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Maximum Principle and Comparison Theorem for Quasi-linear Stochastic PDE's

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Abstract

We prove a comparison theorem and maximum principle for a local solution of quasi-linear parabolic stochastic PDEs, similar to the well known results in the deterministic case. The proofs are based on a version of Ito's formula and estimates for the positive part of a local solution which is non-positive on the lateral boundary. Moreover we shortly indicate how these results generalize for Burgers type SPDEs.

Key words: Stochastic partial differential equation, Ito's formula, Maximum principle, Moser's

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1 Introduction

In the theory of Partial Differential Equations, the maximum principle plays an important role and there is a huge literature on this subject. It permits one to study the local behavior of solutions of PDE since it gives a relation between the bound of the solution on the boundary and a bound on the whole domain. The maximum principle for quasi-linear parabolic equations was proved by Aronson -Serrin (see Theorem 1 of [1]) in the following form.

Theorem 1. *Let u be a weak solution of a quasi-linear parabolic equation of the form*

$$\partial_t u = div \mathscr{A}(t,x,u,\nabla u) + \mathscr{B}(t,x,u,\nabla u)$$

in the bounded cylinder $]0,T[\times\mathscr{O} \subset \mathbb{R}^{d+1}$. *If* $u \leq M$ *on the parabolic boundary* $\{[0,T[\times\partial\mathscr{O}\} \cup \{\{0\}\times\mathscr{O}\}$, *then one has*

$$u \leq M + Cf(\mathscr{A},\mathscr{B}),$$

where C depends only on T, the volume of $\mathscr{O}$ *and the structure of the equation, while* $f(\mathscr{A},\mathscr{B})$ *is directly expressed in terms of some quantities related to the coefficients* $\mathscr{A}$ *and* $\mathscr{B}$.

The method of proof was based on Moser's iteration scheme adapted to the nonlinear case. This method of Aronson and Serrin was further adapted to the stochastic framework in [5], obtaining some L^p a priori estimates for the uniform norm of the solution of the stochastic quasi-linear parabolic equation. However the results of that paper concern only the case of solution with null Dirichlet condition and the method was based on the properties of the semi-group corresponding to null boundary condition. In particular the version of Ito's formula established in ([5], Proposition 10) was for solutions with null Dirichlet condition.

The aim of the present paper is to consider the case of local solutions, which, roughly speaking, are weak solutions without conditions at the boundary. For example a solution obtained in a larger domain $\mathscr{D}$ with null conditions on $\partial\mathscr{D}$, when regarded on $\mathscr{O}$ becomes a local solution. We assume that a local solution is bounded from above by an Ito process on the boundary of the domain and then we deduce a stochastic version of the maximum principle of Aronson -Serrin. This generalization is not a simple consequence of the previous results because the local solutions which do not vanish on the lateral boundary are not directly tractable with the semigroup of null Dirichlet conditions. The main point is that we have to establish an Ito's type formula for the positive part of a local solution which is non-positive on the lateral boundary (see Proposition 1).

More precisely, we study the following stochastic partial differential equation (in short SPDE) for a real -valued random field $u_t(x) = u(t,x)$,

$$du_t(x) = Lu_t(x)\,dt + f_t\left(x,u_t(x),\nabla u_t(x)\right)dt + \sum_{i=1}^{d}\partial_i g_{i,t}\left(x,u_t(x),\nabla u_t(x)\right)dt$$

$$+\sum_{j=1}^{d_1} h_{j,t}\left(x,u_t(x),\nabla u_t(x)\right)dB_t^j \qquad (1)$$

with a given initial condition $u_0 = \xi$, where L is a symmetric, uniformly elliptic, second order differential operator defined in some bounded open domain $\mathscr{O}\subset\mathbb{R}^d$ and $f, g_i, i = 1,...,d, h_j, j = 1,...,d_1$ are nonlinear random functions. Let us note that in order to simplify the appearance of the equation we have chosen to write it as a sum of a linear uniformly parabolic part and two nonlinear terms, expressed by f and g in (1).

The study of the L^p norms w.r.t. the randomness of the space-time uniform norm on the trajectoriesof a stochastic PDE was started by N. V. Krylov in [7]. His aim was to obtain estimates useful for numerical approximations. In [5] we have introduced the method of iteration of Moser (more precisely a version due to Aronson -Serrin for non -linear equations) in the stochastic framework, which allowed us to treat equations with measurable coefficients. The present paper is a continuation of these. One of our motivations is to get Holder continuity properties for the solution of the SPDE in a forthcoming paper. As in the deterministic case we think that an essential step is to establish a stochastic version of a maximum principle. Moreover, our maximum principle allows one to estimate the solution of the Dirichlet problem with random boundary data. For simplicity, let us give a consequence of it. Under suitable assumptions on f, g, h (Lipschitz continuity and integrability conditions), we have

Theorem 2. *Let $(M_t)_{t\geq 0}$ be an Itô process satisfying some integrability conditions, $p \geq 2$ and u be a local weak solution of (1). Assume that $u \leq M$ on the parabolic boundary $\{[0,T[\times \partial\mathcal{O}\} \cup \{\{0\}\times\mathcal{O}\}$, then for all $t \in [0,T]$:*

$$E\left\|(u-M)^+\right\|^p_{\infty,\infty;t} \leq k\left(p,t\right) E\left(\|\xi - M_0\|^p_\infty + \left\|(f^{0,M})^+\right\|^{*p}_{\theta,t} + \left\||g^{0,M}|^2\right\|^{*p/2}_{\theta;t} + \left\||h^{0,M}|^2\right\|^{*p/2}_{\theta;t}\right)$$

where $f^{0,M}(t,x) = f(t,x,M,0)$, $g^{0,M}(t,x) = g(t,x,M,0)$, $h^{0,M}(t,x) = h(t,x,M,0)$ and k is a function which only depends on the structure constants of the SPDE, $\|\cdot\|_{\infty,\infty;t}$ is the uniform norm on $[0,t]\times\mathcal{O}$ and $\|\cdot\|^*_{\theta;t}$ is a certain norm which is precisely defined below.

The paper is organized as follows : in section 2 we introduce notations and hypotheses and we take care to detail the integrability conditions which are used all along the paper. In section 3 we establish Itô's formula for the positive part of the local solution (Proposition 1). In section 4, we prove a comparison theorem (Theorem 5) which yields the maximum principle (Theorem 7). Then in section 5 we prove an existence result for Burgers type SPDE's with null Dirichlet conditions and so we generalize results obtained by Gyöngy and Rovira [6]. Moreover we shortly indicate how the maximum principle and the comparison theorem generalize to this kind of equations. Finally in the appendix we present some technical facts related to solutions in the L^1-sense which are used in the proofs of the preceding sections.

2 Preliminaries

2.1 $L^{p,q}$-spaces

Let $\mathcal{O}$ be an open bounded domain in $\mathbb{R}^d$. The space $L^2(\mathcal{O})$ is the basic Hilbert space of our framework and we employ the usual notation for its scalar product and its norm,

$$(u,v) = \int_{\mathcal{O}} u(x)v(x)\,dx,\ \|u\|_2 = \left(\int_{\mathcal{O}} u^2(x)\,dx\right)^{\frac{1}{2}}.$$

In general, we shall use the notation

$$(u,v) = \int_{\mathcal{O}} u(x)v(x)\,dx,$$

where u, v are measurable functions defined in $\mathscr{O}$ and $uv \in L^1(\mathscr{O})$.
Another Hilbert space that we use is the first order Sobolev space of functions vanishing at the boundary, $H_0^1(\mathscr{O})$. Its natural scalar product and norm are

$$(u,v)_{H_0^1(\mathscr{O})} = (u,v) + \int_{\mathscr{O}} \sum_{i=1}^{d} \left(\partial_i u(x)\right)\left(\partial_i v(x)\right) dx,\ \|u\|_{H_0^1(\mathscr{O})} = \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}}.$$

We shall denote by $H^1_{loc}(\mathscr{O})$ the space of functions which are locally square integrable in $\mathscr{O}$ and which admit first order derivatives that are also locally square integrable.

For each $t > 0$ and for all real numbers p, $q \geq 1$, we denote by $L^{p,q}([0,t]\times\mathscr{O})$ the space of (classes of) measurable functions $u : [0,t]\times\mathscr{O} \longrightarrow \mathbb{R}$ such that

$$\|u\|_{p,q;\,t} := \left(\int_0^t \left(\int_{\mathscr{O}} |u(s,x)|^p\, dx\right)^{q/p} ds\right)^{1/q}$$

is finite. The limiting cases with p or q taking the value ∞ are also considered with the use of the essential sup norm. We identify this space, in an obvious way, with the space $L^q\left([0,t]\,;L^p\left(\mathscr{O}\right)\right)$, consisting of all measurable functions $u : [0,t] \to L^p\left(\mathscr{O}\right)$ such that $\displaystyle\int_0^t \left\|u_s\right\|_p^q ds < \infty$. This identification implies that $\displaystyle\left(\int_0^t \left\|u_s\right\|_p^q ds\right)^{\frac{1}{q}} = \|u\|_{p,q;\,t}$.

The space of measurable functions $u : \mathbb{R}_+ \to L^2\left(\mathscr{O}\right)$ such that $\|u\|_{2,2;t} < \infty$, for each $t \geq 0$, is denoted by $L^2_{loc}\left(\mathbb{R}_+;L^2\left(\mathscr{O}\right)\right)$. Similarly, the space $L^2_{loc}\left(\mathbb{R}_+;H_0^1\left(\mathscr{O}\right)\right)$ consists of all measurable functions $u : \mathbb{R}_+ \to H_0^1\left(\mathscr{O}\right)$ such that

$$\|u\|_{2,2;t} + \|\nabla u\|_{2,2;t} < \infty,$$

for any $t \geq 0$.

Next we are going to introduce some other spaces of functions of interest and to discuss a certain duality between them. They have already been used in [1] and [5] but here intervenes a new case and we change a little bit the notation used before in a way which, we think, make things clearer.

Let $\left(p_1,q_1\right),\left(p_2,q_2\right) \in [1,\infty]^2$ be fixed and set

$$I = I\left(p_1,q_1,p_2,q_2\right) := \Big\{\left(p,q\right) \in [1,\infty]^2 \,/\, \exists\, \rho \in [0,1]\, s.t.$$

$$\frac{1}{p} = \rho\frac{1}{p_1} + \left(1-\rho\right)\frac{1}{p_2}, \frac{1}{q} = \rho\frac{1}{q_1} + \left(1-\rho\right)\frac{1}{q_2}\Big\}.$$

This means that the set of inverse pairs $\left(\frac{1}{p},\frac{1}{q}\right)$, (p,q) belonging to I, is a segment contained in the square $[0,1]^2$, with the extremities $\left(\frac{1}{p_1},\frac{1}{q_1}\right)$ and $\left(\frac{1}{p_2},\frac{1}{q_2}\right)$. There are two spaces of interest associated to I. One is the intersection space

$$L_{I;t} = \bigcap_{(p,q)\in I} L^{p,q}\left([0,t]\times\mathscr{O}\right).$$

Standard arguments based on Hölder's inequality lead to the following inclusion (see e.g. Lemma 2 in [5])

$$L^{p_1,q_1}([0,t]\times\mathcal{O})\cap L^{p_2,q_2}([0,t]\times\mathcal{O})\subset L^{p,q}([0,t]\times\mathcal{O}),$$

for each $(p,q)\in I$, and the inequality

$$\|u\|_{p,q;t}\leq\|u\|_{p_1,q_1;t}\vee\|u\|_{p_2,q_2;t},$$

for any $u\in L^{p_1,q_1}([0,t]\times\mathcal{O})\cap L^{p_2,q_2}([0,t]\times\mathcal{O})$. Therefore the space $L_{I;t}$ coincides with the intersection of the extreme spaces,

$$L_{I;t}=L^{p_1,q_1}([0,t]\times\mathcal{O})\cap L^{p_2,q_2}([0,t]\times\mathcal{O})$$

and it is a Banach space with the following norm

$$\|u\|_{I;t}:=\|u\|_{p_1,q_1;t}\vee\|u\|_{p_2,q_2;t}.$$

The other space of interest is the algebraic sum

$$L^{I;t}:=\sum_{(p,q)\in I}L^{p,q}([0,t]\times\mathcal{O}),$$

which represents the vector space generated by the same family of spaces. This is a normed vector space with the norm

$$\|u\|^{I;t}:=\inf\left\{\sum_{i=1}^{n}\left\|u_i\right\|_{p_i,q_i;\,t}\ /\ u=\sum_{i=1}^{n}u_i, u_i\in L^{p_i,q_i}([0,t]\times\mathcal{O}),\ (p_i,q_i)\in I,\ i=1,...n;\ n\in\mathbb{N}^*\right\}.$$

Clearly one has $L^{I;t}\subset L^{1,1}([0,t]\times\mathcal{O})$ and $\|u\|_{1,1;t}\leq c\|u\|^{I;t}$, for each $u\in L^{I;t}$, with a certain constant $c>0$.

