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Percolation Transition for Some Excursion Sets

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Abstract: We consider a random field $(X_n)_{n\in\mathbb{Z}^d}$ and investigate when the set $A_h = \{k \in \mathbb{Z}^d; |X_k| \geq h\}$ has infinite clusters. The main problem is to decide whether the critical level $h_c = \sup\{h \in \mathbb{R}; P(A_h \text{ has an infinite cluster}) > 0\}$ is neither 0 nor $+\infty$. Thus, we say that a percolation transition occurs. In a first time, we show that weakly dependent Gaussian fields satisfy to a well-known criterion implying the percolation transition. Then, we introduce a concept of percolation along reasonable paths and therefore prove a phenomenon of percolation transition for reasonable paths even for strongly dependent Gaussian fields. This allows to obtain some results of percolation transition for oriented percolation. Finally, we study some Gibbs states associated to a perturbation of a ferromagnetic quadratic interaction. At first, we show that a transition percolation occurs for superstable potentials. Next, we go to the the critical case and show that a transition percolation occurs for directed percolation when $d \geq 4$. We also note that the assumption of ferromagnetism can be relaxed when we deal with Gaussian Gibbs measures, *i.e.* when there is no perturbation of the quadratic interaction.

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Introduction

In the last twenty years, percolation processes have taken a major place in the modeling of disordered spatial systems, e.g. of inhomogeneous media. Of course, the mathematical study of dependent percolation is not as well advanced as those of Bernoulli percolation. In spite of this, by the appearance of new powerful tools [13] and by its deep relationships with some model of statistical mechanic, dependent percolation became an exciting area of research. We refer the reader to the stimulating book by Georgii, Häggström and Maes [10] for an overview of this large virgin country.

We will concentrate here about the problem of percolation transition for some families of dependent fields. The questions are simple to formulate:

- Given a stationary random field $(X_n)_{n\in\mathbb{Z}^d}$, for which values of h does the so-called excursion set
$$E_h = \{k \in \mathbb{Z}^d; X_k \geq h\}$$
have an infinite connected component with a positive probability ?
- If this happens with positive probability, does it actually happens almost surely ?

It is also natural to introduce the critical level

$$h_c = \sup\{h \in \mathbb{R}; P(\text{the origin belongs to an infinite cluster}) > 0\}.$$

We say that there is a percolation transition if h_c belongs to the interior of the support of the distribution of a single site variable. In this paper, we will deal with a random field which is obtained as the absolute value of an initial random field. It means that we study

$$A_h = \{k \in \mathbb{Z}^d; |X_k| \geq h\},$$

not E_h.

The case of a Gaussian field $(X_n)_{n\in\mathbb{Z}^d}$ is naturally interesting. It is actually used by physicists as a model of composite media. Excursion sets E_h are denoted as one-level cut Gaussian Random Models, whereas excursion sets A_h correspond to two-level cut Gaussian Random Model. We refer the reader to Roberts and Teubner [19], Roberts and Knackstedt [18], and the references therein for more information.

The mathematical treatment of the problem was initiated by Molchanov and Stepanov. At the beginning [14] of a cycle of three papers [14, 15, 16] about dependent percolation, they have formulated a simple criterion to ensure the presence (or absence) of percolation for a low (or high) level h. The study of E_h for weakly correlated Gaussian fields was one of their applications. Later, Bricmont, Lebowitz and Maes [2] provided the first example of a percolation transition for a system with infinite susceptibility. The aim of this paper is the mathematical study of A_h for some random fields $(X_n)_{n\in\mathbb{Z}^d}$. Of course we will deal with Gaussian fields, but our study will not be limited to these fields: we will also study some Gibbs measures associated to a perturbation of a ferromagnetic quadratic interaction.

In section 1, we show how to apply the criterion of Molchanov and Stepanov to a weakly dependent Gaussian fields and obtain the existence of a percolation transition for stationary Gaussian fields with finite susceptibility.

In section 2, we show how some restrictions on the geometry of the percolating cluster allows to replace the Molchanov-Stepanov criterion by a weakened condition.

Then, we can ensure the absence of percolation along "reasonable" clusters even if the dependence of the underlying process is strong.

These results are used in section 3 to prove the existence of a percolation transition for dependent oriented percolation on $\mathbb{Z}^d$ in cases where the Molchanov-Stepanov criterion is not satisfied. An example is also given.

The goal of section 4 is to extend the preceding results to show percolation transition for some Gibbs measures associated to a perturbation of a quadratic interaction. For superstable and ferromagnetic interactions, we show the existence of a percolation transition for (unoriented) site percolation. When the assumption of superstability is not satisfied, we nevertheless obtain the existence of a percolation transition for directed site percolation when $d \geq 4$. Finally, we remark that the assumption of ferromagnetism can also be relaxed when we consider Gaussian Gibbs measures, *i.e.* when there is no perturbation of the quadratic interaction.

For some proofs, we will need some results related to the control of the covariance of stationary Gaussian processes with a spectral density which can have some singular points. For readability, Fourier analytic results have been relegated to the final section.

Notations

0.1. **Graphs and lattices.** A directed graph (or digraph) is a couple $G = (V, E)$ with $E \subset V \times V$. We say that two vertices $x, y \in \mathbb{Z}^d$ are adjacent in G if $(x, y) \in E$.

The neighborhood of a set A is

$$\mathcal{V}_G(A) = \bigcup_{x \in A} \{y; (x, y) \in E\}.$$

A path from x to y is a sequence of vertices with x as the first element and y as the last one such that each element of the sequence is adjacent in G with the next one. The set of points which can be reached from x is denoted by $C_G(x)$.

Let $\Omega = \mathbb{R}^E$ and P be a probability measure on $(\Omega, \mathcal{B}(\Omega))$. As usually, $X_k : \Omega \to \mathbb{R}$ denotes the canonical projection on the k-th component.

Let h be a positive number. For a given digraph (V, E), we will consider the random subgraphs $(V, E^X_{h^+})$ and $(V, E^X_{h^-})$ of (E, V), where $E^X_{h^+}$ and $E^X_{h^-}$ are the subset of V defined by

$$E^X_{h^+} = \{(i, j) \in V \times V; |X_i| \geq h \text{ and } |X_j| \geq h\} \tag{1}$$

and

$$E^X_{h^-} = \{(i, j) \in V \times V; |X_i| < h \text{ and } |X_j| < h\}. \tag{2}$$

For $x \in V$, we define the random set $C^X_{h^+}(x)$ (*resp.* $C^X_{h^-}(x)$) to be the set $C_H(x)$ with $H = (V, E^X_{h^+})(x)$. (*resp.* $H = (V, E^X_{h^-})(x)$).

We will say that a realization of the field $(X_k)_{k \in E}$ percolates over h (*resp.* under h) if $(V, E^X_{h^+})$ (*resp.* $(V, E^X_{h^-})$) contains at least one infinite cluster. For $x \in V$, we say that a realization of the field $(X_k)_{k \in V}$ percolates over h (*resp.* under h) from x if $|C^X_{h^+}(x)| = +\infty$ (*resp.* $|C^X_{h^-}(x)| = +\infty$).

We will work here with classical graphs built on $\mathbb{Z}^d$ or $\mathbb{Z}^d_+$: for $x \in \mathbb{Z}^d$, let us define $\|x\|_1 = \sum_{i=1}^d |x_i|$ and $\|x\|_\infty = \sup\{|x_i|; 1 \leq i \leq d\}$. We will currently work with $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, with

$$\mathbb{E}^d = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d; \|y - x\|_1 = 1\}$$

and $\overrightarrow{\mathbb{L}}^d = (\mathbb{Z}_+^d, \overrightarrow{\mathbb{E}}^d)$, with

$$\overrightarrow{\mathbb{E}}^d = \{(x,y) \in \mathbb{Z}_+^d \times \mathbb{Z}_+^d; \sum_{i=1}^d y_i > \sum_{i=1}^d x_i \text{ and } \|y-x\|_1 = 1\}.$$

We note by μ_d the d-dimensional connective constant, that is

$$\mu_d = \lim_{n \to +\infty} c(d,n)^{1/n},$$

where $c(d,n)$ is the number of injective paths on $\mathbb{L}^d$ starting from the origin and whose length is n.

If A if a subset of $\mathbb{Z}^d$, we denote by $\mathrm{Mod}(A)$ the smallest subgroup of $(\mathbb{Z}^d,+)$ which contains A.

0.2. **Gibbs measures.** Let us recall the concept of Gibbs measure. Each $\omega \in \Omega = \mathbb{R}^{\mathbb{Z}^d}$ can be considered as a map from $\mathbb{Z}^d$ to $\mathbb{R}$. We will denote ω_Λ its restriction to Λ. Then, when A and B are two disjoint subsets of $\mathbb{Z}^d$ and $(\omega,\eta) \in \mathbb{R}^A \times \mathbb{R}^B$, $\omega\eta$ denotes the concatenation of ω and η, that is the element $z \in \mathbb{R}^{A \cup B}$ such that

$$z_i = \begin{cases} \omega_i & \text{if } i \in A \\ \eta_i & \text{if } i \in B. \end{cases}$$

For finite subset Λ of $\mathbb{Z}^d$, we define $\sigma(\Lambda)$ to be the σ-field generated by $\{X_i, i \in \Lambda\}$.

For every finite Λ in $\mathbb{Z}^d$, let Φ_Λ be a real-valued $\sigma(\Lambda)$-measurable function. The family $(\Phi_\Lambda)_\Lambda$, when Λ describes the finite subsets of $\mathbb{Z}^d$, is called an interaction potential, or simply a potential. For a finite subset Λ of $\mathbb{Z}^d$, the quantity

$$H_\Lambda = \sum_{B:\ B\cap\Lambda\neq\emptyset} \Phi_B$$

is called the Hamiltonian on the volume Λ. Usually, H_Λ can be defined only on a subset of $\mathbb{R}^{\mathbb{Z}^d}$. We suppose that there exists a subset $\tilde{\Omega}$ of Ω such that

$$\forall \text{ finite } \Lambda \ \forall \omega \in \tilde{\Omega} \qquad \sum_{B:\ B\cap\Lambda\neq\emptyset} |\Phi_B(\omega)| < +\infty.$$

$(H_\Lambda)_\Lambda$ is called the Hamiltonian.

We now define the so called partition function Z_Λ: denoting by λ the Lebesgue's measure on the real line, we let

$$Z_\Lambda(\omega) = \int_{\mathbb{R}^\Lambda} \exp(-H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})) d\lambda^{\otimes\Lambda}(\eta_\Lambda).$$

By convention, we set $\exp(-H_\Lambda(\eta_\Lambda\omega_{\Lambda^c})) = 0$ when the Hamiltonian is not defined.

We suppose that for each ω in $\tilde{\Omega}$, we have $0 < Z_\Lambda(\omega) < +\infty$. Then, we can define for each bounded measurable function f and for each $\omega \in \tilde{\Omega}$,

$$\Pi_\Lambda f(\omega) = \frac{\int_{\mathbb{R}^\Lambda} \exp(-H_\Lambda(\eta_\Lambda\omega_{\Lambda^c})) f(\eta_\Lambda\omega_{\Lambda^c}) d\lambda^{\otimes\Lambda}(\eta_\Lambda)}{Z_\Lambda(\omega)}.$$

For each ω, we will denote by $\Pi_\Lambda(\omega)$ the measure on Ω which is associated to map $f \mapsto \Pi_\Lambda f(\omega)$.

If a measure μ on Ω is such that $\mu(\tilde{\Omega}) = 1$, we say that μ is a Gibbs measure or a Gibbsian field when for each bounded measurable function f and each finite subset Λ of $\mathbb{Z}^d$, we have

$$E_\mu(f|(X_i)_{i\in\Lambda^c}) = \Pi_\Lambda f \quad \mu \text{ a.s.}$$

Let $J : \mathbb{Z}^d \to \mathbb{R}$ be an even function such that $\sum_{i\in\mathbb{Z}^d} |J(i)| < +\infty$ and V a continuous function.

Given these parameters, we deal with Gibbsian random fields μ associated to the potential $\Phi^{J,V}$ defined on Ω by

$$\Phi_\Lambda^{J,V}(\omega) = \begin{cases} \frac{1}{2}(J(0)\omega_i^2 + V(\omega_i)) & \text{if } \Lambda = \{i\} \\ J(i-j)\omega_i\omega_j & \text{if } \Lambda = \{i,j\}, i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Then, the corresponding Hamiltonian function is equal to

$$H_\Lambda^{J,V}(\omega) = \frac{1}{2}\sum_{i\in\Lambda} V(\omega_i) + \frac{1}{2}\sum_{i,j\in\Lambda} J(i-j)\omega_i\omega_j + \sum_{i\in\Lambda, j\in\Lambda^c} J(i-j)\omega_i\omega_j. \tag{3}$$

We can define

$$\tilde{\Omega} = \{\omega \in \mathbb{R}^{\mathbb{Z}^d} \quad \forall i \in \mathbb{Z}^d \quad \sum_{j\in\mathbb{Z}^d} |J(i-j)\omega_j| < +\infty\}.$$

On $\tilde{\Omega}$, H_Λ is well defined. It is clear that it could not be possible to take a larger $\tilde{\Omega}$, so this is a canonical choice.

For fixed (J,V), we denote by $\mathfrak{G}_{J,V}$ the set of Gibbs measures on $\mathbb{R}^{\mathbb{Z}^d}$ associated to the Hamiltonian given in (3). If $\mathfrak{G}_{J,V}$ contains more than one point, we say that phase transition occurs. $\mathfrak{G}_{J,V}$ is a convex set whose extreme points are called pure phases. (For general results on Gibbs measures, see [9].)

For $z = (z_1, ..., z_d) \in \mathbb{C}^d$ and $n = (n_1, ..., n_d) \in \mathbb{Z}^d$, we set

$$z^n = \prod_{i=1}^{n} z_i^{n_i} \text{ and } |n| = \sum_{i=1}^{d} |n_i|,$$

$$\mathbb{U} = \{z \in \mathbb{C}^d,\ \forall i \in \{1, \dots, d\}\ |z_i| = 1\}.$$

We introduce $\hat{J}$, the dual function of J, defined on a subset of $\mathbb{C}^d$ by

$$\hat{J}(z) = \sum_{n\in\mathbb{Z}^d} J(n)z^n. \tag{4}$$

whenever the considered series is absolutely convergent. Since J is summable, it is clear that $\hat{J}$ always defines a continuous map on $\mathbb{U}$. We denote by dz the normalized Haar measure on $\mathbb{U}$. In other words, if f is a measurable function on $\mathbb{U}$, we have

$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi[^d} f(e^{i\theta_1}, \dots, e^{i\theta_d})\ d\theta_1 \dots d\theta_d = \int_{\mathbb{U}} f(z)\ dz.$$

By the way, $\forall n \in \mathbb{Z}^d \quad \int_{\mathbb{U}} \hat{J}(z)z^{-n} = J(n)$.

