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# A Singular Parabolic Anderson Model

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**Abstract:** We consider the following stochastic partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \kappa u \dot{F},$$

for $x \in \mathbf{R}^d$ in dimension $d \geq 3$, where $\dot{F}(t,x)$ is a mean zero Gaussian noise with the singular covariance

$$E\left[\dot{F}(t,x)\dot{F}(s,y)\right] = \frac{\delta(t-s)}{|x-y|^2}.$$

Solutions $u_t(dx)$ exist as singular measures, under suitable assumptions on the initial conditions and for sufficiently small $\kappa$. We investigate various properties of the solutions using such tools as scaling, self-duality and moment formulae.

**Keywords and phrases:** stochastic partial differential equation, Anderson model, intermittency.

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# 1 Introduction

For readers who want to skip the motivation and definitions, the main results are summarized in Subsection 1.3.

## 1.1 Background and Motivation

The parabolic Anderson problem is modeled by the following stochastic partial differential equation (SPDE):

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \kappa u \dot{F}. \tag{1.1}$$

Here $u(t,x) \geq 0$ for $t \geq 0$ and $x \in \mathbf{R}^d$. $\dot{F} = \dot{F}(t,x)$ is a generalized Gaussian noise whose covariance will be specified later.

The main result of this paper is that for a special choice of covariance structure of the noise $\dot{F}$ the equation (1.1) has solutions that are measures on $\mathbf{R}^d$, and that these measures are singular. Linear equations driven by rough noises may easily have distribution valued solutions. For non-linear equations, or linear equations with multiplicative noise terms as in (1.1), solutions that are distribution valued, but not function valued, are rare since they potentially involve non-linear functions, or products, of distributions. Indeed the only two cases we know of are the following: the Dawson-Watanabe branching diffusions, which can be thought of as solutions to the heat equation with a multiplicative noise $\sqrt{u}\dot{W}$ for a space-time white noise $W$ (see Dawson [Daw93] and comments below); equations modeling stochastic quantization, related to certain quantum fields, which involve a Wick product is used (see [Albeverio+Rockner89]). We are describing equations with solutions that are distributions on $\mathbf{R}^d$. Our comments do not include the literature on equations with solutions that are distributions on Wiener space (see [HØUZ96], [NZ89] and [NR97] which treats the parabolic Anderson model with space-time white noise in high dimensions.). It was surprising to us to find that a noise $\dot{F}$ in (1.1) might have a sufficiently singular spatial correlation as to force the solutions to be singular measures, but not so as to destroy solutions all together.

The parabolic Anderson problem has various modeling interpretations (see Carmona and Molchanov [Car94]). The key behavior of solutions, called intermittency, is that they become concentrated in small regions, often called peaks, separated by large almost dead regions. Except when the covariance of the noise is singular at 0, the linear form of the noise term allows the use of the Feynman-Kac formula to study the solutions. Using this, mostly in the setting of discrete space with a discrete Laplacian and with a time-independent noise, there have been many successful descriptions of the solutions (see [GMK00] and the references there to work of Gärtner, Molchanov, den Hollander, König and others.) There is less work on the equation with space-time noises but the memoir [Car94] considers the case of Gaussian noises with various space and time covariances.

In addition the ergodic theory of such linear models has been independently studied. Discrete versions of the SPDE fit into the framework of interacting particle systems, under the name of linear systems. The reader can consult Liggett [Lig85], Chapter IX, Section 4 where, using the tools of duality and moments, the ergodic behavior of solutions is investigated. This work has been continued for lattice indexed systems of stochastic ODEs (see Cox, Fleischmann and Greven [CFG96] and also Cox, Klenke and Perkins [CKP01]). The basic picture is that in dimensions d=1,2 and $d \geq 3$ if $\kappa$ is large, the dead regions get larger and larger and the solutions become locally extinct. Conversely in $d \geq 3$, if $\kappa$ is small, the diffusion is sufficient to stop the peaks growing and there are non-trivial steady states.

In this paper we study a special case where the noise is white in time and has a space correlation that scales, namely

$$E\left[\dot{F}(t,x)\dot{F}(s,y)\right] = \frac{\delta(t-s)}{|x-y|^p}. \tag{1.2}$$

The presence of slowly decaying covariances is interesting; one interpretation of the equation given in [Car94] is in the setting of temperature changes in fluid flow and the noise arises as a model for the velocities in the fluid, where it is well known that there are slowly decaying covariances (in both space and time). Also the equations might arise as a limit of rescaled models where the covariance scaling law emerges naturally. Mathematically these covariances are convenient since they imply a scaling relation for the solutions that allow us to convert large time behavior into small scale behavior at a fixed time.

For $0 < p < 2$ (in dimensions $d \geq 2$) there are function valued solutions with these scaling covariances. The Kolmogorov criterion can be used to estimate the Hölder continuity of solutions and in Bentley [Ben99] the Hölder continuity is shown to break down as $p \uparrow 2$. In this paper we study just the case $p = 2$ and establish, in dimensions $d \geq 3$ and when $\kappa$ is small, the existence, and uniqueness in law, of measure valued solutions. One can imagine that the regularity of solutions breaks down as $p \uparrow 2$ but that there exists a singular, measure valued solution at $p = 2$ (we do not believe the equation makes sense for the case $p > 2$). Note that measure valued solutions to an SPDE have been successfully studied in the case of Dawson-Watanabe branching diffusions, which can be considered as solutions to the heat equation with the noise term $\sqrt{u}dW$, for a space-time white noise $W$ (see Dawson [Daw93]). Unless $d = 1$, this equation must be understood in terms of a martingale problem.

The special covariance $|x-y|^{-2}$ has two singular features: the blow-up near $x = y$ which causes the local clustering, so that the solutions become singular measures; and the fat tails at infinity which affects large time behavior (for instance we shall prove local extinction in all dimensions). The scaling is convenient in that it allows intuition about large time behavior to be transfered to results on local singularity, and vice-versa. In particular the singularity of the measures can be thought of as a description of the intermittency at large times.

## 1.2 Definitions

Our first task is to give a rigorous meaning to measure valued solutions of (1.1). We shall define solutions in terms of a martingale problem. We do not investigate the possibility of a strong solution for the equation. We do, however, construct solutions as a Wiener chaos expansion with respect to our noise. These solutions are adapted to the same filtration as the noise, and for some purposes provide a replacement for strong solutions. One advantage of working with martingale problem is that passing to the limit in approximations can be easier with this formulation.

We now fix a suitable state space for our solutions. Throughout the paper we consider only dimensions $d \geq 3$. The parameter $\kappa$ will also be fixed to lie in the range

$$0 < \kappa < \frac{d-2}{2}. \tag{1.3}$$

The restrictions on $d$ and $\kappa$ are due to our requirement that solutions have finite second moments. We do not explore the possibility of solutions without second moments.

Let $\mathcal{M}$ denote the non-negative Radon measures on $\mathbf{R}^d$, $\mathcal{C}_c$ the space of continuous functions on $\mathbf{R}^d$ with compact support, and $\mathcal{C}_c^k$ the space of functions in $\mathcal{C}_c$ with k continuous derivatives. We write $\mu(f)$ for the integral $\int f(x)\mu(dx)$, where $\mu \in \mathcal{M}$ and $f$ is integrable. Unless otherwise indicated, the integral is over the full space $\mathbf{R}^d$. We consider $\mathcal{M}$ with the vague topology, that is, the topology generated by the maps $\mu \to \mu(f)$ for $f \in \mathcal{C}_c$.

The class of allowable initial conditions is described in terms of the singularity of the measures. Define

$$\|\mu\|_\alpha^2 = \int\int \left(1 + |x-y|^{-\alpha}\right) \mu(dx)\mu(dy)$$

and let $\mathcal{H}_\alpha^a = \{\mu \in \mathcal{M} : \|\mu(dx)\exp(-a|x|)\|_\alpha < \infty\}$. Note the spaces $\mathcal{H}_\alpha^a$ are decreasing in $\alpha$ and increasing in $a$. Then define

$$\mathcal{H}_\alpha = \bigcup_a \mathcal{H}_\alpha^a, \quad \mathcal{H}_{\alpha+} = \bigcup_a \bigcup_{\beta > \alpha} \mathcal{H}_\beta^a, \quad \mathcal{H}_{\alpha-} = \bigcup_a \bigcap_{\beta < \alpha} \mathcal{H}_\beta^a.$$

The sets $\mathcal{H}_\alpha^a$ are Borel subsets of $\mathcal{M}$. The formula for the second moments of solutions also leads, for each $d$ and $\kappa$, to a distinguished choice of $\alpha$. Throughout the paper we make the choice

$$\alpha = \frac{d-2}{2} - \left[\left(\frac{d-2}{2}\right)^2 - \kappa^2\right]^{1/2}.$$

The restriction (1.3) ensures $\alpha \in (0, (d-2)/2)$. We shall require the initial conditions to lie in $\mathcal{H}_{\alpha+}$, again to guarantee the existence of second moments.

Suppose $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is a filtered probability space. We call an adapted continuous $\mathcal{M}$ valued process $\{u_t(dx) : t \geq 0\}$ a (martingale problem) solution to (1.1) if it satisfies

i.) $P(u_0 \in \mathcal{H}_{\alpha+}) = 1$,

ii.) $\{u_t(dx)\}$ satisfies the first and second moment bounds (1.6), (1.7) given below, and

iii.) $\{u_t(dx)\}$ satisfies the following martingale problem: for all $f \in \mathcal{C}_c^2$

$$z_t(f) = u_t(f) - u_0(f) - \int_0^t \frac{1}{2} u_s(\Delta f) ds \tag{1.4}$$

is a continuous local $\mathcal{F}_t$-martingale with finite quadratic variation given by

$$\langle z(f)\rangle_t = \kappa^2 \int_0^t \int\int \frac{f(x)f(y)}{|x-y|^2} u_s(dy)u_s(dx)ds \tag{1.5}$$

If in addition $P(u_0 = \mu) = 1$, we say that the solution $\{u_t(dx)\}$ has initial condition $\mu$.

Let $G_t(x) = (2\pi t)^{-d/2}\exp(-|x|^2/2t)$ for $x \in \mathbf{R}^d$ and $t > 0$. The moment conditions we require are that for all measurable $f : \mathbf{R}^d \to [0,\infty)$,

$$E\left[u_t(f)\Big|u_0\right] = \int\int G_t(x-x')f(x')u_0(dx'), \tag{1.6}$$

and there exists $C$, depending only on the dimension $d$ and $\kappa$, so that

$$\begin{aligned} & E\left[\left(\int f(x)u_t(dx)\right)^2 \Bigg| u_0\right] \qquad (1.7)\\ \leq\ & C\int_{\mathbf{R}^{4d}} G_t(x-x')G_t(y-y')f(x')f(y')\left(1+\frac{t^\alpha}{|x-y|^\alpha|x'-y'|^\alpha}\right) u_0(dx)u_0(dy)dx'dy'. \end{aligned}$$

The construction of solutions in Section 3 shows that our bound (1.7) on second moments is quite natural. We believe that the moment bounds (1.6) and (1.7) are implied by the martingale problem (1.4) and (1.5), although we do not show this. Since establishing second moment bounds is a normal first step to finding a solution to the martingale problem, we include these bounds as part of the definition of a solution.

We finish this subsection with some simple consequences of the second moment bound (1.7).

**Lemma 1** *Suppose $\{u_t(dx)\}$ is a solution to (1.1) with initial condition $\mu$. Choose $a$ so that $\mu \in \mathcal{H}^a_\alpha$.*

**i)** *For any $f \in \mathcal{C}_c$ and $t \geq 0$, we have*

$$E\left[\int_0^t \int\int \frac{f(x)f(y)}{|x-y|^2} u_s(dx)u_s(dy)ds\right] < \infty$$

*and hence the process $z_t(f)$ defined in (1.4) is a true martingale.*

**ii)** *For any $0 \leq \rho < d-\alpha$ and $t > 0$*

$$E\left[\int\int \left(1+|x-y|^{-\rho}\right) e^{-a|x|-a|y|} u_t(dx)u_t(dy)\right] < \infty$$

*and hence $u_t \in \mathcal{H}_{(d-\alpha)-}$ almost surely.*

**Proof.** For part i) it is sufficient to check that $E[\langle z(f)\rangle_t] < \infty$ to ensure that $z_t(f)$ is a true martingale. Using the second moment bounds (1.7) we have

$$\begin{aligned} & E\left[\int_0^t \int\int \frac{f(x)f(y)}{|x-y|^2} u_s(dx)u_s(dy)ds\right] \qquad (1.8)\\ \leq\ & C\int_0^t \int_{\mathbf{R}^{4d}} G_s(x-x')G_s(y-y')\frac{f(x')f(y')}{|x'-y'|^2}\left(1+\frac{s^\alpha}{|x-y|^\alpha|x'-y'|^\alpha}\right)\mu(dx)\mu(dy)dx'dy'ds. \end{aligned}$$

We now estimate the $dx'dy'$ integral in the above expression by using the simple bound, for $0 \leq r < d$,

$$\int\int G_t(x-x')G_t(y-y')|x'-y'|^{-r} dx'\, dy' \leq C(r)\left(|x-y|^{-r} \wedge t^{-r/2}\right). \tag{1.9}$$

The above estimate follows from the scaling properties of the normal density. For $f$ of compact support and any $a > 0$, we have the bound

$$\int G_s(x-x')f(x')dx' \leq C(a,f,t)e^{-a|x|} \quad \text{for all } s \leq t,\ x \in \mathbf{R}^d. \tag{1.10}$$

Then, applying Hölder's inequality with $1 < p < d/2$ and $p^{-1} + q^{-1} = 1$, we have, for all $s \leq t$,

$$\begin{aligned}
&\int\int G_s(x-x')G_s(y-y')\frac{f(x')f(y')}{|x'-y'|^2}dx'dy' \\
&\leq \left(\int\int G_s(x-x')G_s(y-y')\frac{1}{|x'-y'|^{2p}}dx'dy'\right)^{1/p} \\
&\qquad \cdot \left(\int\int G_s(x-x')G_s(y-y')f^q(x')f^q(y')dx'dy'\right)^{1/q} \\
&\leq C(a,f,t)e^{-a|x|-a|y|}\ (|x-y|^{-2} \wedge s^{-1}) \\
&\leq C(a,f,t)e^{-a|x|-a|y|}\ |x-y|^{-\alpha\wedge 2}\ s^{-(2-\alpha)_+/2} \\
&\leq C(a,f,t)e^{-a|x|-a|y|}\ (1+|x-y|^{-\alpha})\ s^{-(2-\alpha)_+/2}.
\end{aligned}$$

A similar calculation, using $2+\alpha < d$, gives the bound

$$\int\int G_s(x-x')G_s(y-y')\frac{f(x')f(y')}{|x'-y'|^{2+\alpha}}dx'dy' \leq C(a,f,t)s^{-(2+\alpha)/2}.$$

Now we substitute these bounds into (1.8) to obtain

$$\begin{aligned}
&E\left[\int\int \frac{f(x)f(y)}{|x-y|^2}u_s(dx)u_s(dy)ds\right] \\
&\leq C(a,f,t)\int_0^t s^{-(2-\alpha)_+/2}\int\int (1+|x-y|^{-\alpha})\, e^{-a|x|-a|y|}\mu(dx)\mu(dy)ds.
\end{aligned}$$

which is finite since $\|\mu(dx)\exp(-a|x|)\|_\alpha < \infty$.

For part ii), we use the second moment bound (1.7) to see that

$$\begin{aligned}
&E\left[\int\int (1+|x-y|^{-\rho})\, e^{-a|x|-a|y|}u_t(dx)u_t(dy)\right] \\
&\leq C\int_{\mathbf{R}^{4d}} G_t(x-x')G_t(y-y')\,(1+|x'-y'|^{-\rho})\, e^{-a|x'|-a|y'|} \\
&\qquad \cdot \left(1+\frac{t^\alpha}{|x-y|^\alpha|x'-y'|^\alpha}\right)\mu(dx)\mu(dy)dx'dy'.
\end{aligned} \tag{1.11}$$

Using the bound $\int G_t(x-x')\exp(-a|x'|)dx' \leq C(t,a)\exp(-a|x|)$ and (1.9), we estimate the $dx'dy'$ integral in the same way as above. We illustrate this only on the most singular term. For

$p, q > 1$ with $p^{-1} + q^{-1} = 1$,

$$
\begin{aligned}
& \int\int G_t(x-x')G_t(y-y')|x'-y'|^{-(\rho+\alpha)} e^{-a|x'|-a|y'|} dx'dy' \\
\leq \; & \left( \int\int G_t(x-x')G_t(y-y') e^{-ap|x'|-ap|y'|} dx'dy' \right)^{1/p} \\
& \quad \cdot \left( \int\int G_t(x-x')G_t(y-y')|x'-y'|^{-q(\rho+\alpha)} dx'dy' \right)^{1/q} \\
\leq \; & C(a,\rho,p,q) e^{-a|x|-a|y|} \left( |x-y|^{-(\rho+\alpha)} \wedge t^{-(\rho+\alpha)/2} \right) \qquad (1.12) \\
\leq \; & C(t,a,\rho,p,q) e^{-a|x|-a|y|}.
\end{aligned}
$$

provided that $q(\rho+\alpha) < d$. Such a $q > 1$ can be found whenever $\rho + \alpha < d$. Substituting this estimate into (1.11) gives the result. ∎

## 1.3 Main Results

We start with a result on existence and uniqueness.

**Theorem 1** *For any $\mu \in \mathcal{H}_{\alpha+}$ there exists a solution to (1.1) started at $\mu$. Solutions starting at $\mu \in \mathcal{H}_{\alpha+}$ are unique in law. If we denote this law by $Q_\mu$ then the set $\{Q_\mu : \mu \in \mathcal{H}_{\alpha+}\}$ forms a Markov family of laws.*

The existence part of Theorem 1 is proved in Section 3, and the uniqueness in Section 4. The next theorem, which is proved in Section 5, shows death from finite initial conditions and local extinction from certain infinite initial conditions. Write $B(x,r)$ for the open ball or radius $r$ centered at $x$. We say that a random measure $u_0$ has bounded local intensity if $E[u_0(B(x,1))]$ is a bounded function of $x$.

**Theorem 2** *Suppose $\{u_t(dx)\}$ is a solution to (1.1).*

**i) Death from finite initial conditions.** *If $P(u_0 \in \mathcal{H}^0_\alpha) = 1$ then $u_t(\mathbf{R}^d) \to 0$ almost surely as $t \to \infty$.*

**ii) Local extinction from infinite initial conditions.** *If $u_0$ has bounded local intensity and $A \subseteq \mathbf{R}^d$ is a bounded set then $u_t(A) \to 0$ in probability as $t \to \infty$.*

Finally, we state our main results describing the nature of the measures $u_t(dx)$. These are proved in Section 6.

**Theorem 3** *Suppose that $\{u_t(dx)\}$ is a solution to (1.1) satisfying $P(u_0 \neq 0) = 1$. Fix $t > 0$. Then the following properties hold with probability one.*

**i) Dimension of support.** *If a Borel set $A$ supports the measure $u_t(dx)$ then the Hausdorff dimension of $A$ is at least $d - \alpha$.*

**ii) Density of support.** *The closed support of $u_t(dx)$ is $\mathbf{R}^d$.*

**iii) Singularity of solutions.** *The absolutely continuous part of $u_t(dx)$ is zero.*

**Remarks**

**1.** Although Theorem 3 gives an almost sure result for fixed $t$, it leaves open the possibility that there are random times at which the properties fail. In Section 6 we shall show that $P(u_t \in \mathcal{H}_{\alpha+}$ for all $t \geq 0) = 1$. This implies that the weaker lower bound $d - 2 - \alpha$ on the dimension of supporting sets is valid for all times.

**2.** The reader might compare the behavior described in Theorem 3 with that of the Dawson-Watanabe branching diffusion in $\mathbf{R}^d$, for $d \geq 2$. This is a singular measure valued process whose support is two dimensional, and, if started with a finite measure of compact support, has compact support for all time.

**3.** Many of the results go through for the boundary case $\kappa = (d-2)/2$ and for initial conditions in $\mathcal{H}_\alpha$, although we have not stated results in these cases. The chaos expansion in Section 3 holds in both these boundary cases and the second moments are finite. Although our proof that the chaos expansion satisfies (1.5) uses $\kappa < (d-2)/2$ and $\mu \in \mathcal{H}_{\alpha+}$ we do not believe these restrictions are needed for this. However our proof of uniqueness for solutions in Section 4 does seem to require the strict inequalities. This leaves open the possibility that there are solutions with a different law to that constructed via the chaos expansion. Theorems 2 and 3 will hold in the boundary cases for the solutions constructed via chaos expansion solutions. Parts of Theorems 2 and 3 also hold for all solutions, for example Propositions 2 and 3 use only the martingale problem in their proof and hold for any solution in the boundary cases.