We also remark that if $(p,q)\in I$, then the conjugate pair (p',q'), with $\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$, belongs to another set, I', of the same type. This set may be described by

$$I'=I'(p_1,q_1,p_2,q_2):=\left\{(p',q')\ /\ \exists\,(p,q)\in I\ s.t.\ \frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1\right\}$$

and it is not difficult to check that $I'(p_1,q_1,p_2,q_2)=I\left(p_1',q_1',p_2',q_2'\right)$, where p_1',q_1',p_2' and q_2' are defined by $\frac{1}{p_1}+\frac{1}{p_1'}=\frac{1}{q_1}+\frac{1}{q_1'}=\frac{1}{p_2}+\frac{1}{p_2'}=\frac{1}{q_2}+\frac{1}{q_2'}=1$.

Moreover, by Hölder's inequality, it follows that one has

$$\int_0^t\int_{\mathcal{O}}u(s,x)\,v(s,x)\,dxds\leq\|u\|_{I;t}\|v\|^{I';t},\tag{2}$$

for any $u\in L_{I;t}$ and $v\in L^{I';t}$. This inequality shows that the scalar product of $L^2([0,t]\times\mathcal{O})$ extends to a duality relation for the spaces $L_{I;t}$ and $L^{I';t}$.

Now let us recall that the Sobolev inequality states that

$$\|u\|_{2^*}\leq c_S\|\nabla u\|_2,$$

for each $u \in H_0^1(\mathcal{O})$, where $c_S > 0$ is a constant that depends on the dimension and $2^* = \frac{2d}{d-2}$ if $d > 2$, while 2^* may be any number in $]2,\infty[$ if $d = 2$ and $2^* = \infty$ if $d = 1$. Therefore one has

$$\|u\|_{2^*,2;t} \leq c_S \|\nabla u\|_{2,2;t} ,$$

for each $t \geq 0$ and each $u \in L^2_{loc}\left(\mathbb{R}_+; H_0^1(\mathcal{O})\right)$. And if $u \in L^\infty_{loc}\left(\mathbb{R}_+; L^2(\mathcal{O})\right) \bigcap L^2_{loc}\left(\mathbb{R}_+; H_0^1(\mathcal{O})\right)$, one has

$$\|u\|_{2,\infty;t} \vee \|u\|_{2^*,2;t} \leq c_1 \left(\|u\|^2_{2,\infty;t} + \|\nabla u\|^2_{2,2;t}\right)^{\frac{1}{2}},$$

with $c_1 = c_S \vee 1$.

One particular case of interest for us in relation with this inequality is when $p_1 = 2, q_1 = \infty$ and $p_2 = 2^*, q_2 = 2$. If $I = I(2,\infty,2^*,2)$, then the corresponding set of associated conjugate numbers is $I' = I'(2,\infty,2^*,2) = I\left(2,1,\frac{2^*}{2^*-1},2\right)$, where for $d = 1$ we make the convention that $\frac{2^*}{2^*-1} = 1$. In this particular case we shall use the notation $L_{\#;t} := L_{I;t}$ and $L^*_{\#;t} := L^{I';t}$ and the respective norms will be denoted by

$$\|u\|_{\#;t} := \|u\|_{I;t} = \|u\|_{2,\infty;t} \vee \|u\|_{2^*,2;t} ,\ \|u\|^*_{\#;t} := \|u\|^{I';t} .$$

Thus we may write

$$\|u\|_{\#;t} \leq c_1 \left(\|u\|^2_{2,\infty;t} + \|\nabla u\|^2_{2,2;t}\right)^{\frac{1}{2}}, \tag{3}$$

for any $u \in L^\infty_{loc}\left(\mathbb{R}_+; L^2(\mathcal{O})\right) \bigcap L^2_{loc}\left(\mathbb{R}_+; H_0^1(\mathcal{O})\right)$ and $t \geq 0$ and the duality inequality becomes

$$\int_0^t \int_{\mathcal{O}} u(s,x)\, v(s,x)\, dx ds \leq \|u\|_{\#;t} \|v\|^*_{\#;t} ,$$

for any $u \in L_{\#;t}$ and $v \in L^*_{\#;t}$.

2.2 Hypotheses

Let $\{B_t := (B^j_t)_{j\in\{1,\cdots,d_1\}}\}_{t\geq 0}$ be a d_1-dimentional Brownian motion defined on a standard filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$.

Let A be a symmetric second order differential operator given by $A := -L = -\sum_{i,j=1}^d \partial_i (a^{i,j} \partial_j)$. We assume that a is a measurable and symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda |\xi|^2 \leq \sum_{i,j} a^{i,j}(x) \xi^i \xi^j \leq \Lambda |\xi|^2, \quad \forall x \in \mathcal{O},\ \xi \in \mathbb{R}^d , \tag{4}$$

where λ and Λ are positive constants. The energy associated with the matrix a will be denoted by

$$\mathcal{E}(w,v) = \sum_{i,j=1}^d \int_{\mathcal{O}} a^{i,j}(x) \partial_i w(x) \partial_j v(x)\, dx. \tag{5}$$

It's defined for functions $w,\ v \in H_0^1(\mathcal{O})$, or for $w \in H^1_{loc}(\mathcal{O})$ and $v \in H_0^1(\mathcal{O})$ with compact support.

We consider the semilinear stochastic partial differential equation (1) for the real-valued random field $u_t(x)$ with initial condition $u(0,.) = \xi(.)$, where ξ is a $\mathscr{F}_0$-measurable random variable with values in $L^2_{loc}(\mathscr{O})$.

We assume that we have predictable random functions

$$
\begin{aligned}
f \quad &: \quad \mathbb{R}_+ \times \Omega \times \mathscr{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\,,\\
h \quad &: \quad \mathbb{R}_+ \times \Omega \times \mathscr{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d_1}\\
g \quad &= \quad (g_1, ..., g_d) : \mathbb{R}_+ \times \Omega \times \mathscr{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d
\end{aligned}
$$

We define

$$f(\cdot,\cdot,\cdot,0,0) := f^0, \quad h(\cdot,\cdot,\cdot,0,0) := h^0 \quad \text{and} \quad g(\cdot,\cdot,\cdot,0,0) := g^0 = (g_1^0, ..., g_d^0).$$

We considere the following sets of assumptions :

Assumption (H): There exist non negative constants C, α, β such that

- **(i)** $|f(t,\omega,x,y,z) - f(t,\omega,x,y',z')| \leq C\left(|y-y'| + |z-z'|\right)$
- **(ii)** $\left(\sum_{j=1}^{d_1} |h_j(t,\omega,x,y,z) - h_j(t,\omega,x,y',z')|^2\right)^{\frac{1}{2}} \leq C\,|y-y'| + \beta\,|z-z'|$,
- **(iii)** $\left(\sum_{i=1}^{d} |g_i(t,\omega,x,y,z) - g_i(t,\omega,x,y',z')|^2\right)^{\frac{1}{2}} \leq C\,|y-y'| + \alpha\,|z-z'|$.
- **(iv)** the contraction property (as in [5]) : $\alpha + \dfrac{\beta^2}{2} < \lambda\,$.

Moreover we introduce some integrability conditions on f^0, g^0, h^0 and the initial data ξ :

Assumption (HD) *local integrability conditions on f^0, g^0 and h^0* :

$$E\int_0^t \int_K \left(|f_t^0(x)| + |g_t^0(x)|^2 + |h_t^0|^2\right) dx\,dt < \infty$$

for any compact set $K \subset \mathscr{O}$, and for any $t \geq 0$.

Assumption (HI) *local integrability condition on the initial condition* :

$$E\int_K |\xi(x)|^2 dx < \infty$$

for any compact set $K \subset \mathscr{O}$.

Assumption (HD#)

$$E\left(\left(\left\|f^0\right\|_{\#;t}^*\right)^2 + \left\|g^0\right\|_{2,2;t}^2 + \left\|h^0\right\|_{2,2;t}^2\right) < \infty,$$

for each $t \geq 0$.

Sometimes we shall consider the following stronger forms of these conditions:

Assumption (HD2)

$$E\left(\left\|f^0\right\|_{2,2;t}^2+\left\|g^0\right\|_{2,2;t}^2+\left\|h^0\right\|_{2,2;t}^2\right)<\infty,$$

for each $t \geq 0$.

Assumption (HI2) *integrability condition on the initial condition* :

$$E\|\xi\|_2^2<\infty.$$

Remark 1. *Note that $(2,1)$ is the pair of conjugates of the pair $(2,\infty)$ and so $(2,1)$ belongs to the set I' which defines the space $L^*_{\#;t}$. Since $\|v\|_{2,1;t} \leq \sqrt{t}\,\|v\|_{2,2;t}$ for each $v \in L^{2,2}([0,t]\times\mathcal{O})$, it follows that*

$$L^{2,2}([0,t]\times\mathcal{O})\subset L^{2,1;t}\subset L^*_{\#;t},$$

*and $\|v\|^*_{\#;t} \leq \sqrt{t}\,\|v\|_{2,2;t}$, for each $v \in L^{2,2}([0,t]\times\mathcal{O})$. This shows that the condition **(HD#)** is weaker than **(HD2)**.*

The Lipschitz condition **(H)** is assumed to hold throughtout this paper, except the last section devoted to Burgers type equations. The weaker integrability conditions **(HD)** and **(HI)** are also assumed to hold everywhere in this paper. The other stronger integrability conditions will be mentioned whenever we will assume them.

2.3 Weak solutions

We now introduce $\mathcal{H}=\mathcal{H}(\mathcal{O})$, the space of $H^1_0(\mathcal{O})$-valued predictable processes $(u_t)_{t\geq 0}$ such that

$$\left(E\sup_{0\leq t\leq T}\left\|u_t\right\|_2^2+\int_0^T E\,\mathcal{E}\left(u_t\right)dt\right)^{1/2}<\infty\,,\quad \text{for each } T>0\,.$$

We define $\mathcal{H}_{loc}=\mathcal{H}_{loc}(\mathcal{O})$ to be the set of $H^1_{loc}(\mathcal{O})$-valued predictable processes such that for any compact subset K in $\mathcal{O}$ and all $T>0$:

$$\left(E\sup_{0\leq t\leq T}\int_K u_t(x)^2\,dx+E\int_0^T\int_K|\nabla u_t(x)|^2\,dx\,dt\right)^{1/2}<\infty.$$

The space of test functions is $\mathcal{D}=\mathcal{C}_c^\infty\otimes\mathcal{C}_c^2(\mathcal{O})$, where $\mathcal{C}_c^\infty$ denotes the space of all real infinite differentiable functions with compact support in $\mathbb{R}$ and $\mathcal{C}_c^2(\mathcal{O})$ the set of C^2-functions with compact support in $\mathcal{O}$.

Definition 1. *We say that $u\in\mathcal{H}_{loc}$ is a weak solution of equation (1) with initial condition ξ if the following relation holds almost surely, for each $\varphi\in\mathcal{D}$,*

$$\begin{aligned}\int_0^\infty\Big[(u_s,\partial_s\varphi)-\mathcal{E}\left(u_s,\varphi_s\right)+\left(f\left(s,u_s,\nabla u_s\right),\varphi_s\right)-\sum_{i=1}^d\left(g_i\left(s,u_s,\nabla u_s\right),\partial_i\varphi_s\right)\Big]ds\\+\int_0^\infty\left(h\left(s,u_s,\nabla u_s\right),\varphi_s\right)dB_s+\left(\xi,\varphi_0\right)=0.\end{aligned}\tag{6}$$

We denote by $\mathcal{U}_{loc}(\xi,f,g,h)$ the set of all such solutions u.
If u belongs to $\mathcal{H}$, we say that u solves the SPDE with zero Dirichlet condition on the boundary.

In general we do not know much about the set $\mathscr{U}_{loc}\left(\xi,f,g,h\right)$. It may be empty or may contain several elements. But under the conditions **(H)**, **(HI2)** and **(HD2)** we know from Theorem 9 in [4] that there exists a unique solution in $\mathscr{H}$ and that this solution admits $L^2(\mathscr{O})$-continuous trajectories. As the space $H_0^1(\mathscr{O})$ consists of functions which vanish in a generalized sense at the boundary $\partial\mathscr{O}$, we may say that a solution which belongs to $\mathscr{H}$ satisfies the zero Dirichlet conditions at the boundary of $\mathscr{O}$. Thus we may say that under the assumptions **(H)**, **(HD2)** and **(HI2)** there exists a unique solution with null Dirichlet conditions at the boundary of $\mathscr{O}$. This result will be generalised below. We denote by $\mathscr{U}\left(\xi,f,g,h\right)$ the solution of (1) with zero Dirichlet boundary conditions whenever it exists and is unique.

We should also note that if the conditions **(H)**, **(HD2)** and **(HI2)** are satisfied and if u is a process in $\mathscr{H}$, the relation from this definition holds with any test function $\varphi\in\mathscr{D}$ if and only if it holds with any test function in $\mathscr{C}_c^\infty\left(\mathbf{R}_+\right)\otimes H_0^1(\mathscr{O})$. In fact, in this case, one may use as space of test functions any space of the form $\mathscr{C}_c^\infty\left(\mathbf{R}_+\right)\otimes V$, where V is a dense subspace of $H_0^1(\mathscr{O})$, obtaining equivalent definitions of the notion of solution with null Dirichlet conditions at the boundary of $\mathscr{O}$. In [4] one uses $\mathscr{C}_c^\infty\left(\mathbf{R}_+\right)\otimes\mathscr{D}(A)$ as space of test functions because this is the space which suits better the abstract analytic functional framework of that paper.

