_0 \supset N$. Define $A_t := \kappa[0,t[$, $t \ge 0$, and $S := \inf\{t : A_t = \infty\}$. As in the proof of Theorem (6.21), we will be done once we show that $\mathbf{P}^{m+\rho}[S < \zeta] = 0$. To this end observe that because g is regular and strictly positive on $E \setminus N_0$, we have

$$\inf_{0 \le s \le t} g(X_s) > 0 \qquad \text{on } \{t < \zeta\}, \quad \mathbf{P}^x\text{-a.s.}$$

for each $x \in E \setminus N_0$ and each $t > 0$. But for $x \in E \setminus N_0$,

$$\begin{aligned} 1 \ge Vg(x) &= \mathbf{P}^x \int_{[0,\zeta[} g(X_s)\,\kappa(ds) \\ &\ge \mathbf{P}^x \left[\int_{[0,t[} g(X_s)\,\kappa(ds); t < \zeta \right] \\ &\ge \mathbf{P}^x \left[\inf_{0 \le s < t} g(X_s) \cdot \kappa[0,t[; t < \zeta \right], \end{aligned}$$

from which it follows that $\kappa[0,t[< \infty$, $\mathbf{P}^x$-a.s. on $\{t < \zeta\}$ for each $x \in E \setminus N_0$ and each $t > 0$. This is more than enough to imply that $\mathbf{P}^{m+\rho}[S < \zeta] = 0$.

(b) The proof of this assertion is quite similar to that of part (a), so we omit it. □

(6.28) Remark. The smoothness conditions appearing in Theorems (6.21) and (6.27) are comparable. Thus, if μ is a strongly smooth element of $\mathcal{S}_0^{\#}(m)$ then by (6.15) and (6.21) there is a strictly positive function g that is finely lower semicontinuous and regular on an absorbing set E' with $E \setminus E' \in \mathcal{N}(m)$, such that $\mu(g) < \infty$; *cf.* [**FG96**; §5]. On the other hand, if V is a regular strongly supermedian kernel satisfying the hypothesis of part (a) of Theorem (6.27), then $G_n := \{g > 1/n\}$, $n \ge 1$, defines a strong nest such that, for each n, $V(1_{G_n}) \le n$ off an m-exceptional set. This should be compared to the notion *smooth kernel* used in [**BB01a**]; see especially Theorem 2.1 in [**BB01a**].

The following uniqueness result parallels (5.19).

(6.29) Theorem. *Let A and B be PLAFs (resp. PcNAFs) with characteristic measures μ_A and μ_B, and potential kernels U_A and U_B. The following are equivalent:*

(i) *For all $t \geq 0$, $\mathbf{P}^{m+\rho}(A_t \neq B_t) = 0$ (resp. $\mathbf{P}^m(A_t \neq B_t) = 0$);*

(ii) *$\mu_A = \mu_B$;*

(iii) *$\{x \in E : U_A(x, \cdot) \neq U_B(x, \cdot)\}$ is m-exceptional (resp. m-polar);*

(iv) *$\{x \in E : U_A(x, \cdot) \neq U_B(x, \cdot)\}$ is $(m + \rho)$-null (resp. m-null);*

(v) *There exists a strictly positive function $g \in p\mathcal{E}^n$ such that $U_A g = U_B g < \infty$ off an m-exceptional (resp. m-polar) set.*

(6.30) Remark. Since A and B are left-continuous on $]0, \infty[$ (resp. right-continuous on $[0, \infty[$) $\mathbf{P}^x$-a.s. for $x \notin N_A \cup N_B$, condition (i) is equivalent to

(i') A and B are $\mathbf{P}^{m+\rho}$-indistinguishable (resp. $\mathbf{P}^m$-indistinguishable).

Theorem (6.29) is a direct consequence of Theorem (5.19) and the following result that links the notions of HRM and PLAF (or PcNAF). The reader is invited to extract a proof of Proposition (6.31) from the proof of Theorem (5.8) found in the appendix.

(6.31) Proposition. *Let A be a PLAF (resp. PcNAF) with characteristic measure μ_A and potential kernel U_A. Then there is a unique perfect HRM (resp. co-natural HRM) κ with $\mu_\kappa = \mu_A$ and $U_\kappa(x, \cdot) = U_A(x, \cdot)$ for all x outside an m-exceptional (resp. m-polar) set.*

We end this section with a brief discussion of the *fine support* of a PLAF and of its associated potential function. Let V be a regular strongly supermedian kernel such that $v := V1 < \infty$. Then by Theorem (6.27) and its proof there is a (unique) PLAF A with empty exceptional set such that $V = U_A$. Recall from Section 5 that $\nu \dashv \mu$ (balayage order) provided $\nu U \leq \mu U$. Because X is transient, we have $\nu \dashv \mu$ if and only if $\nu(u) \leq \mu(u)$ for every excessive function u. We follow Feyel [**Fe83**] in defining the *fine support* $\delta(v)$ of the regular strongly supermedian function v as in the theory of Choquet boundaries: $\delta(v)$ is the set of points $x \in E$ such that the only measure ν on E with $\nu \dashv \epsilon_x$ and $\nu(v) = v(x)$ is ϵ_x itself. Recall the notation $H_B f(x) := \mathbf{P}^x[f(X_{D_B})]$.

(6.32) Proposition. *(a) $\delta(v)$ is a finely closed element of $\mathcal{E}^n$;*

(b) $H_{\delta(v)} v = v$, and consequently $U_A(1_{\delta(v)}) = v$;

(c) If B is a finely closed nearly Borel set with $H_B v = v$ (or with $U_A 1_B = v$), then $\delta(v) \subset B$.

An analytic proof of this proposition can be found in [**Fe83**] or in [**BB02**]. In our setting the proposition is an immediate consequence of the following description of $\delta(v)$ and the subsequent discussion.

