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Random directed trees and forest – drainage networks with dependence*

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Abstract

Consider the d-dimensional lattice $\mathbb{Z}^d$ where each vertex is 'open' or 'closed' with probability p or $1-p$ respectively. An open vertex v is connected by an edge to the closest open vertex w in the 45° (downward) light cone generated at v. In case of non-uniqueness of such a vertex w, we choose any one of the closest vertices with equal probability and independently of the other random mechanisms. It is shown that this random graph is a tree almost surely for $d = 2$ and 3 and it is an infinite collection of distinct trees for $d \geq 4$. In addition, for any dimension, we show that there is no bi-infinite path in the tree.

Key words: Random Graph, Random Oriented Trees, Random Walk.

AMS 2000 Subject Classification: Primary 05C80, 60K35.

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1 Introduction

During the last two decades there has been a considerable amount of study to understand the structure of random spanning trees. In particular, for the uniform spanning tree model the tree/forest dichotomy according to the dimension of the lattice was established by Pemantle [13]. Also, for the Euclidean minimal weight spanning tree/forest model Alexander [1] showed that the two dimensional structure of the random graph is that of a tree and Newman and Stein [12] through a study of the fractal dimension of the incipient cluster in the Bernoulli bond percolation problem suggest that the random graph is a forest in suitably high dimensions.

Lately there has been an interest in studying these random spanning trees where the edges have a preferred direction of propagation. These studies have been motivated by studies of Alpine drainage patterns (see e.g., Leopold and Langbein [10], Scheidegger [15], Howard [9]). In a survey of such models, Rodriguez-Iturbe and Rinaldo [14] have explored (non-rigorously) power law structures and other physical phenomenon, while Nandi and Manna [11] obtained relations between these 'river networks' and scale-free networks. Also, in connection with the small-world phenomenon, Baccelli and Bordenave [3] and Bordenave [4] have studied the navigation properties of such directed spanning trees.

Gangopadhyay, Roy and Sarkar [8] studied a random graph motivated by Scheidegger river networks. They considered the d-dimensional lattice $\mathbb{Z}^d$ where each vertex is 'open' or 'closed' with probability p or $1-p$ respectively. The open vertices representing the water sources. An open vertex v was connected by an edge to the closest open vertex w such that the dth co-ordinates of v and w satisfy $w(d) = v(d) - 1$. In case of non-uniqueness of such a vertex w, any one of the closest open vertices was chosen with equal probability and independent of the other random mechanisms. They established that for $d = 2$ and 3, the random graph constructed above is a tree, while for $d \geq 4$, the graph is a forest (i.e. infinitely many trees). Ferrari, Landim and Thorisson [6] have obtained a similar dichotomy for a continuous version of this model which they termed Poisson trees. In this model, the vertices are Poisson points in $\mathbb{R}^d$ and, given a Poisson point u, it is connected to another Poisson point v by an edge if (i) the first $(d-1)$ co-ordinates of v lie in a $(d-1)$ dimensional ball of a fixed radius r centred at the first $(d-1)$ co-ordinates of u and (ii) if v is the first such point from u in the direction specified by the dth co-ordinate.

Mathematically these models are also attractive by their obvious connection to the Brownian web as described by Fontes, Isopi, Newman and Ravishankar [7]. In particular, Ferrari, Fontes and Wu [5] have shown that, when properly rescaled, Poisson trees converge to the Brownian web. Tóth and Werner [17] considered coalescing oriented random walks on $\mathbb{Z}^2$, oriented so as to allow steps only upwards or rightwards. Wilson's method of 'rooted at infinity' associates to this Markov chain a wired random spanning tree on $\mathbb{Z}^2$. Tóth and Werner also obtained an invariance principle for the Markov chain they studied.

Motivated by the above, we consider a general class of such models. Here a source of water is connected by an edge to the nearest (see Figure 1) source lying downstream in a 45 degree light cone generating from the source. Like above a uniform choice is adopted in case of non-uniqueness. The random graph obtained by this construction is the object of study in this paper. We establish that for $d = 2$ and 3, all the tributaries connect to form a single riverine delta, while for $d \geq 4$, there are infinitely many riverine deltas, each with its own distinct set of tributaries. Further we also show that there are no bi-infinite paths in this oriented random graph. The angle of the light cone is unimportant here, and our results are valid for any angle.

These results are similar to those obtained by Gangopadhyay, Roy and Sarkar [8] and as in that paper the results are obtained by a combination of a Lyapunov method and a random walk comparison. However, unlike the Gangopadhyay, Roy and Sarkar [8] or the Ferrari, Landim and Thorisson [6], there is no 'longitudinal' or 'lateral' independence in the generalised set-up we study here. As such, the dependence structure brings its own complicacies resulting in more intricate arguments in both the Lyapunov and random walk methods. In particular, the dependence induces a 'history' in the processes, which has to be carefully incorporated to obtain a suitable Markov model for comparison.

In the next subsection we define the model precisely, state our main results and compare it with the existing results in the literature.

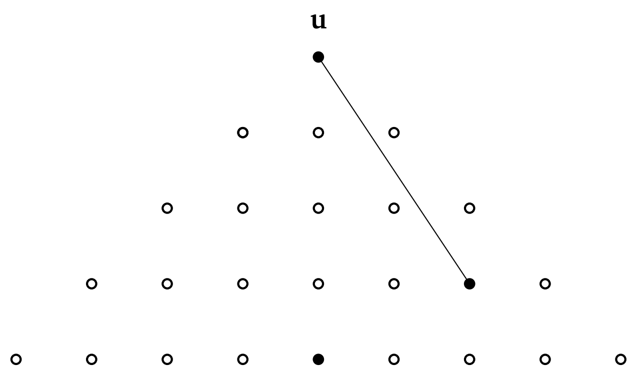


Figure 1: In the 45 degree light cone generated at the open point **u** each level is examined for an open point and a connection is made. If there are more than one choice at a certain level then the choice is made uniformly. As illustrated above the connection could be made to the open point that is not the closest in the conventional graph distance on $\mathbb{Z}^d$.

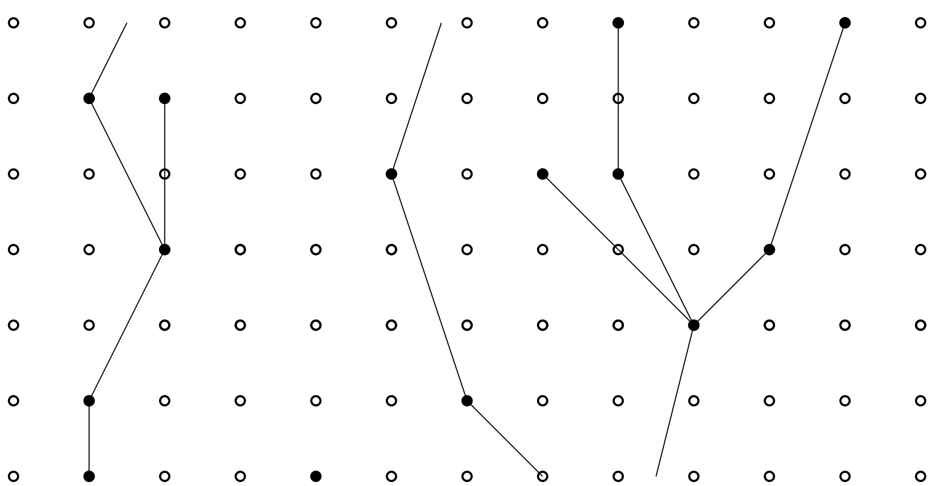


Figure 2: Trees on $\mathbb{Z}^2$. The shaded points are open, while the others are closed.

1.1 Main Results

Before we describe the model we fix some notation which describe special regions in $\mathbb{Z}^d$. For $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{Z}^d$ and $k \geq 1$ let $m_k(\mathbf{u}) = (u_1, \dots, u_{d-1}, u_d - k)$.
Also, for $k, h \geq 1$ define the regions
$H(\mathbf{u}, k) = \{\mathbf{v} \in \mathbb{Z}^d : v_d = u_d - k \text{ and } ||\mathbf{v} - m_k(\mathbf{u})||_{L_1} \leq k\}$,
$\Lambda(\mathbf{u}, h) = \{\mathbf{v} : \mathbf{v} \in H(\mathbf{u}, k) \text{ for some } 1 \leq k \leq h\}$, $\Lambda(\mathbf{u}) = \cup_{h=1}^{\infty} \Lambda(\mathbf{u}, h)$ and
$B(\mathbf{u}, h) = \{\mathbf{v} : \mathbf{v} \in H(\mathbf{u}, k) \text{ and } ||\mathbf{v} - m_k(\mathbf{u})||_{L_1} = k \text{ for some } 1 \leq k \leq h\}$.
We set $H(\mathbf{u}, 0) = \Lambda(\mathbf{u}, 0) = \emptyset$.

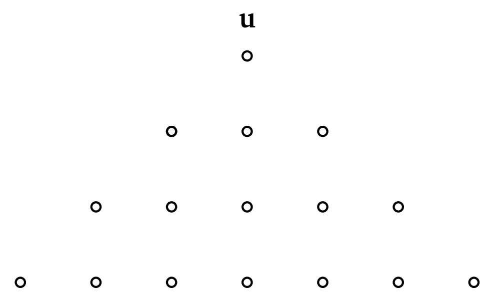


Figure 3: The region $\Lambda(\mathbf{u}, 3)$. The seven vertices at the bottom constitute $H(\mathbf{u}, 3)$ while the six vertices on the two linear 'boundary' segments containing $\mathbf{u}$ constitute $B(\mathbf{u}, 3)$

We equip $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ with the σ-algebra $\mathscr{F}$ generated by finite-dimensional cylinder sets and a product probability measure P_p defined through its marginals as

$$P_p\{\omega : \omega(\mathbf{u}) = 1\} = 1 - P_p\{\omega : \omega(\mathbf{u}) = 0\} = p \text{ for } \mathbf{u} \in \mathbb{Z}^d \text{ and } 0 \leq p \leq 1. \tag{1}$$

On another probability space $(\Xi, \mathscr{S}, \mu)$ we accommodate the collection $\{U_{\mathbf{u},\mathbf{v}} : \mathbf{v} \in \Lambda(\mathbf{u}), \mathbf{u} \in \mathbb{Z}^d\}$ of i.i.d. uniform $(0, 1)$ random variables. The random graph, defined on the product space $(\Omega \times \Xi, \mathscr{F} \times \mathscr{S}, \mathbb{P} := P_p \times \mu)$, is given by the vertex set

$$\mathscr{V} := \mathscr{V}(\omega, \xi) = \{\mathbf{u} \in Z^d : \omega(\mathbf{u}) = 1\} \text{ for } (\omega, \xi) \in \Omega \times \Xi,$$

and the (almost surely unique) edge set

$$\begin{aligned} \mathscr{E} = \Big\{ &< \mathbf{u}, \mathbf{v} >: \mathbf{u}, \mathbf{v} \in \mathscr{V}, \text{ and for some } h \geq 1, \mathbf{v} \in \Lambda(\mathbf{u}, h), \\ &\Lambda(\mathbf{u}, h-1) \cap \mathscr{V} = \emptyset \text{ and } U_{\mathbf{u},\mathbf{v}} \leq U_{\mathbf{u},\mathbf{w}} \text{ for all } \mathbf{w} \in \Lambda(\mathbf{u}, h) \cap \mathscr{V} \Big\}. \end{aligned} \tag{2}$$

The graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is the object of our study here. The construction of the edge-set ensures that, almost surely, there is exactly one edge going 'down' and, as such, each connected component of the graph is a tree.

Our first result discusses the structure of the graph and the second result discusses the structure of each connected component of the graph.

Theorem 1. *For $0 < p < 1$ we have, almost surely*

(i) *for $d = 2, 3$, the graph $\mathscr{G}$ is almost surely connected and consists of a connected tree*

(ii) *for $d \geq 4$ the graph $\mathscr{G}$ is almost surely disconnected and consists of infinitely components each of which is a tree.*

While the model guarantees that no river source terminates in the downward direction, this is not the case in the upward direction. This is our next result.

Theorem 2. *For $d \geq 2$, the graph $\mathscr{G}$ contains no bi-infinite path almost surely.*

Our specific choice of 'right-angled' cones is not important for the results. Thus if, for some $1 < a < \infty$ we had $\Lambda_a(\mathbf{u}, h) = \cup_{a=1}^{h} H_a(\mathbf{u}, k)$ where $H_a(\mathbf{u}, k) = \{\mathbf{v} \in \mathbb{Z}^d : v_d = u_d - k$, and $||\mathbf{v} - m_k(\mathbf{u})||_{L_1} \leq ak\}$ then also our results hold. In the case $a = \infty$, this corresponds to the model considered in [8]. The results would also generalise to the model considered in [6].

Using the notation as in (2), we chose the "nearest" vertex at level h uniformly among all open vertices available at that level to connect to the vertex $\mathbf{u}$. One could relax the latter and choose among all open vertices available at level h in any random fashion. If the random fashion is symmetric in nature then our results will still hold.

For proving Theorem 1 we first observe that the river flowing down from any open point $\mathbf{u}$ is a random walk on $\mathbb{Z}^d$. The walk jumps downward only in the d-th coordinate and also conditional on this jump the new position in the first $d-1$ coordinates are given by a symmetric distribution. Then the broad idea of establishing the results for the case $d = 2, 3$ is that we show that two random walks starting from two arbitrary open points $\mathbf{u}$ and $\mathbf{v}$ meet in finite time almost surely. The random walks are dependent and as oppposed to the model considered in [8] they also carry information,(which we call history), as they traverse downwards in $\mathbb{Z}^d$. The second fact makes the problem harder to work with. Consequently, we construct a suitable Markov chain which carries the "history set" along with it and then find a suitable Lyaponov function to establish the Foster's criterion for recurrence (See Lemma 2.2). We also benefit from the observation that this "history set" is always a triangle. The precise definitions and the complete proof of Theorem 1 (i), is presented in Section 2.

For proving Theorem 1 (ii) one shows that two random walks starting from two open points $\mathbf{u}$ and $\mathbf{v}$ which are far away do not ever meet. Once the starting points are far away one is able to couple these dependent random walks with a system of independent random walks. The coupling has to be done carefully because of the "history set" information in the two walks. To finish the proof one needs estimates regarding intersections of two independent random walks and these may be of intrinsic interest (See Lemma 3.2 and Lemma 3.3).The details are worked out in Section 3.

The proof of Theorem 2 requires a delicate use of the Burton–Keane argument regarding the embedding of trees in an Euclidean space. This is carried out in Section 4.

2 Dimensions 2 and 3

In this section we prove Theorem 1(i). We present the proof for $d = 2$ and later outline the modifications required to prove the theorem for $d = 3$. To begin with we have a collection $\{U_{\mathbf{u},\mathbf{v}} : \mathbf{v} \in \Lambda(\mathbf{u}, h), h \geq 1, \mathbf{u} \in \mathbb{Z}^d\}$ of i.i.d. uniform $(0, 1)$ random variables.

Consider two distinct vertices $\mathbf{u} := (u_1, u_2)$ and $\mathbf{v} := (v_1, v_2)$ where

$$\mathbf{u}, \mathbf{v} \text{ are such that } |u_1 - v_1| > 1 \text{ and } v_2 = u_2 - 1; \tag{3}$$

This ensures that $\mathbf{u} \notin \Lambda(\mathbf{v},h)$, $\mathbf{v} \notin \Lambda(\mathbf{u},h)$ for any $h \geq 1$. We will show that *given* $\mathbf{u}$ and $\mathbf{v}$ open, they are contained in the same component of $\mathscr{G}$ with probability 1.

This suffices to prove the theorem, because if two open vertices $\mathbf{u}$ and $\mathbf{v}$ do not satisfy the condition (3) then, almost surely, we may get open vertices $\mathbf{w_1}, \dots, \mathbf{w_n}$ such that each of the pairs $\mathbf{w_i}$ and $\mathbf{w_{i+1}}$ as well as the pairs $\mathbf{u}$ and $\mathbf{w_1}$, and $\mathbf{v}$ and $\mathbf{w_n}$ satisfy the condition (3). This ensures that all the above vertices, and hence both $\mathbf{u}$ and $\mathbf{v}$ belong to the same component of $\mathscr{G}$.