Remark 2. *It is proved in [4] that under **(HI2)** and **(HD2)** the solution with null Dirichlet conditions at the boundary of $\mathscr{O}$ has a version with $L^2(\mathscr{O})$-continuous trajectories and, in particular, that $\lim_{t\to 0}\|u_t-\xi\|_2=0$, a.s. This property extends to the local solutions in the sense that any element of $\mathscr{U}_{loc}(\xi,f,g,h)$ has a version with the property that a.s. the trajectories are $L^2(K)$-continuous, for each compact set $K\subset\mathscr{O}$ and*

$$\lim_{t\to 0}\int_K\left(u_t(x)-\xi(x)\right)^2dx=0.$$

In order to see this it suffices to take a test function $\phi\in\mathscr{C}_c^\infty(\mathscr{O})$ and to verify that $v=\phi u$ satifies the equation

$$dv_t=\left(Lv_t+\overline{f}_t+div\overline{g}_t\right)+\overline{h}_t dB_t,$$

with the initial condition $v_0=\phi\xi$, where

$$\begin{aligned}
&\overline{f}_t(x)=\phi(x)f\left(t,x,u_t(x),\nabla u_t(x)\right)-\langle\nabla\phi(x),a(x)\nabla u_t(x)\rangle-\langle\nabla\phi(x),g\left(t,x,u_t(x),\nabla u_t(x)\right)\rangle,\\
&\overline{g}_t(x)=\phi(x)g\left(t,x,u_t(x),\nabla u_t(x)\right)-u_t(x)a(x)\nabla\phi(x)\quad\text{and}\\
&\overline{h}_t(x)=\phi(x)h\left(t,x,u_t(x),\nabla u_t(x)\right).
\end{aligned}$$

Thus $v=\mathscr{U}\left(\phi\xi,\overline{f},\overline{g},\overline{h}\right)$ and the results of [4] hold for v.

Remark 3. *Let us now precise the sense in which a solution is dominated on the lateral boundary. Assume that v belongs to $H^1_{loc}(\mathscr{O}')$ where $\mathscr{O}'$ is a larger open set such that $\overline{\mathscr{O}}\subset\mathscr{O}'$. Then it is well known that the condition $v^+_{|\mathscr{O}}\in H_0^1(\mathscr{O})$ expresses the boundary relation $v\le 0$ on $\partial\mathscr{O}$. Similarly, if a process u belongs to $\mathscr{H}_{loc}(\mathscr{O}')$, then the condition $u^+_{|\mathscr{O}}\in\mathscr{H}(\mathscr{O})$ ensures the inequality $u\le 0$ on the lateral boundary $\{[0,\infty[\times\partial\mathscr{O}\}$.*

3 Itô's formula

3.1 Estimates for solutions with null Dirichlet conditions

Now we are going to improve the existence theorem and the estimates satisfied by the solution obtained in the general framework of [4]. Though strictly speaking this improvement is not indispensable for the main subject, it is interesting because it shows the minimal integrability conditions one should impose to the functions f^0, g^0, h^0. Namely, taking into account the advantage of uniform ellipticity, we replace the condition **(HD2)** with the weaker one **(HD#)**.

Theorem 3. *Under the conditions* **(H)**, **(HD#)** *and* **(HI2)** *there exists a unique solution of (1) in* $\mathcal{H}$. *Moreover, this solution has a version with* $L^2(\mathcal{O})$*-continuous trajectories and it satisfies the following estimates*

$$E\left(\|u\|_{2,\infty;t}^2 + \|\nabla u\|_{2,2;t}^2\right) \le k(t) E\left(\|\xi\|_2^2 + \left(\left\|f^0\right\|_{\#;t}^*\right)^2 + \left\|g^0\right\|_{2,2;t}^2 + \left\|h^0\right\|_{2,2;t}^2\right),$$

for each $t \ge 0$, *where* $k(t)$ *is a constant that only depends on the structure constants and* t.

Proof:

Theorem 9 of [4] ensures the existence of the solution under the stronger condition **(HD2)**. So we now assume this condition and we shall next prove that then the solution $u = \mathcal{U}\left(\xi, f, g, h\right)$ satisfies the estimates asserted by our theorem. We start by writing Ito's formula for the solution in the form

$$\begin{aligned}
&\left\|u_t\right\|_2^2 + 2\int_0^t \mathcal{E}\left(u_s, u_s\right) ds = \|\xi\|_2^2 + 2\int_0^t \left(u_s, f_s\left(u_s, \nabla u_s\right)\right) ds \\
&-2\int_0^t \sum_{i=1}^d \left(\partial_i u_s, g_{i,s}\left(u_s, \nabla u_s\right)\right) ds + \int_0^t \left\|h_s\left(u_s, \nabla u_s\right)\right\|_2^2 ds \\
&+2\sum_{j=1}^{d_1} \int_0^t \left(u_s, h_{j,s}\left(u_s, \nabla u_s\right)\right) dB_s^j,
\end{aligned} \tag{7}$$

equality which holds a.s. (See (ii) of the Proposition 7 in [4]). This is in fact a stochastic version of Cacciopoli's identity, well-known for deterministic parabolic equations.

The Lipschitz condition and the inequality (2) lead to the following estimate

$$\int_0^t \left(u_s, f_s\left(u_s, \nabla u_s\right)\right) ds \le \varepsilon \|\nabla u\|_{2,2;t}^2 + c_\varepsilon \|u\|_{2,2;t}^2 + \delta \|u\|_{\#;t}^2 + c_\delta \left(\left\|f^0\right\|_{\#;t}^*\right)^2,$$

where $\varepsilon, \delta > 0$ are two small parameters to be chosen later and c_ε, c_δ are constants depending of them. Similar estimates hold for the next two terms

$$-\int_0^t \sum_{i=1}^d \left(\partial_i u_s, g_{i,s}\left(u_s, \nabla u_s\right)\right) ds \le (\alpha + \varepsilon) \|\nabla u\|_{2,2;t}^2 + c_\varepsilon \|u\|_{2,2;t}^2 + c_\varepsilon \left\|g^0\right\|_{2,2;t}^2,$$

$$\int_0^t \left\|h_s\left(u_s, \nabla u_s\right)\right\|_2^2 ds \le \left(\beta^2 + \varepsilon\right) \|\nabla u\|_{2,2;t}^2 + c_\varepsilon \|u\|_{2,2;t}^2 + c_\varepsilon \left\|h^0\right\|_{2,2;t}^2.$$

Since $\mathscr{E}\left(u_s,u_s\right)\geq\lambda\left\|\nabla u_s\right\|_2^2$, we deduce from the equality (7),

$$\left\|u_t\right\|_2^2+2\left(\lambda-\alpha-\frac{\beta^2}{2}-\frac{5}{2}\varepsilon\right)\|\nabla u\|_{2,2;t}^2\leq\delta\|u\|_{\#;t}^2+\|\xi\|_2^2+2c_\delta\left(\left\|f^0\right\|_{\#;t}^*\right)^2$$

$$+2c_\varepsilon\left\|g^0\right\|_{2,2;t}^2+c_\varepsilon\left\|h^0\right\|_{2,2;t}^2+5c_\varepsilon\|u\|_{2,2;t}^2+2M_t, \qquad (8)$$

a.s., where $M_t:=\sum_{j=1}^{d_1}\int_0^t\left(u_s,h_{j,s}\left(u_s,\nabla u_s\right)\right)dB_s^j$ represents the martingale part. Further, using a stopping procedure while taking the expectation, the martingale part vanishes, so that we get

$$E\left\|u_t\right\|_2^2+2\left(\lambda-\alpha-\frac{\beta^2}{2}-\frac{5}{2}\varepsilon\right)E\|\nabla u\|_{2,2;t}^2\leq\delta E\|u\|_{\#;t}^2$$

$$+E\left(\|\xi\|_2^2+2c_\delta\left(\left\|f^0\right\|_{\#;t}^*\right)^2+2c_\varepsilon\left\|g^0\right\|_{2,2;t}^2+c_\varepsilon\left\|h^0\right\|_{2,2;t}^2\right)+5c_\varepsilon\int_0^t E\left\|u_s\right\|_2^2ds.$$

Then we choose $\varepsilon=\frac{1}{5}\left(\lambda-\alpha-\frac{\beta^2}{2}\right)$, set $\gamma=\lambda-\alpha-\frac{\beta^2}{2}$ and apply Gronwall's lemma obtaining

$$E\left\|u_t\right\|_2^2+\gamma E\|\nabla u\|_{2,2;t}^2\leq\left(\delta E\|u\|_{\#;t}^2+EF\left(\delta,\xi,f^0,g^0,h^0,t\right)\right)e^{5c_\varepsilon t}, \qquad (*)$$

where $F\left(\delta,\xi,f^0,g^0,h^0,t\right)=\left(\|\xi\|_2^2+2c_\delta\left(\left\|f^0\right\|_{\#;t}^*\right)^2+2c_\varepsilon\left\|g^0\right\|_{2,2;t}^2+c_\varepsilon\left\|h^0\right\|_{2,2;t}^2\right)$. As a consequence one gets

$$E\|u\|_{2,2;t}^2\leq\frac{1}{5c_\varepsilon}\left(\delta E\|u\|_{\#;t}^2+EF\left(\delta,\xi,f^0,g^0,h^0,t\right)\right)\left(e^{5c_\varepsilon t}-1\right). \qquad (**)$$

We now return to the inequality (8) and estimate a.s. the supremum for the first term, obtaining

$$\|u\|_{2,\infty;t}^2\leq\delta\|u\|_{\#;t}^2+F\left(\delta,\xi,f^0,g^0,h^0,t\right)+5c_\varepsilon\|u\|_{2,2;t}^2+2\sup_{s\leq t}M_s.$$

We would like to take the expectation in this relation and for that reason we need to estimate the bracket of the martingale part,

$$\langle M\rangle_t^{\frac{1}{2}}\leq\|u\|_{2,\infty;t}\|h(u,\nabla u)\|_{2,2;t}\leq\eta\|u\|_{2,\infty;t}^2+c_\eta\left(\|u\|_{2,2;t}^2+\|\nabla u\|_{2,2;t}^2+\left\|h^0\right\|_{2,2;t}^2\right),$$

with η another small parameter to be properly chosen. Using this estimate and the inequality of Burkholder-Davis-Gundy we deduce from the preceding inequality

$$\left(1-2C_{BDG}\eta\right)E\|u\|_{2,\infty;t}^2\leq\delta E\|u\|_{\#;t}^2+EF\left(\delta,\xi,f^0,g^0,h^0,t\right)$$

$$+\left(5c_\varepsilon+2C_{BDG}c_\eta\right)E\|u\|_{2,2;t}^2+2C_{BDG}c_\eta E\|\nabla u\|_{2,2;t}^2+2C_{BDG}c_\eta E\left\|h^0\right\|_{2,2;t}^2,$$

where C_{BDG} is the constant corresponding to the Burkholder-Davis-Gundy inequality. Further we choose the parameter $\eta=\frac{1}{4C_{BDG}}$ and combine this estimate with (*) and (**) to deduce an estimate of the form

$$E\left(\|u\|_{2,\infty;t}^2+\|\nabla u\|_{2,2;t}^2\right)\leq\delta c_2(t)E\|u\|_{\#;t}^2+c_3(\delta,t)R\left(\xi,f^0,g^0,h^0,t\right),$$

where $R\left(\xi,f^0,g^0,h^0,t\right) := \|\xi\|_2^2+\left(\left\|f^0\right\|_{\#;t}^*\right)^2+\left\|g^0\right\|_{2,2;t}^2+\left\|h^0\right\|_{2,2;t}^2$, and $c_3(\delta,t)$ is a constant that depends of δ and t, while $c_2(t)$ is independent of δ. Dominating the term $E\|u\|_{\#;t}^2$ by using the estimate (3) and then choosing $\delta=\frac{1}{2c_1^2c_2(t)}$ we obtain the estimate asserted by our theorem.

The existence of the solution in the general case, when only condition **(HD#)** is fulfilled, follows by an approximation procedure. The function f is approximated by $f_n := f - f^0 + f_n^0$, where $f_n^0, n\in\mathbb{N}$, is a sequence of bounded functions such that $E\left(\left\|f^0-f_n^0\right\|_{\#;t}^*\right)^2 \to 0$, as $n\to 0$. The solutions, $u_n, n\in\mathbb{N}$, of the equation (1) corresponding to the functions $f_n, n\in\mathbb{N}$, form a Cauchy sequence in the sense of the following relation

$$\lim_{n,m\to\infty} E\left(\left\|u_n-u_m\right\|_{2,\infty;t}^2+\left\|\nabla\left(u_n-u_m\right)\right\|_{2,2;t}^2\right)=0,$$

which follows from the estimate already proven. The limit $u=\lim_{n\to\infty}u_n$ represents the solution associated with f. It clearly satisfies the estimate asserted by the theorem.

