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Reflected diffusions defined via the extended Skorokhod map*

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Abstract

This work introduces the extended Skorokhod problem (ESP) and associated extended Skorokhod map (ESM) that enable a pathwise construction of reflected diffusions that are not necessarily semimartingales. Roughly speaking, given the closure G of an open connected set in $\mathbb{R}^J$, a non-empty convex cone $d(x) \subset \mathbb{R}^J$ specified at each point x on the boundary ∂G, and a càdlàg trajectory ψ taking values in $\mathbb{R}^J$, the ESM $\bar{\Gamma}$ defines a constrained version ϕ of ψ that takes values in G and is such that the increments of $\phi - \psi$ on any interval $[s,t]$ lie in the closed convex hull of the directions $d(\phi(u)), u \in (s,t]$. When the graph of $d(\cdot)$ is closed, the following three properties are established: (i) given ψ, if (ϕ,η) solve the ESP then (ϕ,η) solve the corresponding Skorokhod problem (SP) if and only if η is of bounded variation; (ii) given ψ, any solution (ϕ,η) to the ESP is a solution to the SP on the interval $[0,\tau_0)$, but not in general on $[0,\tau_0]$, where τ_0 is the first time that ϕ hits the set $\mathcal{V}$ of points $x \in \partial G$ such that $d(x)$ contains a line; (iii) the graph of the ESM $\bar{\Gamma}$ is closed on the space of càdlàg trajectories (with respect to both the uniform and the J_1-Skorokhod topologies).

The paper then focuses on a class of multi-dimensional ESPs on polyhedral domains with a non-empty $\mathcal{V}$-set. Uniqueness and existence of solutions for this class of ESPs is established

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and existence and pathwise uniqueness of strong solutions to the associated stochastic differential equations with reflection is derived. The associated reflected diffusions are also shown to satisfy the corresponding submartingale problem. Lastly, it is proved that these reflected diffusions are semimartingales on $[0, \tau_0]$. One motivation for the study of this class of reflected diffusions is that they arise as approximations of queueing networks in heavy traffic that use the so-called generalised processor sharing discipline

Key words: reflected diffusions,reflected Brownian motion,Skorokhod map, skorokhod problem,reflection map,extended Skorokhod map,extended Skorohod problem, stochastic differential equations with reflection,submartingale problem, semimartingales, generalised processor sharing,strong solutions

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1 Introduction

1.1 Background and Motivation

Let G be the closure of an open, connected domain in $\mathbb{R}^J$. Let $d(\cdot)$ be a set-valued mapping defined on the boundary ∂G of G such that for every $x \in \partial G$, $d(x)$ is a non-empty, closed and convex cone in $\mathbb{R}^J$ with vertex at the origin $\{0\}$, and the graph $\{(x, d(x)) : x \in \partial G\}$ of $d(\cdot)$ is closed. For convenience, we extend the definition of $d(\cdot)$ to all of G by setting $d(x) = \{0\}$ for x in the interior G° of G. In this paper we are concerned with reflected deterministic and stochastic processes, and in particular reflected Brownian motion, associated with a given pair $(G, d(\cdot))$. Loosely speaking, reflected Brownian motion behaves like Brownian motion in the interior G° of the domain G and, whenever it reaches a point $x \in \partial G$, is instantaneously restricted to remain in G by a constraining process that pushes along one of the directions in $d(x)$. For historical reasons, this constraining action is referred to as instantaneous reflection, and so we will refer to $d(\cdot)$ as the reflection field. There are three main approaches to the study of reflected diffusions – the Skorokhod Problem (SP) approach, first introduced in [42] and subsequently developed in numerous papers such as [1, 12, 18, 27, 32, 41, 44], the submartingale problem formulation, introduced in [43], and Dirichlet form methods (see, for example, [9, 26] and references therein).

In the SP approach, the reflected process is represented as the image of an unconstrained process under a deterministic mapping referred to as the Skorokhod Map (SM). A rigorous definition of the SP is given below. Let $\mathcal{D}[0,\infty)$ be the space of $\mathbb{R}^J$-valued, right-continuous functions on $[0,\infty)$ that have left limits in $(0,\infty)$. Unless stated otherwise, we endow $\mathcal{D}[0,\infty)$ with the topology of uniform convergence on compact intervals, and note that the resulting space is complete [5, 35]. Let $\mathcal{D}_G[0,\infty)$ (respectively, $\mathcal{D}_0[0,\infty)$) be the subspace of functions f in $\mathcal{D}[0,\infty)$ with $f(0) \in G$ (respectively, $f(0) = 0$) and let $\mathcal{BV}_0[0,\infty)$ be the subspace of functions in $\mathcal{D}_0[0,\infty)$ that have finite variation on every bounded interval in $[0,\infty)$. For $\eta \in \mathcal{BV}_0[0,\infty)$ and $t \in [0,\infty)$, we use $|\eta|(t)$ to denote the total variation of η on $[0,t]$. Also, for $x \in G$, let $d^1(x)$ denote the intersection of $d(x)$ with $S_1(0)$, the unit sphere in $\mathbb{R}^J$ centered at the origin. A precise formulation of the SP is given as follows.

Definition 1.1. (Skorokhod Problem) *Let $(G, d(\cdot))$ and $\psi \in \mathcal{D}_G[0,\infty)$ be given. Then $(\phi, \eta) \in \mathcal{D}_G[0,\infty) \times \mathcal{BV}_0[0,\infty)$ solve the SP for ψ if $\phi(0) = \psi(0)$, and if for all $t \in [0,\infty)$, the following properties are satisfied:*

1. $\phi(t) = \psi(t) + \eta(t)$;
2. $\phi(t) \in G$;
3. $|\eta|(t) < \infty$;
4. $|\eta|(t) = \int_{[0,t]} \mathbb{1}_{\{\phi(s) \in \partial G\}} d|\eta|(s)$;
5. *There exists a measurable function $\gamma : [0,\infty) \to S_1(0)$ such that $\gamma(t) \in d^1(\phi(t))$ ($d|\eta|$-almost everywhere) and*
$$\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s).$$

Note that properties 1 and 2 ensure that η constrains ϕ to remain within G. Property 3 requires that the constraining term η has finite variation (on every bounded interval). Property 4 allows η to change only at times s when $\phi(s)$ is on the boundary ∂G, in which case property 5 stipulates that the change be along one of the directions in $d(\phi(s))$. If $(\phi, \phi-\psi)$ solve the SP for ψ, then we write $\phi \in \Gamma(\psi)$, and refer to Γ as the Skorokhod Map (henceforth abbreviated as SM). Observe that in general the SM could be multi-valued. With some abuse of notation we write $\phi = \Gamma(\psi)$ when Γ is single-valued and $(\phi, \phi - \psi)$ solve the SP for ψ. The set of $\psi \in \mathcal{D}_G[0,\infty)$ for which there exists a solution to the SP is defined to be the domain of the SM Γ, denoted dom (Γ).

The SP was first formulated for the case $G = \mathbb{R}_+$, the non-negative real line, and $d(0) = e_1$ by A.V. Skorokhod [42] in order to construct solutions to one-dimensional stochastic differential equations with reflection (SDERs), with a Neumann boundary condition at 0. As is well-known (see, for example, [1]), the associated one-dimensional SM, which we denote by Γ_1, admits the following explicit representation (here $a \vee b$ denotes the maximum of a and b):

$$\Gamma_1(\psi)(t) \doteq \psi(t) + \sup_{s\in[0,t]} [-\psi(s)] \vee 0. \tag{1.1}$$

If W is an adapted, standard Brownian motion defined on a filtered probability space $((\Omega, \mathcal{F}, P), \{\mathcal{F}_t\})$, then the map Γ_1 can be used to construct a reflected Brownian motion Z by setting $Z(\omega) \doteq \Gamma_1(W(\omega))$ for $\omega \in \Omega$. Since the SP is a pathwise technique, it is especially convenient for establishing existence and pathwise uniqueness of strong solutions to SDERs. Another advantage of the SP is that, unlike the submartingale problem, it can be used to construct reflected stochastic processes that are not necessarily diffusions or even Markov processes. On the other hand, any reflected stochastic process defined as the image of a semimartingale under the SM must itself necessarily be a semimartingale (this is an immediate consequence of property 3 of the SP). Thus the SP formulation does not allow the construction of reflected Brownian motions that are not semimartingales.

A second, probabilistic, approach that is used to analyse reflected diffusions is the submartingale problem, which was first formulated in [43] for the analysis of diffusions on smooth domains with smooth boundary conditions and later applied to nonsmooth domains (see, for example, [15, 16, 31, 45, 46]). The submartingale problem associated with a class of reflected Brownian motions (RBMs) in the J-dimensional orthant that are analysed in this paper is described in Definition 4.5. The submartingale formulation has the advantage that it can be used to construct and analyse reflected diffusions that are not necessarily semimartingales. A drawback, however, is that it only yields weak existence and uniqueness of solutions to the associated SDERs. The third, Dirichlet form, approach, has an analytic flavor and is particularly well-suited to the study of symmetric Markov processes (e.g. Brownian motion with normal reflection) in domains with rough boundaries. However, once again, this approach only yields weak existence and uniqueness of solutions [9, 26].

In this paper we introduce a fourth approach, which we refer to as the Extended Skorokhod Problem (ESP), which enables a pathwise analysis of reflected stochastic processes that are not necessarily semimartingales. As noted earlier, the inapplicability of the SP for the construction of non-semimartingale reflected diffusions is a consequence of property 3 of the SP, which requires that the constraining term, η, be of bounded variation. This problem is further compounded by the fact that properties 4 and 5 of the SP are also phrased in terms of the total variation measure $d|\eta|$. It is thus natural to ask if property 3 can be relaxed, while still imposing conditions that suitably restrict (in the spirit of properties 4 and 5 of the SP) the times at and directions in which η can constrain ϕ. This motivates the following definition.

Definition 1.2. (Extended Skorokhod Problem) *Suppose $(G, d(\cdot))$ and $\psi \in \mathcal{D}_G[0,\infty)$ are given. Then $(\phi,\eta) \in \mathcal{D}_G[0,\infty) \times \mathcal{D}[0,\infty)$ solve the ESP for ψ if $\phi(0) = \psi(0)$, and if for all $t \in [0,\infty)$, the following properties hold:*

1. *$\phi(t) = \psi(t) + \eta(t)$;*
2. *$\phi(t) \in G$;*
3. *For every $s \in [0,t]$*
$$\eta(t) - \eta(s) \in \overline{\text{co}}\left[\cup_{u\in(s,t]} d(\phi(u))\right], \tag{1.2}$$
where $\overline{\text{co}}[A]$ represents the closure of the convex hull generated by the set A;
4. *$\eta(t) - \eta(t-) \in \overline{\text{co}}\left[d(\phi(t))\right]$.*

Observe that properties 1 and 2 coincide with those of the SP. Property 3 is a natural generalisation of property 5 of the SP when η is not necessarily of bounded variation. However, note that it only guarantees that

$$\eta(t) - \eta(t-) \in \overline{\text{co}}[d(\phi(t)) \cup d(\phi(t-))] \quad \text{for } t \in [0,\infty).$$

In order to ensure uniqueness of solutions under reasonable conditions for paths that exhibit jumps, it is necessary to impose property 4 as well. Since $d(x) = \{0\}$ for $x \in G^\circ$, properties 3 and 4 of the ESP together imply that if $\phi(u) \in G^\circ$ for $u \in [s,t]$, then $\eta(t) = \eta(s-)$, which is a natural generalisation of property 4 of the SP. As in the case of the SP, if (ϕ,η) solve the ESP for ψ, we write $\phi \in \bar{\Gamma}(\psi)$, and refer to $\bar{\Gamma}$ as the Extended Skorokhod Map (ESM), which could in general be multi-valued. The set of ψ for which the ESP has a solution is denoted $\text{dom}\,(\bar{\Gamma})$. Once again, we will abuse notation and write $\phi = \bar{\Gamma}(\psi)$ when $\psi \in \text{dom}\,(\bar{\Gamma})$ and $\bar{\Gamma}(\psi) = \{\phi\}$ is single-valued.

The first goal of this work is to introduce and prove some general properties of the ESP, which show that the ESP is a natural generalisation of the SP. These (deterministic) properties are summarised in Theorem 1.3. The second objective of this work is to demonstrate the usefulness of the ESP for analysing reflected diffusions. This is done by focusing on a class of reflected diffusions in polyhedral domains in $\mathbb{R}^J$ with piecewise constant reflection fields (whose data $(G, d(\cdot))$ satisfy Assumption 3.1). As shown in [21, 23, 36, 37], ESPs in this class arise as models of queueing networks that use the so-called generalised processor sharing (GPS) service discipline. For this class of ESPs, existence and pathwise uniqueness of strong solutions to the associated SDERs is derived, and the solutions are shown to also satisfy the corresponding submartingale problem. In addition, it is shown that the J-dimensional reflected diffusions are semimartingales on the closed interval $[0,\tau_0]$, where τ_0 is the first time to hit the origin. These (stochastic) results are presented in Theorem 1.4. It was shown in [48] that when $J = 2$, RBMs in this class are not semimartingales on $[0,\infty)$. In subsequent work, the results derived in this paper are used to study the semimartingale property on $[0,\infty)$ of higher-dimensional reflected diffusions in this class. The applicability of the ESP to analyse reflected diffusions in curved domains will also be investigated in future work. In this context, it is worthwhile to note that the ESP coincides with the Skorokhod-type lemma introduced in [8] for the particular two-dimensional thorn domains considered there (see Section 1.3 for further discussion). The next section provides a more detailed description of the main results.

1.2 Main Results and Outline of the Paper

The first main result characterises deterministic properties of the ESP on general domains G with reflection fields $d(\cdot)$ that have a closed graph. As mentioned earlier, the space $\mathcal{D}[0,\infty)$ is endowed with the topology of uniform convergence on compact sets (abbreviated u.o.c.). For notational conciseness, throughout the symbol $\rightarrow$ is used to denote convergence in the u.o.c. topology. On occasion (in which case this will be explicitly mentioned), we will also consider the Skorokhod J_1 topology on $\mathcal{D}[0,\infty)$ (see, for example, Section 12.9 of [47] for a precise definition) and use $\stackrel{J_1}{\rightarrow}$ to denote convergence in this topology. Recall $S_1(0)$ is the unit sphere in $\mathbb{R}^J$ centered at the origin. The following theorem summarises the main results of Section 2. Properties 1 and 2 of Theorem 1.3 correspond to Lemma 2.4, property 3 is equivalent to Theorem 2.9 and property 4 follows from Lemma 2.5 and Remark 2.11.

Theorem 1.3. *Given $(G, d(\cdot))$ that satisfy Assumption 2.1, let Γ and $\bar{\Gamma}$ be the corresponding SM and ESM. Then the following properties hold.*

1. $\mathrm{dom}\,(\Gamma) \subseteq \mathrm{dom}\,(\bar{\Gamma})$ *and for* $\psi \in \mathrm{dom}\,(\Gamma)$, $\phi \in \Gamma(\psi) \Rightarrow \phi \in \bar{\Gamma}(\psi)$.

2. *Suppose $(\phi,\eta) \in \mathcal{D}_G[0,\infty) \times \mathcal{D}_0[0,\infty)$ solve the ESP for $\psi \in \mathrm{dom}\,(\bar{\Gamma})$. Then (ϕ,η) solve the SP for ψ if and only if $\eta \in \mathcal{BV}_0[0,\infty)$.*

3. *If (ϕ,η) solve the ESP for some $\psi \in \mathrm{dom}\,(\bar{\Gamma})$ and $\tau_0 \doteq \inf\{t \geq 0 : \phi(t) \in \mathcal{V}\}$, where*

$$\mathcal{V} \doteq \{x \in \partial G : \text{ there exists } d \in S_1(0) \text{ such that } \{d, -d\} \subseteq d(x)\},$$

   then (ϕ,η) also solve the SP for ψ on $[0,\tau_0)$. In particular, if $\mathcal{V} = \emptyset$, then (ϕ,η) solve the SP for ψ.

4. *Given a sequence of functions $\{\psi_n\}$ such that $\psi_n \in \mathrm{dom}\,(\bar{\Gamma})$, for $n \in \mathbb{N}$, and $\psi_n \rightarrow \psi$, let $\{\phi_n\}$ be a corresponding sequence with $\phi_n \in \bar{\Gamma}(\psi_n)$ for $n \in \mathbb{N}$. If there exists a limit point ϕ of the sequence $\{\phi_n\}$ with respect to the u.o.c. topology, then $\phi \in \bar{\Gamma}(\psi)$. The statement continues to hold if $\psi_n \rightarrow \psi$ is replaced by $\psi_n \stackrel{J_1}{\rightarrow} \psi$ and ϕ is a limit point of $\{\phi_n\}$ with respect to the Skorokhod J_1 topology.*

The first three results of Theorem 1.3 demonstrate in what way the ESM $\bar{\Gamma}$ is a generalisation of the SM Γ. In addition, Corollary 3.9 proves that the ESM is in fact a strict generalisation of the SM Γ for a large class of ESPs with $\mathcal{V} \neq \emptyset$. Specifically, for that class of ESPs it is shown that there always exists a continuous function ψ and a pair (ϕ,η) that solve the ESP for ψ such that $|\eta|(\tau_0) = \infty$. The fourth property of Theorem 1.3, stated more succinctly, says that if $d(\cdot)$ has a closed graph on $\mathbb{R}^J$, then the corresponding (multi-valued) ESM $\bar{\Gamma}$ also has a closed graph (where the closure can be taken with respect to either the u.o.c. or Skorokhod J_1 topologies). As shown in Lemma 2.6, the closure property is very useful for establishing existence of solutions – the corresponding property does not hold for the SM without the imposition of additional conditions on $(G, d(\cdot))$. For example, the completely-$\mathcal{S}$ condition in [4, 33], or generalisations of it introduced in [12] and [18], were imposed in various contexts to establish that the SM Γ has a closed graph. However, all these conditions imply that $\mathcal{V} = \emptyset$. Thus properties 3 and 4 above together imply and generalise (see Corollary 2.10 and Remark 2.12) the closure property results for the SM established in [4, 12, 18, 33].

While Theorem 1.3 establishes some very useful properties of the ESP under rather weak assumptions on $(G, d(\cdot))$, additional conditions are clearly required to establish existence and uniqueness of solutions to the ESP (an obvious necessary condition for existence of solutions is that for each $x \in \partial G$, there exists a vector $d \in d(x)$ that points into the interior of G). Here we do not attempt to derive general conditions for existence and uniqueness of solutions to the ESP on arbitrary domains. Indeed, despite a lot of work on the subject (see, for example, [1, 4, 12, 18, 22, 23, 27, 32, 44]), necessary and sufficient conditions for existence and uniqueness of solutions on general domains are not fully understood even for the SP. Instead, in Section 3 we focus on a class of ESPs in polyhedral domains with piecewise constant $d(\cdot)$. We establish sufficient conditions for existence and uniqueness of solutions to ESPs in this class in Section 3.1, and in Theorem 3.6 verify these conditions for the GPS family of ESPs described in Section 3.2. This class of ESPs is of interest because it characterises models of networks with fully cooperative servers (see, for instance, [21, 23, 24, 36, 37]). Applications, especially from queueing theory, have previously motivated the study of many polyhedral SPs with oblique directions of constraint (see, for example, [11, 13, 27]).

In Section 4 we consider SDERs associated with the ESP. The next main theorem summarises the results on properties of reflected diffusions associated with the GPS ESP, which has as domain $G = \mathbb{R}_+^J$, the non-negative J-dimensional orthant. To state these results we need to first introduce some notation. For a given integer $J \geq 2$, let Ω_J be the set of continuous functions ω from $[0, \infty)$ to $\mathbb{R}_+^J = \{x \in \mathbb{R}^J : x_i \geq 0, i = 1, \ldots, J\}$. For $t \geq 0$, let $\mathcal{M}_t$ be the σ-algebra of subsets of Ω_J generated by the coordinate maps $\pi_s(\omega) \doteq \omega(s)$ for $0 \leq s \leq t$, and let $\mathcal{M}$ denote the associated σ-algebra $\sigma\{\pi_s : 0 \leq s < \infty\}$. The definition of a strong solution to an SDER associated with an ESP is given in Section 4.1.

Theorem 1.4. *Consider drift and dispersion coefficients $b(\cdot)$ and $\sigma(\cdot)$ that satisfy the usual Lipschitz conditions (stated as Assumption 4.1(1)) and suppose that a J-dimensional, adapted Brownian motion, $\{X_t, t \geq 0\}$, defined on a filtered probability space $((\Omega, \mathcal{F}, P), \{\mathcal{F}_t\})$ is given. Then the following properties hold.*

1. *For every $z \in \mathbb{R}_+^J$, there exists a pathwise unique strong solution Z to the SDER associated with the GPS ESP with initial condition $Z(0) = z$. Moreover, Z is a strong Markov process.*
2. *Suppose, in addition, that the diffusion coefficient is uniformly elliptic (see Assumption 4.1(2)). If for each $z \in \mathbb{R}_+^J$, Q_z is the measure induced on $(\Omega_J, \mathcal{M})$ by the law of the pathwise unique strong solution Z with initial condition z, then for $J = 2$, $\{Q_z, z \in \mathbb{R}_+^J\}$ satisfies the submartingale problem associated with the GPS ESP (described in Definition 4.5).*
3. *Furthermore, if the diffusion coefficient is uniformly elliptic, then Z is a semimartingale on $[0, \tau_0]$, where τ_0 is the first time to hit the set $\mathcal{V} = \{0\}$.*

The first statement of Theorem 1.4 follows directly from Corollary 4.4, while the second property corresponds to Theorem 4.6. As can be seen from the proofs of Theorem 4.3 and Corollary 4.4, the existence of a strong solution Z to the SDER associated with the GPS ESP (under the standard assumptions on the drift and diffusion coefficients) and the fact that Z is a semimartingale on $[0, \tau_0)$ are quite straightforward consequences of the corresponding deterministic results (specifically, Theorem 3.6 and Theorem 2.9). In turn, these properties can be shown to imply the first two properties of the associated submartingale problem. The proof of the remaining third condition of the submartingale problem relies on geometric properties of the GPS ESP (stated in Lemma 3.4) that reduce the problem to the verification of a property of one-dimensional reflected Brownian motion, which is carried out in Corollary 3.5.

The most challenging result to prove in Theorem 1.4 is the third property, which is stated as Theorem 5.10. As Theorem 3.8 demonstrates, this result does not carry over from a deterministic analysis of the ESP, but instead requires a stochastic analysis. In Section 5, we first establish this result in a more general setting, namely for strong solutions Z to SDERs associated with general (not necessarily polyhedral) ESPs. Specifically, in Theorem 5.2 we identify sufficient conditions (namely inequalities (5.38) and (5.39) and Assumption 5.1) for the strong solution Z to be a semimartingale on $[0, \tau_0]$. The first inequality (5.38) requires that the drift and diffusion coefficients be uniformly bounded in a neighbourhood of $\mathcal{V}$. This automatically holds for the GPS ESP with either bounded or continuous drift and diffusion coefficients since, for the GPS ESP, $\mathcal{V} = \{0\}$ is bounded. The second inequality (5.39) is verified in Corollary 5.6. As shown there, due to a certain relation between Z and an associated one-dimensional reflected diffusion (see Corollary 3.5 for a precise statement) the verification of the relation (5.39) essentially reduces to checking a property of an ordinary (unconstrained) diffusion. The key condition is therefore Assumption 5.1, which requires the existence of a test function that satisfies certain oblique derivative inequalities on the boundary of the domain. Section 6 is devoted to the construction of such a test function for (a slight generalisation of) the GPS family of ESPs. This construction may be of independent interest (for example, for the construction of viscosity solutions to related partial differential equations [19]).

A short outline of the rest of the paper is as follows. In Section 2, we derive deterministic properties of the ESP on general domains (that satisfy the mild hypothesis stated as Assumption 2.1) – the main results of this section were summarised above in Theorem 1.3. In Section 3, we specialise to the class of so-called polyhedral ESPs (described in Assumption 3.1). We introduce the class of GPS ESPs in Section 3.2 and prove some associated properties. In Section 4, we analyse SDERs associated with ESPs. We discuss the existence and uniqueness of strong solutions to such SDERs in Section 4.1, and show that the pathwise unique strong solution associated with the GPS ESP solves the corresponding submartingale problem in Section 4.2. In Section 5, we state general sufficient conditions for the reflected diffusion to be a semimartingale on $[0, \tau_0]$ and then verify them for non-degenerate reflected diffusions associated with the GPS ESP. This entails the construction of certain test functions that satisfy Assumption 5.1, the details of which are relegated to Section 6.

1.3 Relation to Some Prior Work

When $J = 2$, the data $(G, d(\cdot))$ for the polyhedral ESPs studied here corresponds to the two-dimensional wedge model of [46] with $\alpha = 1$ and the wedge angle less than π. In [46], the submartingale problem approach was used to establish weak existence and uniqueness of the associated reflected Brownian motions (RBMs). Corollary 4.4 of the present paper (specialised to the case $J = 2$) establishes strong uniqueness and existence of associated reflected diffusions (with drift and diffusion coefficient satisfying the usual Lipschitz conditions, and the diffusion coefficient possibly degenerate), thus strengthening the corresponding result (with $\alpha = 1$) in

Theorem 3.12 of [46]. The associated RBM was shown to be a semimartingale on $[0, \tau_0]$ in Theorem 1 of [48] and this result, along with additional work, was used to show that the RBM is not a semimartingale on $[0, \infty)$ in Theorem 5 of [48]. An explicit semimartingale representation for RBMs in certain two-dimensional wedges was also given in [14]. Here we employ different techniques, that are not restricted to two dimensions, to prove that the J-dimensional GPS reflected diffusions (for all $J \geq 2$) are semimartingales on $[0, \tau_0]$. This result is used in a forthcoming paper to study the semimartingale property of this family of J-dimensional reflected diffusions on $[0, \infty)$. Investigation of the semimartingale property is important because semimartingales comprise the natural class of integrators for stochastic integrals (see, for example, [3]) and the evolution of functionals of semimartingales can be characterized using Itô's formula.

Although this paper concentrates on reflected diffusions associated with the class of GPS ESPs, or more generally on ESPs with polyhedral domains having piecewise constant reflection fields, as elaborated below, the ESP is potentially also useful for analysing non-semimartingale reflected diffusions in curved domains. In view of this fact, many results in the paper are stated in greater generality than required for the class of polyhedral ESPs that are the focus of this paper. Non-semimartingale RBMs in 2-dimensional cusps with normal reflection fields were analysed using the submartingale approach in [15, 16]. In [8], a pathwise approach was adopted to examine properties of reflected diffusions in downward-pointing 2-dimensional thorns with horizontal vectors of reflection. Specifically, the thorns G considered in [8] admit the following description in terms of two continuous real functions L, R defined on $[0, \infty)$, with $L(0) = R(0) = 0$ and $L(y) < R(y)$ for all $y > 0$: $G = \{(x, y) \in \mathbb{R}^2 : y \geq 0, L(y) \leq x \leq R(y)\}$. The deterministic Skorokhod-type lemma introduced in Theorem 1 of [8] can easily be seen to correspond to the ESP associated with $(G, d(\cdot))$, where $d(\cdot)$ is specified on the boundary ∂G by $d((x, y)) = \{\alpha e_1, \alpha \geq 0\}$ when $x = L(y), y \neq 0$, $d((x, y)) = \{-\alpha e_1, \alpha \geq 0\}$ when $x = R(y), y \neq 0$, $d((0, 0)) = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ and, as usual, $d(x) = \{0\}$ for $x \in G^\circ$. The Skorokhod-type lemma of [8] can thus be viewed as a particular two-dimensional ESP, and existence and uniqueness for solutions to this ESP for continuous functions ψ (defined on $[0, \infty)$ and taking values in $\mathbb{R}^2$) follows from Theorem 1 of [8]. While the Skorokhod-type lemma of [8] was phrased in the context of the two-dimensional thorns considered therein, the ESP formulation is applicable to more general reflection fields and domains in higher dimensions. The Skorokhod-type lemma was used in [8] to prove an interesting result on the boundedness of the variation of the constraining term η during a single excursion of the reflected diffusions in these thorns. Other works that have studied the existence of a semimartingale decomposition for symmetric, reflected diffusions associated with Dirichlet spaces on possibly non-smooth domains include [9, 10].