0.3. **Miscellaneous.** We recall that J_ν is the Bessel function of first order with index ν, that is

$$J_\nu(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(i(x\sin\theta - \nu\theta))d\theta = \frac{(x/2)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\int_{-1}^{+1}(1-t^2)^{\nu-\frac{1}{2}}e^{-itx}\ dt.$$

If f is a C^N-smooth function on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, we denote by $D_x^N f$ the N^{th} derivative of f at point x: it is a linear map from $(\mathbb{R}^d)^{\otimes N}$ to $\mathbb{R}$.

1. Weakly Dependent Gaussian Fields

A natural approach to generate some dependent random fields is to use Gaussian fields. In their pioneering paper [14], Molchanov and Stepanov consider Gaussian variables with a bounded spectral density as an illustration of their criterion. They proved that for large h, $\{k \in \mathbb{Z}^d; X_k \geq h\}$ does not percolate. By a symmetry argument, their result also implies the existence of a percolation transition.

In the present paper, we will consider the problem of percolation for $\{k \in \mathbb{Z}^d; |X_k| \geq h\}$.

At first, let us recall the Molchanov-Stepanov criterion. In this proposition, two vertices i and j are said to be adjacent if $\|i-j\|_1 = 1$ and to be $*$-adjacent if $\|i-j\|_\infty = 1$.

Proposition 1 (Molchanov and Stepanov). *There exist two finite constants c_d^{dis} and c_d^{agr} only depending from the dimension such that for each $\{0,1\}$-valued random field $(X_k)_{k\in\mathbb{Z}^d}$, we have the following results:*

- *If there exists $C > 0$ such that for each connected set A, we have*
$$P(\forall k \in A; X_k = 1) \leq C\exp(-c_d^{dis}|A|),$$
then $\{k \in \mathbb{Z}^d; X_k = 1\}$ has almost surely only finite clusters.
- *If there exists $C > 0$ such that for each $*$-connected set A, we have*
$$P(\forall k \in A; X_k = 0) \leq C\exp(-c_d^{agr}|A|),$$
then $\{k \in \mathbb{Z}^d; X_k = 1\}$ has almost surely at least one infinite cluster.

We well need some lemmas. Note that some of them (that is Lemma 2 and Lemma 3) were (at least implicitly) used by Molchanov and Stepanov in their proof of the absence of an infinite cluster in $\{k; X_k \geq h\}$ for large h. We recall that we consider here $\{k; |X_k| \geq h\}$, not $\{k; X_k \geq h\}$.

1.1. A percolation transition result.

Theorem 1. *Let $(X_n)_{n\in\mathbb{Z}}$ be a centered stationary Gaussian field with finite susceptibility,* i.e. *such that*

$$\sum_{k\in\mathbb{Z}^d} |c_k| < +\infty, \text{ with } c_k = \mathbb{E}\ X_0X_k.$$

Then, let us define

$$h_+ = \inf\{a \geq 0; P(|C_{a^+}^X(0)| = +\infty) = 0\}$$

and

$$h_- = \sup\{a \geq 0; P(|C_{a^-}^X(0)| = +\infty) = 0\}.$$

Then,

- $0 < h_+ < +\infty$ *and* $0 < h_- < +\infty$.
- *For each* $a > h_+$ *there is almost surely not percolation over level* a, *whereas there is almost surely percolation over level* a *for* $a < h_+$.
- *For each* $a < h_-$ *there is almost surely not percolation under level* a, *whereas there is almost surely percolation under level* a *for* $a > h_-$.

1.2. **Proof of theorem 1.** The proof of theorem 1 will need some lemmas. Some of these will be used again to get further results.

We begin with an elementary but useful lemma:

Lemma 1. *Let* X *be a* $\mathbb{R}^n$*-valued centered Gaussian vector with covariance matrix* C.

- *For each real* α *with* $\alpha < \rho(C)^{-1}$, *one has*
$$\mathbb{E}\ \exp(\frac{\alpha}{2}\|X\|_2^2) = \prod_i (1-\alpha\lambda_i)^{-1/2}.$$
where the λ_i*'s are the eigenvalues of* C.
- *Moreover, if* $0 \leq \alpha < \rho(C)^{-1}$, *then*
$$\mathbb{E}\ \exp(\frac{\alpha}{2}\|X\|_2^2) \leq (1-\alpha\rho(C))^{-n/2}, \tag{5}$$
where $\rho(C)$ *is the spectral radius of* C.

Lemma 2. *Let* X *be a* $\mathbb{R}^n$*-valued centered Gaussian vector with covariance matrix* C *and* $a^2 > \rho(C)$. *Then*

$$P(\|X\|^2 \geq na^2) \leq e^{-nh(\frac{a^2}{\rho(C)})}, \tag{6}$$

where $h(x) = \frac{1}{2}(x - \ln x - 1)$. h *is increasing and positive on* $(1,+\infty)$, *with* $+\infty$ *as limit at* $+\infty$.

Proof. For each $\alpha > 0$, we have

$$\begin{aligned} P(\|X\|^2 \geq na^2) &= P(\exp(\frac{\alpha}{2}\|X\|^2) \geq \exp(\frac{\alpha}{2}na^2)) \\ &\leq \frac{\mathbb{E}\ \exp(\frac{\alpha}{2}\|X\|^2)}{\exp(\frac{\alpha}{2}na^2)}. \end{aligned}$$

If moreover $\alpha < \rho(C)^{-1}$, it follows from lemma 1 that

$$\begin{aligned} P(\|X\|^2 \geq na^2) &\leq \frac{(1-\alpha\rho(C))^{-n/2}}{\exp(\frac{\alpha}{2}na^2)} \\ &\leq ((1-\alpha\rho(C))\exp(\alpha a^2))^{-n/2}. \end{aligned}$$

We choose $\alpha = \frac{1}{\rho(C)} - \frac{1}{a^2}$ and get

$$\begin{aligned} P(\|X\|^2 \geq na^2) &\leq (\frac{\rho(C)}{a^2}\exp(\frac{a^2}{\rho(C)} - 1)^{-n/2} \\ &\leq e^{-nh(\frac{a^2}{\rho(C)})}. \end{aligned}$$

□

Note that the proof of Lemma 2 follows the standard of the theory of large deviations. h naturally appears as the function associated to a χ^2 distribution in Chernof's theorem.

Now, we can claim the lemma which contains the half part of our first result about percolation transition.

Lemma 3. *Let $(X_n)_{n\in\mathbb{Z}}$ be a centered stationary Gaussian field with bounded spectral density g Then, for each $x^2 > \|g\|_\infty$, we have*

$$P(\{\forall k \in A; |X_k| \geq x\}) \leq \exp(-h(\frac{x^2}{\|g\|_\infty})|A|). \tag{7}$$

Proof. Let $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^d$ and M_g be the Toeplitz operator: $\ell^2(\mathbb{T}) \to \ell^2(\mathbb{T})$ defined by $M_g(f) = gf$. If $A \subset \mathbb{Z}^d$ and P_A is the orthogonal projection from $\ell^2(\mathbb{T})$ into $L = \mathrm{Lin}\{\exp(i\langle.|k\rangle); k \in A\}$, then the matrix of covariance of the vector $\tilde{X} = (X_k)_{k\in A}$ is also the matrix of the restriction of $P_A M_g$ to L. Therefore

$$\rho(C) = \sup_{\{\substack{x \in L \\ \|x\|_2 = 1}} \|P_A M_g x\|_2 \leq \sup_{\{\substack{x \in L \\ \|x\|_2 = 1}} \|M_g x\|_2 \leq \|g\|_\infty$$

Since $\{\forall k \in A; |X_k| \geq x\} \subset \{\|\tilde{X}\|^2 \geq |A|x^2\}$, it just remains to apply lemma 2 $\square$

We will now turn to the reverse side of the percolation transition.

Lemma 4. *Let X be a n-dimensional Gaussian vector with positive definite covariance matrix C. Let us denote by $\Upsilon(C)$ the spectral gap* i.e. *the smallest eigenvalue of C. Then, for each $a^2 < \Upsilon(C)$, we have*

$$P(\|X\|^2 \leq na^2) \leq e^{-nh(\frac{a^2}{\Upsilon(C)})}, \tag{8}$$

where $h(x) = \frac{1}{2}(x - \ln x - 1)$. h is positive and decreasing on $(0,1)$, with an infinite limit at 0.

Proof. For each $\alpha > 0$, we have

$$\begin{aligned} P(\|X\|^2 \leq na^2) &= P(\exp(-\frac{\alpha}{2}\|X\|^2) \geq \exp(\frac{-\alpha}{2}na^2)) \\ &\leq \frac{\mathbb{E}\ \exp(-\frac{\alpha}{2}\|X\|^2)}{\exp(-\frac{\alpha}{2}na^2)}. \end{aligned}$$

By lemma 1, it follows that

$$\begin{aligned} P(\|X\|^2 \leq na^2) &\leq (1+\alpha\Upsilon(C))^{-n/2}\exp(\frac{\alpha}{2}na^2) \\ &\leq ((1+\alpha\Upsilon(C))\exp(\alpha a^2))^{-n/2}. \end{aligned}$$

Then, we choose $\alpha = \frac{1}{a^2} - \frac{1}{\Upsilon(C)}$ and get

$$P(\|X\|^2 \leq na^2) \quad \leq \quad e^{-nh(\frac{a^2}{\Upsilon(C)})}.$$

$\square$

Lemma 5. *Let $(X_n)_{n\in\mathbb{Z}}$ be a centered stationary Gaussian field with variance $\sigma^2 > 0$ and with finite susceptibility,* i.e. *such that*

$$\sum_{k\in\mathbb{Z}^d} |c_k| < +\infty, \text{ with } c_k = \mathbb{E}\ X_0 X_k.$$

Then, there exists $f : (0,\sigma^2) \to (0,+\infty)$ such that for each finite set $A \subset \mathbb{Z}^d$, we have

(9) $$P(\{\forall k \in A; |X_k| \le x\}) \le \exp(-f(x)|A|).$$

f is positive and decreasing on $(0,1)$, with an infinite limit at 0.

Proof. Let $\varepsilon \in (0, \sigma^2)$. Since

$$\sum_{i \in N_\varepsilon \mathbb{Z}^d \setminus \{0\}} |c_i| \quad \le \quad \sum_{\|i\| \ge N_\varepsilon} |c_i|,$$

we can find $N_\varepsilon \in \mathbb{Z}_+$ such that

$$\sum_{i \in N_\varepsilon \mathbb{Z}^d \setminus \{0\}} |c_i| \le \varepsilon.$$

For each $k \in \{0, N_\varepsilon - 1\}^d$, we can define $A_k = A \cap (k + N_\varepsilon \mathbb{Z}^d)$. By the pigeonhole principle, there exists k such that $|A_k| \ge \frac{|A|}{N_\varepsilon^d}$ Let $\tilde{X}$ be the $|A_k|$-dimensional Gaussian vector composed by the $(X_i)_{i \in A_k}$, it is obvious that

$$P(\{\forall k \in A; |X_k| \le x\}) \le P(\|\tilde{X}\|_2^2 \le nx^2).$$

By lemma 4,

$$\forall x \in (0, \Upsilon(C)) \quad P(\|\tilde{X}\|_2^2 \le nx^2) \le \exp(-|A_k| h(\frac{x^2}{\Upsilon(C)})),$$

where $C = (c_{i-j})_{(i,j) \in A_k \times A_k}$. But

$$\begin{aligned} \Upsilon(C) &\ge \sigma^2 - \sup_{j \in A_k} \sum_{i \in A_k; i \ne j} |c_{i-j}| \\ &\ge \sigma^2 - \sum_{i \in N_\varepsilon \mathbb{Z}^d \setminus \{0\}} |c_i| \\ &\ge \sigma^2 - \varepsilon. \end{aligned}$$

It follows that

(10) $$\forall x \in (0, \sigma^2 - \varepsilon) \quad P(\{\forall k \in A; |X_k| \le x\}) \le \exp(-\frac{h(\frac{x^2}{\sigma^2 - \varepsilon})}{N_\varepsilon^d} |A|).$$

□

We now dispose from the tools needed to prove Theorem 1 itself.

Proof of Theorem 1. Since $(X_n)_{n \in \mathbb{Z}^d}$ has a finite susceptibility, it has a spectral density, and therefore has a spectral measure without atoms. Then, by a result of Maruyama and Fomin (see for example [22], lecture 13), it follows that the law of $(X_n)_{n \in \mathbb{Z}^d}$ is ergodic under the group of translations of $\mathbb{Z}^d$. Since the existence of an infinite cluster is a translation-invariant event, it follows that the existence of a percolating cluster is a deterministic event. By monotonicity, $\{a \ge 0; P(|C^X_{a^+}(0)| = +\infty) = 0\}$ and $\{a \ge 0; P(|C^X_{a^-}(0)| = +\infty) = 0\}$ are intervals. It follows that when $a < a_+$ (*resp.* when $a > a_-$), we have $P(|C^X_{a^+}(0)| = +\infty) > 0$ (*resp.* $P(|C^X_{a^-}(0)| = +\infty) > 0$). In both cases, the probability of percolating is positive, and then equal to one. As in the case of independent percolation, the almost sure absence of percolation from the origin imply the almost sure absence of percolation from everywhere using the stationarity of $(X_n)_{n \in \mathbb{Z}^d}$ and the denumerability of $\mathbb{Z}^d$.

By lemma 5, if a is so small enough that $f(a) > c_d^{dis}$, then the Criterion of Molchavov and Stepanov says that there is no percolation under a. Again by lemma 5, if a is so small enough that $f(a) > c_d^{agr}$, then the Criterion of Molchavov and Stepanov says that there is percolation over a. Since X has a spectral density, one can apply lemma 3: if a is so large enough that $h(\frac{a^2}{\|g\|_\infty}) > c_d^{dis}$, then the Criterion of Molchavov and Stepanov says that there is no percolation over a. Similarly, if a is so large enough that $h(\frac{a^2}{\|g\|_\infty}) > c_d^{agr}$, then the Criterion of Molchavov and Stepanov says that there is percolation under a.

□

2. Reasonable percolating sets

2.1. The concept of reasonable sets. We will define some families of subset of $\mathbb{Z}^d$. For $s \in [1, d]$, let

$$\mathcal{M}_{s,K} = \{A \subset \mathbb{Z}^d \quad \sup_{x\in A} \sup_{r\geq 1} \frac{|A \cap B(x,r)|}{(2r+1)^s} \leq K\}, \tag{11}$$

where

$$B(x,r) = \{y \in \mathbb{Z}^d; \|x-y\|_\infty \leq r\}.$$

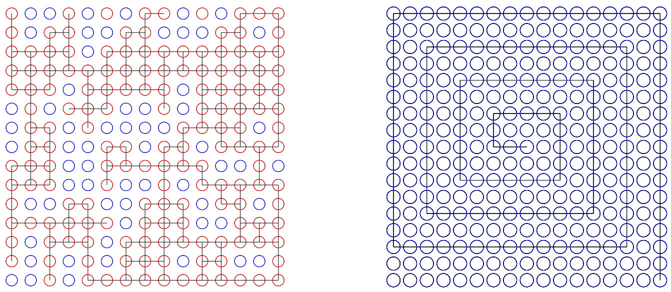
The elements of $\mathcal{M}_{s,K}$ are said to be (s,K) - reasonable sets. Similarly, we say that a path in $\mathbb{Z}^d$ is (s,K) - reasonable if and only if its support is a (s,K) - reasonable sets.