It remains to check the uniqueness assertion. Let u,u' be two solutions in $\mathscr{H}$. Then their difference $\overline{u}=u-u'$ is a solution of a similar equation $\overline{u}=\mathscr{U}\left(0,\overline{f},\overline{g},\overline{h}\right)$, where

$$\overline{f}(t,x,y,z)=f(t,x,y+u'(t,x),z+\nabla u'(t,x))-f(t,x,u'(t,x),\nabla u'(t,x)),$$

$$\overline{g}(t,x,y,z)=g(t,x,y+u'(t,x),z+\nabla u'(t,x))-g(t,x,u'(t,x),\nabla u'(t,x)),$$

$$\overline{h}(t,x,y,z)=h(t,x,y+u'(t,x),z+\nabla u'(t,x))-h(t,x,u'(t,x),\nabla u'(t,x)).$$

Since $\overline{f}^0=\overline{g}^0=\overline{h}^0=0$ and $\bar{u}_0=0$ we may apply the above established estimates to deduce that $\overline{u}=0$.

□

3.2 Estimates of the positive part of the solution

In this section we shall assume that the conditions **(H)**, **(HI2)** and **(HD#)** are fulfilled. By Theorem 3 we know that the equation (1) has a unique solution with null Dirichlet boundary conditions which we denote by $\mathscr{U}\left(\xi,f,g,h\right)$. Next we are going to apply Proposition 2 of the appendix to the solution u. In fact we have in mind to apply it with $\varphi(y)=(y^+)^2$. In the following corollary we make a first step and relax the hypotheses on φ.

Corollary 1. *Let us assume the hypotheses of the preceding Theorem with the same notations. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a function of class $\mathscr{C}^2$ and assume that φ'' is bounded and $\varphi'(0)=0$. Then the following relation holds a.s. for all $t\geq 0$:*

$$\int_{\mathscr{O}}\varphi\left(u_t(x)\right)dx+\int_0^t\mathscr{E}\left(\varphi'\left(u_s\right),u_s\right)ds=\int_{\mathscr{O}}\varphi\left(\xi(x)\right)dx+\int_0^t\left(\varphi'\left(u_s\right),f_s(u_s,\nabla u_s\right)ds$$

$$-\int_0^t \sum_{i=1}^d \left(\partial_i\left(\varphi'\left(u_s\right)\right), g_{i,s}(u_s,\nabla u_s\right) ds + \frac{1}{2}\int_0^t \left(\varphi''\left(u_s\right), \left|h_s(u_s,\nabla u_s)\right|^2\right) ds$$

$$+\sum_{j=1}^{d_1}\int_0^t \left(\varphi'\left(u_s\right), h_{j,s}(u_s,\nabla u_s)\right) dB_s^j.$$

Proof: Thanks to the estimate obtained in Theorem 3 and the inequality (3) we deduce that the process $\varphi'(u)$ belongs to $\mathscr{H}\bigcap L_{\#;t}$ and that $f(u,\nabla u)$ belongs to $L^*_{\#;t}$, for all $t>0$. From this we get the desired result by approximating φ and passing to the limit in Proposition 2. □

We next prove an estimate for the positive part u^+ of the solution $u=\mathscr{U}\left(\xi,f,g,h\right)$. For this we need the following notation:

$$\begin{aligned}
&f^{u,0}=1_{\{u>0\}}f^0,\ g^{u,0}=1_{\{u>0\}}g^0,\ h^{u,0}=1_{\{u>0\}}h^0,\\
&f^u=f-f^0+f^{u,0},\ g^u=g-g^0+g^{u,0},\ h^u=h-h^0+h^{u,0}\\
&f^{u,0+}=1_{\{u>0\}}\left(f^0\vee 0\right),\ \xi^+=\xi\vee 0.
\end{aligned}\tag{9}$$

Theorem 4. *The positive part of the solution satisfies the following estimate*

$$E\left(\left\|u^+\right\|_{2,\infty;t}^2+\left\|\nabla u^+\right\|_{2,2;t}^2\right)\leq k(t)E\left(\left\|\xi^+\right\|_2^2+\left(\left\|f^{u,0+}\right\|_{\#;t}^*\right)^2+\left\|g^{u,0}\right\|_{2,2;t}^2+\left\|h^{u,0}\right\|_{2,2;t}^2\right),$$

with the same constant $k(t)$ as in the Theorem 3.

Proof:

We first show that the relation (7) appearing in the proof of the Theorem 3 still holds with u replaced by u^+ and with f^u, g^u, h^u, ξ^+ in the respective places of f, g, h, ξ.

The idea is to apply Ito's formula to the function ψ defined by $\psi\left(y\right)=\left(y^+\right)^2$, for any $y\in\mathbb{R}$. Since this function is not of the class $\mathscr{C}^2$ we shall make an approximation as follows. Let φ be a $\mathscr{C}^\infty$ function such that $\varphi\left(y\right)=0$ for any $y\in]-\infty,1]$ and $\varphi\left(y\right)=1$ for any $y\in[2,\infty[$. We set $\psi_n\left(y\right)=y^2\varphi\left(ny\right)$, for each $y\in\mathbb{R}$ and all $n\in\mathbb{N}^*$. It is easy to verify that $\left(\psi_n\right)_{n\in\mathbb{N}^*}$ converges uniformly to the function ψ and that

$$\lim_{n\to\infty}\psi_n'\left(y\right)=2y^+,\ \lim_{n\infty}\psi_n''\left(y\right)=2\cdot 1_{\{y>0\}},$$

for any $y\in\mathbb{R}$. Moreover we have the estimates

$$0\leq\psi_n\left(y\right)\leq\psi\left(y\right),\ 0\leq\psi'\left(y\right)\leq Cy,\ \left|\psi_n''\left(y\right)\right|\leq C,$$

for any $y\geq 0$ and all $n\in\mathbb{N}^*$, where C is a constant. Thanks to Corallary 1 we have for all $n\in\mathbb{N}^*$ and each $t\geq 0$, a.s.,

$$\begin{aligned}
&\int_{\mathscr{O}}\psi_n\left(u_t\left(x\right)\right)dx+\int_0^t\mathscr{E}\left(\psi_n{}'\left(u_s\right),u_s\right)ds=\int_{\mathscr{O}}\psi_n\left(\xi\left(x\right)\right)dx+\int_0^t\left(\psi_n{}'\left(u_s\right),f_s\left(u_s,\nabla u_s\right)\right)ds\\
&-\int_0^t\sum_{i=1}^d\left(\psi_n{}''\left(u_s\right)\partial_i u_s, g_{i,s}\left(u_s,\nabla u_s\right)\right)ds+\frac{1}{2}\int_0^t\left(\psi_n{}''\left(u_s\right),\left|h_s\left(u_s,\nabla u_s\right)\right|^2\right)ds\\
&\quad+\sum_{j=1}^{d_1}\int_0^t\left(\psi_n{}'\left(u_s\right),h_{j,s}\left(u_s,\nabla u_s\right)\right)dB_s^j.
\end{aligned}\tag{10}$$

As a consequence of the local property of the Dirichlet form, $\psi_n'(u)$ converges to u^+ in $L^2_{loc}\left(\mathbb{R}_+;H^1_0(\mathcal{O})\right)$. (see Theorem 5.2 in [3] or [2]). Therefore, letting $n\to\infty$, the relation becomes

$$\int_{\mathcal{O}}\left(u_t^+(x)\right)^2dx+2\int_0^t\mathcal{E}\left(u_s^+,u_s^+\right)ds=\int_{\mathcal{O}}\left(\xi^+(x)\right)^2dx+2\int_0^t\left(u_s^+,f_s\left(u_s,\nabla u_s\right)\right)ds$$

$$-2\int_0^t\sum_{i=1}^d\left(1_{\{u_s>0\}}\partial_i u_s,g_{i,s}\left(u_s,\nabla u_s\right)\right)ds+\int_0^t\left(1_{\{u_s>0\}},\left|h_s\left(u_s,\nabla u_s\right)\right|^2\right)ds$$

$$+2\sum_{j=1}^{d_1}\int_0^t\left(u_s^+,h_{j,s}\left(u_s,\nabla u_s\right)\right)dB_s^j.$$

This turns out to be exactly the relation (7) with u^+,f^u,g^u,h^u,ξ^+ in the respective places of u,f,g,h,ξ. Then one may do the same calculation as in the preceding proof with only one minor modification concerning the term which contains f^u, namely one has

$$\int_0^t\left(u_s^+,f_s\left(u_s,\nabla u_s\right)\right)ds=\int_0^t\left(u_s^+,f_s^u\left(u_s^+,\nabla u_s^+\right)\right)ds$$

$$\le\varepsilon\left\|\nabla u^+\right\|_{2,2;t}^2+c_\varepsilon\left\|u^+\right\|_{2,2;t}^2+\delta\left\|u^+\right\|_{\#;t}^2+c_\delta\left(\left\|f^{u,0+}\right\|_{\#;t}^*\right)^2.$$

Thus one has a relation analogous to (8), with $u^+,f^{u,0+},g^{u,0},h^{u,0},\xi^+$ in the respective places of u,f,g,h,ξ and with the corresponding martingale given by

$$\sum_{j=1}^{d_1}\int_0^t\left(u_s^+,h_{j,s}^u\left(u_s^+,\nabla u_s^+\right)\right)dB_s^j.$$

The remaining part of the proof follows by repeating word by word the proof of Theorem 3. $\square$

3.3 The case without lateral boundary conditions

In this subsection we are again in the general framework with only conditions **(H)**, **(HD)** and **(HI)** being fulfilled. The following proposition represents a key technical result which leads to a generalization of the estimates of the positive part of a local solution. Let $u\in\mathcal{U}_{loc}\left(\xi,f,g,h\right)$, denote by u^+ its positive part and let the notation (9) be considered with respect to this new function.

Proposition 1. *Assume that u^+ belongs to $\mathcal{H}$ and assume that the data satisfy the following integrability conditions*

$$E\left\|\xi^+\right\|_2^2<\infty,\ E\left(\left\|f^{u,0}\right\|_{\#;t}^*\right)^2<\infty,\ E\left\|g^{u,0}\right\|_{2,2;t}^2<\infty,\ E\left\|h^{u,0}\right\|_{2,2;t}^2<\infty,$$

for each $t\ge 0$. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a function of class $\mathcal{C}^2$, which admits a bounded second order derivative and such that $\varphi'(0)=0$. Then the following relation holds, a.s., for each $t\ge 0$,

$$\int_{\mathcal{O}}\varphi\left(u_t^+(x)\right)dx+\int_0^t\mathcal{E}\left(\varphi'\left(u_s^+\right),u_s^+\right)ds=\int_{\mathcal{O}}\varphi\left(\xi^+(x)\right)dx+\int_0^t\left(\varphi'\left(u_s^+\right),f_s\left(u_s^+,\nabla u_s^+\right)\right)ds$$

$$-\int_0^t \sum_{i=1}^d \left(\varphi''\left(u_s^+\right)\partial_i u_s^+, g_{i,s}\left(u_s^+, \nabla u_s^+\right)\right) ds + \frac{1}{2}\int_0^t \left(\varphi''\left(u_s^+\right), \left|h_s\left(u_s^+, \nabla u_s^+\right)\right|^2\right) ds$$

$$+\sum_{j=1}^{d_1}\int_0^t \left(\varphi'\left(u_s^+\right), h_{j,s}\left(u_s^+, \nabla u_s^+\right)\right) dB_s^j.$$

Proof:

The version of Ito's formula proved in [5] (Lema 7) works only for solutions with null Dirichlet conditions. In this subsection only the positive part u^+ vanishes at the boundary, but it is not a solution. So we are going to make an approximation of x^+ by some smoother functions $\psi_n(x)$ such that $\psi_n(u)$ satisfy a SPDE and also converges, as n goes to infinity, in a good sense to u^+. The essential point is to prove that the integrability conditions satisfied by our local solution ensure the passage to the limit.

We start with some notation. Let $n \in \mathbb{N}^*$ be fixed and define $\psi = \psi_n$ to be the real function determined by the following conditions

$$\psi(0) = \psi'(0) = 0, \ \psi'' = n 1_{\left[\frac{1}{n}, \frac{2}{n}\right]}.$$

Then clearly ψ is increasing, $\psi(x) = 0$ if $x < \frac{1}{n}$, $\psi(x) = x - \frac{3}{2n}$ for $x > \frac{2}{n}$, and

$$0 \vee \left(x - \frac{3}{2n}\right) \le \psi \le x \vee 0,$$

for any $x \in \mathbb{R}$. The derivative satisfies the inequalities $0 \le \psi' \le 1$ and $\psi'(x) = 1$ for $x \ge \frac{2}{n}$. We set $v_t = \psi\left(u_t\right)$ and prove the following lemma.