1.4 Notation

Here we collect some notation that is commonly used throughout the paper. Given any subset E of $\mathbb{R}^J$, $\mathcal{D}([0, \infty) : E)$ denotes the space of right continuous functions with left limits taking values in E, and $\mathcal{BV}([0, \infty) : E)$ and $\mathcal{C}([0, \infty) : E)$, respectively, denote the subspace of functions that have bounded variation on every bounded interval and the subspace of continuous functions. Given $G \subset E \subset \mathbb{R}^J$, $\mathcal{D}_G([0, \infty) : E) \doteq \mathcal{D}([0, \infty) : E) \cap \{f \in \mathcal{D}([0, \infty) : E) : f(0) \in G\}$ and $\mathcal{C}_G([0, \infty) : E)$ is defined analogously. Also, $\mathcal{D}_0([0, \infty) : E)$ and $\mathcal{BV}_0([0, \infty) : E)$ are defined to be the subspace of functions f that satisfy $f(0) = 0$ in $\mathcal{D}([0, \infty) : E)$ and $\mathcal{BV}([0, \infty) : E)$, respectively. When $E = \mathbb{R}^J$, for conciseness we denote these spaces simply by $\mathcal{D}[0,\infty)$, $\mathcal{D}_0[0,\infty)$, $\mathcal{D}_G[0,\infty)$, $\mathcal{BV}[0,\infty)$, $\mathcal{BV}_0[0,\infty)$, $\mathcal{C}[0,\infty)$ and $\mathcal{C}_G[0,\infty)$, respectively. Unless specified otherwise, we assume that all the function spaces are endowed with the topology of uniform convergence (with respect to the Euclidean norm) on compact sets, and the notation $f_n \to f$ implies that f_n converges to f in this topology, as $n \to \infty$. For $f \in \mathcal{BV}([0,\infty) : E)$ and $t \in [0,\infty)$, let $|f|(t)$ be the total variation of f on $[0,t]$ with respect to the Euclidean norm on E, which is denoted by $|\cdot|$. For $f \in \mathcal{D}([0,\infty) : E)$ and $t \in [0,\infty)$, , as usual $f(t-) \doteq \lim_{s\uparrow t} f(s)$. For $U \subseteq \mathbb{R}^J$, we use $\mathcal{C}(U)$ and $\mathcal{C}^i(U)$, respectively, to denote the space of real-valued functions that are continuous and i times continuously differentiable on some open set containing U. Let $\mathrm{supp}[f]$ represent the support of a real-valued function f and for $f \in \mathcal{C}^1(E)$, let ∇f denote the gradient of f.

We use $\mathbb{K}$ and $\mathbb{J}$ to denote the finite sets $\{1,\ldots,K\}$ and $\{1,\ldots,J\}$, respectively. Given real numbers a, b, we let $a \wedge b$ and $a \vee b$ denote the minimum and maximum of the two numbers respectively. For $a \in \mathbb{R}$, as usual $\lceil a \rceil$ denotes the least integer greater than or equal to a. Given vectors $u, v \in \mathbb{R}^J$, both $\langle u, v\rangle$ and $u \cdot v$ will be used to denote inner product. For a finite set S, we use $\#[S]$ to denote the cardinality of the set S. For $x \in \mathbb{R}^J$, $d(x, A) \doteq \inf_{y\in A} |x - y|$ is the Euclidean distance of x from the set A. Moreover, given $\delta > 0$, we let $N_\delta(A) \doteq \{y \in \mathbb{R}^J : d(y, A) \leq \delta\}$, be the closed δ-neighbourhood of A. With some abuse of notation, when $A = \{x\}$ is a singleton, we write $N_\delta(x)$ instead of $N_\delta(\{x\})$ and write $N^\circ(\delta)$ to denote the interior $(N(\delta))^\circ$ of N_δ. $S_\delta(x) \doteq \{y \in \mathbb{R}^J : |y - x| = \delta\}$ is used to denote the sphere of radius δ centered at x. Given any set $A \subset \mathbb{R}^J$ we let A°, $\overline{A}$ and ∂A denote its interior, closure and boundary respectively, $\mathbb{1}_A(\cdot)$ represents the indicator function of the set A, $\overline{\mathrm{co}}[A]$ denotes the (closure of the) convex hull generated by the set A and $\overline{\mathrm{cone}}[A]$ represents the closure of the non-negative cone $\{\alpha x : \alpha \geq 0, x \in A\}$ generated by the set A. Given sets $A, M \subset \mathbb{R}^J$ with $A \subset M$, A is said to be open relative to M if A is the intersection of M with some open set in $\mathbb{R}^J$. Furthermore, a point $x \in A$ is said to be a relative interior point of A with respect to M if there is some $\varepsilon > 0$ such that $N_\varepsilon(x) \cap M \subset A$, and the collection of all relative interior points is called the relative interior of A, and denoted as $\mathrm{rint}(A)$.

2 Properties of the Extended Skorokhod Problem

As mentioned in the introduction, throughout the paper we consider pairs $(G, d(\cdot))$ that satisfy the following assumption.

Assumption 2.1. (General Domains) *G is the closure of a connected, open set in $\mathbb{R}^J$. For every $x \in \partial G$, $d(x)$ is a non-empty, non-zero, closed, convex cone with vertex at $\{0\}$, $d(x) \doteq \{0\}$ for $x \in G^\circ$ and the graph $\{(x, d(x)), x \in G\}$ of $d(\cdot)$ is closed.*

Remark 2.2. Recall that by definition, the graph of $d(\cdot)$ is closed if and only if for every pair of convergent sequences $\{x_n\} \subset G$, $x_n \to x$ and $\{d_n\} \subset \mathbb{R}^J$, $d_n \to d$ such that $d_n \in d(x_n)$ for every $n \in \mathbb{N}$, it follows that $d \in d(x)$. Now let

$$d^1(x) \doteq d(x) \cap S_1(0) \quad \text{for } x \in G \tag{2.3}$$

and consider the map $d^1(\cdot) : \partial G \to S_1(0)$. Since ∂G and $S_1(0)$ are closed, Assumption 2.1 implies that the graph of $d^1(\cdot)$ is also closed. In turn, since $S_1(0)$ is compact, $d^1(x)$ is compact for every $x \in G$, and so this implies that $d^1(\cdot)$ is upper-semicontinuous (see Proposition 1.4.8 and Definition 1.4.1 of [2]). In other words, this means that for every $x \in \partial G$, given $\delta > 0$ there exists $\theta > 0$ such that

$$\cup_{y \in N_\theta(x) \cap \partial G} d^1(y) \subseteq N_\delta(d^1(x)) \cap S_1(0). \tag{2.4}$$

Since $d(x) = \{0\}$ for $x \in G^\circ$, this implies in fact that given $\delta > 0$, there exists $\theta > 0$ such that

$$\overline{\text{co}}\left[\cup_{y \in N_\theta(x)} d(y)\right] \subseteq \overline{\text{cone}}\left[N_\delta\left(d^1(x)\right)\right]. \tag{2.5}$$

In fact, since each $d(x)$ is a non-empty cone, the closure of the graph of $d(\cdot)$ is in fact equivalent to the upper semicontinuity (u.s.c.) of $d^1(\cdot)$. The latter characterisation will sometimes turn out to be more convenient to use.

In this section, we establish some useful (deterministic) properties of the ESP under the relatively mild condition stated in Assumption 2.1. In Section 2.1, we characterise the relationship between the SP and the ESP. Section 2.2 introduces the concept of the $\mathcal{V}$-set, which plays an important role in the analysis of the ESP, and establishes its properties.

2.1 Relation to the SP

The first result is an elementary non-anticipatory property of solutions to the ESP, which holds when the ESM is single-valued. A map $\Lambda : \mathcal{D}[0,\infty) \to \mathcal{D}[0,\infty)$ will be said to be non-anticipatory if for every $\psi, \psi' \in \mathcal{D}[0,\infty)$ and $T \in (0,\infty)$, $\psi(u) = \psi'(u)$ for $u \in [0,T]$ implies that $\Lambda(\psi)(u) = \Lambda(\psi')(u)$ for $u \in [0,T]$.

Lemma 2.3. (Non-anticipatory property) *Suppose (ϕ,η) solve the ESP $(G, d(\cdot))$ for $\psi \in \mathcal{D}_G[0,\infty)$ and suppose that for $T \in [0,\infty)$,*

$$\phi^T(\cdot) \doteq \phi(T+\cdot), \quad \psi^T(\cdot) \doteq \psi(T+\cdot) - \psi(T), \quad \eta^T(\cdot) \doteq \eta(T+\cdot) - \eta(T).$$

Then (ϕ^T, η^T) solve the ESP for $\phi(T)+\psi^T$. Moreover, if (ϕ,η) is the unique solution to the ESP for ψ then for any $[T,S] \subset [0,\infty)$, $\phi(S)$ depends only on $\phi(T)$ and the values $\{\psi(s), s \in [T,S]\}$. In particular, in this case the ESM and the map $\psi \mapsto \eta$ are non-anticipatory.

Proof. The proof of the first statement follows directly from the definition of the ESP. Indeed, since (ϕ,η) solve the ESP for ψ, it is clear that for any $T < \infty$ and $t \in [0,\infty)$, $\phi^T(t) - \eta^T(t)$ is equal to

$$\phi(T+t) - \eta(T+t) + \eta(T) = \psi(T+t) + \phi(T) - \psi(T) = \phi(T) + \psi^T(t),$$

which proves property 1 of the ESP. Property 2 holds trivially. Finally, for any $0 \le s \le t < \infty$, $\eta^T(t) - \eta^T(s)$ is equal to

$$\eta(T+t) - \eta(T+s) \in \overline{\text{co}}\left[\cup_{u \in (T+s,T+t]} d(\phi(u))\right] = \overline{\text{co}}\left[\cup_{u \in (s,t]} d(\phi^T(u))\right],$$

which establishes property 3. Property 4 follows analogously, thus proving that (ϕ^T, η^T) solve the ESP for $\phi(T) + \psi^T$.

If (ϕ,η) is the unique solution to the ESP for ψ, then the first statement of the lemma implies that for every $T \in [0,\infty)$ and $S > T$, $\phi(S) = \bar{\Gamma}(\phi(T) + \psi^T)(S-T)$ and $\eta(S) = \phi(S) - \psi(S)$. This immediately proves the second and third assertions of the lemma. ∎

The next result describes in what sense the ESP is a generalisation of the SP. It is not hard to see from Definition 1.2 that any solution to the SP is also a solution to the ESP (for the same input ψ). Lemma 2.4 shows in addition that solutions to the ESP for a given ψ are also solutions to the SP for that ψ precisely when the corresponding constraining term η is of finite variation (on bounded intervals).

Lemma 2.4. (Generalisation of the SP) *Given data $(G, d(\cdot))$ that satisfies Assumption 2.1, let Γ and $\bar{\Gamma}$, respectively, be the associated SM and ESM. Then the following properties hold.*

1. $\mathrm{dom}\,(\Gamma) \subseteq \mathrm{dom}\,(\bar{\Gamma})$ *and for* $\psi \in \mathrm{dom}\,(\Gamma)$, $\phi \in \Gamma(\psi) \Rightarrow \phi \in \bar{\Gamma}(\psi)$.

2. *Suppose* $(\phi, \eta) \in \mathcal{D}_G\,[0, \infty) \times \mathcal{D}_0\,[0, \infty)$ *solve the ESP for* $\psi \in \mathrm{dom}\,(\bar{\Gamma})$. *Then* (ϕ, η) *solve the SP for* ψ *if and only if* $\eta \in \mathcal{BV}_0\,[0, \infty)$.

Proof. The first assertion follows directly from the fact that properties 1 and 2 are common to both the SP and the ESP, and properties 3-5 in Definition 1.1 of the SP imply properties 3 and 4 in Definition 1.2 of the ESP.

For the second statement, first let $(\phi, \eta) \in \mathcal{D}_G\,[0, \infty) \times \mathcal{D}_0\,[0, \infty)$ solve the ESP for $\psi \in \mathrm{dom}\,(\bar{\Gamma})$. If $\eta \notin \mathcal{BV}_0\,[0, \infty)$, then property 3 of the SP is violated, and so clearly (ϕ, η) do not solve the SP for ψ. Now suppose $\eta \in \mathcal{BV}_0\,[0, \infty)$. Then (ϕ, η) automatically satisfy properties 1–3 of the SP. Also observe that η is absolutely continuous with respect to $|\eta|$ and let γ be the Radon-Nikodỳm derivative $d\eta/d|\eta|$ of $d\eta$ with respect to $d|\eta|$. Then γ is $d|\eta|$-measurable, $\gamma(s) \in S_1(0)$ for $d|\eta|$ a.e. $s \in [0, \infty)$ and

$$\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s). \tag{2.6}$$

Moreover, as is well-known (see, for example, Section X.4 of [17]), for $d|\eta|$ a.e. $t \in [0, \infty)$,

$$\gamma(t) = \lim_{n \to \infty} \frac{d\eta[t, t + \varepsilon_n]}{d|\eta|[t, t + \varepsilon_n]} = \lim_{n \to \infty} \frac{\eta(t + \varepsilon_n) - \eta(t-)}{|\eta|(t + \varepsilon_n) - |\eta|(t-)}, \tag{2.7}$$

where $\{\varepsilon_n, n \in \mathbb{N}\}$ is a sequence (possibly depending on t) such that $|\eta|(t + \varepsilon_n) - |\eta|(t-) > 0$ for every $n \in \mathbb{N}$ and $\varepsilon_n \to 0$ as $n \to 0$ (such a sequence can always be found for $d|\eta|$ a.e. $t \in [0, \infty)$). Fix $t \in [0, \infty)$ such that (2.7) holds. Then properties 3 and 4 of the ESP, along with the right-continuity of ϕ, show that given any $\theta > 0$, there exists $\varepsilon_t > 0$ such that for every $\varepsilon \in (0, \varepsilon_t)$,

$$\eta(t + \varepsilon) - \eta(t-) \in \overline{\mathrm{co}} \left[\cup_{u \in [t, t+\varepsilon]} d(\phi(u)) \right] \subseteq \overline{\mathrm{co}} \left[\cup_{y \in N_\theta(\phi(t))} d(y) \right]. \tag{2.8}$$

If $\phi(t) \in G^\circ$, then since G° is open, there exists $\theta > 0$ such that $N_\theta(\phi(t)) \subset G^\circ$, and hence the fact that $d(y) = \{0\}$ for $y \in G^\circ$ implies that the right-hand side of (2.8) is equal to $\{0\}$. When combined with (2.7) this implies that $\gamma(t) = 0$ for $d|\eta|$ a.e. t such that $\phi(t) \in G^\circ$, which establishes property 4 of the SP. On the other hand, if $\phi(t) \in \partial G$ then the u.s.c. of $d^1(\cdot)$ (in particular, relation (2.5)) shows that given $\delta > 0$, there exists $\theta > 0$ such that

$$\overline{\mathrm{co}} \left[\cup_{y \in N_\theta(\phi(t))} d(y) \right] \subseteq \overline{\mathrm{cone}} \left[N_\delta \left(d^1(\phi(t)) \right) \right]. \tag{2.9}$$

Combining this inclusion with (2.8), (2.7) and the fact that $|\eta|(t+\varepsilon_n) - |\eta|(t-) > 0$ for all $n \in \mathbb{N}$, we conclude that

$$\gamma(t) \in \overline{\mathrm{cone}} \left[N_\delta \left(d^1(\phi(t)) \right) \right] \cap S_1(0).$$

Since $\delta > 0$ is arbitrary, taking the intersection of the right-hand side over $\delta > 0$ shows that $\gamma(t) \in d^1(\phi(t))$ for $d|\eta|$ a.e. t such that $\phi(t) \in \partial G$. Thus (ϕ, η) satisfy property 5 of the SP and the proof of the lemma is complete. $\square$

Lemma 2.5 proves a closure property for solutions to the ESP: namely that the graph $\{(\psi, \phi) : \phi \in \bar{\Gamma}(\psi), \psi \in \mathcal{D}_G[0,\infty)\}$ of the set-valued mapping $\bar{\Gamma}$ is closed (with respect to both the uniform and Skorokhod J_1 topologies). As discussed after the statement of Theorem 1.3, such a closure property is valid for the SP only under certain additional conditions, which are in some instances too restrictive (since they imply $\mathcal{V} = \emptyset$). Indeed, one of the goals of this work is to define a suitable pathwise mapping $\psi \mapsto \phi$ for all $\psi \in \mathcal{D}_G[0,\infty)$ even when $\mathcal{V} \neq \emptyset$.

Lemma 2.5. (Closure Property) *Given an ESP $(G, d(\cdot))$ that satisfies Assumption 2.1, suppose for $n \in \mathbb{N}$, $\psi_n \in \operatorname{dom}(\bar{\Gamma})$ and $\phi_n \in \bar{\Gamma}(\psi_n)$. If $\psi_n \to \psi$ and ϕ is a limit point (in the u.o.c. topology) of the sequence $\{\phi_n\}$, then $\phi \in \bar{\Gamma}(\psi)$.*

Remark. For the class of polyhedral ESPs, in Section 3.1 we establish conditions under which the sequence $\{\phi_n\}$ in Lemma 2.5 is precompact, so that a limit point ϕ exists.

Proof of Lemma 2.5. Let $\{\psi_n\}$, $\{\phi_n\}$ and ϕ be as in the statement of the lemma and set $\eta_n \doteq \phi_n - \psi_n$ and $\eta \doteq \phi - \psi$. Since ϕ is a limit point of $\{\phi_n\}$, there must exist a subsequence $\{n_k\}$ such that $\phi_{n_k} \to \phi$ as $k \to \infty$. Property 1 and (since G is closed) property 2 of the ESP are automatically satisfied by (ϕ, η). Now fix $t \in [0,\infty)$. Then given $\delta > 0$, there exists $k_0 < \infty$ such that for all $k \geq k_0$,

$$\eta_{n_k}(t) - \eta_{n_k}(t-) \in d(\phi_{n_k}(t)) \subseteq \overline{\text{cone}}\left[N_\delta\left(d^1(\phi(t))\right)\right],$$

where the first relation follows from property 4 of the ESP and the second inclusion is a consequence of the u.s.c. of $d^1(\cdot)$ (see relation (2.5)) and the fact that $\phi_{n_k}(t) \to \phi(t)$ as $k \to \infty$. Sending first $k \to \infty$ and then $\delta \to 0$ in the last display, we conclude that

$$\eta(t) - \eta(t-) \in d(\phi(t)) \quad \text{for every } t \in (0,\infty), \tag{2.10}$$

which shows that (ϕ, η) also satisfy property 4 of the ESP for ψ.

Now, let $J_\phi \doteq \{t \in (0,\infty) : \phi(t) \neq \phi(t-)\}$ be the set of jump points of ϕ. Then J_ϕ is a closed, countable set and so $(0,\infty) \setminus J_\phi$ is open and can hence be written as the countable union of open intervals $(s_i, t_i), i \in \mathbb{N}$. Fix $i \in \mathbb{N}$ and let $[s,t] \subseteq [s_i, t_i]$. Then for $\varepsilon \in (0, (t-s)/2)$, property 3 of the ESP shows that

$$\eta_{n_k}(t-\varepsilon) - \eta_{n_k}(s+\varepsilon) \in \overline{\text{co}}\left[\cup_{u \in (s+\varepsilon, t-\varepsilon]} d\left(\phi_{n_k}(u)\right)\right].$$

We claim (and justify the claim below) that since $\phi_{n_k} \to \phi$ and ϕ is continuous on $[s+\varepsilon, t-\varepsilon]$, given $\delta > 0$ there exists $k_* = k_*(\delta) < \infty$ such that for every $k \geq k_*$,

$$\cup_{u \in [s+\varepsilon, t-\varepsilon]} d^1\left(\phi_{n_k}(u)\right) \subseteq N_\delta\left(\cup_{u \in [s+\varepsilon, t-\varepsilon]} d^1(\phi(u))\right) \cap S_1(0). \tag{2.11}$$

If the claim holds, then the last two displays together show that

$$\eta_{n_k}(t-\varepsilon) - \eta_{n_k}(s+\varepsilon) \in \overline{\text{cone}}\left[\{0\} \cup N_\delta\left(\cup_{u \in [s+\varepsilon, t-\varepsilon]} d^1(\phi(u))\right)\right].$$

Taking limits first as $k \to \infty$, then $\delta \to 0$ and lastly $\varepsilon \to 0$, we obtain

$$\eta(t-) - \eta(s) \in \overline{\text{co}}\left(\cup_{u\in(s,t)} d(\phi(u))\right) \text{ if } s_i \leq s < t \leq t_i \text{ for some } i \in \mathbb{N}.$$

Now, for arbitrary $(a,b) \subset (0,\infty)$, $\eta(b) - \eta(a)$ can be decomposed into a countable sum of terms of the form $\eta(t) - \eta(t-)$ for some $t \in (a,b]$ and $\eta(t-) - \eta(s)$ for s,t such that $[s,t] \subseteq [s_i,t_i]$ for some $i \in \mathbb{N}$. Thus the last display, together with (2.10), shows that (ϕ,η) satisfy property 3 of the ESP for ψ.

Thus to complete the proof of the lemma, it only remains to justify the claim (2.11). For this we use an argument by contradiction. Suppose there exists some $i \in \mathbb{N}$, $[s,t] \subseteq [s_i,t_i]$, $\varepsilon \in (0,(t-s)/2)$ and $\delta > 0$ such that the relation (2.11) does not hold. Then there exists a further subsequence of $\{n_k\}$ (which we denote again by $\{n_k\}$), and corresponding sequences $\{u_k\}$ and $\{d_k\}$ with $u_k \in [s+\varepsilon, t-\varepsilon]$, $d_k \in d^1(\phi_{n_k}(u_k))$ and $d_k \notin N_\delta\left(\cup_{u\in[s+\varepsilon,t-\varepsilon]} d^1(\phi(u))\right)$ for $k \in \mathbb{N}$. Since $S_1(0)$ and $[s+\varepsilon, t-\varepsilon]$ are compact, there exist $d_* \in S_1(0)$ and $u_* \in [s+\varepsilon, t-\varepsilon]$ such that $d_k \to d_*$, $u_k \to u_*$ (along a common subsequence, which we denote again by $\{d_k\}$ and $\{u_k\}$). Moreover, it is clear that

$$d_* \notin N_{\delta/2}\left(\cup_{u\in[s+\varepsilon,t-\varepsilon]} d^1(\phi(u))\right). \tag{2.12}$$

On the other hand, since $u_k \to u_*$, $\phi_{n_k} \to \phi$ (uniformly) on $[s+\varepsilon, t-\varepsilon]$ and ϕ is continuous on (s,t), this implies that $\phi_{n_k}(u_k) \to \phi(u_*)$. By the u.s.c. of $d^1(\cdot)$ at $\phi(u_*)$, this means that there exists $\tilde{k} < \infty$ such that for every $k \geq \tilde{k}$ the inclusion $d^1(\phi_{n_k}(u_k)) \subseteq N_{\delta/3}(d^1(\phi(u_*)))$ is satisfied. Since $d_k \in d^1(\phi_{n_k}(u_k))$ and $d_k \to d_*$ this implies $d_* \in N_{\delta/3}(d^1(\phi(u_*)))$. However, this contradicts (2.12), thus proving the claim (2.11) and hence the lemma. ■

The three lemmas given above establish general properties of solutions to a broad class of ESPs (that satisfy Assumption 2.1), assuming that solutions exist. Clearly, additional conditions need to be imposed on $(G, d(\cdot))$ in order to guarantee existence of solutions to the ESP (an obvious necessary condition for existence is that for every $x \in \partial G$, there exists $d \in d(x)$ that points into the interior of G). In [22] conditions were established for a class of polyhedral ESPs (of the form described in Assumption 3.1) that guarantee the existence of solutions for $\psi \in \mathcal{D}_{c,G}[0,\infty)$, the space of piecewise constant functions in $\mathcal{D}_G[0,\infty)$ having a finite number of jumps. In the next lemma, the closure property of Lemma 2.5 is invoked to show when existence of solutions to the ESP on a dense subset of $\mathcal{D}_G[0,\infty)$ implies existence on the entire space $\mathcal{D}_G[0,\infty)$. This is used in Section 3 to establish existence and uniqueness of solutions to the class of GPS ESPs.

Lemma 2.6. (Existence and Uniqueness) *Suppose $(G, d(\cdot))$ is such that the domain $\text{dom}(\Gamma)$ of the associated SM Γ contains a dense subset $\mathcal{S}$ of $\mathcal{D}_G[0,\infty)$ (respectively, $\mathcal{C}_G[0,\infty)$). Then the following properties hold.*

1. *If Γ is uniformly continuous on $\mathcal{S}$, then there exists a solution to the ESP for all $\psi \in \mathcal{D}_G[0,\infty)$ (respectively, $\psi \in \mathcal{C}_G[0,\infty)$).*
2. *If $\bar{\Gamma}$ is uniformly continuous on its domain $\text{dom}(\bar{\Gamma})$, then $\bar{\Gamma}$ is defined, single-valued and uniformly continuous on all of $\mathcal{D}_G[0,\infty)$. Moreover, in this case $\psi \in \mathcal{C}_G[0,\infty)$ implies that $\phi = \bar{\Gamma}(\psi) \in \mathcal{C}_G[0,\infty)$.*

In particular, if there exists a projection $\pi : \mathbb{R}^J \to G$ *that satisfies*

$$\pi(x) = x \quad \text{for } x \in G \qquad \text{and} \qquad \pi(x) - x \in d(\pi(x)) \quad \text{for } x \in \partial G, \tag{2.13}$$

and the ESM is uniformly continuous on its domain, then there exists a unique solution to the ESP for all $\psi \in \mathcal{D}_G[0,\infty)$ *and the ESM is uniformly continuous on* $\mathcal{D}_G[0,\infty)$*.*

Proof. Fix $\psi \in \mathcal{D}_G[0,\infty)$. The fact that $\mathcal{S}$ is dense in $\mathcal{D}_G[0,\infty)$ implies that there exists a sequence $\{\psi_n\} \subset \mathcal{S}$ such that $\psi_n \to \psi$. Since $\mathcal{S} \subset \mathrm{dom}(\Gamma)$ and Γ is uniformly continuous on $\mathcal{S}$, there exists a unique solution to the SP for every $\psi \in \mathcal{S}$. For $n \in \mathbb{N}$, let $\phi_n \doteq \Gamma(\psi_n)$. The uniform continuity of Γ on $\mathcal{S}$ along with the completeness of $\mathcal{D}_G[0,\infty)$ with respect to the u.o.c. metric implies that $\phi_n \to \phi$ for some $\phi \in \mathcal{D}_G[0,\infty)$. Since $\phi_n = \Gamma(\psi_n)$, property 1 of Lemma 2.4 shows that $\phi_n \in \bar{\Gamma}(\psi_n)$. Lemma 2.5 then guarantees that $\phi \in \bar{\Gamma}(\psi)$, from which we conclude that $\mathrm{dom}(\bar{\Gamma}) = \mathcal{D}_{\mathrm{G}}[0,\infty)$. This establishes the first statement of the lemma.