Of course, every subset of $\mathbb{Z}^d$ belongs to $\mathcal{M}_{d,1}$.

The following remark is fundamental: if A is the support of a path in the oriented graph $\overrightarrow{\mathbb{E}}^d$, then $A \in \mathcal{M}_{1,d}$

Roughly speaking, s represents the dimension that the path is allowed to take and K/s represents an upper bound for the density of the path in a s-dimensional space.

The next pictures try to give a feeling of what is a reasonable or an improper path in $\mathbb{Z}^2$, with $s = 1$.



a reasonable path an improper path

In the left picture, it is not difficult to extract from the big cluster a fine path joining the center of the picture to its border. It is clear that such a thread is very far from filling any portion of the plane: it can be designed as a a reasonable path. The right picture shows a long path, looking like a spiral. It is manifestly the only path from the center of the picture to its border: if any link is broken, the origin become isolated from the border of the picture. In this case, the thread is like a ball of wool, filling a portion of plane – not at all a reasonable path.

When the decay of the correlation of the Gaussian process is too slow, one is not able to prove the absence of percolation. Nevertheless, we will see that we can sometimes prove the absence of an infinite reasonable path.

Before this, we want to motivate the introduction of reasonable sets by a comparison with what happens in Bernoulli percolation. Obviously, we are preserved from the disaster of an empty concept by the possibility of oriented site percolation, which is always $(1,d)$-reasonable.

One can also note that Bernoulli supercritical percolation enjoys from a property which is a bit weaker than $(1,K)$-reasonable percolation: when $p>p_c$, there exists $K<+\infty$ such that we almost always have an infinite connected subset A of $\mathbb{Z}^d$ with

$$\sup_{x\in A}\ \overline{\lim_{r\geq 1}}\ \frac{|A\cap B_1(x,r)|}{r}\leq K, \tag{12}$$

with $B_1(x,r)=\{y\in\mathbb{Z}^d;\|x-y\|_1\leq r\}$.

Proof. By a classical compactness argument, one can build a semi-infinite geodesic in the infinite cluster, that is a sequence $(x_n)_{n\geq 1}$ of open sites with $\|x_n-x_{n+1}\|_1=1$ for each n, and such that for each $1\leq k\leq n$, the sequence $(x_k,x_{k+1},\dots,x_n)$ realizes a minimal path from x_k to x_n using only open edges. Let us now denote $A=\{x_n;n\geq 1\}$ and consider $x\in A$ and $r\geq 1$: it is easy do see that there exist a maximal k and a minimal n such that $A\cap B_1(x,r)\subset\{x_k,x_{k+1},\dots,x_n\}$. It follow that $|A\cap B(x,r)|\leq D(x,x_k)+D(x,x_n)+1$. On one hand, we have $D(x,x_k)\leq D(x,x_1)$. On the other hand, by definition of n, we necessarily have $\|x-x_n\|_\infty=r$. We can now use a result of Antal and Pisztora ([1], corollary 1.3), which gives a bound for the asymptotic ratio between the chemical distance D and the $\|.\|_1$ distance on $\mathbb{Z}^d$: there exists a constant $\rho(p,d)$ such that

$$\overline{\lim_{\|y\|_1\to+\infty}}\ \frac{D(0,y)}{\|y\|_1}1\!\!1_{\{0\leftrightarrow y\}}\leq\rho(p,d)\ \text{a.s.} \tag{13}$$

It follows that

$$\overline{\lim_{r\geq 1}}\ \frac{|A\cap B_1(x,r)|}{r}\leq\rho(p,d)\ \text{a.s.}\ .$$

Since A is denumerable, it follows that

$$\sup_{x\in A}\ \overline{\lim_{\|y\|_1\to+\infty}}\ \frac{D(0,y)}{\|y\|_1}1\!\!1_{\{0\leftrightarrow y\}}\leq\rho(p,d)\ \text{a.s.}$$

□

We conjecture that $\rho(p,d)$ is not the best value for K: if one could prove, for $x\in\mathbb{Z}^d$, the existence of a semi-infinite geodesic with asymptotic direction $\hat{x}$ – that is, with $\lim_{n\to+\infty}\frac{x_n}{\|x_n\|_1}=\hat{x}=\frac{x}{\|x\|_1}$ –, it would lead to the existence of a percolating cluster which satisfy to equation (12) with $K=\mu(\hat{x})=\frac{1}{\|x\|}\mu(x)$, where $\mu(x)$ is the non-random limit:

$$\lim_{\substack{n\to+\infty\\ 0\leftrightarrow ny}}\frac{D(0,ny)}{n}=\mu(y)\ \ \text{a.s.}$$

The map $x \mapsto \mu(x)$ is a norm. It plays the same role than the homonym function in first-passage percolation. The complete proof of the last assertions can be done using the asymptotic shape theorem for the chemical distance [8]. Now, if μ is not proportional to $\|.\|_1$ – this is at least the case when $d = 2$ and $p \neq \overrightarrow{p_c}$, see Theorem 6.3 in [8] – one can choose x such that $\mu(\hat{x}) < \rho(p, d)$, which proves that $\rho(p, d)$ is not the smallest convenient value for K.

The existence of semi-infinite geodesics in the asymptotic directions $\hat{x}$ seems to be a difficult and reasonable conjecture. There exists an analogue conjecture in classical first-passage percolation, see for instance Newman [17]. Apart from the facts that it would imply that $\rho(p, d)$ is not the smallest value for K, the existence of an infinite family of geodesics can be considered as an heuristic argument to guess that one the these geodesics allow to replace the supremum limit which appears in equation (12) by the supremum which appears in equation (11). Clearly, the fact that in the Bernoulli case percolation would be equivalent to $(1, K)$-reasonable percolation would be a decisive argument in favor to the concept of reasonable percolation.

2.2. Exponential control on reasonable sets.

Theorem 2. *Let $(X_n)_{n\in\mathbb{Z}^d}$ be a centered Gaussian process, $\sigma^2 \in (0, +\infty)$, $s \in [1, +\infty)$ and $\phi : \mathbb{Z}_+ \to \mathbb{R}^+$ such that the following assumptions hold:*

- *ϕ is non-increasing.*
- $\forall i \in \mathbb{Z}^d \quad \mathbb{E}\ X_i^2 \geq \sigma^2$.
- $\forall i, j \in \mathbb{Z}^d \quad |\mathbb{E}\ X_i X_j| \leq \phi(\|i - j\|)$.
- $\sum_{n=1}^{+\infty} n^{s-1}\phi(n) < +\infty$.

Then, for each $K > 0$, one can find two functions $f : (\phi(0), +\infty) \to (0, +\infty)$ and $g : (0, \sigma^2) \to (0, +\infty)$ such that

$$\forall A \in \mathcal{M}_{s,K} \quad P(\{\forall k \in A \quad |X_k| \geq a\}) \leq e^{-f(a)|A|}$$

and

$$\forall A \in \mathcal{M}_{s,K} \quad P(\{\forall k \in A \quad |X_k| \leq a\}) \leq e^{-g(a)|A|},$$

with $\lim_{a\to+\infty} f(a) = +\infty$ *and* $\lim_{a\to 0^+} g(a) = +\infty$.

f and g only depend from σ^2,ϕ, s and K.

Proof. Let ε be a non-negative number and r be any positive integer. The precise choices will be done later. Again by the pigeon-hole principle, we can find $\tilde{A} \subset A$ with $|\tilde{A}| \geq \frac{1}{r^d}|A|$ and such that for each distinct points x and y, one always has $\|x - y\| \geq r$.

The next step consists in bounding from below the spectral gap of the covariance matrix associated to $\tilde{X} = (X_k)_{x\in\tilde{A}}$. It will be done by a classical Hadamard's-like argument: we will bound $\sum_{x\in\tilde{A}\backslash\{y\}} |\mathbb{E}\ X_x X_y|$.

For this purpose, we will use a discrete integration by parts – sometimes called a Abel's transform–: for each $y \in \tilde{A}$ and $k \in \mathbb{Z}_+$, we will define $s_k(y) = |\{x \in \tilde{A}\backslash\{y\}; \|x - y\| = k\}|$ and $b_k(y) = |\{x \in \tilde{A}\backslash\{y\}; \|x - y\| \leq k\}|$.

Of course, $s_k = b_k - b_{k-1}$, $s_0 = 0$ and $b_0 = 0$. Now,

$$\begin{aligned}\sum_{x\in \tilde{A}\backslash\{y\}} |\mathbb{E}\, X_x X_y| &\leq \sum_{x\in \tilde{A}\backslash\{y\}} \phi(\|x-y\|)\\ &= \sum_{n=1}^{+\infty} \phi(n)s_n(x)\\ &= \sum_{n=1}^{+\infty} \phi(n)(b_n(x)-b_{n-1}(x))\\ &= \sum_{n=1}^{+\infty} (\phi(n)-\phi(n+1))b_n(x).\end{aligned}$$

By the definition of $\tilde{A}$, $b_n(x) = 0$ for $n < r$. Then,

$$\begin{aligned}\sum_{x\in \tilde{A}\backslash\{y\}} |\mathbb{E}\, X_x X_y| &\leq \sum_{n=r}^{+\infty} (\phi(n)-\phi(n+1))b_n(x)\\ &\leq \sum_{n=r}^{+\infty} (\phi(n)-\phi(n+1))K(2n+1)^s\\ &= K\big(\phi(r)(2r+1)^s + \sum_{n=r+1}^{+\infty} \phi(n)((2n+1)^s-(2n-1)^s)\big)\\ &\leq K\big(\phi(r)(2r+1)^s + 2s\sum_{n=r+1}^{+\infty} \phi(n)(2n+1)^{s-1}\big).\end{aligned}$$

Since $\sum_{r/2\leq n\leq r} n^{s-1}\phi(n) \geq \phi(r) \sum_{r/2\leq n\leq r} n^{s-1} \geq \phi(r)\frac{r}{2}(\frac{r}{2}-1)^{s-1}$, it follows that $\lim_{r\to+\infty} \phi(r)r^s = 0$. Then, r can be chosen such that

$$\phi(r)(2r+1)^s + 2s\sum_{n=r+1}^{+\infty} \phi(n)(2n+1)^{s-1} \leq \frac{\varepsilon}{K}.$$

We will denote by r_ε the smallest r which can enjoy this property and $\tilde{A}_\varepsilon$ the relative set.

We can now prove the existence of g. We take $\varepsilon \in (0,\sigma^2)$. If C is the covariance matrix associated to $\tilde{X} = (X_k)_{x\in\tilde{A}}$, it is easy to see that $\sigma^2 - \varepsilon \leq \Upsilon(C)$.

Let $x^2 \in (0, \sigma^2-\varepsilon)$: by lemma 4, we have

$$\begin{aligned}P(\forall k\in A; |X_k|\leq x) &\leq P(\|\tilde{X}\|_2^2 \leq |\tilde{A}_\varepsilon|x^2)\\ &\leq \exp(-|\tilde{A}_\varepsilon|h(\frac{x^2}{\Upsilon(C)}))\\ &\leq \exp(-|A|r_\varepsilon^{-d}h(\frac{x^2}{\sigma^2-\varepsilon})).\end{aligned}$$

Then, we can define g by

$$\begin{aligned}g:(0,\sigma^2) &\to (0,+\infty)\\ x &\mapsto \sup_{\varepsilon\in(0,\sigma^2-x^2)} r_\varepsilon^{-d}h(\frac{x^2}{\sigma^2-\varepsilon}).\end{aligned}$$

Similarly, using the fact that $\rho(C) \leq \phi(0)+\varepsilon$, we can define f by

$$\begin{array}{rcl} f:(\phi(0),+\infty) & \to & (0,+\infty) \\ x & \mapsto & \sup_{\varepsilon\in(0,x^2-\phi(0))} r_\varepsilon^{-d} h(\dfrac{x^2}{\phi(0)+\varepsilon}). \end{array}$$

□

2.3. **Absence of reasonable percolation.** Let $x \in \mathbb{Z}^d$, $s \geq 1$ and $K \geq 0$.

We will say that a realization of the field $(X_k)_{k\in E}$ exhibits a (s,K)-reasonable percolation over h (*resp.* under h) if (V, E^X_{h+}) (*resp.* (V, E^X_{h-})) has at least one infinite connected set which belongs to $\mathcal{M}_{s,K}$. We denote by $R^{h,+}_{s,K}$ (*resp.* $R^{h,-}_{s,K}$) this event . For $x \in E$, we say that a realization of the field $(X_k)_{k\in E}$ exhibits a (s,K)-reasonable percolation over h (*resp.* under h) from x if (V, E^X_{h+}) (*resp.* (V, E^X_{h-})) has at least one infinite connected set which belongs to $\mathcal{M}_{s,K}$ and contains x. We also denote by $R^{h,+}_{s,K}(x)$ (*resp.* $R^{h,-}_{s,K}(x)$) this event.

We begin by a general lemma.

Lemma 6. *Let $(X_k)_{k\in\mathbb{Z}^d}$ be a $\{0,1\}$-valued random field. We suppose that there exist $q \in (0, \frac{1}{\mu_d})$ and $C > 0$ such that for each finite set $A \in \mathcal{M}_{s,K}$, we have*

$$P(\forall k \in A; X_k = 1) \leq Cq^{|A|}.$$

Then, there is almost surely no (s,K) reasonable infinite percolating cluster for $\{k \in \mathbb{Z}^d; X_k = 1\}$.

Proof. Let $x \in \mathbb{Z}^d$. Let $n \in \mathbb{Z}_+$. If there is (s,K) reasonable percolation over 1 from x, there exists a self avoiding walk starting from x and whose support S is such that

- $\forall y \in S \quad X_y = 1$.
- $S \in \mathcal{M}_{s,K}$.

Let $\varepsilon > 0$ be such that $(\mu_d+\varepsilon)q < 1$. There exists k_ε such that for each $n \in \mathbb{Z}_+$ and each $x \in \mathbb{Z}^d$, the number of self-avoiding walks starting from x is less $k_\varepsilon(\mu_d+\varepsilon)^n$. Then, we have

$$\forall x \in \mathbb{Z}^d \quad \forall n \in \mathbb{Z}_+ \quad P(R^{1,+}_{s,K}(x)) \leq k_\varepsilon(\mu_d+\varepsilon)^n C q^n,$$

Since n is arbitrary, it follows that $P(R^{1,+}_{s,K}(x)) = 0$ holds for each x and then that $P(R^{1,+}_{s,K}) = 0$. □

Together with the the exponential control on reasonable sets, the previous lemma allows to prove a result related to the absence of reasonable percolation.

Theorem 3. *Let $(X_n)_{n\in\mathbb{Z}^d}$ be a centered Gaussian process, $\sigma^2 \in (0,+\infty)$, $s \in [1,+\infty)$ and $\phi : \mathbb{Z}_+ \to \mathbb{R}^+$ such that the following assumptions hold:*

- *ϕ is non-increasing.*
- $\forall i \in \mathbb{Z}^d \quad \mathbb{E}\, X_i^2 \geq \sigma^2$.
- $\forall i,j \in \mathbb{Z}^d \quad |\mathbb{E}\, X_iX_j| \leq \phi(\|i-j\|)$.
- $\sum_{n=1}^{+\infty} n^{s-1}\phi(n) < +\infty$.