(6.33) Proposition. *Define*

$$S := \inf\{t \geq 0 : A_t > 0\},$$

and let $F = F_A := \{x \in E : \mathbf{P}^x[S = 0] = 1\}$ *denote the fine support of the PLAF* A*. Then* $F = \delta(v)$*.*

Proof. Suppose $x \in \delta(v)$ and define $\nu := \epsilon_x P_S$. Clearly $\nu \dashv \epsilon_x$ while

$$\nu(v) = \mathbf{P}^x[A_\infty \circ \theta_S] = \mathbf{P}^x \int_{[S,\infty[} dA_t = \mathbf{P}^x[A_\infty] = v(x).$$

Consequently $\nu = \epsilon_x$, hence $\epsilon_x P_S U = \epsilon_x U$. This forces $\mathbf{P}^x[S = 0] = 1$, so $x \in F$.

Conversely, suppose that $x \in F$. Fix $\nu \dashv \epsilon_x$ with $\nu(v) = v(x)$. By Rost's theorem ([**Ro71**] or [**G90**; (5.23)]) there is a (randomized) stopping time T with $\nu = \epsilon_x P_T$. Then

$$\mathbf{P}^x[A_\infty] = v(x) = \nu(v) = \mathbf{P}^x[A_\infty \circ \theta_T] = \mathbf{P}^x[A_\infty - A_T],$$

so $\mathbf{P}^x[A_T = 0] = 1$. When coupled with the fact that $A_t > 0$ for all small $t > 0$, $\mathbf{P}^x$-a.s. (because $\mathbf{P}^x[S = 0] = 1$), this yields $\mathbf{P}^x[T = 0] = 1$, whence $\nu = \epsilon_x$. □

If we represent A as

$$A_t = A_t^c + \sum_{0 \leq s < t} a(X_s),$$

where $a \in p\mathcal{E}^n$ with $J := \{a > 0\}$ semipolar, then clearly $S = S_c \wedge D_J$, where $S_c := \inf\{t : A_t^c > 0\}$. It is well known that $S_c = T_{F_c}$ almost surely, where F_c is the fine support (in the usual sense) of the CAF A^c; see [**BG68**; p. 213]. Also, since the début of a nearly Borel set is almost surely equal to the début of its fine closure, we have $F = F_c \cup \bar{J}^f$, where $\bar{J}^f$ is the fine closure of J.

Conversely, suppose that F is a given finely closed nearly Borel set. Recall that $b \in p\mathcal{E}$ is strictly positive and $Ub \leq 1$. Consider the function $v := H_F U b$. It is easy to check that v is a (bounded) regular strongly supermedian function, so by [**Sh88**; (38.2)] there is a PLAF A with empty exceptional set such that $U_A 1 = v$. Furthermore, using Rost's theorem as in the proof of Proposition (6.33), we can show that $\delta(v) = F$. Thus, each finely closed nearly Borel set is the fine support of a PLAF.

The fine support F_A of a positive CAF A is *finely perfect*: F_A is finely closed and each point of F_A is regular for F_A. The known converses to this assertion (*e.g.* [**Az72, FG95**]) are more involved than the construction suggested in the preceding paragraph. The paper [**DG71**], especially Example (4.4) on pp. 543–544, provides an instructive discussion of these matters.

7. Resolvents.

One of the mains results of [**BB01a**] is that if V is a regular strongly supermedian kernel, then V satisfies the hypothesis of Theorem (6.27) (equivalently, V agrees off an m-exceptional set with the potential kernel of a PLAF) if and only if V is, off an m-exceptional set, the initial kernel of a subMarkovian resolvent. Because the regularity of a strongly supermedian kernel amounts to a form of the domination principle, this assertion is closely related to the work of Hunt [**Hu57**], Taylor [**T72,T75**], and Hirsch [**Hi74**] on the existence of subMarkovian resolvents with given initial kernel.

Our aim in this section is to give an explicit representation of the resolvent $(V^q)_{q\geq 0}$ such that with $V^0 = U_A$ for a given PLAF A. We even construct a (simple) Markov process possessing the given resolvent.

Throughout this section we suppose that A satisfies the conditions listed in (6.26). For $q \geq 0$ define

$$M_t^q := e^{qA_t^c} \prod_{0\leq s\leq t} (1 + q\Delta A_s), \tag{7.1}$$

with the convention that $M_{0-}^q = 1$. Then $M_t^0 = 1$ for all $t \geq 0$, $t \mapsto M_t^q$ is right-continuous, increasing, and finite valued on $[0,\zeta[$, and $M_0^q = 1 + q\Delta A_0 = 1 + qA_{0+}$. Moreover, for $t \geq 0$,

$$\begin{aligned} &(i) & dM_t^q &= qM_{t-}^q\, dA_t; \\ &(ii) & d(1/M_t^q) &= -q(M_t^q)^{-1}\, dA_t; \\ &(iii) & M_{t+s}^q &= M_{t-}^q M_s^q \circ \theta_t. \end{aligned} \tag{7.2}$$

(7.3) Theorem. *With the above notation, define*

$$V^q f(x) := \mathbf{P}^x \int_{[0,\zeta[} (M_t^q)^{-1} f(X_t)\, dA_t, \qquad x \in E \setminus N_A, f \in p\mathcal{E}.$$

Then $(V^q)_{q\geq 0}$ is a subMarkovian resolvent on $E \setminus N_A$ with $V^0 = U_A$.