Now suppose $\bar{\Gamma}$ is uniformly continuous on $\mathrm{dom}(\bar{\Gamma})$. Then it is automatically single-valued on its domain and so Lemma 2.4 implies that $\Gamma(\psi) = \bar{\Gamma}(\psi)$ for $\psi \in \mathrm{dom}(\Gamma)$. Thus, by the first statement just proved, we must have $\mathrm{dom}(\bar{\Gamma}) = \mathcal{D}_{\mathrm{G}}[0,\infty)$. In fact, in this case the proof of the first statement shows that $\bar{\Gamma}$ is equal to the unique uniformly continuous extension of Γ from $\mathcal{S}$ to $\mathcal{D}_G[0,\infty)$ (which exists by p. 149 of [40]). In order to prove the second assertion of statement 2 of the lemma, fix $\psi \in \mathcal{C}_G[0,\infty)$ and let $\phi \doteq \bar{\Gamma}(\psi)$. Since ϕ is right-continuous, it suffices to show that for every $\varepsilon > 0$ and $T < \infty$,

$$\lim_{\delta \downarrow 0} \sup_{t \in [\varepsilon, T]} |\phi(t) - \phi(t-\delta)| = 0. \tag{2.14}$$

Fix $T < \infty$ and $\varepsilon > 0$, and choose $\tilde{\varepsilon} \in (0,\varepsilon)$ and $\delta \in (0,\tilde{\varepsilon})$. Define $\psi_1 \doteq \psi, \phi_1 \doteq \phi$,

$$\psi_2^\delta(s) \doteq \phi(\tilde{\varepsilon} - \delta) + \psi(s-\delta) - \psi(\tilde{\varepsilon}-\delta) \quad \text{and} \quad \phi_2^\delta(s) \doteq \phi(s-\delta) \quad \text{for } s \in [\varepsilon, T].$$

By Lemma 2.3, it follows that $\phi_2^\delta = \bar{\Gamma}(\psi_2^\delta)$. Moreover, by the uniform continuity of $\bar{\Gamma}$, for some function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(\eta) \downarrow 0$ as $\eta \downarrow 0$, we have for each $\delta \in (0,\tilde{\varepsilon})$,

$$\begin{aligned}
&\sup_{t\in[\varepsilon,T]} |\phi(t) - \phi(t-\delta)| \\
&\qquad = \sup_{t\in[\varepsilon,T]} |\phi_1(t) - \phi_2^\delta(t)| \\
&\qquad \le h\left(\sup_{t\in[\varepsilon,T]} |\psi_1(t) - \psi_2^\delta(t)|\right) \\
&\qquad = h\left(\sup_{t\in[\varepsilon,T]} |\psi(t) - \psi(t-\delta) + \psi(\tilde{\varepsilon}-\delta) - \phi(\tilde{\varepsilon}-\delta)|\right).
\end{aligned}$$

Sending first $\delta \to 0$ and then $\tilde{\varepsilon} \to 0$, and using the continuity of ψ, the right-continuity of ϕ and the fact that $\phi(0) = \psi(0)$, we obtain (2.14).

Finally, we use the fact that the existence of a projection is equivalent to the existence of solutions to the SP for $\psi \in \mathcal{D}_{c,G}[0,\infty)$ (see, for example, [12, 18, 22]). The last statement of the lemma is then a direct consequence of the first assertion in statement 2 and the fact that $\mathcal{D}_{c,G}[0,\infty)$ is dense in $\mathcal{D}_G[0,\infty)$. ∎

2.2 The $\mathcal{V}$-set of an ESP

In this section we introduce a special set $\mathcal{V}$ associated with an ESP, which plays an important role in characterising the semimartingale property of reflected diffusions defined via the ESP (see Section 5). We first establish properties of the set $\mathcal{V}$ in Lemma 2.8. Then, in Theorem 2.9, we show that solutions to the ESP satisfy the SP until the time to hit an arbitrary small neighbourhood of $\mathcal{V}$. When $\mathcal{V} = \emptyset$, this implies that any solution to the ESP is in fact a solution to the SP. A stochastic analogue of this result is presented in Theorem 4.3.

Definition 2.7. (The $\mathcal{V}$-set of the ESP) *Given an ESP $(G, d(\cdot))$, we define*

$$\mathcal{V} \doteq \{x \in \partial G : \text{ there exists } d \in S_1(0) \text{ such that } \{d, -d\} \subseteq d^1(x)\}. \tag{2.15}$$

Thus the $\mathcal{V}$-set of the ESP is the set of points $x \in \partial G$ such that the set of directions of constraint $d(x)$ contains a line. Note that $x \in G \setminus \mathcal{V}$ if and only if $d^1(x)$ is contained in an open half space of $\mathbb{R}^J$, which is equivalent to saying that

$$\max_{u \in S_1(0)} \min_{d \in d^1(x)} \langle d, u \rangle > 0 \quad \text{for} \quad x \in G \setminus \mathcal{V}. \tag{2.16}$$

Following the convention that the minimum over an empty set is infinity, the above inequality holds trivially for $x \in G^\circ$. We now prove some useful properties of the set $\mathcal{V}$. Below $N^\circ_\delta(A)$ denotes the open δ-neighbourhood of the set A.

Lemma 2.8. (Properties of the $\mathcal{V}$-set) *Given an ESP $(G, d(\cdot))$ that satisfies Assumption 2.1, let the associated $\mathcal{V}$-set be defined by (2.15). Then $\mathcal{V}$ is closed. Moreover, given any $\delta > 0$ and $L < \infty$ there exist $\rho > 0$ and a finite set $\mathbb{K} \doteq \{1, \ldots, K\}$ and collection $\{\mathcal{O}_k, k \in \mathbb{K}\}$ of open sets and associated vectors $\{v_k \in S_1(0), k \in \mathbb{K}\}$ that satisfy the following two properties.*

1. $[\{x \in G : |x| \leq L\} \setminus N^\circ_\delta(\mathcal{V})] \subseteq [\cup_{k \in \mathbb{K}} \mathcal{O}_k]$.
2. *If $y \in \{x \in \partial G : |x| \leq L\} \cap N_\rho(\mathcal{O}_k)$ for some $k \in \mathbb{K}$ then*

$$\langle d, v_k \rangle > \rho \quad \textit{for every} \quad d \in d^1(y).$$

Proof. The fact that $\mathcal{V}$ is closed follows directly from Assumption 2.1, which ensures that the graph of $d^1(\cdot)$ is closed (as observed in Remark 2.2). Fix $\delta > 0$ and for brevity of notation, define $G_L \doteq \{x \in G : |x| \leq L\}$. To establish the second assertion of the theorem, we first observe that by (2.16), for any $x \in G \setminus \mathcal{V}$ there exist $\rho_x, \kappa_x > 0$ and $v_x \in S_1(0)$ such that

$$\min_{d \in N_{\kappa_x}(d^1(x))} \langle d, v_x \rangle > \rho_x. \tag{2.17}$$

Due to the upper semicontinuity of $d^1(\cdot)$ (in particular, property (2.5)), given any $x \in G_L \setminus N_\delta(\mathcal{V})$ there exists $\varepsilon_x > 0$ such that

$$y \in N_{3\varepsilon_x}(x) \quad \Rightarrow \quad d^1(y) \subseteq N^\circ_{\kappa_x}(d^1(x)). \tag{2.18}$$

Since $G_L \setminus N^\circ_\delta(\mathcal{V})$ is compact, there exists a finite set $\mathbb{K} = \{1, \ldots, K\}$ and a finite collection of points $\{x_k, k \in \mathbb{K}\} \subset G_L \setminus N^\circ_\delta(\mathcal{V})$ such that the corresponding open neighbourhoods $\mathcal{O}_k \doteq N^\circ_{\varepsilon_{x_k}}(x_k)$, $k \in \mathbb{K}$, form a covering, i.e.,

$$G_L \setminus N^\circ_\delta(\mathcal{V}) \subset [\cup_{k \in \mathbb{K}} \mathcal{O}_k].$$

This establishes property 1 of the lemma.

Next, note that if $\tilde{\rho} \doteq \min_{k\in\mathbb{K}} \rho_{x_k}$ and $v_k \doteq v_{x_k}$ for $k \in \mathbb{K}$, by (2.17) we obtain the inequality

$$\min_{d\in N_{\kappa_{x_k}}(d^1(x_k))} \langle d, v_k\rangle > \tilde{\rho} > 0 \quad \text{for} \quad k \in \mathbb{K}.$$

Let $\rho \doteq \tilde{\rho} \wedge \min_{k\in\mathbb{K}} \varepsilon_{x_k} > 0$. Then combining (2.18) with the last inequality we infer that for $k \in \mathbb{K}$,

$$y \in \partial G \cap G_L \cap N_\rho(\mathcal{O}_k) \Rightarrow y \in N_{2\varepsilon_{x_k}}(x_k) \Rightarrow d^1(y) \subseteq N_{\kappa_{x_k}}(d^1(x_k)),$$

which implies that $\min_{d\in d^1(y)}\langle d, v_k\rangle > \rho$. This establishes property 2 and completes the proof of the lemma. ∎

Theorem 2.9. *Suppose the ESP $(G, d(\cdot))$ satisfies Assumption 2.1. Let (ϕ, η) solve the ESP for $\psi \in \mathcal{D}_G[0,\infty)$ and let the associated $\mathcal{V}$-set be as defined in (2.15). Then (ϕ, η) solve the SP on $[0, \tau_0)$, where*

$$\tau_0 \doteq \inf\{t \geq 0 : \phi(t) \in \mathcal{V}\}. \tag{2.19}$$

Proof. For $\delta > 0$, define

$$\tau_\delta \doteq \inf\{t \geq 0 : \phi(t) \in N_\delta(\mathcal{V})\}. \tag{2.20}$$

Since $[0, \tau_0) \subseteq \cup_{\delta>0}[0, \tau_\delta)$ (in fact equality holds if ϕ is continuous) and (ϕ, η) solve the ESP for ψ, in order to show that (ϕ, η) solve the SP for ψ on $[0, \tau_0)$, by Lemma 2.4(2) it suffices to show that

$$|\eta|(T \wedge \tau_\delta-) < \infty \qquad \text{for every} \quad \delta > 0 \quad \text{and} \quad T < \infty. \tag{2.21}$$

Fix $\delta > 0$ and $T < \infty$ and let $L \doteq \sup_{t\in[0,T]} |\phi(t)| \vee |\psi(t)|$. Note that $L < \infty$ since $\phi, \psi \in \mathcal{D}_G[0,\infty)$, and define $G_L \doteq \{x \in G : |x| \leq L\}$. Let $\mathbb{K} = \{1, \ldots, K\}$, $\rho > 0$, $\{\mathcal{O}_k, k \in \mathbb{K}\}$ and $\{v_k, k \in \mathbb{K}\}$ satisfy properties 1 and 2 of Lemma 2.8. If $\phi(0) \in N_\delta(\mathcal{V})$, $\tau_\delta = 0$ and (2.21) follows trivially. So assume that $\phi(0) \notin N_\delta(\mathcal{V})$, which in fact implies that $\phi(0) \in G_L \setminus N_\delta(\mathcal{V})$. Then by Lemma 2.8(1), there exists $k_0 \in \mathbb{K}$ such that $\phi(0) \in \mathcal{O}_{k_0}$. Let $T_0 \doteq 0$ and consider the sequence $\{T_m, k_m\}$ generated recursively as follows. For $m = 0, 1, \ldots$, whenever $T_m < \tau_\delta$, define

$$T_{m+1} \doteq \inf\{t > T_m : \phi(t) \notin N^\circ_\rho(\mathcal{O}_{k_m}) \text{ or } \phi(t) \in N_\delta(\mathcal{V})\}.$$

If $T_{m+1} < T \wedge \tau_\delta$, it follows that $\phi(T_{m+1}) \in G_L \setminus N_\delta(\mathcal{V})$ and so by Lemma 2.8(1), there exists k_{m+1} such that $\phi(T_{m+1}) \in \mathcal{O}_{k_{m+1}}$. Since $\phi \in \mathcal{D}_G[0,\infty)$ and $\rho > 0$, there exists a smallest integer $M < \infty$ such that $T_M \geq T \wedge \tau_\delta$. We redefine $T_M \doteq T \wedge \tau_\delta$. For $m = 1, \ldots, M$, let $\mathcal{J}_m$ be the jump points of η in $[T_{m-1}, T_m)$ and define $\mathcal{J} \doteq \cup_{m=1}^M \mathcal{J}_m$. Given any finite partition $\pi_m = \{T_{m-1} = t_0^m < t_1^m < \ldots t_{j_m}^m = T_m\}$ of $[T_{m-1}, T_m]$, we claim (and justify below) that

$$\begin{aligned}
4L &\geq \langle \eta(T_m-) - \eta(T_{m-1}-), v_{k_{m-1}}\rangle \\
&= \left\langle \sum_{i=1}^{j_m} \left[\eta(t_i^m-) - \eta(t_{i-1}^m)\right] + \sum_{t\in\mathcal{J}_m\cap\pi_m} \left[\eta(t) - \eta(t-)\right], v_{k_{m-1}} \right\rangle \\
&\geq \rho \left[\sum_{i=1}^{j_m} |\eta(t_i^m-) - \eta(t_{i-1}^m)| + \sum_{t\in\mathcal{J}_m\cap\pi_m} |\eta(t) - \eta(t-)|\right].
\end{aligned}$$

The first inequality above follows from the relation $\eta(t) = \phi(t) - \psi(t)$ and the definition of L, while the last inequality uses properties 3 and 4 of the ESP, the fact that $\phi(t) \in N_\rho(\mathcal{O}_{k_{m-1}})$ for $t \in [T_{m-1}, T_m)$ and Lemma 2.8(2). In turn, this bound implies that

$$\begin{aligned}
|\eta|(T \wedge \tau_\delta -) &= \sup_\pi \left[\sum_{i=1}^{j_\pi} |\eta(t_i-) - \eta(t_{i-1})| + \sum_{t \in \mathcal{J} \cap \pi} |\eta(t) - \eta(t-)| \right] \\
&= \sum_{m=1}^{M} \sup_{\pi_m} \left[\sum_{i=1}^{j_m} |\eta(t_i^m -) - \eta(t_{i-1}^m)| + \sum_{t \in \mathcal{J}_m \cap \pi_m} |\eta(t) - \eta(t-)| \right] \\
&\leq \frac{4LM}{\rho},
\end{aligned}$$

where the supremum in the first line is over all finite partitions $\pi = \{0 = t_0 < t_1 < \ldots < t_{j_\pi} = T_M\}$ of $[0, T_M]$ and the supremum in the second line is over all finite partitions $\pi_m = \{T_{m-1} = t_0^m < \ldots t_1^m < \ldots < t_{j_m}^m = T_m\}$ of $[T_{m-1}, T_m]$. This establishes (2.21) and thus proves the theorem. ☐

Corollary 2.10. *Suppose the ESP $(G, d(\cdot))$ satisfies Assumption 2.1 and has an empty $\mathcal{V}$-set, where $\mathcal{V}$ is defined by (2.15). If (ϕ, η) solve the ESP for $\psi \in \mathcal{D}_G[0,\infty)$, then (ϕ, η) solve the SP for ψ. Moreover, if there exists a sequence $\{\psi_n\}$ with $\psi_n \to \psi$ such that for every $n \in \mathbb{N}$, (ϕ_n, η_n) solve the SP for ψ_n, then any limit point (ϕ, η) of $\{(\phi_n, \eta_n)\}$ solves the SP for ψ.*

Proof. The first statement follows from Theorem 2.9 and the fact that $\tau_0 = \infty$ when $\mathcal{V} = \emptyset$. By property 1 of Lemma 2.4, for every $n \in \mathbb{N}$, (ϕ_n, η_n) also solve the ESP for ψ_n. Since (ϕ, η) is a limit point of $\{(\phi_n, \eta_n)\}$, the closure property of Lemma 2.5 then shows that (ϕ, η) solve the ESP for ψ. The first statement of the corollary then shows that (ϕ, η) must in fact also solve the SP for ψ. ☐

Remark 2.11. From the definitions of the SP and ESP it is easy to verify that given any time change λ on $\mathbb{R}_+$ (i.e., any continuous, strictly increasing function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lambda(0) = 0$ and $\lim_{t\to\infty} \lambda(t) = \infty$), the pair (ϕ, η) solve the SP (respectively, ESP) for $\psi \in \mathcal{D}_G[0,\infty)$ if and only if $(\phi \circ \lambda, \eta \circ \lambda)$ solve the SP (respectively, ESP) for $\psi \circ \lambda$. From the definition of the J_1-Skorokhod topology (see, for example, Section 12.9 of [47]), it then automatically follows that the statements in Lemmas 2.5 and 2.6 and Corollary 2.10 also hold when $\mathcal{D}_G[0,\infty)$ is endowed with the J_1-Skorokhod topology and an associated metric that makes it a complete space, in place of the u.o.c. topology and associated metric.

Remark 2.12. The second statement of Corollary 2.10 shows that solutions to the SP are closed under limits when $\mathcal{V} = \emptyset$, and thus is a slight generalisation of related results in [4], [12] and [18]. The closure property for polyhedral SPs of the form $\{(d_i, e_i, 0), i = 1, \ldots, J\}$ was established in [4] under what is known as the completely-$\mathcal{S}$ condition, which implies the condition $\mathcal{V} = \emptyset$. In Theorem 3.4 of [18], this result was generalised to polyhedral SPs under a condition (Assumption 3.2 of [18]) that also implies that $\mathcal{V} = \emptyset$. In Theorem 3.1 of [12], the closure property was established for general SPs that satisfy Assumption 2.1, have $\mathcal{V} = \emptyset$ and satisfy the additional conditions (2.15) and (2.16) of [12]. The proof given in this paper uses the ESP and is thus different from those given in [4], [12] and [18].

3 Polyhedral Extended Skorokhod Problems

In this section, we focus on the class of ESPs $(G, d(\cdot))$ with polyhedral domains and piecewise constant reflection fields described in Assumption 3.1 below. Henceforth, we refer to this class as polyhedral ESPs.

Assumption 3.1. *There exists a finite set* $\mathbb{I} = \{1, \ldots, I\}$ *and* $\{(d_i, n_i, c_i) \in \mathbb{R}^J \times S_1(0) \times \mathbb{R} : \langle d_i, n_i \rangle = 1, i \in \mathbb{I}\}$ *such that*

$$G \doteq \cap_{i \in \mathbb{I}} \{x \in \mathbb{R}^J : \langle x, n_i \rangle \geq c_i\},$$

$d(x) = \{0\}$ *and for* $x \in \partial G$,

$$d(x) \doteq \left\{ \sum_{i \in I(x)} \alpha_i d_i : \alpha_i \geq 0 \text{ for } i \in I(x) \right\}, \tag{3.22}$$

where $I(x) \doteq \{i \in \mathbb{I} : \langle x, n_i \rangle = c_i\}$.

Note that a polyhedral ESP has a polyhedral domain G, which consequently admits a representation as the intersection of a finite number of half-spaces. Moreover, it has a reflection field $d(\cdot)$ that is constant and equal to the ray along the positive d_i direction for points in the relative interior of the $(J-1)$-dimensional face $\partial G^i \doteq \{x \in G : \langle x, n_i \rangle = c_i\}$ and, at the intersections of multiple ∂G^i, is equal to the convex cone generated by the corresponding d_i. Thus polyhedral ESPs are completely characterized by a finite set of triplets $\{(d_i, n_i, c_i), i \in \mathbb{I}\}$ (though this representation need not be unique). The condition $\langle d_i, n_i \rangle > 0$ is clearly necessary for existence of solutions: the conditions that $n_i \in S_1(0)$ and $\langle n_i, d_i \rangle = 1$ are just convenient normalizations.

In Section 3.1, we present sufficient conditions for existence and uniqueness of solutions to polyhedral ESPs and Lispchitz continuity of the associated ESMs. In Section 3.2, we introduce the family of multi-dimensional GPS ESPs, and show in Theorem 3.6 that they satisfy the conditions of Section 3.1. The $\mathcal{V}$-set associated with an ESP was shown in Section 2 to play an important role in determining its properties – particularly in determining its relation to the SP. In Section 3.3, we study the implications of the structure of $\mathcal{V}$-sets for the properties of solutions to polyhedral ESPs.

3.1 Existence and Uniqueness of Solutions

Lemma 2.6 showed that the existence of unique solutions for the ESP on all of $\mathcal{D}_G[0, \infty)$ follows if there exists a projection operator for the corresponding SP and the associated ESM is Lispchitz continuous on its domain. In this section, we describe sufficient conditions for existence of the projection operator and Lipschitz continuity of the ESM associated with a polyhedral ESP. These conditions are verified for the GPS ESP in Theorem 3.6.

Given data $\{(d_i, n_i, c_i), i \in \mathbb{I}\}$, the condition stated below as Assumption 3.2 was first introduced as Assumption 2.1 of [18], where it was shown (in Theorem 2.2 of [18]) to be sufficient for Lipschitz continuity of the SM corresponding to the associated (polyhedral) SP. In Theorem 3.3 below we show that this is also a sufficient condition for Lipschitz continuity of the corresponding ESM. Assumption 3.2 is expressed in terms of the existence of a convex set B whose inward normals satisfy certain geometric properties dictated by the data. We will see in Section 5.2 that the existence of such a set B also plays a crucial role in establishing a semimartingale property for reflected diffusions in polyhedral domains defined via the ESP (see the proof of Theorem 5.7 given in Section 6.1). A more easily verifiable "dual" condition that implies the existence of a set B satisfying Assumption 3.2 was introduced in [22, 23]. As demonstrated in [23, 24], this dual condition is often more convenient to use in practice.

We now state the condition. Given a convex set $B \subset \mathbb{R}^J$ and $z \in \partial B$, we let $n(z)$ denote the set of unit inward normals to the set at the point z. In other words,

$$\nu(z) \doteq \{\nu \in S_1(0) : \langle \nu, x - z \rangle \geq 0 \text{ for all } x \in B\}.$$

Assumption 3.2. (Set B) *There exists a compact, convex, symmetric set B with $0 \in B^\circ$, and $\delta > 0$ such that for $i \in \mathbb{I}$,*

$$\left\{ \begin{array}{c} z \in \partial B \\ |\langle z, n_i \rangle| < \delta \end{array} \right\} \Rightarrow \langle \nu, d_i \rangle = 0 \text{ for all } \nu \in \nu(z). \tag{3.23}$$

Theorem 3.3. *If Assumption 3.2 is satisfied for the ESP $\{(d_i, n_i, c_i), i \in \mathbb{I}\}$, then the associated ESM is Lipschitz continuous on its domain of definition.*

Proof. This proof involves a straightforward modification of the proof of Theorem 2.2 in [18] – the only difference being that here we need to allow for constraining terms of unbounded variation. Thus we only provide arguments that differ from those used in [18] and refer the reader to [18] for the remaining details. Let B be the convex set associated with the ESP that satisfies Assumption 3.2. Suppose for $i = 1, 2$, $\psi_i \in \text{dom}(\bar{\Gamma})$ and (ϕ_i, η_i) solve the ESP for ψ_i. Moreover, let $c \doteq \sup_{t \in [0,T]} |\psi_1(t) - \psi_2(t)|$. Then we argue by contradiction to show that $\eta_1(t) - \eta_2(t) \in cB$ for all $t \in [0, T]$. Suppose there exists $a \in (c, \infty)$ such that $\eta_1(t) - \eta_2(t) \notin (aB)^\circ$ for some $t \in [0, T]$ and let $\tau \doteq \inf\{t \in [0, T] : \eta_1(t) - \eta_2(t) \notin (aB)^\circ\}$. As in [18] we consider two cases.

Case 1. Suppose $\eta_1(\tau-) - \eta_2(\tau-) \in \partial(aB)$. In this case, let $z \doteq \eta_1(\tau-) - \eta_2(\tau-)$ and $\nu \in \nu(z/a)$. Then for every $t \in (0, \tau)$, the fact that $\eta_1(t) - \eta_2(t) \in (aB)^\circ$, ν is an inward normal to aB at z and B is convex implies that

$$\langle z - \eta_1(t) + \eta_2(t), \nu \rangle < 0 \quad \text{ for every } t \in (0, \tau).$$

Since $z - \eta_1(t) + \eta_2(t) = (\eta_1(\tau-) - \eta_1(t)) - (\eta_2(\tau-) - \eta_2(t))$, this implies that there must exist a sequence $\{t_n\}$ with $t_n \uparrow \tau$ along which at least one of the following relations must hold:

$$(i) \qquad \langle \eta_1(\tau-) - \eta_1(t_n), \nu \rangle < 0 \quad \text{ for every } n \in \mathbb{N};$$

$$(ii) \qquad \langle \eta_2(\tau-) - \eta_2(t_n), \nu \rangle > 0 \quad \text{ for every } n \in \mathbb{N}.$$

By property 3 of the ESP, for any $n \in \mathbb{N}$,

$$\eta_1(\tau-) - \eta_1(t_n) \in \overline{\text{co}} \left[\cup_{u \in (t_n, \tau)} d(\phi_1(u)) \right].$$

Also, note that $d(\phi_1(u)) = \{0\}$ if $\phi_1(u) \in G^\circ$ and by (3.22), we have

$$d(\phi_1(u)) = \left\{ \sum_{i \in \mathbb{I}: \langle \phi_1(u), n_i \rangle = c_i} \alpha_i d_i : \alpha_i \geq 0 \right\} \qquad \text{if } \phi_1(u) \in \partial G.$$

Therefore, if (i) holds, there must exist $u_n \in (t_n, \tau)$ and $i_n \in \mathbb{I}$, for $n \in \mathbb{N}$, such that $\langle \phi_1(u_n), n_{i_n} \rangle = c_{i_n}$ and $\langle d_{i_n}, \nu \rangle < 0$. This implies that there exists $i \in \mathbb{I}$ such that (possibly along a subsequence, which we relabel again by n) $u_n \in (t_n, \tau)$, $\langle \phi_1(u_n), n_i \rangle = c_i$ and $\langle d_i, \nu \rangle < 0$. Taking limits as $n \to \infty$ and using the fact that $t_n \uparrow \tau$, it follows that $\lim_{n\to\infty} \phi_1(u_n) = \phi_1(\tau-)$ and $\langle \phi_1(\tau-), n_i \rangle = c_i$. This establishes relations (19) and (20) in [18]. The remaining argument given there can then be used to arrive at a contradiction for Case 1(i). By symmetry, Case 1(ii) is proved analogously.

Case 2. Now suppose $\eta_1(\tau-) - \eta_2(\tau-) \in (aB)^\circ$. In this case, set $z \doteq \eta_1(\tau) - \eta_2(\tau)$ and let $r \in [a, \infty)$ be such that $\eta_1(\tau) - \eta_2(\tau) \in \partial(rB)$. Let $\nu \in \nu(z/r)$ and note that by the convexity of B we have $\langle z - \eta_1(\tau-) + \eta_2(\tau-), \nu \rangle < 0$. Noticing that $z - \eta_1(\tau-) + \eta_2(\tau-) = (\eta_1(\tau) - \eta_1(\tau-)) - (\eta_2(\tau) - \eta_2(\tau-))$ observe that at least one of the inequalities below must hold:

$$
\begin{array}{lll}
(i) & \langle \eta_1(\tau) - \eta_1(\tau-), \nu \rangle < 0 & \text{for every } n \in N \\[2ex]
(ii) & \langle \eta_2(\tau) - \eta_2(\tau-), \nu \rangle > 0 & \text{for every } n \in N.
\end{array}
$$

By property 4 of the ESP, this correspondingly implies that at least one of the following two relations must be satisfied:

$$(i) \text{ there exists } i \in \mathbb{I} \text{ such that } \langle \phi_1(\tau), n_i \rangle = c_i \text{ and } \langle d_i, \nu \rangle < 0$$

$$(ii) \text{ there exists } j \in \mathbb{I} \text{ such that } \langle \phi_2(\tau), n_j \rangle = c_j \text{ and } \langle d_j, \nu \rangle > 0.$$

The rest of the argument leading to a contradiction in Case (ii) now follows as in [18] (from relations (24) and (25) onwards). □

Sufficient conditions for the existence of projections for SPs were derived in [4, 12, 33]. However, these are not applicable in the present context since they all assume conditions that imply $\mathcal{V} = \emptyset$. So, instead, we refer to the general results on existence of a projection for polyhedral SPs with $\mathcal{V} \neq \emptyset$ that were obtained in Section 4 of [22] (also see [24] and Section 3 of [23] for application of these methods to concrete SPs).