Then, let us define for $K \in (0,+\infty)$

$$h_+(s,K) = \inf\{a \geq 0; P(R^{a,+}_{s,K}) = 0\}$$

and

$$h_-(s,K) = \sup\{a \geq 0; P(R^{a,-}_{s,K}) = 0\}.$$

Then,

- $h_+(s,K) < +\infty$ *and* $0 < h_-(s,K)$.
- *For each* $a > h_+(s,K)$ *there is almost surely no* (s,K)*-reasonable percolation over level* a.
- *For each* $a < h_-(s,K)$ *there is almost surely no* (s,K)*-reasonable percolation under level* a.

Proof. It follows from lemmas 6 and 2 that $h_+(s,K) \leq \inf\{x; f(x) > \ln \mu_d\} < +\infty$ and $h_-(s,K) \geq \sup\{x; g(x) > \ln \mu_d\} > 0$, where f and g have been defined in Theorem 2. □

Remarks

- Be in mind that $(d,1)$-reasonable percolation does not differ from percolation. In this case, Theorem 3 is a consequence of Theorem 1.
- We must confess that we are a bit disappointed that the result about the absence of percolation is limited to reasonable clusters. Roughly speaking, one exchanged a relaxation of the control of the covariance against an enforcement of the "wisdom" of the cluster. Naturally, one can ask whether it is possible to have percolation without (s,K)-reasonable percolation, especially for $s=1$. The heuristic arguments developed at the beginning of this section intend to make plausible that it is not possible in the case of Bernoulli percolation. Nevertheless, we don't want to be so affirmative in the case of a strong dependence for the following reason: if one want that several random variables simultaneously take big (or small) values, it is better that they have a large positive correlation. Now, if (X,Y) is a Gaussian vector whose law is $\mathcal{N}\left(0, \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}\right)$, an elementary calculus gives $\mathrm{Cov}(X^2,Y^2) = 2\cos^2\theta = 2(\mathrm{Cov}(X,Y))^2$. But when the covariance slowly decreases, it can have a non-monotone behavior, typically a $\frac{\sin x}{x}$ - like behavior – see the example in the next section and also the Fourier analytic results of the last section. So, if we consider two paths from a point to another one, it is not sure that the path which has a maximal probability to be open is the shortest one. Then, it is more hazardous than in the independent case to conjecture that geodesics should look like straight lines.

3. Oriented site percolation for Gaussian fields

We will consider here the problem of percolation on the oriented lattice $\overrightarrow{\mathbb{L}}^d$.

3.1. **A sufficient condition for the existence of oriented percolation.** We begin with a general criterion for the existence of oriented percolation.

Lemma 7. *Let $(X_k)_{k\in\mathbb{Z}^d}$ be a $\{0,1\}$-valued random field. We suppose that there exists $q \in (0, \frac{1}{81})$ and $C > 0$ such that for each finite set A, we have*

$$P(\forall k \in A; X_k = 0) \leq Cq^{|A|}$$

and that $Cw(q) < 1$, where w is defined by

$$\begin{aligned} w : [0, \frac{1}{81}) &\to \mathbb{R} \\ x &\mapsto w(x) = x + \frac{9x}{1 - 9\sqrt{x}}. \end{aligned}$$

Then there is a strictly positive probability that $\{k \in \mathbb{Z}^d; X_k = 1\}$ contains an infinite oriented cluster.

Proof. We will consider the restriction of X to a two-dimensional quarter plane: let us denote $(Y_{k,l})_{k,l\in\mathbb{Z}_+} = (X_{k,l,0,\ldots,0})_{k,l\in\mathbb{Z}_+}$. We also put $Y_{k,l} = 0$ when $k < 0$ or $l < 0$. Of course, percolation in the quarter plane will imply percolation in the whole space for the initial process.

Let $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$. For a finite subset A of $\mathbb{Z}^2$, let us recall a notion of Peierls contours associated to A. Let a, b be two neighbors in $\mathbb{Z}^2$ and i and j be two points $i, j \in \mathbb{Z}^2_*$ such that the quadrangle $aibj$ is a square. We say that the segment joining a and b is drawn if $|A \cap \{a, b\}| = 1$. Drawn segments form a finite family of closed, non self-intersecting, piecewise linear curves, that are called *Peierls contours.*

If i and j are two neighbors in $\mathbb{Z}^2$ separated by a contour γ, say that $i \in \partial_-\gamma$ and $j \in \partial_+\gamma$ if j is in the unbounded connected component of $\mathbb{R}^2\backslash\gamma$.

If A is a finite $\mathbb{L}^d$-connected set, then there exists a unique Peierls contour γ_A such that A remains in the bounded connected component of $\mathbb{R}^2\backslash\gamma_A$.

For each contour γ, let us also define

$$F_\gamma = \{y \in \partial_+\gamma; x - (1,0) \in \partial_-\gamma \text{ or } x - (0,1) \in \partial_-\gamma\}.$$

We can see that if γ is just a simple closed curve with length $l(\gamma)$, we have

$$|F_\gamma| \geq l(\gamma)/4.$$

Proof. On each vertex of the dual lattice $\mathbb{Z}^2_*$ which is a piece of the curve γ, let us draw an arrow in such a way that γ is described with the inside of γ on the left, and the outside of γ on the right – thus, the arrows indicate how to draw the curve anti-clockwise. Since γ is a simple closed curve, there is as many $\uparrow$ and $\leftarrow$ as $\downarrow$ and $\rightarrow$. Then, there is exactly $l(\gamma)/2$ $\uparrow$ and $\rightarrow$. Each point at the right of a $\uparrow$ or over a $\rightarrow$ belongs to F_γ. Therefore, since every point is surrounded by at most one $\uparrow$ and one $\rightarrow$, it follows that $|F_\gamma| \geq l(\gamma)/4$. □

Let us consider the random set $D = C^Y_{1+}(0)$. It is easy to see that $l(\gamma_D)$ is finite as soon as $C^Y_{1+}(0)$ is.

Since $0 \in \partial_- D$, it follows that

$$(Y_0 = 1) \Longrightarrow (\forall k \in F_D \quad Y_k = 0).$$

Thus,

$$
\begin{aligned}
P(C^Y_{1^+}(0) < +\infty) &\le P(l(\gamma_D) < +\infty) \\
&\le P(Y_0 = 0) + \sum_{\gamma} P(\{\forall k \in F_\gamma; Y_k = 0\}) \\
&\le P(Y_0 = 0) + \sum_{\gamma} Cq^{|F_\gamma|} \\
&\le P(Y_0 = 0) + \sum_{\gamma} Cq^{l(\gamma)/4} \\
&\le Cq + \sum_{n=2}^{+\infty} 3^{2n-2} Cq^{(2n)/4} \\
&\le Cw(q).
\end{aligned}
$$

Then, $P(C^Y_{1^+}(0) = +\infty) > 0$ as soon as $Cw(q) < 1$.

□

3.2. **A percolation transition result.**

Theorem 4. *Let $(X_n)_{n\in\mathbb{Z}^d}$ be a centered stationary Gaussian field with bounded spectral density g. Then, the covariance function is*

$$c_n = \frac{1}{(2\pi)^d} \iint_{[-\pi,+\pi[^d} g(x_1,\dots,x_n) e^{i\langle x,n\rangle}\, dx_1 \dots dx_n.$$

We suppose moreover that

$$\sum_{n=1}^{+\infty} \sup\{|c_k|; \|k\| \ge n\} < +\infty.$$

Then, let us define

$$h_- = \sup\{a \ge 0; P(\{\textit{directed percolation happens under } a\}) = 0\}.$$

Then,

- $0 < h_- < +\infty$.
- *For each $a < h_-$ there is almost surely no directed percolation under level a, whereas there is almost surely directed percolation under level a for $a > h_-$.*

Proof. The fact that $0 < h_-(1,d)$ has already be proved in Theorem 3. Since every directed cluster is $(1,d)$- reasonable, it follows that $h_- \ge h_-(1,d) > 0$. Putting together lemma 7 and lemma 3, we get that $P(C^Y_{a^-}(0) < +\infty) < 1$ as soon as $(w \circ \exp(-h))(\frac{a^2}{|g|_\infty}) < 1$. Then, it follows from a straightforward computation that that $h_- \le 3.57\|g\|_\infty^{1/2}$. The existence of a percolating oriented cluster is a translation-invariant event for $(X_n)_{n\in\mathbb{Z}^d}$. As seen in the proof of theorem 1 $(X_n)_{n\in\mathbb{Z}^d}$ is ergodic. Then, the probability of the existence of oriented percolation is full as soon as it is not null. □

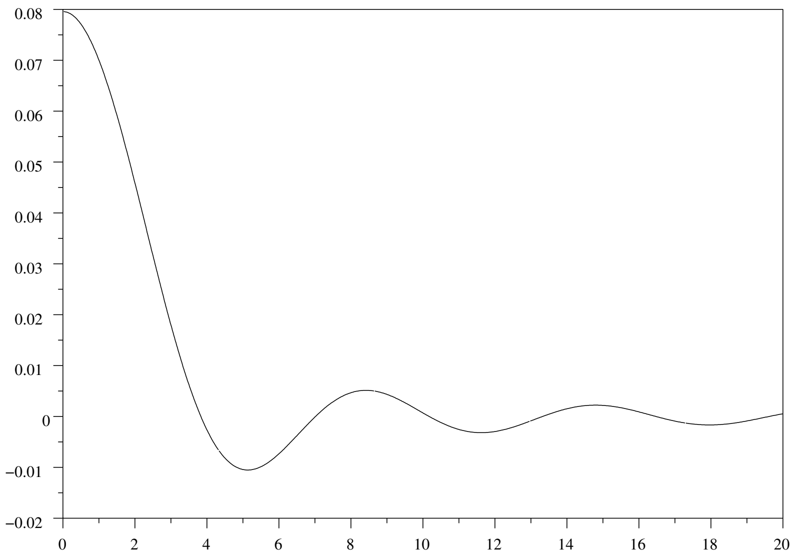
3.3. **An example.** Let A be a symmetric positive definite matrix with spectral gap $\Upsilon(A) \ge \frac{1}{\pi}$ and consider the ellipsoid

$$\mathcal{E} = \{x \in \mathbb{R}^d; \|Ax\|_2 \le 1\}.$$

Let $(X_n)_{n\in\mathbb{Z}^d}$ be a stationary Gaussian process with the indicatrix of $\mathcal{E}$ as spectral density.

We claim that the assumptions of theorem 4 are fulfilled by the process $(X_n)_{n\in\mathbb{Z}^d}$ and that, moreover we have

- $\mathbb{E}\ X_iX_j = c(\|A^{-1}(i-j)\|)$, with
$$c(x) = \frac{1}{\det A}(2\pi)^{-\frac{d}{2}}\frac{1}{x^d}\int_0^x J_{\frac{d-2}{2}}(t)t^{d/2}\ \mathrm{d}t.$$
- For $d=2$
$$c(x) = \frac{1}{\det A}\frac{2}{(2\pi)^{3/2}}\frac{1}{x^{\frac{3}{2}}}\cos(\|A^{-1}n\| - \frac{3\pi}{4}) + O(\frac{1}{x^2}).$$
- For $d\geq 3$
$$c(x) = \frac{1}{\det A}\frac{2}{(2\pi)^{\frac{d+1}{2}}}\frac{1}{x^{\frac{d+1}{2}}}\cos(\|A^{-1}x\| - \frac{(d+1)\pi}{4}) + O(\frac{1}{x^{\frac{d+3}{2}}}).$$



The graph of c when $A = \mathrm{Id}_{\mathbb{R}^2}$.

Proof. Since X is a stationary process, we only computes $\mathbb{E}\ X_0X_n$.

$$\begin{aligned}
\mathbb{E}\ X_0X_n &= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{i\langle n,x\rangle}1\!\!1_{\|Ax\|\leq 1}\ dx_1\ldots dx_n\\
&= \frac{1}{\det A}\frac{1}{(2\pi)^d}\int_{B(0,1)} e^{i\langle A^{-1}n,y\rangle}\ dy_1\ldots dy_d\\
&= \frac{1}{\det A}C(A^{-1}n),
\end{aligned}$$

with

$$C(n) = \frac{1}{(2\pi)^d} \int_{B(0,1)} e^{i\langle n,x\rangle}\, dx_1 \dots dx_d.$$

The result follows of lemmas 12 and 13 of section 5 with $d_0 = 0$.

□

4. Oriented and unoriented percolation for Gibbs measures

We will now prove some results of percolation transition for some Gibbs measures on $\mathbb{R}^{\mathbb{Z}^d}$ corresponding to a potential associated to a function V and a sequence J which satisfy to the assumptions described in section 0. Various results will be proved considering that some of the following assumptions hold:

(H_1) V is even.
(H_2) $\hat{J} \geq 0$ and $\frac{1}{\hat{J}}$ is integrable with respect to the Haar measure on $\mathbb{U}$.
(H_3) V is non-decreasing on $[0, +\infty)$.
(H_4) There exists $A, B \geq 0$ such that $\forall x \in \mathbb{R} \quad V(x) \leq Ax^2 + B$.
(H_5) *Ferromagnetism* $\forall k \in \mathbb{Z}^d \backslash \{0\}; J(k) \leq 0$.
(H_6) *Superstability* $\gamma = \inf\{\hat{J}(z); z \in \mathbb{U}\} > 0$.
(H_7) $\sum_{n=1}^{+\infty} \sup\{|c_k|; \|k\| \geq n\} < +\infty$, with $c_k = \int_{\mathbb{U}} \frac{z^{-k}}{\hat{J}(z)}\, dz$.

The main idea of this section is to compare non-Gaussian Gibbs measures to Gaussian Gibbs measures for which the results of the preceding sections apply.

To understand the signification of the preceding assumptions, one must note that when the state space is not compact, there does not always exist a Gibbs measure for a given Hamiltonian. Fortunately, the optimal conditions for the existence of a Gibbs measure associated to a quadratic Hamiltonian is well known thanks to the independent works by Dobrushin [5] and Künsch [12, 11]. In the case of a stationary Hamiltonian, these conditions are summarized by assumption (H_2).

The strongest assumption (H_7), named here *superstability*, is equivalent to the uniqueness of the Gibbs measure in the class of measures whose support is contained in the set of slowly increasing sequences – for details, see Dobrushin [5] and Garet [7]. As it is usually observed for Gibbs measures with finite state space (*e.g.* in the Ising model), the uniqueness of the Gibbs measures frequently occurs together with a rapid (sometimes exponential) decreasing of the covariance, whereas a phase transition frequently occurs together with a slow decreasing of the covariance

As the non-Gaussian Gibbs measures that we want to study are obtained as perturbations of Gaussian Gibbs measure, it is clear that we can't hope better results than those that are allowed by the speed of decreasing of the covariance of the Gaussian Gibbs measure.