In proving Theorem (7.3), by the device of restricting X to $E \setminus N_A$, it suffices to suppose that N_A is empty, and this we shall do. The key computation is contained in the following

(7.4) Lemma. *Fix $s > 0$ and define $I(q,r) := \int_{[0,s]} (M_t^q)^{-1} M_{t-}^r\, dA_t$ for $q, r \geq 0$. Then $I(q,r) = (r-q)^{-1} \left[(M_s^q)^{-1} M_s^r - 1\right]$ provided $q \neq r$.*

Proof. If $q > 0$, then by (7.2)(ii)

$$\begin{aligned} I(q,0) &= q^{-1} \int_{[0,s]} q(M_t^q)^{-1}\, dA_t \\ &= -q^{-1} \int_{[0,s]} d(1/M_t^q) \\ &= -q^{-1} \left[(M_s^q)^{-1} - 1 \right]. \end{aligned}$$

Now suppose $r > 0$ and $q \neq r$. Then

$$\begin{aligned} I(q,r) &= r^{-1} \int_{[0,s]} (M_t^q)^{-1} M_{t-}^r \, d(rA_t) \\ &= r^{-1} \int_{[0,s]} (M_t^q)^{-1}\, d(M_t^r) \\ &= r^{-1} \left[(M_s^q)^{-1} M_s^r - 1 \right] - r^{-1} \int_{[0,s]} M_{t-}^r \, d(1/M_t^q), \end{aligned}$$

where (7.2)(i) was used for the second equality. Because of (7.2)(ii), the final integral above is equal to

$$-q \int_{[0,s]} (M_t^q)^{-1} M_{t-}^r \, dA_t = -qI(q,r).$$

Consequently,

$$(1 - q/r) I(q,r) = r^{-1} \left[(M_s^q)^{-1} M_s^r - 1 \right],$$

which implies (7.4). □

We now prove Theorem (7.3). First note that $V^q f \in p\mathcal{E}^n$ provided $f \in p\mathcal{E}^n$, and that $qV^q 1(x) = \mathbf{P}^x[1 - (M_{\zeta-}^q)^{-1}] \leq 1$ for all $x \in E$. Also, if $0 < g \leq 1$ with $U_A g \leq 1$, then $V^q g \leq 1$. Now suppose that $q \neq r$ and $f \in p\mathcal{E}^n$ with $U_A f < \infty$. Then, for $x \in E$,

$$\begin{aligned} V^q V^r f(x) &= \mathbf{P}^x \int_{[0,\zeta[} (M_t^q)^{-1} \int_{[0,\zeta\circ\theta_t[} (M_s^r \circ \theta_t)^{-1} f(X_{t+s})\, dA_s \circ \theta_t \, dA_t \\ &= \mathbf{P}^x \int_{[0,\zeta[} (M_t^q)^{-1} \int_{[t,\zeta[} M_{t-}^r (M_s^r)^{-1} f(X_s)\, dA_s\, dA_t \\ &= \mathbf{P}^x \int_{[0,\zeta[} (M_s^r)^{-1} f(X_s) \int_{[0,s]} M_{t-}^r (M_t^q)^{-1}\, dA_t\, dA_s \\ &= (r-q)^{-1} \mathbf{P}^x \int_{[0,\zeta[} \left[(M_s^q)^{-1} - (M_s^r)^{-1} \right] f(X_s)\, dA_s \\ &= (r-q)^{-1} \left[V^q f(x) - V^r f(x) \right]. \end{aligned}$$

This proves that $(V^q)_{q \geq 0}$ is a subMarkovian resolvent of proper kernels, with $V^0 = U_A$. □

(7.5) Question. Suppose that u is a bounded strongly supermedian function, and define $f := u - qV^q u$. Using the resolvent equation for (V^q) we find that $Vf = Vu - qVV^q u = V^q u < q^{-1}u$ on $\{f > 0\}\setminus N_A$. Therefore $V^q u \le q^{-1}u$ on all of $E\setminus N_A$. A routine truncation argument now establishes the following assertion: If u is strongly supermedian then u is supermedian with respect to the resolvent $(V^q)_{q\ge 0}$. Is the converse true? That is, suppose that a nearly Borel function u is supermedian with respect to the resolvent associated with each PLAF of X. Must u then be strongly supermedian?

If A is continuous, then (V^q) is the well-known resolvent of the strong Markov process obtained by time-changing X using the strictly increasing right-continuous inverse of A. In the general case one must, in addition to the time change, make each x in the semipolar set $\{j > 0\}$ (see (6.21)) an exponentially distributed holding point with mean holding time equal to $j(x)$. This will be made more precise in the next result.

Let A be a PLAF satisfying the conditions in (6.26). Since $J := \{j > 0\}$ is semipolar, there is a sequence $(T_n)_{n\ge 1}$ of stopping times with disjoint graphs such that $\cup_n [\![T_n]\!]$ is indistinguishable from $\{(t,\omega) : X_t(\omega) \in J\}$; see [**De88**; p. 70] or [**Sh88**; (41.3)]. (Here, for a stopping time T, $[\![T]\!] := \{(t,\omega) : t = T(\omega) < \infty\}$ denotes the graph of T.) Let (U_n) be a sequence of independent unit exponential random variables that is also independent of X. (The existence of such a sequence may require a product-space augmentation of the original sample space; the details of such a construction are left to the reader.) Now define

$$B_t := A_t^c + \sum_{n:0\le T_n<t} j(X_{T_n})U_n, \qquad t \ge 0. \tag{7.6}$$

Note that conditional on $\mathcal{F}$, the random variable $j(X_{T_n})U_n$ has the exponential distribution with mean $j(X_{T_n})$; since $\sum_{n:T_n<t} j(X_{T_n}) \le A_t$, it follows that $t \mapsto B_t$ is finite on $[0,\zeta[$ and left-continuous on $]0,\zeta[$, $\mathbf{P}^x$-a.s. for all $x \in E\setminus N_A$. Also, $B_0 = 0$. Let $(\tau_t)_{t\ge 0}$, the right-continuous process inverse to B, be defined by

$$\tau_t = \tau(t) := \inf\{s > 0 : B_s > t\}, \qquad t \ge 0.$$

Notice that if $B_{0+} > 0$ then $\tau_t = 0$ for $0 \le t < B_{0+}$. Of course, $\tau_t = \infty$ if $t > B_\infty = B_{\zeta-}$.