3.2 The GPS Family of ESPs

Generalized processor sharing (GPS) is a service discipline used in high-speed networks that allows for the efficient sharing of a single resource amongst traffic of different classes. The GPS SP was introduced in [21, 23] to analyse the behaviour of the GPS discipline. The GPS SP admits a representation of the form $\{(d_i, n_i, 0), i = 1, \ldots, J+1\}$, where $n_i = e_i$ for $i \in \mathbb{J} \doteq \{1, \ldots, J\}$ (here $\{e_i, i \in \mathbb{J}\}$ is the standard orthonormal basis in $\mathbb{R}^J$), $n_{J+1} = \sum_{i=1}^J e_i/\sqrt{J}$, $d_{J+1} = \sum_{i=1}^J e_i/\sqrt{J}$ and the reflection directions $\{d_i, i \in \mathbb{J}\}$ are defined as follows in terms of a "weight" vector $\bar{\alpha} \in (\mathbb{R}^J_+)^\circ$ that satisfies $\sum_{i=1}^J \bar{\alpha}_i = 1$: for $i, j \in \mathbb{J}$,

$$
(d_i)_j = \begin{cases} -\dfrac{\bar{\alpha}_j}{1 - \bar{\alpha}_i} & \text{for } j \neq i, \\ \hfill 1 & \text{for } j = i. \end{cases}
$$

We now recall a property of the GPS ESP proved in [23], and establish a useful corollary that will be used in Section 4 (see Theorem 4.6 and Section 5.2.1) to analyse properties of RBMs associated with the GPS ESP. A $J \times J$ matrix A is said to be completely-$\mathcal{S}$ if for every principal submatrix $\tilde{A}$ of A, there exists a vector $\tilde{y} \geq 0$ such that $\tilde{A}\tilde{y} > 0$ (here the inequalities hold componentwise). Completely-$\mathcal{S}$ matrices were studied in the context of the Skorokhod Problem and semimartingale reflecting diffusions in [4, 33, 38],

Lemma 3.4. *The GPS ESP has $\mathcal{V} = \{0\}$, and the vector d_{J+1} is perpendicular to* $\mathrm{span}[d(x), x \in \partial G \setminus \{0\}]$. *Moreover, for every $j \in \mathbb{J}$, the $J \times J$ matrix A_j, whose columns are given by vectors in the set $\{d_i, i = 1, \ldots, J+1\} \setminus \{d_j\}$, is completely-$\mathcal{S}$.*

Proof. The first statement was proved in Lemma 3.1 of [23] and the second statement follows from the proof of Theorem 3.8 of [23]. □

Corollary 3.5. *Suppose (ϕ, η) solve the GPS ESP for $\psi \in \mathcal{D}_G[0,\infty)$. Then $\langle \phi, d_{J+1}\rangle = \Gamma_1(\langle \psi, d_{J+1}\rangle)$, where Γ_1 is the one-dimensional SM defined in (1.1).*

Proof. Since (ϕ, η) solve the GPS ESP for ψ, by properties 3 and 4 of the ESP we know that

$$\eta(t) - \eta(s) \in \overline{\mathrm{co}}\left[\cup_{u\in(s,t]} d(\phi(u))\right] \quad \text{and} \quad \eta(t) - \eta(t-) \in \overline{\mathrm{co}}[\phi(t)]. \tag{3.24}$$

By Lemma 3.4, it follows that d_{J+1} is perpendicular to $\mathrm{span}[d(x), x \in \partial\mathbb{R}^J_+ \setminus \{0\}]$. Also, for every $u \in [0,\infty)$, since $\phi(u) \in \mathbb{R}^J_+$ we have $\langle \phi(u), d_{J+1}\rangle \geq 0$ and $\langle \phi(u), d_{J+1}\rangle = 0$ if and only if $\phi(u) = 0$. Define the (set-valued) mapping $\hat{d}$ that takes $\mathbb{R}_+$ to subsets of $\mathbb{R}$ by $\hat{d}(x) = \{0\}$ if $x \neq 0$ and $\hat{d}(0) = \mathbb{R}_+$. Then taking the inner product of both sides of both terms in (3.24) with d_{J+1} we obtain

$$\langle \eta(t) - \eta(s), d_{J+1}\rangle \in \overline{\mathrm{co}}\left[\cup_{u\in(s,t]}\langle d(\phi(u)), d_{J+1}\rangle\right] = \overline{\mathrm{co}}\left[\cup_{u\in(s,t]}\hat{d}\left(\langle \phi(u)), d_{J+1}\rangle\right)\right]$$

and, similarly,

$$\langle \eta(t) - \eta(t-), d_{J+1}\rangle \in \langle \overline{\mathrm{co}}(d(\phi(t)), d_{J+1}\rangle = \hat{d}\left(\langle \phi(t), d_{J+1}\rangle\right)$$

(where for a set $A \subset \mathbb{R}^J$, we let $\langle A, d_{J+1}\rangle$ denote the set $\{\langle x, d_{J+1}\rangle : x \in A\}$). The above properties imply that $(\langle \phi, d_{J+1}\rangle, \langle \eta, d_{J+1}\rangle)$ solve the ESP $(\mathbb{R}_+, \hat{d}(\cdot))$ for $\langle \psi, d_{J+1}\rangle$. Moreover, this also shows that $\langle \eta, d_{J+1}\rangle$ is non-decreasing and so, in particular, lies in $\mathcal{BV}_0[0,\infty)$. It then follows from Lemma 2.4(2) that $(\langle \phi, d_{J+1}\rangle, \langle \eta, d_{J+1}\rangle)$ solve the one-dimensional SP for $\langle \psi, d_{J+1}\rangle$. The lemma then follows from the well-known fact that solutions to the one-dimensional SP are unique [42]. □

Theorems 3.7 and 3.8 of [23] show that Assumption 3.2 is satisfied and a projection exists for the GPS ESP. Combining these results with Lemma 2.6 and Theorem 3.3 we obtain the following result.

Theorem 3.6. *For every integer $J \geq 2$ and $\psi \in \mathcal{D}_G[0,\infty)$, there exists a unique solution (ϕ, η) associated with the J-dimensional GPS ESP. Moreover, the associated GPS ESM is Lipschitz continuous on $\mathcal{D}_G[0,\infty)$.*

Remark 3.7. Polyhedral data $(G, d(\cdot))$ with more complicated $\mathcal{V}$-sets arise in the analysis of queueing networks with cooperative servers. In particular, in [24] a family of SPs was introduced to analyse the fluid model of a two-station queueing network with each station serving two classes and in which the GPS discipline is used at each station. This SP has domain $\mathbb{R}_+^4$ and an unbounded $\mathcal{V}$-set equal to $\{x \in \mathbb{R}_+^4 : x_1 = x_4 = 0\} \cup \{x \in \mathbb{R}_+^4 : x_2 = x_3 = 0\}$. For a certain parameter regime it was shown in Theorems 5 and 10 of [24] that a set B satisfying Assumption 3.2 and a projection exists for these ESPs. As in the GPS case, an application of Theorem 3.3 and Lemma 2.6 then yields existence and uniqueness of solutions on $\mathcal{D}_G[0,\infty)$ for the associated ESPs.

3.3 Structure of $\mathcal{V}$-sets for Polyhedral ESPs

In Theorem 2.9, we showed that any solution to an ESP is also a solution to the associated SP on $[0, \tau_0)$. Here we show that this result does not in general hold if $[0, \tau_0)$ is replaced by $[0, \tau_0]$. Specifically, Theorem 3.8 and Corollary 3.9 below show that for a large class of polyhedral ESPs with $\mathcal{V} \neq \emptyset$, there exist (ϕ, η) that solve the ESP for some $\psi \in \mathcal{C}_G[0,\infty)$, but do not solve the SP on $[0, \tau_0]$ for ψ. In Section 5, we consider the stochastic analogue of this question.

Theorem 3.8. *Let $J = 2$ and let $\{(d_i, n_i, 0), i = 1, 2, 3\}$ be the 2-dimensional polyhedral GPS ESP. Then there exists $\psi \in \mathcal{C}_G[0,\infty)$ such that (ϕ, η) solve the ESP for ψ and $|\eta|(\tau_0) = \infty$, where $\tau_0 = \inf\{t \geq 0 : \phi(t) = 0\}$.*

Proof. Fix $J = 2$. From the definition of the GPS data given in Section 3.2, it is easy to see that regardless of the value of the weight vector $\bar{\alpha}$, the 2-dimensional GPS ESP has $n_1 = e_1, d_1 = (1, -1)$, $n_2 = e_2, d_2 = (-1, 1)$ and $n_3 = (1, 1)/\sqrt{2}$, $d_3 = (1, 1)/\sqrt{2}$. Also, we clearly have $\mathcal{V} = \{0\}$. For notational conciseness, let $t_0 \doteq 0$, $\beta \doteq 1/2$, and define $t_n \doteq \sum_{i=1}^n \beta^i$. Note that then $\lim_{n\to\infty} t_n = 1$. Now, consider the piecewise linear function $\psi \in \mathcal{C}_G[0,\infty)$ defined as follows: $\psi(t_0) = \psi(0) \doteq (0, 1)$ and for $n \in \mathbb{N}$, recursively define

$$
\begin{aligned}
\psi(t_{n-1} + \beta^n/4) &= \psi(t_{n-1}) + (-1, 1)/n \\
\psi(t_{n-1} + \beta^n/2) &= \psi(t_{n-1} + \beta^n/4) + \beta^{2n-2}(\beta, -1) \\
\psi(t_{n-1} + 3\beta^n/4) &= \psi(t_{n-1} + \beta^n/2) + (1, -1)/n \\
\psi(t_n) &= \psi(t_{n-1} + 3\beta^n/4) + \beta^{2n-1}(-1, \beta),
\end{aligned}
$$

define $\psi(t)$ by linear interpolation for $t \in [0, 1)$ and set $\psi(t) \doteq (0, 0)$ for $t \geq 1$ (see Figure 1 for an illustration of ψ). It is immediate from the construction that ψ is continuous on $(0, 1)$. Using the fact that $1 - \beta^2 = 3\beta^2$ (since $\beta = 1/2$) we see that for $i \in \mathbb{N}$,

$$\psi(t_i) - \psi(t_{i-1}) = \beta^{2i-2}(\beta, -1) + \beta^{2i-1}(-1, \beta) = -3\beta^{2i}(0, 1),$$

and so for $n \in \mathbb{N}$,

$$\psi(t_n) = \psi(t_0) + \sum_{i=1}^{n} (\psi(t_i) - \psi(t_{i-1})) = \left[1 - 3\sum_{i=1}^{n} \beta^{2i}\right](0, 1).$$

Thus $|\psi(t_n)| = 1/4^n$ and $\lim_{n\to\infty} \psi(t_n) = (0, 0) = \psi(1)$ and so $\psi \in \mathcal{C}_G[0,\infty)$. Also elementary calculations show that for $n \in \mathbb{N}$,

$$\sup_{t\in[t_{n-1},t_n]} |\psi(t) - \psi(t_{n-1})| \leq \frac{\sqrt{5}}{n}.$$

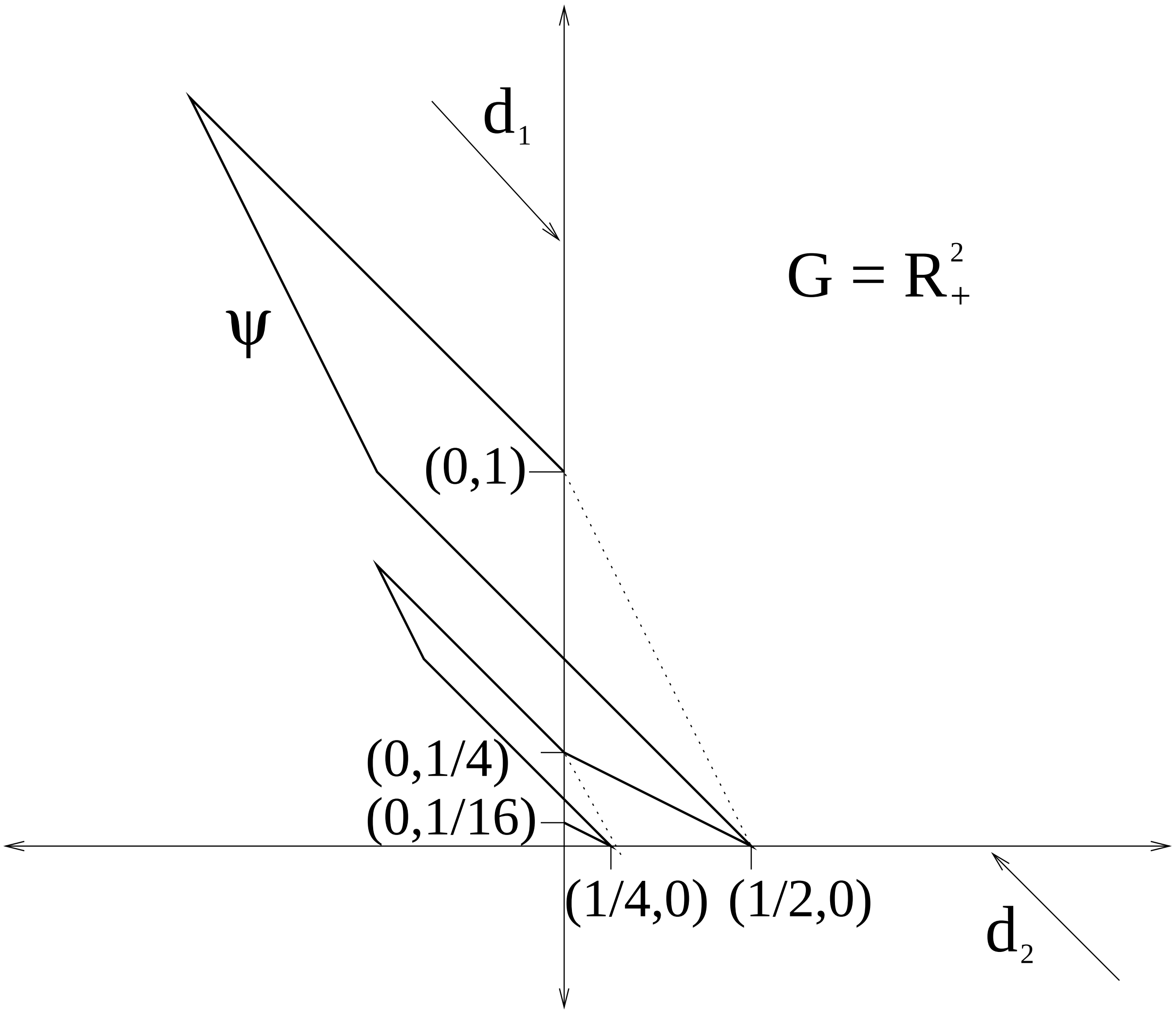

Figure 1: An input trajectory ψ for the 2-dimensional GPS ESP

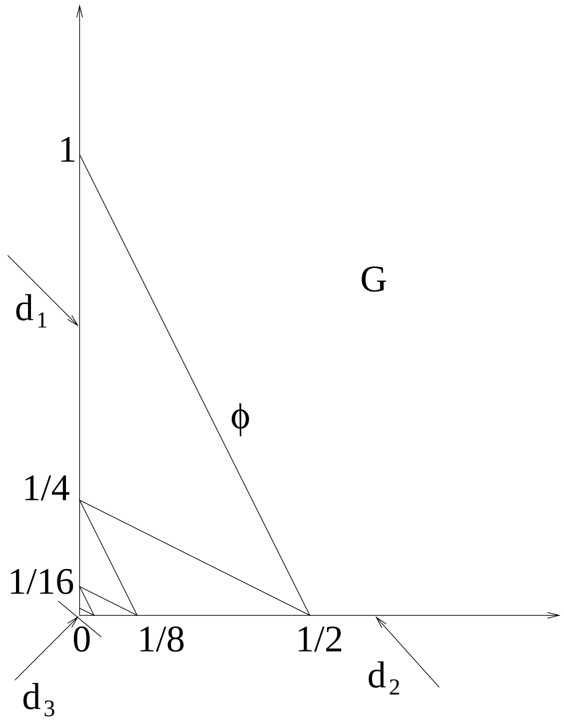

Figure 2: The solution (ϕ, η) to the 2-dimensional GPS ESP with $|\eta|(\tau_0) = \infty$

Hence if we define $k(t)$ to be the unique $k \in \mathbb{N}$ such that $t \in [t_{k-1}, t_k)$, we deduce that

$$\sup_{t\in[t_n,1)} |\psi(t) - \psi(t_n)| \le \sup_{t\in[t_n,1)} \left[|\psi(t) - \psi(t_{k(t)})| + |\psi(t_{k(t)}) - \psi(t_n)|\right] \le \frac{\sqrt{5}}{n} + \frac{1}{4^n}.$$

Now, let ϕ be the piecewise linear trajectory defined as follows: $\phi(t_0) = \phi(0) \doteq (0, 1)$ and, for $n \in \mathbb{N}$, let

$$\begin{aligned} \phi(t_{n-1} + \beta^n/4) &= \beta^{2n-2}(0, 1), \\ \phi(t_{n-1} + \beta^n/2) &= \beta^{2n-2}(0, 1) + \beta^{2n-2}(\beta, -1) = \beta^{2n-1}(1, 0), \\ \phi(t_{n-1} + 3\beta^n/4) &= \beta^{2n-1}(1, 0), \\ \phi(t_n) &= \beta^{2n-1}(1, 0) + \beta^{2n-1}(-1, \beta) = \beta^{2n}(0, 1), \end{aligned}$$

with $\phi(t)$ defined by linear interpolation for all other $t \in (0, 1)$ and $\phi(t) \doteq (0, 0)$ for $t \ge 1$ (see Figure 2 for an illustration of ϕ). It is easy to verify that $\lim_{n\to\infty} \phi(t_n) = \phi(1) = (0, 0)$ and so ϕ is continuous and $\tau_0 = 1$. In addition, it is also straightforward to check that $\eta \doteq \phi - \psi$ is a piecewise linear trajectory that satisfies $\eta(0) \doteq (0, 0)$ and for $n \in \mathbb{N}$,

$$\begin{aligned} \eta(t_{n-1} + \beta^n/4) &= \eta(t_{n-1}) + (1, -1)/n, \\ \eta(t_{n-1} + \beta^n/2) &= \eta(t_{n-1} + \beta^n/4), \\ \eta(t_{n-1} + 3\beta^n/4) &= \eta(t_{n-1} + \beta^n/2) + (-1, 1)/n, \\ \eta(t_n) &= \eta(t_{n-1} + 3\beta^n/4), \end{aligned}$$

with $\eta(t)$ defined by linear interpolation for all other $t \in (0, 1)$ and $\eta(t) = (0, 0)$ for $t \ge 1$.

Next, define $\psi_n(\cdot) \doteq \psi(\cdot \wedge t_n)$ and, likewise, let $\phi_n(\cdot) \doteq \phi(\cdot \wedge t_n)$ and $\eta_n(\cdot) \doteq \eta(\cdot \wedge t_n)$. From the properties of ψ_n and ϕ_n stated above, it is clear that $\phi_n = \Gamma(\psi_n)$, $\psi_n \to \psi$, $\phi_n \to \phi$ and hence $\eta_n \to \eta$. Thus by the closure property established in Lemma 2.5 it follows that (ϕ, η) satisfy the ESP for ψ. Note that $\tau_0 = 1$ and

$$
\begin{aligned}
&|\eta|(1) \\
&\geq \sum_{n=1}^{\infty} \left[\left| \eta\left(t_{n-1} + \frac{\beta^n}{4}\right) - \eta(t_{n-1}) \right| + \left| \eta\left(t_{n-1} + \frac{3\beta^n}{4}\right) - \eta\left(t_{n-1} + \frac{\beta^n}{2}\right) \right| \right] \\
&\geq \sum_{n=1}^{\infty} \frac{2\sqrt{2}}{n} \\
&= \infty,
\end{aligned}
$$

which completes the proof. $\square$

We now show that the argument in the last proof can be generalised to a large class of polyhedral ESPs with $\mathcal{V} \neq \emptyset$. Given any ESP, define $\tilde{\mathcal{V}}$ to be the set of points $x \in \partial G$ such that there exist $x \in \partial G$, $\rho_0 > 0$ and $d \in S_1(0)$ such that for all $\rho \in (0, \rho_0]$,

$$\{d, -d\} \subseteq \overline{\text{cone}} \left[\cup_{z \in N_\rho(x) \setminus \{x\}} d^1(z) \right]. \tag{3.25}$$

We then have the following result.

Corollary 3.9. *Given any polyhedral ESP $\{(d_i, n_i, c_i), i \in \mathbb{I}\}$, if $\tilde{\mathcal{V}} \neq \emptyset$, then there exists $\psi \in \mathcal{C}_G[0, \infty)$ such that (ϕ, η) solve the ESP for ψ and $|\eta|(\tau_0) = \infty$, where $\tau_0 = \inf\{t > 0 : \phi(t) \in \mathcal{V}\}$.*

Proof. Let $x \in \partial G$ be such that (3.25) holds. Since we are dealing with polyhedral SPs, this implies that there exist vectors $z^{(l)}$ and $v^{(l)}, l = 1, \dots, L$, such that $\rho z^{(l)}/\rho_0 \in N_\rho(x) \setminus \{x\} \cap \partial G$, $v^{(l)} \in d(z^{(l)}) \setminus \{0\}$ and

$$\sum_{i=1}^{L} v^{(l)} = 0.$$

Define $z^{(L+1)} \doteq z^{(1)}$. Define $\kappa \doteq 1/2L$ and, as in the proof of Theorem 3.8, let $\beta \doteq 1/2$, $t_0 \doteq 0$, $t_n \doteq \sum_{i=1}^{n} \beta^i$ and note that $\lim_{n\to\infty} t_n = 1$. Now define $\psi \in \mathcal{C}_G[0, \infty)$ recursively as follows: let $\psi(t_0) = \psi(0) \doteq z^{(1)}$ and for $n \in \mathbb{N}$ and $i = 0, \dots, L-1$, let

$$
\begin{aligned}
\psi\left(t_{n-1} + (2i+1)\kappa\beta^n\right) &= \psi\left(t_{n-1} + 2i\kappa\beta^n\right) - v^{(i)}/n \\
\psi\left(t_{n-1} + (2i+2)\kappa\beta^n\right) &= \psi\left(t_{n-1} + (2i+1)\kappa\beta^n\right) \\
&\quad + \beta^{L(n-1)+i}\left(\beta z^{(i+2)} - z^{(i+1)}\right)
\end{aligned}
$$

with $\psi(t)$ defined by linear interpolation for $t \in (0, 1)$ and $\psi(t) \doteq 0$ for $t \geq 1$. It is easy to see that $\lim_{n\to\infty} \psi(t_n) = \psi(t)$ and hence $\psi \in \mathcal{C}_G[0, \infty)$. Moreover, let $\phi(0) \doteq z^{(1)}$, for $n \in \mathbb{N}$ and $i = 0, \dots, L-1$, define

$$
\begin{aligned}
\phi(t_{n-1} + (2i+1)\kappa\beta^n) &= \phi(t_{n-1} + 2i\kappa\beta^n) = \beta^{L(n-1)+i} z^{(i+1)} \\
\phi(t_{n-1} + (2i+2)\kappa\beta^n) &= \phi(t_{n-1} + (2i+1)\kappa\beta^n) \\
&\quad + \beta^{L(n-1)+i}\left(\beta z^{(i+2)} - z^{(i+1)}\right),
\end{aligned}
$$

define $\phi(t)$ by linear interpolation for $t \in (0, 1)$ and let $\phi(t) \doteq 0$ for $t \geq 1$. Finally, let $\eta \doteq \phi - \psi$. Then arguments analogous to those used in Theorem 3.8 show that (ϕ, η) solve the ESP for ψ, $\tau_0 = 1$ and $|\eta|(\tau_0) = \infty$. $\square$

Remark 3.10. It is easy to see that for polyhedral ESPs, $\tilde{\mathcal{V}} \subseteq \mathcal{V}$. Indeed, if $x \in \partial G$ satisfies (3.25), then by the definition (3.22) of $d(\cdot)$ for polyhedral ESPs, there exists $\rho > 0$ such that

$$\left[\cup_{y \in N_\rho(x)\setminus\{x\}} d^1(y)\right] \subseteq d^1(x).$$

Along with (3.25), this implies that $\{d, -d\} \subseteq d^1(x)$ (and hence that $x \in \mathcal{V}$).

However, there exist polyhedral ESPs for which $\mathcal{V} \neq \emptyset$ but $\tilde{\mathcal{V}} = \emptyset$, i.e., (3.25) does not hold for any $x \in \mathcal{V}$. For example, consider $\{(d_i, n_i, 0), i = 1,2,3\}$, where $n_i = e_i$ for $i = 1,2$, $n_3 = (1,2)/\sqrt{5}$, $d_1 = (1,-1), d_2 = (0,1)$ and $d_3 = \sqrt{5}(-1,1)$. Then $G = \mathbb{R}^2_+$, $\{(1,-1),(-1,1)\} \subset d(0)$ and so $\mathcal{V} = \{0\}$. On the other hand, for any $\rho > 0$,

$$\overline{\mathrm{cone}}[\cup_{y \in N_\rho(0)\setminus\{0\}} d(y)] = \{\alpha_1(1,-1)/\sqrt{2} + \alpha_2(0,1) : \alpha_1, \alpha_2 \geq 0\}$$

and so $\langle (1,1), d\rangle > 0$ for all $d \in \overline{\mathrm{cone}}[\cup_{y \in N_\rho(0)\setminus\{0\}} d(y)]$. Thus (3.25) is not satisfied when $x = 0$. However, the ESM is still a strict generalisation of the SM for such ESPs. Indeed, given an input trajectory ψ that satisfies $\psi(0) = 0$, is of unbounded variation and has an image that lies exclusively on the line $\{\alpha(1,-1), \alpha \in \mathbb{R}\}$, it is clear that $(0, -\psi)$ satisfies the ESP for ψ. However, $\eta = -\psi$ is of unbounded variation and so does not solve the SP for ψ. Nevertheless, arguments similar to that used in the proof of Theorem 2.9 can be used to show that $|\eta|(\tau_0) < \infty$.

4 Stochastic Differential Equations with Reflection

In this section, we construct and analyse properties of a general class of reflected diffusions. Throughout, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with the right continuous filtration $\{\mathcal{F}_t, t \geq 0\}$ and assume $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible sets. We will refer to $((\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\})$ as a filtered probability space. In Section 4.1, we use the ESP to construct solutions to a class of stochastic differential equations with reflection (SDER). In Section 4.2, we show that reflected diffusions associated with the GPS family of ESPs satisfy the associated submartingale problem. When $J = 2$, this was proved in [46] for the case of reflected Brownian motion.

4.1 Existence and Uniqueness of Solutions to SDERs

Let $W = \{W_t, t \geq 0\}$ be a K-dimensional, $\{\mathcal{F}_t\}$-adapted, standard Brownian motion, and consider functions $b(\cdot)$ and $\sigma(\cdot)$ on G that take values in $\mathbb{R}^J$ and $\mathbb{R}^J \otimes \mathbb{R}^K$, respectively, and satisfy the following standard Lipschitz continuity and uniform ellipticity conditions.

Assumption 4.1. *The functions $b(\cdot)$ and $\sigma(\cdot)$ satisfy the following conditions.*

1. *There exists a constant $\tilde{L} < \infty$ such that for all $x, y \in G$,*

$$|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \leq \tilde{L}|x - y|. \tag{4.26}$$

2. *The covariance function $a : G \to \mathbb{R}^J \otimes \mathbb{R}^J$ defined by $a(\cdot) \doteq \sigma(\cdot)\sigma^T(\cdot)$ is uniformly elliptic, i.e., there exists $\lambda > 0$ such that*

$$\sum_{i,j=1}^{J} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \textit{for all } \xi \in \mathbb{R}^J \textit{ and } x \in G.$$

We now introduce the definition of a strong solution Z to an SDER associated with an ESP $(G, d(\cdot))$, drift coefficient $b(\cdot)$ and dispersion coefficient $\sigma(\cdot)$. This is a straightforward generalisation of the standard definitions used for SDERs defined via the SP (as, for example, in Section 5 of [12]).