When assumption (H_7) is not fulfilled, the control of the decay of the covariance – that is, the control of the Fourier coefficients $c_k = \int_{\mathbb{U}} \frac{z^{-k}}{\hat{J}(z)}\, dz$ is rather tedious. The proofs of those estimates is relegated to the final section.

The goal of the assumptions $(H_1), (H_3), (H_4), (H_5)$ is to allow the comparison between the Gaussian Gibbs measure and its perturbation. The Assumption (H_5), called ferromagnetism, has the same meaning than in a discrete context, *i.e.* the tendency of the spins to align together. Having in mind the classical domination techniques of comparison for Gibbs measures – *e.g.* Holley's lemma – , the introduction of such an assumption should not be surprising.

We will use a lemma which in the spirit of a lemma due to van Beijeren and Sylvester [24] related to stochastic domination for finite Gibbs measures with the same ferromagnetic interaction and different reference measures.

Let us first recall the concept of domination for finite measures on a partially ordered set E . We say that a measure μ dominates a measure ν, if

$$\frac{\int f\ d\nu}{\nu(E)} \leq \frac{\int f\ d\mu}{\mu(E)}$$

holds as soon as f in an increasing function. We also write $\nu \prec \mu$.

4.1. **First results.**

Lemma 8. *Let $J=(J(i,j))_{i,j\in\Lambda}$ be a symmetric positive definite matrix satisfying to*

- $\exists c>0 \quad \forall i\in\Lambda \quad J(i,i)=c.$
- $\forall (i,j)\in\Lambda^2 \quad i\neq j \Longrightarrow J(i,j)\leq 0.$

Let also be ν_1 and ν_2 two even measures which have a bounded density with respect to Lebesgue's measure

Then, we can define for each bounded function $f:\mathbb{R}^\Lambda\to\mathbb{R}$:

$$\langle f\rangle_{\nu_i}=\frac{\int_{\mathbb{R}^\Lambda} f(\omega)\exp(-\langle J\omega,\omega\rangle)\ d\nu_i^{\otimes\Lambda}(\omega)}{\int_{\mathbb{R}^\Lambda}\exp(-\langle J\omega,\omega\rangle)\ d\nu_i^{\otimes\Lambda}(\omega)}.$$

Let us also suppose that

$$\tilde{\nu_1}\prec\tilde{\nu_2},$$

where $\tilde{\nu}$ is the measure on $(0,+\infty)$ defined by $d\tilde{\nu}(x)=\exp(-\frac{c}{2}x^2)d\nu(x)$.

Then, it follows that for all even bounded functions $F_i:\mathbb{R}\to\mathbb{R}$, nonnegative, and monotone increasing on $[0,+\infty)$, we have

$$\langle\prod_{i\in\Lambda}F_i(\omega_i)\rangle_{\nu_1}\leq\langle\prod_{i\in\Lambda}F_i(\omega_i)\rangle_{\nu_2}.$$

Since the reader can found in [24] the proof of an analogous lemma in a more general context, we will omit these one.

Please note that the conclusion of lemma 8 is *not* the domination of ν'_1 by ν'_2, where ν'_i is the image of $\langle.\rangle_{\nu_i}$ by $(\omega_i)_{i\in\Lambda}\mapsto(|\omega_i|)_{i\in\Lambda}$. To see this, define ν'_i to be the measure on $\{0,1\}$ such that for each (k,l) $\nu'_i(\{(k,l)\})=M^i_{k,l}$, with

$$M^1=\begin{pmatrix}\frac{3}{10}&\frac{3}{10}\\ \frac{2}{10}&\frac{2}{10}\end{pmatrix}\text{ and }M^2=\begin{pmatrix}\frac{4}{10}&\frac{2}{10}\\ \frac{1}{10}&\frac{3}{10}\end{pmatrix}.$$

Let us define F on $\{0,1\}^2$ by $F(x,y)=\max(x,y)$. F is clearly non-decreasing. Since $\int Fd\nu'_1=\frac{7}{10}$ and $\int Fd\nu'_2=\frac{6}{10}$, it follows that ν'_1 is not dominated by ν'_2. However, $\int Fd\nu'_1\leq\int Fd\nu'_2$ holds for each function F which is a product of non-negative and non-decreasing functions, because such a function is a non-negative combination of the functions $(F_{k,l})_{(k,l)\in\{0,1\}^2}$, where $F_{k,l}(x,y)=1\!\!1_{\{x\geq k\}}1\!\!1_{\{y\geq l\}}$.

Theorem 5. *We suppose here that $(H_1),(H_2),(H_3),(H_5)$ are fulfilled. Let $\Lambda_n=\{-n,\dots,n\}^d$. Under the previous assumptions, the sequence $\Pi_{\Lambda_n^c}(0)$ is tight. Then each limit point μ belongs to $\mathfrak{G}_{J,V}$, which is therefore not empty.*

When moreover (H_6) holds, the Hamiltonian is superstable in the sense used by Doss and Royer [6] or Cassandro *et al.* [3]. Then, the existence of a Gibbs measure can be derived from their works. Note that our assumption (H_6) named "superstability" does not imply what they call superstability if no more assumption is done.

However, we will see that in our case, lemma 8 and the assumptions of ferromagnetism will be sufficient and will allow some comparisons which give both the existence of a Gibbs measure and the results of percolation.

Since we will have to compare some measures associated to the same two-body interaction J but with different self-interactions, we will denote by $\Pi_\Lambda^{J,V}(\omega)$ the Gibbs measure on Λ associated to the Hamiltonian defined in (3) and with boundary condition ω.

Proof. Let $\mu_n^V = \Pi_{\Lambda_n}^{J,V}(0)$, where $\Lambda_n = \{-n, \dots, n\}^d$. It is not difficult to see that μ_n^V is a measure such that those that are considered in lemma 8, with $J(i,j) = J(i-j)1\!\!1_\Lambda(i)1\!\!1_\Lambda(j)$ and $d\nu_1 = e^{-V} d\lambda$. We also set $\nu_2 = \lambda$.

It is easy easy to see that the matrix $J_\Lambda = (J(i,j))_{i,j\in\Lambda}$ is positive definite.

Let us also prove that $\tilde{\nu_1} \prec \tilde{\nu_2}$. It is equivalent to proof (see for example [24]) that

$$f : x \mapsto \frac{\tilde{\nu_1}([x, +\infty])}{\tilde{\nu_2}([x, +\infty])}$$

is non-increasing.

Since $f(u) = \frac{\int_u^{+\infty} e^{-\frac{cx^2}{2} - V(x)}\ dx}{\int_u^{+\infty} e^{-\frac{cx^2}{2}}\ dx}$, it follows that

$$f'(u) = \frac{e^{-\frac{cx^2}{2}}}{(\int_u^{+\infty} e^{-\frac{cx^2}{2}}\ dx)^2} \int_x^{+\infty} \left(e^{-V(x)} - e^{-V(u)}\right) e^{-\frac{cx^2}{2}}\ dx.$$

which is non-positive because V is non-decreasing on $[0, +\infty)$.

It is known (see for example [5], chapter 13 or [11, 12]) that the sequence $(\mu_n^0)_{n\geq 1}$ converges to the stationary centered Gaussian measure with spectral density $\frac{1}{\hat{J}}$. It follows that this sequence is tight. Let K be a compact subset of $\mathbb{R}^{\mathbb{Z}^d}$ such that

$$\forall n \geq 1 \quad \mu_n^0(K^c) \leq \varepsilon.$$

We can assume without loss of generality that K writes $K = \prod_{n\in\mathbb{Z}^d}[-a_n, a_n]$. Then it follows from lemma 8 that

$$\forall n \geq 1 \quad \mu_n^V(K^c) \leq \mu_n^0(K^c) \leq \varepsilon.$$

It means that $(\mu_n^V)_{n\geq 1}$ is tight. Then, it follows from the general theory of Gibbs measure – see for example Georgii [9] – that every limit point of this sequence is an extremal Gibbs measure for the Hamiltonian $H_\Lambda^{J,V}$ with Lebesgue's as reference measure. $\square$

Lemma 9. *We suppose here that $(H_1), (H_3), (H_5), (H_6)$ are fulfilled. Let μ^V be an extremal Gibbs measure which is obtained as a limit point of the sequence considered in theorem 5. Then, for each finite set Λ, we have*

Then, for each $x^2 \geq \frac{1}{\gamma}$, we have

$$\mu^V(\{\forall k \in \Lambda; |X_k| \geq x\}) \leq \exp(-h(\gamma x^2)|\Lambda|). \tag{14}$$

Proof. By lemma 8, we have

$$\forall n \geq 1 \quad \mu_n^V(\{\forall k \in \Lambda; |X_k| \geq x\}) \leq \mu_n^0(\{\forall k \in \Lambda; |X_k| \geq x\}).$$

It follows that

$$\mu^V(\{\forall k \in \Lambda; |X_k| \geq x\}) \leq \mu^0(\{\forall k \in \Lambda; |X_k| \geq x\})$$

where μ^0 is the stationary centered Gaussian measure with spectral density $\frac{1}{\hat{J}}$. Now, the result follow from lemma 3. □

The goal of the next lemma is to compare (when a is small) the random field $1\!\!1_{\{|X_k| \leq a\}}$ with a product of Bernoulli measures. It is a classical method in the study of dependent percolation.

For $p \in (0,1)$ we note $\mathrm{Ber}(p)$ the Bernoulli measure $\mathrm{Ber}(p) = p\delta_1 + (1-p)\delta_0$.

Lemma 10. *We suppose here that $(H_1), (H_2), (H_3), (H_4)$ are fulfilled. Let μ be a Gibbs measure for the considered Hamiltonian and $(X_n)_{n \in \mathbb{Z}^d}$ be a random field with $P_X = \mu$.*

We define $\chi = \sqrt{\frac{2}{\pi}(J(0)+A)}\exp(\frac{B-V(0)}{2})$. Then, for fixed $a \in (0, \frac{1}{\chi})$, let us define $Y_k = 1\!\!1_{\{|X_k| \leq a\}}$.

Then,

$$P_Y \prec Ber(a\chi)^{\otimes \mathbb{Z}^d}.$$

Proof. Let $k \in \mathbb{Z}^d, \omega \in \mathbb{R}^{\mathbb{Z}^d}$ and define $\eta = -\sum_{i \neq k} J(i-k)\omega_i$. We have

$$\begin{aligned}
\mathbb{E}\,[Y_k|\sigma(\{k\}^c)](\omega) &= \frac{\int_{[-a,a]} \exp(-\frac{J(0)x^2+V(x)}{2})e^{\eta x}\,dx}{\int_{\mathbb{R}} \exp(-\frac{J(0)x^2+V(x)}{2})e^{\eta x}\,dx} \\
&= \frac{\int_{[0,a]} \exp(-\frac{J(0)x^2+V(x)}{2})\cosh \eta x\,dx}{\int_{\mathbb{R}^+} \exp(-\frac{J(0)x^2+V(x)}{2})\cosh \eta x\,dx} \\
&= 1 - \frac{\int_{\mathbb{R}^+} 1\!\!1_{(a,+\infty)}(x) \exp(-\frac{J(0)x^2+V(x)}{2})\cosh \eta x\,dx}{\int_{\mathbb{R}^+} \exp(-\frac{J(0)x^2+V(x)}{2})\cosh \eta x\,dx} \\
&\leq 1 - \frac{\int_{\mathbb{R}^+} 1\!\!1_{(a,+\infty)}(x) \exp(-\frac{J(0)x^2+V(x)}{2})\,dx}{\int_{\mathbb{R}^+} \exp(-\frac{J(0)x^2+V(x)}{2})\,dx},
\end{aligned}$$

where the last inequality follows from the stochastic domination of $\exp(-\frac{J(0)x^2+V(x)}{2})\,d\lambda(x)$ by $\exp(-\frac{J(0)x^2+V(x)}{2})\cosh \eta x\,d\lambda(x)$ – the same arguments than in theorem 5 apply.

Then

$$\begin{aligned}
\mathbb{E}\,[Y_k|\sigma(\{k\}^c)](\omega) &\leq \frac{\int_{[-a,a]} \exp(-\frac{J(0)x^2+V(x)}{2})\,dx}{\int_{\mathbb{R}} \exp(-\frac{J(0)x^2+V(x)}{2})\,dx} \\
&\leq \frac{\int_{[-a,a]} \exp(-\frac{V(0)}{2})\,dx}{\int_{\mathbb{R}} \exp(-\frac{J(0)x^2+Ax^2+B}{2})\,dx} \\
&\leq \frac{2a\sqrt{J(0)+A}}{\sqrt{2\pi}} \exp(\frac{B-V(0)}{2}).
\end{aligned}$$

Since $\mathbb{E}\left[Y_k|(Y_i)_{i\in\mathbb{Z}^d\backslash\{k\}}\right] = \mathbb{E}\left[\mathbb{E}\left[Y_k|\sigma(\{k\}^c)\right]|(Y_i)_{i\in\mathbb{Z}^d\backslash\{k\}}\right]$, it follows that

$$\forall k\in\mathbb{Z}^d \quad E[Y_k|(Y_i)_{i\in\mathbb{Z}^d\backslash\{k\}}] \leq \chi a. \tag{15}$$

By an usual coupling technique (see, for example, Russo [21], lemma 1), (15) implies that

$$P_Y \prec \mathrm{Ber}(a\chi)^{\otimes\mathbb{Z}^d}.$$

Note that some important results about the domination by a product measure have been obtained by Liggett, Schonmann and Stacey [13].

□

4.2. Percolation transition for superstable Hamiltonians.

Theorem 6. *We suppose here that $(H_1),(H_3),(H_4),(H_5),(H_6)$ are fulfilled. Let $P=\mu^V$ be an extremal Gibbs measure which is obtained as a limit point of the sequence considered in theorem 5. Then, let us define*

$$h_+ = \inf\{a\geq 0; P(\text{ percolation happens over } a)=0\}$$

and

$$h_- = \sup\{a\geq 0; P(\text{ percolation happens under } a)=0\}.$$

Then,

- $0<h_+<+\infty$ *and* $0<h_-<+\infty$.
- *For each $a>h_+$ there is almost surely not percolation over level a, whereas there is almost surely percolation over level a for $a<h_-$.*
- *For each $a<h_-$ there is almost surely not percolation under level a, whereas there is almost surely percolation under level a for $a>h_-$.*

Proof. Since the existence of percolation is a tail event, the $0-1$ behavior follows from the fact that the tail σ-field is trivial under extremal Gibbs measures. If p_c denotes the critical probability for independent Bernoulli site percolation, it follows from lemma 10 that $h_- \geq \frac{p_c}{\chi} > 0$ and $h_+ \geq \frac{1-p_c}{\chi} > 0$.

The facts that $h_- < +\infty$ and $h_+ < +\infty$ follows from lemma 9 together with the criterion of Molchanov and Stepanov.

□

4.3. Transition of directed percolation in the critical case.

We will consider here the problem of percolation transition on the oriented lattice $\overrightarrow{\mathbb{L}}^d$.

The first theorem is related to Gaussian and non-Gaussian Gibbs measures associated to ferromagnetic Hamiltonians.