To prove our contention we construct the process *dynamically* from the two vertices $\mathbf{u}$ and $\mathbf{v}$ as given in (3). The construction will guarantee that the process obtained is equivalent in law to the marginal distributions of the 'trunk' of the trees as seen from $\mathbf{u}$ and $\mathbf{v}$ in $\mathscr{G}$. Without loss of generality we take

$$\mathbf{u} := (0,0) \text{ and } \mathbf{v} := (l_0, -1). \tag{4}$$

Note that all the processes we construct in this section are independent of those constructed in Section 1.1.

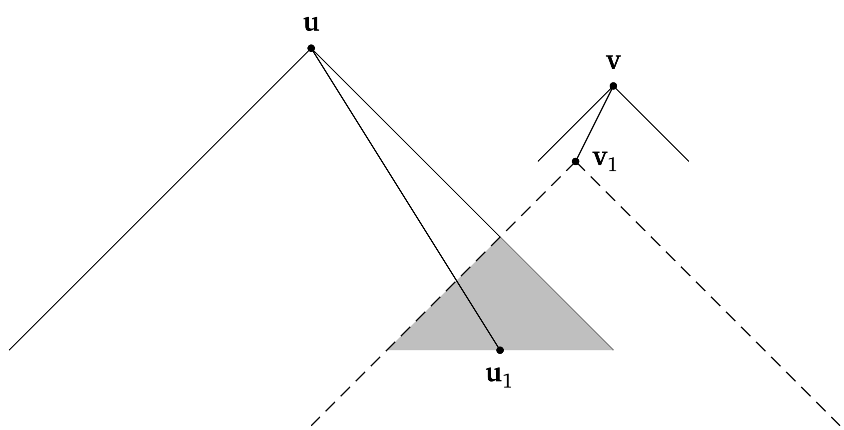


Figure 4: The construction of the process from $\mathbf{u}$ and $\mathbf{v}$.

Before we embark on the formal details of the construction we present the main ideas. From the points $\mathbf{u}$ and $\mathbf{v}$ we look at the region $\Lambda(\mathbf{u}) \cup \Lambda(\mathbf{v})$. On this region we label the vertices *open* or *closed* independently and find the vertices $\mathbf{u}_1$ and $\mathbf{v}_1$ to which $\mathbf{u}$ and $\mathbf{v}$ connect (respectively) according to the mechanism of constructing edges given in Section 1.1. Having found $\mathbf{u}_1$ and $\mathbf{v}_1$ we do the same process again for these vertices. However now we have to remember that we are carrying a *history*, i.e., there is a region, given by the shaded triangle in Figure 4, whose configuration we know. In case the vertical distance between $\mathbf{u}_1$ and $\mathbf{v}_1$ is much larger than the horizontal distance or the history set is non-empty we move the vertex on the top (see Figures 5 and 6), otherwise we move both the vertices simultaneously (see Figure 7).

Note that in this construction, the history will always be a triangular region. Also, at every move during the construction of the process we will keep a track of the vertical height, the horizontal length and the history. This triplet will form a Markov process whose recurrence properties will be used to establish Theorem 1(i).

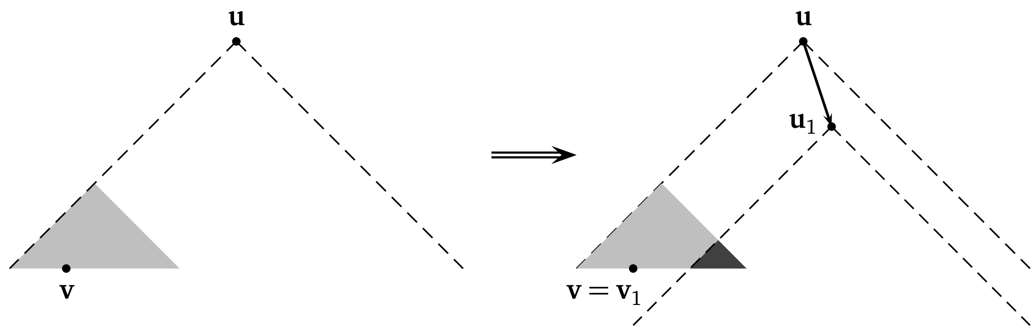


Figure 5: The movement of the vertices : case non-empty history (the lightly shaded region), only the top vertex moves, the darker shaded region in (b) is the new history.

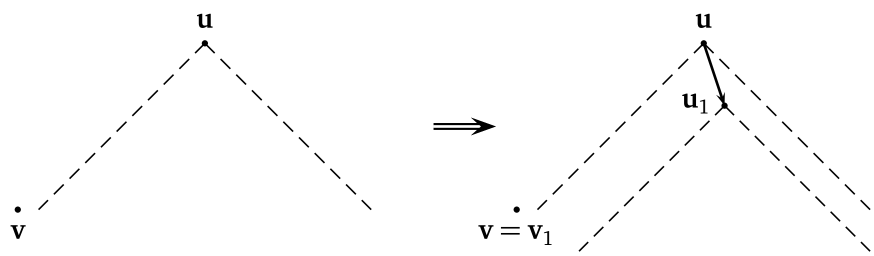


Figure 6: The movement of the vertices : case empty history, ratio of height to length is large. Only the top vertex moves.

We now present a formal construction of the model. In Section 3, we will give a simpler construction, however for the purposes of establishing the Markovian structure as stated in the preceding paragraph, the following construction comes in handy.

Let

$$H_0 = 1,\ L_0 = |l_0| \text{ and } V_0 = \emptyset.$$

Now let $\omega_{\mathbf{u},\mathbf{v}} \in \{0,1\}^{\Lambda(\mathbf{u})\cup\Lambda(\mathbf{v})}$. Let $h_{\mathbf{u}} := \inf\{h : \omega_{\mathbf{u},\mathbf{v}}(\mathbf{w}) = 1 \text{ for some } \mathbf{w} \in \Lambda(\mathbf{u}, h)\}$ and $h_{\mathbf{v}} := \inf\{h : \omega_{\mathbf{u},\mathbf{v}}(\mathbf{w}) = 1 \text{ for some } \mathbf{w} \in \Lambda(\mathbf{v}, h)\}$. Note that under the product measure $\mathbb{P}$ as defined earlier via the marginals (1) on $\{0,1\}^{\Lambda(\mathbf{u})\cup\Lambda(\mathbf{v})}$, the quantities $h_{\mathbf{u}}$ and $h_{\mathbf{v}}$ are finite for $\mathbb{P}$-almost all $\omega_{\mathbf{u},\mathbf{v}}$.

Let $\mathbf{u}_1 := (u_1(1), u_1(2)) \in H(\mathbf{u}, h_{\mathbf{u}})$ be such that $\omega_{\mathbf{u},\mathbf{v}}(\mathbf{u_1}) = 1$ and $U_{\mathbf{u},\mathbf{u}_1} \leq U_{\mathbf{u},\mathbf{w}}$ for all $\mathbf{w} \in H(\mathbf{u}, h_{\mathbf{u}})$ with $\omega_{\mathbf{u},\mathbf{v}}(\mathbf{w}) = 1$. Similarly let $\mathbf{v}_1 := (v_1(1), v_1(2)) \in H(\mathbf{v}, h_{\mathbf{v}})$ be such that $\omega_{\mathbf{u},\mathbf{v}}(\mathbf{v_1}) = 1$ and $U_{\mathbf{v},\mathbf{v}_1} \leq U_{\mathbf{v},\mathbf{w}}$ for all $\mathbf{w} \in H(\mathbf{v}, h_{\mathbf{v}})$ with $\omega_{\mathbf{u},\mathbf{v}}(\mathbf{w}) = 1$. Further define, $H_1 = |u_1(2) - v_1(2)|$, $L_1 = |u_1(1) - v_1(1)|$ and $V_1 = (\Lambda(\mathbf{v}_1) \cap \Lambda(\mathbf{u}, h_{\mathbf{u}})) \cup (\Lambda(\mathbf{u}_1) \cap \Lambda(\mathbf{v}, h_{\mathbf{v}}))$.

Having obtained $\mathbf{u}_k := (u_k(1), u_k(2))$, $\mathbf{v}_k := (v_k(1), v_k(2))$ and H_k, L_k and V_k we consider the following cases

(i) if $V_k \neq \emptyset$ or if $V_k = \emptyset$ and $H_k/L_k \geq C_0$ for a constant C_0 to be specified later in (15) and

  (a) if $u_k(2) \geq v_k(2)$ (see Figure 5 for $V_k \neq \emptyset$ and Figure 6 for $V_k = \emptyset$) then we set

$$\mathbf{v}_{k+1} := \mathbf{v}_k$$

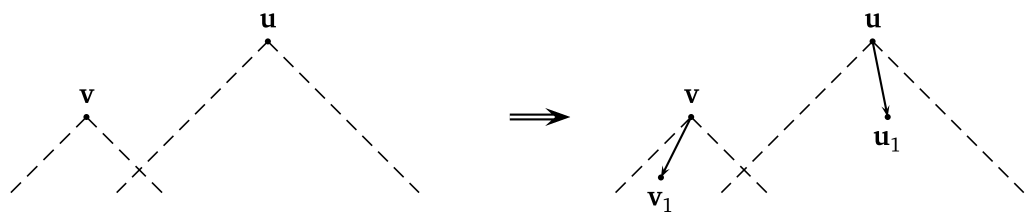


Figure 7: The movement of the vertices : case empty history, ratio of height to length is small. Both the vertices move.

and consider $\omega \in \{0,1\}^{\Lambda(\mathbf{u}_k)\setminus V_k}$. Let

$$\omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) := \begin{cases} \omega(\mathbf{w}) & \text{if } \mathbf{w} \in \Lambda(\mathbf{u}_k) \setminus V_k, \\ \omega_{\mathbf{u}_{k-1},\mathbf{v}_{k-1}}(\mathbf{w}) & \text{if } \mathbf{w} \in V_k \cap \Lambda(\mathbf{u}_k) \end{cases}$$

and let $h_{\mathbf{u}_k} := \inf\{h : \omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) = 1 \text{ for some } \mathbf{w} \in \Lambda(\mathbf{u}_k, h)\}$. Again under the product measure $\mathbb{P}$ on $\Lambda(\mathbf{u}_k)$ such a $h_{\mathbf{u}_k}$ is finite almost surely.

Now let $\mathbf{u}_{k+1} := (u_{k+1}(1), u_{k+1}(2))$ be such that $\omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{u_{k+1}}) = 1$ and $U_{\mathbf{u}_k,\mathbf{u}_{k+1}} \leq U_{\mathbf{u}_k,\mathbf{w}}$ for all $\mathbf{w} \in H(\mathbf{u}_k, h_{\mathbf{u}_k})$ with $\omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) = 1$.

We take

$H_{k+1} = |u_{k+1}(2) - v_{k+1}(2)|$, $L_{k+1} = |u_{k+1}(1) - v_{k+1}(1)|$ and

$V_{k+1} = (\Lambda(\mathbf{v}_{k+1}) \cap \Lambda(\mathbf{u}_k, h_{\mathbf{u}_k})) \cup (\Lambda(\mathbf{u}_{k+1}) \cap V_k))$;

(before we proceed further we note that either $\Lambda(\mathbf{v}_{k+1}) \cap \Lambda(\mathbf{u}_k)$ or $\Lambda(\mathbf{u}_{k+1}) \cap V_k$ is empty – the set which is non-empty is necessarily a triangle)

(b) if $u_k(2) < v_k(2)$ then we set

$$\mathbf{u}_{k+1} := \mathbf{u}_k$$

and consider $\omega \in \{0,1\}^{\Lambda(\mathbf{v}_k)\setminus V_k}$. Let

$$\omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) := \begin{cases} \omega(\mathbf{w}) & \text{if } \mathbf{w} \in \Lambda(\mathbf{v}_k) \setminus V_k, \\ \omega_{\mathbf{u}_{k-1},\mathbf{v}_{k-1}}(\mathbf{w}) & \text{if } \mathbf{w} \in V_k \cap \Lambda(\mathbf{v}_k) \end{cases}$$

and let $h_{\mathbf{v}_k} := \inf\{h : \omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) = 1 \text{ for some } \mathbf{w} \in \Lambda(\mathbf{v}_k, h)\}$. Again under the product measure $\mathbb{P}$ on $\Lambda(\mathbf{v}_k)$ such a $h_{\mathbf{v}_k}$ is finite almost surely.

Now let $\mathbf{v}_{k+1} := (v_{k+1}(1), v_{k+1}(2))$ be such that $\omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{v_{k+1}}) = 1$ and $U_{\mathbf{v}_k,\mathbf{v}_{k+1}} \leq U_{\mathbf{v}_k,\mathbf{w}}$ for all $\mathbf{w} \in H(\mathbf{v}_k, h_{\mathbf{v}_k})$ with $\omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) = 1$. We take

$H_{k+1} = |u_{k+1}(2) - v_{k+1}(2)|$, $L_{k+1} = |u_{k+1}(1) - v_{k+1}(1)|$ and

$V_{k+1} = (\Lambda(\mathbf{u}_{k+1}) \cap \Lambda(\mathbf{v}_k, h_{\mathbf{v}_k})) \cup (\Lambda(\mathbf{v}_{k+1}) \cap V_k))$

(again, note that either $\Lambda(\mathbf{u}_{k+1}) \cap \Lambda(\mathbf{v}_k)$ or $\Lambda(\mathbf{v}_{k+1}) \cap V_k$ is empty – the set which is non-empty is necessarily a triangle);

(ii) if $V_k = \emptyset$, and $H_k/L_k < C_0$ (See Figure 7) then we take $\omega_{\mathbf{u}_k,\mathbf{v}_k} \in \{0,1\}^{\Lambda(\mathbf{u}_k)\cup\Lambda(\mathbf{v}_k)}$. Let $h_{\mathbf{u}_k} := \inf\{h : \omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) = 1 \text{ for some } \mathbf{w} \in \Lambda(\mathbf{u}_k, h)\}$ and $h_{\mathbf{v}_k} := \inf\{h : \omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) = 1 \text{ for some } \mathbf{w} \in \Lambda(\mathbf{v}_k, h)\}$. Again under the product measure $\mathbb{P}$ the quantities $h_{\mathbf{u}_k}$ and $h_{\mathbf{v}_k}$ are finite for $\mathbb{P}$-almost all $\omega_{\mathbf{u}_k,\mathbf{v}_k}$.

Let $\mathbf{u}_{k+1} := (u_{k+1}(1), u_{k+1}(2))$ be such that $\omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{u_{k+1}}) = 1$ and $U_{\mathbf{u}_k,\mathbf{u}_{k+1}} \leq U_{\mathbf{u}_k,\mathbf{w}}$ for all $\mathbf{w} \in H(\mathbf{u}_k, h_{\mathbf{u}_k})$ with $\omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) = 1$.

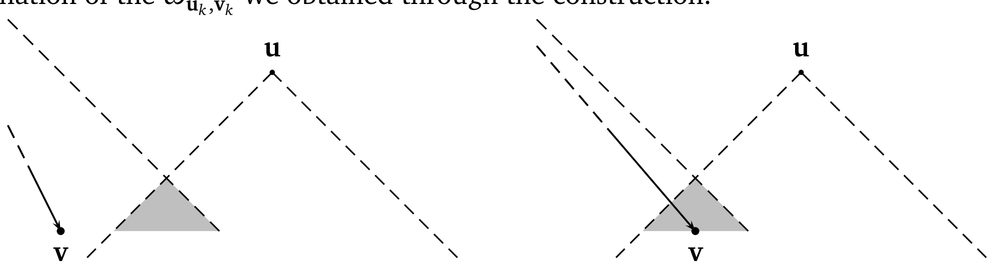
Similarly let $\mathbf{v}_{k+1} := (v_{k+1}(1), v_{k+1}(2))$ be such that $\omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{v_{k+1}}) = 1$ and $U_{\mathbf{v}_k,\mathbf{v}_{k+1}} \le U_{\mathbf{v}_k,\mathbf{w}}$ for all $\mathbf{w} \in H(\mathbf{v}_k, h_{\mathbf{v}_k})$ with $\omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) = 1$.