Definition 4.2. *Given $(G, d(\cdot))$, $b(\cdot)$, $\sigma(\cdot)$ and an $\{\mathcal{F}_t\}$-adapted K-dimensional Brownian motion W on a filtered probability space $((\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\})$, a continuous $\{\mathcal{F}_t\}$-adapted process $Z(\cdot)$ is a strong solution to the associated SDER if $\mathbb{P}$ a.s. for all $t \in [0, \infty)$, $Z(t) \in G$ and*

$$Z(t) = Z(0) + \int_0^t b(Z(s))\, ds + \int_0^t \sigma(Z(s)) \cdot dW(s) + Y(t), \tag{4.27}$$

where

$$Y(t) - Y(s) \in \overline{\text{co}} \left[\cup_{u \in (s,t]} d(Z(u)) \right].$$

In other words, $(Z(\cdot), Y(\cdot))$ should solve (on a $\mathbb{P}$ a.s. pathwise basis) the ESP for

$$X(\cdot) \doteq Z(0) + \int_0^{\cdot} b(Z(s))\, ds + \int_0^{\cdot} \sigma(Z(s)) \cdot dW(s). \tag{4.28}$$

We will use $\mathbb{P}_z$ and $\mathbb{E}_z$ to denote probability and expectation, respectively, conditioned on $Z(0) = z$.

In the next theorem, we state sufficient conditions for the existence of a pathwise unique, strong solution to the SDER. Since the proof employs standard arguments that are used to construct reflected diffusions defined via the SP (see, for instance, [20, 44]) or, more generally, to construct solutions to SDEs (as in Section 5.2 of [30]), we provide only a rough sketch of the proof. We also show that the solution satisfies a semimartingale decomposition property – this is essentially a direct consequence of the corresponding property for the deterministic ESP that was proved in Theorem 2.9. Recall the definition (2.15) of the $\mathcal{V}$-set associated with an ESP. The theorem below shows, in particular, that given a strong solution to the SDER, it is a semimartingale if $\mathcal{V} = \emptyset$.

Theorem 4.3. *Suppose the ESP $(G, d(\cdot))$ satisfies Assumption 2.1 and the associated ESM is well-defined and Lipschitz continuous on $\mathcal{C}_G[0, \infty)$. If the coefficients $b(\cdot), \sigma(\cdot)$ satisfy Assumption 4.1(1), then there exists a pathwise unique, strong solution to the associated SDER. In addition, the process is strong Markov. Furthermore, if we define*

$$\tau_0 \doteq \inf\{t \geq 0 : Z(t) \in \mathcal{V}\}, \tag{4.29}$$

then Z is a semimartingale on $[0, \tau_0)$, which $\mathbb{P}$ a.s. admits the decomposition

$$Z(\cdot) = Z(0) + M(\cdot) + A(\cdot), \tag{4.30}$$

where for $t \in [0, \tau_0)$,

$$M(t) \doteq \int_0^t \sigma(Z(s)) \cdot dW(s) \quad \text{and} \quad A(t) \doteq \int_0^t b(Z(s))\, ds + Y(t), \tag{4.31}$$

where Y has bounded variation on $[0, t]$ and satisfies

$$Y(t) = \int_0^t \gamma(s)\, d|Y|(s), \tag{4.32}$$

and $\gamma(s) \in d^1(Z(s))$ $d|Y|$ a.e. $s \in [0, t]$.

Proof. Given that $b(\cdot)$ and $\sigma(\cdot)$ satisfy Assumption 4.1(1) and the ESM is Lipschitz continuous, we use the same standard approximations as those used to prove existence and uniqueness of solutions to SDERs defined via a Lipschitz continuous SM (see [1, 44]). For $i = 1, 2$, given a continuous $\{\mathcal{F}_t\}$-adapted process Z^i, define $\tilde{X}^i$ by the right hand side of (4.28) with Z replaced by Z^i, and let $X^i \doteq \bar{\Gamma}(\tilde{X}^i)$. Then the martingale property of the stochastic integral, and the Lipschitz continuity of the ESM, $b(\cdot)$ and $\sigma(\cdot)$ guarantee the existence of $C < \infty$ such that

$$\mathbb{E}\left[\sup_{s\in[0,t]} |X^1(s) - X^2(s)|^2\right] \leq C \int_0^t \mathbb{E}\left[\sup_{r\in[0,s]} |X^1(r) - X^2(r)|^2\right] ds. \tag{4.33}$$

(See Section 5.2 of [30] for more details of the derivation of this inequality.) Applying the usual Picard iteration technique along with this bound, and using Gronwall's inequality, Čebysev's inequality and the Borel-Cantelli lemma, one can show the existence of a pathwise unique solution to the SDER (following the same arguments as, for example, in page 17 of [20]).

Given any $z^1, z^2 \in G$, let Z^1 and Z^2 be the associated unique strong solutions and for $i = 1, 2$, let $\tilde{X}^i$ be equal to the right hand side of (4.28) with $Z(0)$ replaced by z^i. If we also define $X^i \doteq \bar{\Gamma}(\tilde{X}^i)$, for $i = 1, 2$, then, just as in (4.33), one can obtain the estimate

$$\mathbb{E}[|Z^1(t) - Z^2(t)|^2] \leq \tilde{C}|z^1 - z^2|^2.$$

This establishes that any strong solution is a Feller process and is therefore strong Markov.

The decomposition in (4.30)-(4.31) for the strong solution Z is an immediate consequence of Definition 4.2, which requires that $\mathbb{P}$ a.s. (Z, Y) satisfy the ESP for X, where X is given by (4.28). From Theorem 2.9 we then conclude that $\mathbb{P}$ a.s. (Z, Y) satisfy the SP for X on $[0, \tau_0)$ and therefore Y is an $\{\mathcal{F}_t\}$-adapted process of bounded variation on $[0, \tau_0)$ that satisfies (4.32). Lastly, since $b(\cdot)$ and $\sigma(\cdot)$ satisfy (4.26), M is a continuous local $\{\mathcal{F}_t\}$-martingale and $\int_0^t b(Z_s)ds$ is $\{\mathcal{F}_t\}$-adapted and of bounded variation. This shows that A is $\{\mathcal{F}_t\}$-adapted and $\mathbb{P}$ a.s. of bounded variation, which completes the proof. □

Combining the above result with Theorem 3.6 yields the following result for the GPS ESP.

Corollary 4.4. *Fix $J \geq 2$ and suppose that the coefficients $b(\cdot)$ and $\sigma(\cdot)$ satisfy Assumption 4.1(1). Then given $z \in \mathbb{R}^J_+$, the SDER associated with the GPS ESP has a pathwise unique, strong solution Z with initial condition z. Moreover, Z is a strong Markov process and is a semimartingale on $[0, \tau_0)$ with decomposition (4.30).*

4.2 The Submartingale Problem and the GPS ESP

Fix an integer $J \geq 2$ and let $\{(d_i, n_i, 0), i = 1, \ldots, J+1\}$ be the representation for the GPS ESP given in Section 3.2. Recall that the domain of the GPS ESP is $G = \mathbb{R}^J_+$ and for $i = 1, \ldots, J+1$, define $\partial G^i \doteq \{x \in \mathbb{R}^J_+ : x_i = 0\}$ and let $\mathcal{S} \doteq \cup_{i=1}^J \text{rint}[\partial G^i]$ be the smooth part of the boundary ∂G. As shown in Corollary 4.4, under Assumption 4.1(1), the SDER associated with the GPS ESP has a unique strong solution. In this section we show that this solution also solves the corresponding submartingale problem.

Recall the definitions of $\mathcal{C}^2(G)$ and $\mathcal{C}^2_b(G)$ given in Section 1.4 and, given drift and dispersion coefficients $b(\cdot)$ and $\sigma(\cdot)$, recall the definition of $a(\cdot)$ stated in Assumption 4.1. Consider the operator $\mathcal{L}$ defined by

$$\mathcal{L}f(x) = \sum_{i=1}^{J} b_i(x)\frac{\partial f(x)}{\partial x_i} + \sum_{i,j=1}^{J} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \quad \text{for } f \in \mathcal{C}^2(G). \tag{4.34}$$

We now define the submartingale problem corresponding to the GPS polyhedral ESP $\{(d_i, n_i, 0), i = 1, \ldots, J+1\}$ and operator $\mathcal{L}$ defined above. For $J = 2$, $b(\cdot) = 0$ and $a(\cdot) = 1$, this corresponds to the problem analysed in [46] with parameter $\alpha = 1$. The definition below refers to the canonical filtered probability space $(\Omega_J, \mathcal{M}, \{\mathcal{M}_t\})$ that was introduced before the statement of Theorem 1.4.

Definition 4.5. *(Submartingale Problem) A family $\{\mathbb{Q}_z, z \in G\}$ of probability measures on $(\Omega_J, \mathcal{M})$ is a solution to the submartingale problem associated with the GPS ESP $\{(d_i, n_i, c_i), i = 1, \ldots, J+1\}$, drift $b(\cdot)$ and dispersion $\sigma(\cdot)$ if and only if for each $z \in \mathbb{R}^J_+$, $\mathbb{Q}_z$ satisfies the following three properties.*

1. *$\mathbb{Q}_z(\omega(0) = z) = 1$;*
2. *For every $t \in [0,\infty)$ and $f \in \mathcal{C}^2_b(\mathbb{R}^J_+)$ such that f is constant in a neighbourhood of the set $\partial G \setminus \mathcal{S}$ and $\langle d_i, \nabla f(x)\rangle \geq 0$ for $x \in \mathrm{rint}[\partial G^i]$, $i = 1, \ldots, J$,*

$$f(\omega(t)) - \int_0^t \mathcal{L}f(\omega(u))\,du \tag{4.35}$$

   is a $\mathbb{Q}_z$-submartingale on $(\Omega_J, \mathcal{M}, \{\mathcal{M}_t\})$;
3.

$$\mathbb{E}_{\mathbb{Q}_z}\left[\int_0^\infty \mathbb{1}_{\{\partial G \setminus \mathcal{S}\}}(\omega(s))\,ds\right] = 0. \tag{4.36}$$

In this case, $\mathbb{Q}_z$ is said to be the solution of the GPS submartingale problem starting at z.

The following theorem shows that the family of laws induced by the unique strong solutions of the GPS ESP satisfy the associated submartingale problem.

Theorem 4.6. *Given drift and dispersion coefficients $b(\cdot)$ and $\sigma(\cdot)$ that satisfy Assumption 4.1 and an adapted K-dimensional Brownian motion W defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, for $z \in \mathbb{R}^J_+$ let $\mathbb{Q}_z$ be the measure on $(\Omega_J, \mathcal{M}, \{\mathcal{M}_t\})$ induced by the unique strong solution Z to the SDER associated with the GPS ESP that has initial condition z. Then for $J = 2$, $\{\mathbb{Q}_z, z \in \mathbb{R}^J_+\}$ satisfies the GPS submartingale problem.*

Proof. Fix $J \geq 2$ and $z \in \mathbb{R}^J_+$, and let Z be the pathwise unique, strong solution associated with the GPS ESP that has initial condition z (which exists by Corollary 4.4). By definition, $\mathbb{Q}_z(\omega(0) = z) = \mathbb{P}_z(Z(0) = z) = 1$ and so the first property of Definition 4.5 is trivially satisfied. Now, let $f \in \mathcal{C}^2_b(\mathbb{R}^J_+)$ be as in the statement of property 2 of the submartingale problem stated in Definition 4.5, and fix $\varepsilon > 0$ such that $f(x) = 0$ for $x \in N_\varepsilon(0) \cap G$. Define $\theta_0 \doteq 0$ and for $n \in \mathbb{N}$, let

$$\sigma_n \doteq \inf\{t > \theta_{n-1} : |Z(t)| \leq \varepsilon/2\} \quad \text{and} \quad \theta_n \doteq \inf\{t > \sigma_n : |Z(t)| \geq \varepsilon\}$$

where, by the usual convention, the infimum over an empty set is taken to be ∞. Since $[0,\varepsilon/2]$ and $[\varepsilon,\infty)$ are closed sets, σ_n and θ_n are $\{\mathcal{F}_t\}$-stopping times. Consider the case when $\sigma_n,\theta_n \to \infty$ as $n\to\infty$ (the other case can be dealt with in a similar manner and is thus left to the reader). Then for $t\in[0,\infty)$, we can write

$$\begin{aligned} f(Z(t)) - f(Z(0)) &= \sum_{n=1}^{\infty} \Big[1\!\!1_{\{\sigma_n \le t\}} [f(Z(t\wedge\theta_n)) - f(Z(\sigma_n))] \\ &\qquad + 1\!\!1_{\{\theta_{n-1}\le t\}} [f(Z(t\wedge\sigma_n)) - f(Z(\theta_{n-1}))]\Big] \\ &= \sum_{n=1}^{\infty} 1\!\!1_{\{\theta_{n-1}\le t\}} [f(Z(t\wedge\sigma_n)) - f(Z(\theta_{n-1}))], \end{aligned}$$

where the last equality is a result of the fact that $Z(t)\in N_\varepsilon(0)$ for $t\in[\sigma_n,\theta_n]$ on the set $\{\theta_n<\infty\}$ (and for $t\in[\sigma_n,\infty)$ on the set $\{\sigma_n<\infty,\theta_n=\infty\}$) and f is constant on $N_\varepsilon(0)$. Fix $n\in\mathbb{N}$. Then the uniqueness of the GPS ESP proved in Theorem 3.6 along with Lemma 2.3 and Theorem 2.9 show that on the set $\{\theta_{n-1}\le t\}$, the process $Z(\cdot\wedge\sigma_n) - Z(\theta_n)$ admits the decomposition (4.30) with 0 replaced by θ_{n-1} and t replaced by $t\wedge\sigma_n$. Therefore, applying Itô's formula, we obtain the following equality $\mathbb{P}_z$ a.s. on the set $\{\theta_{n-1}\le t\}$:

$$\begin{aligned} f(Z(t\wedge\sigma_n)) - f(Z(\theta_{n-1})) &= \int_{\theta_{n-1}}^{t\wedge\sigma_n} \mathcal{L}f(Z(s))\,ds \\ &\quad + \int_{\theta_{n-1}}^{t\wedge\sigma_n} \nabla f(Z(s))\cdot dM(s) \\ &\quad + \int_{\theta_{n-1}}^{t\wedge\sigma_n} \nabla f(Z(s))\cdot\gamma(s)\,d|Y|(s) \end{aligned}$$

with $\gamma(s)\in d(Z(s))$ $d|Y|$ a.e. $s\in[\theta_{n-1},\sigma_n\wedge t]$. Multiplying both sides of the last display by $1\!\!1_{\{\theta_{n-1}\le t\}}$, summing over $n\in\mathbb{N}$ and observing that due to the fact that ∇f and $\mathcal{L}f$ are identically zero in an ε-neighbourhood of 0 (since f is constant there), we have the equalities

$$\sum_{n=1}^{\infty} 1\!\!1_{\{\theta_{n-1}\le t\}} \int_{\theta_{n-1}}^{t\wedge\sigma_n} \nabla f(Z(s))\cdot dM(s) = \int_0^t \nabla f(Z(s))\cdot dM(s)$$

and

$$\sum_{n=1}^{\infty} 1\!\!1_{\{\theta_{n-1}\le t\}} \int_{\theta_{n-1}}^{t\wedge\sigma_n} \mathcal{L}f(Z(s))\,ds = \int_0^t \mathcal{L}f(Z(s))\,ds.$$

Thus we conclude that

$$\begin{aligned} f(Z(t)) - f(Z(0)) &= \int_0^t \mathcal{L}f(Z(s))\,ds + \int_0^t \nabla f(Z(s))\cdot dM(s) \\ &\quad + \sum_{n=1}^{\infty} 1\!\!1_{\{\theta_{n-1}\le t\}} \int_{\theta_{n-1}}^{t\wedge\sigma_n} \nabla f(Z(s))\cdot\gamma(s)\,d|Y|(s). \end{aligned}$$

Since ∇f is bounded, the second term on the right-hand side is an $\{\mathcal{F}_t\}$-martingale. On the other hand, the last term is non-negative (for every $t\ge 0$) due to the assumed derivative condition on f and the fact that $\mathbb{P}_z$ a.s. $\gamma(s)\in d(Z(s))$ $d|Y|$ a.e. $s\in[0,t]$. Rearranging the terms above, we see that the process $\{f(Z(t)) - f(Z(0)) - \int_0^t \mathcal{L}f(Z(s))ds, t\in[0,\infty)\}$ is a $\mathbb{P}_z$-submartingale. By the definition of $\mathbb{Q}_z$, this in turn immediately implies the second property (4.35).

We now show that the time the process spends at the origin $\{0\}$ has zero Lebesgue measure. We start with the observation that $Z(s) \in \mathbb{R}_+^J$ implies $Z(s) = 0$ if and only if $Z(s) \cdot d_{J+1} = 0$. Thus by the definition of $\mathbb{Q}_z$, in order to prove property (4.36) with $\partial \setminus S$ replaced by $\{0\}$, it suffices to show that for every $T < \infty$,

$$\mathbb{E}_z \left[\int_0^T \mathbb{1}_{\{0\}}(Z(s) \cdot d_{J+1})\, ds \right] = 0. \tag{4.37}$$

Since Z is a strong solution, it satisfies the ESP for X and so by Corollary 3.5 we know that for every $s \in [0, \infty)$, $Z(s) \cdot d_{J+1} = \Gamma_1(X \cdot d_{J+1})(s)$. Thus $Z \cdot d_{J+1}$ is a one-dimensional reflected diffusion with diffusion coefficient $\sum_{i,j=1}^J a_{ij}/J > 0$, where the strict inequality is due to Assumption 4.1(2). Hence (4.37) is a consequence of the well-known result that a non-degenerate one-dimensional reflected diffusion spends a.s. zero Lebesgue time at the origin 0 (see, for example, page 90 of [25]). When $J = 2$, $\partial G \setminus \mathcal{S} = \{0\}$, and thus the third property follows immediately. This completes the proof of the theorem. $\square$

Remark 4.7. The proof of Theorem 4.6 immediately shows that for any ESP that admits a strong Markov, strong solution to the associated SDER, and whose $\mathcal{V}$-set is contained in the non-smooth parts of the boundary, the first two properties of (the obvious generalisation of) Definition 4.5 are satisfied. Note, in particular, that this holds for every J-dimensional GPS ESP for $J \geq 2$ and, also, that the the set $\mathcal{V}$ is always contained in the non-smooth part of the domain for any polyhedral ESP. In addition, for any J-dimensional ESP, $J \geq 2$, as shown in Theorem 4.6 the corresponding process spends zero Lebesgue time at the origin. In order to complete the verification of Definition 4.5 for an arbitrary J-dimensional GPS ESP, it thus only remains to show that the reflected diffusion spends zero Lebesgue time on the k-dimensional faces, $1 \leq k \leq J - 2$, of the boundary. This can be done using the fact that the local restriction of the GPS reflection matrix to such a face satisfies the completely-$\mathcal{S}$ condition. The details are deferred to a forthcoming paper in which the study of a more general boundary property of these diffusions is carried out.

5 The Semimartingale Property on $[0, \tau_0]$

In this section, we show that the reflected diffusion Z obtained as a strong solution of the SDER associated with the GPS ESP is a semimartingale on $[0, \tau_0]$. It was shown in Theorem 4.3 that it is a semimartingale on the interval $[0, \tau_0)$. However, as demonstrated below, the transition from establishing the property on the half-open interval $[0, \tau_0)$ to the closed interval $[0, \tau_0]$ is more subtle. First, in Section 5.1, we formulate general sufficient conditions under which the strong solution Z of an SDER associated with a general ESP has the required semimartingale property. In Section 5.2, these sufficient conditions are verified for the GPS ESP. This verification involves establishing the existence of certain test functions that satisfy Assumption 5.1. The details of this proof are deferred to Section 6.

5.1 Sufficient Conditions for General ESPs

The first condition, Assumption 5.1, is the existence of a sufficiently smooth function that satisfies certain oblique derivative conditions and whose second derivatives satisfy a certain growth condition. Recall that $\mathcal{V}$ is defined by (2.15) and let $\mathcal{U} \doteq \partial G \setminus \mathcal{V}$. Also recall that supp$[g]$ denotes the support of the function g.

Assumption 5.1. *There exist constants $L, R < \infty$ and $\beta > 0$, and a function $g \in \mathcal{C}^2(G^\circ \cup \mathcal{U})$ that satisfy the following properties.*

1. $\text{supp}[g] \subset N_R(\mathcal{V})$.
2. *There exist $\theta > 0$ and $r \in (0, R)$ such that*

$$\begin{array}{ll} a) \ \langle \nabla g(x), d \rangle \geq 0 & \text{for } d \in d^1(x) \text{ and } x \in \mathcal{U}, \\ b) \ \langle \nabla g(x), d \rangle \geq \theta & \text{for } d \in d^1(x) \text{ and } x \in \mathcal{U} \cap N_r(\mathcal{V}). \end{array}$$

3. $\sup_{x \in G} |g(x)| \vee |\nabla g(x)| \leq L$ *and for* $x \in G^\circ \cup \mathcal{U}$

$$\frac{1}{2} \sum_{i,j=1}^{J} \left| \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \right| \leq \frac{L}{[d(x, \mathcal{V})]^\beta}.$$

We now state the main theorem of this section.

Theorem 5.2. *Suppose the drift and dispersion coefficients $b(\cdot)$ and $\sigma(\cdot)$ satisfy Assumption 4.1. If there exist constants $L, R < \infty$ and $\beta > 0$ such that*

1. *the ESP $(G, d(\cdot))$ satisfies Assumption 5.1 with those constants;*
2. *the drift and dispersion coefficients satisfy the following bound:*

$$\sup_{x \in N_R(\mathcal{V})} \left[\sum_{i,j=1}^{J} |a_{i,j}(x)| \vee |b(x)| \right] < \infty; \tag{5.38}$$

3. *given an $\{\mathcal{F}_t\}$-adapted K-dimensional Brownian motion W, for every $z \in G$, there exists a strong Markov, strong solution Z to the associated SDER, which has initial condition z and satisfies*

$$\mathbb{E}_z \left[\int_0^{t \wedge \tau_0} \frac{\mathbb{1}_{(0,R]} \left(d(Z(s), \mathcal{V}) \right)}{[d(Z(s), \mathcal{V})]^\beta} \, ds \right] < \infty, \tag{5.39}$$

for $t \in [0, \infty)$, where $\tau_0 \doteq \inf\{t \geq 0 : Z(t) \in \mathcal{V}\}$.

Then $Z(\cdot \wedge \tau_0)$ is an $\{\mathcal{F}_t\}$-semimartingale under $\mathbb{P}_z$.

Remark 5.3. Note that since $b(\cdot)$ and $\sigma(\cdot)$ satisfy the Lipschitz condition (4.26), the inequality (5.38) is automatically satisfied if $\mathcal{V}$ is bounded. Although our main application of this result to the GPS ESP in Section 5.2 has bounded $\mathcal{V}$, in anticipation of applications for which $\mathcal{V}$ is unbounded (see Remark 3.7 in Section 3.3), we consider the general unbounded case here.

Proof of Theorem 5.2. Due to (5.38), by taking the constant L in the statement larger, if necessary, we can assume that

$$\sup_{x \in N_R(\mathcal{V})} \left[\sum_{i,j=1}^{J} |a_{i,j}(x)| \vee |b(x)| \right] < L. \tag{5.40}$$

Since Z solves the SDER associated with the ESP $(G, d(\cdot))$, by (4.27) it follows that $\mathbb{P}_z$ a.s.,

$$Z(t \wedge \tau_0) = z + M(t \wedge \tau_0) + A(t \wedge \tau_0),$$

where M and A are defined by

$$M(t) \doteq \int_0^t \sigma(Z(s)) \cdot dW(s) \qquad A(t) \doteq \int_0^t b(Z(s))\, ds + Y(t),$$

and for $0 \leq s \leq t < \infty$, Y satisfies

$$Y(t) - Y(s) \in \overline{\text{co}} \left[\cup_{u \in (s,t]} d(Z(u)) \right].$$

If $z \in \mathcal{V}$, the theorem follows trivially from the fact that $\tau_0 = 0$ and $Y(0) = |Y|(0) = 0$ $\mathbb{P}_z$ a.s. Hence suppose that $z \in G \setminus \mathcal{V} = G^\circ \cup \mathcal{U}$. Due to Assumption 4.1(1), M is a martingale and $\int_0^{t \wedge \tau_0} b(Z(s))\, ds$ has bounded variation for every $t \in [0, \infty)$. Thus to prove the theorem it suffices to show that for every $z \in G^\circ \cup \mathcal{U}$,

$$|Y|(t \wedge \tau_0) < \infty \qquad \text{for every } t \in [0, \infty) \qquad \mathbb{P}_z \text{ a.s.} \tag{5.41}$$

For notational conciseness, we introduce the operator $\mathcal{A}$ defined by

$$\mathcal{A}g(x) = \frac{1}{2} \sum_{i,j=1}^{J} a_{ij}(x) \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \qquad \text{for } g \in \mathcal{C}_b^2(G).$$

Also, for $\delta \in (0, d(z, \mathcal{V}))$ and $K \in (|z|, \infty)$, define

$$\sigma_{\delta K} \doteq \inf\{t \geq 0 : Z(t) \in N_\delta(\mathcal{V}) \text{ or } |Z(t)| \geq K\}.$$

Let the function g and constants r, R, θ, β and L satisfy Assumption 5.1. Since Z is continuous, $t \wedge \sigma_{\delta K} < \tau_0$ $\mathbb{P}_z$ a.s. Due to the semimartingale decomposition for Z on $[0, \tau_0)$ established in Theorem 4.3 and the fact that $Z \in G^\circ \cup \mathcal{U}$ on $[0, \tau_0)$ $\mathbb{P}_z$ a.s., and $g \in \mathcal{C}^2(G^\circ \cup \mathcal{U})$, Itô's formula yields

$$\begin{aligned} g(Z(t \wedge \sigma_{\delta K})) &- g(z) \\ &= \int_0^{t \wedge \sigma_{\delta K}} \nabla g(Z(s)) \cdot b(Z(s))\, ds + \int_0^{t \wedge \sigma_{\delta K}} \nabla g(Z(s)) \cdot \gamma(s)\, d|Y|(s) \\ &\quad + \int_0^{t \wedge \sigma_{\delta K}} \nabla g(Z(s)) \cdot \sigma(Z(s))\, dW(s) + \frac{1}{2} \int_0^{t \wedge \sigma_{\delta K}} \mathcal{A}g(Z(s))\, ds, \end{aligned} \tag{5.42}$$

where, $\mathbb{P}_z$ a.s., $\gamma(s) \in d^1(Z(s))$ $d|Y|$ a.e. on $[0, t \wedge \sigma_{\delta K}]$. Using Assumption 5.1(2) note that

$$\begin{aligned} \int_0^{t \wedge \sigma_{\delta K}} & \nabla g(Z(s)) \cdot \gamma(s)\, d|Y|(s) \\ & \geq \int_0^{t \wedge \sigma_{\delta K}} \mathbb{1}_{[0,r]} (d(Z(s), \mathcal{V}))\, \nabla g(Z(s)) \cdot \gamma(s)\, d|Y|(s) \\ & \geq \theta \int_0^{t \wedge \sigma_{\delta K}} \mathbb{1}_{[0,r]}(d(Z(s), \mathcal{V}))\, d|Y|(s). \end{aligned}$$

In addition, (5.40) and properties 1 and 3 of Assumption 5.1 imply that

$$\int_0^{t\wedge\sigma_{\delta K}} \mathcal{A}g(Z(s))\,ds \leq L^2 \int_0^{t\wedge\sigma_{\delta K}} \frac{1\!\!1_{(0,R]}\left(d(Z(s),\mathcal{V})\right)}{[d(Z(s),\mathcal{V})]^\beta}\,ds,$$

and also that

$$\begin{aligned}\int_0^{t\wedge\sigma_{\delta K}} |\nabla g(Z(s))\cdot b(Z(s))|\,ds &\leq L^2\int_0^{t\wedge\sigma_{\delta K}} 1\!\!1_{(0,R]}\left(d(Z(s),\mathcal{V})\right)\,ds \\ &\leq L^2R^\beta \int_0^{t\wedge\sigma_{\delta K}} \frac{1\!\!1_{(0,R]}\left(d(Z(s),\mathcal{V})\right)}{[d(Z(s),\mathcal{V})]^\beta}\,ds.\end{aligned}$$

Taking expectations (conditioned on $Z_0 = z$) of both sides of (5.42), the stochastic integral on the right hand side vanishes since ∇g and σ are bounded on $\{x \in G : |x| \leq K\}$. Rearranging terms, using the last three displays and the bound on g and ∇g in Assumption 5.1(3), we obtain

$$\begin{aligned}&\theta\mathbb{E}_z\left[\int_0^{t\wedge\sigma_{\delta K}} 1\!\!1_{[0,r]}\left(d(Z(s),\mathcal{V})\right)\,d|Y|(s)\right] \\ &\quad\leq \mathbb{E}_z\left[g(Z(t\wedge\sigma_{\delta K})) - g(z)\right] + \mathbb{E}_z\left[\int_0^{t\wedge\sigma_{\delta K}} |\nabla g(Z(s))\cdot b(Z(s))|\,ds\right] \\ &\qquad\quad + \frac{1}{2}\mathbb{E}_z\left[\int_0^{t\wedge\sigma_{\delta K}} |\mathcal{A}g(Z(s))|\,ds\right] \\ &\quad\leq 2L + L^2(R^\beta+1)\mathbb{E}_z\left[\int_0^{t\wedge\sigma_{\delta K}} \frac{1\!\!1_{(0,R]}\left(d(Z(s),\mathcal{V})\right)}{[d(Z(s),\mathcal{V})]^\beta}\,ds\right].\end{aligned}$$

Let $\tilde{L} \doteq L^2(R^\beta + 1)$. Using the fact that P_z a.s. $t\wedge\sigma_{\delta K} \uparrow t\wedge\tau_0$ as $K \uparrow \infty$ and $\delta \downarrow 0$, we first let $K \uparrow \infty$ and then $\delta \downarrow 0$, and use the monotone convergence theorem to obtain

$$\mathbb{E}_z\left[\int_0^{t\wedge\tau_0} 1\!\!1_{[0,r]}\left(d(Z(s),\mathcal{V})\right)\,d|Y|(s)\right] \leq \frac{2L}{\theta} + \frac{\tilde{L}}{\theta}\mathbb{E}_z\left[\int_0^{t\wedge\tau_0} \frac{1\!\!1_{(0,R]}\left(d(Z(s),\mathcal{V})\right)}{[d(Z(s),\mathcal{V})]^\beta}\,ds\right].$$

When combined with (5.39), this shows that

$$\int_0^{t\wedge\tau_0} 1\!\!1_{[0,r]}(d(Z(s),\mathcal{V})\,d|Y|(s) < \infty \qquad \mathbb{P}_z \text{ a.s.} \tag{5.43}$$

Now define the random time

$$\kappa \doteq \sup\{t \leq \tau_0 : d(Z(t),\mathcal{V}) > r\}.$$

Since Z has continuous paths, $\kappa < \tau_0$ $\mathbb{P}_z$ a.s. on $\{\tau_0 < \infty\}$. Also, trivially, we have $t < \tau_0$ $\mathbb{P}_z$ a.s. on $\{\tau_0 = \infty\}$. Together, this implies $t\wedge\kappa < \tau_0$ $\mathbb{P}_z$ a.s. Therefore from Theorem 4.3, it follows that $|Y|(t\wedge\kappa) < \infty$ $\mathbb{P}_z$ a.s. When combined with (5.39) and (5.43), this yields for every $t \in [0,\infty)$,

$$\begin{aligned}|Y|(t\wedge\tau_0) &= |Y|(t\wedge\kappa) + \int_{t\wedge\kappa}^{t\wedge\tau_0} d|Y|(s) \\ &= |Y|(t\wedge\kappa) + \int_{t\wedge\kappa}^{t\wedge\tau_0} 1\!\!1_{[0,r]}(d(Z(s),\mathcal{V})\,d|Y|(s) \\ &< \infty \qquad \mathbb{P}_z \text{ a.s.,}\end{aligned}$$

which establishes (5.41) and thus proves the theorem. ■

5.2 Verification of Sufficient Conditions for GPS RBMs

In this section, we verify condition (5.39) and Assumption 5.1 for reflected diffusions associated with (a slight generalization of) the GSP ESP. When $b(\cdot)$ and $\sigma(\cdot)$ are either continuous or locally bounded, note that condition (5.38) holds trivially since the set $\mathcal{V} = \{0\}$ is bounded for the GPS ESP.