## 1.4 Tools

We briefly introduce the main tools that we use. The first tool, scaling for the equation, is summarized in the following lemma.

**Lemma 2** *Suppose that $\{u_t(dx)\}$ is a solution to (1.1). Let $a, b, c > 0$ and define*

$$v_t(A) = a u_{bt}(cA) \quad \textit{for Borel } A \subseteq \mathbf{R}^d$$

*where $cA = \{cx : x \in A\}$. Then $\{v_t(dx)\}$ is a solution to the equation*

$$\frac{\partial v_t}{\partial t} = \frac{b}{2c^2}\Delta v + \kappa \frac{b^{1/2}}{c} v \dot{F}_{b,c}(t,x)$$

*where $\dot{F}_{b,c}(t,x)$ is a Gaussian noise identical in law to $\dot{F}(t,x)$.*

The equation for $\{v_t(dx)\}$ is interpreted via a martingale problem, as in (1.1). The easy proof of this lemma is omitted.

The next tool is our equation for the second moments. The linear noise term implies that the solutions have closed moment equations. By this we mean that the moment densities

$$H_t(x_1, x_2, \ldots, x_n)dx_1 \ldots dx_n = E\left[u_t(dx_1)u_t(dx_2)\ldots u_t(dx_n)\right]$$

satisfy an autonomous PDE. Formally assuming the solution has a smooth density $u_t(x)$, applying Ito's formula to the product $u_t(x_1)\ldots u_t(x_n)$ and taking expectations suggests that $H_t$ satisfies

$$\frac{\partial H_t}{\partial t} = \frac{1}{2}\Delta H_t + \kappa^2 H_t \sum_{1\le i<j\le n} \frac{1}{|x_i - x_j|^2}.$$

Then the Feynman-Kac representation for this linear equation suggests that

$$H_t(x_1, \ldots, x_n) = E_{x_1,\ldots,x_n}\left[u_0(X_t^1)\ldots u_0(X_t^n)\exp\left(\int_0^t \sum_{1\le i<j\le n} \frac{\kappa^2}{|X_s^i - X_s^j|^2} ds\right)\right]$$

where $E_{x_1,\ldots,x_n}$ denotes expectation with respect to $n$ independent d-dimensional Brownian motions $X_t^1, \ldots, X_t^n$. This formula makes sense when $u_0$ has a density. But more generally, we can expect the following. For solutions $\{u_t(dx)\}$ to (1.1), started at $\mu$, and when $f_i \in \mathcal{C}_c$ for $i = 1, \ldots, k$,

$$E\left[\prod_{i=1}^n u_t(f_i)\right] \tag{1.13}$$

$$= \int_{\mathbf{R}^{2nd}} E_{0,x_1,\ldots,x_n}^{t,y_1,\ldots,y_n}\left[\exp\left(\sum_{1\le j<k\le n}\int_0^t \frac{\kappa^2}{\left|X_s^{(j)} - X_s^{(k)}\right|^2} ds\right)\right]\prod_{i=1}^n G_t(x_i - y_i) f_i(y_i)\mu(dx_i)dy_i$$

where $E_{0,x_1,\ldots,x_n}^{t,y_1,\ldots,y_n}$ is expectation with respect to $n$ independent d-dimensional Brownian bridges $(X_t^{(1)}, \ldots, X_t^{(n)})$ started at $(x_i)$ at time zero and ending at $(y_i)$ at time $t$. In Section 2 we investigate the values of $\kappa$ for which this expectation is finite.

The next tool is the expansion of the solution in terms of Wiener chaos, involving multiple integrals over the noise $F(t,x)$. Wiener chaos expansions have been used before for linear equations; for example see Dawson and Salehi [Daw80], Nualart and Zakai [NZ89], or Nualart and Rozovskii [NR97]. The idea is to start with the Green's function representation, assuming (falsely) that a function valued solution exists:

$$u_t(y) = G_t\mu(y) + \kappa \int_0^t \int G_{t-s}(y-z)u_s(z)F(dz, ds). \tag{1.14}$$

The first term on the right hand side of this representation uses the notation $G_t\mu(y) = \int G_t(y-z)\mu(dz)$. The second term again involves the the non-existent density $u_s(z)$. However, we can use the formula for $u_t(y)$ given in (1.14) to substitute for the term $u_s(z)$ which appears on the right hand side. The reader can check that if we keep repeating this substitution, and assume the remainder term vanishes, we will arrive at the following formula: for a test function $f \in \mathcal{C}_c$,

$$u_t(f) = \sum_{n=0}^{\infty} I_t^{(n)}(f, \mu) \tag{1.15}$$

where

$$I_t^{(n)}(f,\mu) = \int\int f(y) I_t^{(n)}(y,z)\mu(dz)dy \tag{1.16}$$

and where the $I^{(n)}$ are defined as follows: $I_t^{(0)}(y,z) = G_t(y-z)$ and for $n \geq 1$

$$I_{s_{n+1}}^{(n)}(y_{n+1},z) = \kappa^n \int_0^{s_{n+1}} \int_0^{s_n} \cdots \int_0^{s_2} \int_{\mathbf{R}^{nd}} G_{s_1}(y_1-z) \prod_{i=1}^n G_{s_{i+1}-s_i}(y_{i+1}-y_i) F(dy_i, ds_i). \tag{1.17}$$

In Section 3 we shall show that the stochastic integrals in (1.17) are well defined, and the series (1.15) converges in $L^2$ and defines a solution. The point is that the series $\sum_n I_t^{(n)}(y,z)$ does not converge pointwise, but after smoothing by integrating against the initial measure and the test function the series does converge. The restriction (1.3) on $\kappa$ and the choice of space $\mathcal{H}_{\alpha+}$ for the initial conditions is exactly what we need to ensure this $L^2$ convergence. For larger values of $\kappa$ it is possible that the series converges in $L^p$ for some $p < 2$. It is also always possible to consider the chaos expansion (1.15) itself as a solution, if we interpret solutions in a suitably weak fashion, for example as a linear functionals on Wiener space. We do not investigate either of these possibilities.

The symmetry of the functions $I_t^{(n)}(y,z)$ in $y$ and $z$ makes it clear that a time reversal property should hold. This is well known for linear systems and for the parabolic Anderson model, and is often called self duality. Suppose that $\{u_t(x)\}$, $\{v_t(x)\}$ are two solutions of (1.1) started from suitable absolutely continuous initial conditions $u_0(x)dx$ and $v_0(x)dx$. We expect that $u_t(v_0)$ has the same distribution as $u_0(v_t)$. In Section 4 we shall use this equality to establish uniqueness of solutions.

The Feynman-Kac formula is a standard tool in analogous discrete space models. In the continuous space setting of the parabolic Anderson equation (1.1), we shall replace the noise $F$ by a noise $\bar{F}$ that is Gaussian, white in time and with a smooth, translation invariant covariance $\Gamma(x-y)$ in space. Then the Feynman-Kac representation is

$$\begin{aligned} u_t(x) &= E_x\left[u_0(X_t)e^{-\Gamma(0)t}\exp\left(\kappa\int_0^t \bar{F}(ds, X_{t-s})\right)\right] \\ &= \int G_t(x-y)u_0(y)e^{-\Gamma(0)t}E_{0,y}^{t,x}\left[\exp\left(\kappa\int_0^t \bar{F}(ds, X_s)\right)\right]. \end{aligned} \tag{1.18}$$

A proof of this representation can be found in Kunita [Kun90] Theorem 6.2.5 and we make use of it in Section 6. Since our covariance blows up at the origin, the factor $\Gamma(0)$ which appears in the exponential is infinite, and the representation can only be used for approximations.

Finally a remark on notation: we use $C(t,p,\ldots)$ for a constant whose exact value is unimportant and may change from line to line, that may depend on the dimension $d$ and the parameter $\kappa$ (and hence also on $\alpha$), but whose dependence on other parameters will be indicated. $c$ or $c_k$ will also denote constants which can change from line to line.

## 2 A Brownian exponential moment

As indicated in the introduction, the second moments of solutions $\{u_t(dx)\}$ to (1.1) can be expressed in terms of the expectation of a functional of a Brownian bridge. An upper bound for these expectations is a key estimate in the construction of our solutions. In this section we show the following bound.

**Lemma 3** *Let $X_s$ be a standard Brownian motion. For all $0 \le \eta \le (d-2)^2/8$ there exists $C(\eta) < \infty$ so that for all $x, y, t$*

$$E_{0,x}^{t,y}\left[\exp\left(\eta \int_0^t \frac{ds}{|X_s|^2}\right)\right] \le C(\eta)\left(1 + \frac{t}{|x|\,|y|}\right)^{\alpha(\eta)}$$

*where*

$$\alpha(\eta) = \frac{d-2}{2} - \left[\left(\frac{d-2}{2}\right)^2 - 2\eta\right]^{1/2}.$$

We first treat the case of Bessel processes and Bessel bridges (see Revuz and Yor [RY91] chapter XI for the basic definitions). The reason for this is that the laws of two Bessel processes, of two suitable different dimensions, are mutually absolutely continuous and the Radon-Nikodym derivative involves exactly the exponential functional we wish to estimate.

Let $C[0,t]$ be the space of real-valued continuous paths up to time $t$, and let $\{R_t\}$ be the canonical path variables. For $d \in [2, \infty)$ and $a, b > 0$, we write $E_a^{(d)}$ for expectation under the law of the $d$-dimensional Bessel process started at $a$, and $q_t^{(d)}(a,b)$ for the transition density. We write $E_{a,b,t}^{(d)}$ for expectation under the law of the $d$-dimensional Bessel bridge starting at $a$ and ending at $b$ at time $t$. Suppose that $Y$ is a non-negative random variable on the space $C[0,t]$, measurable with respect to $\sigma(R_s : s \le t)$. Lemma 4.5 of Yor [Yor80], (or Revuz and Yor [RY91], Chapter XI, exercise 1.22), expressed in our notation, states that the following relationship holds: if $\lambda, \mu \ge 0$ then

$$E_a^{(2\lambda+2)}\left[Y \exp\left(-\frac{\mu^2}{2}\int_0^t \frac{ds}{R_s^2}\right)\left(\frac{R_t}{a}\right)^{-\lambda}\right] = E_a^{(2\mu+2)}\left[Y \exp\left(-\frac{\lambda^2}{2}\int_0^t \frac{ds}{R_s^2}\right)\left(\frac{R_t}{a}\right)^{-\mu}\right]. \tag{2.1}$$

Now for $0 \le \eta \le (d-2)^2/8$ we choose values for $\lambda, \mu, Y$ in this identity as follows:

$$\lambda = \frac{d-2}{2}, \quad \mu = \left[\left(\frac{d-2}{2}\right)^2 - 2\eta\right]^{1/2}, \quad Y = \exp\left(\left[\eta + \frac{\mu^2}{2}\right]\int_0^t \frac{ds}{R_s^2}\right)\left(\frac{R_t}{a}\right)^{\lambda} \mathbf{1}(R_t \in db).$$

Note with these choices that $\alpha(\eta) = \lambda - \mu$, $2\eta + \mu^2 - \lambda^2 = 0$, and $d = 2\lambda + 2$. Applying (2.1) we find

$$\begin{aligned} & E_a^{(d)}\left[\exp\left(\eta \int_0^t \frac{ds}{R_s^2}\right)\mathbf{1}(R_t \in db)\right] \\ = \; & E_a^{(2\lambda+2)}\left[Y \exp\left(-\frac{\mu^2}{2}\int_0^t \frac{ds}{R_s^2}\right)\left(\frac{R_t}{a}\right)^{-\lambda}\right] \end{aligned}$$

$$
\begin{aligned}
&= E_a^{(2\mu+2)}\left[Y\exp\left(-\frac{\lambda^2}{2}\int_0^t\frac{ds}{R_s^2}\right)\left(\frac{R_t}{a}\right)^{-\mu}\right] \\
&= E_a^{(2\mu+2)}\left[\exp\left(\left[\eta+\frac{\mu^2}{2}-\frac{\lambda^2}{2}\right]\int_0^t\frac{ds}{R_s^2}\right)\left(\frac{R_t}{a}\right)^{\lambda-\mu}\mathbf{1}(R_t\in db)\right] \\
&= E_a^{(2\mu+2)}\left[\left(\frac{R_t}{a}\right)^{\alpha(\eta)}\mathbf{1}(R_t\in db)\right] \\
&= a^{-\alpha(\eta)}b^{\alpha(\eta)}q_t^{(2\mu+2)}(a,b)db.
\end{aligned}
$$

Hence

$$
E_{a,b,t}^{(d)}\left[\exp\left(\eta\int_0^t\frac{ds}{R_s^2}\right)\right]=a^{-\alpha(\eta)}b^{\alpha(\eta)}\frac{q_t^{(2\mu+2)}(a,b)}{q_t^{(d)}(a,b)}. \tag{2.2}
$$

There is an exact formula for the Bessel transition density

$$
q_t^{(d)}(a,b)=t^{-1}a^{-(d-2)/2}b^{d/2}\exp(-(a^2+b^2)/2t)I_{(d/2)-1}(ab/t)
$$

in terms of the (modified) Bessel functions $I_\nu$ of index $\nu=(d/2)-1$. The Bessel functions $I_\nu(z)$ are continuous and strictly positive for $z\in(0,\infty)$ and satisfy the asymptotics, for $c_1,c_2>0$,

$$
I_\nu(z)\sim c_1 z^\nu \quad \text{as } z\downarrow 0, \quad I_\nu(z)\sim c_2 z^{-1/2}e^z \quad \text{as } z\uparrow\infty.
$$

By $f(x)\sim g(x)$, we mean that the ratio tends to 1. Using these asymptotics, we find that

$$
E_{a,b,t}^{(d)}\left[\exp\left(\eta\int_0^t\frac{ds}{R_s^2}\right)\right]\le C(\eta)\left(1+\frac{t}{ab}\right)^{\alpha(\eta)} \quad \text{for all } a,b,t>0. \tag{2.3}
$$

We now wish to obtain a similar estimate for a Brownian bridge. Recall the skew product representation for a $d$-dimensional Brownian motion $X_t$, started from $x\neq 0$. There is a Brownian motion $W(t)$ on the sphere $\mathbf{S}^{d-1}$, started at $x/|x|$ and independent of $X$, so that

$$
X_t/|X_t|=W\left(\int_0^t|X_s|^{-2}ds\right).
$$

We may find a constant $C$ so that $P_x(W(t)\in d\theta)\le Cd\theta$ for all $x\in\mathbf{S}^{d-1}$ and $t\ge 1$. We now consider the exponential moment for a $d$-dimensional Brownian bridge running from $x\neq 0$ to $y\neq 0$ in time $t$.

$$
E_{x,0}^{y,t}\left[\exp\left(\eta\int_0^t\frac{ds}{|X_s|^2}\right)\right]\le e^\eta+E_{x,0}^{y,t}\left[\exp\left(\eta\int_0^t\frac{ds}{|X_s|^2}\right)\mathbf{1}\left(\int_0^t\frac{ds}{|X_s|^2}\ge 1\right)\right]. \tag{2.4}
$$

Now we estimate the second term on the right hand side of (2.4).

$$
\begin{aligned}
&E_{x,0}^{y,t}\left[\exp\left(\eta\int_0^t\frac{ds}{|X_s|^2}\right)\mathbf{1}\left(\int_0^t\frac{ds}{|X_s|^2}\ge 1\right)\right] \\
&= \frac{1}{G_t(x-y)}E_x\left[\exp\left(\eta\int_0^t\frac{ds}{|X_s|^2}\right)\mathbf{1}\left(\int_0^t\frac{ds}{|X_s|^2}\ge 1, X_t\in dy\right)\right] \\
&= \frac{C|y|^{1-d}}{G_t(x-y)}E_x\left[\exp\left(\eta\int_0^t\frac{ds}{|X_s|^2}\right)\mathbf{1}\left(\int_0^t\frac{ds}{|X_s|^2}\ge 1, |X_t|\in d|y|, W\left(\int_0^t\frac{ds}{|X_s|^2}\right)\in d(y/|y|)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C|y|^{1-d}}{G_t(x-y)} E_x \left[ \exp\left( \eta \int_0^t \frac{ds}{|X_s|^2} \right) \mathbf{1} \left( \int_0^t \frac{ds}{|X_s|^2} \geq 1, |X_t| \in d|y| \right) \right] \\
&\leq \frac{C|y|^{1-d} q_t^{(d)}(|x|,|y|)}{G_t(x-y)} E^{(d)}_{|x|,|y|,t} \left[ \exp\left( \eta \int_0^t \frac{ds}{|R_s|^2} \right) \right].
\end{aligned}
$$

Using the explicit representation for the Bessel density given above we find that

$$
\frac{|y|^{1-d} q_t^{(d)}(|x|,|y|)}{G_t(x-y)} \leq C(R), \quad \text{whenever } \frac{|x||y|}{t} \leq R.
$$

Combining this with (2.4) and our estimate (2.3) for the Bessel bridge, we obtain the desired bound for $(x,y,t)$ in any region where $\{|x||y|/t \leq R\}$.

We felt there should be a short way to treat the remaining case where $\{|x||y|/t > R\}$, but we seem to need a more complicated argument. Our only aim is to show that in this region, the exponential moment is bounded by a constant. Define

$$
F_K(x,y,t) := E^{t,y}_{0,x} \left[ \exp\left( \eta \int_0^t \frac{ds}{|X_s|^2} \wedge K \right) \right] \leq E^{t,y}_{0,x} \left[ \exp\left( \eta \int_0^t \frac{ds}{|X_s|^2} \right) \right] =: F(x,y,t).
$$

Brownian scaling implies that $F(x,y,t) = F(c^{1/2}x, c^{1/2}y, ct)$ for any $c > 0$. So we may scale time away, and it is enough to control $F(x,y,1)$. We have proved above, for any $R$,

$$
F(x,y,1) \leq C(R,\eta) \left( 1 + \frac{1}{|x|\,|y|} \right)^{\alpha(\eta)} \text{ whenever } |x||y| \leq R. \tag{2.5}
$$

We claim that it is enough to consider the case where $|x| = |y|$. Suppose that $|x||y| \geq 1$ and $|x| > |y|$. Define stopping times

$$
\sigma_1 = \inf\{t : |X_t| \leq |y|\}, \quad \sigma_2 = \inf\{t : |X_t||y| \leq 1-t\},
$$

and let $\sigma = \sigma_1 \wedge \sigma_2$. Note that for $t < \sigma_1$ we have $1/|X_t| \leq 1/|y|$, and for $t < \sigma_2$ we have $1/|X_t| \leq |y|/(1-t)$. So we can bound the integral in $F(x,y,1)$ by

$$
\begin{aligned}
\int_0^1 \frac{dt}{|X_t|^2} &\leq \int_0^\sigma \frac{dt}{|X_t|^2} + \int_\sigma^1 \frac{dt}{|X_t|^2} \\
&\leq \int_0^\sigma \left( \frac{|y|^2}{(1-t)^2} \wedge \frac{1}{|y|^2} \right) dt + \int_\sigma^1 \frac{dt}{|X_t|^2} \\
&\leq \int_0^{(1-|y|^2)_+} \frac{|y|^2}{(1-t)^2} dt + \int_{(1-|y|^2)_+}^1 \frac{1}{|y|^2} dt + \int_\sigma^1 \frac{dt}{|X_t|^2} \\
&\leq 2 + \int_\sigma^1 \frac{dt}{|X_t|^2}.
\end{aligned}
$$

Conditioned on the values of $\sigma$ and $X_\sigma$, the path between $t \in [\sigma, 1]$ is a new Brownian bridge. Hence

$$
F(x,y,1) \leq e^{2\eta} E\left( F(X_\sigma, y, 1-\sigma) \right) = E\left[ F\left( \frac{X_{\sigma_1}}{(1-\sigma_1)^{1/2}}, \frac{y}{(1-\sigma_1)^{1/2}}, 1 \right) \right].
$$

By definition, $|X_{\sigma_2}||y|/(1-\sigma_2)=1$, so that $F(X_{\sigma_2}/(1-\sigma_2)^{1/2}, y/(1-\sigma_1)^{1/2}, 1)$ can be bounded by a constant using (2.5). Also, on the set $\{\sigma_1 < \sigma_2\}$, we know that $|X_{\sigma_1}||y|/(1-\sigma_1) \geq 1$ and $|X_{\sigma_1}| = |y|$. So if we can bound the $F(x,y,1)$ on the diagonal where $\{|x|=|y|, |x||y| \geq 1\}$, then we can bound $F(X_{\sigma_1}/(1-\sigma_1)^{1/2}, y/(1-\sigma_1)^{1/2}, 1)$, and in consequence also bound $F(x,y,1)$.