We take
$H_{k+1} = |u_{k+1}(2) - v_{k+1}(2)|$, $L_{k+1} = |u_{k+1}(1) - v_{k+1}(1)|$ and
$V_{k+1} = (\Lambda(\mathbf{v}_{k+1}) \cap \Lambda(\mathbf{u}_k, h_{\mathbf{u}_k})) \cup (\Lambda(\mathbf{u}_{k+1}) \cap \Lambda(\mathbf{v}_k, h_{\mathbf{v}_k}))$.
(again, note that either $\Lambda(\mathbf{v}_{k+1}) \cap \Lambda(\mathbf{u}_k, h_{\mathbf{u}_k})$ or $\Lambda(\mathbf{u}_{k+1}) \cap \Lambda(\mathbf{v}_k, h_{\mathbf{v}_k})$ is empty – the set which is non-empty is necessarily a triangle).

We will now briefly sketch the connection between the above construction and the model as described in Section 1.1. For this with a slight abuse of notation we take $\omega_{\mathbf{u}_k,\mathbf{v}_k}$ to be the sample point $\omega_{\mathbf{u}_k,\mathbf{v}_k}$ restricted to the set $\Lambda(\mathbf{u}_k, h_{\mathbf{u}_k}) \cup \Lambda(\mathbf{v}_k, h_{\mathbf{v}_k})$. Also let $\omega_1 \in \{0,1\}^{\mathbb{Z}^2 \setminus \cup_{k=0}^{\infty}(\Lambda(\mathbf{u}_k, h_{\mathbf{u}_k}) \cup \Lambda(\mathbf{v}_k, h_{\mathbf{v}_k}))}$, where $\mathbf{u}_0 = \mathbf{u}$ and $\mathbf{v}_0 = \mathbf{v}$. Now define $\omega \in \Omega$ as

$$\omega(\mathbf{w}) = \begin{cases} \omega_{\mathbf{u}_k,\mathbf{v}_k}(\mathbf{w}) & \text{if } \mathbf{w} \in \Lambda(\mathbf{u}_k, h_{\mathbf{u}_k}) \cup \Lambda(\mathbf{v}_k, h_{\mathbf{v}_k}) \text{ for some } k \\ \omega_1(\mathbf{w}) & \text{otherwise.} \end{cases}$$

Thus for every sample path from $\mathbf{u}$ and $\mathbf{v}$ obtained by our construction we obtain a realisation of our graph with $\mathbf{u}$ and $\mathbf{v}$ open. Conversely if $\omega \in \Omega$ gives a realisation of our graph with $\mathbf{u}$ and $\mathbf{v}$ open then we label vertices $\mathbf{u}_i$ and $\mathbf{v}_i$ as vertices such that $< \mathbf{u}_{i-1}, \mathbf{u}_i >$ is an edge in the realisation with $h_{\mathbf{u}_i} := \mathbf{u}_i(2) - \mathbf{u}_{i+1} > 0$, and $h_{\mathbf{v}_i} := \mathbf{v}_i(2) - \mathbf{v}_{i+1}(2) > 0$ is an edge in the realisation with $\mathbf{v}_i(2) < \mathbf{v}_{i-1}(2)$. Now the restriction of ω on $\cup_{i=0}^{\infty}(\Lambda(\mathbf{u}_i, h_{\mathbf{u}_i}) \cup \Lambda(\mathbf{v}_i, h_{\mathbf{v}_i}))$ will correspond to the concatenation of the $\omega_{\mathbf{u}_k,\mathbf{v}_k}$ we obtained through the construction.



(a) **(b)**

Figure 8: The geometry of the history region.

Lemma 2.1. *The process $\{(H_k, L_k, V_k) : k \ge 0\}$ is a Markov process with state space $\mathscr{S} = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \times \{\Lambda(\mathbf{w}, h)), \mathbf{w} \in \mathbb{Z}^2, h \ge 0\}$.*

Proof: If $\mathbf{u}_k = \mathbf{u}$ and $\mathbf{v}_k = \mathbf{v}$ are as in Figure 8 (a) or (b) (i.e. $\mathbf{u}_k(2) \ge \mathbf{v}_k(2)$) then we make the following observations about the history region V_k.

Observations:

(i) V_k is either empty or a triangle (i.e. the shaded region in the figure),

(ii) all vertices in the triangle V_k, except possibly on the base of the triangle, are closed under $\omega_{\mathbf{u}_k,\mathbf{v}_k}$,

(iii) the base of the triangle V_k must be on the horizontal line containing the vertex $\mathbf{v}$,

(iv) one of the sides of the triangle V_k must lie on the boundary $B(\mathbf{u}, H_k)$ of $\Lambda(\mathbf{u}, H_k)$, while the other side does *not* lie on $B(\mathbf{u}, H_k)$ unless $\mathbf{u}$ and $\mathbf{v}$ coincide,

(v) the side which does not lie on $B(\mathbf{u}, H_k)$ is determined by the location of $\mathbf{v}_{k-1}$,

(vi) if $\mathbf{u}_k(2) = \mathbf{v}_k(2)$ then $V_k = \emptyset$,

(vii) the vertex $\mathbf{u}_{k+1}$ may lie on the base of the triangle, but not anywhere else in the triangle.

While observations (ii) – (vi) are self-evident, the reason for (i) above is that if the history region has two or more triangles, then there must necessarily be a fourth vertex $\mathbf{w}$ besides the vertices under consideration, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{v}_{k-1}$ which initiated the second triangle. This vertex $\mathbf{w}$ must either be a vertex $\mathbf{u}_j$ for some $j \leq k-1$ or a vertex $\mathbf{v}_j$ for some $j \leq k-2$. In the former case, the history region due to $\mathbf{w}$ must lie above $\mathbf{u}_k$ and in the latter case it must lie above $\mathbf{v}_{k-1}$. In either case it cannot have any non-empty intersection with the region $\Lambda(\mathbf{u})$.

In Figure 8 (a) where the vertex $\mathbf{v}$ does *not* lie on the base of the shaded triangle, however it lies on the horizontal line containing the base, there may be open vertices on the base of the triangle. If that be the case, the vertex $\mathbf{u}_{k+1}$ may lie anywhere in the triangle subtended by the sides emanating from the vertex $\mathbf{u}$ and the horizontal line; otherwise it may lie anywhere in the region $\Lambda(\mathbf{u})$.

In Figure 8 (b) where the vertex $\mathbf{v}$ lies on the base of the shaded triangle, the vertex $\mathbf{u}_{k+1}$ may lie anywhere in the triangle subtended by the sides emanating from the vertex $\mathbf{u}$ and the horizontal line.

Finally having obtained $\mathbf{u}_{k+1}$, if $V_k \neq \emptyset$ or if $V_k = \emptyset$ and $H_k/L_k \geq C_0$ then we take $\mathbf{v}_{k+1} = \mathbf{v}_k$; otherwise we obtain $\mathbf{v}_{k+1}$ by considering the region $\Lambda(\mathbf{v})$ and remembering that in obtaining $\mathbf{u}_{k+1}$ we may have already specified the configuration of a part of the region in $\Lambda(\mathbf{v})$.

The new history region V_{k+1} is now determined by the vertices $\mathbf{u}_k$, $\mathbf{v}_k$, $\mathbf{u}_{k+1}$ and $\mathbf{v}_{k+1}$.

This justifies our claim that $\{(H_k, L_k, V_k) : k \geq 0\}$ is a Markov process. $\square$

We will now show that

$$\mathbb{P}\{(H_k, L_k, V_k) = (0, 0, \emptyset) : \text{ for some } k \geq 0\} = 1. \tag{5}$$

We use the Foster-Lyapunov criterion (see Asmussen [2], Proposition 5.3, Chapter I) to establish this result.

Proposition 2.1. *(Foster-Lyapunov criterion) An irreducible Markov chain with stationary transition probability matrix $((p_{ij}))$ is recurrent if there exists a function $f : E \to \mathbb{R}$ such that $\{i : f(i) < K\}$ is finite for each K and $\sum_{k \in E} p_{jk} f(k) \leq f(j)$, $j \notin E_0$, for some finite subset E_0 of the state space E.*

For this we change the Markov process slightly. We define a new Markov process with state space $\mathscr{S}$ which has the same transition probabilities as $\{(H_k, L_k, V_k) : k \geq 0\}$ except that instead of $(0, 0, \emptyset)$ being an absorbing state, we introduce a transition

$$\mathbb{P}\{(0, 0, \emptyset) \to (0, 1, \emptyset)\} = 1.$$

We will now show using Lyapunov's method that this modified Markov chain is recurrent. This will imply that $\mathbb{P}\{(H_k, L_k, V_k) = (0, 0, \emptyset) : \text{ for some } k \geq 0\} = 1$. With a slight abuse of notation we let the modified Markov chain be denoted by $\{(H_k, L_k, V_k) : k \geq 0\}$.

Define a function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ by :

$$g(x) = \log(1+x). \tag{6}$$

Some properties of g :

$$\begin{aligned} g^{(1)}(x) &= \frac{1}{1+x}, \\ g^{(2)}(x) &= \frac{-1}{(1+x)^2} < 0, \\ g^{(3)}(x) &= \frac{2}{(1+x)^3}, \\ g^{(4)}(x) &= \frac{-6}{(1+x)^4} < 0 \text{ for all } x > 0. \end{aligned}$$

Thus, using Taylor's expansion and above formulae for derivatives, we have two inequalities: for $x, x_0 \in \mathbb{R}_+$

$$g(x) - g(x_0) \leq \frac{(x-x_0)}{(1+x_0)} \tag{7}$$

and

$$\begin{aligned} g(x) - g(x_0) &\leq \frac{(x-x_0)}{(1+x_0)} - \frac{(x-x_0)^2}{2(1+x_0)^2} + \frac{(x-x_0)^3}{3(1+x_0)^3} \\ &= \frac{1}{6(1+x_0)^3}\Big[6(1+x_0)^2(x-x_0) - 3(1+x_0)(x-x_0)^2 + (x-x_0)^3\Big]. \end{aligned} \tag{8}$$

Now, define $f : \mathscr{S} \mapsto \mathbb{R}_+$ by,

$$f(h,l,V) = h^4 + l^4. \tag{9}$$

Also, we define $u : \mathscr{S} \mapsto \mathbb{R}_+$ by

$$u(h,l,V) = g(f(h,l,V)) + (|V|) = \log\Big[1 + (l^4 + h^4)\Big] + (|V|) \tag{10}$$

where $|V|$ denotes the cardinality of V.

Lemma 2.2. *For all but finitely many $(h,l,V) \in \mathscr{S}$, we have*

$$\mathbb{E}\Big(u(H_{n+1}, L_{n+1}, V_{n+1}) \mid (H_n, L_n, V_n) = (h,l,V)\Big) \leq u(h,l,V). \tag{11}$$

Proof : Before we embark on the proof we observe that using the inequality (7) and the expression

(9), we have that

$$
\begin{aligned}
& \mathbb{E}\Big[u(H_{n+1},L_{n+1},V_{n+1})-u(h,l,V)|(H_n,L_n,V_n)=(h,l,V)\Big] \\
&= \mathbb{E}\Big[g(f(H_{n+1},L_{n+1},V_{n+1})-g(f(h,l,V))|(H_n,L_n,V_n)=(h,l,V)\Big] \\
&\quad +\mathbb{E}\Big[|V_{n+1}|-|V||(H_n,L_n,V_n)=(h,l,V)\Big] \\
&= \mathbb{E}\Big[\log(1+H_{n+1}^4+L_{n+1}^4)-\log(1+h^4+l^4)|(H_n,L_n,V_n)=(h,l,V)\Big] \\
&\quad +\mathbb{E}\Big[|V_{n+1}|-|V||(H_n,L_n,V_n)=(h,l,V)\Big] \\
&\le \frac{1}{1+h^4+l^4}\mathbb{E}\Big[H_{n+1}^4+L_{n+1}^4-(h^4+l^4)|(H_n,L_n,V_n)=(h,l,V)\Big] \\
&\quad +\mathbb{E}\Big[|V_{n+1}|-|V||(H_n,L_n,V_n)=(h,l,V)\Big]. \qquad (12)
\end{aligned}
$$

Let $a_0=0$ and for $k\ge 1$ let $b_k=2k+1$ and $a_k=\sum_{i=1}^k b_i$. We define two integer valued random variables T and D whose distributions are given by:

$$
\begin{aligned}
\mathbb{P}(T=k) &= (1-p)^{a_{k-1}}(1-(1-p)^{b_k}) \text{ for } k\ge 1 \qquad (13)\\
\mathbb{P}(D=j|T=k) &= \frac{1}{2k+1} \text{ for } -k\le j\le k \qquad (14)
\end{aligned}
$$

For $i=1,2$, let T_i,D_i be i.i.d. copies of T,D. Let

$$
\begin{aligned}
\beta_1 &= \mathbb{E}(D_1-D_2)^2\\
\beta_2 &= \mathbb{E}(T_1-T_2)^2
\end{aligned}
$$

Let $0<C_0<1$ be small enough so that

$$36(1+C_0^4)(\beta_1+C_0^2\beta_2)-48\beta_1<0 \qquad (15)$$

We shall consider three cases for establishing (11)

Case 1: V is non-empty.

We will prove the following inequalities:

$$\sup_{(h,l,V):V\neq\emptyset,h\ge 1}\frac{\mathbb{E}\Big[H_{n+1}^4-h^4|(H_n,L_n,V_n)=(h,l,V)\Big]}{h^3}\le C_1, \qquad (16)$$

$$\sup_{(h,l,V):V\neq\emptyset,l\ge 1}\frac{\mathbb{E}\Big[L_{n+1}^4-l^4|(H_n,L_n,V_n)=(h,l,V)\Big]}{l^3}\le C_1 \qquad (17)$$

and

$$\mathbb{E}\Big[|V|-|V_{n+1}||(H_n,L_n,V_n)=(h,l,V)\Big]\ge C_2-C_3\exp(-C_4(h+l)/2) \qquad (18)$$

where C_1, C_2, C_3 and C_4 are positive constants. Putting the inequalities (16), (17) and (18) in (12), for all $(h, l, V) \in \mathscr{S}$ with V non-empty, we have

$$\begin{aligned}&\mathbb{E}\Big[u(H_{n+1}, L_{n+1}, V_{n+1}) - u(h, l, V) | (H_n, L_n, V_n) = (h, l, V)\Big] \\ &\leq \frac{C_1 h^3 + C_1 l^3}{1 + h^4 + l^4} - C_2 + C_3 \exp(-C_4(h + l)/2) < 0\end{aligned}$$

for all (h, l) such that $h + l$ sufficiently large. Therefore, outside a finite number of choices of (h, l), we have the above inequality. Now, for these finite choices for (h, l), there are only finitely many possible choices of V which is non-empty. This takes care of the case when the history is non-empty.

Now, we prove (16), (17) and (18). Define the random variables

$$T_{n+1} = \max(\mathbf{u}_n(2) - \mathbf{u}_{n+1}(2), \mathbf{v}_n(2) - \mathbf{v}_{n+1}(2)) \tag{19}$$

and

$$D_{n+1} = (\mathbf{v}_n(1) - \mathbf{v}_{n+1}(1)) - (\mathbf{u}_n(1) - \mathbf{u}_{n+1}(1)). \tag{20}$$

It is easy to see that the conditional distribution of H_{n+1} given $(H_n, L_n, V_n) = (h, l, V)$ is same as that of the conditional distribution of $|T_{n+1} - h|$ given $\mathbf{u}_0, \dots, \mathbf{u}_n$, $\mathbf{v}_0, \dots, \mathbf{v}_n$ and $h = |\mathbf{u}_n(2) - \mathbf{v}_n(2)|$.