5.2.1 Verification of condition (5.39) for GPS reflected diffusions

Given drift and dispersion coefficients $b(\cdot)$ and $\sigma(\cdot)$ that satisfy Assumption 4.1 and a J-dimensional GPS ESP with $J \geq 2$, let Z be the unique strong solution to the corresponding SDER (which exists by Corollary 4.4). Since $G = \mathbb{R}_+^J$ and $\mathcal{V} = \{0\}$, $d(Z, \mathcal{V})$ is proportional to $\langle Z, d_{J+1} \rangle$. On the other hand, by Corollary 3.5, $\langle Z, d_{J+1} \rangle$ is a non-degenerate, one-dimensional reflected diffusion. Thus verifying the condition (5.39) reduces to checking a property (see (5.50) below) of one-dimensional reflected diffusions. We first prove an estimate for one-dimensional RBMs in Lemma 5.4 and then extend the result in Lemma 5.5 to one-dimensional reflected diffusions using Girsanov transormations and a time-change argument. Below, Γ_1 is the one-dimensional SM defined in (1.1).

Lemma 5.4. *Given a standard one-dimensional BM W_1 defined on a filtered probability space $((\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\})$, let*

$$\tau_0 \doteq \inf\{t \geq 0 : W_1(t) = 0\}. \tag{5.44}$$

Then for any $z \in \mathbb{R}_+$, $R \in (0, \infty)$ and $\varepsilon \in [0, 1)$,

$$\mathbb{E}_z \left[\int_0^{t \wedge \tau_0} \frac{\mathbb{1}_{(0,R]}(W_1(s))}{W_1^{1+\varepsilon}(s)} ds \right] < \infty \quad \text{for } t \in [0, \infty) \tag{5.45}$$

where $\mathbb{E}_z$ denotes expectation with respect to $\mathbb{P}$, conditioned on $W_1(0) = z$.

Proof. When $z = 0$, $\tau_0 = 0$ and the lemma holds trivially. Fix $z, R \in (0, \infty)$ and $\varepsilon \in [0, 1)$. In order to prove (5.45) we will use the well-known fact (see page 379 of [29]) that for any Borel measurable function f,

$$\mathbb{E}_z \left[\int_0^{\tau_c \wedge \tau_a} f(W_1(s)) ds \right] = \int_a^c g_{a,c}(z, y) f(y) m(dy), \tag{5.46}$$

where τ_a and τ_c are defined by (5.44) with 0 replaced by a and c, respectively, $m(dy) = 2dy$ is the speed measure for BM (see Section II.4 in Appendix I.13 of [6]) and $g_{a,c}(\cdot, \cdot)$ is the Green's function for standard one-dimensional BM on (a, c), which is given by

$$g_{a,c}(z, y) \doteq \frac{2(z \wedge y - a)(c - z \vee y)}{c - a} \quad \text{for } 0 < a \leq c.$$

Let $a \doteq 0$ and choose $c \geq R \vee z$. Substituting the measurable function $f(y) = \mathbb{1}_{(0,R]}(y)/y^{1+\varepsilon}$ into (5.46), we then obtain

$$\mathbb{E}_z \left[\int_0^{\tau_c \wedge \tau_0} \frac{\mathbb{1}_{(0,R]}(W_1(s))}{W_1^{1+\varepsilon}(s)} ds \right] = 2 \int_0^R \frac{g_{0,c}(z, y)}{y^{1+\varepsilon}} dy.$$

On the other hand, by the definition of $g_{0,c}(\cdot,\cdot)$, we see that

$$\frac{1}{2}\int_0^R \frac{g_{0,c}(z,y)}{y^{1+\varepsilon}}\,dy = \begin{cases} \displaystyle\int_z^R \frac{z(c-y)}{cy^{1+\varepsilon}}\,dy + \int_0^z \frac{y(c-z)}{cy^{1+\varepsilon}}\,dy & \text{if } z \le R \\ \displaystyle\int_0^R \frac{y(c-z)}{cy^{1+\varepsilon}}\,dy & \text{if } R \le z, \end{cases}$$

and elementary calculations show that

$$\frac{1}{2}\int_0^R \frac{g_{0,c}(z,y)}{y^{1+\varepsilon}}\,dy = \begin{cases} \dfrac{z^{1-\varepsilon}}{\varepsilon(1-\varepsilon)} - \dfrac{zR^{-\varepsilon}}{\varepsilon} - \dfrac{zR^{1-\varepsilon}}{c(1-\varepsilon)} & \text{if } \varepsilon \in (0,1), z \le R \\ \dfrac{R^{1-\varepsilon}}{1-\varepsilon} - \dfrac{zR^{1-\varepsilon}}{c(1-\varepsilon)} & \text{if } \varepsilon \in (0,1), z \ge R \\ z\ln\left(\dfrac{R}{z}\right) + z - \dfrac{zR}{c} & \text{if } \varepsilon = 0, z \le R \\ R\left(1 - \dfrac{z}{c}\right) & \text{if } \varepsilon = 0, z \ge R. \end{cases}$$

Invoking the monotone convergence theorem, using the non-negativity of the integrand and referring to the last display, we conclude that

$$\begin{aligned} \mathbb{E}_z\left[\int_0^{t\wedge\tau_0} \frac{\mathbb{1}_{(0,R]}(W_1(s))}{W_1^{1+\varepsilon}(s)}\,ds\right] &= \lim_{c\uparrow\infty} \mathbb{E}_z\left[\int_0^{t\wedge\tau_c\wedge\tau_0} \frac{\mathbb{1}_{(0,R]}(W_1(s))}{W_1^{1+\varepsilon}(s)}\,ds\right] \\ &\le 2\lim_{c\uparrow\infty}\int_0^R \frac{g_{0,c}(z,y)}{y^{1+\varepsilon}}\,dy \\ &< \infty, \end{aligned}$$

which establishes (5.45). ☐

Lemma 5.5. *Given a filtered probability space $((\Omega,\mathcal{F},\mathbb{P}),\{\mathcal{F}_t\})$, suppose b^* is an $\{\mathcal{F}_t\}$-adapted, real-valued process and M^* is a continuous $\{\mathcal{F}_t\}$-martingale whose quadratic variation process V is $\mathbb{P}$ a.s. continuously differentiable, and there exist constants $\lambda > 0$ and $\Lambda < \infty$ such that $\mathbb{P}$ a.s.,*

$$\sup_{s\in[0,\infty)} b^*(s) \le \Lambda \quad \text{and} \quad \lambda \le \inf_{s\in[0,\infty)} V'(s) < \sup_{s\in[0,\infty)} V'(s) \le \Lambda, \tag{5.47}$$

where V' denotes the process obtained as the pathwise derivative of V. If

$$B(t) \doteq B(0) + \int_0^t b^*(s)\,ds + M^*(t) \quad \text{for } t \in [0,\infty) \tag{5.48}$$

and

$$\tau_0 \doteq \inf\{t \ge 0 : B(t) = 0\}, \tag{5.49}$$

then for any $z \in \mathbb{R}_+$ and $R \in (0,\infty)$,

$$\mathbb{E}_z\left[\int_0^{t\wedge\tau_0} \frac{\mathbb{1}_{(0,R]}(B(s))}{B(s)}\,ds\right] < \infty \quad \text{for } t \in [0,\infty), \tag{5.50}$$

where $\mathbb{E}_z$ denotes expectation with respect to $\mathbb{P}$, conditioned on $B(0) = z$.

Proof. We first use a time-change argument to show that we can restrict ourselves, without loss of generality, to the case when M^* is a one-dimensional standard BM. Define the "inverse" T of V by

$$T(t) \doteq \inf\{s \geq 0 : V(s) > t\} \quad \text{ for } t \in [0, \infty).$$

The assumed properties of V ensure that $\mathbb{P}$ a.s., both T and V are strictly increasing, continuously differentiable functions on $[0, \infty)$ and $V(T(t)) = T(V(t)) = t$ for every $t \in [0, \infty)$. Now let $W_1 \doteq M^*(T(t))$, $\tilde{\mathcal{F}} = \mathcal{F}$, $\tilde{\mathcal{F}}_t \doteq \mathcal{F}_{T(t)}$, $\tilde{B}(t) \doteq B(T(t))$ for $t \in [0, \infty)$, and define $\tilde{\tau}_0 \doteq \inf\{t \geq 0 : \tilde{B}(t) = 0\}$. Then W_1 is an $\{\tilde{\mathcal{F}}_t\}$-adapted, standard one-dimensional BM (see Theorem 4.6 of [30]) and $\tilde{B}$ is given by

$$\tilde{B}(t) = B(0) + \int_0^t \tilde{b}(s)\, ds + W_1(t), \tag{5.51}$$

where $\tilde{b}$ is an $\{\tilde{\mathcal{F}}_t\}$-adapted process that satisfies

$$\tilde{b}(s) = \frac{b^*(T(s))}{V'(T(s))} \leq \frac{\Lambda}{\lambda} \quad \text{ for } s \in [0, \infty).$$

Moreover, $\tau_0 = T(\tilde{\tau}_0)$ and

$$\begin{aligned}
\mathbb{E}_z \left[\int_0^{t \wedge \tau_0} \frac{\mathbb{1}_{(0,R]}(B(s))}{B(s)}\, ds \right] &= \mathbb{E}_z \left[\int_0^{t \wedge T(\tilde{\tau}_0)} \frac{\mathbb{1}_{(0,R]}(\tilde{B}(V(s)))}{\tilde{B}(V(s))}\, ds \right] \\
&= \mathbb{E}_z \left[\int_0^{V(t) \wedge \tilde{\tau}_0} \frac{\mathbb{1}_{(0,R]}(\tilde{B}(r))}{\tilde{B}(r)\; V'(T(r))}\, dr \right] \\
&\leq \frac{1}{\lambda} \mathbb{E}_z \left[\int_0^{V(t) \wedge \tilde{\tau}_0} \frac{\mathbb{1}_{(0,R]}(\tilde{B}(r))}{\tilde{B}(r)}\, dr \right].
\end{aligned}$$

This shows that in order to prove the theorem, it suffices to establish (5.50) for processes $\tilde{B}$ of the form (5.51), with $\tilde{b}$ uniformly bounded.

We shall now simplify the problem further by applying a Girsanov transformation to remove the drift $\tilde{b}$ from the process $\tilde{B}$. Fix $t \in [0, \infty)$ and define

$$H(s) \doteq \exp\left(- \int_0^s \tilde{b}(r)\, dW_1(r) - \frac{1}{2} \int_0^s \tilde{b}^2(r)\, dr \right) \quad \text{ for } s \in [0, t].$$

Since $\tilde{b}$ is bounded, the process $H = \{H(s), s \in [0, t]\}$ is an $\{\tilde{\mathcal{F}}_s\}$-martingale with expectation 1. Then by Girsanov's theorem (see, for example, Theorem 5.1 of [30]), the process $\tilde{B} = \{\tilde{B}_s, s \in [0, t]\}$ is a standard, one-dimensional BM on $((\Omega, \tilde{\mathcal{F}}, \mathbb{Q}), \{\tilde{\mathcal{F}}_s, s \in [0, t]\})$, where $\mathbb{Q}$ is the probability measure on $(\Omega, \tilde{\mathcal{F}})$ defined by

$$\mathbb{Q}(A) = \mathbb{P}(H(s)A) \quad \text{ for every } A \in \tilde{\mathcal{F}}_s, \quad s \in [0, t].$$

Also, consider the process $N = \{N(s), s \in [0, t]\}$ defined by

$$N(s) \doteq \exp\left(\int_0^s \tilde{b}(r)\, d\tilde{B}(r) - \frac{1}{2} \int_0^t \tilde{b}^2(r)\, dr \right) \quad \text{ for } s \in [0, t] \tag{5.52}$$

and note that $N(\cdot) = 1/H(\cdot)$. So, N is an $\{\tilde{\mathcal{F}}_s\}$-martingale under $\mathbb{Q}$ and $\mathbb{P}(A) = \mathbb{Q}(N(s)A)$ for $A \in \tilde{\mathcal{F}}_s$, $s \in [0,t]$. Let $\mathbb{E}_z^{\mathbb{Q}}$ denote expectation with respect to $\mathbb{Q}$, conditioned on $\tilde{B}(0) = z$ and, for greater clarity, we denote the corresponding expectation $\mathbb{E}_z$ with respect to $\mathbb{P}$ by $\mathbb{E}_z^{\mathbb{P}}$. Then, using Fubini's theorem, the properties of $\mathbb{Q}$ and $\mathbb{P}$ stated above and Hölder's inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}_z^{\mathbb{P}}\left[\int_0^{t\wedge\tau_0} \frac{\mathbb{1}_{(0,R]}(\tilde{B}(s))}{\tilde{B}(s)}\,ds\right] \\
&\quad= \int_0^t \mathbb{E}_z^{\mathbb{P}}\left[\mathbb{1}_{[0,\tau_0]}(s)\frac{\mathbb{1}_{(0,R]}(\tilde{B}(s))}{\tilde{B}(s)}\right]\,ds \\
&\quad= \int_0^t \mathbb{E}_z^{\mathbb{Q}}\left[\mathbb{1}_{[0,\tau_0]}(s)\frac{\mathbb{1}_{(0,R]}(\tilde{B}(s))}{\tilde{B}(s)}N(s)\right]\,ds \\
&\quad\leq \left(\int_0^t \mathbb{E}_z^{\mathbb{Q}}\left[\mathbb{1}_{[0,\tau_0]}(s)\frac{\mathbb{1}_{(0,R]}(\tilde{B}(s))}{\tilde{B}^{3/2}(s)}\right]\,ds\right)^{2/3}\left(\int_0^t \mathbb{E}_z^{\mathbb{Q}}\left[(N(s))^3\right]\,ds\right)^{1/3} \\
&\quad= \left(\int_0^t \mathbb{E}_z^{\mathbb{P}}\left[\mathbb{1}_{[0,\tau_0]}(s)\frac{\mathbb{1}_{(0,R]}(W_1(s))}{W_1^{3/2}(s)}\right]\,ds\right)^{2/3}\left(\int_0^t \mathbb{E}_z^{\mathbb{Q}}\left[(N(s))^3\right]\,ds\right)^{1/3}.
\end{aligned}
$$

It only remains to show that each of the two terms in the last line are finite. An application of Lemma 5.4 with $\varepsilon = 1/2$ immediately shows that the first term is finite. For the second term, let $N^{3\tilde{b}(\cdot)}$ be defined like N in (5.52), but with $\tilde{b}(\cdot)$ replaced everywhere by $3\tilde{b}(\cdot)$. Then it is easy to see that $N^{3\tilde{b}(\cdot)}$ is an $\{\tilde{\mathcal{F}}_s\}$-martingale under $\mathbb{Q}$ with expectation 1. When combined with the fact that for $s \in [0,t]$,

$$
(N(s))^3 = N^{3\tilde{b}(\cdot)}(s)\exp\left(2\int_0^s \tilde{b}^2(r)\,dr\right) \leq N^{3\tilde{b}(\cdot)}(s)\exp\left(2t\left[\sup_{r\in[0,t]}\tilde{b}^2(r)\right]\right),
$$

this shows that the second term is also finite and thus completes the proof. $\square$

Corollary 5.6. *Given an $\{\mathcal{F}_t\}$-adapted, K-dimensional Brownian motion defined on a filtered probability space $((\Omega,\mathcal{F},\mathbb{P}),\mathcal{F}_t)$, the J-dimensional GPS ESP and drift and dispersion coefficients $b(\cdot)$ and $\sigma(\cdot)$ satisfying Assumption 4.1, let Z be the unique, strong solution to the associated SDER and let τ_0 be given by (4.29). Then for every $z \in \mathbb{R}_+^J$ and $R \in (0,\infty)$, it follows that*

$$
\mathbb{E}_z\left[\int_0^{t\wedge\tau_0}\frac{\mathbb{1}_{(0,R]}(d(Z(s),\mathcal{V}))}{d(Z(s),\mathcal{V})}ds\right] < \infty \qquad \text{for } t \in [0,\infty), \tag{5.53}
$$

where $\mathbb{E}_z$ is expectation with respect to $\mathbb{P}$, conditioned on $Z(0) = z$.

Proof. By Lemma 3.4, we know that $\mathcal{V} = \{0\}$ for the family of GPS ESPs and therefore $d(z,\mathcal{V}) = |z|$. Now let $f(z) \doteq \langle z, d_{J+1}\rangle$. Then $f(z) = 0$ for $z \in \mathbb{R}_+^J$ if and only if $|z| = 0$, and there exist $0 < k_1 < k_2 < \infty$ such that

$$
k_1 f(z) \leq d(z,\mathcal{V}) = |z| \leq k_2 f(z) \qquad \text{for every } z \in \mathbb{R}_+^J.
$$

Hence if $B \doteq f(Z)$ then $\tau_0 = \inf\{t > 0 : B(t) = 0\}$ and (5.53) holds if

$$\tilde{\mathbb{E}}_{z^*}\left[\int_0^{t\wedge\tau_0} \frac{\mathbb{1}_{(0,\tilde{R}]}(B(s))}{B(s)} ds\right] < \infty \qquad \text{for every } t \in [0,\infty), \tag{5.54}$$

where $\tilde{R} \doteq R/k_1$, $z^* \doteq \langle z, d_{J+1}\rangle$ and for $z^* \in \mathbb{R}_+$, $\tilde{\mathbb{E}}_{z^*}$ is expectation with respect to $\mathbb{P}$, conditioned on $B(0) = z^*$. According to Definition 4.2, $Z = \bar{\Gamma}(X)$ $\mathbb{P}$ a.s., where X is given by (4.28); by Corollary 3.5, $B \doteq f(Z) = \Gamma_1(\langle X, d_{J+1}\rangle)$, where Γ_1 is the 1-dimensional SM. Thus for $t \in [0, \tau_0)$, $B(t) = \langle X(t), d_{J+1}\rangle$, which can be written explicitly as

$$B(t) = \int_0^t b^*(s)ds + M^*(t),$$

where b^* is the $\{\mathcal{F}_t\}$-adapted process defined by $b^*(s) = \sum_{i=1}^J b_i(X(s))/\sqrt{J}$ and M^* is an $\{\mathcal{F}_t\}$-adapted continuous martingale with quadratic variation process V given by

$$V(t) = \sum_{i,j=1}^J \int_0^t \frac{a_{ij}(X(s))}{J},$$

where $a(\cdot) = \sigma(\cdot)\sigma^T(\cdot)$ is the diffusion coefficient defined in Assumption 4.1(2). The fact that $b(\cdot)$ and $\sigma(\cdot)$ satisfy Assumption 4.1 ensures that the processes b^* and V satisfy the conditions of Lemma 5.5. So the estimate (5.54) follows from Lemma 5.5, which in turn establishes (5.53). $\square$

5.2.2 Verification of Assumption 5.1

In this section, we consider ESPs on conical polyhedral domains with vertex at the origin that satisfy Assumption 3.2 (the "set B" condition) and have $\mathcal{V} = \{0\}$. The main result of this section is Theorem 5.9, which proves the existence of a function that satisfies Assumption 5.1 for this class of ESPs. The theorem relies on two results. The first result, stated as Theorem 5.7, establishes the existence of a family of "local" functions $\{g_{z,r}, z \in \mathcal{U}\}$ where, roughly speaking, each $g_{z,r}$ satisfies the properties of Assumption 5.1 in a bounded, convex neighbourhood of z. This result holds for any polyhedral ESP that satisfies Assumption 3.2. The second result is Lemma 5.8, which establishes a covering property that shows that the local functions constructed in Theorem 5.7 can be patched together to yield a function g that satisfies Assumption 5.1. We first introduce some notation, then state Theorem 5.7 and Lemma 5.8, and then present the proof of the main result, Theorem 5.9. The proofs of Theorem 5.7 and Lemma 5.8 are relegated to Section 6.

Given $\mathbb{I} = \{1, \ldots, I\}$ and a polyhedral ESP $\{(d_i, n_i, c_i), i \in \mathbb{I}\}$ with domain G, recall that $I(x) \doteq \{i \in \mathbb{I} : \langle x, n_i\rangle = c_i\}$ for $x \in \partial G$. For $C \subseteq \mathbb{I}$, let F_C be defined by

$$F_C \doteq \{x \in \partial G : I(x) = C\},$$

and note that ∂G is the disjoint union of $F_C, C \subseteq \mathbb{I}$, $C \neq \emptyset$. Given $C \subseteq \mathbb{I}$, we shall refer to F_C as a facet, to distinguish it from its closure

$$\{x \in G : \langle x, n_i\rangle = c_i \text{ for every } i \in C\},$$

which we will refer to as a face (the two definitions coincide if and only if F_C is a point). For $z \in F_C \subset \partial G$, define

$$r_z \doteq d\left(z, \partial G \setminus [\cup_{C' \subseteq C} F_{C'}]\right), \tag{5.55}$$

with the convention that the distance of z to the empty set $\emptyset$ is equal to zero. If $z \in F_C$, r_z is the minimum distance from z to any face on which it does not lie. It is not hard to see that F_C is relatively open and $r_z > 0$ as long as z is not a vertex. On the other hand, $r_z = 0$ when z is the sole vertex because in that case $C = \mathbb{I}$ and so $\partial G \setminus [\cup_{C' \subseteq C} F_{C'}] = \emptyset$. From the definition of polyhedral ESPs it follows that

$$x \in N^{\circ}_{r_z}(z) \cap G \quad \text{implies that} \quad d(x) \subseteq d(z). \tag{5.56}$$

Given any subset $U \subset \partial G$, we define

$$\mathcal{P}(U) \doteq \{C \subseteq \mathbb{I} : x \in F_C \text{ for some } x \in U\} \tag{5.57}$$

to be the collection of sets C such that U has a non-empty intersection with the corresponding facet F_C. Observe that then the set $\mathcal{U} \doteq \partial G \setminus \mathcal{V}$ can be written in the form $\mathcal{U} = \cup_{C \in \mathcal{P}(\mathcal{U})} F_C$. For example, in the case of the GPS ESP with $\mathcal{V} = \{0\}$, we have $\mathcal{P}(\mathcal{U}) = \{C \subset \mathbb{I} : C \neq \mathbb{I}, C \neq \emptyset\}$. We can now state the main results of the section. Recall that $N_1(0)$ is the open unit ball centered at 0.

Theorem 5.7. *Suppose the polyhedral ESP $\{(d_i, n_i, c_i), i \in \mathbb{I}\}$ satisfies Assumption 3.2, and let $\mathcal{U} = \partial G \setminus \mathcal{V}$, where the $\mathcal{V}$-set is defined by (2.15). There exists a function $A : \mathcal{U} \to [0, \infty)$, constants $A' < \infty$ and $\theta > 0$, bounded convex sets $Q^+_C, Q^-_C, C \in \mathcal{P}(\mathcal{U})$, such that $0 \in (Q^-_C)^{\circ}$ and $Q^-_C \subset (Q^+_C)^{\circ} \subset N_1(0)$, and a family of functions $\{g_{z,r} : z \in \mathcal{U}, r \in (0, r_z)\}$ satisfying the following properties.*

1. *$g_{z,r} \in \mathcal{C}^{\infty}(G)$ and $g_{z,r}(x) = \nabla g_{z,r}(x) = 0$ for $x \in \mathcal{V}$;*
2. *$\operatorname{supp}[g_{z,r}] \cap G \subset \left(z + rQ^+_{I(z)}\right)^{\circ}$;*
3. *$\sup_{x \in G} |g_{z,r}(x)| \le A'r$ for every $z \in \mathcal{U}, r \in (0, r_z)$;*
4. *$\langle \nabla g_{z,r}(x), d\rangle \ge 0$ for $d \in d^1(x)$ and $x \in \mathcal{U}$;*
5. *$\langle \nabla g_{z,r}(x), d\rangle \ge \theta$ for $d \in d^1(x)$ and $x \in \left[z + rQ^-_{I(z)}\right] \cap \partial G$;*
6. *$\sup_{x \in G} \sum_{i,j=1}^{J} \left|\frac{\partial^2 g_{z,r}(x)}{\partial x_i \partial x_j}\right| \le \frac{A(z)}{r}$;*
7. *For every $R < \infty$, $\tilde{A}_R \doteq \sup_{z \in N_R(\mathcal{V}) \cap \mathcal{U}} A(z) < \infty$.*

Lemma 5.8. *Suppose the polyhedral ESP $\{(d_i, n_i, 0), i \in \mathbb{I}\}$ has a conical domain with angle less than π and has $\mathcal{V} = \{0\}$, where $\mathcal{V}$ is defined by (2.15). Let Q^+_C, Q^-_C, $C \in \mathcal{P}(\mathcal{U})$, be convex sets such that $0 \in (Q^-_C)^{\circ}$ and $Q^-_C \subset (Q^+_C)^{\circ} \subset N_1(0)$. Then, given $0 < r < R < \infty$, there exists a countable set of vectors $S \subset \mathcal{U}$ and a corresponding set of scalars $\{\rho_z : z \in S, \rho_z < r_z\}$ such that the sets $Q^-_z \doteq z + \rho_z Q^-_{I(z)}$ and $Q^+_z \doteq z + \rho_z Q^+_{I(z)}$, $z \in S$, satisfy the following five properties.*

1. There exists $N < \infty$ such that for every $x \in G$

$$\#\left[\{z \in S : x \in Q_z^+\}\right] \leq N\,; \tag{5.58}$$

2. There exists $\nu \in (0,\infty)$ such that if $x \in Q_z^+$ then $\nu\rho_z \geq d(x,\mathcal{V})$;

3. If $x \in \partial G \cap Q_z^+$ then $d^1(x) \subseteq d^1(z)$;

4. $\mathcal{U} \cap N_r(\mathcal{V}) \subset [\cup_{z\in S} Q_z^-]$;

5. $[\cup_{z\in S} Q_z^+] \subset N_R(\mathcal{V})$.