Lemma 1. *The process $v = \left(v_t\right)_{t>0}$ satisfies the following SPDE*

$$dv_t = Lv_t dt + \check{f}_t dt + \widehat{f}_t dt + \sum_{i=1}^d \partial_i \check{g}_{i,t} dt + \sum_{j=1}^{d_1} \check{h}_{j,t} dB_t^j$$

with the initial condition $v_0 = \psi(\xi)$ and zero Dirichlet conditions at the boundary of $\mathcal{O}$, where the processes intervening in the equation are defined by

$$\check{f}_t(x) = \psi'\left(u_t(x)\right) f_t\left(x, u_t^+(x), \nabla u_t^+(x)\right),$$

$$\check{g}_t(x) = \psi'\left(u_t(x)\right) g_t\left(x, u_t^+(x), \nabla u_t^+(x)\right),$$

$$\check{h}_t(x) = \psi'\left(u_t(x)\right) h_t\left(x, u_t^+(x), \nabla u_t^+(x)\right),$$

$$\widehat{f}_t(x) = -\psi''\left(u_t(x)\right)\Big(\sum_{i,j=1}^d \left(a^{ij}\left(\partial_i u_t^+\right)\left(\partial_j u_t^+\right)\right)(x) + \sum_{i=1}^d \left(\partial_i u_t^+\right) g_{i,t}\left(u_t^+, \nabla u_t^+\right)(x)$$

$$-\frac{1}{2}\left|h_t\left(u_t^+, \nabla u_t^+\right)\right|^2(x)\Big).$$

The assumptions on u^+ ensure that v belong to $\mathcal{H}$. We also note that the functions $\check{f}, \widehat{f}, \check{g}$ and $\check{h}$ vanish on the set $\left\{u_t \leq \frac{1}{n}\right\}$ and they satisfy the following integrability conditions:

$$E\left\|\check{f}\right\|_{1,1;t}^{2} \leq E\left(\left\|\check{f}\right\|_{\#;t}^{*}\right)^{2},\ E\left\|\check{g}\right\|_{2,2;t}^{2},\ E\left\|\check{h}\right\|_{2,2;t}^{2},\ E\left\|\widehat{f}\right\|_{1,1;t} < \infty,$$

for each $t \geq 0$. The equation from the statement should be considered in the weak L^1 sense of Definition 4 introduced in the Appendix .

Proof of the Lemma :

Let $\phi \in \mathcal{C}_c^{\infty}(\mathcal{O})$ and set $v_t = \phi u_t$, which defines a process in $\mathcal{H}$. A direct calculation involving the definition relation shows that this process satisfies the following equation with $\phi\xi$ as initial data and zero Dirichlet boundary conditions,

$$dv_t = \left(Lv_t + \widetilde{f}_t + \sum_{i=1}^{d} \partial_i \widetilde{g_{i,t}}\right) dt + \sum_{j=1}^{d_1} \widetilde{h_{j,t}} dB_t^j,$$

where

$$\widetilde{f}_t = \phi f_t\left(u_t, \nabla u_t\right) - \sum_{i,j=1}^{d} a^{i,j}\left(\partial_i \phi\right)\left(\partial_j u_t\right) - \sum_{i=1}^{d}\left(\partial_i \phi\right) g_{i,t}\left(u_t, \nabla u_t\right),$$

$$\widetilde{g_{i,t}} = \phi g_{i,t}\left(u_t, \nabla u_t\right) - u_t \sum_{j=1}^{d} a^{i,j} \partial_j \phi,\ i = 1,...d,\ \ \widetilde{h_{j,t}} = \phi h_{j,t}\left(u_t, \nabla u_t\right),\ j = 1,...,d_1.$$

Then we may write Ito's formula in the form

$$\left(\psi\left(v_t\right), \varphi_t\right) + \int_0^t \mathcal{E}\left(\psi'\left(v_s\right)\varphi_s, v_s\right) ds = \left(\psi\left(\phi\xi\right), \varphi_0\right) + \int_0^t \left(\psi\left(v_s\right), \partial_s \varphi_s\right) ds$$

$$+\int_0^t \left(\psi'\left(v_s\right)\varphi_s, \widetilde{f}_s\right) ds - \int_0^t \sum_{i=1}^{d}\left(\partial_i\left(\psi'\left(v_s\right)\varphi_s\right), \widetilde{g_{i,s}}\right) ds + \frac{1}{2}\int_0^t \left(\psi''\left(v_s\right)\varphi_s, \left|\widetilde{h}_s\right|^2\right) ds$$

$$+\sum_{j=1}^{d_1}\int_0^t \left(\psi'\left(v_s\right)\varphi_s, \widetilde{h_{js}}\right) dB_s^j.$$

where $\varphi \in \mathcal{D}$. (The proof of this relation follows from the same arguments as the proof of Lemma 7 in [5].) Now we take ϕ such that $\phi = 1$ in an open subset $\mathcal{O}' \subset \mathcal{O}$ and such that supp$(\varphi_t) \subset \mathcal{O}'$ for each $t \geq 0$, so that this relation becomes

$$\left(v_t, \varphi_t\right) + \int_0^t \mathcal{E}\left(\psi'\left(u_s\right)\varphi_s, u_s\right) ds = \left(\psi\left(\xi\right), \varphi_0\right) + \int_0^t \left(v_s, \partial_s \varphi_s\right) ds$$

$$+\int_0^t \left(\varphi_s, f_s\left(u_s, \nabla u_s\right)\right) ds - \int_0^t \sum_{i=1}^{d}\left(\partial_i\left(\psi'\left(u_s\right)\varphi_s\right), g_{i,s}\left(u_s, \nabla u_s\right)\right) ds$$

$$+\frac{1}{2}\int_0^t \left(\psi''\left(u_s\right)\varphi_s, \left|h_s\left(u_s, \nabla u_s\right)\right|^2\right) ds + \sum_{j=1}^{d_1}\int_0^t \left(\psi'\left(u_s\right)\varphi_s, h_{j,s}\left(u_s, \nabla u_s\right)\right) dB_s^j.$$

By remarking for example that

$$\mathcal{E}\big(\psi'(u_s)\varphi_s, u_s\big) = \sum_{i,j=1}^{d} \int_{\mathcal{O}} a^{ij} \partial_i(\psi'(u_s)\varphi_s)\partial_j(u_s)\, dx$$

$$= \sum_{i,j=1}^{d} \int_{\mathcal{O}} a^{ij} \psi''(u_s)\partial_i(\varphi_s)\partial_j(u_s)\, dx + \mathcal{E}\big(\psi(u_s), \varphi_s\big),$$

an inspection of this relation reveals that this is in fact the definition equality of the equation of the lemma in the sense of the Definition 4 in the Appendix. □

Proof of Proposition 1 :

It is easy to see that the proof can be reduced to the case where the function φ has both first and second derivatives bounded. Then we write the formula of Proposition 2 of the Appendix to the process v and obtain

$$\int_{\mathcal{O}} \varphi\left(v_t\right) + \int_0^t \mathcal{E}\left(\varphi'\left(v_s\right), v_s\right) = \int_{\mathcal{O}} \varphi\left(v_0\right) dx + \int_0^t \left(\varphi'\left(v_s\right), \check{f}_s + \widehat{f}_s\right) ds$$

$$- \int_0^t \sum_{i=1}^{d} \left(\partial_i\left(\varphi'\left(v_s\right)\right), \check{g}_{i,s}\right) ds + \frac{1}{2} \int_0^t \left(\varphi''\left(u_s\right), \left|\check{h}_s\right|^2\right) ds$$

$$+ \sum_{j=1}^{d_1} \int_0^t \left(\varphi'\left(v_s\right), \check{h}_{j,s}\right) dB_s^j.$$

Further we change the notation taking into account the fact that the function ψ depends on the natural number n. So we write ψ_n for ψ, v_t^n for $\psi_n\left(u_t\right) = v_t$ and $\check{f}^n, \widehat{f^n}, \check{g}^n, \check{h}^n$ for the corresponding functions denoted before by $\check{f}, \widehat{f}, \check{g}, \check{h}$. Then we pass to the limit with $n \to \infty$. Obviously one has

$$\left\|v^n - u^+\right\|_{2,2;t} \to 0, \ \left\|\left|\nabla v^n - \nabla u^+\right|\right\|_{2,2;t} \to 0,$$

for each $t \geq 0$, a.s. and $\psi_n'(u) \to 1_{\{u>0\}}$. Then one deduces that

$$\left\|\check{f}^n - f\left(u^+, \nabla u^+\right)\right\|_{\#;t}^{*} \to 0,$$

$$\left\|\left|\check{g}^n - g\left(u^+, \nabla u^+\right)\right|\right\|_{2,2;t} \to 0,$$

$$\left\|\left|\check{h}^n - h\left(u^+, \nabla u^+\right)\right|\right\|_{2,2;t} \to 0,$$

for each $t \geq 0$, a.s.

On the other hand, since the assumptions on φ ensure that $\left|\varphi'(x)\right| \leq K|x|$ for any $x \in \mathbf{R}$, with some constant K, we deduce that $|\varphi'(v^n)\psi_n''(u)| \leq 2K\, \mathbf{1}_{[\frac{1}{n}, \frac{2}{n}]}(u)$. Therefore by the dominated convergence theorem we get that

$$\left\|\varphi'(v^n)\widehat{f^n}\right\|_{1,1;t} \to 0,$$

for each $t \geq 0$, a.s. Finally we deduce that the above relation passes to the limit and implies the relation stated by the theorem. $\square$

The above proposition immediately leads to the following generalization of the estimates of the positive part obtained in the previous section, with the same proof.

Corollary 2. *Under the hypotheses of the above Proposition with same notations, one has the following estimates*

$$E\left(\left\|u^{+}\right\|_{2,\infty;t}^{2}+\left\|\nabla u^{+}\right\|_{2,2;t}^{2}\right) \leq k(t) E\left(\left\|\xi^{+}\right\|_{2}^{2}+\left(\left\|f^{u,0+}\right\|_{\#;t}^{*}\right)^{2}+\left\|g^{u,0}\right\|_{2,2;t}^{2}+\left\|h^{u,0}\right\|_{2,2;t}^{2}\right).$$

4 Main results : comparison theorem and maximum principle

In this section we are still in the general framework and we consider $u \in \mathscr{U}_{loc}(\xi, f, g, h)$ a local solution of our SPDE. We first give the following comparison theorem.

Theorem 5. *Assume that f^1, f^2 are two functions similar to f which satisfy the Lipschitz condition **(H)-(i)** and such that both triples $\left(f^1, g, h\right)$ and $\left(f^2, g, h\right)$ satisfy **(HD)**. Assume that ξ^1, ξ^2 are random variables similar to ξ and that both satisfy **(HI)**. Let $u^i \in \mathscr{U}_{loc}\left(\xi^i, f^i, g, h\right), i=1,2$ and suppose that the process $\left(u^1-u^2\right)^{+}$ belongs to $\mathscr{H}$ and that one has*

$$E\left(\left\|f^1\left(.,.,u^2,\nabla u^2\right)-f^2\left(.,.,u^2,\nabla u^2\right)\right\|_{\#;t}^{*}\right)^{2}<\infty, \quad \text{for all} \quad t \geq 0.$$

If $\xi^1 \leq \xi^2$ a.s. and $f^1\left(t,\omega,u^2,\nabla u^2\right) \leq f^2\left(t,\omega,u^2,\nabla u^2\right)$, $dt \otimes dx \otimes dP$-a.e., then one has $u^1(t,x) \leq u^2(t,x)$, $dt \otimes dx \otimes dP$-a.e.

Proof:

The difference $v = u^1 - u^2$ belongs to $\mathscr{U}_{loc}\left(\overline{\xi}, \overline{f}, \overline{g}, \overline{h}\right)$, where $\overline{\xi} = \xi^1 - \xi^2$,

$$\overline{f}(t,\omega,x,y,z) = f^1\left(t,\omega,x,y+u_t^2(x),z+\nabla u_t^2(x)\right) - f^2\left(t,\omega,x,u_t^2(x),\nabla u_t^2(x)\right),$$

$$\overline{g}(t,\omega,x,y,z) = g\left(t,\omega,x,y+u_t^2(x),z+\nabla u_t^2(x)\right) - g\left(t,\omega,x,u_t^2(x),\nabla u_t^2(x)\right),$$

$$\overline{h}(t,\omega,x,y,z) = h\left(t,\omega,x,y+u_t^2(x),z+\nabla u_t^2(x)\right) - h\left(t,\omega,x,u_t^2(x),\nabla u_t^2(x)\right).$$

The result follows from the preceding corollary, since $\overline{\xi} \leq 0$ and $\overline{f}^0 \leq 0$ and $\overline{g}^0 = \overline{h}^0 = 0$.
$\square$

Before presenting the next application we are going to recall some notation used in [5]. For $d \geq 3$ and some parameter $\theta \in [0,1[$ we used the notation

$$\Gamma_\theta^* = \left\{(p,q) \in [1,\infty]^2 \,/\, \frac{d}{2p} + \frac{1}{q} = 1-\theta\right\},$$

$$L_\theta^* = \sum_{(p,q)\in\Gamma_\theta^*} L^{p,q}([0,t]\times\mathscr{O})$$

$$\|u\|_{\theta;t}^{*} := \inf\Bigg\{ \sum_{i=1}^{n} \left\| u_i \right\|_{p_i,q_i;\,t} \,/\, u = \sum_{i=1}^{n} u_i, u_i \in L^{p_i,q_i}\left([0,t]\times\mathcal{O}\right),$$

$$\left(p_i,q_i\right) \in \Gamma_\theta^*,\ i = 1,...n;\ n \in \mathbf{N}^* \Bigg\}.$$

Remark 4. *In the paper [5] we have omitted the cases $d = 1, 2$. In fact, one can cover these cases by setting*

$$\Gamma_\theta = \left\{ \left(p,q\right) \in [1,\infty]^2 \,/\, \frac{2^*}{2^*-2}\frac{1}{p} + \frac{1}{q} = \frac{2^*}{2^*-2} + \theta \right\},$$

$$\Gamma_\theta^* = \left\{ \left(p,q\right) \in [1,\infty]^2 \,/\, \frac{2^*}{2^*-2}\frac{1}{p} + \frac{1}{q} = 1-\theta \right\}$$

and by using similar calculations with the convention $\frac{2^*}{2^*-2} = 1$ if $d = 1$.