We now give a brief sketch to motivate the final argument. Consider the "worst case" of a bridge from $x = Ne_1$ to $y = -Ne_1$ over time one. Run both ends of the bridge until they first hit the ball of radius $N/2$. When $N$ is large, the bridge will enter the ball near $x/2$ and exit near $y/2$. Also, it will spend close to time $1/2$ inside the ball. We may therefore approximately bound the exponential as

$$\exp\left(\int_0^1 \frac{1}{|X_s|^2} ds\right) \leq \exp(4N^{-2}) \exp\left(\int_{3/4}^{1/4} \frac{1}{|X_s|^2} ds\right).$$

Using the scaling of $F(x,y,t)$, we see that $F(Ne_1, -Ne_1, 1)$ is approximately bounded by

$$\exp(4N^{-2}) F(Ne_1/2, -Ne_1/2, 1/2) = \exp(4N^{-2}) F(Ne_1/2^{1/2}, -Ne_1/2^{1/2}, 1).$$

By iterating this argument, we can bound $F(Ne_1, -Ne_1, 1)$ for large values $N$, by $F(Ne_1, -Ne_1, 1)$ for small values of $N$. Then, using (2.5), we get a bound for the small values of $N$ as well.

We now give the basic iterative construction. Suppose that $|x| = |y| = R \geq 1$, and consider the Brownian bridge $\{X_t\}$ from $x$ to $y$ in time 1. Define random times

$$\sigma = \inf\{t : |X_t| \leq R/2\}, \quad \tau = \sup\{t : |X_t| \leq R/2\}$$

on the set $\{\inf_t |X_t| < R/2\} = \{\sigma < \tau\}$. On the set $\{|X_t| > R/2, \forall t \in [0,1]\}$, we have the bound $\int_0^1 |X_s|^{-2} ds \leq 4R^{-2}$. On $\{\sigma < \tau\}$, we have the bound

$$\int_0^1 \frac{dt}{|X_t|^2} \leq 4R^{-2} \int_\sigma^\tau \frac{dt}{|X_t|^2}.$$

Conditioned on $\sigma, \tau, X_\sigma, X_\tau$, the path $\{X_t : t \in [\sigma, \tau]\}$ is a new Brownian bridge. So we may estimate

$$F(x,y,1) \leq \exp(4\eta R^{-2}) \left(P(\{|X_t| > R/2, \forall t \in [0,1]\}) + E\left(F(X_\sigma, X_\tau, \tau - \sigma)\mathbf{1}(\sigma < \tau)\right)\right). \quad (2.6)$$

The same bound holds with $F$ replaced by $F_K$.

We will repeat this construction with a new Brownian bridge running from $X_\sigma/(\tau-\sigma)^{1/2}$ to $X_\tau/(\tau-\sigma)^{1/2}$. The following lemma shows that when $R$ is large we have usually made an improvement, in that this bridge is closer to the origin.

**Lemma 4** *There exists $\gamma < 1$ and $c_3 < \infty, c_4 > 0$ so that, when $|x| = |y| = R$,*

$$P\left(\sigma < \tau,\ X_\sigma \cdot X_\tau \leq 0,\ \left|X_\sigma/(\tau-\sigma)^{1/2}\right| \geq \gamma |x|\right) \leq c_3 \exp(-c_4 R), \quad (2.7)$$

*and there exist $c_5 < \infty, c_6 > 0$ so that, if in addition $x \cdot y \geq 0$,*

$$P\left(\inf_t |X_t| < R/2\right) \leq c_5 \exp(-c_6 R), \quad (2.8)$$

**Proof**. We scale the Brownian bridge by defining $\tilde{X}_t^R = X_t/R$. The starting and ending positions $\tilde{x} = \tilde{X}_0^R, \tilde{y} = \tilde{X}_1^R$ now satisfy $|\tilde{x}| = |\tilde{y}| = 1$, and the process $\tilde{X}_t^R$ is stopped upon hitting the ball of radius $1/2$. However, the process $\tilde{X}_t^R$ has smaller variance. Indeed, we have the equality in law:

$$\tilde{X}_t^R \stackrel{\mathcal{L}}{=} (1-t)\tilde{x} + t\tilde{y} + (B_t - tB_1)/R.$$

As $R \to \infty$, the process converges to the straight line $\tilde{X}_t = (1-t)\tilde{x} + t\tilde{y}$. For this limiting process, the basic construction is deterministic. If $\tilde{x} \cdot \tilde{y} \geq 0$, then the straight line never gets closer to the origin than $2^{-1/2}$. For large $R$, a large deviations estimate shows that deviations away from the straight line are exponentially unlikely, and (2.8) follows. To obtain (2.7), one again considers the straight line $\tilde{X}_t$ and maximizes $\tilde{X}_\sigma/(\tau-\sigma)^{1/2}$ over those starting and ending points $\tilde{x}, \tilde{y}$ for which $\tilde{X}_\sigma \cdot \tilde{X}_\tau < 0$. The maximum occurs, for example, when $\tilde{X}_\sigma/(\tau-\sigma)^{1/2} = (1/2)e_1$ and $\tilde{X}_\tau/(\tau-\sigma)^{1/2} = (1/2)e_2$. A little trigonometry shows that either $\tilde{X}_\sigma \cdot \tilde{X}_\tau < 0$ or else $\tilde{X}_\sigma/(\tau-\sigma)^{1/2} \leq \tilde{\gamma}$ for some $\tilde{\gamma} \in (0,1)$. By taking $\gamma \in (\tilde{\gamma}, 1)$, a large deviations argument yields (2.7). ∎

Applying (2.8) to the bound (2.6) we find, when $|x| = |y| = R$ and $x \cdot y \geq 0$,

$$F_K(x,y,1) \leq \exp(4\eta R^{-2}) \left(1 + c_5 e^{-c_6 R} \sup_{|x|=|y|\geq R/2} F_K(x,y,1)\right). \tag{2.9}$$

Now we wish to iterate the basic construction to define a Markov chain $(x(n), y(n))_{n=0,1,\ldots}$ on $\left(\mathbf{R}^d \cup \{\Delta\}\right)^2$. Throughout, $|x(n)| = |y(n)|$ or $x(n) = y(n) = \Delta$ will hold. $\Delta$ is cemetery state from which there is no return. It will be convenient to set $F(\Delta, \Delta, 1) = F_K(\Delta, \Delta, 1) = 1$. We set $x(0) = x$ and $y(0) = y$. Suppose $x(n)$ and $y(n)$ have been defined and are not equal to $\Delta$. Then we repeat the basic construction described above, but started at the radius $R = |x(n)| = |y(n)|$. We define

$$\begin{cases} x(n+1) = X_\sigma/(\tau-\sigma)^{1/2},\ y(n+1) = X_\tau/(\tau-\sigma)^{1/2} & \text{on } \{\sigma < \tau\}, \\ x(n+1) = y(n+1) = \Delta & \text{on } \{|X_t| > R/2,\ \forall t \in [0,1]\}. \end{cases}$$

We will shortly choose a constant $R_0 \in [1, R]$. Define stopping times for $(x(n), y(n))$ as follows.

$$\begin{aligned} N_1 &= \inf\{n : x(n) = y(n) = \Delta\}, \\ N_2 &= \inf\{n : |x(n)| \leq R_0\}, \\ N_3 &= \inf\{n : x(n) \cdot y(n) \geq 0\}, \\ N_4 &= \inf\{n : |x(n)| > \gamma |x(n-1)|\} \end{aligned}$$

Let $N = N_1 \wedge N_2 \wedge N_3 \wedge N_4$. Technically, we should define $|\Delta|$ to make these times well defined. But we adopt the convention that if $N \geq k$ and $N_1 = k$ then $N_2 = N_3 = N_4 = \infty$. Note that $N$ is a bounded stopping time, since if $N_4$ has not occurred then $N_2 \leq N_0$, where $R\alpha^{N_0} \leq R_0$. We now expand $F_K(x,y,1)$ as in (2.6) to find

$$F_K(x,y,1) \leq \sum_{n=1}^{N_0} E\left[\mathbf{1}(N=n) \exp\left(4\eta(|x(0)|^{-2} + \ldots + |x(n-1)|^{-2})\right) F_K(x(n), y(n), 1)\right].$$

On $\{N = n\}$, we know that $R_0 \le |x(n-1)| \le \gamma|x(n-2)| \le \gamma^2|x(n-3)| \le \ldots$. Hence, on this set,

$$\exp\left(4\eta(|x(0)|^{-2} + \ldots + |x(n-1)|^{-2})\right) \le \exp\left(\frac{4\eta}{R_0^2(1-\gamma^2)}\right).$$

We choose $R_0$ large enough that this exponential is bounded by 2. This leads to the simpler bound

$$F_K(x,y,1) \le 2E\left[F_K(x(N), y(N), 1)\right] \tag{2.10}$$

We now find various estimates for $E\left[F_K(x(N), y(N), 1)\right]$, depending on the value of $N$. When $N = N_1$ we have, by definition,

$$\mathbf{1}(N = N_1)F_K(x(N), y(N), 1) = 1. \tag{2.11}$$

When $N = N_2$ we have $|x(n)| \in [R_0/2, R_0]$, and so we can bound

$$\mathbf{1}(N = N_2)F_K(x(N), y(N), 1) \le \sup_{|x|=|y|\in[R_0/2,R_0]} F(x,y,1). \tag{2.12}$$

When $N = N_3 > N_2$ we have $|x(N)| = |y(N)| \ge R_0$, and $x(N) \cdot y(N) \ge 0$. Thus, we may use (2.9) to bound $F_K(x(N), y(N), 1)$. By choosing $R_0$ large enough, this gives the bound

$$\mathbf{1}(N = N_3 > N_2)F_K(x(N), y(N), 1) \le 2\left(1 + \frac{1}{16}\sup_{|x|=|y|\ge R_0/2} F_K(x,y,1)\right). \tag{2.13}$$

Finally, when $N = N_4 < N_2 \wedge N_3$ we simply bound

$$E\left[\mathbf{1}(N = N_4 < N_2 \wedge N_3)F_K(x(N), y(N), 1)\right] \le P(N = N_4 < N_2 \vee N_3) \sup_{|x|=|y|\ge R_0/2} F_K(x,y,1).$$

We now claim that

$$\lim_{R\to\infty} \sup_{|x|=|y|\ge R} P(N = N_4 < N_2 \wedge N_3) = 0. \tag{2.14}$$

Indeed, we may apply Lemma 4 to see that

$$P(N = N_4 = k+1 < N_2 \wedge N_3 | (x(j), y(j))\ j = 0, 1, \ldots, k) \le c_3 \exp(-c_4|x(k)|)\mathbf{1}(N > k).$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{\infty} P(N = N_4 = k < N_2 \wedge N_3) \\
&\le \sum_{k=1}^{\infty} E\left[c_3 \exp(-c_4|x(k-1)|)\mathbf{1}(N > k-1)\right] \\
&\le E\left[c_3 \sum_{k=0}^{N-1} \exp(-c_4|x(k)|)\right] \\
&\le c_3 \sum_{k=0}^{\infty} \exp(-c_4 R_0 \gamma^{-k}) \\
&\to 0 \quad \text{as } R_0 \to \infty.
\end{aligned}$$

Using the claim (2.14), we may choose $R_0$ large enough that

$$E\left[\mathbf{1}(N = N_4 > N_3 \vee N_2) F_K(x(N), y(N), 1)\right] \leq \frac{1}{8} \sup_{|x|=|y|\geq R_0/2} F_K(x,y,1). \tag{2.15}$$

Choosing $R_0$ large enough that all four estimates (2.11), (2.12), (2.13), (2.15) hold, we substitute them into (2.10) to obtain

$$\begin{aligned} F_K(x,y,1) &\leq 6 + 2 \sup_{|x|=|y|\in[R_0/2,R_0]} F(x,y,1) + \frac{1}{2} \sup_{|x|=|y|\geq R_0/2} F_K(x,y,1) \\ &\leq 6 + 3 \sup_{|x|=|y|\in[R_0/2,R_0]} F(x,y,1) + \frac{1}{2} \sup_{|x|=|y|\geq R_0} F_K(x,y,1). \end{aligned}$$

Taking the supremum over $x, y$ in $\{|x| = |y| \geq R_0\}$ of the left hand side, we obtain

$$\sup_{|x|,|y|\geq R_0/2} F_K(x,y,1) \leq 12 + 6 \sup_{|x|=|y|\in[R_0/2,R_0]} F(x,y,1).$$

Letting $K \to \infty$ gives a bound for $F(x,y,1)$ on the set $\{|x| = |y| \geq R_0/2\}$. Together with (2.5), this completes the proof of the main estimate.

**Remarks**

**1.** The moment $E^{(d)}_{a,b,t}\left[\exp(\eta \int_0^t R_s^{-2} ds)\right]$ is infinite for $\eta > (d-2)^2/8$. This follows since the formula (2.2) cannot be analytically extended, as a function of $\eta$, into the region $\{z : Re(z) < r\}$ for any $r > (d-2)^2/8$. This strongly suggests there are no solutions to (1.1) having finite second moments $E[(u_t(f))^2]$ when $\kappa > (d-2)/2$. Similarly, the blow-up of the Brownian exponential moment suggests there should be no solutions to (1.1) with finite second moments for any $\kappa > 0$ when the noise has covariance (1.2) with $p > 2$.

**2.** As indicated in Subsection 1.4, higher moments are controlled by the Brownian exponential moments (1.13). Using Hölder's inequality we find

$$\begin{aligned} &E^{t,y_1,\ldots,y_n}_{0,x_1,\ldots,x_n}\left[\exp\left(\sum_{1\leq j<k\leq n} \int_0^t \frac{\kappa^2}{\left|X^{(j)}_s - X^{(k)}_s\right|^2} ds\right)\right] \\ &\leq \prod_{1\leq j<k\leq n} \left(E^{t,y_j,y_k}_{0,x_j,x_k}\left[\exp\left(\int_0^t \frac{n(n-1)\kappa^2}{\left|X^{(j)}_s - X^{(k)}_s\right|^2} ds\right)\right]\right)^{1/n(n-1)}. \end{aligned}$$

The exponential moment calculated in this section shows that this is finite when $n(n-1)\kappa^2/2 \leq (d-2)^2/8$. This should lead to the solutions to (1.1) having finite moments $E[(u_t(f))^n]$ when $\kappa \leq (d-2)(4n(n-1))^{-1/2}$. We do not think this simple Hölder argument leads to the correct critical values for the existence of higher moments.

# 3 Existence of Solutions

In this section we give a construction of solutions to (1.1) using the chaos expansion (1.15). However, it is hard to show from the series expansion that the resulting solution is a non-negative measure. For that purpose, we give a second construction as a limit of less singular SPDEs. A comparison theorem will show that the approximating equations have solutions which are non-negative functions implying that the limit must also be non-negative. Finally, we show that the two constructions yield the same process and that it is a solution of (1.1).

We first construct a noise $F$ with the desired covariance. Let $g(x) = c_7|x|^{-(d+2)/2}$. A simple calculation shows, for a suitable value of the constant $c_7$, that the convolution $g * g(z) = |z|^{-2}$. Now let $W$ be an adapted space-time white noise on $\mathbf{R}^d \times [0,\infty)$ on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Define, for $f : \mathbf{R}^d \to \mathbf{R}$ that is bounded, measurable and of compact support,

$$F(t,f) = \int_0^t \int (f * g)(z) W(dz, ds). \tag{3.1}$$

It is straightforward to show that $F(t,f)$ is well defined, is a Gaussian martingale, and that

$$\langle F(\cdot, f)\rangle_t = t \int\int \frac{f(y)f(z)}{|y-z|^2} dy\, dz.$$

If we write $F(t,A)$ when $f = I_A$, then $\{F(t,A) : t \geq 0, A \subseteq \mathbf{R}^d\}$ is a martingale measure. Hence (see [Wal86] Chapter 2) we can define a stochastic integral $\int_0^t \int h(s,y)F(dy,ds)$ for suitable predictable integrands $h$ so that

$$\left[\int_0^\cdot \int h(s,y)F(dy,ds)\right]_t = \int_0^t \int\int \frac{h(s,y)h(s,z)}{|y-z|^2} dy\, dz\, ds.$$

Next, we show that the expansion (1.15) converges.

**Lemma 5** *Suppose $\mu \in \mathcal{H}_\alpha$. Then, for $f \in \mathcal{C}_c$, the series $\sum_{n=0}^\infty I_t^{(n)}(f,\mu)$, defined by (1.16) and (1.17), converges in $L^2$. Moreover*

$$\begin{aligned} E\left[\left(\sum_{n=0}^\infty I_t^{(n)}(f,\mu)\right)^2\right] &= \sum_{n=0}^\infty E\left[\left(I_t^{(n)}(f,\mu)\right)^2\right] \\ &= \int_{\mathbf{R}^{4d}} f(y')f(x')G_t(x-x')G_t(y-y') \\ &\qquad \cdot E_{0,x,y}^{t,x',y'}\left[\exp\left(\int_0^t \frac{\kappa^2}{|X_s^1 - X_s^2|^2} ds\right)\right] dx'dy'\mu(dx)\mu(dy). \end{aligned} \tag{3.2}$$

**Proof.** We first check that the right hand side of (3.2) is finite. Using the fact that $(X_t^1 - X_t^2)/\sqrt{2}$ is a Brownian bridge from $x-y$ to $x'-y'$, we may use Lemma 3 to obtain

$$\begin{aligned} &\int_{\mathbf{R}^{4d}} f(y')f(x')G_t(x-x')G_t(y-y')E_{0,x,y}^{t,x',y'}\left[\exp\left(\int_0^t \frac{\kappa^2}{|X_s^1 - X_s^2|^2} ds\right)\right] \mu(dx)\mu(dy)dx'dy' \\ &\leq C(t)\int_{\mathbf{R}^{4d}} f(y')f(x')G_t(x-x')G_t(y-y')\left(1 + |x-y|^{-\alpha}|x'-y'|^{-\alpha}\right)\mu(dx)\mu(dy)dx'dy'. \end{aligned}$$

Now, estimates similar to those in Lemma 1 show this expression is finite.

The multiple Wiener integrals of different orders are orthogonal, if they have finite second moments; that is if $m \neq n$, and if

$$E\left[\left(I_t^{(k)}(f,\mu)\right)^2\right] < \infty \tag{3.3}$$

for $k = m, n$, then

$$E\left[I_t^{(m)}(f,\mu) I_t^{(n)}(f,\mu)\right] = 0.$$

It is therefore enough to establish the second equality in (3.2), since this implies (3.3), and then orthogonality of the terms in the series implies the first equality in (3.2). First note that, with $s_{n+1} = t$,

$$\begin{aligned} E\left[\left(I_t^{(n)}(f)\right)^2\right] &= \kappa^{2n} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \int_{\mathbf{R}^{2(n+1)d}} f(y_{n+1}) f(z_{n+1}) dy_{n+1} dz_{n+1} G_{s_1}\mu(y_1) G_{s_1}\mu(z_1) \\ &\quad \cdot \prod_{i=1}^n \left[G_{s_{i+1}-s_i}(y_{i+1}-y_i) G_{s_{i+1}-s_i}(z_{i+1}-z_i) |y_i - z_i|^{-2} dy_i dz_i ds_i\right]. \end{aligned} \tag{3.4}$$

Expanding the exponential in the final term of (3.2), we have

$$\begin{aligned} E_{0,x,y}^{t,x',y'}\left[\exp\left(\int_0^t \frac{\kappa^2}{|X_s^1 - X_s^2|^2} ds\right)\right] &= 1 + E_{0,x,y}^{t,x',y'}\left[\sum_{n=1}^\infty \frac{1}{n!} \int_0^t \cdots \int_0^t \prod_{i=1}^n \frac{\kappa^2 ds_i}{\left|X_{s_i}^1 - X_{s_i}^2\right|^2}\right] \\ &= 1 + \sum_{n=1}^\infty \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} E_{0,x,y}^{t,x',y'}\left[\prod_{i=1}^n \frac{\kappa^2 ds_i}{\left|X_{s_i}^1 - X_{s_i}^2\right|^2}\right]. \end{aligned}$$

In the final sum, if $n = 1$, then only the integral $\int_0^t$ is present. Substituting this sum into the right hand side of (3.2), one may match, by using the finite dimensional distributions of the Brownian bridge, the nth term with the expression $E[I_t^{(n)}(f)]^2$ in (3.4). ∎

The chaos expansion defines a linear random functional on test functions (in that there is a possible null set for each linear relation). Also this linear random functional satisfies the moment bounds (1.6) and (1.7). The second moment bound (1.7) implies that there is a regularization (see [Ito84] Theorem 2.3.3), ensuring there is a random distribution $u_t$ so that

$$u_t(f) = \sum_{i=0}^\infty I_t^{(n)}(f,\mu) \quad \text{for all } f \in \mathcal{C}_c, \text{ almost surely.} \tag{3.5}$$

To show that $u_t$ is actually a random measure, we now construct a sequence of SPDE approximations to (1.1). We will index our approximations by numbers $\varepsilon > 0$. Recall that $h(x) := |x|^{-2} = (g * g)(x)$, where $g(x) = c_7 |x|^{-(d+2)/2}$. Let

$$g^{(\varepsilon)}(x) = \left(c_7 |x|^{-(d+2)/2}\right) \wedge \varepsilon^{-1}, \quad \text{and} \quad h^{(\varepsilon)}(x) = \left(g^{(\varepsilon)} * g^{(\varepsilon)}\right)(x).$$

As $\varepsilon \downarrow 0$ we have $g^{(\varepsilon)}(x) \uparrow g(x)$ and $h^{(\varepsilon)}(x) \uparrow h(x)$. We can construct, as in (3.1), a mean zero Gaussian field $F^{(\varepsilon)}(t,x)$ with covariance

$$E\left[\dot{F}^{(\varepsilon)}(t,x) \dot{F}^{(\varepsilon)}(s,y)\right] = \delta(t-s) h^{(\varepsilon)}(x-y).$$

We consider the approximating SPDE

$$\frac{\partial u^{(\varepsilon)}}{\partial t} = \frac{1}{2}\Delta u^{(\varepsilon)} + \kappa u^{(\varepsilon)}\dot{F}^{(\varepsilon)}, \quad u_0^{(\varepsilon)} = \mu^{(\delta)}, \tag{3.6}$$

with the initial condition $\mu^{(\delta)} = G_\delta \mu$, for some $\delta = \delta(\varepsilon) > 0$ to be chosen later. Since the correlation is continuous in $x$ and $y$, standard results give existence and uniqueness of a non-negative, continuous, function-valued solution $u_t^{(\varepsilon)}(t, x)$. Moreover we may represent the solutions in terms of a chaos expansion

$$u_t^{(\varepsilon)}(f) = \sum_{n=0}^{\infty} I_t^{(n,\varepsilon)}(f, \mu^{(\delta)})$$

where the terms $I_t^{(n,\varepsilon)}(f, \mu^{(\delta)})$ are defined as in (1.16) and (1.17) except that $\mu, F$ are replaced by $\mu^{(\delta)}, F^{(\varepsilon)}$. We now connect the approximations with the original series construction.