Observe that Theorem 5.7 (in particular, the existence and shape of the convex sets Q_C^- and Q_C^+) is heavily dependent on the geometry of the directions of constraint and is not much concerned with the structure of the set $\mathcal{U}$. On the other hand, the covering result in Lemma 5.8 depends more on the geometry of $\mathcal{U}$. The proofs of Theorem 5.7 and Lemma 5.8 are given in Sections 6.1 and 6.2 respectively. Here, we show how these results can be combined to construct a function g that satisfies Assumption 5.1.

Theorem 5.9. *Suppose the polyhedral ESP $\{(d_i, n_i, 0), i \in \mathbb{I}\}$ has a conical domain with angle less than π, satisfies Assumption 3.2 and has $\mathcal{V} = \{0\}$. Then for any $R < \infty$, there exists $L < \infty$ such that the ESP satisfies Assumption 5.1 with $\beta = 1$.*

Proof. Fix $R < \infty$. Let $\tilde{A}_R < \infty$, $\theta > 0$, $\{g_{z,r} : z \in \mathcal{U}, r \in (0, r_z)\}$ and Q_C^+, Q_C^-, $C \in \mathcal{P}(\mathcal{U})$, satisfy the properties stated in Theorem 5.7. Fix $r \in (0,R)$ and corresponding to Q_C^+, Q_C^-, $C \in \mathcal{P}(\mathcal{U})$, choose a countable set of points $S \subset \mathcal{U}$ and $\rho_z \in (0, r_z)$, $z \in S$, such that the properties stated in Lemma 5.8 are satisfied, and let the corresponding sets Q_z^- and Q_z^+, $z \in S$, be as defined in Lemma 5.8. Also define

$$g(x) \doteq \sum_{z\in S} g_{z,\rho_z}(x) \qquad \text{for } x \in G. \tag{5.59}$$

For $x \in G^\circ \cup \mathcal{U}$, let $J(x) \doteq \{z \in S : x \in (Q_z^+)^\circ\}$. Since, by property 1 of Lemma 5.8, the cardinality of $J(x)$ is finite (it is in fact uniformly bounded by N), there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subset \cap_{z\in J(x)} (Q_z^+)^\circ$, and so for every $y \in N_\varepsilon(x)$, $J(y) = J(x)$. Thus for every $x \in G^\circ \cup \mathcal{U}$, there exists $\varepsilon > 0$ such that

$$g(y) = \sum_{z\in J(x)} g_{z,\rho_z}(y) \qquad \text{for } y \in N_\varepsilon(x). \tag{5.60}$$

Along with properties 1–3 of Theorem 5.7, this guarantees that g lies in $\mathcal{C}^\infty(G^\circ \cup \mathcal{U})$, satisfies $g(x) = 0$ for $x \in \mathcal{V}$ and is a continuous function on G.

Now, property 5 of Lemma 5.8 ensures that $\text{supp}[g] \subset N_R(\mathcal{V})$, which establishes Assumption 5.1(1). On the other hand, (5.60), with $y = x$, combined with properties 4 and 5 of Theorem 5.7, property 4 of Lemma 5.8 and the fact that $\rho_z < r_z$, implies Assumption 5.1(2). Furthermore, (5.60), along with properties 6 and 7 of Theorem 5.7 and properties 1 and 2 of Lemma 5.8, implies that for $x \in G^\circ \cup \mathcal{U} \cap N_R(\mathcal{V})$,

$$\sum_{j,k=1}^{J} \left| \frac{\partial^2 g(x)}{\partial x_j \partial x_k} \right| \leq \sum_{z\in J(x)} \sum_{j,k=1}^{J} \left| \frac{\partial^2 g_{z,\rho_z}(x)}{\partial x_j \partial x_k} \right| \leq \frac{\tilde{A}_R N}{\min_{z\in J(x)} \rho_z} \leq \frac{\tilde{A}_R N \nu}{d(x,\mathcal{V})}.$$

Since g is identically zero outside $N_R(\mathcal{V})$, this shows that Assumption 5.1(3) is satisfied with $L = \tilde{A}_R N\nu$ and $\beta = 1$, thus completing the proof of the theorem. ☐

Theorem 5.10. *Let Z be the pathwise unique, strong solution to the SDER associated with the GPS ESP and drift and dispersion coefficients $b(\cdot)$ and $\sigma(\cdot)$ that satisfy Assumption 4.1. Then $Z(\cdot \wedge \tau_0)$ is a semimartingale, where $\tau_0 = \inf\{t \geq 0 : Z(t) = 0\}$.*

Proof. The bound (5.38) holds by Remark 5.3. Moreover, given any $R < \infty$, Theorem 5.9 shows there exist $L < \infty$ and $\beta = 1$ such that Assumption 5.1 is satisfied for the GPS ESP. The existence of a pathwise unique, strong solution follows from Corollary 4.4, and Corollary 5.6 shows that condition (5.39) is satisfied for any $R < \infty$ and $\beta = 1$. Thus the result follows from Theorem 5.2. ☐

6 Construction of Test Functions for the GPS Family

In this section, we consider a slight generalization of the family of GPS ESPs, namely polyhedral ESPs in convex conical polyhedral domains with vertex at the origin that satisfy Assumption 3.2 and have $\mathcal{V} = \{0\}$. In Section 6.1, we prove Theorem 5.7 and in Section 6.2, we establish Lemma 5.8. Together with Theorem 5.9, this demonstrates the existence of a function g that satisfies Assumption 5.1 for this class of ESPs.

6.1 Proof of Theorem 5.7

We first prove some preliminary results in Lemmas 6.1-6.4. The following notation is used throughout this section. For any set $A \subset \mathbb{R}^J$, $\mathrm{rint}(A)$ is used to denote the relative interior of the set A (see [39] for a precise definition). For conciseness, in this section we will often use A^ε to denote $N_\varepsilon(A) = \{x : d(x, A) \leq \varepsilon\}$ for $\varepsilon > 0$. Recall the definitions given in Section 5.2.2 of F_C, $C \subseteq \mathbb{I}$, and $\mathcal{P}(\mathcal{U})$, where $\mathcal{U} = \partial G \setminus \mathcal{V}$. For $C \subset \mathbb{I}$, $C \neq \emptyset$, let the cone L_C be defined by

$$L_C \doteq \left\{ -\sum_{i \in C} a_i d_i : a_i \geq 0 \right\}, \tag{6.61}$$

and the set K_C by

$$K_C \doteq \left\{ -\sum_{i \in C} a_i d_i / |d_i| : a_i \geq 0, \sum_{i \in C} a_i = 1 \right\}. \tag{6.62}$$

Note that K_C is a convex, compact subset of $\mathbb{R}^J$, and for $x \in F_C$, $L_C = -d(x)$ and $-d^1(x) \subseteq \{tK_C : t \geq 1\}$. We first state a useful consequence of the existence of a set B that satisfies Assumption 3.2 for the ESP. This result was proved in [19].

Lemma 6.1. *Suppose the polyhedral ESP $\{(d_i, n_i, c_i), i \in \mathbb{I}\}$ satisfies Assumption 3.2. For $C \subseteq \mathbb{I}$, $C \neq \emptyset$, if L_C is defined by (6.61) then*

$$\min_{i \in C} \langle n_i, d \rangle < 0 \quad \textit{for every } d \in L_C \setminus \{0\}.$$

Proof. This lemma corresponds to Lemma A.3 in [19] (specialised to the case when the vector field γ_i in [19] is constant and equal to $-d_i$), with the caveat that n_i used in [19] represents an outward normal, while in this paper n_i denotes an inward normal to the domain G. Note that the condition (A.1) specified in [19] follows from Assumption 3.2 due to Lemma 2.1 of [18]. $\square$

Since Assumption 3.2 holds for the polyhedral ESPs under consideration, Lemma 6.1 is applicable. If $x \in \mathcal{U}$, then the vectors $d_i, i \in I(x)$, are linearly independent. Hence for $C \in \mathcal{P}(\mathcal{U})$, $0 \notin K_C$. By Lemma 6.1 this implies that for $C \in \mathcal{P}(\mathcal{U})$,

$$\min_{i \in C} \langle n_i, d \rangle < 0 \quad \text{for all } d \in K_C.$$

Since K_C is compact, there exists $\delta_C > 0$ such that

$$\min_{i \in C} \langle n_i, d \rangle < 0 \quad \text{for all } d \in K_C^{\delta_C}. \tag{6.63}$$

Let K_{C,δ_C} be a closed, convex set that has a $\mathcal{C}^\infty$ boundary and satisfies

$$K_C^{\delta_C/2} \subset (K_{C,\delta_C})^\circ \subset K_{C,\delta_C} \subset K_C^{\delta_C}. \tag{6.64}$$

(Here a convex set $F \subset \mathbb{R}^J$ is said to have a C^∞ boundary if for every point $y \in \partial F$, there exists a (relative) neighbourhood of y in ∂F that is a $\mathcal{C}^\infty$ submanifold of $\mathbb{R}^J$, appropriately modelled on some hyperplane of $\mathbb{R}^J$; for closed convex sets F, this has been shown in [28] to be equivalent to the gauge function of F being C^∞ in a neighbourhood of the boundary of F.) Also, define

$$L_{C,\delta_C} \doteq \cup_{t \geq 0} t K_{C,\delta_C}. \tag{6.65}$$

Then the inequality (6.63) clearly holds with $K_C^{\delta_C}$ replaced by K_{C,δ_C}. This in turn implies that $0 \notin K_{C,\delta_C}$, that L_{C,δ_C} is a (half) cone with vertex at the origin, and that there exists $\beta_C > 0$ such that

$$\min_{i \in C} \langle n_i, d \rangle \leq -2\beta_C |d| \quad \text{for all } d \in L_{C,\delta_C}. \tag{6.66}$$

It is also clear that, since K_{C,δ_C} has a $\mathcal{C}^\infty$ boundary, L_{C,δ_C} is a cone whose boundary is $\mathcal{C}^\infty$ everywhere except at the vertex $\{0\}$.

In Lemma 6.2 below, we construct a family $\{g_C, C \in \mathcal{P}(\mathcal{U})\}$ of functions, where g_C is associated with the facet F_C in $\mathcal{U}$. These functions serve as the basic building blocks for the construction of the family of functions $\{g_{z,r}, z \in \mathcal{U}, r \in (0, r_z)\}$ of Theorem 5.7; indeed the latter will essentially be obtained as suitably scaled translates of the functions g_C, $C \in \mathcal{P}(\mathcal{U})$. Each function g_C is constructed as (a suitable approximation of) the distance function to the cone L_{C,δ_C}. As shown below in Lemma 6.2, the geometry of the directions of constraint (imposed by Assumption 3.2) ensures that this distance function locally satisfies the necessary gradient conditions (see property 2 of the lemma). This observation was first made in [19] when considering an SP with $\mathcal{V} = \emptyset$, and was used there to construct a $\mathcal{C}^1$ function that satisfies gradient conditions similar to those in (6.67) and (6.70). However, the construction here is considerably more involved due to the fact that $\mathcal{V} \neq \emptyset$ and we need a $\mathcal{C}^2$ function whose second-order partial derivatives are uniformly bounded. In particular, since the distance function is not $\mathcal{C}^2$, we need to establish the existence of sufficiently smooth approximations to the distance function that satisfy both the gradient properties and the bound on the second-order derivatives. While discussing approximations, for conciseness we will use the Schwarz notation for multi-indices: given $\alpha \in \mathbb{Z}_+^J$ and a function f on some open set $\Omega \subset \mathbb{R}^J$, recall that

$$D^\alpha f = \frac{\partial^{\alpha_1+\ldots\alpha_J} f}{\partial x_1^{\alpha_1}\ldots\partial x_J^{\alpha_J}} \quad \text{and} \quad D_w^\alpha f = \left(\frac{\partial^{\alpha_1+\ldots\alpha_J} f}{\partial x_1^{\alpha_1}\ldots x_J^{\alpha_J}}\right)_w$$

denote the ordinary and weak derivative operators of the function f of order α on Ω and $|\alpha|$ denotes $\alpha_1+\ldots+\alpha_J$ (see [7, Definition 2, page 19] for the definition of weak derivatives). With some abuse of notation, we will say $h = D_w^\alpha f$ to mean h is a weak derivative of the function f of order α on Ω.

Lemma 6.2. *For every $C \in \mathcal{P}(\mathcal{U})$, given any $0 < \tilde{\eta}_C < \tilde{\lambda}_C < \infty$, $\tilde{\varepsilon}_C > 0$ and*

$$\Lambda_C \doteq \left(L_{C,\delta_C}^{\tilde{\lambda}_C}\right)^\circ \setminus L_{C,\delta_C}^{\tilde{\eta}_C} = \{x \in \mathbb{R}^J : \tilde{\eta}_C < d(x, L_{C,\delta_C}) < \tilde{\lambda}_C\},$$

there exists a function $g_C : \Lambda_C \to \mathbb{R}$, that satisfies the following four properties.

1. *$g_C \in \mathcal{C}^\infty(\Lambda_C)$;*
2. *there exists $\theta_C > 0$ such that*
$$\langle \nabla g_C(x), p\rangle \le -\theta_C \qquad \text{for } p \in K_C^{\delta_C/3} \text{ and } x \in \Lambda_C; \tag{6.67}$$
3. *for every $j, k \in \{1, \ldots, J\}$,*
$$\sup_{x\in\Lambda_C} \left|\frac{\partial^2 g_C(x)}{\partial x_j \partial x_k}\right| < \infty;$$
4. *$\sup_{x\in\Lambda_C} (|g_C(x) - d(x, L_{C,\delta_C})| \vee (|\nabla g_C(x)| - 1)) \le \tilde{\varepsilon}_C$.*

Proof. Fix $C \in \mathcal{P}(\mathcal{U})$, $0 < \tilde{\eta}_C < \tilde{\lambda}_C < \infty$ and $\tilde{\varepsilon}_C > 0$. For ease of notation, for the rest of this proof we will usually just write L_δ, Λ, $\tilde{\lambda}$ and $\tilde{\eta}$ for L_{C,δ_C}, Λ_C, $\tilde{\lambda}_C$ and $\tilde{\eta}_C$, respectively. Define $\tilde{g}_C : \mathbb{R}^J \to \mathbb{R}_+$ to be $\tilde{g}_C(\cdot) \doteq d(\cdot, L_\delta)$, the distance function to the cone L_δ. The proof is comprised of four steps. In the first step, we collect some known properties of $\tilde{g}_C$. In the second step, we establish gradient properties of the type (6.67) for $\tilde{g}_C$ on Λ_C, and in the third step, we obtain bounds on the growth of the second derivatives of $\tilde{g}_C$ on a subset of Λ_C. In the last step we show that there exists a smooth approximation g_C of $\tilde{g}_C$ that satisfies the conditions of the theorem.

Step 1. Let $P_{L_\delta} : \mathbb{R}^J \to L_\delta$ be the metric projection onto the cone L_δ (which assigns to each point $x \in \mathbb{R}^J$ the point on L_δ that is closest to x), and let $H_C \doteq \{x \in \mathbb{R}^J : P_{L_\delta}(x) = 0\}$ be the set of points that get projected to the vertex 0 under the map P_{L_δ}. Since L_δ is a closed convex set, it is well-known (and seems to have been first proved in [34, p. 286]) that on $\mathbb{R}^J \setminus L_\delta$, $\tilde{g}_C$ is $\mathcal{C}^1$, P_{L_δ} is Lipschitz with constant 1, $\tilde{g}_C(\cdot) = |x - P_{L_\delta}(\cdot)|$ and

$$\nabla \tilde{g}_C(x) = \frac{x - P_{L_\delta}(x)}{|x - P_{L_\delta}(x)|} \qquad \text{for } x \in \mathbb{R}^J \setminus L_\delta. \tag{6.68}$$

We now argue that $\tilde{g}_C$ is (at least) $\mathcal{C}^3$ and P_{L_δ} is (at least) $\mathcal{C}^2$ on $\mathbb{R}^J \setminus [L_\delta \cup \partial H_C]$. Indeed, first note that the fact that $\tilde{g}_C(x) = |x|$ and $P_{L_\delta}(x) = 0$ for x in the interior of H_C implies that both functions are C^∞ on H_C°. Next, observe that since the boundary of K_{C,δ_C} is $\mathcal{C}^\infty$ (by construction), the boundary of L_δ is also $\mathcal{C}^\infty$ everywhere except at the vertex 0. Theorem 2 of [28] asserts that for $p \geq 1$, if the boundary of L_δ is $\mathcal{C}^p$ in a neighbourhood of a point $x \in \partial L_\delta$ then the distance function $\tilde{g}_C$ is $\mathcal{C}^p$ and the projection $\mathbb{P}_{L_\delta}$ is $\mathcal{C}^{p-1}$ in a neighbourhood of the open normal ray to L_δ at the point x. In particular, this guarantees that $\tilde{g}_C$ is (at least) $\mathcal{C}^3$ and P_{L_δ} is (at least) $\mathcal{C}^2$ on $\mathbb{R}^J \setminus [L_\delta \cup \partial H_C]$.

Step 2. For any $x \in \mathbb{R}^J \setminus L_\delta$, because $x - P_{L_\delta}(x)$ is an outward normal to L_δ at $P_{L_\delta}(x)$ and L_δ is convex, we have

$$\langle x - P_{L_\delta}(x), w - P_{L_\delta}(x)\rangle \leq 0 \quad \text{for every } w \in L_\delta. \tag{6.69}$$

Since L_δ is a convex cone with vertex at the origin that contains K_{C,δ_C}, we have $P_{L_\delta}(x)+p \in L_\delta$ for $p \in K_{C,\delta_C}$. Setting $w = P_{L_\delta}(x) + p$ in the last inequality we see that $\langle x - P_{L_\delta}(x), p\rangle \leq 0$ for every $p \in K_{C,\delta_C}$. Since $K_C^{\delta_C/2} \subset (K_{C,\delta_C})^\circ$ by (6.64) and $K_C^{\delta_C/2}$ is compact, combining this with the expression for $\nabla \tilde{g}_C$ given in (6.68), we conclude that there must exist θ_C that satisfies

$$\langle \nabla \tilde{g}_C(x), p\rangle \leq -2\theta_C \qquad \text{for } p \in K_C^{\delta_C/2} \text{ and } x \in \mathbb{R}^J \setminus L_{C,\delta_C}^{\tilde{\eta}_C/2}. \tag{6.70}$$

Step 3. Let $\{\delta_{ij}, i,j \in \{1,\dots,J\}\}$ be the Kronecker delta function: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. For $x \in \mathbb{R}^J \setminus [L_\delta \cup \partial H_C]$, using the expression (6.68) we obtain

$$\begin{aligned}
&\frac{\partial^2 \tilde{g}_C}{\partial x_j \partial x_i}(x) = \frac{\partial}{\partial x_j}\langle \nabla \tilde{g}_C(x), e_i\rangle \\
&= \frac{\partial}{\partial x_j}\left(\frac{x_i - \langle P_{L_\delta}(x), e_i\rangle}{|x - P_{L_\delta}(x)|}\right) \\
&= \frac{1}{|x - P_{L_\delta}(x)|}\frac{\partial}{\partial x_j}(x_i - \langle P_{L_\delta}(x), e_i\rangle) + (x_i - \langle P_{L_\delta}(x), e_i\rangle)\frac{\partial}{\partial x_j}\left(\frac{1}{|x - P_{L_\delta}(x)|}\right) \\
&= \frac{1}{\tilde{g}_C(x)}\left(\delta_{ij} - \frac{\partial}{\partial x_j}\langle P_{L_\delta}(x), e_i\rangle\right) - (x_i - \langle P_{L_\delta}(x), e_i\rangle)\frac{\langle \nabla \tilde{g}_C(x), e_j\rangle}{\tilde{g}_C^2(x)}.
\end{aligned}$$

Observing that the maps $F : x \mapsto \langle x, e_i\rangle$ and P_{L_δ} are non-expansive (see, for example, Section 4 of [28] for the latter result), denoting the differential of P_{L_δ} by DP_{L_δ} and using the trivial relations $|x_i - \langle P_{L_\delta}(x), e_i\rangle| \leq |x - P_{L_\delta}(x)| = \tilde{g}_C(x)$ and $|\nabla \tilde{g}_C(x)| = 1$, we obtain the following bound: for $x \in \mathbb{R}^J \setminus [L_\delta \cup \partial H_C]$,

$$\left|\frac{\partial^2 \tilde{g}_C}{\partial x_j \partial x_i}(x)\right| \leq \frac{1 + |\nabla F(P_{L_\delta}(x))||DP_{L_\delta}(x)|}{\tilde{g}_C(x)} + \frac{1}{\tilde{g}_C(x)} \leq \frac{3}{\tilde{g}_C(x)}. \tag{6.71}$$

Step 4. While the constructed function $\tilde{g}_C$ satisfies most of the desired properties, it is not $\mathcal{C}^2$ in a neighbourhood of ∂H_C. We shall use an approximation argument to overcome this problem. Since $\tilde{g}_C \in \mathcal{C}^1(\mathbb{R}^J \setminus L_\delta)$, by Lemma 2 on page 19 of [7], $D^\alpha \tilde{g}_C = D_w^\alpha \tilde{g}_C$ on $\mathbb{R}^J \setminus L_\delta$ whenever $|\alpha| = 1$. When $|\alpha| = 2$, we only know that $D^\alpha \tilde{g}_C$ exists on the open set $\mathbb{R}^J \setminus [L_\delta \cup \partial H_C]$ (this was established in Step 1). We now claim that $D^\alpha \tilde{g}_C$, $|\alpha| = 2$, serve as *weak* derivatives of second order on the larger domain $\mathbb{R}^J \setminus L_\delta$. Although for $|\alpha| = 2$, $D^\alpha \tilde{g}_C$ is defined only on $\mathbb{R}^J \setminus [L_\delta \cup \partial H_C]$, since ∂H_C is a set of Lebesgue measure zero and weak derivatives are only defined upto sets of measure zero, the claim makes sense. Moreover, in order to establish the claim, it clearly suffices to show that weak derivatives of second order exist on $\mathbb{R}^J \setminus L_\delta$. From the expression (6.68) for $\nabla \tilde{g}_C$ and the fact that P_{L_δ} is Lipschitz, we know that $D^\alpha g$ is absolutely continuous on $\mathbb{R}^J \setminus L_\delta$ when $|\alpha| = 1$. A simple integration by parts argument (along the lines, for example, of Lemma 9 of page 34 of [7]), combined with the fact that $D^\alpha \tilde{g}_C = D^\alpha_w \tilde{g}_C$ when $|\alpha| = 1$, then shows that $D^\alpha_w \tilde{g}_C$ exists on $\mathbb{R}^J \setminus L_\delta$ for $|\alpha| = 2$ and the claim follows.

The closure of Λ lies in $\mathbb{R}^J \setminus L_\delta^{\tilde{\eta}/2}$. The claim just proved, along with (6.71), then implies that when $|\alpha| = 2$,

$$\operatorname*{ess\,sup}_{x\in\Lambda} |D^\alpha_w \tilde{g}_C(x)| = \sup_{x\in\Lambda\setminus\partial H_C} |D^\alpha \tilde{g}_C(x)| \leq \frac{3}{\inf_{x\in\Lambda} \tilde{g}_C(x)} = \frac{6}{\tilde{\eta}}. \tag{6.72}$$

Therefore, for $|\alpha| = 2$, the essential supremum of $D^\alpha \tilde{g}_C = D^\alpha_w \tilde{g}_C$ on Λ is finite. Furthermore, $\tilde{g}_C$ is also uniformly bounded on Λ (by $\tilde{\lambda}$). Thus $\tilde{g}_C$ lies in $W^2_\infty(\Lambda)$, the Banach space of uniformly bounded functions on Λ for which all weak derivatives of second order exist, equipped with the norm

$$||f||_{W^2_\infty(\Lambda)} = \operatorname*{ess\,sup}_{x\in\Lambda} |f(x)| + \sum_{\alpha\in\mathbb{Z}^J_+:|\alpha|=2} \operatorname*{ess\,sup}_{x\in\Lambda} |D^\alpha_w f(x)|.$$

Although the space $C^\infty(\Lambda)$ is not dense in $W^2_\infty(\Lambda)$, by Theorem 1 on page 48 of [7] there exists a sequence $\{f_k\}$ of $C^\infty(\Lambda)$ functions such that for $k \in \mathbb{N}$, the following three properties hold:

1. $\sup_{x\in\Lambda} |f_k(x) - \tilde{g}_C(x)| \leq \dfrac{1}{k}$;
2. $\sup_{x\in\Lambda} |D^\alpha f_k(x) - D^\alpha \tilde{g}_C(x)| \leq \dfrac{1}{k}$ if $|\alpha| = 1$;
3. $|\sup_{x\in\Lambda} |D^\alpha f_k(x)| - \sup_{x\in\Lambda} |D^\alpha \tilde{g}_C(x)|| \leq \dfrac{1}{k}$ if $|\alpha| = 2$.

Combining this with (6.70), (6.72) and the fact that $|\nabla \tilde{g}_C(x)| = 1$ for all $x \in \Lambda$ (see (6.68)), it is clear that there exists a large enough integer $k > 1/\tilde{\varepsilon}_C$ such that $g_C \doteq f_k$ satisfies the properties of the lemma. ☐

In the proof of Theorem 5.7 given below, we show that a family of functions $\{g_{z,r}, z \in \mathcal{U}, r \in (0, r_z)\}$ that satisfy the necessary properties can be obtained as localized, scaled versions of the functions $g_C, C \in \mathcal{P}(\mathcal{U})$, constructed in the last lemma. The next two lemmas will be used in the proof in order to characterize the geometry of the supports of these localized functions $\{g_{z,r}\}$ (see Figure 3 for an illustration).