(7.7) Theorem. *Let $(V^q)_{q\ge 0}$ be as in (7.3) and define $Z_t = X_{\tau(t)}$, $t \ge 0$. Then for $x \in E\setminus N_A$ and $f \in p\mathcal{E}$,*

$$V^q f(x) = \mathbf{P}^x \int_0^\infty e^{-qt} f(Z_t)\, dt, \qquad q \ge 0.$$

Proof. Fix $q > 0$ and let $m(t) := 1 - e^{-qt}$ for $t \geq 0$. As in the proof of (7.3) we may suppose that N_A is empty. Define

$$W^q f(x) := \mathbf{P}^x \int_0^\infty e^{-qt} f(Z_t)\, dt = q^{-1} \mathbf{P}^x \int_0^\infty f(X_{\tau(t)})\, dm(t).$$

Adapting the change-of-variable formula [**Sh88**; (A4.7)] to account for the fact that B is left-continuous, one finds that

$$W^q f(x) = q^{-1} \mathbf{P}^x \int_{[0,\infty[} f(X_t)\, d_t m(B_t). \tag{7.8}$$

But

$$d_t m(B_t) = -d(e^{-qA_t^c} e^{-qB_t^d}) = q e^{-qB_t}\, dA_t^c - e^{-qA_t^c}\, d(e^{-qB_t^d}),$$

and

$$d(e^{-qB_t^d}) = \Delta(e^{-qB_t^d}) = e^{-qB_t^d}(e^{-q\Delta B_t^d} - 1).$$

Combining these observations, (7.8) becomes

$$\begin{aligned} W^q f(x) = \mathbf{P}^x &\int_0^\infty f(X_t) e^{-qA_t^c} \prod_{n: T_n < t} e^{-qj(X_{T_n})U_n}\, dA_t^c \\ &+ q^{-1} \mathbf{P}^x \sum_n f(X_{T_n}) e^{-qA_{T_n}^c} \prod_{k<n} e^{-qj(X_{T_k})U_k} (1 - e^{-qj(X_{T_n})U_n}). \end{aligned} \tag{7.9}$$

Using now the fact that (U_n) is independent of $\mathcal{F}$, the first term on the right side of (7.9) is seen to equal

$$\mathbf{P}^x \int_0^\infty f(X_t) e^{-qA_t^c} \prod_{n: T_n < t} (1 + qj(X_{T_n}))^{-1}\, dA_t^c = \mathbf{P}^x \int_0^\infty f(X_t) (M_t^q)^{-1}\, dA_t^c,$$

since A^c is continuous. Similarly, the second term on the right side of (7.9) equals

$$\mathbf{P}^x \sum_n f(X_{T_n}) e^{-qA_{T_n}^c} \prod_{k<n} (1 + qj(X_{T_k}))^{-1} \cdot \frac{j(X_{T_n})}{1 + qj(X_{T_n})} = \mathbf{P}^x \int_{[0,\infty[} f(X_t)(M_t^q)^{-1}\, dA_t^d.$$

Consequently, $W^q f = V^q f$ if $q > 0$. The case $q = 0$ now follows immediately by monotone convergence. ∎

(7.10) Remark. The reader may check that $Z = (Z_t, \mathbf{P}^x)$ is a simple Markov process for each $x \in E \setminus N_A$, but in general it is not a strong Markov process. In the special case in which J is a finite set and $A_t^c = t$, the process Z has a very simple description: Between visits to J, Z behaves like X; each $x \in J$ is a holding point, where Z is delayed for an exponential (mean $j(x)$) time. Also of interest is the special case in which $A^c = 0$ and $\{t : X_t \in J\}$ is almost surely dense in $[0, \zeta[$.

A. Appendix.

In this appendix we collect some important properties of strongly supermedian functions, and we give a direct proof of Theorem (5.8). See [**Mr73, Fe81, Fe83, BB99**] for background on strongly supermedian functions.

First note that if f is strongly supermedian and if (T_n) is a monotone sequence of stopping times, then $\lim_n \mathbf{P}^x[f(X_{T_n})]$ exists. Consequently, the process $t \mapsto f(X_t)$ has left limits on $]0, \infty]$ and right limits on $[0, \infty[$, almost surely. Let $\bar{f} := \lim_{t\downarrow 0} P_t f$ denote the excessive regularization of f. We claim that the processes $(f(X)_{t+})_{t\geq 0}$ and $(\bar{f}(X_t))_{t\geq 0}$ are indistinguishable. It suffices to prove this for bounded f since $\overline{f \wedge c} = \bar{f} \wedge c$ for $c \in \mathbf{R}^+$. Next, since both processes are optional, we need only check that

$$\mathbf{P}^x[f(X)_{T+}; T < \infty] = \mathbf{P}^x[\bar{f}(X_T); T < \infty]$$

for all stopping times T. Because f is bounded,

$$\begin{aligned}
\mathbf{P}^x[f(X)_{T+}; T < \infty] &= \lim_{t\downarrow 0} \mathbf{P}^x[f(X_{T+t}); T < \infty] \\
&= \lim_{t\downarrow 0} \mathbf{P}^x[P_t f(X_T); T < \infty] \\
&= \mathbf{P}^x[\bar{f}(X_T) : T < \infty].
\end{aligned}$$

These facts will be used without special mention in the sequel.

(A.1) Definition. Let $E' \in \mathcal{E}^n$ be an absorbing set. A strongly supermedian function f is *regular on* E' if f is finite on E' and for every increasing sequence (T_n) of stopping times we have

$$\lim_n \mathbf{P}^x[f(X_{T_n})] = \mathbf{P}^x[f(X_T)], \qquad \forall x \in E',$$

where $T := \lim_n T_n$. When $E' = E$ we simply say that f is regular.

The next theorem is a fundamental result of J.-F. Mertens [**Mr73**]. The proof in complete generality is rather complicated; we present a simpler proof for the special case of bounded strongly supermedian functions, which is the only case we shall be using. A strongly supermedian function u dominates another strongly supermedian function v in the specific order provided there is a strongly supermedian function w such that $u = v + w$.