**Lemma 6** *Suppose that $\mu \in \mathcal{H}_{\alpha+}$. Then we may define $I_t^{(n)}(f, \mu)$ and $I_t^{(n,\varepsilon)}(f, \mu^{(\delta)})$ on the same probability space so that, for suitably chosen $\delta(\varepsilon) > 0$, fixed $t \geq 0$ and $f \in \mathcal{C}_c$,*

$$u_t^{(\varepsilon)}(f) \to u_t(f) \quad \text{in } L^2 \text{ as } \varepsilon \to 0.$$

*Hence the chaos expansion (3.5) defines a random measure $u_t(dx)$, for each $\mu \in \mathcal{H}_{\alpha+}$ and $t \geq 0$.*

**Proof.** Let $\dot{W}(t, x)$ be a space-time white noise on $[0, \infty) \times \mathbf{R}^d$, and construct both the noises $F$ and $F^{(\varepsilon)}$ using $W$ as in (3.1). Using the convergence of both the $F$ and $F^{(\varepsilon)}$ chaos expansions and the orthogonality of multiple Wiener integrals of different orders, we find

$$\begin{aligned} & E\left[\left(u_t^{(\varepsilon)}(f) - u_t(f)\right)^2\right] \\ = & \sum_{n=0}^{\infty} E\left[\left(I_t^{(n)}(f,\mu) - I_t^{(n,\varepsilon)}(f,\mu^{(\delta)})\right)^2\right] \\ \leq & \; 2\sum_{n=0}^{\infty} E\left[\left(I_t^{(n)}(f,\mu) - I_t^{(n,\varepsilon)}(f,\mu)\right)^2\right] + 2\sum_{n=0}^{\infty} E\left[\left(I_t^{(n,\varepsilon)}(f,\mu) - I_t^{(n,\varepsilon)}(f,\mu^{(\delta)})\right)^2\right]. \end{aligned} \tag{3.7}$$

We show separately that both sums on the right hand side of (3.7) converge to zero as $\varepsilon \downarrow 0$. We use the telescoping expansion, for $n \geq 1$,

$$\begin{aligned} & I_t^{(n)}(f,\mu) - I_t^{(n,\varepsilon)}(f,\mu) \\ = & \; \kappa^n \int_0^t \cdots \int_0^{s_2} \int_{\mathbf{R}^{(n+1)d}} G_{s_1}\mu(y_1) f(y_{n+1}) dy_{n+1} \prod_{i=1}^n \left[G_{s_{i+1}-s_i}(y_{i+1}-y_i) F(dy_i, ds_i)\right] \\ & -\kappa^n \int_0^t \cdots \int_0^{s_2} \int_{\mathbf{R}^{(n+1)d}} G_{s_1}\mu(y_1) f(y_{n+1}) dy_{n+1} \prod_{i=1}^n \left[G_{s_{i+1}-s_i}(y_{i+1}-y_i) F^{(\varepsilon)}(dy_i, ds_i)\right] \\ = & \sum_{m=1}^n J_t^{(n,m,\varepsilon)}(f,\mu) \end{aligned}$$

where $J_t^{(n,m,\varepsilon)}(f)$ is defined to equal

$$\kappa^n \int_0^t \cdots \int_0^{s_2} \int_{\mathbf{R}^{(n+1)d}} G_{s_1}\mu(y_1) f(y_{n+1}) dy_{n+1} \prod_{i=1}^n G_{s_{i+1}-s_i}(y_{i+1}-y_i)$$
$$\cdot \prod_{i=1}^{m-1} F(dy_i, ds_i) \cdot \left[F(dy_m, ds_m) - F^{(\varepsilon)}(dy_m, ds_m)\right] \cdot \prod_{i=m+1}^{n} F^{(\varepsilon)}(dy_i, ds_i)$$

and where a product over the empty set is defined to be 1. The isometry for the stochastic integral gives

$$\begin{aligned} & E\left[\left(J_t^{(n,m,\varepsilon)}(f,\mu)\right)^2\right] \\ &= \kappa^{2n} \int_0^t \cdots \int_0^{s_2} \int_{\mathbf{R}^{2(n+1)d}} G_{s_1}\mu(y_1) G_{s_1}\mu(z_0) f(y_{n+1}) f(z_{n+1}) dy_{n+1} dz_{n+1} \\ &\quad \cdot \prod_{i=1}^n \left[G_{s_{i+1}-s_i}(y_{i+1}-y_i) G_{s_{i+1}-s_i}(z_{i+1}-z_i) dy_i dz_i ds_i\right] \\ &\quad\quad \cdot \prod_{i=1}^{m-1} h(y_i - z_i) \left[\left(g - g^{(\varepsilon)}\right) * \left(g - g^{(\varepsilon)}\right)\right](y_m - z_m) \prod_{i=m+1}^{n} h^{(\varepsilon)}(y_i - z_i). \end{aligned}$$

Note that $0 \leq [(g - g^{(\varepsilon)}) * (g - g^{(\varepsilon)})](x) \leq h(x)$, and that $[(g - g^{(\varepsilon)}) * (g - g^{(\varepsilon)})](x) \downarrow 0$ as $\varepsilon \to 0$. Using the finiteness of $E\left[(I_t^{(n)}(f,\mu))^2\right]$, the dominated convergence theorem implies that $E\left[\left(J_t^{(n,m,\varepsilon)}(f,\mu)\right)^2\right] \downarrow 0$, and therefore

$$\lim_{\varepsilon \downarrow 0} E\left[\left(I_t^{(n,\varepsilon)}(f,\mu) - I_t^{(n)}(f,\mu)\right)^2\right] = 0. \tag{3.8}$$

The $L^2$ isometry, and the fact that $h^{(\varepsilon)}(x) \leq h(x)$, imply that

$$\begin{aligned} E\left[\left(I_t^{(n)}(f,\mu) - I_t^{(n,\varepsilon)}(f,\mu)\right)^2\right] &\leq 2E\left[\left(I_t^{(n)}(f,\mu)\right)^2\right] + 2E\left[\left(I_t^{(n,\varepsilon)}(f,\mu)\right)^2\right] \\ &\leq 4E\left[\left(I_t^{(n)}(f,\mu)\right)^2\right]. \end{aligned} \tag{3.9}$$

Using (3.8), (3.9), the convergence of the series $\sum_{n=0}^{\infty} E\left[\left(I_t^{(n)}(f,\mu)\right)^2\right]$, and the dominated convergence theorem, we deduce that the first term on the right hand side of (3.7) goes to zero as $\varepsilon \downarrow 0$.

We now show that for fixed $\varepsilon > 0$, the second term on the right hand side of (3.7) converges to zero as $\delta \downarrow 0$. Recall that the initial condition was $\mu^{(\delta)} = G_\delta \mu$ for some $\delta = \delta(\varepsilon) > 0$. But for fixed $\varepsilon$, the $L^2$ isometry shows, as in Lemma 5, that

$$\sum_{n=0}^{\infty} E\left[\left(I_t^{(n,\varepsilon)}(f,\mu) - I_t^{(n,\varepsilon)}(f,\mu^{(\delta)})\right)^2\right]$$

$$= \int_{\mathbf{R}^{4d}} f(y')f(x')G_t(x-x')G_t(y-y')$$
$$\cdot E_{0,x,y}^{t,x',y'}\left[\exp\left(\int_0^t \kappa^2 h^{(\varepsilon)}(X_s^1 - X_s^2)ds\right)\right] dx'dy' \left(\mu - \mu^{(\delta)}\right)(dx)\left(\mu - \mu^{(\delta)}\right)(dy).$$

When $\varepsilon > 0$, the Brownian bridge expectation is a bounded continuous function of $x, y, x', y'$ and the convergence to zero as $\delta \downarrow 0$ is clear. This completes the proof of the $L^2$ convergence stated in the lemma.

The $L^2$ boundedness of $u_t^{(\varepsilon)}(f)$, for each $f \in \mathcal{C}_c$, implies that $\{u_t^{(\varepsilon)}(x)dx\}$ is a tight family of random Radon measures. The $L^2$ convergence of $u_t^{(\varepsilon)}(f)$ implies that there is a random measure $u_t$ satisfying (3.5) and that $u_t^{(\varepsilon)} \to u_t$ in distribution as $\varepsilon \to 0$. ∎

It remains to show that $\{u_t(dx)\}$ is a solution of (1.1), and for this we must show that there is a continuous version of the process $t \to u_t$ and that it satisfies the martingale problem (1.4) and (1.5). Fix $f \in \mathcal{C}_c^2$. From the definition (1.17) we have, for $n \geq 1$,

$$I_s^{(n)}(y,z) = \kappa \int_0^s \int G_{s-r}(y-y')I_r^{(n-1)}(y',z)F(dy',dr).$$

Then using a stochastic Fubini theorem (see [Wal86] Theorem 2.6), and the fact that $G_t * f(y)$ solves the heat equation, we have, for $n \geq 1$,

$$\begin{aligned}
\int_0^t I_s^{(n)}(\frac{1}{2}\Delta f, \mu)ds &= \frac{1}{2}\int_0^t \int\int I_s^{(n)}(y,z)\Delta f(y)\mu(dz)dyds \\
&= \frac{\kappa}{2}\int_0^t \int\int \left(\int_0^s \int G_{s-r}(y-y')I_r^{(n-1)}(y',z)F(dy',dr)\right)\Delta f(y)\mu(dz)dyds \\
&= \frac{\kappa}{2}\int_0^t \int \left(\int_r^t G_{s-r} * \Delta f(y')ds\right)\int I_r^{(n-1)}(y',z)\mu(dz)F(dy',dr) \\
&= \kappa \int_0^t \int (G_{t-r} * f(y') - f(y'))\int I_r^{(n-1)}(y',z)\mu(dz)F(dy',dr) \\
&= I_t^{(n)}(f,\mu) - \kappa\int_0^t \int\int f(y')I_r^{(n-1)}(y',z)\mu(dz)F(dy',dr).
\end{aligned}$$

Rearranging the terms, we see that for each $n \geq 1$, the process

$$z_t^{(n)}(f) := I_t^{(n)}(f,\mu) - \int_0^t I_s^{(n)}(\frac{1}{2}\Delta f,\mu)ds = \kappa\int_0^t \int\int f(y)I_s^{(n-1)}(y,z)\mu(dz)F(dy,ds) \quad (3.10)$$

is a continuous martingale. We now define

$$u_{N,t}(x) = \sum_{k=0}^N I_t^{(k)}(x,x')\mu(dx'), \quad z_{N,t}(f) = \sum_{k=1}^N z_t^{(k)}(f).$$

Then, for $f \in \mathcal{C}_c^2$ and $N \geq 1$,

$$u_{N,t}(f) = \mu(f) + \int_0^t u_{N,s}(\frac{1}{2}\Delta f)ds + z_{N,t}(f). \quad (3.11)$$

Lemma 5 implies that $E[(u_{N,t}(\Delta f) - u_t(\Delta f))^2]$ converges monotonically to zero. Using the domination from Lemma 1, part i), we have

$$E\left[\sup_{t\leq T}\left|\int_0^t u_{N,s}(\frac{1}{2}\Delta f)ds - \int_0^t u_s(\frac{1}{2}\Delta f)ds\right|^2\right] \to 0.$$

Lemma 5 also implies that $z_{N,t}(f)$ converges in $L^2$ to $z_t(f)$. By Doob's inequality,

$$E\left[\sup_{t\leq T}|z_{N,t}(f) - z_t(f)|^2\right] \to 0.$$

This uniform convergence, and (3.11), show that there is a continuous version of both $t \to z_t(f)$ and $t \to u_t(f)$. Using this fact for a suitable countable class of $\mathcal{C}_c^2$ test functions $f$, shows that there is a continuous version (in the vague topology) of $t \to u_t$.

Now we calculate the quadratic variation $z_t(f)$, which is the $L^1$ limit of $\langle z_{N,\cdot}(f)\rangle_t$. It is enough to consider the case $f \geq 0$. Using (3.10) we have

$$\begin{aligned}
\langle z_{N+1,\cdot}(f)\rangle_t &= \sum_{k=1}^{N+1}\sum_{l=1}^{N+1}\kappa^2\int_0^t\int_{\mathbf{R}^{4d}}\frac{f(x)f(y)}{|x-y|^2}I_s^{(k-1)}(x,x')I_s^{(l-1)}(y,y')\mu(dx')\mu(dy')dx\,dy\,ds \\
&= \kappa^2\int_0^t\int_{\mathbf{R}^{2d}}\frac{f(x)f(y)}{|x-y|^2}u_{N,s}(x)u_{N,s}(y)dx\,dy\,ds \\
&\to \kappa^2\int_0^t\int_{\mathbf{R}^{2d}}\frac{f(x)f(y)}{|x-y|^2}u_s(dx)u_s(dy)ds. \qquad (3.12)
\end{aligned}$$

We need to justify this final convergence. To do so, we split the difference between the second and third lines of (3.12) into two terms

$$\begin{aligned}
&\int_0^t\int_{\mathbf{R}^{2d}}\frac{f(x)f(y)}{|x-y|^2}u_{N,s}(x)dx\,(u_{N,s}(y)dy - u_s(dy))\,ds \\
&\quad + \int_0^t\int_{\mathbf{R}^{2d}}\frac{f(x)f(y)}{|x-y|^2}(u_{N,s}(x)dx - u_s(dx))u_s(dy)ds.
\end{aligned}$$

We show that the first term converges to zero in $L^1$; the argument for the second term is the same. We use the fact that $|x-y|^{-2}$ is a convolution of $c_7|z|^{-(d+2)/2}$ with itself to see that

$$\begin{aligned}
&\int_{\mathbf{R}^{2d}}\frac{f(x)f(y)}{|x-y|^2}u_{N,s}(x)dx\,(u_{N,s}(y)dy - u_s(dy)) \\
&= c_7\int_{\mathbf{R}^{3d}}\frac{f(x)f(y)}{|x-z|^{(d+2)/2}|y-z|^{(d+2)/2}}u_{N,s}(x)dx(u_{N,s}(y)dy - u_s(dy))dz \\
&= c_7\int_{\mathbf{R}^d}u_{N,s}(f_z)(u_{N,s}(f_z) - u_s(f_z))dz
\end{aligned}$$

where $f_z(x) = f(x)|x-z|^{-(d+2)/2}$. Hence, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&E\left[\left|\int_0^t\int_{\mathbf{R}^{2d}}\frac{f(x)f(y)}{|x-y|^2}u_{N,s}(x)dx(u_{N,s}(y)dy - u_s(dy))\,ds\right|\right] \qquad (3.13) \\
&\leq c_7\left(\int_0^t\int E\left[(u_{N,s}(f_z))^2\right]dzds\right)^{1/2}\left(\int_0^t\int E\left[(u_{N,s}(f_z) - u_s(f_z))^2\right]dzds\right)^{1/2}
\end{aligned}$$

The argument from Lemma 5 shows that $E\left[(u_{N,s}(f_z))^2\right]$ can be bounded, uniformly in $N$, by

$$E\left[(u_{N,s}(f_z))^2\right] \leq \int_{\mathbf{R}^{4d}} f_z(x')f_z(y')G_s(x-x')G_s(y-y')\left(1+\frac{s^\alpha}{|x-y|^\alpha|x'-y'|^\alpha}\right)\mu(dx)\mu(dy)dx'dy'.$$

The same bound holds for $E\left[(u_s(f_z))^2\right]$. It is straightforward but lengthy to estimate this term. We show how to deal with the most singular term only. The method is to estimate the $dx'dy'$ integral first, using the inequalities (1.9) and (1.10). Applying Hölder's inequality in the same way as in Lemma 1, these inequalities imply that

$$\begin{aligned}
&\int_{\mathbf{R}^{2d}} G_s(x-x')G_s(y-y')\frac{f(x')f(y')}{|x'-z|^{(d+2)/2}|y'-z|^{(d+2)/2}|x'-y'|^\alpha}dx'dy' \\
&\leq \; C(a)e^{-a|x|-a|y|}s^{-\alpha/2}\left(|x-z|^{-(d+2)/2}\wedge s^{-(d+2)/4}\right)\left(|y-z|^{-(d+2)/2}\wedge s^{-(d+2)/4}\right) \\
&\leq \; C(\beta,a)e^{-a|x|-a|y|}s^{-1-\alpha+(\beta/2)}|x-z|^{-(d+\beta-\alpha)/2}|y-z|^{-(d+\beta-\alpha)/2}.
\end{aligned}$$

where we have chosen $\beta \in (\alpha, \alpha+2)$ and $a$ so that $\mu \in \mathcal{H}^a_\beta$. To apply Holder's inequality here, splitting the three factors $f(x')f(y')$, $|x'-z|^{-(d+2)/2}|y'-z|^{-(d+2)/2}$ and $|x'-y'|^\alpha$, we need the bound $\alpha+((d+2)/2) < d$, which is implied by our assumption that $\alpha < (d-2)/2$. Substituting this estimate into (3.13), we find

$$\begin{aligned}
&\int_0^t \int E\left[(u_{N,s}(f_z))^2\right]dzds \\
&\leq \; C(\beta,a)\int_0^t\int_{\mathbf{R}^{3d}} e^{-a|x|-a|y|}s^{-1+(\beta/2)}|x-y|^{-\alpha} \\
&\qquad \cdot|x-z|^{-(d+\beta-\alpha)/2}|y-z|^{-(d+\beta-\alpha)/2}\mu(dx)\mu(dy)dzds \\
&= \; C(\beta,a)\int_0^t\int_{\mathbf{R}^{2d}} e^{-a|x|-a|y|}s^{-1+(\beta/2)}|x-y|^{-\beta}\mu(dx)\mu(dy)dzds
\end{aligned}$$

which is finite since $\mu \in \mathcal{H}^a_\beta$. This bound also gives the domination required to see that $\int_0^t\int E\left[(u_{N,s}(f_z)-u_s(f_z))^2\right]dzds \to 0$ as $N \to \infty$. This finishes the justification of the convergence in (3.12), identifying the quadratic variation $\langle z.(f)\rangle_t$, and completes the construction of a solution $\{u_t(dx)\}$ to (1.1) started at $\mu$.

## 4 Self Duality and Uniqueness

In this section we establish the self duality of solutions in the following form:

**Proposition 1** *Suppose $\{u_t(dx)\}$ and $\{v_t(dx)\}$ are solutions of (1.1), with deterministic initial conditions $u_0(dx) = f(x)dx$ and $v_0(dx) = g(x)dx$. Suppose also that $sup_x e^{-a|x|}f(x) < \infty$ for some $a$ and that $g(x)$ is bounded and has compact support. Then $u_t(g)$ has the same distribution as $v_t(f)$.*

**Remarks**

**1**. The duality formula is immediately clear for the solutions constructed using the chaos expansion in Section, 3 since the expression (1.17) for the nth order of the expansion is symmetric under the interchange of $y$ and $z$. We will show in this section that the self duality relation holds for any solution to (1.1). We then use the self duality relation to show uniqueness in law for solutions.