Note that, for any given non-empty set V, as noted in observations (iv), at least one diagonal line will be unexplored, and consequently the conditional distribution T_{n+1} given $\mathbf{u}_0, \dots, \mathbf{u}_n$, $\mathbf{v}_0, \dots, \mathbf{v}_n$, is dominated by a geometric random variable G with parameter p. Thus, for any $k \geq 1$

$$\mathbb{E}(T_{n+1}^k | \mathbf{u}_0, \dots, \mathbf{u}_n, \ \mathbf{v}_0, \dots, \mathbf{v}_n) \leq \mathbb{E}(G^k) < \infty$$

Therefore, we have

$$\begin{aligned}&\mathbb{E}\Big[H_{n+1}^4 - h^4 \Big| (H_n, L_n, V_n) = (h, l, V)\Big] \\ &= \mathbb{E}\Big[(T_{n+1} - h)^4 - l^4 \Big| \mathbf{u}_0, \dots, \mathbf{u}_n, \mathbf{v}_0, \dots, \mathbf{v}_n, h = |\mathbf{u}_n(2) - \mathbf{v}_n(2)|, l = |\mathbf{u}_n(1) - \mathbf{v}_n(1)|\Big] \\ &\leq \mathbb{E}\Big[(G - h)^4 - h^4\Big] \\ &\leq h^3 C_1\end{aligned} \tag{21}$$

for a suitable choice of C_1.

Similarly, we have that the conditional distribution of L_{n+1} given $(H_n, L_n, V_n) = (h, l, V)$ is same as that of $|D_{n+1} - l|$ given $\mathbf{u}_0, \dots, \mathbf{u}_n$, $\mathbf{v}_0, \dots, \mathbf{v}_n$ and $l = |\mathbf{u}_n(1) - \mathbf{v}_n(1)|$. Since $D_n \leq 2T_n$ for every n, we have,

$$\begin{aligned}&\mathbb{E}\Big[L_{n+1}^4 - l^4 \Big| (H_n, L_n, V_n) = (h, l, V)\Big] \\ &= \mathbb{E}\Big[(D_{n+1} - l)^4 - l^4 \Big| \mathbf{u}_0, \dots, \mathbf{u}_n, \mathbf{v}_0, \dots, \mathbf{v}_n, h = |\mathbf{u}_n(2) - \mathbf{v}_n(2)|, l = |\mathbf{u}_n(1) - \mathbf{v}_n(1)|\Big] \\ &\leq 8l^3\mathbb{E}(G) + 24l^2\mathbb{E}(G^2) + 32l\mathbb{E}(G^3) + 16\mathbb{E}(G^4) \\ &\leq l^3 C_1.\end{aligned}$$

For the inequality (18), we require the following observations:

- If $T_{n+1} \le h$ then $V_{n+1} \subseteq V_n$.
- If $h < T_{n+1} < (h+l)/2$ then $V_{n+1} = \emptyset$.
- If $T_{n+1} \ge (h+l)/2$ then $|V_{n+1}| \le (T_{n+1} - (h+l)/2)^2$.

Further, when $T_{n+1} \le h$, we have

$$\mathbb{P}(|V| - |V_{n+1}| \ge 1) \ge \min\{p(1-p), p/2\} =: \alpha(p).$$

This is seen in the following way. We look at the case when $T_{n+1} = 1$ and connect to the point which will always be unexplored. In that case, $|V| - |V_{n+1}| \ge 1$. Note that if both points on the line $T_{n+1} = 1$ were available, this probability is at least $p(1-p)$. If both points were not available, then there are two possible cases, i.e., the history point on the top line is open or the history point on the top line is closed. In the first case, the probability is $p/2$ while in the second case the probability is p.

Thus, we have,

$$\begin{aligned}
&\mathbb{E}\Big[|V| - |V_{n+1}|\Big|(H_n, L_n, V_n) = (h,l,V)\Big] \\
&= \mathbb{E}\Big[\big[|V| - |V_{n+1}|\big]1(T_{n+1} \le h)\Big|(H_n, L_n, V_n) = (h,l,V)\Big] \\
&\quad + \mathbb{E}\Big[\big[|V| - |V_{n+1}|\big]1(h < T_{n+1} < (h+l)/2)\Big|(H_n, L_n, V_n) = (h,l,V)\Big] \\
&\quad + \mathbb{E}\Big[\big[|V| - |V_{n+1}|\big]1(T_{n+1} \ge (h+l)/2)\Big|(H_n, L_n, V_n) = (h,l,V)\Big] \\
&\ge \mathbb{E}\Big[\big[|V| - |V_{n+1}|\big]1(T_{n+1} \le h)\Big|(H_n, L_n, V_n) = (h,l,V)\Big] \\
&\quad - \mathbb{E}\Big[|V_{n+1}|1(T_{n+1} \ge (h+l)/2)\Big|(H_n, L_n, V_n) = (h,l,V)\Big] \\
&\ge \alpha(p) - \mathbb{E}\Big[(T_{n+1} - (h+l)/2)^2 1(T_{n+1} \ge (h+l)/2)\Big|(H_n, L_n, V_n) = (h,l,V)\Big] \\
&\ge \alpha(p) - \mathbb{E}\Big[T_{n+1}^2 1(T_{n+1} \ge (h+l)/2)\Big|(H_n, L_n, V_n) = (h,l,V)\Big] \\
&\ge \alpha(p) - \mathbb{E}\Big[G^2 1(G \ge (h+l)/2)\Big] \\
&\ge \alpha(p) - C_3 \exp(-C_4(h+l)/2)
\end{aligned}$$

where C_3 and C_4 are positive constants. This completes the proof in the case when history is non-null.

Case 2: $V = \emptyset$, $\frac{h}{|l|} \ge C_0$

For this case, we use the inequality (12) with $|V| = 0$. We will show that, for all h large enough,

$$\mathbb{E}\Big[H_{n+1}^4 - h^4|(H_n, L_n, V_n) = (h,l,\emptyset)\Big] = -C_5 h^3 + O(h^2), \tag{22}$$

$$\mathbb{E}\Big[L_{n+1}^4 - l^4|(H_n, L_n, V_n) = (h,l,\emptyset)\Big] \le C_6 h^2, \tag{23}$$

$$\mathbb{E}\Big[|V_{n+1}||(H_n, L_n, V_n) = (h,l,\emptyset)\Big] \le C_7 \exp(-C_8 h) \tag{24}$$

where C_5, C_6, C_7 and C_8 are positive constants.

Using the above estimates (22), (23) and (24) in (12) with $|V|=0$, we have,

$$\begin{aligned}
&\mathbb{E}\Big[u(H_{n+1},L_{n+1},V_{n+1})-u(h,l,\emptyset)|(H_n,L_n,V_n)=(h,l,\emptyset)\Big]\\
&\leq \frac{-C_5h^3+C_6h^2}{1+h^4+l^4}+C_7\exp(-C_8h)<0 \text{ for all } h \text{ large enough.}
\end{aligned}$$

Now, we prove the estimates (22), (23) and (24). It is easy to see that the conditional distribution of H_{n+1} given $(H_n,L_n,V_n)=(h,l,\emptyset)$ is same as that of $|T_{n+1}-h|$ where T_{n+1} is as defined in (19). We note here that $V=\emptyset$ ensures that T_{n+1} does not depend on the path $\mathbf{u}_0,\dots,\mathbf{u}_n,\mathbf{v}_0,\dots,\mathbf{v}_n$. Therefore, we have

$$\begin{aligned}
&\mathbb{E}\Big[H_{n+1}^4-h^4|(H_n,L_n,V_n)=(h,l,\emptyset)\Big]\\
=\ &\mathbb{E}\Big[(T_{n+1}-h)^4-h^4\Big]\\
=\ &-4h^3\mathbb{E}(T_{n+1})+6h^2\mathbb{E}(T_{n+1}^2)-4h\mathbb{E}(T_{n+1}^3)+\mathbb{E}(T_{n+1}^4)\\
=\ &-h^3C_5+O(h^2)
\end{aligned}$$

where $C_5=4\mathbb{E}(T_{n+1})>0$.

Again, we have that the conditional distribution of L_{n+1} given $(H_n,L_n,V_n)=(h,l,\emptyset)$ is same as that of $|D_{n+1}-l|$. Here too we note here that $V=\emptyset$ ensures that D_{n+1} does not depend on the path $\mathbf{u}_0,\dots,\mathbf{u}_n,\mathbf{v}_0,\dots,\mathbf{v}_n$. Therefore, we have,

$$\begin{aligned}
&\mathbb{E}\Big[L_{n+1}^4-l^4|(H_n,L_n,V_n)=(h,l,\emptyset)\Big]\\
=\ &\mathbb{E}\Big[(D_{n+1}-l)^4-l^4\Big]\\
=\ &4l^3\mathbb{E}(D_{n+1})+6l^2\mathbb{E}(D_{n+1}^2)+4l\mathbb{E}(D_{n+1}^3)+\mathbb{E}(D_{n+1}^4)\\
\leq\ &6l^2\mathbb{E}(|D_{n+1}|^2)+4l\mathbb{E}(|D_{n+1}|^3)+\mathbb{E}(|D_{n+1}|^4)\\
\leq\ &6h^2\mathbb{E}(|D_{n+1}|^2)/C_0^2+4h\mathbb{E}(|D_{n+1}|^3)/C_0+\mathbb{E}(|D_{n+1}|^4)\\
\leq\ &C_6h^2
\end{aligned}$$

for suitable choice of $C_6>0$.

Finally, to prove (24), we observe that if $T_{n+1}<(l+h)/2$ then $V_{n+1}=\emptyset$. If $T_{n+1}\geq(l+h)/2$ then $|V_{n+1}|\leq(T_{n+1}-(l+h)/2)^2$. Therefore, we have

$$\begin{aligned}
&\mathbb{E}\Big[|V_{n+1}||(H_n,L_n,V_n)=(h,l,\emptyset)\Big]\\
\leq\ &\mathbb{E}\Big[(T_{n+1}-(h+l)/2))^2 1(T_{n+1}\geq(h+l)/2)\Big]\\
\leq\ &\mathbb{E}\Big[T_{n+1}^2 1(T_{n+1}\geq(h+l)/2)\Big]\\
\leq\ &\mathbb{E}\Big[T_{n+1}^2 1(T_{n+1}\geq h/2)\Big]\\
\leq\ &C_7\exp(-C_8h)
\end{aligned}$$

for suitable choices of positive constants C_7 and C_8.

Case 3 $V = \emptyset, \frac{h}{l} < C_0$. Using (8), we have,

$$\begin{aligned} & u(H_{n+1}, L_{n+1}, V_{n+1}) - u(h, l, \emptyset) \\ \leq \; & \frac{1}{(1+h^4+l^4)} \Big[6(1+h^4+l^4)^2 \Big[(H_{n+1}^4 + L_{n+1}^4) - (h^4+l^4) \Big] \\ & \quad -3(1+h^4+l^4) \Big[(H_{n+1}^4 + L_{n+1}^4) - (h^4+l^4) \Big]^2 \\ & \quad + \Big[(H_{n+1}^4 + L_{n+1}^4) - (h^4+l^4) \Big]^3 \Big] + |V_{n+1}|. \end{aligned}$$

Taking conditional expectation and denoting, $(H_{n+1}^4 + L_{n+1}^4) - (h^4+l^4)$ by R_n, we have,

$$\begin{aligned} & \mathbb{E}\big(u(H_{n+1}, L_{n+1}, V_{n+1}) - u(h, l, \emptyset) \mid (H_n, L_n, V_n) = (h, l, \emptyset)\big) \\ \leq \; & \frac{1}{(1+h^4+l^4)} \Big[6(1+h^4+l^4)^2 \mathbb{E}\big(R_n \mid (H_n, L_n, V_n) = (h, l, \emptyset)\big) \\ & \quad -3(1+h^4+l^4) \mathbb{E}\big(R_n^2 \mid (H_n, L_n, V_n) = (h, l, \emptyset)\big) \\ & \quad + \mathbb{E}\big(R_n^3 \mid (H_n, L_n, V_n) = (h, l, \emptyset)\big) \Big] \\ & + \mathbb{E}\big(|V_{n+1}| \mid (H_n, L_n, V_n) = (h, l, \emptyset)\big). \end{aligned} \tag{25}$$

We want to show that: for all l large enough,

$$\mathbb{E}\big(R_n \mid (H_n, L_n, V_n) = (h, l, \emptyset)\big) \leq 6l^2[\beta_1 + C_0^2\beta_2] \tag{26}$$

$$\mathbb{E}\big(R_n^2 \mid (H_n, L_n, V_n) = (h, l, \emptyset)\big) \geq 16 l^6 \beta_1 \tag{27}$$

$$\mathbb{E}\big(R_n^3 \mid (H_n, L_n, V_n) = (h, l, \emptyset)\big) \leq C_{10} l^9 \tag{28}$$

$$\mathbb{E}\big(|V_{n+1}| \mid (H_n, L_n, V_n) = (h, l, \emptyset)\big) \leq C_{11} \exp(-C_{12} l) \tag{29}$$

where β_1, β_2 are as defined earlier and C_9, C_{10}, C_{11} and C_{12} are positive constants.
Using the above inequalities in (25), and the definition of C_0, we have,

$$\begin{aligned} & \mathbb{E}\Big[u(H_{n+1}, L_{n+1}, V_{n+1}) - u(h, l, \emptyset) | (H_n, L_n, V_n) = (h, l, \emptyset) \Big] \\ \leq \; & \frac{1}{(1+h^4+l^4)^3} \Big[36(1+h^4+l^4)^2 l^2 (\beta_1 + C_0^2 \beta_2) \\ & \quad -48(1+h^4+l^4) l^6 \beta_1 + C_{10} l^9 \Big] + C_{11} \exp(-C_{12} l) \\ < \; & 0 \text{ for all } l \text{ large enough,} \end{aligned}$$

because the leading term in the first inequality is of the order l^{-2} and it appears with the coefficient $[36(\beta_1 + C_0^2\beta_2) - 48\beta_1]$ which is negative by our choice of C_0 given in (15).

Proof of (26) and (27): Let us assume without loss of generality that $v_n(1) = u_n(1) + l$ and $v_n(2) = u_n(2) + h$. Let $T_{n+1}^v = v_n(2) - v_{n+1}(2)$, $D_{n+1}^v = v_n(1) - v_{n+1}(1)$ and $T_{n+1}^u = u_n(2) - u_{n+1}(2)$, $D_{n+1}^u = u_n(1) - u_{n+1}(1)$. Using this notation, we have

$$
\begin{aligned}
R_n &= [h+(T^v_{n+1}-T^u_{n+1})]^4 - h^4 + (l+(D^v_{n+1}-D^u_{n+1}))^4 - l^4 \\
&= 4h^3(T^v_{n+1}-T^u_{n+1}) + 6h^2(T^v_{n+1}-T^u_{n+1})^2 + 4h(T^v_{n+1}-T^u_{n+1})^3 \\
&\quad +(T^v_{n+1}-T^u_{n+1})^4 + 4l^3(D^v_{n+1}-D^u_{n+1}) + 6l^2(D^v_{n+1}-D^u_{n+1})^2 \\
&\quad +4l(D^v_{n+1}-D^u_{n+1})^3 + (D^v_{n+1}-D^u_{n+1})^4. \qquad (30)
\end{aligned}
$$

Now, it is easy to observe that if both $T^u_{n+1} < (l+h)/2$ and $T^v_{n+1} < (l-h)/2$, both of them will behave independently and the distribution on that set is same as that of T. Further, the tail probabilities of the height distribution T decays exponentially. Thus,

$$\mathbb{E}\left(T^j 1(T > \frac{l-h}{2})\right) = O(\exp(-C_{13}(l-h)) \quad \text{for some constant } C_{13} > 0 \text{ and for all } j \geq 1.$$

Since $0 < C_0 < 1$ we have for all $j \geq 1$ and suitable constants $C_{14}, C_{15} > 0$,

$$
\begin{aligned}
\mathbb{E}((T^v_{n+1}-T^u_{n+1})^j \mid (H_n, L_n, V_n) = (h, l, \emptyset)) &= \mathbb{E}((T_2-T_1)^j) + O(\exp(-C_{14}l)) \\
\mathbb{E}((D^v_{n+1}-D^u_{n+1})^j \mid (H_n, L_n, V_n) = (h, l, \emptyset)) &= \mathbb{E}((D_2-D_1)^j) + O(\exp(-C_{15}l))
\end{aligned}
$$

where T_1, T_2 are i.i.d. copies of T and D_1, D_2 are i.i.d copies of D.