(A.2) Theorem. [Mertens] *A strongly supermedian function f may be decomposed uniquely as $h+g$, where h is the largest (in the specific order) excessive function specifically dominated by f, and g is a regular strongly supermedian function.*

Proof. We give a proof only in the special case of bounded f. As before, $\bar f$ denotes the excessive regularization of f. The process $(f(X_t))_{t\ge 0}$ is a bounded optional strong supermartingale (under each measure $\mathbf{P}^x$, $x \in E$), so by Theorem 20 on page 429 of [**DM80**] there exists, for each $x \in E$, an increasing predictable process $(A^x_t)_{t\ge 0}$ such that $\mathbf{P}^x[A^x_\infty] < \infty$ and

$$(A.3) \qquad f(X_T) = \mathbf{P}^x[A^x_\infty|\mathcal{F}_T] - A^x_T, \qquad \mathbf{P}^x\text{-a.s.}$$

for each stopping time T. It follows from the proof given in [**DM80**] that $A^x_t = A^{x,-}_t + A^{x,+}_{t-}$, with $A^{x,-}$ increasing and predictable, and $A^{x,+}$ increasing and optional, both processes being right-continuous. Moreover, the process $A^{x,+}$ is purely discontinuous. In view of the footnote on page 430 of [**DM80**],

$$(A.4) \qquad A^{x,+}_{t-} = \sum_{0\le s<t} [f(X_s) - f(X)_{s+}] = \sum_{0\le s<t} [f(X_s) - \bar f(X_s)]$$

up to $\mathbf{P}^x$ evanescence, for each $x \in E$. Defining B_t to be the sum on the far right of (A.4), noting that $B_0 = 0$, equation (A.3) may be re-written as

$$(A.5) \qquad f(X_T) + B_T = \mathbf{P}^x[A^x_\infty|\mathcal{F}_T] - A^{x,-}_T.$$

Since $A^{x,-}$ is increasing, it follows that $(f(X_t) + B_t)_{t\ge 0}$ is a strong supermartingale under each measure $\mathbf{P}^x$, $x \in E$. Following Mertens we now define $g(x) := \mathbf{P}^x[B_\infty]$. Then $P_T g(x) = \mathbf{P}^x[B_\infty - B_T] \le g(x)$, and if $T_n \uparrow T$ then $P_{T_n} g(x) \downarrow P_T g(x)$ since B is left-continuous. Thus g is a regular strongly supermedian function. Also,

$$g(x) = \mathbf{P}^x[B_\infty - B_{0+}] + \mathbf{P}^x[B_{0+}] = \bar g(x) + f(x) - \bar f(x),$$

so $f - g = \bar f - \bar g$. Define $h := f - g$, and note that $P_t h \to \bar f - \bar g = f - g = h$ as $t \downarrow 0$. Moreover,

$$P_t h(x) = \mathbf{P}^x[f(X_t)] - g(x) + \mathbf{P}^x[B_t] \le f(x) - g(x) + \mathbf{P}^x[B_0] = h(x),$$

since $B_0 = 0$ and $(f(X_t) + B_t)_{t\ge 0}$ is a supermartingale. Thus h is excessive.

Now suppose that $f = h_0 + g_0$ is a second decomposition of f into excessive and strongly supermedian components. Because g_0 is strongly supermedian, by what has already been proved we can write $g_0 = h_1 + g_1$ with h_1 excessive and g_1 a regular strongly supermedian function. Because h_0 is excessive, $g_0 - \overline{g_0} = f - \bar{f} = g - \bar{g}$. From this and the construction of the "purely discontinuous" component g_1 it follows that $g_1 = g$. Therefore, $g_0 = h_1 + g$, so $h = h_0 + h_1$ specifically dominates h_0. This establishes the case of (A.2) that we shall need. ☐

(A.6) Remark. This proof works just as well if $(f(X_t))_{t \geq 0}$ is of class (D) relative to each $\mathbf{P}^x$, and this holds if and only if $P_{\{f>n\}} f(x) \to 0$ as $n \to \infty$ for each $x \in E$; see [**Sh88**; (33.3)].

The following is an analog of the classical approximation of excessive functions by potentials; it seems to have gone unnoticed in the literature.

(A.7) Corollary. *If X is transient, then a strongly supermedian function f is the increasing limit of a sequence of regular strongly supermedian functions.*

Proof. Use Mertens' theorem to write f as $h+g$, where h is an excessive function and g is a regular strongly supermedian function. Since X is transient, there is an increasing sequence (Ub_n) of potentials with $Ub_n \uparrow h$. Now each potential Ub_n is regular as is $f_n := Ub_n + g$, which increases pointwise to f. ☐

We come now to a key fact concerning strongly supermedian kernels.

(A.8) Proposition. *Let V be a regular strongly supermedian kernel. Let $f \in p\mathcal{E}^n$ with $h = Vf$ bounded. If X is transient, then h is a regular strongly supermedian function.*

Proof. Define the réduite Rg of g by

$$Rg(x) := \inf\{u(x) : u \geq g, u \text{ is strongly supermedian}\}, \qquad x \in E.$$

Clearly $Rg_1 \leq Rg_2$ if $g_1 \leq g_2$. Replacing V by the kernel $g \mapsto V(fg)$, we may suppose that $f = 1$ in the proof. We begin by proving the following assertion, in which $h := V1$.

(A.9). *If (h_n) is an increasing sequence of strongly supermedian functions with $h_n \uparrow h$, then $R(h - h_n) \downarrow 0$.*

This assertion is proved in [**BB01b**], and we repeat that proof here for the convenience of the reader. Given $\epsilon > 0$ let $A_{n,\epsilon} := \{h < h_n + \epsilon\}$. Then $A_{n,\epsilon} \uparrow E$ as $n \to \infty$ for each $\epsilon > 0$. Now $V1_{A_{n,\epsilon}} \leq h$, so $V1_{A_{n,\epsilon}} \leq h_n + \epsilon$ on $A_{n,\epsilon}$, hence everywhere by the regularity of V. Consequently,

$$h - h_n = V1_{A_{n,\epsilon}} + V1_{A^c_{n,\epsilon}} - h_n \leq V1_{A^c_{n,\epsilon}} + \epsilon;$$

since the function $V1_{A^c_{n,\epsilon}} + \epsilon$ is strongly supermedian, we have $R(h - h_n) \le V1_{A^c_{n,\epsilon}} + \epsilon$. But $V1_{A^c_{n,\epsilon}} \downarrow 0$ as $n \to \infty$ because $h < \infty$, so $\lim_n R(h - h_n) \le \epsilon$ for all $\epsilon > 0$. This establishes (A.9).