Theorem 7. *We suppose here that $(H_1),(H_2),(H_3),(H_4),(H_5),(H_7)$ are fulfilled.*

Then, there exists at least one extremal Gibbs measure which is obtained as a limit point of the sequence considered in theorem 5. Let $P=\mu^V$ be such a measure and define

$$h_+ = \inf\{a\geq 0; P(\text{ directed percolation happens over } a)=0\}.$$

Then,

- $0<h_+<+\infty$.
- *For each $a>h_+$ there is almost surely no directed percolation over level a, whereas there is almost surely directed percolation over level a for $a<h_+$.*

Under the assumption of ferromagnetism (H_5), each of the following sets of assumption implies (H_2) and (H_7).

$$(I_1): \begin{cases} \hat{J} > 0 \text{ on } \mathbb{U} \text{ (Superstability).} \\ \exists \alpha > 1 \quad \sum_{n\in\mathbb{Z}^d} \|n\|^\alpha |J(n)| < +\infty. \end{cases} \quad \text{or } (I_2): \begin{cases} d \geq 4. \\ \hat{J} \geq 0 \text{ on } \mathbb{U}. \\ \sum_{n\in\mathbb{Z}^d} \|n\|^3 |J(n)| < +\infty. \\ Mod(\{n, J(n) \neq 0\}) = \mathbb{Z}^d. \end{cases}$$

Proof. Since the existence of percolation is a tail event, the $0-1$ behavior follows from the fact that the tail σ-field is trivial under extremal Gibbs measures. If p_c denotes the critical probability for independent Bernoulli directed site percolation, it follows from lemma 10 that $h_+ \geq \frac{1-p_c}{\chi} > 0$.

Let us define

$$c_n = \int_{\mathbb{U}} \frac{z^{-n}}{\hat{J}(z)}\, dz.$$

Since (H_2) and (H_7) hold, it follows from theorem 2 that there exists a function $f : (c_0, +\infty) \to (0, +\infty)$ having an infinite limit at $+\infty$ and such that

$$\forall A \in \mathcal{M}_{1,d} \quad \mu^0(\{\forall k \in A \quad |X_k| \geq a\}) \leq e^{-f(a)|A|}.$$

By lemma 8, it follows that

$$\forall A \in \mathcal{M}_{1,d} \quad \mu^V(\{\forall k \in A \quad |X_k| \geq a\}) \leq \mu^0(\{\forall k \in A \quad |X_k| \geq a\}) \leq e^{-f(a)|A|}.$$

Now, it follows from lemma 6 that there is no $(1,d)$ reasonable percolation over a for large (but finite) a. Then, there is also no oriented percolation over a for large (but finite) a. Precisely, $h_+ \leq \inf\{x; f(x) > \ln \mu_d\}$.

Now, it suffices to prove that (H_2) and (H_7) hold under each of the two systems of assumptions to complete the proof.

- If $\hat{J}$ does not vanish on $\mathbb{U}$, it is clear that (H_2) holds. Moreover, it follows that J is an invertible element in the Banach Algebra

$$\mathbb{A}_\alpha = \{x \in \mathbb{C}^{\mathbb{Z}^d}; \sum_{n\in\mathbb{Z}^d} (1+\|n\|)^\alpha |x_n| < +\infty\}$$

  – for a proof, see [7]. It follows that the Fourier coefficients of $\frac{1}{\hat{J}}$ form a sequence which belongs to $\mathbb{A}_\alpha$ we have

$$\sum_{n\in\mathbb{Z}^d} (1+\|n\|)^\alpha |c_n| < +\infty.$$

- For the proof of the second case, we state a lemma which will be useful later.

Lemma 11. *Let $(J(n))_{n\in\mathbb{Z}^d}$ be a real sequence. We suppose that*

$$\begin{cases} J(0) = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |J(n)| \\ \sum_{n\in\mathbb{Z}^d} \|n\|^3 |J(n)| < +\infty. \\ Mod(A) = \mathbb{Z}^d \end{cases}$$

We define $f(\theta_1, \dots, \theta_d) = \hat{J}(e^{i\theta_1}, \dots, e^{i\theta_d})$.

Then

- $\forall z \in \mathbb{U} \quad \hat{J}(z) \geq 0$.
- $Z = \{z \in \mathbb{U}; \hat{J}(z) = 0\} \subset \{-1,+1\}^d$. *If, moreover, J is ferromagnetic (that is, if $J(n) \leq 0$ holds for each $n \neq 0$), then $Z = \{(1,\dots,1)\}$.*
- *If $\theta \in \mathbb{R}^d$ is such that $z = (e^{i\theta_1},\dots,e^{i\theta_d}) \in Z$, then $D^2_\theta f(h \otimes h) = Q(h)$, where*

$$Q(h) = \sum_{n \in \mathbb{Z}^d} |J(n)| \langle n, h \rangle^2,$$

which is a definite positive quadratic form.

Proof of lemma 11. –

$$\begin{aligned}
\hat{J}(z) &= J(0) + \sum_{n \in \mathbb{Z}^d \backslash \{0\}} J(n) z^n \\
&= J(0) + \sum_{n \in \mathbb{Z}^d \backslash \{0\}} |J(n)| \frac{J(n)}{|J(n)|} z^n \\
&\geq J(0) - \sum_{n \in \mathbb{Z}^d \backslash \{0\}} |J(n)||z^n| \\
&\geq J(0) - \sum_{n \in \mathbb{Z}^d \backslash \{0\}} |J(n)| \\
&\geq 0.
\end{aligned}$$

- Let z be a root of $\hat{J}$. For the triangular inequality being an equality, it requires that $-\frac{J(n)}{|J(n)|} z^n \in \mathbb{R}_+$ as soon as $J(n) \neq 0$. In facts, it means that

$$\forall n \in \mathbb{Z}^d \quad (J(n) \neq 0) \Longrightarrow (z^n = -\frac{|J(n)|}{J(n)}).$$

Since $J(n)$ is a real number, $\frac{|J(n)|}{J(n)} \in \{-1,+1\}$. Then

$$\forall n \in \mathbb{Z}^d \quad (J(n) \neq 0) \Longrightarrow (z^n \in \{-1,+1\}).$$

Let $t = (z_1^2,\dots,z_d^2)$. For each $n \in A$ $t^n = (z^n)^2 = 1$. It follows that $A = \{n \in \mathbb{Z}^d; J(n) \neq 0\} \subset M_z = \{n \in \mathbb{Z}^d; t^n = 1\}$. Clearly, M_z is a subgroup of $\mathbb{Z}^d$. Since we have supposed that $\mathrm{Mod}(A) = \mathbb{Z}^d$, we get $M_z = \mathbb{Z}^d$. For each $i \in \{1,\dots d\}$, $t_1 = t^{e_i} = 1$: we have $t = (1,\dots,1)$. It follows that $z \subset \{-1,+1\}^d$

When J is non-positive, we have

$$\forall n \in \mathbb{Z}^d \quad (J(n) \neq 0) \Longrightarrow (z^n = 1).$$

In this case, we work directly with z: let $M_z = \{n \in \mathbb{Z}^d; z^n = 1\}$. $A \subset M_z$ and M_z is always a subgroup of $\mathbb{Z}^d$. Since $\mathrm{Mod}(A) \subset M_z$ and $\mathrm{Mod}(A) = \mathbb{Z}^d$, we get $M_z = \mathbb{Z}^d$. For each $i \in \{1,\dots d\}$, $z_1 = z^{e_i} = 1$: we have $z = (1,\dots,1)$.

- For each $k \leq 3$, we have

$$D^k_\theta f(h^{\otimes k}) = \sum_{n \in \mathbb{Z}^d} J(n) e^{i\langle n,\theta \rangle} (i \langle n, h \rangle)^k.$$

Particularly, if $z = (e^{i\theta_1}, \dots, e^{i\theta_d})$ is a root of $\hat{J}$,

$$\begin{aligned} D^2_\theta f(h \otimes h) &= -\sum_{n\in\mathbb{Z}^d} J(n) e^{i\langle n,\theta\rangle} \langle n, h\rangle^2 \\ &= -\sum_{n\in\mathbb{Z}^d} J(n) z^n \langle n, h\rangle^2 \\ &= \sum_{n\in\mathbb{Z}^d} |J(n)| \langle n, h\rangle^2 = Q(h). \end{aligned}$$

Clearly, Q is a non-negative quadratic form. If h is such that $Q(h) = 0$, then $\langle n, h\rangle = 0$ as soon as $J(n) \neq 0$. It means that h is orthogonal to A. Since Mod(A) $= \mathbb{Z}^d$, h is orthogonal to $\mathbb{Z}^d$; it follows that $h = 0$ and Q is positive definite.

□

Now we go back to the study of the second set of assumptions. We can suppose that $\hat{J}$ vanishes on $\mathbb{U}$, because if it is not the case, the first family of assumptions is fulfilled. Then, it follows from lemma 11 that $(1, \dots, 1)$ is the only root of $\hat{J}$ on $\mathbb{U}$.

Let us define $f(\theta_1, \dots, \theta_d) = \hat{J}(e^{i\theta_1}, \dots, e^{i\theta_d})$ and the positive definite quadratic form Q as in lemma 11.

Since $(0, \dots, 0)$ is a minimum for f, it follows that $D^1_0 f = 0$. We have

$$f(\theta) = \frac{1}{2} Q(\theta) + R(\theta),$$

with $D^k_0 R = 0$ for $k \in \{0, 1, 2\}$.

The polar form associated to Q is

$$q(x, y) = \sum_{n\in\mathbb{Z}^d} |J(n)| \langle n, x\rangle \langle n, y\rangle.$$

Then, the matrix $B = (b_{i,j})_{1\le i,j\le d}$ of Q in the canonical basis of $\mathbb{R}^d$ is defined by $b_{i,j} = q(e_i, e_j) = \sum_{n\in\mathbb{Z}^d} |J(n)| n_i n_j$. Since Q is positive definite, B is a symmetric positive definite matrix. If we denote by A a square root of B, A is a symmetric positive definite matrix which is such that

$$||A\theta||^2 = \langle A\theta, A\theta\rangle = \langle A^* A\theta, \theta\rangle = \langle B\theta, \theta\rangle = q(\theta, \theta) = Q(\theta).$$

Since $(1, \dots 1)$, is the only root of $\hat{J}$ on $\mathbb{U}$, it follows that f only vanishes on $(2\pi\mathbb{Z})^d$. Since A is positive definite, it is invertible. Then, we can apply Theorem 9 with $d_0 = 2$ and $N = 3$. When $d = 4$, equation (17) applies, whence equation (18) applies for $d \ge 5$. In both cases, the biggest part of the equation is $O(\frac{1}{\|n\|^\beta})$. It follows that $c_n = O(\frac{1}{\|n\|^\beta})$, with $\beta = \frac{3(d-1)}{d+3} \ge \frac{9}{7}$, which ensures that (H_7) holds.

□

Remark: The assumption Mod$(\{n, J(n) \neq 0\}) = \mathbb{Z}^d$ can seem to be mysterious, but it is actually natural. Indeed, if $M =$ Mod$(\{n, J(n) \neq 0\})$ is a proper sub-module of $\mathbb{Z}^d$, the expression of the Hamiltonian can be splitted into $[\mathbb{Z}^d : M]$ parts so that it appears that sites which are inside different classes of $\mathbb{Z}^d/M$ do not interact. Hence, the particle system can be decomposed into $[\mathbb{Z}^d : M]$ non-interacting sub-systems. It is now convenient to study one of these sub-systems using another parameterization – which is always possible, since every sub-module of $\mathbb{Z}^d$ is free.

Example: Let $m \geq 0$ and define J by

$$J(i) = \begin{cases} 1+m & \text{if } i = 0 \\ -\frac{1}{2d} & \text{if } |i| = 1 \\ 0 & \text{else.} \end{cases}$$

In this case, the Hamiltonian formally writes:

$$H = \sum_i V(X_i) + \frac{1+m}{2} \sum_i X_i^2 - \frac{1}{2d} \sum_{\|i-j\|_1=1} X_i X_j,$$

or also with the more intuitive form:

$$H = \sum_i V(X_i) + \frac{m}{2} \sum_i X_i^2 + \frac{1}{2d} \sum_{\|i-j\|_1=1} (X_i - X_j)^2.$$

When $V = 0$, these models are called *harmonic models.* For $m > 0$ and $V = 0$, it is the so-called *harmonic model with mass*, whereas for $m = 0$ and $V = 0$, it is the so-called *massless harmonic model.*

By a direct computation, we see that $\hat{J} \geq m$ on $\mathbb{U}$. Then, the assumptions of Theorems 6 and 7 – by the way of set of assumptions (I_1) – are fulfilled as soon as $m > 0$.

If $m = 0$, the assumptions of Theorems 6 are not fulfilled. Nevertheless, for $m = 0$ and $d \geq 4$, the set of assumptions (I_2) is fulfilled, so Theorem 7 apply.

As an example of an allowed perturbation V, we can take $V(x) = \ln(\cosh x)$: it is clearly even, increasing on $[0, +\infty)$, and satisfy to $V(x) \leq \frac{x^2}{2}$.

Remarks:

- *Hidden superstability*

  Consider the Hamiltonian formally written as

  $$H = \sum_i V(X_i) + \frac{1}{2d} \sum_{\|i-j\|_1=1} (X_i - X_j)^2,$$

  with $V(x) = (1+|x|)\ln(\cosh x)$. At first sight, one could think that we are, as in the preceding example, in the critical case, and that the superstability assumption is not fulfilled. In fact, there is superstability, but it is hidden in the perturbating term. Indeed, we can rewrite the Hamiltonian as

  $$H = \sum_i W(X_i) + \frac{1}{2} \sum_i X_i^2 + \frac{1}{2d} \sum_{\|i-j\|_1=1} (X_i - X_j)^2,$$

  with $W(x) = V(x) - \frac{x^2}{2}$. Now, for each $x \geq 0$, we have

  $$W'(x) = \ln(\cosh x) + \frac{(\cosh x - 1) + (1 - (1+x)e^{-x})}{x \cosh x} \geq 0.$$

  Similarly, we prove that $W(x) \leq \frac{x^2}{2}$. Now, it follows that Theorems 6 and 7 apply.

- *The massless harmonic model*
  It is interesting to compare the results for the massless harmonic model with those that were obtained by Bricmont, Lebowitz and Maes in [2]. Using some results of potential theory, they have shown the existence of a (unoriented) percolation transition for $\{k \in \mathbb{Z}^d; X_k \geq h\}$ for the harmonic model with mass in each dimension and for the massless harmonic model when $d = 3$.
- *Gaussian Gibbs measure and random walks*
  In the ferromagnetic case, c_n is proportional to the Green function associated to the random walk associated to the measure μ defined by $\mu(0) = 0$ and $\mu(n) = -\frac{J(n)}{J(0)}$ for $n \neq 0$. If we know that this random walk is aperiodic and that $\mu^{*k}(n) = o(\|n\|^{2-d})$ holds for each $k \geq 1$, then it follows from a result by Spitzer [23] that for $d \geq 3$, we have $c_n \sim \langle n, Q^{-1}n\rangle^{1-d/2}$ for a suitable definite matrix Q. Then,

$$\sum_{k=1}^{+\infty} \sup\{|c_n|; \|n\| \geq k\} < +\infty$$

  holds for $d \geq 4$ and not for $d = 3$. Note that the estimate of Spitzer is more precise that ours, but it requires an assumption of aperiodicity that sometimes fails, for example for one-range interactions.