**2**. Even when working with the martingale problem, the self duality relation is heuristically clear, as can be seen by applying the technique of Markov process duality (see Ethier and Kurtz [EK86] chapter 4). Take $\{u_t(dx)\}$ and $\{v_t(dx)\}$ to be independent solutions to (1.1). Suppose (falsely) that the solutions are function valued and have suitable behavior at infinity such that the integrals $u_s(v_{t-s})$ and $v_{t-s}(u_s)$ are finite and equal by integration by parts. Take a twice differentiable $h : [0, \infty) \to \mathbf{R}$. Applying Ito's formula formally, using the martingale problem (1.4), leads to

$$\frac{d}{ds} h(u_s(v_{t-s})) = (1/2) h'(u_s(v_{t-s}))\,(u_s(\Delta v_{t-s}) - v_{t-s}(\Delta u_s)) + \text{martingale terms.}$$

Here we have used the cancellation of the two second derivative terms involving $h''$ after applying Ito's formula for $u_s$ and for $v_{t-s}$. Applying integration by parts, the term $(u_s(\Delta v_{t-s}) - v_{t-s}(\Delta u_s))$ vanishes, and this leaves only martingale terms. Taking expectations and integrating over $s \in [0, t]$ leads to

$$E\,[h(u_t(g))] = E\,[h(v_t(f))] \tag{4.1}$$

which implies the self duality. To make this argument rigorous, we shall use a smoother approximate duality relation.

**3**. The self duality relation can be extended to hold for more general initial conditions, and to be symmetric in the requirements on the initial conditions $\mu$ and $\nu$. This would be expected by the symmetry of the chaos expansion. One needs to define certain collision integrals $(\mu, \nu)$ between measures in $\mathcal{H}_{\alpha+}$. For example, suppose $\mu, \nu \in \mathcal{H}_{\alpha+}$, and for simplicity suppose that $\mu, \nu$ are supported in the ball $B(0, R)$. Define $f_\varepsilon(x) = \int \phi_\varepsilon(x-y)\nu(dy)$, so that $f_\varepsilon$ is the density of the measure $\phi_\varepsilon * \nu$. Then, if $\{u_t(dx)\}$ is a solution started at $\mu$, we claim that the random variables

$$u_t(f_\varepsilon) = \int\!\!\int \phi_\varepsilon(x-y) u_t(dx) \nu(dy), \qquad \varepsilon > 0$$

form a Cauchy sequence in $L^2$ as $\varepsilon \to 0$. Indeed, using the second moment formula (1.7), a short calculation leads to

$$\begin{aligned}
&E\left[(u_t(f_\varepsilon) - u_t(f_{\varepsilon'}))^2\right] \\
&\quad = E\left[(u_t(f_\varepsilon - f_{\varepsilon'}))^2\right] \\
&\quad \le C(t, R, \mu) \int\!\!\int (f_\varepsilon(x) - f_{\varepsilon'}(x))(f_\varepsilon(y) - f_{\varepsilon'}(y))(1 + |x-y|^{-\alpha})\,dx\,dy \\
&\quad = C(t, R, \mu) \|\phi_\varepsilon * \nu - \phi_{\varepsilon'} * \nu\|_\alpha^2.
\end{aligned}$$

Here we are extending the use of the norm $\|\mu\|_\alpha$ to signed measures. Now it is not difficult to show that $\|\phi_\varepsilon * \nu - \nu\|_\alpha \to 0$ as $\varepsilon \to 0$, which completes the proof of the Cauchy property. Denote the $L^2$ limit as $u_t(\nu)$, and construct $v_t(\mu)$ analogously. Then the duality relation holds in this extended setting when $\mu, \nu \in \mathcal{H}^0_{\alpha+}$, although we make no use of it in this paper.

In the rest of this section we give the proof of Proposition 1, and deduce uniqueness in law and the Markov property. The proof follows from two lemmas. The first of these is an approximate duality relation, where we smooth the measure valued solutions.

**Lemma 7** *Suppose $\{u_t(dx)\}$ is a solution of (1.1) with initial condition $\mu$ and $\{v_t(dx)\}$ is an independent solution with a compactly supported initial condition $\nu$. Suppose $h : [0,\infty) \to \mathbf{R}$ has two bounded continuous derivatives and $\phi : \mathcal{R}^d \to [0,\infty)$ is continuous with compact support. Fix $0 < t_0 < t_1$ and a bounded $\sigma(u_s(dx) : 0 \le s \le t_0)$ variable $Z_{t_0}$. Then*

$$\begin{aligned} & E\left[Z_{t_0} h\left(\int\!\!\int \phi(x-y) u_{t_1}(dx)\nu(dy)\right)\right] - E\left[Z_{t_0} h\left(\int\!\!\int \phi(x-y) u_{t_0}(dx) v_{t_1-t_0}(dy)\right)\right] \\ = \; & \frac{\kappa^2}{2} E\left[Z_{t_0} \int_{t_0}^{t_1} \int_{\mathbf{R}^{4d}} h''\left(\int\!\!\int \phi(x-y) u_s(dx) v_{t_1-s}(dy)\right) \phi(x_1-y_1)\phi(x_2-y_2)\right. \\ & \left. \cdot \left(\frac{1}{|x_1-x_2|^2} - \frac{1}{|y_1-y_2|^2}\right) u_s(dx_1) v_{t_1-s}(dy_1) u_s(dx_2) v_{t_1-s}(dy_2) ds\right]. \qquad (4.2) \end{aligned}$$

**Proof.** We first establish that the expectation on the right hand side of (4.2) is finite. Using the independence of $\{u_t(dx)\}$ and $\{v_t(dx)\}$, the compact support of $\phi$ and the bound on second moments in (1.7), a lengthy but straightforward calculation, similar to that in Lemma 1, yields

$$\begin{aligned} & E\left[\int_{\mathbf{R}^{4d}} \phi(x_1-y_1)\phi(x_2-y_2) \left(\frac{1}{|x_1-x_2|^2} + \frac{1}{|y_1-y_2|^2}\right) u_s(dx_1) v_t(dy_1) u_s(dx_2) v_t(dy_2)\right] \\ \le \; & C(\phi,\mu,\nu,T)\left(s^{-(2-\alpha)_+/2} + t^{-(2-\alpha)_+/2}\right) \quad \text{for all } s,t \le T. \qquad (4.3) \end{aligned}$$

Furthermore, using the formula for first moments (1.6), an easy calculation shows that

$$E\left[\int\!\!\int \phi(x-y) u_s(dx) v_t(dy)\right] \le C(\phi,\mu,\nu,T) \quad \text{for all } 0 \le s,t \le T. \qquad (4.4)$$

We now follow the standard method of duality, as explained in Ethier and Kurtz [EK86] Section 4.4. Taking $f \in \mathcal{C}_c^2$, applying Ito's formula using the martingale problem for $u_t(f)$, and then taking expectations, we obtain, for $s \ge t_0$,

$$\begin{aligned} & E[Z_{t_0} h(u_s(f))] - E[Z_{t_0} h(u_{t_0}(f))] \\ = \; & \int_{t_0}^{s} E\left[Z_{t_0}\left(h'(u_r(f)) u_r(\tfrac{1}{2}\Delta f) + \frac{\kappa^2}{2} h''(u_r(f)) \int\!\!\int \frac{f(x_1)f(x_2)}{|x_1-x_2|^2} u_r(dx_1) u_r(dx_2)\right)\right] dr. \end{aligned}$$

Here Lemma 1 implies that the local martingale arising from Ito's formula is a true martingale. Now take $\psi : \mathbf{R}^{2d} \to \mathbf{R}$ to be twice continuously differentiable with compact support. Replace the deterministic function $f(x)$ by the random $\mathcal{C}_c^2$ function, independent of $\{u_t(dx)\}$, given by $f(x) = \int \psi(x,y) v_t(dy)$. Fubini's theorem and the integrability in (4.3) and (4.4) imply that, for $s \ge t_0$,

$$\begin{aligned} & E\left[Z_{t_0} h\left(\int\!\!\int \psi(x,y) u_s(dx) v_t(dy)\right)\right] - E\left[Z_{t_0} h\left(\int\!\!\int \psi(x,y) u_{t_0}(dx) v_t(dy)\right)\right] \\ = \; & \int_{t_0}^{s} E\left[Z_{t_0} h'\left(\int\!\!\int \psi(x,y) u_r(dx) v_t(dy)\right) \int\!\!\int \frac{1}{2}\Delta^{(x)} \psi(x,y) u_r(dx) v_t(dy)\right] dr \end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa^2}{2}\int_{t_0}^{s} E\left[Z_{t_0}h''\left(\int\int \psi(x,y)u_r(dx)v_t(dy)\right)\right.\\
&\qquad \left.\cdot\int_{\mathbf{R}^{4d}} \frac{\psi(x_1,y_1)\psi(x_2,y_2)}{|x_1-x_2|^2} u_r(dx_1)u_r(dx_2)v_t(dy_1)v_t(dy_2)\right] dr.
\end{aligned}
$$

In a similar way, applying Ito's formula to $v_t(f)$, we obtain the decomposition

$$
\begin{aligned}
&E\left[Z_{t_0}h\left(\int\int \psi(x,y)u_s(dx)v_t(dy)\right)\right] - E\left[Z_{t_0}h\left(\int\int \psi(x,y)u_s(dx)\nu(dy)\right)\right]\\
&= \int_0^t E\left[Z_{t_0}h'\left(\int\int \psi(x,y)u_s(dx)v_r(dy)\right)\int\int \frac{1}{2}\Delta^{(y)}\psi(x,y)u_s(dx)v_r(dy)\right] dr\\
&\quad +\frac{\kappa^2}{2}\int_0^t E\left[Z_{t_0}h''\left(\int\int \psi(x,y)u_s(dx)v_r(dy)\right)\right.\\
&\qquad \left.\cdot\int_{\mathbf{R}^{4d}} \frac{\psi(x_1,y_1)\psi(x_2,y_2)}{|y_1-y_2|^2} u_s(dx_1)u_s(dx_2)v_r(dy_1)v_r(dy_2)\right] dr.
\end{aligned}
$$

If we let

$$
F(s,t) = E\left[Z_{t_0}h\left(\int\int \psi(x,y)u_s(dx)v_t(dy)\right)\right]
$$

then the last two decompositions show that $s \to F(s,t)$ and $t \to F(s,t)$ are both absolutely continuous, and gives expressions for their derivatives $\partial_1 F(s,t)$ and $\partial_2 F(s,t)$. Then applying Lemma 4.4.10 from [EK86] we obtain

$$
\begin{aligned}
&E\left[Z_{t_0}h\left(\int\int \psi(x,y)u_{t_1}(dx)\nu(dy)\right)\right] - E\left[Z_{t_0}h\left(\int\int f(x,y)u_{t_0}(dx)v_{t_1-t_0}(dy)\right)\right]\\
&= F(t_1,0) - F(t_0,t_1-t_0)\\
&= \int_{t_1-t_0}^{t_1} \partial_1 F(s,t_1-s) - \partial_2 F(s,t_1-s)ds\\
&= \int_{t_0}^{t_1} E\left[Z_{t_0}h'\left(\int\int \psi(x,y)u_s(dx)v_{t_1-s}(dy)\right)\int\int \frac{1}{2}(\Delta^{(x)}-\Delta^{(y)})\psi(x,y)u_s(dx)v_{t_1-s}(dy)\right] ds\\
&\quad +\frac{\kappa^2}{2}\int_{t_0}^{t_1}\int_{\mathbf{R}^{4d}} E\left[Z_{t_0}h''\left(\int\int \psi(x,y)u_s(dx)v_{t_1-s}(dy)\right)\psi(x_1,y_1)\psi(x_2,y_2)\right.\\
&\qquad \left.\cdot\left(\frac{1}{|x_1-x_2|^2} - \frac{1}{|y_1-y_2|^2}\right)u_s(dx_1)v_{t_1-s}(dy_1)u_s(dx_2)v_{t_1-s}(dy_2)\right] ds. \qquad (4.5)
\end{aligned}
$$

Now suppose that $\phi : \mathbf{R}^d \to [0,\infty)$ is smooth and has compact support. Choose a series of smooth, compactly support functions $\psi_n(x,y)$ satisfying $0 \le \psi_n \uparrow 1$ as $n \to \infty$ and with $\partial_x\psi_n$, $\partial_y\psi_n$, $\partial_{xx}\psi_n$, $\partial_{yy}\psi_n$ converging uniformly to zero. Apply (4.5) to the function $\psi(x,y) = \psi_n(x,y)\phi(x-y)$. Using $(\Delta^{(x)}-\Delta^{(y)})\phi(x-y) = 0$ we may, using the integrability in (4.3) and (4.4), pass to the limit in (4.5) to get (4.2). Finally, we obtain the result for general continuous $\phi$ by taking smooth approximations. ∎

Now we take $\phi(x)$ a smooth, non-negative function on $\mathbf{R}^d$, supported on the unit ball $\{x \in \mathbf{R}^d : |x| \le 1\}$, and satisfying $\int_{\mathbf{R}^d}\phi(x)dx = 1$. Define an approximate identity by $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$. We may, and shall, suppose that $0 \le \phi(x) \le 2G_\varepsilon(x)$, and hence that $\phi_\varepsilon \le G_{\varepsilon^2}$. We shall use this test function along with Lemma 7. In order to do so, we need the following lemma, which controls the right hand side of (4.2).

**Lemma 8** *Suppose $\{u_t(dx)\}$ and $\{v_t(dx)\}$ are independent solutions of (1.1), with initial conditions $\mu, \nu$, where $\nu$ compactly supported. Then*

$$E\left[\int_0^t \int_{\mathbf{R}^{4d}} \phi_\varepsilon(x_1-y_1)\phi_\varepsilon(x_2-y_2)\left|\frac{1}{|x_1-x_2|^2}-\frac{1}{|y_1-y_2|^2}\right| u_s(dx_1)v_{t-s}(dy_1)u_s(dx_2)v_{t-s}(dy_2)ds\right]$$

*converges to zero as $\varepsilon \to 0$.*

**Proof** This lemma is a straightforward but lengthy consequence of the second moment bounds (1.7). Since it is this proof that requires the strict inequality $\kappa < (d-2)/2$ and also the requirement that $\mu, \nu \in \mathcal{H}_\beta$ for some $\beta > \alpha$, we give some of the details.

The second moment bounds (1.7) show that show that the expectation in the statement of the lemma is bounded by

$$\begin{aligned} C\int_0^t \int_{\mathbf{R}^{8d}} & \phi_\varepsilon(x_1'-y_1')\phi_\varepsilon(x_2'-y_2')G_s(x_1-x_1')G_s(x_2-x_2')G_{t-s}(y_1-y_1')G_{t-s}(y_2-y_2') \\ & \cdot\left(1+\frac{s^\alpha}{|x_1-x_2|^\alpha|x_1'-x_2'|^\alpha}\right)\left(1+\frac{(t-s)^\alpha}{|y_1-y_2|^\alpha|y_1'-y_2'|^\alpha}\right) \\ & \cdot\left|\frac{1}{|x_1'-x_2'|^2}-\frac{1}{|y_1'-y_2'|^2}\right|\mu(dx_1)\mu(dx_2)\nu(dy_1)\nu(dy_2)dx_1'dx_2'dy_1'dy_2'ds. \end{aligned} \tag{4.6}$$

The idea is to first bound the $dx_1'dx_2'dy_1'dy_2'$ integral. We can split the $dx_1'dx_2'dy_1'dy_2'$ integral into four terms by expanding the brackets

$$\left(1+\frac{s^\alpha}{|x_1-x_2|^\alpha|x_1'-x_2'|^\alpha}\right)\left(1+\frac{(t-s)^\alpha}{|y_1-y_2|^\alpha|y_1'-y_2'|^\alpha}\right).$$

We shall only show how to treat the worst of these terms, namely

$$\begin{aligned} \int_{\mathbf{R}^{4d}} & \phi_\varepsilon(x_1'-y_1')\phi_\varepsilon(x_2'-y_2')G_s(x_1-x_1')G_s(x_2-x_2')G_{t-s}(y_1-y_1')G_{t-s}(y_2-y_2') \\ & \cdot\left(\frac{s^\alpha(t-s)^\alpha}{|x_1-x_2|^\alpha|x_1'-x_2'|^\alpha|y_1-y_2|^\alpha|y_1'-y_2'|^\alpha}\right)\left|\frac{1}{|x_1'-x_2'|^2}-\frac{1}{|y_1'-y_2'|^2}\right|dx_1'dx_2'dy_1'dy_2'ds. \end{aligned} \tag{4.7}$$

This is the term that requires the restriction on $\kappa$. The other three terms are similar but easier.

We split the domain of integration in (4.7) into two regions. First we consider $x_1', y_1', x_2', y_2'$ lying in the set $A_\varepsilon = \{|x_1'-x_2'| \geq \varepsilon^\gamma, |y_1'-y_2'| \geq \varepsilon^\gamma\}$, where $\gamma \in (0,1)$ will be chosen later in the proof. On this set, we may restrict ourselves to the support of $\phi_\varepsilon$. That is, we may also suppose that $|x_1'-y_1'| \leq \varepsilon$ and $|x_2'-y_2'| \leq \varepsilon$. We have, arguing using the mean value theorem,

$$\left|\frac{1}{|x_1'-x_2'|^2}-\frac{1}{|y_1'-y_2'|^2}\right| \leq C(\gamma)\varepsilon^{1-3\gamma} \quad \text{for all } \varepsilon < 1/3.$$

Therefore, the integral (4.7), over the set $A_\varepsilon$, can be bounded by

$$C(\gamma)\varepsilon^{1-3\gamma-2\alpha\gamma}\frac{s^\alpha(t-s)^\alpha}{|x_1-x_2|^\alpha|y_1-y_2|^\alpha}G_{t+\varepsilon^2}(x_1-y_1)G_{t+\varepsilon^2}(x_2-y_2).$$

We shall choose $\gamma > 0$ so that $1 - 3\gamma - 2\alpha\gamma > 0$. It is easy to show that this bound, substituted into (4.6) will vanish as $\varepsilon \downarrow 0$. To estimate the integral (4.7) over the complimentary set $A_\varepsilon^c$, we simply bound

$$\left| \frac{1}{|x_1' - x_2'|^2} - \frac{1}{|y_1' - y_2'|^2} \right| \leq \frac{1}{|x_1' - x_2'|^2} + \frac{1}{|y_1' - y_2'|^2}$$

and it becomes

$$\begin{aligned}
& \frac{s^\alpha (t-s)^\alpha}{|x_1 - x_2|^\alpha |y_1 - y_2|^\alpha} \int_{A_\varepsilon^c} G_s(x_1 - x_1') G_s(x_2 - x_2') G_{t-s}(y_1 - y_1') G_{t-s}(y_2 - y_2') \\
& \qquad \cdot \phi_\varepsilon(x_1' - y_1') \phi_\varepsilon(x_2' - y_2') |x_1' - x_2'|^{-\alpha} |y_1' - y_2'|^{-\alpha} \left( |x_1' - x_2'|^{-2} + |y_1' - y_2'|^{-2} \right) dx_1' dx_2' dy_1' dy_2' \\
\leq \; & C \frac{s^\alpha (t-s)^\alpha}{|x_1 - x_2|^\alpha |y_1 - y_2|^\alpha} \int_{A_\varepsilon^c} G_s(x_1 - x_1') G_s(x_2 - x_2') G_{t-s}(y_1 - y_1') G_{t-s}(y_2 - y_2') \\
& \qquad \cdot \phi_\varepsilon(x_1' - y_1') \phi_\varepsilon(x_2' - y_2') \left( |x_1' - x_2'|^{-(2+2\alpha)} + |y_1' - y_2'|^{-(2+2\alpha)} \right) dx_1' dx_2' dy_1' dy_2'.
\end{aligned}$$

We only show how to treat the integral with the term $|x_1' - x_2'|^{-(2+2\alpha)}$, since the term $|y_1' - y_2'|^{-(2+2\alpha)}$ is entirely similar. Note that the restriction $\kappa < (d-2)/2$ is present simply to ensure that $2 + 2\alpha < d$, so the pole $|z|^{-(2+2\alpha)}$ is integrable on $\mathbf{R}^d$. We may choose $\delta \in (2 + 2\alpha, (2 + \alpha + \beta) \wedge d)$. Then, using the bound $\phi_\varepsilon \leq 2G_{\varepsilon^2}$, we can do the $dy_1' dy_2'$ integrals to see that

$$\begin{aligned}
& \int_{A_\varepsilon^c} G_s(x_1 - x_1') G_s(x_2 - x_2') G_{t-s}(y_1 - y_1') G_{t-s}(y_2 - y_2') \\
& \qquad \cdot \phi_\varepsilon(x_1' - y_1') \phi_\varepsilon(x_2' - y_2') |x_1' - x_2'|^{-(2+2\alpha)} dx_1' dx_2' dy_1' dy_2' \\
\leq \; & \int_{\{|x_1' - x_2'| \leq \varepsilon\}} G_s(x_1 - x_1') G_s(x_2 - x_2') \\
& \qquad \cdot G_{t-s+\varepsilon^2}(y_1 - x_1') G_{t-s+\varepsilon^2}(y_2 - x_2') |x_1' - x_2'|^{-(2+2\alpha)} dx_1' dx_2' \\
\leq \; & C \varepsilon^{(\delta - 2 - 2\alpha)\gamma} \int_{\mathbf{R}^{2d}} G_s(x_1 - x_1') G_s(x_2 - x_2') \\
& \qquad \cdot G_{t-s+\varepsilon^2}(y_1 - x_1') G_{t-s+\varepsilon^2}(y_2 - x_2') |x_1' - x_2'|^{-\delta} dx_1' dx_2'.
\end{aligned}$$