Now, to conclude (26), we just need the observation that all odd moments of the terms $T_1 - T_2$ and $D_1 - D_2$ are 0. Thus in the conditional expectation of (30), we see that the terms involving h^3 and l^3 do not contribute, the coefficient of l^2 in the second term contributes $6\mathbb{E}(D_1 - D_2)^2$, the coefficient of h^2 contributes $6\mathbb{E}(T_1 - T_2)^2$ and all other terms have smaller powers of h and l. From the fact that $h/l < C_0$ and our choice of C_0 given by (15), we conclude the result.

To show (27), studying R_n^2, we note that there are only three terms which are important, (a) coefficient of l^6, (b) coefficient of h^6 and (c) coefficient of h^3l^3. All other terms, are of type h^il^j have $i + j < 6$ and, since $h/l < C_0$, these terms are of order smaller than l^6. The coefficient of l^6 is $16\mathbb{E}\big((D_1 - D_2)^2\big) = 16\beta_1$, the coefficient of h^6 is $16\mathbb{E}\big((T_1 - T_2)^2\big) > 0$ while the coefficient of h^3l^3 is $16\mathbb{E}\big((T_1 - T_2)(D_1 - D_2)\big) = 16\mathbb{E}((T_1 - T_2)\mathbb{E}(D_1 - D_2|T_1, T_2)) = 0$. Thus (27) holds.

Proof of (28): In the expansion of R_n^3, all the terms are of type h^il^j with $i + j \leq 9$. Thus, using the fact that $h/l < C_0$, and all terms have finite expectation, we conclude the result.

Proof of (29) : Finally, the history set V_{n+1} is empty if $T^u_{n+1} < (l+h)/2$ and $T^v_{n+1} < (l-h)/2$. Otherwise, the history is bounded by $(T^u_{n+1}-(l+h)/2)^2 1_{T^u_{n+1}>(l+h)/2} + (T^v_{n+1}-(l-h)/2)^2 1_{T^v_{n+1}>(l-h)/2}$. Again, since the tails probabilities decay exponentially, the above expectations decay exponentially with l.

This completes the proof of Lemma 2.2 and we obtain that

$$\mathbb{E}\Big(u(H_{n+1}, L_{n+1}, V_{n+1}) \mid (H_n, L_n, V_n) = (h, l, V)\Big) \leq u(h, l, V)$$

holds outside $l \geq l_0, h \leq h_0$ and $|V| \leq k$ for some constants l_0, h_0 and k.

Our Markov chain being irreducible on its state space, by the Foster-Lyapunov criteria (Lemma 2.1) we have that it is recurrent. Therefore, we have

$$\mathbb{P}\big[(H_n, L_n, V_n) = (0, 0, \emptyset) \text{ for some } n \geq 1 | (H_0, L_0, V_0) = (h, l, v)\big] = 1$$

for any $(h,l,v) \in \mathscr{S}$.

For $d = 3$ we need to consider the 'width', i.e. the displacement in the third dimension. Thus we have now a process $(H_n, L_n, W_n, V_n), n \geq 0$ where H_n is the displacement in the direction of propagation of the tree (i.e. the third coordinate), and L_n and W_n are the lateral displacements in the first and the second coordinates respectively. Now the history region V_n would be a tetrahedron. Instead of (10) we now consider the Lyapunov function

$$u(h,l,w,V) = g(f(h,l,w,V)) + (|V|) = \log\left[1 + (l^4 + h^4 + w^4)\right] + (|V|)$$

where

$$f(h,l,w,V) = l^4 + h^4 + w^4$$

and $g(x) = \log(1+x)$ is as in (6). This yields the required recurrence of the state $(0,0,0,\emptyset)$ for the Markov process (H_n, L_n, W_n, V_n) thereby completing the proof the first part of Theorem 1.

3 $d \geq 4$

For notational simplicity we present the proof only for $d = 4$. We first claim that on $\mathbb{Z}^4$ the graph $\mathscr{G}$ admits two distinct trees with positive probability, i.e.,

$$\mathbb{P}\{\mathscr{G} \text{ is disconnected}\} > 0. \tag{31}$$

We start with two distinct open vertices and follow the mechanism described below to generate the trees emanating from these vertices. Given two vertices $\mathbf{u}$ and $\mathbf{v}$, we say $\mathbf{u} \succ \mathbf{v}$ if $\mathbf{u}(4) > \mathbf{v}(4)$ or if $\mathbf{u}(4) = \mathbf{v}(4)$, and $\mathbf{u}$ is larger than $\mathbf{v}$ in the lexicographic order. Starting with two distinct open vertices, $\mathbf{u}$ and $\mathbf{v}$ with $\mathbf{u} \succ \mathbf{v}$ we set

$$T_{\mathbf{u},\mathbf{v}}(0) = (\mathbf{u}_0, \mathbf{v}_0, V_0) \text{ where } \mathbf{u}_0 = \mathbf{u}, \mathbf{v}_0 = \mathbf{v}, \text{ and } V_0 = \emptyset.$$

Let $R(\mathbf{u}) \in \mathbb{Z}^4$ be the unique open vertex such that $\langle \mathbf{u}, R(\mathbf{u}) \rangle \in \mathscr{G}$. We set $\mathbf{u}_1 = \max\{R(\mathbf{u}), \mathbf{v}\}$ and $\mathbf{v}_1 = \min\{R(\mathbf{u}), \mathbf{v}\}$ where the maximum and minimum is over vectors and henceforth understood to be with respect to the ordering $\succ$. We define,

$$T_{\mathbf{u},\mathbf{v}}(1) = (\mathbf{u}_1, \mathbf{v}_1, V_1)$$

where

$$\begin{aligned} V_1 &= \big(\Lambda(\mathbf{v}) \cap \Lambda(\mathbf{u}, \mathbf{u}(4) - R(\mathbf{u})(4))\big) \cup \big(V_0 \cap \Lambda(\mathbf{u}_1)\big). \\ &= \Lambda(\mathbf{v}) \cap \Lambda(\mathbf{u}, \mathbf{u}(4) - R(\mathbf{u})(4)) \end{aligned}$$

The set V_1 is exactly the history set in this case.

Having defined, $\{T_{\mathbf{u},\mathbf{v}}(k) : k = 0,1,\ldots,n\}$ for $n \geq 1$, we define $T_{\mathbf{u},\mathbf{v}}(n+1)$ in the same manner. Let $R(\mathbf{u}_n) \in \mathbb{Z}^4$ be the unique open vertex such that $\langle \mathbf{u}_n, R(\mathbf{u}_n) \rangle \in \mathscr{G}$. Define $\mathbf{u}_{n+1} = \max\{R(\mathbf{u}_n), \mathbf{v}_n\}, \mathbf{v}_{n+1} = \min\{R(\mathbf{u}_n), \mathbf{v}_n\}$ and

$$T_{\mathbf{u},\mathbf{v}}(n+1) = (\mathbf{u}_{n+1}, \mathbf{v}_{n+1}, V_{n+1})$$

where

$$V_{n+1} = \big(\Lambda(\mathbf{v}_n) \cap \Lambda(\mathbf{u}_n, \mathbf{u}_n(4) - R(\mathbf{u}_n)(4))\big) \cup \big(V_n \cap \Lambda(\mathbf{u}_{n+1})\big).$$

The process $\{T_{\mathbf{u},\mathbf{v}}(k) : k \geq 0\}$ tracks the position of trees after the k^{th} step of the algorithm defined above along with the history carried at each stage. Clearly, if $\mathbf{u}_k = \mathbf{v}_k$ for some $k \geq 1$, the trees emanating from $\mathbf{u}$ and $\mathbf{v}$ meet while the event that the trees emanating from $\mathbf{u}$ and $\mathbf{v}$ never meet corresponds to the event that $\{\mathbf{u}_k \neq \mathbf{v}_k : k \geq 1\}$.

A formal construction of the above process is achieved in the following manner. We start with an independent collection of i.i.d. random variables, $\{W_1^{\mathbf{w}}(\mathbf{z}), W_2^{\mathbf{w}}(\mathbf{z}) : \mathbf{z} \in \Lambda(\mathbf{w}), \mathbf{w} \in \mathbb{Z}^4\}$, defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with each of these random variables being uniformly distributed on $[0,1]$. Starting with two open vertices $\mathbf{u}$ and $\mathbf{v}$, set $\mathbf{u}_0 = \max(\mathbf{u}, \mathbf{v}), \mathbf{v}_0 = \min(\mathbf{u}, \mathbf{v})$ and $V_0 = \emptyset$. For $\omega \in \Omega$, we define $k_{\mathbf{u}_0} = k_{\mathbf{u}_0}(\omega)$ as

$$k_{\mathbf{u}_0} := \min\{k : W_1^{\mathbf{u}_0}(\mathbf{z}) < p \text{ for some } \mathbf{z} \in H(\mathbf{u}_0, k)\}$$

and

$$N_{\mathbf{u}_0} := \{\mathbf{z} \in H(\mathbf{u}_0, k_{\mathbf{u}_0}) : W_1^{\mathbf{u}_0}(\mathbf{z}) < p\}.$$

We pick

$$R(\mathbf{u}_0) \in N_{\mathbf{u}_0} \text{ such that } W_2^{\mathbf{u}_0}(R(\mathbf{u}_0)) = \min\{W_2^{\mathbf{u}_0}(\mathbf{z}) : \mathbf{z} \in N_{\mathbf{u}_0}\}.$$

Define, as earlier,

$$\mathbf{u}_1 = \max(R(\mathbf{u}_0), \mathbf{v}_0),\ \mathbf{v}_1 = \min(R(\mathbf{u}_0), \mathbf{v}_0)$$

and

$$V_1 = \big(\Lambda(\mathbf{v}_0) \cap \Lambda(\mathbf{u}_0, k_{\mathbf{u}_0})\big) \cup \big(V_0 \cap \Lambda(\mathbf{u}_1)\big) = \big(\Lambda(\mathbf{v}_0) \cap \Lambda(\mathbf{u}_0, k_{\mathbf{u}_0})\big).$$

Further, for $\mathbf{z} \in V_1$, define

$$W_H^1(\mathbf{z}) = W_1^{\mathbf{u}_0}(\mathbf{z}).$$

Having defined $\{(\mathbf{u}_k, \mathbf{v}_k, V_k, \{W_H^k(\mathbf{z}) : \mathbf{z} \in V_k\}) : 1 \leq k \leq n\}$, we define $\mathbf{u}_{n+1}, \mathbf{v}_{n+1}, V_{n+1}$, $\{W_H^{n+1}(\mathbf{z}) : \mathbf{z} \in V_{n+1}\}$ as follows: for $\omega \in \Omega$, we define $k_{\mathbf{u}_n} = k_{\mathbf{u}_n}(\omega)$ as

$$\begin{aligned} k_{\mathbf{u}_n} := \min\{k : & W_H^n(\mathbf{z}) < p \text{ for some } \mathbf{z} \in V_n \cap H(\mathbf{u}_n, k) \\ & \text{or } W_1^{\mathbf{u}_n}(\mathbf{z}) < p \text{ for some } \mathbf{z} \in H(\mathbf{u}_n, k) \setminus V_n\} \end{aligned}$$

and

$$\begin{aligned} N_{\mathbf{u}_n} := \{\mathbf{z} \in H(\mathbf{u}_n, k_{\mathbf{u}_n}) : & W_H^n(\mathbf{z}) < p \text{ if } \mathbf{z} \in V_n \cap H(\mathbf{u}_n, k) \\ & \text{or } W_1^{\mathbf{u}_n}(\mathbf{z}) < p \text{ if } \mathbf{z} \in H(\mathbf{u}_n, k) \setminus V_n\}. \end{aligned}$$

We pick

$$R(\mathbf{u}_n) \in N_{\mathbf{u}_n} \text{ such that } W_2^{\mathbf{u}_n}(R(\mathbf{u}_n)) = \min\{W_2^{\mathbf{u}_n}(\mathbf{z}) : \mathbf{z} \in N_{\mathbf{u}_n}\}.$$

Finally, define

$$\mathbf{u}_{n+1} = \max(R(\mathbf{u}_n), \mathbf{v}_n),\ \mathbf{v}_{n+1} = \min(R(\mathbf{u}_n), \mathbf{v}_n)$$

and

$$V_{n+1} = \big(\Lambda(\mathbf{v}_n) \cap \Lambda(\mathbf{u}_n, k_{\mathbf{u}_n})\big) \cup \big(V_n \cap \Lambda(\mathbf{u}_n)\big).$$

For $\mathbf{z} \in V_{n+1}$, define

$$W_H^{n+1}(\mathbf{z}) = \begin{cases} W_H^n(\mathbf{z}) & \text{if } z \in V_n \cap \Lambda(\mathbf{u}_n) \\ W_1^{\mathbf{u}_n}(\mathbf{z}) & \text{if } \Lambda(\mathbf{v}_n) \cap \Lambda(\mathbf{u}_n, k_{\mathbf{u}_n}). \end{cases}$$

This construction shows that the $\{(\mathbf{u}_k, \mathbf{v}_k, V_k, \{W_H^k(\mathbf{z}) : \mathbf{z} \in V_k\}) : k \geq 0\}$ is a Markov chain starting at $(\mathbf{u}_0, \mathbf{v}_0, \emptyset, \emptyset)$. A formal proof that this Markov chain describes the joint distribution of the trees emanating from the vertices $\mathbf{u}$ and $\mathbf{v}$ can be given in the same manner as in Lemma 2.1.

For $\mathbf{z} \in \mathbb{Z}^4$, define

$$\|\mathbf{z}\|_1 = |\mathbf{z}(1)| + |\mathbf{z}(2)| + |\mathbf{z}(3)|$$

where $\mathbf{z}(i)$ is the i^{th} co-ordinate of $\mathbf{z}$. Fix $n \geq 1$, $0 < \epsilon < 1/3$ and two open vertices $\mathbf{u}, \mathbf{v}$ and consider the trees emanating from $\mathbf{u}$ and $\mathbf{v}$. Define the event,

$$A_{n,\epsilon} = A_{n,\epsilon}(\mathbf{u}_0, \mathbf{v}_0) := \left\{ \begin{array}{l} \mathbf{u}_k \neq \mathbf{v}_k \text{ for } 1 \leq k \leq n^4 - 1, V_{n^4} = \emptyset, \\ n^{2(1-\epsilon)} \leq \|\mathbf{u}_{n^4} - \mathbf{v}_{n^4}\|_1 \leq n^{2(1+\epsilon)}, \\ 0 \leq \mathbf{u}_{n^4}(4) - \mathbf{v}_{n^4}(4) < \log(n^2) \end{array} \right\}$$

for which we show that the following Lemma holds:

Lemma 3.1. *For $0 < \epsilon < 1/3$ there exist constants $C_1, \beta > 0$ and $n_0 \geq 1$ such that, for all $n \geq n_0$,*

$$\inf_{\substack{n^{1-\epsilon} \leq \|\mathbf{u}-\mathbf{v}\|_1 \leq n^{1+\epsilon}, \\ 0 \leq \mathbf{u}(4)-\mathbf{v}(4) < \log n}} \mathbb{P}\big(A_{n,\epsilon} \mid (\mathbf{u}_0, \mathbf{v}_0, V_0) = (\mathbf{u}, \mathbf{v}, \emptyset)\big) \geq 1 - C_1 n^{-\beta}.$$

First we prove the result using the Lemma 3.1. First, for fixed $0 < \epsilon < 1/3$, we choose n_0 from the above Lemma. Now, fix any $n \geq n_0$ and $\mathbf{u}$ such that $n^{1-\epsilon} \leq \|\mathbf{u}\|_1 \leq n^{1+\epsilon}, 0 \leq \mathbf{u}(4) < \log n$. With positive probability, the vertices $\mathbf{0}$ and $\mathbf{u}$ are both open. On the event that both the vertices $\mathbf{0}$ and $\mathbf{u}$ are both open, we consider the trees emanating from $\mathbf{u}$ and $\mathbf{v} = \mathbf{0}$. We want to show that $\mathbb{P}\{\mathbf{u}_k \neq \mathbf{v}_k$ for $k \geq 1\} > 0$.