Now suppose that X is transient. By (A.7) there exists an increasing sequence (h_n) of regular strongly supermedian functions with $h_n \uparrow h$ and, by (A.9), $R(h-h_n) \downarrow 0$. Let (T_k) be an increasing sequence of stopping times with limit T. Let g be a strongly supermedian function dominating $h - h_n$. Then

$$P_T h \le P_{T_k} h \le P_{T_k} g + P_{T_k} h_n \le g + P_{T_k} h_n,$$

and, since h_n is regular, $\lim_k P_{T_k} h \le g + P_T h_n$. Therefore

$$(A.10) \qquad P_T h \le \lim_k P_{T_k} h \le \inf_g \{g + P_T h_n\} = R(h - h_n) + P_T h_n,$$

where the infimum is taken over all strongly supermedian majorants g of $h - h_n$. Letting $n \to \infty$ in (A.10) we see that $\lim_k P_{T_k} h = P_T h$, proving that h is regular. □

(A.11) Remark. In comparing (A.8) with [**BB01b**; Thm. 2.5] one must bear in mind that the definitions of regularity are different. In fact, Beznea and Boboc use the property (A.9) as the defining property of regularity for strongly supermedian functions. The second part of the proof of (A.8) shows that a strongly supermedian function satisfying (A.9) is regular as defined in (A.1)—one needs the full power of Mertens theorem for finite h here. It can be shown that the two definitions are in fact equivalent.

As a final bit of preparation for the proof of Theorem (5.8), we record the following result. A parallel result for regular strongly supermedian kernels appears as Theorem 2.2 in [**BB01b**]; see also [**A73**; Thm. 5.2, p. 509].

(A.12) Proposition. *Let A^1 and A^2 be PLAFs with exceptional sets N_1 and N_2, and define $B := A^1 + A^2$. Then there exist $g_1 \in p\mathcal{E}^n$ and $g_2 \in p\mathcal{E}^n$ with $g_1 + g_2 \le 1$ such that A^j and $g_j * B$ are $\mathbf{P}^x$-indistinguishable for $j = 1, 2$ and all $x \notin N_1 \cup N_2$.*

Proof. We can write $A^j_t = A^{j,c}_t + \sum_{0 \le s < t} a_j(X_s)$, where $A^{j,c}_t$ is a PCAF, $a_j \in p\mathcal{E}^n$, and $\{a_j > 0\}$ is semipolar. Then $B^c := A^{1,c} + A^{2,c}$ is the continuous part of B, and it is well known that this implies the existence of $f_1, f_2 \in p\mathcal{E}^n$ with $f_1 + f_2 \le 1$ such that $A^{j,c}$ and $f_j * B$ are $\mathbf{P}^x$-indistinguishable for all $x \notin N_1 \cup N_2$, where N_j is an exceptional set for A^j. See, for example, [**Sh88**; (66.2)]. Since $\{a_1 + a_2 > 0\}$ is semipolar, we may also assume that $f_1 = f_2 = 0$ on $\{a_1 + a_2 > 0\}$. Then, setting $g_j := f_j + a_j(a_1 + a_2)^{-1} 1_{\{a_1 + a_2 > 0\}}$ for $j = 1, 2$, it is clear that A^j and $g_j * B$ are $\mathbf{P}^x$-indistinguishable for $x \notin N_1 \cup N_2$. □

Proof of Theorem (5.8). As in the statement of the theorem, X is transient and V is a regular strongly supermedian kernel that is $(m+\rho)$-proper. Suppose, for the moment, that $V1$ is bounded. If $f \in bp\mathcal{E}$ then Vf is a bounded regular strongly supermedian function by (A.8). Hence [**Sh88**; Thm. 38.2] implies that there is a unique PLAF A^f, with empty exceptional set, such that $Vf(x) = \mathbf{P}^x[A^f_\infty]$ for all $x \in E$. In the present context, "unique" means up to $\mathbf{P}^x$-indistinguishability for all $x \in E$. Theorem (38.2) in [**Sh88**] states that the PLAF A^f is perfect as defined in [**Sh88**], but the proof only shows that it is "almost perfect" as defined there. In our terminology, this means that the exceptional set is empty although the defining set need not be all of Ω. The uniqueness implies that $f \mapsto A^f$ is additive and positive-homogeneous. In particular, if $(f_n) \subset bp\mathcal{E}$ is an increasing sequence with limit $f \in bp\mathcal{E}$, then $A^{f_n} \uparrow A^f$. Define $A := A^1$. If $f \in bp\mathcal{E}$ satisfies $0 \le f \le 1$, then $A = A^f + A^{1-f}$. These PLAFs have empty exceptional sets, so (A.12) implies that there exists $\tilde{f} \in bp\mathcal{E}^n$ such that $A^f = \tilde{f} * A$. Define an operator T by $Tf := \tilde{f}$, so that $A^f = (Tf) * A$. If $c_j \in \mathbf{R}^+$ and $f_j \in bp\mathcal{E}$ for $j = 1, 2$, then (by uniqueness) $T(c_1 f_1 + c_2 f_2) = c_1 T f_1 + c_2 T f_2$, U_A-a.e.; that is, off a set H with $U_A(x, H) = 0$ for all $x \in E$. Also, if $f_n \uparrow f$ then $Tf_n \uparrow Tf$, U_A-a.e. Now regard T as a map from $bp\mathcal{E}$ into equivalence classes of $bp\mathcal{E}^n$ functions agreeing U_A-a.e., and extend T to $b\mathcal{E}$ by linearity. Then T is a pseudo-kernel from $(E, \mathcal{E}^n)$ to $(E, \mathcal{E})$ as defined in [**DM83**; IX.11], and by the theorem of IX.11, there exists a kernel K from $(E, \mathcal{E}^n)$ to $(E, \mathcal{E})$ such that $Tf = Kf$, U_A-a.e., for all $f \in b\mathcal{E}$. Thus, for $f \in bp\mathcal{E}$, we have $A^f = (Kf) * A$.