We want to express these quantities in the new notation introduced in the subsection 2.1 and to compare the norms $\|u\|_{\theta;t}^*$ and $\|u\|_{\#;t}^*$. So, we first remark that $\Gamma_\theta^* = I\left(\infty, \frac{1}{1-\theta}, \frac{d}{2(1-\theta)}, \infty\right)$ and that the norm $\|u\|_{\theta;t}^*$ coincides with $\|u\|^{\Gamma_\theta^*;t} = \|u\|^{I\left(\infty,\frac{1}{1-\theta},\frac{d}{2(1-\theta)},\infty\right);t}$. On the other hand, we recall that the norm $\|u\|_{\#;t}^*$ is associated to the set $I\left(2,1,\frac{2^*}{2^*-1},2\right)$, i.e. $\|u\|_{\#;t}^*$ coincides with $\|u\|^{I\left(2,1,\frac{2^*}{2^*-1},2\right);t}$. Then we may prove the following result.

Lemma 2. *One has $\|u\|_{\#;t}^* \le c\,\|u\|_{\theta;t}^*$, for each $u \in L_\theta^*$, with some constant $c > 0$.*

Proof:
The points defining the sets $I\left(\infty, \frac{1}{1-\theta}, \frac{d}{2(1-\theta)}, \infty\right)$ and $I\left(2,1,\frac{2^*}{2^*-1},2\right)$ obviously satisfy the inequalities

$$\infty \ge 2, \frac{1}{1-\theta} \ge 1, \frac{d}{2(1-\theta)} \ge \frac{2^*}{2^*-1} = \frac{2d}{d+2}, \infty \ge 2,$$

and hence for each pair $\left(p,q\right) \in \Gamma_\theta^*$, there exists a pair $\left(\widehat{p},\widehat{q}\right) \in I\left(2,1,\frac{2^*}{2^*-1},2\right)$ such that $p \le \widehat{p}$ and $q \le \widehat{q}$. This implies the inclusion

$$L_\theta^* = \sum_{\left(p,q\right)\in\Gamma_\theta^*} L^{p,q}\left([0,t]\times\mathcal{O}\right) \subset L^{I\left(2,1,\frac{2^*}{2^*-1},2\right);t} = \sum_{\left(p,q\right)\in I\left(2,1,\frac{2^*}{2^*-1},2\right)} L^{p,q}\left([0,t]\times\mathcal{O}\right),$$

and the asserted inequality. $\square$

We now consider the following assumption:

Assumption (HDθp)

$$E\left(\left(\left\|f^0\right\|_{\theta;t}^*\right)^p + \left(\left\|\left|g^0\right|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\left\|\left|h^0\right|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}}\right) < \infty,$$

for each $t \ge 0$, where $\theta \in [0,1[$ and $p \ge 2$ are fixed numbers. By the preceding Lemma and since in general one has $\|u\|_{1,1;t} \le c\,\|u\|_{\theta;t}^*$, it follows that this property is stronger than **(HD#)**.

As now we want to establish a maximum principle, we have to assume that ξ is bounded with respect to the space variable, so we introduce the following:

Assumption (HI∞p)

$$E\left\|\xi\right\|_{\infty}^{p}<\infty,$$

where $p\geq 0$ is a fixed number.

Then we have the following result which generalizes the maximum principle to the stochastic framework.

Theorem 6. *Assume* **(H)**, ***(HDθp)***, ***(HI∞p)*** *for some $\theta\in[0,1[$, $p\geq 2$, and that the constants of the Lipschitz conditions satisfy $\alpha+\frac{\beta^2}{2}+72\beta^2<\lambda$. Let $u\in\mathcal{U}_{loc}\left(\xi,f,g,h\right)$ be such that $u^+\in\mathcal{H}$. Then one has*

$$E\left\|u^+\right\|_{\infty,\infty;t}^{p}\leq k(t)\,E\left(\left\|\xi^+\right\|_{\infty}^{p}+\left(\left\|f^{0,+}\right\|_{\theta;t}^{*}\right)^{p}+\left(\left\|\left|g^{0}\right|^{2}\right\|_{\theta;t}^{*}\right)^{\frac{p}{2}}+\left(\left\|\left|h^{0}\right|^{2}\right\|_{\theta;t}^{*}\right)^{\frac{p}{2}}\right),$$

where $k(t)$ is constant that depends of the structure constants and $t\geq 0$.

Proof:

Set $v=\mathcal{U}\left(\xi^+,\widehat{f},g,h\right)$ the solution with zero Dirichlet boundary conditions, where the function $\widehat{f}$ is defined by $\widehat{f}=f+f^{0,-}$, with $f^{0,-}=0\vee\left(-f^{0}\right)$. The assumption on the Lipschitz constants ensure the applicability of the theorem 11 of [5], which gives the estimate

$$E\left\|v\right\|_{\infty,\infty;t}^{p}\leq k(t)\,E\left(\left\|\xi^+\right\|_{\infty}^{p}+\left(\left\|f^{0,+}\right\|_{\theta;t}^{*}\right)^{p}+\left(\left\|\left|g^{0}\right|^{2}\right\|_{\theta;t}^{*}\right)^{\frac{p}{2}}+\left(\left\|\left|h^{0}\right|^{2}\right\|_{\theta;t}^{*}\right)^{\frac{p}{2}}\right),$$

because $\widehat{f}^{0}=f^{0,+}$. Then $(u-v)^+\in\mathcal{H}$ and we observe that all the conditions of the preceding theorem are satisfied so that we may apply it and deduce that $u\leq v$. This implies $u^+\leq v^+$ and the above estimate of v leads to the asserted estimate. □

Remark 5. *As noted in Subsection 2.3 the condition $u^+\in\mathcal{H}$ means that $u\leq 0$ on the lateral boundary $[0,\infty[\times\partial\mathcal{O}$. Similarly, concerning the next theorem, we observe that the condition $(u-M)^+\in\mathcal{H}$ means that $u\leq M$ on the lateral boundary $[0,\infty[\times\partial\mathcal{O}$.*

Let us generalize the previous result by considering a real Itô process of the form

$$M_t=m+\int_0^t b_s ds+\sum_{j=1}^{d_1}\int_0^t\sigma_{j,s}dB_s^j,$$

where m is a real random variable and $b=\left(b_t\right)_{t\geq 0}$, $\sigma=\left(\sigma_{1,t},...,\sigma_{d,t}\right)_{t\geq 0}$ are adapted processes.

Theorem 7. *Assume* **(H)**, ***(HDθp)***, ***(HI∞p)*** *for some $\theta\in[0,1[$, $p\geq 2$, and that the constants of the Lipschitz conditions satisfy $\alpha+\frac{\beta^2}{2}+72\beta^2<\lambda$. Assume also that m and the processes b and σ satisfy the following integrability conditions*

$$E\left|m\right|^{p}<\infty,\ E\left(\int_0^t\left|b_s\right|^{\frac{1}{1-\theta}}ds\right)^{p(1-\theta)}<\infty,\ E\left(\int_0^t\left|\sigma_s\right|^{\frac{2}{1-\theta}}ds\right)^{\frac{p(1-\theta)}{2}}<\infty,$$

for each $t \geq 0$. Let $u \in \mathcal{U}_{loc}\left(\xi, f, g, h\right)$ be such that $(u-M)^+$ belongs to $\mathcal{H}$. Then one has

$$E\left\|(u-M)^+\right\|_{\infty,\infty;t}^p \leq k(t)E\Big[\left\|(\xi-m)^+\right\|_\infty^p + \left(\left\|\left(f(\cdot,\cdot,M,0)-b\right)^+\right\|_{\theta;t}^*\right)^p + \\ \left(\left\|\left|g(\cdot,\cdot,M,0)\right|^2\right\|_{\theta;T}^*\right)^{\frac{p}{2}} + \left(\left\||h(\cdot,\cdot,M,0)-\sigma|^2\right\|_{\theta;T}^*\right)^{\frac{p}{2}}\Big]$$

where $k(t)$ is the constant from the preceding corollary. The right hand side of this estimate is dominated by the following quantity which is expressed directly in terms of the characteristics of the process M,

$$k(t)E\Big[\left\|(\xi-m)^+\right\|_\infty^p + |m|^p + \left(\left\|f^{0,+}\right\|_{\theta;t}^*\right)^p + \left(\left\|\left|g^0\right|^2\right\|_{\theta;T}^*\right)^{\frac{p}{2}} + \left(\left\|\left|h^0\right|^2\right\|_{\theta;T}^*\right)^{\frac{p}{2}} \\ + \left(\int_0^t \left|b_s\right|^{\frac{1}{1-\theta}} ds\right)^{p(1-\theta)} + \left(\int_0^t \left|\sigma_s\right|^{\frac{2}{1-\theta}} ds\right)^{\frac{p(1-\theta)}{2}}\Big].$$

Proof:

One immediately observes that $u-M$ belongs to $\mathcal{U}_{loc}\left(\xi-m, \overline{f}, \overline{g}, \overline{h}\right)$, where

$$\overline{f}\left(t,\omega,x,y,z\right) = f\left(t,\omega,x,y+M_t(\omega),z\right) - b_t(\omega),$$

$$\overline{g}\left(t,\omega,x,y,z\right) = g\left(t,\omega,x,y+M_t(\omega),z\right),$$

$$\overline{h}\left(t,\omega,x,y,z\right) = h\left(t,\omega,x,y+M_t(\omega),z\right) - \sigma_t(\omega).$$

In order to apply the preceding theorem we only have to estimate the zero terms. So we see that $\overline{f}_t^0 = f_t\left(M_t,0\right) - b_t, \overline{g}_t^0 = g_t\left(M_t,0\right), \overline{h}_t^0 = h_t\left(M_t,0\right) - \sigma_t$, and hence we get the first estimate from the statement. Further we may write

$$\overline{f}_t^{0,+} \leq C\left|M_t\right| + f_t^{0,+} + \left|b_t\right|,$$

$$\left|\overline{g}_t^0\right|^2 \leq 2C^2\left|M_t\right|^2 + 2\left|g_t^0\right|^2,$$

$$\left|\overline{h}_t^0\right|^2 \leq 3C^2\left|M_t\right|^2 + 3\left|h_t^0\right|^2 + 3\left|\sigma_t\right|^2.$$

Then we have the estimates

$$\left\|\overline{f}^{0,+}\right\|_{\theta;t}^* \leq \left\|f^{0,+}\right\|_{\theta;t}^* + C\sup_{s\leq t}\left|M_t\right| + \left(\int_0^t \left|b_s\right|^{\frac{1}{1-\theta}} ds\right)^{1-\theta},$$

$$\left\|\left|\overline{g}^0\right|^2\right\|_{\theta;t}^* \leq 2\left\|\left|g^0\right|^2\right\|_{\theta;t}^* + 2C^2\sup_{s\leq t}\left|M_t\right|^2,$$

$$\left\|\overline{h}^0\right\|_{\theta;t}^* \leq 3\left\|\left|h^0\right|^2\right\|_{\theta;t}^* + 3C^2\sup_{s\leq t}\left|M_t\right|^2 + 3\left(\int_0^t \left|\sigma_s\right|^{\frac{2}{1-\theta}} ds\right)^{1-\theta}.$$

On the other hand, one has

$$\sup_{s\leq t}\left|M_t\right| \leq |m| + \int_0^t \left|b_s\right| ds + \sup_{s\leq t}\left|N_t\right|,$$

where we have denoted by N_t the martingale $\sum_{j=1}^{d_1}\int_0^t \sigma_{j,s}dB_s^j$. The inequality of Burkholder -Davis -Gundy implies

$$E\sup_{s\leq t}\left|M_t\right|^p \leq cE\left[|m|^p + \left(\int_0^t \left|b_s\right| ds\right)^p + \left(\int_0^t \left|\sigma_s\right|^2 ds\right)^{\frac{p}{2}}\right],$$

and this allows us to conclude the proof. □

5 Burgers type equations

All along this section, we relax the hypothesis on the predictable random function g which is assumed to be locally Lipschitz with polynomial growth with respect to y. We shall generalize some results from Gyöngy and Rovira [6]. Indeed, we shall assume that the assumption **(H)** holds, but instead of the condition (iii) we assume the following:

Assumption (G): there exists two constants $C > 0$ and $r \geq 1$, and two functions $\bar{g}$, $\hat{g}$ such that

- **(i)** the function g can be expressed by : $g(t,\omega,x,y,z) = \bar{g}(t,\omega,x,y,z) + \hat{g}(t,\omega,y)$, $\forall (t,\omega,x,y,z) \in \mathbb{R}_+ \times \Omega \times \mathscr{O} \times \mathbb{R} \times \mathbb{R}^d$.
- **(ii)** $\left(\sum_{i=1}^d |g_i(t,\omega,x,y,z) - g_i(t,\omega,x,y',z')|^2\right)^{\frac{1}{2}} \leq C\left(1+|y|^r+|y'|^r\right)|y-y'| + \alpha|z-z'|$,
- **(iii)** $\left(\sum_{i=1}^d |\bar{g}_i(t,\omega,x,y,z) - \bar{g}_i^0(t,\omega,x)|^2\right)^{\frac{1}{2}} \leq C|y| + \alpha|z|$,
  where α is the constant which appears in assumption **(H)**.