We now split into two cases: $s \leq t/2$ and $s \geq t/2$. When $s \leq t/2$ we have the bound

$$G_{t-s+\varepsilon^2}(y_1 - x_1') G_{t-s+\varepsilon^2}(y_2 - x_2') \leq C(a, \nu, t) \exp(-a|x_1'| - a|x_2'|), \quad \text{for } y_1, y_2 \in \mathrm{supp}(\nu).$$

So, when $s \leq t/2$,

$$\begin{aligned}
& \int_{\mathbf{R}^{2d}} G_s(x_1 - x_1') G_s(x_2 - x_2') G_{t-s+\varepsilon}(y_1 - x_1') G_{t-s+\varepsilon}(y_2 - x_2') |x_1' - x_2'|^{-\delta} dx_1' dx_2' \\
\leq \; & C(a, \nu, t) \int_{\mathbf{R}^{2d}} G_s(x_1 - x_1') G_s(x_2 - x_2') \exp(-a|x_1'| - a|x_2'|) |x_1' - x_2'|^{-\delta} dx_1' dx_2' \\
\leq \; & C(a, \nu, t) \exp(-a|x_1| - a|x_2|) \left( |x_1 - x_2|^{-\delta} \wedge s^{-\delta/2} \right) \\
\leq \; & C(a, \nu, t) \exp(-a|x_1| - a|x_2|) |x_1 - x_2|^{-\beta + \alpha} s^{-(\delta - \beta + \alpha)/2},
\end{aligned}$$

using the tricks from Lemma 1 for this last inequality. Combining all these bounds one has, when substituting the integral (4.7) over the region $A_\varepsilon^c$ into (4.6), and considering only the time interval $[0, t/2]$, the estimate

$$C(a, \nu, t) \varepsilon^{(\delta - 2 - 2\alpha)\gamma} \int_0^{t/2} \int_{\mathbf{R}^{4d}} \left( \frac{s^{-(\delta - \beta - \alpha)/2}}{|x_1 - x_2|^\beta |y_1 - y_2|^\alpha} \right) e^{-a|x_1| - a|x_2|} \mu(dx_1) \mu(dx_2) \nu(dy_1) \nu(dy_2) ds.$$

Choosing $a$ so that $e^{-a|x|}\mu(dx) \in \mathcal{H}^0_\beta$, the integral is finite. and so this expression vanishes as $\varepsilon \downarrow 0$. The integral over $[t/2, t]$ is treated in a similar way, using the assumption that $\nu \in \mathcal{H}_\beta$. ∎

To deduce Proposition 1 from Lemmas 7 and 8 is easy. By a simple approximation argument it is enough to prove (4.1) for $h$ with two bounded continuous derivatives. We apply the approximate duality relation (4.2), using $0 = t_0 < t_1 = t$ and $Z_{t_0} = 1$, to the function $\phi_\varepsilon$. Then take $\varepsilon \to 0$ and use the control on the error term in Lemma 8 to obtain the result.

We show two consequences of the duality relation and its proof.

**Corollary 1** *Solutions to (1.1) are unique in law and we let $Q_\mu$ denote the law of solutions started at $\mu \in \mathcal{H}_{\alpha+}$.*

**Proof** First suppose that $\{u_t(dx)\}$ and $\{v_t(dx)\}$ are two solutions with the same deterministic initial condition $\mu$. Construct a third solution $\{w_t(dx)\}$, independent of $\{u_t(dx)\}$ and $\{v_t(dx)\}$ and with initial condition $w_0(dx) = f(x)dx$ for some non-negative, continuous, compactly supported function $f$. Then apply the approximate duality relation (4.2), with $0 = t_0 < t_1 = t$ and $Z_{t_0} = 1$, to the pair $\{u_t(dx)\}$ and $\{w_t(dx)\}$ and to the pair $\{v_t(dx)\}$ and $\{w_t(dx)\}$, using the function $\phi_\varepsilon$. Subtracting the two approximate duality relations we see that

$$E\left[h\left(\int\int \phi_\varepsilon(x-y)u_t(dx)f(y)dy\right)\right] - E\left[h\left(\int\int \phi_\varepsilon(x-y)v_t(dx)f(y)dy\right)\right]$$

equals the sum of two error terms, both of which converge to zero as $\varepsilon \to 0$ by Lemma 8. Hence $E[h(u_t(f))] = E[h(v_t(f))]$ for all such $f$ and for all suitable $h$. Choosing $h(z) = \exp(-\lambda z)$ we obtain equality of the Laplace functionals of $u_t(dx)$ and $v_t(dx)$ and hence equality of the one dimensional distributions.

Now we use an induction argument to show that the finite dimensional distributions agree. Suppose the $n$-dimensional distributions have been shown to agree. Choose $0 \le s_1 < s_2 \ldots < s_{n+1}$ and set $t_1 = s_{n+1}, t_0 = s_n$. Then apply the approximate duality relation (4.2) to the pair $\{u_t(dx)\}$ and $\{w_t(dx)\}$ with $Z_{t_0} = \prod_{i=1}^n \exp(-u_{s_i}(f_i))$ for compactly supported $f_i \ge 0$. Also apply the approximate duality relation (4.2) to the pair $\{v_t(dx)\}$ and $\{w_t(dx)\}$ with $Z_{t_0} = \prod_{i=1}^n \exp(-v_{s_i}(f_i))$. We subtract the two approximate duality relations, use the equality of the $n$-dimensional distributions, and let $\varepsilon \downarrow 0$, to obtain equality of the $n+1$-dimensional distributions. This completes the induction. Since the processes have continuous paths, the finite dimensional distributions determine the law.

For general initial conditions $u_0$, we let $P_\mu$ be a regular conditional probability given that $u_0 = \mu$. It is not difficult to check that for almost all $\mu$ (with respect to the law of $u_0$) the process $\{u_t(dx)\}$ is a solution to (1.1) started at $\mu$ under $P_\mu$. (The moment conditions carry over under the regular conditional probability and these allow one to reduce to a countable family of test functions in the martingale problem). By the argument above the law of $\{u_t(dx)\}$ under the conditional probability $P_\mu$ is uniquely determined (for almost all $\mu$). This in turn determines the law of $\{u_t(dx)\}$. ∎

**Corollary 2** *For any bounded Borel measurable* $H : C([0,\infty),\mathcal{M}) \to \mathbf{R}$ *the map* $\mu \to Q_\mu[H]$, *the integral of* $H$ *with respect to* $Q_\mu$, *is measurable from* $\mathcal{H}_{\alpha+}$ *to* $\mathbf{R}$.

*The set of laws* $\{Q_\mu : \mu \in \mathcal{H}_{\alpha+}\}$ *forms a Markov family, in that for any solution* $\{u_t(dx)\}$ *to (1.1), for any bounded measurable* $H : C([0,\infty),\mathcal{M}) \to \mathbf{R}$, *and for any* $t \geq 0$

$$E\left[H(u_{t+\cdot})|\mathcal{F}_t\right] = Q_{u_t}\left[H\right], \quad \textit{almost surely.}$$

**Proof.** We use the methods of Theorem 4.4.2 of Ethier and Kurtz [EK86]. We were unable to directly apply these results, but with a little adjustment the methods apply to our case. We point out the key changes needed.

We only allow initial conditions in the strict subset $\mathcal{H}_{\alpha+}$ of all Radon measures, and do not yet know that the process takes values in this subset. But by restricting to the ordinary Markov property it is enough to know that $P(u_t(dx) \in \mathcal{H}_{\alpha+}) = 1$ for each fixed $t$, and this follows from Lemma 1 part ii).

The measurability of $\mu \to Q_\mu[H]$ can often be established for martingale problems by constructing it as the inverse of a suitable Borel bijection (see [EK86] Theorem 4.4.6). We do not use this method, as $\mathcal{H}_{\alpha+}$ is not complete under the vague topology. However the measurability can be established directly as follows. By a monotone class argument, it is enough to consider $H$ of the form $H(\omega) = \prod_{i=1}^n h_i(\omega_{t_i}(f_i))$ for bounded continuous functions $h_i$, for $f_i \in \mathcal{C}_c$, for $0 \leq t_1 < t_2 \ldots t_n$, and for $n \geq 1$. But for such $H$ we can write, using the construction of solutions from Section 3,

$$Q_\mu[H] = E\left[\prod_{i=1}^n h_i\left(\sum_{n=0}^\infty I_{t_i}^{(n)}(f_i,\mu)\right)\right] = \lim_{N\to\infty} E\left[\prod_{i=1}^n h_i\left(\sum_{n=0}^N I_{t_i}^{(n)}(f_i,\mu)\right)\right].$$

For each $N < \infty$, the integrands $\sum_{n=0}^N I_{t_i}^{(n)}(f_i,\mu)$ are, by the definition of the maps $I^{(n)}(f,\mu)$, continuous in $\mu$. So $Q_\mu[H]$ is the limit of continuous maps on $\mathcal{H}_{\alpha+}$.

We can now follow the method of in Theorem 4.4.2 part c) in Ethier and Kurtz [EK86] in the proof of the Markov property. The only important change in the argument from Ethier and Kurtz is that we have uniqueness in law for solutions to (1.1), and this requires the moment bounds (1.6) and (1.7) to hold as well the martingale problem (1.4) and (1.5). The key point is to show that, for any $t > 0$, the process $\{u_{t+\cdot}(dx)\}$ satisfies these moment bounds. For this it is enough to show for all $f : \mathbf{R}^d \to [0,\infty)$ and $0 < s < t$

$$E\left[u_t(f)|\sigma(u_r(dx) : r \leq s)\right] = \int\int G_{t-s}(x-x')f(x')u_s(dx),$$

and there exists $C$, depending only on the dimension $d$ and $\kappa$, so that

$$\begin{aligned} & E\left[\left(\int f(x)u_t(dx)\right)^2 \middle| \sigma(u_r(dx) : r \leq s)\right] \\ \leq \; & C\int_{\mathbf{R}^{4d}} G_{t-s}(x-x')G_{t-s}(y-y')f(x')f(y')\left(1+\frac{(t-s)^\alpha}{|x-y|^\alpha|x'-y'|^\alpha}\right)u_s(dx)u_s(dy)dx'dy'. \end{aligned}$$

By uniqueness in law it is enough to prove these bounds for the solutions constructed via chaos expansions in Section 3. It is also enough to prove these bounds for $f \in \mathcal{C}_c$. The first moment follows from the fact that $E[I_t^{(n)}(f,\mu)|\mathcal{F}_s] = I_s^{(n)}(G_{t-s}f,\mu)$, and the convergence of the series (3.2). For the second moment bound we use the approximations $u_t^{(\varepsilon)}$ introduced in Section 3, for which we know $u^{(\varepsilon)}(f) \to u_t(f)$ in $L^2$. Fix $0 < s_1 < \ldots < s_n \le s$, $f_1, \ldots, f_n \in \mathcal{C}_c$ and a bounded continuous function $h : \mathbf{R}^n \to \mathbf{R}$. Then, using the Markov property of the approximations $u_t^{(\varepsilon)}$,

$$
\begin{aligned}
&E\left[(u_t(f))^2 h(u_{s_1}(f_1), \ldots, u_{s_n}(f_n))\right] \\
&= \lim_{\varepsilon \downarrow 0} E\left[(u_t^{(\varepsilon)}(f))^2 h(u_{s_1}(f_1), \ldots, u_{s_n}(f_n))\right] \\
&\le C \lim_{\varepsilon \downarrow 0} E\left[\int_{\mathbf{R}^{4d}} G_{t-s}(x-x')G_{t-s}(y-y')f(x')f(y') \right.\\
&\qquad \left. \cdot \left(1 + \frac{(t-s)^\alpha}{|x-y|^\alpha |x'-y'|^\alpha}\right) u_s^{(\varepsilon)}(dx)u_s^{(\varepsilon)}(dy)dx'dy'\ h(u_{s_1}(f_1), \ldots, u_{s_n}(f_n))\right] \\
&= CE\left[\int_{\mathbf{R}^{4d}} G_{t-s}(x-x')G_{t-s}(y-y')f(x')f(y') \right. \qquad (4.8)\\
&\qquad \left. \cdot \left(1 + \frac{(t-s)^\alpha}{|x-y|^\alpha |x'-y'|^\alpha}\right) u_s(dx)u_s(dy)dx'dy'\ h(u_{s_1}(f_1), \ldots, u_{s_n}(f_n))\right].
\end{aligned}
$$

The last equality follows by the convergence $u_t^{(\varepsilon)}(f) \to u_t(f)$ for compactly supported $f$ and an approximation argument using the uniform second moment bounds on $u_s^{(\varepsilon)}$ and $u_s$. The inequality (4.8) implies the desired second moment bound, and this completes the proof. ∎

# 5 Death of solutions

To study questions of extinction, we adapt a method from the particle systems literature. Liggett and Spitzer used this technique, described in Chapter IX, Section 4 of [Lig85], to study analogous questions for linear particle systems. The corresponding result for linear particle systems, indexed on $\mathbf{Z}^d$ and with noise that is white in space, is that death of solutions occurs in dimensions $d = 1, 2$ for all $\kappa$, and in dimensions $d \ge 3$ for sufficiently large $\kappa$. The long range correlations of our noise leads to different behavior, to an increased chance of death. Death occurs for all the values of $d \ge 3$ and for all values of $\kappa$ that we are considering. However, our basic estimate in the proof of Proposition 2 below leaves open the possibility that the death is extremely slow.

We start by considering initial conditions with finite total mass. To study the evolution of the total mass we want to use the test function $f = 1$ in the martingale problem. The next lemma shows this is possible by approximating $f$ by suitable compactly supported test functions.

**Lemma 9** *Suppose that $\{u_t(dx)\}$ is a solution to (1.1) started at $\mu \in \mathcal{H}_\alpha^0$. Then the total mass $\{u_t(1) : t \ge 0\}$ is a continuous martingale with*

$$\langle u(1)\rangle_t = \int_0^t \int\int \frac{u_s(dx)\, u_s(dy)}{|x-y|^2} ds.$$

**Proof.** We first check that the assumptions on the initial condition imply that $E[u_t(1)^2] < \infty$. The bound on second moments (1.7) implies that

$$\begin{aligned}
E\left[u_t(1)^2\right] &\leq C\int_{\mathbf{R}^{4d}} G_t(x-x')G_t(y-y')\left(1+\frac{t^\alpha}{|x-y|^\alpha|x'-y'|^\alpha}\right)dx'dy'\mu(dx)\mu(dy) \\
&\leq C\int_{\mathbf{R}^{2d}}\left(1+\frac{t^{\alpha/2}}{|x-y|^\alpha}\right)\mu(dx)\mu(dy) \quad \text{using (1.9)} \\
&\leq C(1+t^{\alpha/2})\|\mu\|_\alpha^2. \qquad (5.1)
\end{aligned}$$

We may find $f_n \in C_c^2(\mathbf{R}^d)$ so that $0 \leq f_n \uparrow 1$ and $\|\Delta f_n\|_\infty \downarrow 0$ as $n \to \infty$. Applying Doob's inequality we have, for any $T \geq 0$,

$$\begin{aligned}
&E\left[\sup_{t\leq T}|z_t(f_n)-z_t(f_m)|^2\right] \\
&\leq CE\left[|z_T(f_n)-z_T(f_m)|^2\right] \\
&= CE\left[\left|u_T(f_n-f_m)-\mu(f_n-f_m)-\frac{1}{2}\int_0^T u_s(\Delta f_n-\Delta f_m)ds\right|^2\right] \\
&\leq C\left(E\left[u_T(f_n-f_m)^2\right]+\mu(f_n-f_m)^2+\frac{1}{2}(\|\Delta f_n\|_\infty+\|\Delta f_m\|_\infty)^2E\left[\left(\int_0^T u_s(1)ds\right)^2\right]\right).
\end{aligned}$$

This expression is seen to converge to zero as $n, m \to \infty$ by using dominated convergence and the bound in (5.1). From this we can deduce that, along a subsequence, $z_t(f_n)$ converges uniformly on compacts to a continuous martingale. Also,

$$E\left[\sup_{t\leq T}\left|\int_0^t u_s(\Delta f_n)ds\right|^2\right] \leq \|\Delta f_n\|_\infty^2 T\int_0^T E[u_s^2(1)]ds \to 0.$$

Since

$$z_t(f_n)+\int_0^t u_s(\frac{1}{2}\Delta f_n)ds = u_t(f_n)-\mu(f_n) \to u_t(1)-\mu(1)$$

we can conclude that $u_t(1)$ is a continuous martingale. Moreover, we claim that

$$\begin{aligned}
&E\left[\sup_{t\leq T}\left|\int_0^t\int\int\frac{u_s(dx)\,u_s(dy)}{|x-y|^2}ds-\langle z(f_n)\rangle_t\right|\right] \qquad (5.2)\\
&= E\left[\int_0^T\int\int\frac{1-f_n(x)f_m(y)}{|x-y|^2}u_s(dx)u_s(dy)ds\right] \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

This follows by dominated convergence and the bound

$$\begin{aligned}
E\left[\int_0^T\int\int\frac{u_s(dx)\,u_s(dy)}{|x-y|^2}ds\right] &= \lim_{n\to\infty}E\langle z(f_n)\rangle_T \\
&= \lim_{n\to\infty}E\left[\left(u_T(f_n)-\mu(f_n)-\int_0^t u_s(\Delta f_n)ds\right)^2\right] \\
&= E[u_T(1)^2]-\mu(1)^2 < \infty.
\end{aligned}$$

Using (5.2), it is now straightforward to identify the quadratic variation of $u_t(1)$, as in the statement of the lemma. ∎

**Proposition 2** *Suppose that $\{u_t(dx)\}$ is a solution to (1.1) started at $\mu \in \mathcal{H}_{\alpha+} \cap \mathcal{H}^0_\alpha$. Then*

$$\lim_{t \to \infty} u_t(1) = 0 \quad \textit{almost surely.}$$

**Proof.** The previous lemma shows that the process $u_t(1)$ is a non-negative martingale and hence converges almost surely. We will show that

$$\lim_{t\to\infty} E\left[u_t(1)^{1/2}\right] = 0 \tag{5.3}$$

which then implies that the limit of $u_t(1)$ must be zero. First we consider the case that $\mu$ is compactly supported inside the ball $B(0,K)$. Let $C_t = B(0,R_t)$ be the closed ball with radius

$$R_t = K + [c_8 (t \vee 1) \log\log(t \vee 4)]^{1/2}$$

where $c_8$ is a fixed constant satisfying $c_8 > 4$. We write $C_t^c$ for the complement of this ball. Let $\tau_0$ be the first time $t \geq 0$ that $u_t(1) = 0$. (In a later section we shall show that $P(\tau_0 = \infty) = 1$ whenever $\mu(1) > 0$ but we do not need to assume this here.) Using Ito's formula, and labeling any local martingale terms by $dM$, we find that for $t < \tau_0$,

$$\begin{aligned}
du_t(1)^{1/2} &= dM_t - \frac{\kappa^2}{8} u_t(1)^{1/2} \int\int \frac{u_t(dx)u_t(dy)}{u_t(1)^2 |x-y|^2}\, dt \\
&\leq dM_t - \frac{\kappa^2}{32R_t^2} u_t(1)^{1/2} \int_{C_t}\int_{C_t} \frac{u_t(dx)u_t(dy)}{u_t(1)^2}\, dt \\
&= dM_t - \frac{\kappa^2}{32R_t^2} u_t(1)^{1/2} \left(1 - \int_{C_t^c} \frac{u_t(dx)}{u_t(1)}\right)^2 dt \\
&\leq dM_t - \frac{\kappa^2}{32R_t^2} u_t(1)^{1/2} \left(1 - 2\int_{C_t^c} \frac{u_t(dx)}{u_t(1)}\right) dt \\
&\leq dM_t - \frac{\kappa^2}{32R_t^2} u_t(1)^{1/2} \left(1 - 2\left[\int_{C_t^c} \frac{u_t(dx)}{u_t(1)}\right]^{1/2}\right) dt \\
&= dM_t - \frac{\kappa^2}{32R_t^2} u_t(1)^{1/2} dt + \frac{\kappa^2}{16R_t^2} u_t(C_t^c)^{1/2} dt.
\end{aligned} \tag{5.4}$$