We are now going to show that $Kf = f$, U_A-a.e. To this end let $f = 1_B$ where $B \in \mathcal{E}$, and write A^B for A^{1_B} and k_B for $K1_B$. Then $A^B = k_B * A$, so $V1_B = U_A k_B$. Recall that $H_B f(x) := \mathbf{P}^x[f(X_{D_B})]$ for $x \in E$. Then, since V is regular and $H_B V1_B$ is strongly supermedian,

$$V1_B(x) = H_B V1_B(x) = \mathbf{P}^x \int_{[D_B, \infty[} k_B(X_t)\, dA_t.$$

Therefore

$$0 = \mathbf{P}^x \int_{[0, D_B[} k_B(X_t)\, dA_t = \mathbf{P}^x \int_{[0, D_B[} 1_{B^c}(X_t) k_B(X_t)\, dA_t. \tag{A.13}$$

We claim that

$$\mathbf{P}^x \int_{[0, \infty[} 1_{B^c}(X_t) k_B(X_t)\, dA_t = 0, \tag{A.14}$$

provided B is closed.

To establish (A.14) we consider the excursions of X from the closed set B. Let $G = G(\omega)$ be the set of strictly positive left endpoints of the maximal open intervals of the complement in $]0,\infty[$ of the closure of $\{t : X_t(\omega) \in B\}$—the excursion intervals from B. Since $T_B = D_B$ if $D_B > 0$, (A.13) implies

$$\mathbf{P}^x \int_{[0,\infty[} 1_{B^c}(X_t) k_B(X_t)\, dA_t = \mathbf{P}^x \sum_{s\in G} \int_{[s,s+D_B\circ\theta_s[} 1_{B^c}(X_t) k_B(X_t)\, dA_t.$$

Given $\epsilon > 0$, let $g_1^\epsilon < g_2^\epsilon < \cdots$ be the left endpoints of the successive excursion intervals exceeding ϵ in length. Then $T_n := g_n^\epsilon + \epsilon$ is a stopping time for each n; see [**De72**; VI-T2]. Also, $T_n + D_B \circ \theta_{T_n}$ is the right endpoint of the n^{th} such excursion interval. From (A.13),

$$\begin{aligned}\mathbf{P}^x \sum_n \int_{[T_n, T_n + D_B\circ\theta_{T_n}[} & 1_{B^c}(X_t) k_B(X_t)\, dA_t \\ &= \mathbf{P}^x \sum_n \mathbf{P}^{X(T_n)} \int_{[0,D_B[} 1_{B^c}(X_t) k_B(X_t)\, dA_t = 0.\end{aligned}$$

Recalling the dependence on ϵ and summing over $\epsilon = 1/k$ for $k = 1, 2, \ldots$, we obtain

$$\mathbf{P}^x \sum_{s\in G} \int_{]s,s+D_B\circ\theta_s[} 1_{B^c}(X_t) k_B(X_t)\, dA_t = 0.$$

In order to complete the proof it remains to show that

$$(A.15) \qquad \mathbf{P}^x \sum_{s\in G} 1_{B^c}(X_s) k_B(X_s)\, \Delta A_s = 0.$$

Now $G = G^i \cup G^r$, almost surely, where G^i is a countable union of graphs of stopping times and G^r meets the graph of no stopping time. Using the strong Markov property as before, the portion of the sum in (A.15) corresponding to G^i vanishes. Finally, the process $(\Delta A_t)_{t\ge 0}$ is indistinguishable from $(a(X_t))_{t\ge 0}$, where $\{a > 0\}$ is semipolar. In fact, if $u = U_A 1$, then one may take $a := u - \bar{u}$, by [**Sh88**; (37.7)]. But the set $\{t : a(X_t) > 0\}$ is also the countable union of graphs of stopping times, so the portion of the sum in (A.15) corresponding to G^r also vanishes. This establishes the claim (A.14).

Now fix $x \in E$. Then $U_A(x,\cdot)$ is a finite measure. Let B be an open subset of E with $U_A(x,\partial B) = 0$. Formula (A.14) implies that $\int_{\overline{B}^c} k_{\overline{B}}(y)\, U_A(x,dy) = 0$, and so $k_B(y) \le k_{\overline{B}}(y) = 0$ on $\overline{B}^c$, $U_A(x,\cdot)$-a.e. Hence $K(\cdot,B) = 0$, $U_A(x,\cdot)$-a.e. on B^c since $U_A(x,\partial B) = 0$. Using (A.14) with B replaced by B^c we obtain $\int_B k_{B^c}(y)\, U_A(x,dy) = 0$, and so $K(\cdot,B^c) = 0$ on B, $U_A(x,\cdot)$-a.e. Recalling the meaning of K, we see that $K(\cdot,B) + K(\cdot,B^c) = 1$, $U_A(x,\cdot)$-a.e. Therefore $K(\cdot,B) = 1_B$, $U_A(x,\cdot)$-a.e. The class of open sets $B \subset E$ with $U_A(x,\partial B) = 0$ contains a countable subcollection that generates the topology of E and is closed under finite intersections, hence $Kf = f$, $U_A(x,\cdot)$-a.e., for each

$f \in b\mathcal{E}$ and each $x \in E$. Thus, if $f \in b\mathcal{E}$, we have $Vf = U_A K f = U_A f$; that is, $V = U_A$. Now A is a PLAF with empty exceptional set, and $A_t = A_t^c + \sum_{0 \le s < t} a(X_s)$, where A^c is continuous and $a \ge 0$ is nearly Borel measurable with $\{a > 0\}$ semipolar. Clearly $A_t^d := \sum_{0 \le s < t} a(X_s)$ is perfectly homogeneous; that is, $A_{t+s}^d(\omega) = A_t^d(\omega) + A_s^d(\theta_t \omega)$ for all $s, t \ge 0$ and $\omega \in \Omega$. But Meyer's master perfection theorem [**G90**; Thm. A.33] implies that A^c is indistinguishable from a perfectly homogeneous AF, and so we may suppose that A^c is perfectly homogeneous; thus A can be taken to be perfectly homogeneous as well. In brief, A is a PLAF with $N_A = \emptyset$ and $\Omega_A = \Omega$.