The last theorem shows that for Gaussian Gibbsian fields, the assumption of ferromagnetism can be relaxed.

Theorem 8. *We suppose here that (H_2) and (H_7) are fulfilled. Let $P = \mu^0$ be the extremal Gibbs measure which is obtained as a limit point of the sequence considered in theorem 5,* i.e. *the centered Gaussian measure with spectral density $\frac{1}{\hat{J}}$.*
Then, let us define

$$h_+ = \inf\{a \geq 0; P(\text{ directed percolation happens over } a) = 0\}.$$

Then,

- $0 < h_+ < +\infty$.
- *For each $a > h_+$ there is almost surely not percolation over level a, whereas there is almost surely percolation over level a for $a < h_+$.*

Moreover each the following assumptions implies that (H_2) and (H_7) hold:

$$(I_1): \begin{cases} \hat{J} > 0 \text{ on } \mathbb{U} \text{ (Superstability).} \\ \exists \alpha > 1 \quad \sum_{n \in \mathbb{Z}^d} \|n\|^\alpha |J(n)| < +\infty. \end{cases} \quad or\ (I_3): \begin{cases} d \geq 4 \\ \hat{J} \geq 0 \text{ on } \mathbb{U}. \\ \sum_{n \in \mathbb{Z}^d} \|n\|^3 |J(n)| < +\infty. \\ Z = \{z \in \mathbb{U}; \hat{J}(z) = 0\} \text{ is finite} \\ \forall z \in Z \quad h \mapsto -\sum_{n \in \mathbb{Z}^d} J(n) z^n \langle n, h\rangle^2 \\ \text{is a definite positive quadratic form.} \end{cases}$$

$$
or\ (I_4) : \begin{cases} d \geq 4 \\ \sum_{n \in \mathbb{Z}^d} \|n\|^3 |J(n)| < +\infty. \\ J(0) = \sum_{n \in \mathbb{Z}^d \backslash \{0\}} |J(n)| \\ Mod(A) = \mathbb{Z}^d. \end{cases}
$$

Proof. Since the skeleton of this proof is essentially the same as those of the previous theorem, we will only sketch it, employing the same notations. Of course, since we directly deal with a Gaussian field, there is no need of stochastic comparison and the proof of the first assumption is more simple than those of Theorem 9. Indeed, we have essentially to prove that (H_2) and (H_7) hold.

- The fact that the first family of assumptions implies (H_2) and (H_7) has already be proved in Theorem 9.
- Let us study the second family of assumptions. Since the first family of assumptions is fulfilled when $Z = \varnothing$, we can assume without loss that $Z \neq \varnothing$. Since Z is a finite set, there exists $r > 0$ such that $\|x-y\| > r$ holds if x and y are two distinct roots of f. Let R be the finite subset of $[0, 2\pi)^d$ such that $Z = \{(e^{i\theta_1}, \ldots, e^{i\theta_d}); \theta \in R\}$ and let $F = (2\pi\mathbb{Z})^d + B(0, r/2)$. For each $\theta \in R$, define

$$O_\theta = \mathbb{R}^d \backslash (F + (R \backslash \{\theta\})).$$

Since $(O_\theta)_{\theta \in R}$ is a covering of $\mathbb{R}^d$ by periodic open sets, one can build infinitely smooth non-negative periodic functions $(f_\theta)_{\theta \in R}$such that
  - For each $\theta \in R$, the support of f_θ holds in O_θ.
  - $1 = \sum_{\theta \in R} f_\theta$. (partition of the unity)

Then,

$$\frac{1}{f} = \sum_{\theta \in R} \frac{f_\theta}{f}.$$

The set of poles of $\frac{f_\theta}{f}$ is $\theta + (2\pi\mathbb{Z})^d$ and f_θ is identically equal to 1 on $B(\theta, r/2)$. Let us write

$$\frac{f_\theta}{f} = (\frac{f_\theta}{f} + (1 - f_\theta)) - (1 - f_\theta)$$

Since f_θ is infinitely smooth, the Fourier coefficients of $1 - f_\theta$ form a rapidly decreasing sequence. By construction, $\frac{f_\theta}{f} + (1 - f_\theta)$ does not vanish; let us define $g_\theta = (\frac{f_\theta}{f} + (1 - f_\theta))^{-1}$. $g(\theta)$ only vanishes on $\theta + (2\pi\mathbb{Z})^d$ and g_θ is C^3 in the neighborhood of θ. We can write

$$g_\theta(\theta + x) = Q(x) + R_\theta(x),$$

with $R_\theta(0) = 0$, $D_0 R_\theta(0) = 0$ and $D_0^2 R_\theta(0) = 0$. Now, as in the proof of Theorem 7, we apply Theorem 9 to $g_\theta(\theta + .)$ with $d_0 = 2, N = 3$ and get the desired decay for (c_n): $c_n = O(\frac{1}{\|n\|^\beta})$, with $\beta = \frac{3(d-1)}{d+3} \geq \frac{9}{7}$.
- It is an immediate consequence of lemma 11 that $(I_4) \Longrightarrow (I_3)$.

□

An example of an antiferromagnetic model.
Let us define J by

$$J(i) = \begin{cases} 1 & \text{if } i = 0 \\ \frac{1}{2d} & \text{if } |i| = 1 \\ 0 & \text{else.} \end{cases}$$

In this case, the Hamiltonian formally writes:

$$H = \frac{1}{2}\sum_i X_i^2 + \frac{1}{2d}\sum_{\|i-j\|_1=1} X_i X_j,$$

or also with the more intuitive form:

$$H = \frac{1}{2d}\sum_{\|i-j\|_1=1} (X_i + X_j)^2.$$

It is not difficult to see that the set of assumptions (I_4) is fulfilled as soon as $d \geq 4$. Then, Theorem 8 applies, but not Theorem 6.

5. Some Fourier asymptotics

In this section, we will prove a theorem to estimate the asymptotic behavior of the Fourier coefficients of some quite smooth functions which only have a singularity in a single point.

Lemma 12. *Let $d \geq 2$ and $d_0 \in (-\infty, d)$. Let*

$$C(n) = \frac{1}{(2\pi)^d}\int_{B(0,1)} \frac{1}{\|x\|^{d_0}} e^{i\langle n,x\rangle}\ dx_1 \dots dx_d.$$

Then,

$$C(n) = (2\pi)^{-\frac{d}{2}} \frac{1}{\|n\|^\alpha}\int_0^{\|n\|} J_{\frac{d-2}{2}}(t)t^\gamma\ dt,$$

with $\alpha = d - d_0$ and $\gamma = \frac{d}{2} - d_0$.

Proof. By invariance under the group of isometries, it is easy to see that $C(n)$ only depends on $\|n\|_2$.

Using some methods of integration such as those which are described in Rudin [20], it is not so hard to proof that if $g \circ \|.\|$ is integrable, we have

$$\int_{\mathbb{R}^d} e^{i\lambda x_1}\ g(\|x\|)\ dx = G_d \int_0^{+\infty} r^{d-1} g(r) \int_{-1}^{+1} (1-t^2)^{\frac{d-1}{2}-1} e^{ir\lambda t}\ dt\ dr,$$

where G_d, which only depends from the dimension, can be computed with the choice $\lambda = 0$ and $g(r) = \exp(-\frac{r^2}{2})$.

Now, if we take $g(r) = r^{-d_0}1\!\!1_{[0,1]}(x)$, the desired formula follows from a last change of variable.

□

Now, we have to estimate $C(\|x\|)$ for large x.

Lemma 13. *Let*

$$I_{d,d_0}(x) = \int_0^x J_{\frac{d-2}{2}}(t)t^\gamma\ dt,$$

with $\gamma = d - d_0$. Then,

- *if* $\gamma < \frac{1}{2}$, *then* $\lim_{x \to +\infty} I_{d,d_0}(x) = \frac{\Gamma(\frac{d-d_0}{2})}{2^{\frac{d}{2}-d_0}}$.
- *if* $\gamma \geq \frac{1}{2}$, *then*

$$I_{d,d_0}(x) = \sqrt{\frac{2}{\pi}} x^{\gamma - \frac{1}{2}} \cos\left(x - \frac{(d+1)\pi}{4}\right) + O(x^{\max(\gamma - \frac{3}{2}, 0)}).$$

Proof. When $\gamma < \frac{1}{2}$, the integral

$$\int_0^{+\infty} J_{\frac{d-2}{2}}(t) t^\gamma \, dt \tag{16}$$

is semi-convergent an can be computed. More generally, if ν and μ are complex numbers such that

$$0 < \text{Re } \mu < \text{Re } \nu + \frac{3}{2},$$

we have

$$\int_0^{+\infty} \frac{J_\nu(t)}{t^{\nu - \mu + 1}} \, dt = \frac{\Gamma(\frac{\mu}{2})}{2^{\nu - \mu + 1} \Gamma(\nu - \frac{1}{2}\mu + 1)}.$$

This integral is sometimes called *Weber's integral*, who computed its values when ν is an integer. For a proof or an historic, see [25], § 13.24, page 391.

Here, we have $\nu = \frac{d-2}{2}$ and $\mu = d - d_0 = \alpha$, then

$$\int_0^{+\infty} J_{\frac{d-2}{2}}(t) t^\gamma \, dt = \frac{\Gamma(\frac{d-d_0}{2})}{2^{\frac{d}{2} - d_0}}.$$

Let us consider the case $\gamma \geq \frac{1}{2}$. In this case, (16) diverges, but $I_\alpha(x)$ can be estimated using some asymptotics for Bessel's functions. Bessel's functions are related to Hankel's functions H_ν^1 and H_ν^2 by the formula

$$J_\nu = \frac{1}{2}(H_\nu^1 + H_\nu^2).$$

We have on $\mathbb{C} \backslash \mathbb{R}^-$ the following asymptotics (see for example [4], chap. XV.).

$$\begin{cases} H_\nu^1(z) = \sqrt{\frac{2}{\pi}} z^{-\frac{1}{2}} e^{i(z - \frac{\pi}{4} - \frac{\nu\pi}{2})} (1 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} + O(\frac{1}{|z|^{n+1}})) \\ H_\nu^2(z) = \sqrt{\frac{2}{\pi}} z^{-\frac{1}{2}} e^{-i(z - \frac{\pi}{4} - \frac{\nu\pi}{2})} (1 + \frac{b_1}{z} + \cdots + \frac{b_n}{z^n} + O(\frac{1}{|z|^{n+1}})) \end{cases}$$

Let γ satisfy to $\gamma \geq \frac{1}{2}$. Then,

$$\begin{aligned} \int_0^x H^1_{\frac{d-2}{2}}(t) t^\gamma \, dt &= \int_0^1 H^1_{\frac{d-2}{2}}(t) t^\gamma \, dt \\ &+ \sqrt{\frac{2}{\pi}} \int_1^x t^{\gamma - \frac{1}{2}} e^{i(t - (d-1)\frac{\pi}{4})} \, dt \\ &+ \sqrt{\frac{2}{\pi}} a_1 \int_1^x t^{\gamma - \frac{3}{2}} e^{i(t - (d-1)\frac{\pi}{4})} \, dt \\ &+ \int_1^x t^{\gamma - \frac{1}{2}} e^{i(t - (d-1)\frac{\pi}{4})} \phi(t) \, dt, \end{aligned}$$

with $\phi(t) = O(\frac{1}{t^2})$. It immediately comes that

$$\begin{aligned}\int_0^x H^1_{\frac{d-2}{2}}(t)t^\gamma \, dt &= \sqrt{\frac{2}{\pi}} \int_1^x t^{\gamma-\frac{1}{2}} e^{i(t-(d-1)\frac{\pi}{4})} \, dt \\ &+ \sqrt{\frac{2}{\pi}} a_1 \int_1^x t^{\gamma-\frac{3}{2}} e^{i(t-(d-1)\frac{\pi}{4})} \, dt \\ &+ O(x^{\max(\gamma-\frac{3}{2},0)}).\end{aligned}$$

A partial integration gives

$$\int_1^x t^{\gamma-\frac{3}{2}} e^{i(t-(d-1)\frac{\pi}{4})} \, dt = [t^{\gamma-\frac{3}{2}} e^{i(t-\frac{(d+1)\pi}{4})}]_1^x \int_1^x (\gamma - \frac{3}{2}) t^{\gamma-\frac{5}{2}} e^{i(t-(d-1)\frac{\pi}{4})} \, dt,$$

Hence

$$\int_1^x t^{\gamma-\frac{3}{2}} e^{i(t-(d-1)\frac{\pi}{4})} \, dt = O(x^{\max(\gamma-\frac{3}{2},0)}).$$

Similarly

$$\int_1^x t^{\gamma-\frac{1}{2}} e^{i(t-(d-1)\frac{\pi}{4})} \, dt = [t^{\gamma-\frac{1}{2}} e^{i(t-\frac{(d+1)\pi}{4})}]_1^x - \int_1^x (\gamma - \frac{1}{2}) t^{\gamma-\frac{3}{2}} e^{i(t-(d-1)\frac{\pi}{4})} \, dt,$$

The norm of the integral appearing in the last part of the preceding identity can be bounded as previously, and we finally get

$$\int_0^x H^1_{\frac{d-2}{2}}(t)t^\gamma = \sqrt{\frac{2}{\pi}} x^{\gamma-\frac{1}{2}} e^{i(x-\frac{(d+1)\pi}{4})} + O(x^{\max(\gamma-\frac{3}{2},0)}).$$

Similarly

$$\int_0^x H^2_{\frac{d-2}{2}}(t)t^\gamma = \sqrt{\frac{2}{\pi}} x^{\gamma-\frac{1}{2}} e^{-i(x-\frac{(d+1)\pi}{4})} + O(x^{\max(\gamma-\frac{3}{2},0)}).$$

Hence

$$\int_0^x J_{\frac{d-2}{2}}(t)t^\gamma = \sqrt{\frac{2}{\pi}} x^{\gamma-\frac{1}{2}} \cos(x - \frac{(d+1)\pi}{4}) + O(x^{\max(\gamma-\frac{3}{2},0)}).$$

$\square$

Theorem 9. *Let $d \geq 2$ and $f : \mathbb{R}^d \to \mathbb{R}^+$ with $(2\pi\mathbb{Z})^d$ as period and such that the following holds:*

- *f only vanishes on $(2\pi\mathbb{Z})^d$.*
- *There exists $N \in \mathbb{Z}_+, d_0 \in \mathbb{R}$ such that $0 < d_0 < \min(d, N)$, a matrix $A \in Gl_d(\mathbb{R})$ and a C^N-smooth function R such that*

$$f(x) = \|Ax\|^{d_0} + R(x),$$

with for each $k \in \{0, \ldots, N-1\}$:

$$D_0^k R = 0.$$

Let

$$c_n = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} e^{i\langle n,x\rangle} \frac{1}{f(x)} \, dx_1 \ldots dx_n.$$

We also define $\alpha = d - d_0, \gamma = \frac{d}{2} - d_0$, $\beta = \frac{N(N+d-2d_0)}{N+d} = N - 2d_0 + \frac{2d_0 d}{N+d}$.