Let $\tau_0 = 0$ and for $i \geq 1$, let $\tau_i := n^4 + (n^2)^4 + \cdots + (n^{2^{i-1}})^4$. For $i \geq 1$, define the event

$$B_i = \left\{ \begin{array}{l} \mathbf{u}_k \neq \mathbf{v}_k, \text{ for } \tau_{i-1} < k \leq \tau_i, V_{\tau_i} = \emptyset, \\ \qquad n^{2^i(1-\epsilon)} \leq \|\mathbf{u}_{\tau_i} - \mathbf{v}_{\tau_i}\|_1 \leq n^{2^i(1+\epsilon)}, \\ \qquad \text{and } 0 \leq \mathbf{u}_{\tau_i}(4) - \mathbf{v}_{\tau_i}(4) < \log n^{2^i} \end{array} \right\}.$$

Then, we have

$$\begin{aligned} \mathbb{P}\{\text{for all } k \geq 1, \mathbf{u}_k \neq \mathbf{v}_k\} &= \lim_{i\to\infty} \mathbb{P}\{\text{for } 1 \leq k \leq \tau_i, \mathbf{u}_k \neq \mathbf{v}_k\} \\ &\geq \limsup_{i\to\infty} \mathbb{P}\big\{\text{for } 1 \leq k \leq \tau_i, \mathbf{u}_k \neq \mathbf{v}_k, \text{ and for } 1 \leq l \leq i, V_{\tau_l} = \emptyset, \\ &\qquad n^{2^l(1-\epsilon)} \leq \|\mathbf{u}_{\tau_l} - \mathbf{v}_{\tau_l}\|_1 \leq n^{2^l(1+\epsilon)} \text{ and } 0 \leq \mathbf{u}_{\tau_l}(4) - \mathbf{v}_{\tau_l}(4) < \log n^{2^l}\big\} \\ &= \limsup_{i\to\infty} \mathbb{P}\big(\cap_{l=1}^i B_l\big) \\ &= \limsup_{i\to\infty} \prod_{l=2}^i \mathbb{P}\big(B_l \mid \cap_{j=1}^{l-1} B_j\big) \mathbb{P}(B_1). \end{aligned} \tag{32}$$

For $l \geq 2$, B_l is a event which involves the random variables $(\mathbf{u}_k, \mathbf{v}_k, V_k)$ for $k = \tau_{l-1}+1, \ldots, \tau_l$. Using the Markov property, we have that the probability $\mathbb{P}\big(B_l \mid \cap_{j=1}^{l-1} B_j\big)$ depends only on $(\mathbf{u}_{\tau_{l-1}}, \mathbf{v}_{\tau_{l-1}}, V_{\tau_{l-1}})$. Furthermore, on the set $\cap_{j=1}^{l-1} B_j$, we note that $n^{2^{l-1}(1-\epsilon)} \leq \|\mathbf{u}_{\tau_{l-1}} - \mathbf{v}_{\tau_{l-1}}\|_1 \leq n^{2^{l-1}(1+\epsilon)}, 0 \leq \mathbf{u}_{\tau_{l-1}}(4) - \mathbf{v}_{\tau_{l-1}}(4) < \log n^{2^{l-1}}$ and $V_{\tau_{l-1}} = \emptyset$. Therefore we have that, for $l \geq 2$,

$$
\begin{aligned}
& \mathbb{P}\big(B_l \mid \cap_{j=1}^{l-1} B_j\big) \\
\geq \; & \inf_{\substack{n^{2^{l-1}(1-\epsilon)} \leq \|\mathbf{z}_1 - \mathbf{z}_2\|_1 \leq n^{2^{l-1}(1+\epsilon)}, \\ 0 \leq \mathbf{z}_1(4) - \mathbf{z}_2(4) < \log n}} \mathbb{P}\big(B_l \mid (\mathbf{u}_{\tau_{l-1}}, \mathbf{v}_{\tau_{l-1}}, V_{\tau_{l-1}}) = (\mathbf{z}_1, \mathbf{z}_2, \emptyset)\big) \\
= \; & \inf_{\substack{n^{2^{l-1}(1-\epsilon)} \leq \|\mathbf{z}_1 - \mathbf{z}_2\|_1 \leq n^{2^{l-1}(1+\epsilon)}, \\ 0 \leq \mathbf{z}_1(4) - \mathbf{z}_2(4) < \log n}} \mathbb{P}\big(A_{n^{2^{l-1}}, \epsilon} \mid (\mathbf{u}_0, \mathbf{v}_0, V_0) = (\mathbf{z}_1, \mathbf{z}_2, \emptyset)\big) \\
\geq \; & 1 - C_1 n^{-2^{l-1}\beta} \qquad (33)
\end{aligned}
$$

and since $n^{1-\epsilon} \leq \|\mathbf{u}\|_1 \leq n^{1+\epsilon}, 0 \leq \mathbf{u}(4) < \log n$.

$$
\mathbb{P}(B_1) = \mathbb{P}\big(A_{n,\epsilon} \mid (\mathbf{u}_0, \mathbf{v}_0, V_0) = (\mathbf{u}, \mathbf{0}, \emptyset)\big) \geq 1 - C_1 n^{-\beta} \qquad (34)
$$

Therefore, from (32), (33) and (34), we have,

$$
\mathbb{P}\{\mathscr{G} \text{ is disconnected}\} \geq \mathbb{P}\{\mathbf{u}, \mathbf{0} \text{ are both open}\} \times \lim_{i \to \infty} \prod_{l=1}^{i} \big(1 - C_1 n^{-2^{l-1}\beta}\big) > 0.
$$

This completes the proof of the claim.

We will work towards the proof of Lemma 3.1. Towards that, we introduce an independent version of the above process. In the same probability space, starting with two vertices $\mathbf{u}$ and $\mathbf{v}$, and the same set of uniformly distributed random variables, define $\mathbf{u}_0^{(\mathrm{I})} = \max\{\mathbf{u}_0, \mathbf{v}_0\}$ and $\mathbf{v}_0^{(\mathrm{I})} = \min\{\mathbf{u}_0, \mathbf{v}_0\}$. As in the construction at the beginning of the Section, we define $\mathbf{u}_{n+1}^{(\mathrm{I})} = \max\{R^{(\mathrm{I})}(\mathbf{u}_n^{(\mathrm{I})}), \mathbf{v}_n^{(\mathrm{I})}\}$ and $\mathbf{v}_{n+1}^{(\mathrm{I})} = \min\{R^{(\mathrm{I})}(\mathbf{u}_n^{(\mathrm{I})}), \mathbf{v}_n^{(\mathrm{I})}\}$ where $R^{(\mathrm{I})}$ is similar to R defined in the earlier construction except that the history part is completely ignored.

The independent version tracks the two trees, emanating from the vertices $\mathbf{u}$ and $\mathbf{v}$, with the condition that the trees do not depend on the information (history) carried. The only constraint is that while growing the tree from a vertex, it waits for the tree from the other vertex to catch up, before taking the next step. Note that if the history set is empty, then both constructions match exactly.

We define an event similar to $A_{n,\epsilon}$ but in terms of $\{(\mathbf{u}_k^{(\mathrm{I})}, \mathbf{v}_k^{(\mathrm{I})}) : 1 \leq k \leq n^4\}$. Fix $n \geq 1$, $0 < \epsilon < 1/3$ and two open vertices $\mathbf{u}, \mathbf{v} \in \mathbb{Z}$ and define the event,

$$
B_{n,\epsilon}(\mathbf{u}_0^{(\mathrm{I})}, \mathbf{v}_0^{(\mathrm{I})}) := \left\{ \begin{array}{l} \|\mathbf{u}_k^{(\mathrm{I})} - \mathbf{v}_k^{(\mathrm{I})}\|_1 \geq \log(n^2) \text{ for } 1 \leq k \leq n^4 - 1, \\ 0 \leq \mathbf{u}_k^{(\mathrm{I})}(4) - \mathbf{v}_k^{(\mathrm{I})}(4) < \log(n^2) \text{ for } 1 \leq k \leq n^4, \\ n^{2(1-\epsilon)} \leq \|\mathbf{u}_{n^4}^{(\mathrm{I})} - \mathbf{v}_{n^4}^{(\mathrm{I})}\|_1 \leq n^{2(1+\epsilon)}. \end{array} \right\}
$$

We will show that the following Lemma holds:

Lemma 3.2. *For $0 < \epsilon < 1/3$ there exist constants $C_2, \gamma > 0$ and $n_0 \geq 1$ such that, for all $n \geq n_0$,*

$$
\inf_{\substack{n^{1-\epsilon} \leq \|\mathbf{u} - \mathbf{v}\|_1 \leq n^{1+\epsilon}, \\ 0 \leq \mathbf{u}(4) - \mathbf{v}(4) < \log n}} \mathbb{P}\big(B_{n,\epsilon}(\mathbf{u}, \mathbf{v})\big) \geq 1 - C_2 n^{-\gamma}.
$$

First we prove Lemma 3.1, assuming Lemma 3.2.

Proof of Lemma 3.1: Given $0 < \epsilon < 1/3$, fix $n_0 \geq 1$ from Lemma 3.2. Now fix $n \geq n_0$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^4$ such that $n^{1-\epsilon} \leq \|\mathbf{u}-\mathbf{v}\|_1 \leq n^{1+\epsilon}$ and $0 \leq \mathbf{u}(4) - \mathbf{v}(4) < \log n$. Now note that both the events $A_{n,\epsilon}(\mathbf{u},\mathbf{v})$ and $B_{n,\epsilon}(\mathbf{u},\mathbf{v})$ are defined on the same probability space.

We claim that $A_{n,\epsilon}(\mathbf{u},\mathbf{v}) \supseteq B_{n,\epsilon}(\mathbf{u},\mathbf{v})$. To prove this, we show that, on the set $B_{n,\epsilon}(\mathbf{u},\mathbf{v})$, we have that $\{(\mathbf{u}_k,\mathbf{v}_k,V_k) = (\mathbf{u}_k^{(\mathrm{I})},\mathbf{v}_k^{(\mathrm{I})},\emptyset) : 1 \leq k \leq n^4\}$. This follows easily from the observation that if $V_k = \emptyset$, for some $k \geq 0$, then the two constructions, given before, match exactly. That is, if $(\mathbf{u}_k,\mathbf{v}_k,V_k) = (\mathbf{u}_k^{(\mathrm{I})},\mathbf{v}_k^{(\mathrm{I})},\emptyset)$ for $k \leq i, i \geq 0$, we have, $R(\mathbf{u}_i) = R^{(\mathrm{I})}(\mathbf{u}_i^{(\mathrm{I})})$. Thus, we have $\mathbf{u}_{i+1} = \mathbf{u}_{i+1}^{(\mathrm{I})}$ and $\mathbf{v}_{i+1} = \mathbf{v}_{i+1}^{(\mathrm{I})}$. Furthermore, on the event $B_{n,\epsilon}(\mathbf{u},\mathbf{v})$, we have that $\|\mathbf{u}_{i+1} - \mathbf{v}_{i+1}\|_1 \geq \log(n^2)$ and $0 \leq \mathbf{u}_{i+1}(4) - \mathbf{v}_{i+1}(4) < \log n^2$. From the definition of the history set, the separation of $\mathbf{u}$ and $\mathbf{v}$ implies that $V_{i+1} = \emptyset$. Therefore the claim follows by an induction argument. Thus, we have

$$\mathbb{P}\big(A_{n,\epsilon} | (\mathbf{u}_0,\mathbf{v}_0,V_0) = (\mathbf{u},\mathbf{v},\emptyset)\big) \geq \mathbb{P}\big(B_{n,\epsilon}(\mathbf{u},\mathbf{v})\big) \geq 1 - C_2 n^{-\gamma}.$$

Hence, Lemma 3.1 follows by choosing $C_1 = C_2$ and $\beta = \gamma$. □

Now define for $k \geq 0$, $S_k^{(\mathrm{I})} = \mathbf{u}_k - \mathbf{v}_k$. Then, the event $B_{n,\epsilon}$ can be restated in terms of $S_k^{(\mathrm{I})}$. Indeed, define $\mathbf{s} = \mathbf{u} - \mathbf{v}$ where $\mathbf{u} \succ \mathbf{v}$ and

$$C_{n,\epsilon}(\mathbf{s}) := \left\{ \begin{array}{l} \|S_k^{(\mathrm{I})}\|_1 \geq \log(n^2) \text{ for } 1 \leq k \leq n^4 - 1, \\ 0 \leq S_k^{(\mathrm{I})}(4) < \log(n^2) \text{ for } 1 \leq k \leq n^4, \\ n^{2(1-\epsilon)} \leq \|S_{n^4}^{(\mathrm{I})}\|_1 \leq n^{2(1+\epsilon)}. \end{array} \right\}$$

Lemma 3.2 now can be restated as

Lemma 3.3. *For $0 < \epsilon < 1/3$ there exist constants $C_2, \gamma > 0$ and $n_0 \geq 1$ such that, for all $n \geq n_0$,*

$$\inf_{\substack{n^{1-\epsilon} \leq \|\mathbf{s}\|_1 \leq n^{1+\epsilon}, \\ 0 \leq \mathbf{s}(4) < \log n}} \mathbb{P}\big(C_{n,\epsilon}(\mathbf{s})\big) \geq 1 - C_2 n^{-\gamma}.$$

In order to study the event $C_{n,\epsilon}(\mathbf{s})$, we have to look at the steps taken by $\mathbf{u}_k^{(\mathrm{I})}$ for each $k \geq 1$. Let $X_k = R^{(\mathrm{I})}(\mathbf{u}_k^{(\mathrm{I})}) - \mathbf{u}_k^{(\mathrm{I})}$ for $k \geq 0$. The construction clearly shows that each $\{X_k : k \geq 1\}$ is a sequence of i.i.d. random variables.

The distribution of X_k can be easily found. Let $a_0 = 0$, and for $i \geq 1$, $a_i = |\Lambda(\mathbf{0}, i)|$ and $b_i = |H(\mathbf{0}, i)|$. Define a random variable T on $\{1, 2, \dots\}$, given by

$$\mathbb{P}(T = i) = (1-p)^{a_{i-1}}\big(1 - (1-p)^{b_i}\big). \tag{35}$$

Now, define D on $\mathbb{Z}^3$ as follows:

$$\mathbb{P}(D = z \mid T = i) = \begin{cases} \frac{1}{b_i} & \text{for } (z, -i) \in H(\mathbf{0}, i) \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

Note T and D are higher dimensional equivalents of T and D in (19) and (20). It is easy to see that X_k and $(D, -T)$ are identical in distribution. Let $\{(D_i, -T_i) : i \geq 1\}$ be independent copies of $(D, -T)$. Then, $\{S_k^{(\mathrm{I})} : k \geq 0\}$ can be represented as follows : set $S_0^{(\mathrm{I})} = \mathbf{s}$ and for $k \geq 1$,

$$S_k^{(\mathrm{I})} \stackrel{d}{=} \begin{cases} S_{k-1}^{(\mathrm{I})} + (D_k, -T_k) & \text{if } S_{k-1}^{(\mathrm{I})} + (D_k, -T_k) \succ \mathbf{0} \\ -\big(S_{k-1}^{(\mathrm{I})} + (D_k, -T_k)\big) & \text{otherwise.} \end{cases}$$

Note that, from the definition of the order relation, $S_k^{(\mathrm{I})}(4) \stackrel{d}{=} |S_{k-1}^{(\mathrm{I})}(4) - T_k| \geq 0$ for each $k \geq 1$. Now, we define a random walk version of the above process in the following way: given $\mathbf{s} \succ \mathbf{0}$ and the collection $\{(D_i, -T_i) : i \geq 1\}$ of i.i.d. steps, define : $S_0^{(\mathrm{RW})} = \mathbf{s}$ and for $k \geq 1$,

$$S_k^{(\mathrm{RW})} = \big(\mathbf{s}(1), \mathbf{s}(2), \mathbf{s}(3), 0\big) + \Big(\sum_{i=1}^{k} D_i, |S_{k-1}^{(\mathrm{RW})}(4) - T_k|\Big).$$

The random walk $S_k^{(\mathrm{RW})}$ executes a three dimensional random walk in its first three co-ordinates, starting at $\big(\mathbf{s}(1), \mathbf{s}(2), \mathbf{s}(3)\big)$ with step size distributed as D on $\mathbb{Z}^3$. The fourth co-ordinate follows the fourth co-ordinate of the process $S_k^{(\mathrm{I})}$.