Next we suppose only that $V1$ is bounded off an m-exceptional set N; we can (and do) assume that $N \in \mathcal{E}$ and that $E \setminus N$ is absorbing. We apply what we have already proved to X^N, the restriction of X to $E \setminus N$, to obtain a PLAF A of X^N with empty exceptional set and defining set all of Ω^N (the sample space of X^N) such that $V(x, \cdot) = U_A(x, \cdot)$ for all $x \in E \setminus N$. Since we are interested in perfection we must be careful in choosing the realization of the restricted process. We take X^N to be the canonical realization of the restriction of (P_t) to $E \setminus N$. Thus the state space of X^N is $E' := E \setminus N$ and Ω^N may be identified with

$$\Omega' := \{\omega \in \Omega : \omega(t) \in E' \cup \{\Delta\} \text{ for all } t \ge 0\},$$

while X_t^N and θ_t^N are the restrictions of X_t and θ_t to Ω'.

We are now going to use the PLAF A of X^N to define an AF of (X, D_N) — X killed at the début $D_N := \inf\{t \ge 0 : X_t \in N\}$ of N. Since E' is absorbing, $\mathbf{P}^x[D_N = \infty] = 1$ for all $x \in E'$. Also, D_N is a perfect terminal time, non-exact in general. Define a map $k^A : \Omega \to \Omega'$ by setting $k^A\omega(t) = \omega(t)$ if $0 \le t < D_N(\omega)$, $k^A\omega(t) = \Delta$ if $t \ge D_N(\omega)$. That is, $k^A\omega = k_{D_N(\omega)}\omega$. Next, define B_t^c on Ω by $B_t^c(\omega) = A_t^c(k^A\omega)$. One checks easily that $\theta_t^N k^A = k^A \theta_t$ on $\{t < D_N\}$, which implies that $B_{t+s}^c = B_t^c + 1_{\{t<D_N\}} B_s^c \circ \theta_t$. Thus B^c is a perfectly homogeneous, right-continuous AF of (X, D_N) that is a.s. continuous. Clearly $\mathbf{P}^x[B_t^c = A_t^c, \forall t \ge 0] = 1$ for all $x \in E'$. In particular, $U_{B^c}(x, \cdot) = U_{A^c}(x, \cdot)$ (on E'). We now appeal to [**GS74**; (3.9)], which asserts the existence of a perfectly homogeneous optional RM λ^c of X such that $\lambda^c(dt) = dB_t^c$ on $[0, D_N[$; the proof even shows that λ^c is a.s. diffuse. See also [**Sh88**; (38.6)]. (Here, *perfectly homogeneous* means that $\lambda^c(\theta_t\omega, B) = \lambda^c(\omega, B+t)$ for all $t \ge 0$, $\omega \in \Omega$, and $B \in \mathcal{B}_{[0,\infty[}$.) Since $\mathbf{P}^x[D_N = \infty] = 1$ for all $x \in E'$, we have $U_{\lambda^c} 1 = U_{B^c} 1$ on E'. Finally, $\lambda := \lambda^c + \sum_{t \ge 0} a(X_t)\epsilon_t$ is a perfectly homogeneous optional RM of X with $U_\lambda(x, \cdot) = U_A(x, \cdot)$ for all $x \in E'$.

As on pp. 89–90 of [**G90**] there is a perfectly homogeneous optional RM γ^c carried by $]\alpha, \beta[$ such that $\lambda^c = \gamma^c|_\Omega$. (*Now* perfectly homogeneous means that $\gamma^c(\theta_t w, B) = \gamma^c(w, B + t)$ for all $t \in \mathbf{R}$, $w \in W$, and $B \in \mathcal{B}_{\mathbf{R}}$.) Define $\gamma := \gamma^c + \sum_{t \ge \alpha} a(Y_t^*)\epsilon_t$.

Then γ is perfectly homogeneous, optional, and is carried by Λ^*. Also, it is not hard to check that γ satisfies property (3.11)(iii); consequently γ is $\mathbf{Q}_m$ co-predictable. Since $\gamma|_\Omega = \lambda$, we have $U_\gamma(x,\cdot) = U_A(x,\cdot)$ for all $x \in E'$. By Theorem (6.15) the characteristic measure $\mu_\gamma = \mu_A$ is σ-finite and charges no m-exceptional set. Thus, by (3.11) and (3.12) there is a perfect HRM κ that is $\mathbf{Q}_m$-indistinguishable from γ. Since κ is perfect and γ is perfectly homogeneous, an application of the strong Markov property (2.5) and the section theorem (2.6) (as in the proof of the implication (i)$\Longrightarrow$(iii) of (5.19)) show that the set $\{x \in E : U_\kappa(x,\cdot) \neq U_\gamma(x,\cdot)\}$ is m-exceptional. It follows that $U_\kappa(x,\cdot) = U_A(x,\cdot)$ for all x outside an m-exceptional set. Since $V(x,\cdot) = U_A(x,\cdot)$ for $x \in E'$, this proves Theorem (5.8) when $V1$ is bounded off an m-exceptional set.

In the general case we use (5.5) to find $g \in p\mathcal{E}^n$ with $0 < g \leq 1$, such that $Vg \leq 1$ off an m-exceptional set. Define $V^g f := V(gf)$ for $f \in p\mathcal{E}$. Then we can apply what is proved above to find a perfect HRM κ^g such that $V^g = U_{\kappa^g}$ off and m-exceptional set. The perfect HRM κ defined by $\kappa(dt) := g(Y_t^*)^{-1}\,\kappa^g(dt)$ evidently has potential kernel equal to V off an m-exceptional set. This establishes Theorem (5.8) in full generality. ☐

Added Note. L. Beznea and N. Boboc have recently given a positive answer to Question (7.5). Their paper "On the strongly supermedian functions and kernels" also contains new analytic proofs of (4.7) and (5.11).

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