The following asymptotics holds:

$$c_n = \frac{1}{\det A} \frac{1}{(2\pi)^{d/2}} \frac{1}{\|A^{-1}n\|^{d-d_0}} I_{d,d_0}(\|A^{-1}n\|) + O(\|n\|^{-\beta}),$$

where I_{d,d_0} has been defined in lemma 13. It can be specified as follows:

- *If $d_0 > \frac{d-1}{2}$, we have, when $\|n\|$ goes to $+\infty$:*

$$c_n = \frac{1}{\det A} \frac{2^{d_0}\Gamma(\frac{d-d_0}{2})}{(4\pi)^{d/2}} \frac{1}{\|A^{-1}n\|^{d-d_0}} + O(\|n\|^{-\beta}). \tag{17}$$

- *If $d_0 \leq \frac{d-1}{2}$, we have*

(18)
$$c_n = \frac{1}{\det A} \frac{1}{(2\pi)^{d/2}} \sqrt{\frac{2}{\pi}} \frac{1}{\|A^{-1}n\|^{\frac{d+1}{2}}} \cos(\|A^{-1}n\| - \frac{(d+1)\pi}{4}) + O(\|n\|^{-\min(\alpha,\beta,\frac{d+3}{2})}).$$

Proof. By the separation theorem, there exists an infinitely smooth function h which is identically 0 on $\{x \in [-\pi,\pi]^d\ ,\ \|Ax\| \geq 1\}$ and identically 1 on $\{x \in [-\pi,\pi]^d\ ,\ \|Ax\| \leq \frac{1}{2}\}$. Then $\frac{1}{f} = \frac{h}{f} + \frac{1-h}{f}$, it follows that

$$\begin{aligned} c_n &= \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} e^{i\langle n,x\rangle} \frac{h(x)}{f(x)}\ dx_1 \dots dx_n \\ &+ \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} e^{i\langle n,x\rangle} \frac{1-h(x)}{f(x)}\ dx_1 \dots dx_n. \end{aligned} \tag{19}$$

The function

$$\frac{1-h}{f}$$

can be periodized into a C^N -smooth function, because it coincides with the periodic function $\frac{1}{f}$ on the boundary of $[-\pi,\pi]^d$. Its Fourier coefficients, which represent the second part of the sum in (19) are $O(\|n\|^{-N})$.

Let us now study the first term of the sum. Since the support of h resides inside $A^{-1}B(0,1)$, this integral can be written as

$$\frac{1}{(2\pi)^d} \frac{1}{\det A} \int_{B(0,1)} e^{i\langle A^{-1}n,y\rangle} \frac{h_1(y)}{f_1(y)}\ dy_1 \dots dy_n,$$

when we put $h_1(y) = h(A^{-1}y)$ et $f_1(y) = f(A^{-1}y)$. Then h_1 is C^∞-smooth, is identically 1 on $B(0,\frac{1}{2})$, whereas f_1 only vanishes at 0 and satisfies to

$$f_1(x) = \|x\|^{d_0} + R_1(x), \tag{20}$$

where R_1 follows the same assumptions as R.

We put

$$s(x) = \frac{h_1(x)}{f_1(x)} - \frac{1}{\|x\|^{d_0}}$$

and also

$$C(u) = \frac{1}{(2\pi)^d} \int_{B(0,1)} e^{i\langle u,x\rangle} \frac{1}{\|x\|^{d_0}}\ dx_1 \dots dx_n,$$

in such a way that

$$\begin{aligned} &\frac{1}{(2\pi)^d} \frac{1}{\det A} \int_{B(0,1)} e^{i\langle A^{-1}n,y\rangle} \frac{h_1(y)}{f_1(y)}\ dy_1 \dots dy_n = \\ &\quad \frac{1}{\det A} C(A^{-1}n) \\ &\quad + \frac{1}{(2\pi)^d} \frac{1}{\det A} \int_{B(0,1)} e^{i\langle A^{-1}n,x\rangle} s(x)\ dx_1 \dots dx_n. \end{aligned}$$

The control of the first part is done by using lemmas 12 and 13. It remains to control

$$\int_{B(0,1)} e^{i\langle A^{-1}n,x\rangle}dx.$$

We will use Green's formula, which will be used here as a multidimensional integration by parts.

If we suppose that V is a volume with an infinitely smooth orientable boundary, and that u is C^2-smooth and ϕ C^1-smooth from V into $\mathbb{R}$, we have

$$\int_V \phi\Delta u\ dx = \int_{\partial V} \phi\langle \text{grad } u, \vec{N}(x)\rangle\ d\sigma(x) - \int_V \langle \text{grad } u, \text{grad } \phi\rangle dx.$$

$\vec{N}(x)$ denotes the unitary vector which is normal to ∂V and is oriented to the outside of V and σ is the surface measure on ∂V.

Taking $u(x) = -\frac{1}{\|n\|^2}e^{i\langle n,x\rangle}$, we get grad $u = -\frac{in}{\|n\|^2}e^{i\langle n,x\rangle}$ and $\Delta u = e^{i\langle n,x\rangle}$, hence

(21)

$$\int_V \phi e^{i\langle n,x\rangle}\ dx = -\frac{i}{\|n\|}\int_{\partial V} \phi\langle v, \vec{N}(x)\rangle e^{i\langle n,x\rangle}\ d\sigma(x) + \frac{i}{\|n\|}\int_V D_x^1\phi(v)e^{i\langle n,x\rangle}\ dx,$$

with $v = \frac{n}{\|n\|}$. Iterating this process, we get for a C^N-smooth ϕ:

$$\begin{aligned}\int_V \phi e^{i\langle n,x\rangle}\ dx &= \sum_{k=0}^{N-1} \frac{i^{k-1}}{\|n\|^{k+1}}\int_{\partial V} D_x^k\phi(v^{\otimes k})\langle v, \vec{N}(x)\rangle e^{i\langle n,x\rangle}\ d\sigma(x)\\ &+ \frac{i^N}{\|n\|^N}\int_V D_x^N\phi(v^{\otimes N})\langle v, \vec{N}(x)\rangle e^{i\langle n,x\rangle}\ dx.\end{aligned}$$

We will take here $\phi = s$, $V = V_{r_n} = \{x \in \mathbb{R}^d \quad r_n \leq \|x\| \leq 1\}$, where r_n is a sequence which is to determinate and will have a null limit.

To control the partial derivatives, we will need the following lemma:

Lemma 14. *The exist a constant K such that*

$$(22) \qquad 0 < \|x\| < \frac{1}{2} \Rightarrow \quad \forall k \leq N \quad \sup_{\|h\|=1} |D_x^k s(h^{\otimes k})| \leq K\|x\|^{N-2d_0-k}.$$

For readability, we will prove it later and admit it for a short time.

Then,

$$\begin{aligned}|\int_{V_{r_n}} s(x)e^{i\langle n,x\rangle}\ dx \quad | &\leq \quad K\ S_d\ r_n^{d-1}\frac{r_n^{N-2d_0}}{\|n\|}\sum_{k=0}^{N-1}(\|n\|r_n)^{-k}\\ &+ \quad \frac{K\ B_d}{\|n\|^N}r_n^{-2d_0}.\end{aligned}$$

where S_d is the area of the d-dimensional unit sphere and B_d the volume of the d-dimensional unit ball.

Also, integrating the inequality (14) with $k = 0$, we get

$$\int_{B(0,r_n)} |s(x)\ dx| \leq \frac{K}{d+N-2d_0}r_n^{d+N-2d_0}.$$

Thus, combining these two inequalities, we have

$$\begin{aligned}
|\int_{B(0,1)} s(x)e^{i\langle n,x\rangle}\ dx\quad| \leq &\quad K\ S_d r_n^{d-1}\frac{r_n^{N-2d_0}}{\|n\|}\sum_{k=0}^{N-1}(\|n\|r_n)^{-k}\\
+ &\quad \frac{K\ B_d}{\|n\|^N}r_n^{-2d_0}\\
+ &\quad \frac{K}{d+N-2d_0}r_n^{d+N-2d_0}.
\end{aligned}$$

Then, we choose r_n in such a way that the two last terms – which we suppose to be the biggest – have save order, that is

$$r_n = \frac{1}{\|n\|^{\frac{N}{N+d}}}.$$

Finally, we get

$$|\int_{B(0,1)} s(x)e^{i\langle n,x\rangle}\ dx| = O(\frac{1}{\|n\|^{\frac{N(N+d-2d_0)}{N+d}}}). \tag{23}$$

□

It is now time to prove lemma 14.

Proof of lemma 14. In order to compute and control the successive derivatives at x, we will make a development of d in the neighborhood x. We know that

$$s(x+h) = \sum_{k=0}^{N}\frac{1}{k!}D_x^k s(h^{\otimes k}) + o(\|h\|^N).$$

Since $h_1 = 1$ on $B(0,\frac{1}{2})$, it comes that

$$s(x+h) = F_{\|x\|^{d_0},R_1(x)}(u_x(h),v_x(h)),$$

where

$$\begin{cases}
F_{A,B}(h_1,h_2) &= (A+B+h_1+h_2)^{-1}-(A+h_1)^{-1},\\
u_x(h) &= \|x+h\|^{d_0}-\|x\|^{d_0},\\
v_x(h) &= R_1(x+h)-R_1(x).
\end{cases}$$

We have

$$F_{A,B}(h_1,h_2) = (B+h_2)(A+h_1)^{-1}((A+B)+(h_1+h_2))^{-1}.$$

The following identities holds in the ring of formal series $\mathbb{R}[[h_1,h_2]]$:

$$\frac{1}{A+h_1} = \sum_{k\geq 0}\frac{1}{A^{k+1}}(-1)^k h_1^k.$$

$$\begin{aligned}
\frac{1}{A+B+h_1+h_2} &= \sum_{k\geq 0}\frac{1}{(A+B)^{k+1}}(-1)^k(h_1+h_2)^k\\
&= \sum_{k\geq 0}\frac{1}{(A+B)^{k+1}}(-1)^k\sum_{a+b=k}\binom{a+b}{a}h_1^a h_2^b\\
&= \sum_{a,b\geq 0}\binom{a+b}{a}\frac{1}{(A+B)^{a+b+1}}(-1)^{a+b}h_1^a h_2^b.
\end{aligned}$$

Making the product, we get

$$F_{A,B}(h_1,h_2) = \sum_{a,b,k\geq 0} \binom{a+b}{a} (-1)^{a+b+k+1} \times$$
$$\left(\frac{B}{A^{k+1}(A+B)^{a+b+1}} h_1^{a+k} h_2^b + \frac{1}{A^{k+1}(A+B)^{a+b+1}} h_1^{a+k} h_2^{b+1}\right).$$

Let us now expand $u_x(h)$: we have

$$\|x+h\|^{d_0} = \|x\|^{d_0}(1 + 2\frac{\langle x,h\rangle}{\|x\|^2} + \frac{\langle h,h\rangle}{\|x\|^2})^{\frac{d_0}{2}}.$$

In $\mathbb{R}[[x]]$ holds

$$(1+x)^{\frac{d_0}{2}} = \sum_{i\geq 0} \binom{\frac{d_0}{2}}{i} x^i.$$

We have also

$$(2\frac{\langle x,h\rangle}{\|x\|^2} + \frac{\langle h,h\rangle}{\|x\|^2})^i = \sum_{l=0}^{i} \binom{i}{l} \frac{1}{\|x\|^{2i}} (2\langle x,h\rangle)^{i-l} \langle h,h\rangle^l.$$

We can also deduce that for each $n \geq 1$

$$\frac{D_0^n u_x(h^{\otimes n})}{n!} = \sum_{i\leq \frac{n}{2}} \binom{\frac{d_0}{2}}{i} \binom{i}{n-i} \frac{1}{\|x\|^{2i}} (2\langle x,h\rangle)^{2i-n} \langle h,h\rangle^{n-i} \|x\|^{d_0}.$$

Then, by the inequality of Cauchy-Schwarz, we have

$$\left|\frac{D_0^n u_x(h^{\otimes n})}{n!}\right| \leq \sum_{i\leq \frac{n}{2}} \left|\binom{\frac{d_0}{2}}{i}\right| \binom{i}{n-i} 2^{2i-n} \|x\|^{d_0-n} \|h\|^n.$$

It follows that we can find K_u in order to have for each $k \leq N$

$$\left|\frac{D_0^n u_x(h^{\otimes n})}{n!}\right| \leq K_u \|x\|^{d_0-n} \|h\|^n. \tag{24}$$

In another way, since $R_1(0) = D_0^1 R_1 = \ldots D_0^{N-1} R_1 = 0$, Taylor's formula gives

$$\forall k \leq N \quad \|D_x^k R_1\| = O(\|x\|^{N-k}),$$

or, equivalently, there exists a constant K such that

$$\forall x \quad \|x\| < \frac{1}{2} \forall k \leq N \quad \left|\frac{D_0^k v_x}{k!}(h^{\otimes k})\right| \leq K_v \|x\|^{N-k} \|h\|^k. \tag{25}$$

The homogeneous component whose degree related to h is n writes $u_x(h)^{a+k} v_x(h)^b$ is

$$\Phi_{a,b,k}^n(h) = \sum \prod_{p=1}^{a+k} \frac{D_0^{i_p} u_x(h^{\otimes i_p})}{i_p!} \prod_{p=1}^{b} \frac{D_0^{j_p} v_x(h^{\otimes j_p})}{j_p!},$$

where the sum runs over the positive integers satisfying to

$$\sum_{p=1}^{a+k} i_p + \sum_{p=1}^{b} j_p = n.$$

$n \geq a+k+b$ is a necessary condition for the sum not to be zero. If it holds , the inequalities (24) and (25) show that we can find K such that for $|x| < \frac{1}{2}$, we can write:

$$|\Phi^n_{a,b,k}(h)| \leq K\|x\|^{(a+k)d_0+bN-n}\|h\|^n. \tag{26}$$

Composing these two developments, we get

$$\frac{D^n_x(h^{\otimes n})}{n!} = \sum_{\substack{a,b,k\geq 0\\ a+b+k=n}} \binom{a+b}{a}(-1)^{a+b+k+1}\frac{R_1(x)\Phi^n_{a,b,k}(h)}{\|x\|^{(k+1)d_0}(\|x\|^{d_0}+R_1(x))^{a+b+1}}$$
$$+\sum_{\substack{a,b,k\geq 0\\ a+b+k+1=n}} \binom{a+b}{a}(-1)^{a+b+k+1}\frac{\Phi^n_{a,b+1,k}(h)}{\|x\|^{(k+1)d_0}(\|x\|^{d_0}+R_1(x))^{a+b+1}}.$$

Applying the bounds found in (26) and the fact that $R_1(x)$ is $o(\|x\|^{d_0})$, we can see that each term of the sum if bounded by

$$K'\|x\|^{N-2d_0+b(N-d_0)-n} \leq K''\|x\|^{N-2d_0-n},$$

which is manifestly what we wanted to prove. $\square$

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