We first consider equation (1) with null Dirichlet boundary condition

$$u_t(x) = 0, \quad \text{for all } t > 0,\ x \in \partial\mathscr{O}.$$

and the initial condition $u(0,.) = \xi(.)$

The effect of the polynomial growth contained in the term $\hat{g}$ will be canceled by the following simple lemma

Lemma 3. *Let $u \in H_0^1(\mathscr{O})$, $\psi \in \mathscr{C}^1(\mathbb{R})$ with bounded derivative and F a real-valued bounded measurable function. Then*

$$\int_{\mathscr{O}} \partial_i\big(\psi(u(x))\big)F(u)(x)\,dx = 0, \quad \forall i = 1,\cdots,d.$$

Proof: We define
$$G(y)=\int_0^y \psi'(z)F(z)\,dz.\ \ \forall y\in\mathbb{R},$$
so that $\partial_i G(u)=G'(u)\partial_i u=\partial_i\big(\psi(u)\big)F(u)$. Then, we deduce that the integral from the statement becomes $\int_{\mathcal{O}}\partial_i\big(G(u(x))\big)\,dx$, which is null because $u\in H_0^1(\mathcal{O})$. $\square$

The natural idea is to approximate the coefficient g by a sequence of globally Lipschitz functions. To this end we define, for all $n\geq 1$, the coefficient g^n by:
$$\forall(t,w,x,y,z)\in\mathbb{R}_+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d,\ g^n(t,w,x,y,z)=g(t,w,x,((-n)\vee y)\wedge n,z).$$
In the same way, we define $\bar{g}^n$, $\hat{g}^n$, so that $g^n=\bar{g}^n+\hat{g}^n$.
One can easily check that for all $n\in\mathbb{N}$, $g^{n,0}=g^0$ and that the following relations hold:
$$\begin{aligned}&\Big(\sum_{i=1}^d|g_i^n(t,\omega,x,y,z)-g_i^n(t,\omega,x,y',z')|^2\Big)^{\frac12}\leq C\big(1+2n^r\big)\,|y-y'|+\alpha\,|z-z'|\,,\\&\Big(\sum_{i=1}^d|\bar{g}_i^n(t,\omega,x,y,z)-\bar{g}_i^0(t,\omega,x)|^2\Big)^{\frac12}\leq C\big(1+|y|\big)+\alpha\,|z-z'|\,,\end{aligned}\tag{11}$$
with the same constants C, α, r as in hypothesis (**G**), so we are able to apply Theorem 11 of [5] (or Theorem 3 above) and get the solutions $u_n=\mathcal{U}(\xi,f,g^n,h)$ for all $n=1,2,\ldots$. We know that for t fixed, $E\,\|u^n\|^p_{2,\infty;t}$ is finite. The key point is that this quantity does not depend on n. This is the aim of the following

Lemma 4. *Assume that conditions **(H)(i)-(ii)**, **(G)**, **(HDθp)** and **(HI∞p)** are fulfilled for some $\theta\in[0,1[$ and $p\geq 2$, and that the constants of the Lipschitz conditions satisfy $\alpha+\frac{\beta^2}{2}+72\beta^2<\lambda$. Then, for fixed $t>0$,*
$$E\,\|u^n\|^p_{\infty,\infty;t}\leq k(t)\,E\left(\|\xi\|^p_\infty+\left\|f^0\right\|^{*p}_{\theta,t}+\left\||\bar{g}^0|^2\right\|^{*p/2}_{\theta;t}+\left\||h^0|^2\right\|^{*p/2}_{\theta;t}\right),$$
where $k(t)$ only depends on C, α and β.

Proof: Thanks to the Itô's formula (see Lemma 7 in [5]) , we have for all $l\geq 2$, $n\in\mathbb{N}$ and $t>0$:
$$\begin{aligned}\int_{\mathcal{O}}|u_t^n(x)|^l\,dx+\int_0^t\mathcal{E}\left(l\,(u_s^n)^{l-1}sgn(u_s^n),\,u_s^n\right)ds=\int_{\mathcal{O}}|\xi(x)|^l\,dx\\+\ l\int_0^t\int_{\mathcal{O}}sgn(u_s^n)|u_s^n(x)|^{l-1}f(s,x,u_s^n,\nabla u_s^n)\,dx\,ds\\-\ \ l(l-1)\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}|u_s^n(x)|^{l-2}\partial_i u_s^n(x)\,g_i(s,x,u_s^n,\nabla u_s^n)\,dx\,ds\\+\,l\sum_{j=1}^{d_1}\int_0^t\int_{\mathcal{O}}sgn(u_s^n)|u_t^n(x)|^{l-1}h_j(s,x,u_s^n,\nabla u_s^n)\,dx\,dB_s^j\\+\frac{l(l-1)}{2}\sum_{j=1}^{d_1}\int_0^t\int_{\mathcal{O}}|u_t^n(x)|^{l-2}h_j^2(s,x,u_s^n,\nabla u_s^n)\,dx\,ds\,,\end{aligned}$$

P-almost surely.
The midle term in the right hand side can be written as

$$\sum_{i=1}^{d}\int_0^t\int_{\mathcal{O}}|u_s^n(x)|^{l-2}\partial_i u_s^n(x)\, g_i^n(s,x,u_s^n,\nabla u_s^n)\,dx\,ds$$
$$=\sum_{i=1}^{d}\int_0^t\int_{\mathcal{O}}|u_s^n(x)|^{l-2}\partial_i u_s^n(x)\, \bar{g}_i^n(s,x,u_s^n,\nabla u_s^n)\,dx\,ds$$

because by Lemma 3 we have

$$\int_0^t\int_{\mathcal{O}}|u_s^n(x)|^{l-2}\partial_i u_s^n(x)\,\hat{g}_i^n(s,u_s^n)\,dx\,ds=0.$$

Now, as

$$|\bar{g}(t,\omega,x,u_s^n,\nabla u_s^n)|\leq|\bar{g}^0(t,\omega,x)|+C|u_s^n|+\alpha|\nabla u_s^n|,$$

and as f and h satisfy similar inequalities with constants which do not depend on n, we can follow exactly the same arguments as the ones in [5] (Lemmas 12, 14, 16 and 17) replacing g by $\bar{g}$ and this yields the result.
Let us remark that in [5], we first assume that initial conditions are bounded and then pass to the limit. Here, it is not necessary since *a priori* we know that $E\,\|u^n\|_{\infty,\infty;t}^p$ is finite. $\square$

We need to introduce the following

Definition 2. *We denote by $\mathcal{H}_b$ the subset of processes u in $\mathcal{H}$ such that for all $t>0$*

$$E\parallel u\parallel_{\infty,\infty;t}^2<+\infty.$$

We are now able to enounce the following existence result which gives also uniform estimates for the solution :

Theorem 8. *Assume that conditions **(H)(i)-(ii)**, **(G)**, **(HDθp)** and **(HI∞p)** are fulfilled for some $\theta\in[0,1[$ and $p\geq 2$, and that the constants of the Lipschitz conditions satisfy $\alpha+\frac{\beta^2}{2}+72\beta^2<\lambda$. Then the equation (1) admits a unique solution $u\in\mathcal{H}_b$. Moreover*

$$E\,\|u\|_{\infty,\infty;t}^p\leq k(t)\,E\left(\|\xi\|_\infty^p+\left\|f^0\right\|_{\theta;t}^{*p}+\left\||\bar{g}^0|^2\right\|_{\theta;t}^{*p/2}+\left\||h^0|^2\right\|_{\theta;t}^{*p/2}\right),$$

where k is a function which only depends on structure constants.

Proof: We keep the notations of previous Lemma and so consider the sequence $(u_n)_{n\in\mathbb{N}}$. For all $n\in\mathbb{N}$, we introduce the following stopping time:

$$\tau_n=\inf\{t\geq 0,\ \|u^n\|_{\infty,\infty;t}>n\}.$$

Now, let $n\in\mathbb{N}$ be fixed, we set $\tau=\tau_n\wedge\tau_{n+1}$. Define now for $i=n,n+1$

$$v_t^i=\begin{cases}u_t^i & \text{if } t<\tau\\ P_{t-\tau}u_\tau^i & \text{elsewhere,}\end{cases}$$

where $(P_t)_{t\geq 0}$ is the semigroup associated to A with zero Dirichlet condition.
One can verify that $v^i = \mathcal{U}(\xi, \mathbf{1}_{\{t\leq\tau\}}\cdot f, \mathbf{1}_{\{t\leq\tau\}}\cdot g^{n+1}, \mathbf{1}_{\{t\leq\tau\}}\cdot h)$. It is clear that the coefficients of the equation satisfied by v^i fulfill hypotheses **(H)** and that moreover $\mathbf{1}_{\{t\leq\tau\}}\cdot g^{n+1}$ is globally Lipschitz continuous. Hence, by Theorem 3 (or Theorem 11 of [5]) this equation admits a unique solution. So, we conclude that $v^n = v^{n+1}$ which implies that $\tau_{n+1}\geq\tau_n$ and $u^n = u^{n+1}$ on $[0,\tau_n]$. Thanks to previous Lemma, we have

$$\lim_{n\to+\infty}\tau_n = +\infty,\ \ P-a.e.$$

We define $u_t = \lim_{n\to\infty} u_t^n$. It is easy to verify that u is a weak solution of (1) and that it satisfies the announced estimate.
Let us prove that u is unique. Let v be another solution in $\mathcal{H}_b$. By the same reasoning as the one we have just made, one can prove that $u = v$ on each $[0,\nu_n]$ where for all $n\in\mathbb{N}$,

$$\nu_n = \inf\{t\geq 0,\ \|v\|_{\infty,\infty;t} > n\}.$$

As $v\in\mathcal{H}_b$, $\lim_{n\to+\infty}\nu_n = +\infty$ a.e. and this leads to the conclusion. □

Remark 6. *The function k which appears in the above theorem only depends on structure constants but not on r.*

In the setting of this section, with (**H**) (iii) replaced by (**G**), one may define local solutions without lateral boundary conditions by restricting the attention to processes $u\in\mathcal{H}_{loc}$ such that $\|u\|_{\infty,\infty;t} < \infty$ a.s. for any $t\geq 0$ and such the relation 6 of the definition is satisfied. Then Proposition 1, Corollary 2 and Theorems 5, 6, 7 of the preceding section still hold for such bounded solutions. The proof follows from the stopping procedure used in the proof of Theorem 8.

6 Appendix

As we have relaxed the hypothesis on f^0 which does not necessarily satisfy an L^2-condition but only L^1, we need to introduce another notion of solution with null Dirichlet conditions at the boundary of $\mathcal{O}$, which is a solution in the L^1 sense.

6.1 Weak L^1 -solution

Since this notion intervenes only as a technical tool, we develop only the stricly necessary aspects related to it. It is defined by using the duality of L^1 with L^∞. To this end we introduce a few notations concerning the extension of our operator to $L^1(\mathcal{O})$.
Let $(P_t)_{t\geq 0}$ be the semi-group (in $L^2(\mathcal{O})$) whose generator is $L = -A$. It is well-known that for all $t\geq 0$, P_t can be extended to a sub-Markovian contraction of $L^1(\mathcal{O})$ that we denote by $P_t^{(1)}$. Following [2], Proposition 2.4.2, we know that $(P_t^{(1)})_{t\geq 0}$ is a strongly continuous contraction semi-group in $L^1(\mathcal{O})$, whose generator $L^{(1)}$ is the smallest closed extension on $L^1(\mathcal{O})$ of $(L,\mathcal{D}(A))$. We set $A^{(1)} = -L^{(1)}$ and denote by $\mathcal{D}(A^{(1)})$ its domain.
Let us also put the following notation:

$$\mathcal{D}_\infty(A) = \{u\in\mathcal{D}(A)\cap L^\infty(\mathcal{O})\,/Au\in L^\infty(\mathcal{O})\},$$

$$[u]_\infty = \|u\|_\infty + \|Au\|_\infty \,,$$

for each $u \in \mathscr{D}_\infty(A)$. It is not difficult to see that the space $\mathscr{D}_\infty(A)$ endowed with the norm $[\cdot]_\infty$ is a Banach space and that it is dense both in $\mathscr{D}(A)$ and $\mathscr{D}\left(A^{(1)}\right)$. Then a suitable space of test functions is defined by

$$\mathscr{D}_0 = \mathscr{C}_c^\infty([0,\infty[) \otimes \mathscr{D}_\infty(A).$$

We start presenting some facts in the deterministic setting. Analogous to Lemma 2 of [4] one has the following result.

Lemma 5. *If $u : \mathbb{R}_+ \to L^1(\mathscr{O})$ is such that*

$$\int_0^t \int_{\mathscr{O}} \left|u_s(x)\right| dx ds < \infty$$

and

$$\int_0^t \int_{\mathscr{O}} \left(u_s, \partial_t \varphi - A\varphi_s\right) ds = 0,$$

for any $\varphi \in \mathscr{D}_0$, then $u = 0$, as an element of $L^1_{loc}\left(\mathbb{R}_+; L^1(\mathscr{O})\right)$.