Note that, we have constructed both the processes using the same steps $\{(D_i, -T_i) : i \geq 1\}$ and on the same probability space. Therefore, it is clear that the fourth co-ordinate of both the processes are the same, i.e., $S_k^{(\mathrm{I})}(4) = S_k^{(\mathrm{RW})}(4)$ for $k \geq 1$. We will show that the first three co-ordinates of both the processes have the same norm. In other words,

Lemma 3.4. *For $k \geq 1$ and $\alpha_i, \beta_i \geq 0$ for $1 \leq i \leq k$,*

$$\begin{aligned} &\mathbb{P}\big\{\|S_i^{(RW)}\|_1 = \alpha_i, S_i^{(RW)}(4) = \beta_i \text{ for } 1 \leq i \leq k\big\} \\ &= \mathbb{P}\big\{\|S_i^{(I)}\|_1 = \alpha_i, S_i^{(I)}(4) = \beta_i \text{ for } 1 \leq i \leq k\big\}. \end{aligned}$$

We postpone the proof of Lemma 3.4 for the time being. We define a random walk version of the event $C_{n,\epsilon}(\mathbf{s})$. For $n \geq 1$ and $0 < \epsilon < 1/3$ and $\mathbf{s} \succ \mathbf{0}$, define

$$D_{n,\epsilon}(\mathbf{s}) := \left\{ \begin{array}{l} \|S_k^{(\mathrm{RW})}\|_1 \geq \log(n^2) \text{ for } 1 \leq k \leq n^4 - 1, \\ 0 \leq S_k^{(\mathrm{RW})}(4) < \log(n^2) \text{ for } 1 \leq k \leq n^4, \\ n^{2(1-\epsilon)} \leq \|S_{n^4}^{(\mathrm{RW})}\|_1 \leq n^{2(1+\epsilon)}. \end{array} \right\}$$

In view of Lemma 3.4, it is enough to prove the following Lemma:

Lemma 3.5. *For $0 < \epsilon < 1/3$ there exist constants $C_2, \gamma > 0$ and $n_0 \geq 1$ such that, for all $n \geq n_0$,*

$$\inf_{\substack{n^{1-\epsilon} \leq \|\mathbf{s}\|_1 \leq n^{1+\epsilon}, \\ 0 \leq \mathbf{s}(4) < \log n}} \mathbb{P}\big(D_{n,\epsilon}(\mathbf{s})\big) \geq 1 - C_2 n^{-\gamma}.$$

We first prove Lemma 3.5 and then return to proof of Lemma 3.4.

Proof of Lemma 3.5: For $\mathrm{z} \in \mathbb{Z}^3$, define $\|\mathrm{z}\| = |\mathrm{z}(1)| + |\mathrm{z}(2)| + |\mathrm{z}(3)|$, the usual L^1 norm in $\mathbb{Z}^3$. Define, $\Delta_k = \{\mathrm{z} \in \mathbb{Z}^3 : \|\mathrm{z}\| \leq k\}$. Let $\mathbf{s}^{(1)} = (\mathbf{s}(1), \mathbf{s}(2), \mathbf{s}(3))$ be the first three co-ordinates of the starting point $\mathbf{s}$. Let $\mathbf{r}_k$ represent the random walk part (first three co-ordinates) of $S_k^{(\mathrm{RW})}$, i.e., $\mathbf{r}_k = \mathbf{s}^{(1)} + \sum_{i=1}^{k} D_i$. Now note that

$$\Big(D_{n,\epsilon}(\mathbf{s})\Big)^c \subseteq E_{n,\epsilon} \cup F_{n,\epsilon} \cup G_{n,\epsilon} \cup H_{n,\epsilon}$$

where

$$
\begin{aligned}
E_{n,\epsilon} &:= \bigcup_{k=1}^{n^4-1} \Big\{ \|\mathbf{r}_k\| < \log(n^2) \Big\}, \\
F_{n,\epsilon} &:= \Big\{ \|\mathbf{r}_{n^4}\| > n^{2(1+\epsilon)} \Big\} = \Big\{ \mathbf{r}_{n^4} \notin \Delta_{n^{2}(1+\epsilon)} \Big\}, \\
G_{n,\epsilon} &:= \Big\{ \|\mathbf{r}_{n^4}\| \leq n^{2(1-\epsilon)} \Big\} = \Big\{ \mathbf{r}_{n^4} \in \Delta_{n^{2}(1-\epsilon)} \Big\},
\end{aligned}
$$

and

$$
H_{n,\epsilon} := \bigcup_{k=1}^{n^4} \Big\{ S_k^{(\mathrm{RW})}(4) \geq \log(n^2) \Big\}.
$$

Note that the events $E_{n,\epsilon}, F_{n,\epsilon}$ and $G_{n,\epsilon}$ depend on the random walk part while $H_{n,\epsilon}$ depends on the fourth co-ordinate of $S_k^{(\mathrm{RW})}$. Also note that $\sum_{j=1}^{k} D_j$ is an aperiodic, isotropic, symmetric random walk whose steps are i.i.d. with each step having the same distribution as D where $\mathbb{V}\mathrm{ar}(D) = \sigma^2 I$, for some $\sigma > 0$ and $\sum_{\mathrm{z}\in\mathbb{Z}^3} \|\mathrm{z}\|^2 \mathbb{P}(D = \mathrm{z}) < \infty$. The events $F_{n,\epsilon}$ and $G_{n,\epsilon}$ are exactly as in Lemma 3.3 of Gangopadhyay, Roy and Sarkar [8]. Hence, we conclude that there exist constants $C_3, C_4 > 0$ and $\alpha > 0$ such that for all n sufficiently large,

$$
\sup_{\substack{n^{1-\epsilon} \leq \|\mathbf{s}\|_1 \leq n^{1+\epsilon}, \\ 0 \leq \mathbf{s}(4) < \log n}} \mathbb{P}\big(F_{n,\epsilon}\big) = \sup_{\mathbf{s}^{(1)} \in \Delta_{n^{1+\epsilon}} \setminus \Delta_{n^{1+\epsilon}}} \mathbb{P}\big(F_{n,\epsilon}\big) \leq C_3 n^{-\alpha}
$$

and

$$
\sup_{\substack{n^{1-\epsilon} \leq \|\mathbf{s}\|_1 \leq n^{1+\epsilon}, \\ 0 \leq \mathbf{s}(4) < \log n}} \mathbb{P}\big(G_{n,\epsilon}\big) = \sup_{\mathbf{s}^{(1)} \in \Delta_{n^{1+\epsilon}} \setminus \Delta_{n^{1+\epsilon}}} \mathbb{P}\big(G_{n,\epsilon}\big) \leq C_4 n^{-\alpha}.
$$

The probability of the event $E_{n,\epsilon}$ can be computed in the same fashion as in Lemma 3.3 of Gangopadhyay, Roy and Sarkar [8]. Indeed, we have,

$$
\begin{aligned}
\mathbb{P}\big(E_{n,\epsilon}\big) &= \mathbb{P}\Big(\|\mathbf{r}_k\| \leq 2\log n \text{ for some } k = 1, 2, \ldots, n^4 - 1 \Big) \\
&= \mathbb{P}\Big(\sum_{i=1}^{k} D_i \in (-\mathbf{s}^{(1)} + \Delta_{2\log n}) \text{ for some } k = 1, 2, \ldots, n^4 - 1 \Big) \\
&\leq \mathbb{P}\Big(\sum_{i=1}^{k} D_i \in (-\mathbf{s}^{(1)} + \Delta_{2\log n}) \text{ for some } k \geq 1 \Big) \\
&\leq \mathbb{P}\Big(\bigcup_{\mathrm{z} \in -\mathbf{s}^{(1)} + \Delta_{2\log n}} \{ \sum_{i=1}^{k} D_i = \mathrm{z} \text{ for some } k \geq 1 \} \Big) \\
&\leq C_5 (2\log n)^3 \sup_{\mathrm{z} \in -\mathbf{s}^{(1)} + \Delta_{2\log n}} \mathbb{P}\{ \sum_{i=1}^{k} D_i = \mathrm{z} \text{ for some } k \geq 1 \} \qquad (37)
\end{aligned}
$$

for some suitable positive constant C_5.

From Proposition P26.1 of Spitzer [16] (pg. 308),

$$\lim_{|z|\to\infty} |z|\,\mathbb{P}\left\{\sum_{j=1}^{i} D_j = z \text{ for some } i \ge 1\right\} = (4\pi\sigma^2)^{-1} > 0. \tag{38}$$

For $\mathbf{s}^{(1)} \in (\Delta_{n^{(1+\epsilon)}} \setminus \Delta_{n^{(1-\epsilon)}})$ and $z \in -\mathbf{s}^{(1)} + \Delta_{2\log n}$, we must have that $\|z\| \ge n^{1-\epsilon}/2$ for all n sufficiently large. Thus, for all n sufficiently large, we have, using (38) and (37),

$$\mathbb{P}\big(E_{n,\epsilon}\big) \le C_5 (2\log n)^3 C_6 n^{-(1-\epsilon)} \le C_7 n^{-\frac{(1-\epsilon)}{2}}$$

where C_5, C_6 and C_7 are suitably chosen positive constants.

Finally, for the event $H_{n,\epsilon}$, let $E_k = \left\{S_k^{(\mathrm{RW})}(4) \ge \log(n^2)\right\}$. Then,

$$H_{n,\epsilon} = E_1 \cup \bigcup_{k=2}^{n^4} E_k \cap_{j=1}^{k-1} E_k^c,$$

and we have

$$\mathbb{P}\big(H_{n,\epsilon}\big) = \mathbb{P}(E_1) + \sum_{k=2}^{n^4} \mathbb{P}\big(E_k \cap_{j=1}^{k-1} E_k^c\big) \le \mathbb{P}(E_1) + \sum_{k=2}^{n^4} \mathbb{P}\big(E_k \mid \cap_{j=1}^{k-1} E_k^c\big).$$

On the set $\cap_{j=1}^{k-1} E_k^c$, we have $0 \le S_{k-1}^{(\mathrm{RW})}(4) < \log(n^2)$ and $S_k^{(\mathrm{RW})}(4) = |S_{k-1}^{(\mathrm{RW})}(4) - T_k| \ge \log(n^2)$ implies that $T_k \ge \log(n^2)$. Hence, $\mathbb{P}\big(E_k \mid \cap_{j=1}^{k-1} E_k^c\big) \le \mathbb{P}(T_k \ge \log(n^2))$. Similarly, $\mathbb{P}(E_1) \le \mathbb{P}(T_1 \ge \log(n^2))$. Thus, we get

$$\mathbb{P}\big(H_{n,\epsilon}\big) \le n^4 \mathbb{P}(T \ge \log(n^2)) \le n^4 (1-p)^{(2\log(n))^4} \le C_8 \exp(-C_9 \log n)$$

for some positive constants $C_8, C_9 > 0$. This completes the proof of the Lemma 3.5. □

Finally, we are left with the proof of Lemma 3.4.

Proof of Lemma 3.4: We define an intermediate process on $\mathbb{Z}^4 \times \{-1, 1\}$ in the following way. Given $\mathbf{s} \succ \mathbf{0}$ and the steps $\{(D_i, -T_i) : i \ge 1\}$, define $(\tilde{S}_0, F_0) = (\mathbf{s}, 1)$ and for $k \ge 1$,

$$(\tilde{S}_k, F_k) = \begin{cases} \Big(\tilde{S}_{k-1} + (F_{k-1}D_k, -T_k), F_{k-1}\Big) & \text{if } \tilde{S}_{k-1} + (F_{k-1}D_k, -T_k) \succ \mathbf{0} \\ -\Big(S_{k-1}^{(\mathrm{I})} + (F_{k-1}D_k, -T_k), F_{k-1}\Big) & \text{otherwise.} \end{cases}$$

Using induction, it is easy to check that

$$\tilde{S}_k^{(1)} = F_k\Big(\tilde{S}_0^{(1)} + \sum_{i=1}^{k} D_i\Big) = F_k\Big(\mathbf{s}^{(1)} + \sum_{i=1}^{k} D_i\Big)$$

where $z^{(1)}$ is the first three co-ordinates of z. Therefore, we have that for $k \ge 1$, $\|\tilde{S}_k\|_1 = \|\mathbf{s}^{(1)} + \sum_{i=1}^{k} D_i\| = \|S_k^{(RW)}\|_1$ since $F_k \in \{-1, 1\}$. From the definition, we have $S_k^{(I)}(4) = \tilde{S}_k(4) = S_k^{(RW)}(4)$ for each $k \ge 1$. Furthermore, a straight forward calculation shows that

$$\{S_i^{(I)} : i = 0, 1, \dots, k\} \text{ and } \{\tilde{S}_i : i = 0, 1, \dots, k\} \tag{39}$$

are identical in distribution. Note that, from the definition of $S_k^{(I)}$ and $\tilde{S}_k$, we can write, for $k \geq 0$,

$$S_{k+1}^{(I)} = f(S_k^{(I)}, D_{k+1}, T_k) \text{ and } \tilde{S}_{k+1} = f(\tilde{S}_k, F_k D_{k+1}, T_k) \tag{40}$$

where $f : \mathbb{Z}^4 \times \mathbb{Z}^3 \times \mathbb{N} \to \mathbb{Z}^4$ is a suitably defined function. The exact form of f is unimportant, the only observation that is crucial is that we can use the same f for both the cases. This establishes (39) and completes the proof of Lemma 3.4. $\square$

Hence we have shown that

$$\mathbb{P}\{\mathscr{G} \text{ is disconnected}\} > 0$$

and, by the inherent ergodicity of the process, this implies that

$$\mathbb{P}\{\mathscr{G} \text{ is disconnected}\} = 1.$$

A similar argument along with the ergodicity of the random graph, may further be used to establish that for any $k \geq 1$

$$\mathbb{P}\{\mathscr{G} \text{ has at least } k \text{ trees}\} = 1.$$

Consequently, we have that

$$\mathbb{P}\Big\{\bigcap_{k\geq 1}\{\mathscr{G} \text{ has at least } k \text{ trees }\}\Big\} = 1$$

and thus

$$\mathbb{P}\{\mathscr{G} \text{ has infinitely many trees}\} = 1.$$

4 Geometry of the graph $\mathscr{G}$

We now show that the tree(s) are not bi-infinite almost surely. For this argument, we consider, $d = 2$. Similar arguments, with minor modifications go through for any dimensions.

For $t \in \mathbb{Z}$, define the set of all open points on the line $L_t := \{(u, t) : -\infty < u < \infty\}$ by N_t. In other words, $N_t := \{\mathbf{y} \in \mathscr{V} : \mathbf{y} = (y_1, t)\}$. Fix $\mathbf{x} \in N_t$ and $n \geq 0$, set $B_n(\mathbf{x}) := \{\mathbf{y} \in \mathscr{V} : h^n(\mathbf{y}) = \mathbf{x}\}$, where $h^n(\mathbf{y})$ is the (unique) n^{th} generation offspring of the vertex $\mathbf{y}$. Thus, $B_n(\mathbf{x})$ stands for the set of the n^{th} generation ancestors of the vertex $\mathbf{x}$.