The local martingale term in (5.4) is given by $dM_t = (1/2)u_t(1)^{-1/2} du_t(1)$ and is reduced by the stopping times $\tau_{1/n} = \inf\{t : u_t(1) \leq 1/n\}$. So applying (5.4) at the time $t \wedge \tau_{1/n}$ and taking expectations we obtain

$$E[u_{t\wedge\tau_{1/n}}(1)^{1/2}] \leq \mu(1)^{1/2} - E\left[\int_0^{t\wedge\tau_{1/n}} \frac{\kappa^2}{32R_s^2} u_s(1)^{1/2} ds\right] + E\left[\int_0^{t\wedge\tau_{1/n}} \frac{\kappa^2}{16R_s^2} u_s(C_s^c)^{1/2} ds\right].$$

Letting $n \to \infty$, using monotone convergence and the moments established in (5.1), we obtain the same inequality with $\tau_{1/n}$ replaced by $\tau_0$. Since the paths of a non-negative local martingale must remain at zero after hitting zero we may further replace $t \wedge \tau_0$ by $t$ in the inequality. Defining $\eta_t = E\left[u_t(1)^{1/2}\right]$ we therefore have

$$\eta_t \leq \eta_0 - \int_0^t \frac{\kappa^2}{32R_s^2}\eta_s ds + \int_0^t \frac{\kappa^2}{16R_s^2} E\left[u_s(C_s^c)^{1/2}\right] ds \tag{5.5}$$

The aim is to estimate the expectation in this inequality and to show that it implies that $\eta_t \to 0$. Let

$$\Xi(s,t) = \exp\left( - \int_s^t \frac{\kappa^2}{32 R_r^2} dr \right).$$

It follows from the definition of $R_t$ that $\int_s^\infty \frac{\kappa^2}{32R_r^2} dr = \infty$ for any $s \geq 0$ and so, for any $s \geq 0$,

$$\lim_{t \to \infty} \Xi(s,t) = 0. \tag{5.6}$$

Applying Gronwall's inequality to (5.5), we obtain

$$\eta_t \leq \Xi(0,t)\eta_0 + \int_0^t \Xi(s,t) \frac{\kappa^2}{16R_s^2} E\left[u_s(C_s^c)^{1/2}\right] ds. \tag{5.7}$$

If we show that $\int_0^\infty \frac{\kappa^2}{16R_s^2} E\left[u_s(C_s^c)^{1/2}\right] ds < \infty$, it then follows that $\eta_t \downarrow 0$ as $t \to \infty$ (use $0 \leq \Xi(s,t) \leq 1$, (5.6) and dominated convergence). Using the Cauchy-Schwarz inequality and the formula for first moments, we obtain

$$\begin{aligned}
\left(E\left[u_t(C_t^c)^{1/2}\right]\right)^2 &\leq E\left[u_t(C_t^c)\right] \\
&\leq \int_{C_t^c} \int_{\mathbf{R}^d} G_t(x-y)\mu(dx)dy \\
&\leq C\mu(1) \int_{R_t - K}^\infty (2\pi t)^{-d/2} \exp(-r^2/2t) r^{d-1} dr \\
&= C\mu(1) \int_{(R_t-K)/\sqrt{t}}^\infty \exp(-s^2/2) s^{d-1} ds \\
&\leq C\mu(1) \exp(-(R_t-K)^2/2t) \left(1 + (R_t - K/\sqrt{t})^{d-1}\right) \\
&\leq C\mu(1) \left[\log(t \vee 4)\right]^{-c_8/2} \left[\log\log(t \vee 4)\right]^{(d-1)/2}.
\end{aligned} \tag{5.8}$$

Here we have used the following standard inequality: by the change of variables $y = x + z$ we find

$$\begin{aligned}
\int_x^\infty \exp(-y^2/2) y^{d-1} dy &\leq C\exp(-x^2/2) \int_0^\infty \exp(-z^2/2)(x^{d-1} + z^{d-1}) dz \\
&\leq C(1 + x^{d-1}) \exp(-x^2/2), \quad \text{when } x \geq 0.
\end{aligned}$$

Finally we use (5.8) to derive the following:

$$\int_0^\infty \frac{\kappa^2}{16R_s^2} E\left[u_s(C_s^c)\right]^{1/2} ds \leq C(\kappa,\mu) \left(1 + \int_4^\infty \frac{(\log\log s)^{(d-5)/4}}{s(\log s)^{c_8/4}} ds\right) < \infty.$$

This completes the proof in the case that $\mu$ is compactly supported. In the general case, we fix $\varepsilon > 0$, and split the initial condition so that $\mu = \mu^{(1)} + \mu^{(2)}$ where $\mu^{(1)}(1) \leq \varepsilon$ and $\mu^{(2)}$ is compactly supported. By uniqueness in law, we may consider any solution with initial condition $\mu$ and we choose to construct one as follows: let $u^{(1)}, u^{(2)}$ be solutions, as constructed in Section 3, with respect to the same noise and with initial conditions $\mu^{(1)}, \mu^{(2)}$ and set $u = u^{(1)} + u^{(2)}$. It is easy to check that $u$ is a solution starting at $\mu$, which is a statement of the linearity of the equation. Using the Cauchy-Schwarz inequality and the formula for first moments (1.6), we have

$$\begin{aligned} E\left[u_t(1)^{1/2}\right] &= E\left[\left(u_t^{(1)}(1)+u_t^{(2)}(1)\right)^{1/2}\right] \\ &\leq E\left[u_t^{(1)}(1)^{1/2}\right]+E\left[u_t^{(2)}(1)^{1/2}\right] \\ &\leq E\left[u_t^{(1)}(1)\right]^{1/2}+E\left[u_t^{(2)}(1)^{1/2}\right] \\ &= \varepsilon^{1/2}+E\left[u_t^{(2)}(1)^{1/2}\right]. \end{aligned}$$

Thus, (5.3) follows from the compactly supported case, and the proposition is proved. ∎

**Proof of Theorem 2.** Firstly, we deal with the case of an initial condition with finite total mass. If $P(u_0 \in \mathcal{H}^0_{\alpha+}) = 1$, then

$$P(u_t(1) \to 0) = \int_{\mathcal{H}^0_{\alpha+}} Q_\mu(U_t(1) \to 0) P(u_0 \in d\mu) = 1.$$

Secondly, we treat the case of an initial condition that has locally bounded intensity. For such $u_0$ we have, using the first moment formula,

$$E[u_1(\phi)] = E\left(\int\int G_1(x-z)\phi(x)u_0(dz)dx\right) \leq C\int \phi(x)dx \tag{5.9}$$

for some constant $C < \infty$. That is, $E[u_1(dx)] \leq Cdx$, where we write $dx$ for Lebesgue measure. Fix a bounded set $A$. By the linearity of the equation, the map $\mu \to Q_\mu(U_t(A)\wedge 1)$ is increasing in $\mu$. Moreover, it is concave in $\mu$. Indeed, if $u_t^\mu(dx)$ and $u_t^\nu(dx)$ are solutions started from $\mu$ and $\nu$, with respect to the same noise, then by linearity and the concavity of $f(z) = z \wedge 1$,

$$\begin{aligned} Q_{\theta\mu+(1-\theta)\nu}(U_t(A)\wedge 1) &= E\left[(\theta u_t^\mu(A)+(1-\theta)u_t^\nu(A))\wedge 1\right] \\ &\geq E\left[\theta(u_t^\mu(A)\wedge 1)+(1-\theta)(u_t^\nu(A)\wedge 1)\right] \\ &= \theta Q_\mu(U_t(A)\wedge 1)+(1-\theta)Q_\nu(U_t(A)\wedge 1). \end{aligned}$$

Thus

$$\begin{aligned} E[u_{t+1}(A)\wedge 1] &= \int_{\mathcal{H}_{\alpha+}} Q_\mu(U_t(A)\wedge 1)P(u_1 \in d\mu) \quad \text{(by the Markov property)} \\ &\leq Q_{E[u_1(dx)]}(U_t(A)\wedge 1) \quad \text{(by Jensen's inequality)} \\ &\leq Q_{CL(dx)}(U_t(A)\wedge 1) \quad \text{(using (5.9))} \\ &= Q_{I_A}(CU_t(1)\wedge 1) \quad \text{(by self duality)} \end{aligned}$$

which converges to zero by Proposition 2. This completes the proof of Theorem 2. ∎

## 6 Support Properties

In this section we establish the various properties listed in Theorem 3.

## 6.1 Dimension of Support

We can apply Frostman's Lemma (see [F85] Corollary 6.6) to obtain a lower bound on the Hausdorff dimension of supporting sets for solutions $u_t(dx)$. Indeed, Lemma 1 part ii) and Frostman's Lemma imply that any non-empty Borel supporting set for the measure $u_t(dx)$, at a fixed $t > 0$, must, almost surely, have dimension at least $d - \alpha$. We prove in Subsection 6.2 that if $\mu \neq 0$, then $u_t \neq 0$ almost surely. This establishes the result for fixed $t$ in Theorem 3 i). We now show a weaker lower bound that holds at all times.

**Proposition 3** *Suppose that $\{u_t(dx)\}$ is a solution to (1.1).*

**i.)** *If $P(u_0 \in \mathcal{H}_{(d-2-\alpha)-}) = 1$ then $P(u_t \in \mathcal{H}_{(d-2-\alpha)-}$ for all $t \geq 0) = 1$. Indeed, for some $a$,*

$$P\left(\text{There exists } a \text{ so that } \sup_{s \leq t} \|u_s e^{-a|x|}\|_\beta < \infty\right) = 1 \qquad \text{for all } t \geq 0 \text{ and } \beta < d - 2 - \alpha.$$

**ii.)** *For any initial condition we have $P(u_t \in \mathcal{H}_{(d-2-\alpha)-}$ for all $t > 0) = 1$.*

**Remarks**

**1.** Since $\mathcal{H}_{(d-2-\alpha)-} \subseteq \mathcal{H}_{\alpha+}$ (which requires $\kappa < (d-2)/2)$) we also have, for any initial condition, $P(u_t \in \mathcal{H}_{\alpha+}$ for all $t > 0) = 1$.

**2.** Using Frostman's Lemma, part ii) of this proposition implies the following. At all times $t > 0$, a Borel set $A_t$ that supports $u_t(dx)$ must have Hausdorff dimension at least $d - \alpha - 2$.

**3.** The idea behind the proof of Proposition 3 is to show, for suitable values of $\rho$, that the process $S_t^{(\rho)} = \int\int u_t(dx)u_t(dy)/|x-y|^\rho$ is a non-negative supermartingale. Applying Ito's formula formally, ignoring the singularity in $|x-y|^{-\rho}$, and writing $dM$ for any local martingale terms, we find

$$\begin{aligned} dS_t^{(\rho)} &= dM + \int\int u_t(dx)u_t(dy)\left(\frac{1}{2}\Delta\left(|x-y|^{-\rho}\right) + \kappa^2|x-y|^{-(\rho+2)}\right)dt \\ &= dM + \left(\rho^2 - (d-2)\rho + \kappa^2\right)\int\int u_t(dx)u_t(dy)|x-y|^{-(\rho+2)}dt \end{aligned}$$

where $\Delta$ is the Laplacian on $\mathbf{R}^{2d}$, acting on both variables $x$ and $y$. The solution to the inequality $\rho^2 - (d-2)\rho + \kappa^2 \leq 0$ gives the condition $\alpha \leq \rho \leq d - 2 - \alpha$. The rigorous calculation below, namely Lemma 10 and the proof of Proposition 3, does not quite apply to the boundary value of $\rho = d - 2 - \alpha$.

First we prove a lemma extending the martingale problem to test functions on $\mathbf{R}^{2d}$.

**Lemma 10** *Suppose that $\{u_t(dx)\}$ is a solution to (1.1) with initial condition $\mu$. Then for twice differentiable function $f : \mathbf{R}^{2d} \to \mathbf{R}$ with compact support*

$$M_t(f) = \int\int f(x,y)u_t(dx)u_t(dy) - \int_0^t \int\int \left(\frac{1}{2}\Delta f(x,y) + \kappa^2 \frac{f(x,y)}{|x-y|^2}\right) u_s(dx)u_s(dy)ds \quad (6.1)$$

*defines a continuous local martingale.*

**Proof.** For $f$ of product form, that is $f(x,y) = \sum_{k=1}^n \phi_k(x)\psi_k(y)$ where $\phi_k, \psi_k \in C_c^2$, this claim is a consequence of the martingale problem (1.4) and (1.5) together with integration by parts. Now we claim that we can choose $f_n(x,y)$ of product form, and with a common compact support, so that $f_n$ and $\Delta f_n$ converge uniformly to $f$ and $\Delta f$. One way to see this is consider the one point compactification $E$ of the open box $\{(x,y) : |x|, |y| < N\}$ and to let $(X_t, Y_t)$ be independent d-dimensional Brownian motions absorbed on hitting the boundary point of $E$. Then consider the algebra $\mathcal{A}$ generated by the constant functions and the product functions $\phi(x)\psi(y)$, where $\phi, \psi$ are compactly supported in $\{x : |x| < N\}$. The Stone-Weierstrass theorem shows that this algebra is dense in the space of continuous functions on $E$ and the transition semigroup $\{T_t\}$ of $(X_t, Y_t)$ maps $\mathcal{A}$ to itself. A lemma of Watanabe (see [EK86] Proposition 3.3) now implies that $\mathcal{A}$ is a core for the generator of $(X_t, Y_t)$ and this implies the above claim.

The continuity of $t \to u_t$, and the calculation in Lemma 1 part ii), imply that $M_t(f_n)$ converges to $M_t(f)$ uniformly on compacts, in probability. So the limit $M_t(f)$ has continuous paths. Also, if $f_n$ and $f$ are supported in the compact set $A$, the stopping times

$$T_k = \inf\{t : u_t(A) + \int_0^t \int_A \int_A \frac{u_s(dx)u_s(dy)}{|x-y|^2} ds \geq k\}$$

satisfy $T_k \uparrow \infty$ and reduce all the local martingales $M_t(f_n)$ to bounded martingales. We may then pass to the limit as $n \to \infty$ to see that $M_t(f)$ is a local martingale reduced by $\{T_k\}$. ∎

**Proof of Proposition 3.** For part i) we may, by conditioning on the initial condition, suppose that $u_0 = \mu \in \mathcal{H}_{(d-2-\alpha)-}$. We may then choose $a$ so that $\mu \in \mathcal{H}_\beta^a$ for all $\beta < d-2-\alpha$.

We shall approximate $|x-y|^{-\rho}$ by a sequence of compactly supported functions as follows. Choose $\phi_n \in \mathcal{C}_c^2$ satisfying and $\phi_n(x) = 1$ for $|x| \leq n$ and with $\phi_n, \partial_{x_i}\phi_n, \partial_{x_i x_j}\phi_n$ uniformly bounded over $x$ and $n$. Define, for $\varepsilon > 0$ and $\rho \in [(d-2)/2, d-2-\alpha]$,

$$f_{n,\varepsilon}(x,y) = \left(1 + (\varepsilon + |x-y|^2)^{-\rho/2}\right) e^{-a(1+|x|^2)^{1/2} - a(1+|y|^2)^{1/2}} \phi_n(x)\phi_n(y).$$

A short calculation shows that

$$\begin{aligned}
&\left(\frac{1}{2}\Delta + \frac{\kappa^2}{|x-y|^2}\right) (\varepsilon + |x-y|^2)^{-\rho/2} \\
= \;& \left(\varepsilon + |x-y|^2\right)^{-(\rho+4)/2} \left((\rho^2 - (d-2)\rho + \kappa^2)|x-y|^2 + (2\kappa^2 - \rho d)\varepsilon + \varepsilon^2\kappa^2|x-y|^{-2}\right) \\
\leq \;& \left(\varepsilon + |x-y|^2\right)^{-(\rho+4)/2} \varepsilon^2\kappa^2|x-y|^{-2} \\
\leq \;& C(\rho)|x-y|^{-(\rho+2)}.
\end{aligned}$$

The penultimate inequality follows from the restriction on the value of $\rho$ and $\kappa$, and the last inequality follows by considering separately the cases $\varepsilon < |x-y|^2$ and $\varepsilon \geq |x-y|^2$. A simple calculation also shows that

$$\left|\Delta\left(e^{-a(1+|x|^2)^{1/2}-a(1+|y|^2)^{1/2}}\right)\right| \leq C(a)e^{-a(1+|x|^2)^{1/2}-a(1+|y|^2)^{1/2}} \leq C(a)e^{-a|x|-a|y|}.$$

Now a lengthy calculation, using the above two bounds as key steps, shows that

$$\left|\left(\frac{1}{2}\Delta + \frac{\kappa^2}{|x-y|^2}\right) f_{n,\varepsilon}(x,y)\right| \leq C(a,\rho)\left(1+|x-y|^{-(\rho+2)}\right)e^{-a|x|-a|y|}.$$

Note that the bound is uniform over $n$ and $\epsilon$. Using the test function $f_{n,\varepsilon}(x,y)$ in Lemma 10 we have that

$$M_t(f_{n,\varepsilon}) = \int\int f_{n,\varepsilon}(x,y)u_t(dx)u_t(dy) - \mathcal{E}(\varepsilon,t) \tag{6.2}$$

is a continuous local martingale and

$$\mathcal{E}(\varepsilon,t) \leq C(a,\rho)\int_0^t\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\left(1+|x-y|^{-(\rho+2)}\right)e^{-a|x|-a|y|}u_s(dx)u_s(dy)ds. \tag{6.3}$$

Now we apply Doob's inequality in the following form

**Lemma 11** *Suppose $\{A_t\}$, $\{M_t\}$, $\{D_t\}$ are continuous processes satisfying $0 \leq A_t = M_t + D_t$ and where $M_t$ is a continuous local martingale with $M_0$ bounded. Then for $\lambda \geq 0$*

$$P\left(\sup_{s\leq t} A_s \geq 2\lambda\right) \leq \frac{1}{\lambda}\left(E[M_0] + 3E[\sup_{s\leq t}|D_s|]\right).$$

**Proof** If $\{T_k\}$ reduce the local martingale $M_t$ then by Doob's inequality for positive submartingales

$$\begin{aligned}
P(\sup_{s\leq t\wedge T_k}|M_s| \geq \lambda) &\leq \frac{1}{\lambda}E[|M_{t\wedge T_k}|] \\
&\leq \frac{1}{\lambda}\left(E[A_{t\wedge T_K}] + E[|D_{t\wedge T_K}|]\right) \\
&\leq \frac{1}{\lambda}\left(E[M_0] + 2E[|D_{t\wedge T_K}|]\right) \\
&\leq \frac{1}{\lambda}\left(E[M_0] + 2E[\sup_{s\leq t}|D_s|]\right).
\end{aligned}$$

Let $k \to \infty$ and combine with the bound $P(\sup_{s\leq t}|D_s| \geq \lambda) \leq E[\sup_{s\leq t}|D_s|]/\lambda$ to complete the lemma. ∎

We apply this lemma to the decomposition (6.2) together with the bound (6.3) to obtain

$$P\left(\sup_{s\leq t}\int\int\left(1+|x-y|^{-\rho}\right)e^{-a|x|-a|y|}u_t(dx)u_t(dy) > 2\lambda\right)$$

$$= \lim_{\varepsilon \to 0, n \to \infty} P\left( \sup_{s \le t} \int \int f_{n,\varepsilon}(x,y) u_t(dx) u_t(dy) > 2\lambda \right)$$
$$\le \frac{C(a,\rho)}{\lambda} \|\mu(dx) \exp(-a|x|)\|_\rho^2$$
$$+ \frac{C(a,\rho)}{\lambda} E\left[ \int_0^t \int \int \left(1 + |x-y|^{-(\rho+2)}\right) e^{-a|x| - a|y|} u_s(dx) u_s(dy) ds \right]. \qquad (6.4)$$

As in Lemma 1 part ii), and using the fact that $\mu \in \mathcal{H}_\beta^a$ for all $\beta < d-2-\alpha$, one shows that the expectation on the right hand side of (6.4) is finite. One needs, however, the strict inequality $\rho < d-2-\alpha$. This ensures that the worst pole in the above expression is $|x'-y'|^{-(\rho+2+\alpha)}$, which is still integrable. So, the bound (1.12) applies. The bound in (6.4) implies part i) of the Proposition.