This last lemma allows us to extend the notion of solution of the equation

$$\partial_t u - Lu = w \qquad (*)$$

to the L^1 framework as follows.

Definition 3. *Let $w \in L^1_{loc}\left(\mathbb{R}_+; L^1(\mathscr{O})\right)$ and $\xi \in L^1(\mathscr{O})$ be given. Then we say that $u \in L^1_{loc}\left(\mathbb{R}_+; L^1(\mathscr{O})\right)$ is a weak L^1 -solution of the equation $(*)$ with the initial condition $u_0 = \xi$ and zero Dirichlet conditions at the boundary of $\mathscr{O}$ provided that one has*

$$\int_0^\infty \left[\left(u_t, \partial_t \varphi - A\varphi_t\right) + \left(w_t, \varphi_t\right)\right] dt + \left(\xi, \varphi_0\right) = 0,$$

for any $\varphi \in \mathscr{D}_0$.

The solution is expressed in terms of the semigroup $\left(P_t^{(1)}\right)_{t\geq 0}$ as stated in the next lemma with same proof as the one of Lemma 3 in [4].

Lemma 6. *If $w \in L^1_{loc}\left(\mathbb{R}_+; L^1(\mathscr{O})\right)$ and $\xi \in L^1(\mathscr{O})$, then there exists a unique weak L^1 -solution of $(*)$ with initial condition $u_0 = \xi$ and zero Dirichlet boundary conditions and it is expressed by*

$$u_t = \int_0^t P^{(1)}_{t-s} w_s ds + P^{(1)}_t \xi,$$

for any $t \geq 0$.

We now turn out to the stochastic case.
The space of all predictable processes with trajectories in $L^i_{loc}\left(\mathbb{R}_+; L^i(\mathscr{O})\right)$, a.s., and such that

$$E\,\|u\|^i_{i,i;t} < \infty,$$

for each $t \geq 0$, will be denoted by $\mathscr{P}\left(L^i\right)$, for $i = 1, 2$.

Definition 4. *Now let* $w \in \mathscr{P}\left(L^1\right), w'^i, w''^j \in \mathscr{P}\left(L^2\right), i = 1, ..., d, j = 1, ..., d_1$ *and* $\xi \in L^1\left(\Omega, \mathscr{F}_0, P; L^1(\mathscr{O})\right)$ *be given and set* , $w' = \left(w'^1, ..., w'^d\right), w'' = \left(w''^1, ..., w''^{d_1}\right)$. *Then we say that a process* $u \in \mathscr{P}\left(L^1\right)$ *represents a weak* L^1 *-solution of the equation*

$$du_t = Lu_t dt + w_t dt + \sum_{i=1}^{d} \partial_i w_t'^i dt + \sum_{j=1}^{d_1} w_t''^j dB_t^j \qquad (**)$$

with initial condition $u_0 = \xi$ *and zero Dirichlet conditions at the boundary of* $\mathscr{O}$ *provided that the following relation holds, a.s.,*

$$\int_0^\infty \left[(u_s, \partial_s \varphi - A\varphi_s) + (w_s, \varphi_s) - \sum_{i=1}^{d} \left(w_s'^i, \partial_i \varphi\right) \right] ds$$

$$+ \sum_{j=1}^{d_1} \int_0^\infty \left(w_s''^j, \varphi_s\right) dB_s^j + (\xi, \varphi_0) = 0,$$

for each test function $\varphi \in \mathscr{D}_0$.

It is easy to see that, in the case where, besides the preceding conditions, the trajectories of the solution u belong a.s. to $L^2_{loc}\left(\mathbb{R}_+; H_0^1(\mathscr{O})\right)$, the above relation is equivalent to

$$\int_0^\infty \left[(u_s, \partial_s \varphi) - \mathscr{E}(u_s, \varphi_s) + (w_s, \varphi_s) - \sum_{i=1}^{d} \left(w_s'^i, \partial_i \varphi\right) \right] ds$$

$$+ \sum_{j=1}^{d_1} \int_0^\infty \left(w_s''^j, \varphi_s\right) dB_s^j + (\xi, \varphi_0) = 0.$$

So, on account of the Proposition 7 of [4] and of the preceding lemma, if $w \in \mathscr{P}\left(L^2\right)$ and $\xi \in L^2\left(\Omega, \mathscr{F}_0, P; L^2(\mathscr{O})\right)$ the notion of a weak L^1-solution of (**) just introduced coincides with the notion of a weak solution previously defined, with $f = f^0 = w, g = g^0 = w'$ and $h = h^0 = w''$. Moreover, we have the following general explicit expression for the solution, similar to Proposition 7 of [4].

Lemma 7. *If* $w \in \mathscr{P}\left(L^1\right), w' = \left(w'^1, ..., w'^d\right), w'' = \left(w''^1, ..., w''^{d_1}\right), w'^k, w''^l \in \mathscr{P}\left(L^2\right), k = 1, ..., d, l = 1, ..., d_1$ *and* $\xi \in L^1\left(\Omega, \mathscr{F}_0, P; L^1(\mathscr{O})\right)$, *then there exists a unique weak* L^1 *-solution of the equation* $(**)$. *The solution is expressed by*

$$u_t = P_t^{(1)} \xi + \int_0^t P_{t-s}^{(1)} w_s ds + \int_0^t P_{t-s} \left(\sum_{i=1}^{d} \partial_i w_s'^i \right) ds + \sum_{j=1}^{d_1} \int_0^t P_{t-s} w_s''^j dB_s^j.$$

6.2 Ito's formula

We now can prove the following version of Ito's formula.

Proposition 2. *Let us assume hypotheses of the preceding Lemma and that u belongs to $\mathscr{H}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathscr{C}^2$, assume that φ' and φ'' are bounded and $\varphi'(0) = 0$. Then the following relation hold a.s. for all $t \geq 0$:*

$$\int_{\mathcal{O}} \varphi\left(u_t(x)\right) dx + \int_0^t \mathscr{E}\left(\varphi'\left(u_s\right), u_s\right) ds = \int_{\mathcal{O}} \varphi\left(\xi(x)\right) dx + \int_0^t \left(\varphi'\left(u_s\right), w_s\right) ds$$

$$-\int_0^t \sum_{i=1}^d \left(\partial_i\left(\varphi'\left(u_s\right)\right), w_s'^i\right) ds + \frac{1}{2}\int_0^t \left(\varphi''\left(u_s\right), \left|w_s''\right|^2\right) ds + \sum_{j=1}^{d_1}\int_0^t \left(\varphi'\left(u_s\right), w_s''^j\right) dB_s^j.$$

Proof: We denote by $v = \left(v_t\right)_{t \geq 0}$ the process defined by

$$v_t = \int_0^t P_{t-s}^{(1)} w_s ds.$$

Let us define for all $n \in \mathbb{N}^*$ and $t \geq 0$,

$$\xi_n = P_{\frac{1}{n}}^{(1)}\xi,\ v_t^n = P_{\frac{1}{n}} v_t, w_t^n = P_{\frac{1}{n}}^{(1)} w_t.$$

Since u belongs to $\mathscr{H}$, then process $\xi + v$ also belongs to $\mathscr{H}$.
We fix n for the moment. It is known that the semigroup has bounded densities, so that there exists some constants $K_t, t > 0$, such that

$$\left\| P_t^{(1)} f \right\|_\infty \leq K_t \left\| f \right\|_1,$$

and

$$\left\| A P_t^{(1)} f \right\|_2 \leq K_t \left\| f \right\|_1,$$

for any $f \in L^1(\mathcal{O})$. The second inequality follows from the well know estimate of spectral calculus $\|AP_t f\|_2 \leq e^{-1} t^{-1} \|f\|_2$. So, it is clear that ξ^n belongs to $L^1\left(\Omega, \mathscr{F}_0, P; L^\infty(\mathcal{O}) \bigcap \mathscr{D}(A)\right)$ and that for all $T > 0$ $(w_t^n)_{t \in [0,T]}$ belongs to $L^1(\Omega \times [0,T]; \mathscr{D}(A))$. As a consequence, v^n is $\mathscr{D}(A)$-differentiable and for all $t > 0$:

$$\partial_t v_t^n = w_t^n + A v_t^n.$$

Consider now sequences $(w'^{i,k})_{k \in \mathbb{N}^*}$, $1 \leq i \leq d$ of adapted processes in $C_c^\infty([0,\infty)) \otimes L^2(\Omega) \otimes \mathscr{D}(A^{3/2})$ which converge to w'^i, $1 \leq i \leq d$, in $\mathscr{P}(L^2)$ and sequences $(w''^{j,k})_{k \in \mathbb{N}^*}$, $1 \leq j \leq d_1$ of adapted processes in $C_c^\infty([0,\infty)) \otimes L^2(\omega) \otimes \mathscr{D}(A)$ which converge to w''^j, $1 \leq j \leq d_1$, in $\mathscr{P}(L^2)$.
We set for all $k \in \mathbb{N}^*$:

$$u^{n,k} = \mathscr{U}(\xi^n, w^n, w'^{,k}, w''^{,k}),$$

then we know that for all t

$$u_t^{n,k} = P_t \xi^n + \int_0^t P_{t-s} w_s^n ds + \int_0^t P_{t-s}\left(\sum_{i=1}^d \partial_i w_s'^{i,k}\right) ds + \sum_{j=1}^{d_1}\int_0^t P_{t-s} w_s''^{j,k} dB_s^j.$$

Lemma 6 in [5] ensures that $u^{n,k} - v^n = \mathscr{U}(\xi^n, 0, w'^{,k}, w''^{,k})$ is an $L^2(\mathcal{O})$-valued semi-martingale hence $u^{n,k}$ is also a semi-martingale since v^n is differentiable.
Thanks to the Ito's formula (see Lemma 7 in [5]), we have

$$\int_{\mathcal{O}} \varphi\left(u_t^{n,k}(x)\right) dx = \int_{\mathcal{O}} \varphi\left(\xi^n(x)\right) dx - \int_0^t \left(\varphi'\left(u_s^{n,k}\right), A u_s^{n,k}\right) ds + \int_0^t \left(\varphi'\left(u_s^{n,k}\right), w_s^n\right) ds$$

$$-\int_0^t \sum_{i=1}^d \left(\partial_i \left(\varphi'\left(u_s^{n,k}\right)\right), w_s'^{i,k}\right) ds + \frac{1}{2}\int_0^t \left(\varphi''\left(u_s^{n,k}\right), \left|w_s''^{,k}\right|^2\right) ds + \sum_{j=1}^{d_1}\int_0^t \left(\varphi'\left(u_s^{n,k}\right), w_s''^{j,k}\right) dB_s^j.$$

As a consequence of Lemma 6 in [5], we know that $u^{n,k}$ tends to u^n in $\mathscr{H}$ so, making k tend to $+\infty$ and using the fact that for all k,

$$-\int_0^t \left(\varphi'\left(u_s^{n,k}\right), Au_s^{n,k}\right) ds = \int_0^t \mathscr{E}\left(\varphi'\left(u_s^{n,k}\right) u_s^{n,k}\right) ds,$$

we get :

$$\int_{\mathscr{O}} \varphi\left(u_t^n(x)\right) dx + \int_0^t \mathscr{E}\left(\varphi'\left(u_s^n\right), u_s^n\right) ds = \int_{\mathscr{O}} \varphi\left(\xi^n(x)\right) dx + \int_0^t \left(\varphi'\left(u_s^n\right), w_s^n\right) ds$$

$$-\int_0^t \sum_{i=1}^d \left(\partial_i \left(\varphi'\left(u_s^n\right)\right), w_s'^i\right) ds + \frac{1}{2}\int_0^t \left(\varphi''\left(u_s^n\right), \left|w_s''\right|^2\right) ds + \sum_{j=1}^{d_1}\int_0^t \left(\varphi'\left(u_s^n\right), w_s''^j\right) dB_s^j.$$

As we assume that $\xi + \nu$ belongs to $\mathscr{H}$, u^n tends to u in $\mathscr{H}$ as n tends to $+\infty$, so

$$\lim_{n\to+\infty}\int_0^t \mathscr{E}\left(\varphi'\left(u_s^n\right), u_s^n\right) ds = \int_0^t \mathscr{E}\left(\varphi'\left(u_s\right), u_s\right) ds$$

Moreover, for all n

$$\begin{aligned}\int_0^t \left(\varphi'\left(u_s^n\right), w_s^n\right) ds &= \int_0^t \left(\varphi'\left(u_s^n\right), P_{\frac{1}{n}}^{(1)} w_s\right) ds \\ &= \int_0^t \left(P_{\frac{1}{n}}\varphi'\left(u_s^n\right), w_s\right) ds\end{aligned}$$

Since φ'' is bounded and u^n tends to u in $\mathscr{H}$, it is easy to prove that $P_{\frac{1}{n}}\varphi'(u^n)$ converges to $\varphi'(u)$ in $\mathscr{P}(L^2)$. Then, thanks to the dominated convergence theorem, we get that for a subsequence:

$$\lim_{n\to+\infty}\int_0^t \left(\varphi'\left(u_s^n\right), w_s^n\right) ds = \int_0^t \left(\varphi'\left(u_s\right), w_s\right) ds.$$

We then obtain the result by making n tend to $+\infty$ in the other terms of the equality without any problem. □

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