Now consider the set of vertices in N_t which have n^{th} order ancestors, i.e., $M_t^{(n)} := \{\mathbf{x} \in N_t : B_n(\mathbf{x}) \neq \emptyset\}$. Clearly, $M_t^{(n)} \subseteq M_t^{(m)}$ for $n > m$ and so $R_t := \lim_{n\to\infty} M_t^{(n)} = \cap_{n\geq 0} M_t^{(n)}$ is well defined and this is the set of vertices in N_t which have bi-infinite paths. Our aim is to show that $\mathbb{P}(R_t = \emptyset) = 1$ for all $t \in \mathbb{Z}$. Since $\{R_t : t \in \mathbb{Z}\}$ is stationary, it suffices to show that $\mathbb{P}(R_0 = \emptyset) = 1$.

We claim, for any $1 \leq k < \infty$,

$$\mathbb{P}(|R_0| = k) = 0. \tag{41}$$

Indeed, if $\mathbb{P}(|R_0| = k) > 0$ for some $1 \leq k < \infty$, we must have, for some $-\infty < x_1 < x_2 < \ldots < x_k < \infty$ such that,

$$\mathbb{P}\big(R_0 = \{(x_1, 0), (x_2, 0), \ldots, (x_k, 0)\}\big) > 0.$$

Clearly, by stationarity again, for any $t \in \mathbb{Z}$,

$$\begin{aligned} & \mathbb{P}\big(R_0 = \{(x_1+t,0),(x_2+t,0),\ldots,(x_k+t,0)\}\big) \\ = \; & \mathbb{P}\big(R_0 = \{(x_1,0),(x_2,0),\ldots,(x_k,0)\}\big) > 0. \end{aligned} \tag{42}$$

However, using (42)

$$\mathbb{P}(|R_0| = k) = \sum_{E=\{(x_1,0),(x_2,0),\ldots,(x_k,0)\}} \mathbb{P}(R_0 = E) = \infty.$$

This is obviously not possible, proving (41) .

Thus, we have that

$$\mathbb{P}(|R_0| = 0) + \mathbb{P}(|R_0| = \infty) = 1.$$

Assume that $\mathbb{P}(|R_0| = 0) < 1$, so that $\mathbb{P}(|R_0| = \infty) > 0$.

Now, call a vertex $\mathbf{x} \in R_t$ a *branching point* if there exist distinct points $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\mathbf{x}_1, \mathbf{x}_2 \in B_1(\mathbf{x})$ and $B_n(\mathbf{x}_1) \neq \emptyset, B_n(\mathbf{x}_2) \neq \emptyset$ for all $n \geq 1$, i.e, $\mathbf{x}$ has at least two distinct infinite branches of ancestors.

We first show that, if $\mathbb{P}(|R_0| = \infty) > 0$,

$$\mathbb{P}(\text{Origin is a branching point}) > 0. \tag{43}$$

Since $\mathbb{P}(|R_0| = \infty) > 0$, we may fix two vertices $\mathbf{x} = (x_1, 0)$ and $\mathbf{y} = (y_1, 0)$ such that

$$\mathbb{P}(\mathbf{x}, \mathbf{y} \in R_0) > 0.$$

Suppose that $x_1 < y_1$ and set $M = y_1 - x_1$ and $\mathbf{x}_M = (0, M)$ and $\mathbf{y}_M = (M, M)$. By translation invariance of the model, we must have,

$$\mathbb{P}(\mathbf{x}_M, \mathbf{y}_M \in R_M) = \mathbb{P}(\mathbf{x}, \mathbf{y} \in R_0) > 0.$$

Thus the event $E_1 := \{B_n(\mathbf{x}_M) \neq \emptyset, B_n(\mathbf{y}_M) \neq \emptyset \text{ for all } n \geq 1\}$ has positive probability. Further this event depends only on sites $\{\mathbf{u} := (u_1, u_2) : u_2 \geq M\}$.

Now, consider the event $E_2 :=$ {origin is open and all sites other than the origin in $\Lambda(\mathbf{x}_M, M) \cup \Lambda(\mathbf{y}_M, M)$ is closed}. Clearly, the event E_2 depends only on a finite subset $\{\mathbf{u} = (u_1, u_2) : 0 \leq u_2 \leq M - 1, |u_1| \leq 2M\}$ and $\mathbb{P}(E_2) > 0$. Since E_1 and E_2 depend on disjoint sets of vertices, we have,

$$\mathbb{P}(\text{Origin is a branching point}) \geq \mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2) > 0.$$

Now, we define C_t as the set of all vertices with their second co-ordinates strictly larger than t, each vertex having infinite ancestry and its immediate offspring having the second co-ordinate at most t, i.e., $C_t = \{\mathbf{y} = (y_1, u) \in \mathscr{V} : u > t, B_n(\mathbf{y}) \neq \emptyset \text{ for } n \geq 1, \text{ and } h(\mathbf{y}) = (x_1, v), \text{ with } v \leq t\}$. Since every open vertex has a unique offspring, for each $\mathbf{y} \in C_t$, the edge joining $\mathbf{y}$ and $h(\mathbf{y})$ intersects the line L_t at a single point only, say $(I_y(t), t)$. We define, for all $n \geq 1$,

$$C_t(n) := \{\mathbf{y} \in C_t : 0 \leq I_y(t) < n\} \text{ and } r_t(n) = |C_t(n)|.$$

We show that $\mathbb{E}(r_t(n)) < \infty$ and consequently $r_t(n)$ is finite almost surely. First, note that

$$C_t(n) = \cup_{j=1}^{n} \{\mathbf{y} \in C_t : j - 1 \leq I_y(t) < j\}.$$

Since the sets on the right hand side of above equality are disjoint, we have

$$r_t(n) = \sum_{j=1}^{n} r_t^{(j)}$$

where $r_t^{(j)} = |\{\mathbf{y} \in C_t : j-1 \le I_y(t) < j\}|$ for $1 \le j \le n$. By the translation invariance of the model, it is clear that the marginal distribution of $r_t^{(j)}$, are the same for $1 \le j \le n$. Thus, it is enough to show that $\mathbb{E}(r_t^{(1)}) < \infty$.

Now, we observe that $\{\mathbf{y} \in C_t : 0 \le I_y(t) < 1\} \subseteq U_t := \{\mathbf{y} = (y_1, u) \in \mathcal{V} : u > t, h(\mathbf{y}) = (x_1, v), v \le t, 0 \le I_y(t) < 1\}$. The second set represents the vertices whose second co-ordinates are strictly larger than t, for each such vertex its immediate offspring having a second co-ordinate at most t and the edge connecting the vertex and its offspring intersecting the line L_t at some point between 0 and 1. Note that we have relaxed the condition of having infinite ancestry of the vertices above.

Set $a_i = 1 + 2i$, for $i = 1, 2, \dots$ and $s_i = \sum_{j=1}^{i} a_j$. Here, a_i is the number of vertices on the line L_{t+i} which can possibly be included in the set U_t. Now, if $s_{i+1} \ge |U_t| > s_i$, then some vertex $\mathbf{x}$ whose second co-ordinate is at least $(t+i+1)$ will connect to some vertex $\mathbf{y}$ whose second co-ordinate is at most t. Thus, the probability of such an event is dominated by the probability of the event that the vertices in the cone $\Lambda(\mathbf{y})$ up to the level $\mathbf{x}$ are closed. Since there are at least $i^2 - 1$ many vertices in this region, we have

$$\mathbb{P}(s_{i+1} \ge |U_t| > s_i) \le (1-p)^{i^2-1}.$$

Thus, $\mathbb{E}(|U_t|) < \infty$.

Now, consider $C_0(n)$ and divide it into two parts, $C_0^{(1)}(n)$ and $C_0^{(2)}(n)$ where $C_0^{(1)}(n) = \{\mathbf{y} \in C_0(n) : \mathbf{y} = (y_1, 1)\}$ and $C_0^{(2)}(n) = \{\mathbf{y} \in C_0(n) : \mathbf{y} = (y_1, u), u > 1\}$. We divide the set $C_0^{(2)}(n)$ into two further subsets. It is clear that for each $\mathbf{y} \in C_0^{(2)}(n)$, the edge joining $\mathbf{y}$ and $h(\mathbf{y})$ intersects both the lines L_0 and L_1. Let us denote by $(I(\mathbf{y}; 1), 1) (= I_y(t+1))$ the point of intersection of the edge $\langle \mathbf{y}, h(\mathbf{y}) \rangle$ with L_1. Define,

$$C_0^{(2)}(n; 1) = \{\mathbf{y} \in C_0^{(2)}(n) : 0 \le I(\mathbf{y}; 1) < n\}$$

and

$$C_0^{(2)}(n; 2) = \{\mathbf{y} \in C_0^{(2)}(n) : I(\mathbf{y}; 1) \ge n \text{ or } I(\mathbf{y}; 1) < 0\}.$$

Thus, by definition of $C_0^{(2)}(n; 1)$, we have $C_0^{(2)}(n; 1) \subseteq C_1(n)$. Further, for any vertex $\mathbf{x}$, $h(\mathbf{x})$ can lie at an angle at most $\frac{\pi}{4}$ from the vertical line passing through $\mathbf{x}$, so we see that any vertex of $C_0^{(2)}(n)$ which intersects L_0 at $(I_t(y), 0)$ with $1 \le I_t(y) < n-1$, must intersect L_1 at some point $(I(\mathbf{y}; 1), 1)$ with $0 \le I(\mathbf{y}; 1) < n$. Therefore, we must have that,

$$C_0^{(2)}(n; 2) \subseteq \{\mathbf{y} \in C_0 : 0 \le I_y(0) < 1\} \bigcup \{\mathbf{y} \in C_0 : n-1 \le I_y(0) < n\}.$$

Thus, we get,

$$|C_0^{(2)}(n; 2)| \le r_0^{(1)} + r_0^{(n)}. \tag{44}$$

Now, consider the set $C_0^{(1)}(n) \setminus \{(-1, 1), (n, 1)\}$ and partition it into two sets, one of which contains only branching points and the other does not contain any branching point, i.e., $C_0^{(1)}(n) \setminus \{(-1, 1), (n, 1)\} = C_0^{(1)}(n; 1) \cup C_0^{(1)}(n; 2)$, where

$$C_0^{(1)}(n; 1) = \left\{\mathbf{y} \in C_0^{(1)}(n) \setminus \{(-1, 1), (n, 1)\} : y \text{ is a not branching point}\right\}$$

and

$$C_0^{(1)}(n;2) = \Big\{ y \in C_0^{(1)}(n) \setminus \{(-1,1),(n,1)\} : y \text{ is a branching point} \Big\}.$$

Now, it is clear that for each $\mathbf{y} \in C_0^{(1)}(n;1)$, there exists a unique ancestor which further has infinite ancestry. Therefore, we can define,

$$D_1 = \{\mathbf{z} : h(\mathbf{z}) \in C_0^{(1)}(n;1) \text{ and } B_n(\mathbf{z}) \neq \emptyset \text{ for } n \geq 1\}.$$

For $\mathbf{y} \in C_0^{(1)}(n;2)$, being a branching point, there exists at least two distinct vertices, both of which has infinite ancestry. Thus, we may define,

$$D_2 = \{\mathbf{z}_1, \mathbf{z}_2 : h(\mathbf{z}_1), h(\mathbf{z}_2) \in C_0^{(1)}(n;2), h(\mathbf{z}_1) = h(\mathbf{z}_2) \text{ and } B_n(\mathbf{z}_1) \neq \emptyset, B_n(\mathbf{z}_2) \neq \emptyset, \text{ for } n \geq 1\}.$$

Since every vertex has a unique offspring, we must have, $D_1 \cap D_2 = \emptyset$. Further, by definition of $C_1(n)$, we have, $D_1 \cup D_2 \subseteq C_1(n)$. Also, it is clear that $(D_1 \cup D_2) \cap C_0^{(2)}(n;1) = \emptyset$ as the offspring of any vertex in D_1 and D_2 lie on the line L_1 while the offspring of any vertex in $C_0^{(2)}(n;1)$ lie on L_u for $u \leq 0$. Thus, we have,

$$\begin{aligned}
|C_1(n)| &\geq |C_0^{(2)}(n;1)| + |C_0^{(1)}(n;1)| + 2(|C_0^{(1)}(n;2)|) \\
&= \big[|C_0^{(2)}(n;1)| + |C_0^{(2)}(n;2)| + |C_0^{(1)}(n;1)| + |C_0^{(1)}(n;2)| \big] \\
&\qquad + |C_0^{(1)}(n;2)| - |C_0^{(2)}(n;2)| \\
&\geq |C_0(n)| + |C_0^{(1)}(n;2)| - 2 - r_0^{(1)} - r_0^{(n)}
\end{aligned} \tag{45}$$

where we have used the inequality (44) in the last step.

But, we have from stationarity, $\mathbb{E}(|C_1(n))| = \mathbb{E}(|C_0(n)|)$ for all $n \geq 1$. Thus, for n sufficiently large, from (43) and (44), we have

$$\begin{aligned}
0 = \mathbb{E}\big(C_1(n) - C_0(n)\big) & \\
&\geq \mathbb{E}\big(|C_0^{(1)}(n;2)|\big) - 2 - 2\mathbb{E}(r_0^{(1)}) \\
&= n\mathbb{P}(\text{Origin is a branching point}) - 2 - 2\mathbb{E}(r_0^{(1)}) \\
&> 0.
\end{aligned}$$

This contradiction establishes Theorem 2.

References

[1] Alexander, K. (1995) Percolation and Minimal Spanning Forests in Infinite Graphs. *Ann. Probab* **23**, 87–104. MR1330762

[2] Asmussen, S. (1987) *Applied probability and queues.* Wiley, New York. MR0889893

[3] Baccelli, F. and Bordenave, C. (2007) The radial spanning tree of a Poisson point process. *Ann. Appl. Probab.* **17**, 305 –359. MR2292589

[4] BORDENAVE, C. (2008) Navigation on a Poisson point process. *Ann. Appl. Probab.* **18**, 708 –756. MR2399710

[5] FERRARI, P. A., FONTES, L.R.G. and WU, X-Y. (2005) Two-dimensional Poisson trees converge to the Brownian web. *Ann. I. H. Poincaré* **41**, 851–858. MR2165253

[6] FERRARI, P. A., LANDIM, C., and THORISSON, H. (2004) Poisson trees, succession lines and coalescing random walks. *Ann. Inst. H. Poincaré Probab. Statist.* **40**, 141–152.

[7] FONTES, L.R.G., ISOPI, M., NEWMAN, C.M. and RAVISHANKAR, K. (2004) The Brownian web: characterization and convergence. *Ann. Probab.* **32**, 2857–2883. MR2094432

[8] GANGOPADHYAY, S., ROY, R. and SARKAR, A. (2004) Random oriented trees: a model for drainage networks *Ann. Appl. Probab.* **14**, 1242–1266. MR2071422

[9] HOWARD, A. D. (1971). Simulation of stream networks by headward growth and branching. *Geogr. Anal.* **3**, 29–50.

[10] LEOPOLD, L. B. and LANGBEIN, W. B. (1962). The concept of entropy in landscape evolution. *U.S. Geol. Surv. Prof. Paper.* 500-A.

[11] NANDI, A.K. and MANNA, S.S. (2007). A transition from river networks to scale-free networks. *New J. Phys.* **9** 30 `http://www.njp.org` `doi:10.1088/1367-2630/9/2/030`.

[12] NEWMAN, C. M. and STEIN, D. L. (1996) Ground-state structure in a highly disordered spin-glass model. *J. Statist. Phys.* **82**, 1113–1132. MR1372437

[13] PEMANTLE, R. (1991) Choosing a spanning tree for the integer lattice uniformly. *Ann. Probab.* **19**, 1559–1574. MR1127715

[14] RODRIGUEZ-ITURBE, I. and RINALDO, A. (1997). *Fractal river basins: chance and self-organization.* Cambridge Univ. Press, New York.

[15] SCHEIDEGGER, A. E. (1967). A stochastic model for drainage pattern into an intramontane trench. *Bull. Ass. Sci. Hydrol.* **12**, 15–20.

[16] SPITZER, F (1964). *Principles of random walk.* Van Nostrand, Princeton. MR0171290

[17] TÓTH, B and WERNER, W. (1998) The true self-repelling motion. *Probability Theory and Related Fields* **111**, 375–452 MR1640799