For part ii), we may suppose, by conditioning on the initial condition, that $u_0 = \mu \in \mathcal{H}_{\alpha+}^0$. But then Lemma 1 part ii) implies, for fixed $t_0 > 0$, that $u_{t_0}(dx) \in \mathcal{H}_{d-2-\alpha}$ almost surely. The Markov property of solutions and part i) then imply that the desired conclusion holds for $t \ge t_0$. Letting $t_0 \downarrow 0$ completes this proof. ∎

**Corollary 3** *The family $\{Q_\mu : \mu \in \mathcal{H}_{(d-2-\alpha)-}\}$ is a strong Markov family.*

**Proof.** Let $\{u_t\}$ be a solution defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and satisfying $P(u_0 \in \mathcal{H}_{(d-2-\alpha)-}) = 1$. Let $\tau < \infty$ be a $\mathcal{F}_t$-stopping time, and let $\tau(n) < \infty$ be discrete stopping times satisfying $\tau(n) \downarrow \tau$. Fix $0 \le t_1 < \ldots < t_n$ and $f_1, \ldots, f_n \in \mathcal{C}_c$, and set $H(u) = \exp(i(u_{t_1}(f_1) + \ldots + u_{t_n}(f_n)))$. Fix a set $\Lambda \in \mathcal{F}_\tau$. Then the ordinary Markov property implies that

$$E\left[H(u_{\tau(n)+\cdot}) \mathbf{1}(\Lambda)\right] = E\left[Q_{u_{\tau(n)}}[H] \mathbf{1}(\Lambda)\right]. \qquad (6.5)$$

If we can pass to the limit as $n \to \infty$ to replace $\tau(n)$ by $\tau$, then this identity will imply the result. By the continuity of paths the left hand side of (6.5) converges as desired. We claim that

$$\text{if } \mu_n \to \mu \text{ vaguely and } \sup_n \|\mu_n(dx) e^{-a|x|}\|_\alpha < \infty \text{ then } Q_{\mu_n}[H] \to Q_\mu[H].$$

Assuming this claim, Proposition 3 part i) allows us to pass to the limit on the right hand side of (6.5). To prove the claim we let $u_t(dx)$ be the solution starting at $\mu$ constructed using the chaos expansion and $u_{N,t}$ the approximation using only the first $N$ terms of the expansion. Then

$$Q_\mu[H] = E\left[\exp(i \sum_{j=1}^n u_{t_j}(f_j))\right]$$
$$= E\left[\exp(i \sum_{j=1}^n u_{N,t_j}(f_j))\right] + \text{Error}(N,\mu)$$

where

$$|\text{Error}(N,\mu)| \le \left( \sum_{j=1}^n E\left[(u_{t_j}(f_j) - u_{N,t_j}(f_j))^2\right] \right)^{1/2}.$$

The function $E\left[\exp(i\sum_{j=1}^n u_{N,t_j}(f_j))\right]$ is continuous in $\mu$ and $\mathrm{Error}(N,\mu)\to 0$ as $N\to\infty$. So the claim follows if we can show $\sup_n|\mathrm{Error}(N,\mu_n)|\to 0$ as $N\to\infty$. Using the isometry as in Lemma 5 we see that

$$E\left[(u_t(f)-u_{N,t}(f))^2\right]=\int\int H_N(x,y)\mu(dx)\mu(dy)$$

where

$$H_N(x,y)=\int\int f(x')f(y')G_t(x-x')G_t(y-y')E_{0,x,y}^{t,x',y'}\left[\sum_{k=N+1}^{\infty}\frac{1}{k!}\left(\int_0^t\frac{\kappa^2 ds}{|X_s^1-X_s^2|^2}\right)^k\right]dx'dy'$$

is bounded by

$$H_N(x,y)\le C(a,t)e^{-a|x|-a|y|}(1+|x-y|^{-\alpha}).$$

Note that $H_N(x,y)$ is nonincreasing but not continuous. The assumptions of the claim allow us, by an approximation argument, to ignore the singularity in the function $H_N(x,y)$ and replace it by a nonincreasing, continuous function $\tilde{H}_N(x,y)$ of compact support. But then the vague convergence $\mu_n\to\mu$ implies that $\sup_n\int\int\tilde{H}_N(x,y)\mu_n(dx)\mu_n(dy)\downarrow 0$ as $N\to\infty$ (for example by the argument used to prove Dini's lemma). This completes the proof of the claim. ∎

## 6.2 Density of Support

In this subsection we give the proof of Theorem 3 ii). We start with an outline of the method. Assume that $u_0(B(a,r))>0$ and fix $T>0$. We wish to show that with probability one, $u_T(B(b,r))>0$. We consider various tubes in $[0,T]\times\mathbf{R}^d$ which connect $\{0\}\times B(a,r)$ with $\{T\}\times B(b,r)$. (By a tube we mean that for any time $t$ the cross section of the tube with the slice $\{t\}\times\mathbf{R}^d$ is a ball of radius $r$.) We consider a subsolution to the equation which has Dirichlet boundary conditions on the edge of the tube. We will show that the probability that the subsolution is non-zero at time $T$ is a constant not depending on the tube. It is possible to construct an infinite family of such tubes such that each pair has very little overlap. Then a zero-one law will guarantee that, with probability one, at least one of the subsolutions will be non-zero. Applying this for a countable family of open balls, we shall obtain the density of the support. This implies that the solution never dies out completely. Note also that for the equation (1.1) posed on a finite region, the above argument fails, as there is not enough room to fit an infinite family of nearly disjoint tubes.

Let us give a rigorous definition of the tubes described above. For a piecewise smooth function $g:[0,T]\to\mathbf{R}^d$ the tube centered on $g$ is defined as

$$\mathbf{T}=\left\{(t,x)\in[0,T]\times\mathbf{R}^d : x\in B(g(t),r)\right\}.$$

If $\mathbf{T}$ is such a tube, let $\partial\mathbf{T}$ be the boundary of $\mathbf{T}$, minus the part of the boundary at $t=0$ and $t=T$. We aim to find a solution $(u_t^{\mathbf{T}}(dx):0\le t\le T)$ to the equation (1.1), but restricted to the tube $\mathbf{T}$, and with Dirichlet boundary conditions. That is,

$$\left\{\begin{array}{rcll}\frac{\partial u_t^{\mathbf{T}}}{\partial t} & = & \Delta u_t^{\mathbf{T}}+\kappa u_t^{\mathbf{T}}\dot{F}(t,x) & \text{for } (x,t)\in\mathbf{T},\\ u_0^{\mathbf{T}}(dx) & = & \nu(dx), & \text{where } \mathrm{supp}(\nu)\subseteq B(g(0),r),\\ u_t^{\mathbf{T}}(dx) & = & 0 & \text{for } (x,t)\in\partial\mathbf{T}.\end{array}\right. \tag{6.6}$$

As in Section 3, a chaos expansion with respect to the noise $F$ yields solutions to (6.6). We do not give the proof. The only changes needed are that the stochastic integrals are restricted to the tube and the Green's function $G_t(x-y)$ must be replaced by the Green's function for the tube $G_t^{\mathbf{T}}(x-y)$, that is the fundamental solution for the heat equation in the tube with Dirichlet boundary conditions. As in Section 3, the convergence of the series is guaranteed by the finiteness of exponential Brownian bridge moments; however the moments that are needed are of the form

$$E_{0,x,y}^{t,x',y'}\left[\exp\left(\int_0^t \frac{\kappa^2}{|X_s^1 - X_s^2|^2}ds\right) I(\sup_{s\le t}|X_s^1 - g(s)| \vee |X_s^2 - g(s)| < r)\right]$$

and so are less than the corresponding moments needed to ensure the solution on the whole space converges.

Fix the noise $F$, on its filtered probability space, and construct via chaos expansions $\{u_t(dx)\}$ the solution to (1.1) started at $\mu \in \mathcal{H}_{\alpha+}$ and $\{u_t^T(dx)\}$ the solution to (6.6) started at $\nu = \mu|_{B(g(0),r)}$. We may also construct approximating solutions $u_t^{(\mathbf{T},\varepsilon)}(x)dx$ to $u_t^{\mathbf{T}}(dx)$, by using the smoother noise $F^\varepsilon$ and the initial condition $\nu^{(\varepsilon)} = G_\varepsilon^{\mathbf{T}}\nu$, exactly as we approximated $u_t(dx)$ by $u_t^{(\varepsilon)}(x)dx$. A standard comparison argument shows that $u_t^{(\mathbf{T},\varepsilon)}(x) \le u_t^{(\varepsilon)}(x)$. Passing to the limit as $\varepsilon \to 0$ we find that, with probability one,

$$u_t^{\mathbf{T}}(dx) \le u_t(dx) \qquad \text{for all } 0 \le t \le T. \tag{6.7}$$

We now start the proof of Theorem 3 ii). As described at the beginning of this section, it is enough to assume that $u_0(B(a,r)) > 0$, for some $a \in \mathbf{R}^d$ and $r > 0$, and to show, for fixed $b \in \mathbf{R}^d$, that $u_T(B(b,r)) > 0$ with probability one. For notational ease we shall take $a = b = 0$ and $r = 1$; the proof needs only small changes for other values of $r, a, b$. Let $e_1$ be the unit vector $(1, 0, \ldots, 0)$. We consider a sequence of piecewise linear functions $g_n(t)$ for $n \in \mathbf{Z}$, given by

$$g_n(t) = \begin{cases} 2nte_1 & \text{if } \ 0 \le t \le T/2 \\ 2n(T-t)e_1 & \text{if } \ T/2 \le t \le T \end{cases}$$

We write $\mathbf{T}_n$ for the tube centered on $g_n$. The Feynman-Kac representation (1.18), adapted for the Dirichlet boundary conditions, gives the following representation for the solution $u_T^{(\mathbf{T}_n,\varepsilon)}(f)$, where $f \ge 0$ is a test function supported in $B(0,1)$.

$$\begin{aligned} u_T^{(\mathbf{T}_n,\varepsilon)}(f) &= e^{-\Gamma_\varepsilon(0)T}\int_{B(0,1)} dx \int_{B(0,1)} \nu^{(\varepsilon)}(dy) G_T(x-y) f(x) \\ &\quad \cdot E_{0,y}^{T,x}\left[\exp\left(\kappa \int_0^T F^{(\varepsilon)}(ds, X_s)\right) I((s,X_s) \in \mathbf{T}_n,\ \forall s \le T)\right] \end{aligned}$$

By conditioning on the position of the Brownian bridge at time $T/2$, we find

$$\begin{aligned} u_T^{(\mathbf{T}_n,\varepsilon)}(f) &= e^{-\Gamma_\varepsilon(0)T}\int dx \int \nu^{(\varepsilon)}(dy) \int dz f(x) \\ &\quad \cdot G_{T/2}(x - nTe_1 - z) G_{T/2}(nTe_1 + z - y) f(x) E_1(y,z) E_2(z,x) \end{aligned} \tag{6.8}$$

where

$$E_1(y,z) = E_{0,y}^{T/2,z+nT}\left[\exp\left(\kappa\int_0^{T/2} F^{(\varepsilon)}(ds,X_s)\right) I((s,X_s)\in \mathbf{T}_n,\ \forall s\le T/2)\right]$$

and

$$E_2(z,x) = E_{0,z+nT}^{T/2,x}\left[\exp\left(\kappa\int_{T/2}^{T} F^{(\varepsilon)}(ds,X_{s-(T/2)})\right) I((s,X_{s-(T/2)})\in \mathbf{T}_n,\ \forall s\le T)\right].$$

All the randomness in the representation (6.8) is contained in the Brownian bridges expectations $E_1(x,z)$ and $E_2(z,y)$. By adding a suitable linear drift to the Brownian bridge we may rewrite

$$E_1(y,z) = E_{0,y}^{T/2,z}\left[\exp\left(\kappa\int_0^{T/2} F^{(n,\varepsilon)}(ds,X_s)\right) I((s,X_s)\in \mathbf{T}_n,\ \forall s\le T/2)\right]$$

where $F^{(n,\varepsilon)}(x,t) = F^{(\varepsilon)}(x+nt,t)$ is a new noise which has the same covariance structure as $F^{(\varepsilon)}$. This shows that the law of $E_1(y,z)$ is independent of $n$, and a similar argument applies to $E_2(z,x)$, which is also independent of $E_1(y,z)$. Also, for $x,z\in B(0,1)$,

$$\frac{G_{T/2}(x-nTe_1-z)}{G_{T/2}(x-z)} = \exp(-n^2T-2ne_1\cdot(x-z)) \ge \exp(-n^2T-4|n|).$$

A similar lower bound holds for $G_{T/2}(nTe_1+z-y)$. Using these bounds in (6.8), we see that the variable $u_T^{(\mathbf{T}_n,\varepsilon)}(f)$ stochastically dominates the variable $C(n,T)u_T^{(\mathbf{T}_0,\varepsilon)}(f)$, where $C(n,T)$ is a strictly positive constant independent of $\varepsilon$. Letting $\varepsilon\downarrow 0$, we obtain the same stochastic dominance for the solutions driven by the singular noise $F$:

$$u_T^{(\mathbf{T}_n)}(f) \overset{s}{\ge} C(n,T)u_T^{(\mathbf{T}_0)}(f),$$

where the inequality stands for stochastic domination. Let $A_n$ be the event $\{u_T^{\mathbf{T}_n}(B(0,1))>0\}$. Then, by this stochastic domination, $P(A_n)\ge P(A_0)$. Also, $P(A_0)>0$. This can be seen from the fact that the first moment of $u_T^{(n)}(B(0,1))$ is given, as are the first moments in (1.6), by the heat flow in the tube. Hence, it is hence non-zero.

Finally, we apply a zero-one law to conclude the result. Consider the sequence of noises defined by

$$F_n = \left(\dot F(t,x+g_n(t)) : 0<t<T,\ |x|<1\right) \quad \text{for n=0,1,...}$$

Since the correlation structure of $F$ is unchanged by piecewise linear shifts, the noises $\{F_k\}$ are identically distributed and form a stationary sequence. We claim this sequence is also strongly mixing. For this, it is enough to show, for all $k$ and bounded measurable $G,H$, that as $n\to\infty$,

$$E\left[G(F_{-k},\ldots,F_k)H(F_{n-k},\ldots,F_{n+k})\right] \to E\left[G(F_{-k},\ldots,F_k)\right]E\left[H(F_{-k},\ldots,F_k)\right]. \qquad (6.9)$$

Suppose that $\phi_{i,j}(x,t)$ are test functions supported in $(0,T)\times B(0,1)$. Suppose $G$ and $H$ are bounded continuous functions of the vector

$$\left(\int_0^T\int_{B(0,1)}\phi_{i,j}dF_i : -k\le i\le k, j=1,\ldots,k\right).$$

Each integral $\int\int \phi_{i,j} dF_i$ is a Gaussian variable. Also, the covariance between $\int\int \phi_{i,j} dF_i$ and $\int\int \phi_{i+n,j'} dF_{i+n}$ converges to zero as $n \to \infty$. This implies that the mixing relation (6.9) holds for $G, H$ of this special type. A monotone class argument then proves the mixing relation for general $G$ and $H$.

Define $\mathcal{S}_n$ to be the $\sigma$-field generated by the noises $(F_n, F_{n+1}, F_{n+2}, \ldots)$. The strong mixing of the sequence implies that the sigma field $\mathcal{S} = \cap_{n=1}^{\infty} \mathcal{S}_n$ is trivial in that $P(S) = 0$ or $1$ for all $S \in \mathcal{S}$. The construction of the solutions by a Wiener chaos expansion shows that the solution $u^{(\mathbf{T}_n)}$ is measurable with respect to the sigma field generated by the noise $\dot{F}(t,x)$ for $(t,x) \in T_n$. Thus, the event $A_n$ is $\mathcal{S}_n$ measurable, and the event $\{A_n \ i.o.\}$ is $\mathcal{S}$-measurable. Since $P(A_n)$ is bounded below uniformly in $n$, the event $\{A_n \ i.o.\}$ must have probability one. Finally, since $u_T(B(0,1)) \geq \sup_n u_T^{(\mathbf{T}_n)}(B(0,1))$ by (6.7), the proof is complete.

### 6.3 Singularity of solutions

In this subsection we prove the singularity assertion in Theorem 3 iii). We first sketch a short argument that suggests the solutions are singular. Fix $T > 0$ and $x \in \mathbf{R}^d$. For $t \in [0,T)$ we consider the process

$$M_t(x) = \int G_{T-t}(x-y) u_t(dy).$$

It is possible to extend the martingale problem (1.4) to test functions that depend on time and that do not have compact support, provided that they decay faster than exponentially at infinity. Using the test function $(t,y) \to G_{T-t}(x-y)$ it follows from this extension that $\{M_t\}$ is a nonnegative continuous local martingale for $t \in [0,T)$. The explosion principle (see [RW00] Corollary IV. 34.13) implies that the quadratic variation must remain bounded as $t \uparrow T$. Therefore, with probability 1,

$$\langle M(x) \rangle_T = \int_0^T \int\int u_t(dy) u_t(dz) G_{T-t}(x-y) G_{T-t}(x-z) |y-z|^{-2} < \infty. \tag{6.10}$$

However, a short calculation shows that if $u_t(y)$ has a continuous, strictly positive density in the neighborhood of $(T,x)$ then the integral in (6.10) is infinite.

Instead of pursuing this argument, we show that the scaling relation can be used to convert the death of solutions at large times to the singularity of solutions at a fixed time. Applying the scaling Lemma 2, with the choices $a = \varepsilon^{-d}$, $b = \varepsilon^2$ and $c = \varepsilon$, we find that, under the initial condition $u_0(dx) = C dx$ (where $dx$ is Lebesgue measure), that $u_t(B(0,\varepsilon))$ has the same distribution as $\varepsilon^d u_{t/\varepsilon^2}(B(0,1))$. Also, as in the proof of Theorem 2 ii), the linearity of the equation and the concavity of the function $z \to \sqrt{z}$ imply that the map $\mu \to Q_\mu[U_t(B(0,\varepsilon))^{1/2}]$ is increasing and concave in $\mu$.

Take a solution $\{u_t(dx)\}$ with $u_0$ of locally bounded intensity. Then, for fixed $t > 0$,

$$E\left[\frac{u_t(B(0,\varepsilon))^{1/2}}{\varepsilon^{d/2}}\right]$$

$$
\begin{aligned}
&= \int_{\mathcal{H}_{\alpha+}} Q_\mu \left[ \frac{U_{t/2}(B(0,\varepsilon))^{1/2}}{\varepsilon^{d/2}} \right] P(u_{t/2} \in d\mu) \quad \text{(by the Markov property)} \\
&\leq Q_{E(u_{t/2}(dx))} \left[ \frac{U_{t/2}(B(0,\varepsilon))^{1/2}}{\varepsilon^{d/2}} \right] \quad \text{(by Jensen's inequality)} \\
&\leq Q_{C(t)L(dx)} \left[ \frac{U_{t/2}(B(0,\varepsilon))^{1/2}}{\varepsilon^{d/2}} \right] \quad \text{(since } u_0 \text{ has bounded intensity)} \\
&= Q_{C(t)L(dx)} \left[ U_{t/2\varepsilon^2}(B(0,1))^{1/2} \right] \quad \text{(by scaling)} \\
&= (C(t))^{1/2} \, Q_{I_{B(0,1)}} \left[ U_{t/2\varepsilon^2}(1)^{1/2} \right] \quad \text{(by self-duality, Proposition 1)} \\
&\to 0 \quad \text{as } \varepsilon \downarrow 0 \quad \text{(by Proposition 2).}
\end{aligned}
$$

The same result holds true if $B(0,\varepsilon)$ is replaced by $B(x,\varepsilon)$ for any $x \in \mathbf{R}^d$. We may decompose the measure $u_t = u_t^{(ac)} + u_t^{(s)}$ into its absolutely continuous and singular parts and write $u_t^{(ac)} = A_t(x)dx$ for a locally $L^1$ function $A_t(x) \geq 0$. Then

$$
\begin{aligned}
E\left[ \int A_t^{1/2}(x)dx \right] &= \int E\left[ \lim_{\varepsilon \downarrow 0} \frac{u_t^{(ac)}(B(x,\varepsilon)^{1/2})}{\varepsilon^{d/2}} \right] dx \quad \text{(Lebesgue differentiation theorem)} \\
&\leq \int \lim_{\varepsilon \downarrow 0} E\left[ \frac{u_t^{(ac)}(B(x,\varepsilon)^{1/2})}{\varepsilon^{d/2}} \right] dx \quad \text{(Fatou's lemma)} \\
&\leq \int \lim_{\varepsilon \downarrow 0} E\left[ \frac{u_t(B(x,\varepsilon)^{1/2})}{\varepsilon^{d/2}} \right] dx = 0.
\end{aligned}
$$

Thus $A_t = 0$ with probability one.

In general we may decompose the initial condition $u_0 \in \mathcal{H}_{\alpha+}$ as a countable sum of measures $u_0 = \sum_n u_0^{(n)}$ where each $u_0^{(n)}$ has locally bounded intensity. Use a single noise to define chaos expansion solutions $u_t^{(n)}(dx)$ with initial conditions $u_0^{(n)}$. It is easy to check that $\sum_n u_t^{(n)}(dx)$ is a solutions started at $u_0$. Then, applying the above argument to each $u_t^{(n)}$ yields the desired result in the general case. This completes the proof of Theorem 3 iii).

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