Lemma 6.3. *For $C \in \mathcal{P}(\mathcal{U})$, let $\delta_C > 0$ satisfy (6.63) and let q_C be a unit vector in K_C. Then there exists $\lambda_C > 0$ such that for every $z \in F_C$ and $r \in (0, r_z)$, the set*

$$M_C(z, r) \doteq z + \lambda_C r q_C + L_{C,\delta_C} \tag{6.73}$$

satisfies the following three properties:

1. *there exists $\eta_C \in (0, \lambda_C)$ such that $M^{\eta_C r}(z, r) \cap G = \emptyset$;*
2. *$M_C^{3\lambda_C r}(z, r) \cap G \subset N_r(z) \cap G$;*

3. for every $\varepsilon > 0$, $\tilde{z} \in F_C$ and $\tilde{r} \in (0, r_{\tilde{z}})$,

$$x \in [M_C^{\varepsilon r}(z,r) - z] \Leftrightarrow \frac{\tilde{r}x}{r} \in \left[M_C^{\varepsilon \tilde{r}}(\tilde{z}, \tilde{r}) - \tilde{z}\right].$$

Proof. Fix $C \in \mathcal{P}(\mathcal{U})$, $\delta_C > 0$ and $q_C \in K_C$ as in the statement of the lemma. Since G is a convex polyhedron and $n_i, i \in C$, are inward normals to G at z, any $x \in G$ satisfies

$$\min_{i \in C} \langle n_i, x - z \rangle \geq 0.$$

Along with (6.66), this shows that the boundary of G separates $z + L_{C,\delta_C}$ from the interior of G. More precisely, for $z \in F_C$, we see that

$$(z + L_{C,\delta_C}) \cap G = \{z\} \tag{6.74}$$

and so the (minimal) angle ϕ_C between the closed convex cone $z + L_{C,\delta_C}$ and the closed polyhedron G at z satisfies $0 < \phi_C < \pi/2$. Therefore for any $r \in (0, r_z)$,

$$\left\{x \in \mathbb{R}^J : d(x, z + L_{C,\delta_C}) \leq \frac{r}{2} \sin \phi_C \right\} \cap \partial G \subset N_r(z) \cap \partial G.$$

Define $\lambda_C \doteq \sin \phi_C / 6 \in (0, 1/3)$. Then the last display implies that for every $r \in (0, r_z)$,

$$\left\{x \in \mathbb{R}^J : d(x, z + L_{C,\delta_C}) \leq 3\lambda_C r \right\} \cap G \cap \partial N_r(z) = \emptyset. \tag{6.75}$$

It is clear from the definition that λ_C is independent of $z \in F_C$ (since the angle ϕ_C does not depend on $z \in F_C$) and thus (6.75) holds for all $z \in F_C$ and $r \in (0, r_z)$.

Now let $M_C(z, r)$ be the cone L_{C,δ_C} shifted to have its vertex at $z + \lambda_C r q_C$, as defined in (6.73). Then, $z + \lambda_C r q_C$ lies inside the cone $z + L_{C,\delta_C}$ because $q_C \in K_C \subset L_{C,\delta_C}$ and $\lambda_C r > 0$. Since $z + L_{C,\delta_C}$ is a convex cone, it follows that

$$M_C(z,r) \subset z + L_{C,\delta_C} \quad \text{and} \quad M_C(z,r) \cap \{z\} = \emptyset.$$

Hence we infer that

$$M_C^{3\lambda_C r}(z,r) \doteq \{x : d(x, M_C(z,r)) \leq 3\lambda_C r\} \subset \{x : d(x, z + L_{C,\delta_C}) \leq 3\lambda_C r\}.$$

The last three displays together with (6.74) show that

$$M_C(z,r) \cap G \cap N_r(z) = \emptyset \quad \text{and} \quad M_C^{3\lambda_C r}(z,r) \cap G \cap \partial N_r(z) = \emptyset. \tag{6.76}$$

Together with the fact that $z \in M_C^{3\lambda_C r}(z,r)$, the second equality above establishes property 2 of the lemma. Moreover, since $M_C(z,r)$ and $G \cap N_r(z)$ are both closed, from the first relation of the last display and property 2, it is clear that property 1 of the lemma also holds.

It only remains to prove property 3. For $z \in F_C$, $x \in M_C^{\varepsilon r}(z,r) - z$ if and only if there exists v with $|v| \leq \varepsilon r$ and $u \in L_{C,\delta_C}$ such that $x = v + \lambda_C r q_C + u$. This implies that $\tilde{r}x/r = \tilde{r}v/r + \lambda_C \tilde{r} q_C + \tilde{r}u/r = \tilde{v} + \lambda_C \tilde{r} q_C + \tilde{u}$, where $|\tilde{v}| = |\tilde{r}v/r| \leq \varepsilon \tilde{r}$ and $\tilde{u} = \tilde{r}u/r$ lies in L_{C,δ_C} since L_{C,δ_C} is a cone with vertex at 0, which concludes the proof. □

We now identify the sets Q_C^-, Q_C^+, $C \in \mathcal{P}(\mathcal{U})$, that arise in Theorem 5.7.

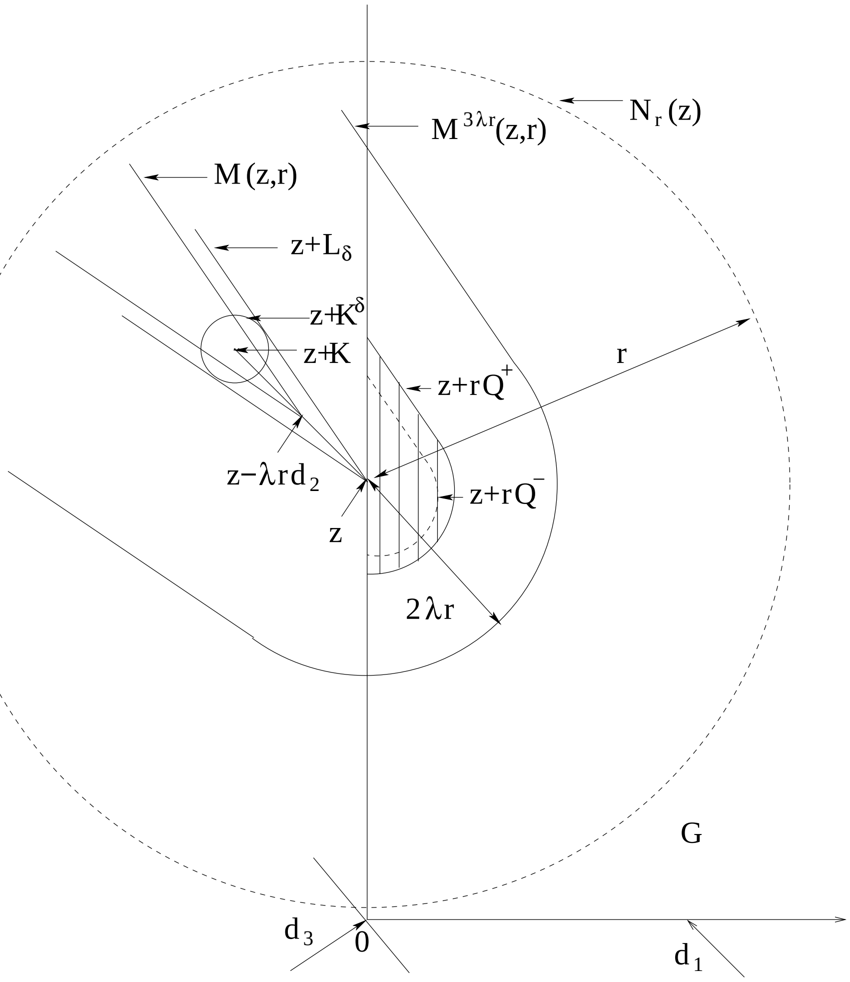

Figure 3: Construction of $g_{z,r}$ for the 2-dimensional GPS ESP

Lemma 6.4. *Fix $C \in \mathcal{P}(\mathcal{U})$, let $\{M_C(z,r), z \in F_C, r \in (0,r_z)\}$ be as in Lemma 6.3. For any $\alpha \in (0,1)$, there exist closed, convex sets Q_C^- and Q_C^+ such that $Q_C^- \subset (Q_C^+)^\circ \subset N_1(0)$ and for every $z \in F_C$ and $r \in (0,r_z)$,*

$$M_C^{2\lambda_C r}(z,r) \cap G = z + rQ_C^+ \qquad M_C^{(1+\alpha)\lambda_C r}(z,r) \cap G = z + rQ_C^- \tag{6.77}$$

and

$$z \in \operatorname{rint}[(z + r'Q_C^-) \cap \partial G]. \tag{6.78}$$

Proof. Fix $C \in \mathcal{P}(\mathcal{U})$, choose a particular $z' \in F_C$ and $r' \in (0, r_{z'})$, and define Q_C^- and Q_C^+ using (6.77), with z and r replaced by z' and r', respectively. Since the set $M_C^{\theta\lambda_C r'}(z',r')$, for $\theta = \alpha, 2$, and the set G are closed and convex, clearly Q_C^+ and Q_C^- are also closed and convex. The inclusion $Q_C^- \subset (Q_C^+)^\circ$ also follows directly from the definition and the fact that $1+\alpha < 2$. Moreover, since property 2 of Lemma 6.3 implies $M_C^{2\lambda_C r'}(z',r') \cap G \subset N_{r'}(z')$, it follows that $z' + r'Q_C^+ \subset N_{r'}(z')$ or, equivalently, that $Q_C^+ \subset N_1(0)$.

Property 3 of Lemma 6.3, the fact that G is a cone and the property that for $z \in F_C$ and $r < r_z$, $d(\mathcal{V}, M_C^{2\lambda_C r}(z,r) \cap G) > d(\mathcal{V}, \partial N_r(z)) > 0$ (where the latter inequality holds since $r < r_z < d(z,\mathcal{V})$), then show that (6.77) is applicable for every $z \in F_C$ and $r \in (0,r_z)$. In addition, since $d(z, M_C(z,r)) = \lambda_C r$ and $\alpha > 0$, it follows that for every $r \in (0,r_z)$,

$$z \in \operatorname{rint}[M_C^{(1+\alpha)\lambda_C r}(z,r) \cap \partial G] = \operatorname{rint}[(z + rQ_C^-) \cap \partial G],$$

which proves (6.78). □

The sets Q_C^+, Q_C^- and $M_C(z,r)$ for the case of the two-dimensional GPS ESP are illustrated in Figure 3. We now combine the above results to establish Theorem 5.7. For $C \in \mathcal{P}(\mathcal{U})$, let $\delta_C > 0$ satisfy (6.63) and let $q_C \in K_C$, $\eta_C \in (0,1)$, $\lambda_C \in (\eta_C/2, \eta_C)$, and $\{M_C(z,r), z \in F_C, r \in (0,r_z)\}$ be as in Lemma 6.3. Choose $\alpha \in (0,1/3)$, and let Q_C^-, Q_C^+ be the corresponding sets described in Lemma 6.4. Lastly, with the choice of $\tilde{\eta}_C = \eta_C$, $\tilde{\lambda}_C = 3\lambda_C$ and $\tilde{\varepsilon}_C = \alpha\lambda_C/2$, let g_C, Λ_C and $\theta_C > 0$ be as in Lemma 6.2.

For $z \in \mathcal{U}$ and $r \in (0,r_z)$, we now construct $g_{z,r}$ as a suitably scaled and localized version of the function $g_{I(z)}$, which vanishes on $\mathcal{V} = \{0\}$, while still maintaining the smoothness and derivative properties of $g_{I(z)}$. First define $\tilde{\alpha} \doteq 1 + 3\alpha/2$ and $\tilde{\kappa} \doteq 2 - \alpha/2$ and note that $\tilde{\alpha} < \tilde{\kappa}$. Then let h_C be a non-decreasing $\mathcal{C}^\infty(\mathbb{R}_+)$ function that satisfies

$$\begin{aligned} h_C'(s) &\geq 0 \quad \text{for } s \in [0,\infty) \\ h_C'(s) &\geq 1 \quad \text{for } s \in (0, \tilde{\alpha}\lambda_C) \\ h_C(s) &= 0 \quad \text{for } s \in [\tilde{\kappa}\lambda_C, \infty). \end{aligned} \tag{6.79}$$

For $C \in \mathcal{P}(\mathcal{U})$, if $z \in F_C$ and $r \in (0,r_z)$, define

$$\Omega_C(z,r) \doteq \left(M_C^{3\lambda_C r}\right)^\circ \setminus M_C^{\eta_C r} \quad \text{for } z \in F_C \text{ and } r \in (0,r_z) \tag{6.80}$$

and let $g_{z,r} : \mathbb{R}^J \to \mathbb{R}_+$ be given by

$$g_{z,r}(x) \doteq \begin{cases} r h_C\left(g_C\left(\dfrac{x - \lambda_C q_C - z}{r}\right)\right) & \text{if } x \in \Omega_C(z,r), \\ 0 & \text{otherwise.} \end{cases} \tag{6.81}$$

We show below that the functions $\{g_{z,r}, z \in F_C, r \in (0, r_z)\}$ above satisfy the properties listed in Theorem 5.7.

Proof of Theorem 5.7. Define

$$A' \doteq \max_{C \in \mathcal{P}(\mathcal{U})} \sup_{s \in [0, 3\lambda_C]} \left[|h_C(s)| \vee |h'_C(s)| \vee |h''_C(s)| \right] \vee 2 \tag{6.82}$$

and $\theta \doteq \min_{C \in \mathcal{P}(\mathcal{U})} \theta_C$ and note that $A' < \infty$ and $\theta > 0$. Since by (6.79), $h_C(s) = 0$ for $s \geq 3\lambda_C \geq \tilde{\kappa}\lambda_C$, the definition of A' in (6.82) immediately implies property 3 of the theorem.

Fix $C \in \mathcal{P}(\mathcal{U}), z \in F_C$ and $r \in (0, r_z)$, let $g_{z,r}$ be defined as in (6.81) and for the rest of the proof, write h for h_C, L_δ for L_{C,δ_C}, q for q_C, η for η_C, λ for λ_C, $\tilde{\lambda}$ for $\tilde{\lambda}_C$, $\tilde{\eta}$ for $\tilde{\eta}_C$, $\tilde{\varepsilon}$ for $\tilde{\varepsilon}_C$, Λ for Λ_C and $\Omega(z, r)$ for $\Omega_C(z, r)$. Note that since $\tilde{\lambda} = 3\lambda$ and $\tilde{\eta} = \eta$,

$$x \in \Omega(z, r) \Rightarrow \frac{x - \lambda q - z}{r} \in \left(L_C^{\tilde{\lambda}} \right)^\circ \setminus L_C^{\tilde{\eta}} = \Lambda. \tag{6.83}$$

This guarantees that whenever $x \in \Omega(z, r)$, $(x - \lambda q - z)/r$ lies in the domain Λ of g_C and so the function $g_{z,r}$ is well-defined.

Now, observe that property 4 of Lemma 6.2, the identity $\tilde{\kappa} + \tilde{\varepsilon}/\lambda = \tilde{\kappa} + \alpha/2 = 2$, the definitions (6.79), (6.80), (6.81) and (6.77) of $\Omega(z, r)$, h, $g_{z,r}$ and Q_C^+, respectively, and properties 1 and 2 of Lemma 6.3, when combined, yield the relation

$$\begin{aligned}
&\operatorname{supp}[g_{z,r}] \cap G \\
&\quad \subseteq \left\{ x \in \mathbb{R}^J : g_C \left(\frac{x - \lambda q - z}{r} \right) \leq \tilde{\kappa}\lambda \right\} \cap G \\
&\quad \subset \left\{ x \in \mathbb{R}^J : d \left(\frac{x - \lambda q - z}{r}, L_\delta \right) \leq \tilde{\kappa}\lambda + \tilde{\varepsilon} \right\} \cap G \\
&\quad = \left\{ x \in \mathbb{R}^J : d(x - \lambda q - z, L_\delta) \leq 2\lambda r \right\} \cap G \\
&\quad = M_C^{2\lambda r}(z, r) \cap G \\
&\quad = z + r Q_C^+ \\
&\quad \subset \Omega(z, r) \cap G.
\end{aligned} \tag{6.84}$$

The inclusion $\operatorname{supp}[g_{z,r}] \cap G \subset \Omega(z, r) \cap G$, the relation (6.83) and the fact that $h \in \mathcal{C}^\infty(\mathbb{R}_+)$ and $g_C \in \mathcal{C}^\infty(\Lambda)$, imply that $g_{z,r} \in \mathcal{C}^\infty(\mathbb{R}^J)$. In addition, since $\Omega(z, r) \cap G \subset N_r(z) \cap G$, the inclusion also implies that for $r < r_z$,

$$x \in \Omega(z, r) \cap G \quad \Rightarrow \quad d^1(x) \subseteq d^1(z) \tag{6.85}$$

and because $N_r(z) \cap \mathcal{V} = \emptyset$, that $g_{z,r}(x) = \nabla g_{z,r}(x) = 0$ for $x \in \mathcal{V}$. The last three assertions show that $g_{z,r}$ satisfies properties 1 and 2 of Theorem 5.7.

Now for $x \in \mathbb{R}^J$,

$$\nabla g_{z,r}(x) = h' \left(g_C \left(\frac{x - \lambda q - z}{r} \right) \right) \nabla g_C \left(\frac{x - \lambda q - z}{r} \right). \tag{6.86}$$

Property 2 of Lemma 6.2, along with the fact that $z \in F_C$ implies $d^1(z) \subset \{aK_C, a \leq -1\}$, shows that

$$\langle \nabla g_C(y), d \rangle \geq \theta \quad \forall d \in d^1(z), y \in \Lambda.$$

When combined with (6.83) and (6.85), this in turn shows that

$$\left\langle \nabla g_C\left(\frac{x-\lambda q-z}{r}\right), d\right\rangle \geq \theta \quad \forall d \in d^1(x), x \in \Omega(z,r)\cap \partial G, \tag{6.87}$$

while property (6.84) ensures that

$$\nabla g_C\left(\frac{x-\lambda q-z}{r}\right) = 0 \qquad \text{for } x \in G\setminus \Omega(z,r).$$

Property 5 of the theorem then follows from the last display, (6.86), (6.87) and the first inequality in (6.79). On the other hand, using the identity $\tilde{\alpha} = 1+\alpha+\alpha/2 = 1+\alpha+\tilde{\varepsilon}/\lambda$, and once again invoking property 4 of Lemma 6.2 and recalling the defintion (6.77) of Q_C^-, we observe that

$$\begin{aligned}
&\left\{x\in\mathbb{R}^J : g_C\left(\frac{x-\lambda q-z}{r}\right) \leq \tilde{\alpha}\lambda\right\}\cap G\\
&\quad = \left\{x\in\mathbb{R}^J : g_C\left(\frac{x-\lambda q-z}{r}\right) \leq (1+\alpha)\lambda+\tilde{\varepsilon}\right\}\cap G\\
&\quad \supset \left\{x\in\mathbb{R}^J : d\left(\frac{x-\lambda q-z}{r}, L_\delta\right) \leq (1+\alpha)\lambda\right\}\cap G\\
&\quad = M_C^{(1+\alpha)\lambda r}(z,r)\cap G\\
&\quad = z+rQ_C^-.
\end{aligned} \tag{6.88}$$

When combined with the second inequality in (6.79), (6.86) and (6.87), this implies property 4 of the theorem.

Differentiating $g_{z,r}$ twice and using the chain rule, we see that for $x\in\Omega(z,r)$,

$$\begin{aligned}
&\frac{\partial^2 g_{z,r}}{\partial x_i \partial x_j}(x)\\
&= \frac{1}{r}h''\left(g_C\left(\frac{x-\lambda qr-z}{r}\right)\right)\frac{\partial g_C}{\partial x_i}\left(\frac{x-\lambda qr-z}{r}\right)\frac{\partial g_C}{\partial x_j}\left(\frac{x-\lambda qr-z}{r}\right)\\
&\quad +h'\left(g_C\left(\frac{x-\lambda qr-z}{r}\right)\right)\frac{\partial^2 g_C}{\partial x_i\partial x_j}\left(\frac{x-\lambda qr-z}{r}\right).
\end{aligned}$$

Let $L'<\infty$ be an upper bound (independent of j,k) for the second derivatives in property 3 of Lemma 6.2. By property 4 of Lemma 6.2, we know that $\sup_{x\in\Lambda}|\nabla g_C(x)| \leq 1+\tilde{\varepsilon}\leq 2$. Along with (6.84) and the definition of A', this implies the bound

$$\sup_{x\in G}\left|\frac{\partial^2 g_{z,r}}{\partial x_i\partial x_j}(x)\right| = \sup_{x\in\Omega(z,r)\cap G}\left|\frac{\partial^2 g_{z,r}}{\partial x_i\partial x_j}(x)\right| \leq \frac{4A'}{r}+A'L' \leq \frac{4A'+A'L'r_z}{r}.$$

Hence properties 6 and 7 of the theorem are satisfied with $A(z) \doteq A'(4+L'r_z)$ and $\tilde{A}_R \doteq A'(4+L'R)$. This completes the proof of the theorem. ■

6.2 Proof of Lemma 5.8

In the last section, we constructed a family of $C^\infty(\mathbb{R}^J)$ functions $\{g_{z,r} : z\in\mathcal{U}, r\in(0,r_z)\}$, with each $g_{z,r}$ satisfying certain gradient and second derivative properties in a neighbourhood of z. Lemma 5.8 below shows that a countable set S and scalars $\rho_z, z \in S$, can be chosen such that any $x \in G$ lies in the support of at most a finite number (independent of x) of functions $g_{z,\rho_z}, z \in S$, and the sets $z + \rho_z Q^-_{I(z)}, z \in S$, cover $N_r(\mathcal{V}) \setminus \mathcal{V}$. This ensures that the function g defined as the countable sum $\sum_{z\in S} g_{z,\rho_z}$ is well-defined on G and satisfies the necessary derivative conditions on all of G. Although the notation in the proof of Lemma 5.8 is a bit involved, the basic idea behind the proof is quite simple. One first identifies a finite number of points z on a hyperplane H_s a distance s away from $\mathcal{V} = \{0\}$, and associated scalars ρ_z such that the union of the corresponding neighbourhoods covers the intersection of G with the fattening of a hyperplane H_s (see the claim below). Using scaling arguments one then identifies a corresponding finite number of points on a suitable translation of that hyperplane along its normal. The covering is then obtained by taking the union of the associated (finite number) of neighbourhoods in each of a countable number of hyperplanes $H_{s_j}, s_j \downarrow 0$.

Proof of Lemma 5.8. By assumption, G is a convex cone with vertex at the origin and angle less than π. Thus there exists $v \in G^\circ$ with $|v| = 1$, such that the maximum angle ω between v and any $x \in \partial G$ is less than $\pi/2$. This implies that there exists $\zeta = \cos\omega > 0$ such that

$$|x| < \frac{\langle x, v\rangle}{\zeta} \quad \text{for every } x \in G \setminus \{0\}. \tag{6.89}$$

Now for $s > 0$, let H_s be the hyperplane defined by

$$H_s \doteq \{x \in \mathbb{R}^J : \langle x, v\rangle = s\}$$

and for $0 \leq s < \tilde{s} < \infty$ define the "slabs" $H[s, \tilde{s}]$ and $H(s, \tilde{s}]$ to be

$$H[s, \tilde{s}] \doteq \{x : s \leq \langle x, v\rangle \leq \tilde{s}\} \quad \text{and} \quad H(s, \tilde{s}] \doteq \{x : s < \langle x, v\rangle \leq \tilde{s}\}.$$

For $C \in \mathcal{P}(\mathcal{U})$, let Q^+_C, Q^-_C be the closed, bounded, convex sets with $0 \in (Q^-_C)^\circ \subset (Q^+_C)^\circ$ specified in the statement of the lemma. Given $0 < r < R < \infty$, choose $s_1 \in (0, R)$ such that

$$G \cap N_r(0) \subset G \cap H[0, s_1] \subset [N_R(0)]^\circ. \tag{6.90}$$

Also, define

$$\kappa \doteq \sup\{\rho > 0 : z + \rho Q^+_{I(z)} \subset N_R(0) \text{ for every } z \in H_{s_1} \cap G\}, \tag{6.91}$$

where we recall that $I(z) \doteq \{i \in \mathbb{I} : \langle z, n_i\rangle = 0\}$.

We now show that the lemma is a consequence of the following claim, and defer the proof of the claim to the end.

Claim. There exist $0 < \beta < \gamma < 1$, a finite set $S_1 = \{z^{(1)}_i, i = 1, \ldots, L\} \subset H_{s_1}$ with associated scalars $\rho_{z^{(1)}_i}$, $i = 1, \ldots, L$, such that the sets defined for $i = 1, \ldots, L$, by

$$Q^+_{z^{(1)}_i} \doteq z^{(1)}_i + \rho_{z^{(1)}_i} Q^+_{I(z^{(1)}_i)} \quad \text{and} \quad Q^-_{z^{(1)}_i} \doteq z^{(1)}_i + \rho_{z^{(1)}_i} Q^-_{I(z^{(1)}_i)} \tag{6.92}$$

satisfy the following two properties:

1. $H[s_1(1-\beta), s_1(1+\beta)] \cap \partial G \subset [\cup_{z\in S_1} Q^-_z] \cap \partial G$;

2. $[\cup_{z\in S_1} Q_z^+]\cap G \subset H[s_1(1-\gamma), s_1(1+\gamma)]\cap G$.

Suppose the claim is true. Then for $k=2,3,\ldots$, and $i=1,\ldots,L$, define

$$z_i^{(k)} \doteq (1-\beta)^{k-1} z_i^{(1)} \quad \text{and} \quad \rho_{z_i^{(k)}} \doteq (1-\beta)^{k-1}\rho_{z_i^{(1)}}, \tag{6.93}$$

let $S_k \doteq \{z_i^{(k)}, i=1,\ldots,L\}$ and define $Q^-_{z_i^{(k)}}$ and $Q^+_{z_i^{(k)}}$ as in (6.92), with (1) replaced everywhere by (k). Then, since G is a conical polyhedron it is clear that for $i=1,\ldots,L$, $I(z_i^{(k)}) = I(z_i^{(1)})$ and therefore we have

$$Q^-_{z_i^{(k)}} = (1-\beta)^{k-1} Q^-_{z_i^{(1)}} \quad \text{and} \quad Q^+_{z_i^{(k)}} = (1-\beta)^{k-1} Q^+_{z_i^{(1)}}.$$

Combining this with the claim it follows that for $k\in\mathbb{N}$,

$$H[s_1(1-\beta)^k, s_1(1-\beta)^{k-1}(1+\beta)]\cap\partial G \subset \left[\cup_{z\in S_k} Q_z^-\right]\cap\partial G \tag{6.94}$$

and

$$\left[\cup_{z\in S_k} Q_z^+\right]\cap G \subset H[s_1(1-\beta)^{k-1}(1-\gamma), s_1(1-\beta)^{k-1}(1+\gamma)]\cap G. \tag{6.95}$$

Define $S \doteq \cup_{k\in\mathbb{N}} S_k$. Since each S_k has L elements, S is countable.
Let n be the smallest integer such that

$$n > \log[(1-\gamma)/(1+\gamma)]/\log(1-\beta). \tag{6.96}$$

Fix $k\in\mathbb{N}$, $z\in S_k$ and $x\in Q_z^+$ for this paragraph. We first show that

$$\tilde{z} \notin \bigcup_{j=(k-n+1)\vee 1}^{k+n-1} S_j \quad \Rightarrow \quad x\notin Q^+_{\tilde{z}}. \tag{6.97}$$

Indeed, note that (6.95) implies that

$$x\in H[s_1(1-\beta)^{k-1}(1-\gamma), s_1(1-\beta)^{k-1}(1+\gamma)]\cap G.$$

If $i\geq k+n$ then $(i-1)-(k-1)\geq n$ and so (6.96) yields the inequality

$$(1-\beta)^{i-1}(1+\gamma) < (1-\beta)^{k-1}(1-\gamma).$$

The inclusion (6.95) then implies that $x\notin \cup_{y\in S_i} Q_y^+$. Likewise, if $1\leq i\leq (k-n)\vee 1$ then for $k\geq n+1$, $(k-1)-(i-1)\geq n$, and so (6.96) implies that

$$(1-\beta)^{k-1}(1+\gamma) < (1-\beta)^{i-1}(1-\gamma).$$

Once again, we have $x\notin\cup_{y\in S_i} Q_y^+$ by (6.95) and so (6.97) follows. Since each S_i has L elements, this establishes property 1 with $N=2(n-1)L$. Next, observe that since $z\in S_k$, from (6.95) and (6.93) we obtain

$$\frac{\langle z,\nu\rangle}{\rho_z} = \frac{(1-\beta)^{k-1}s_1}{(1-\beta)^{k-1}\rho^{(1)}_{I(z)}} \leq \frac{s_1}{\rho_*}$$

where $\rho_* \doteq \min_{z\in S_1} \rho^z > 0$. Moreover, since $x \in Q_z^+$ by (6.89) and (6.95), we have

$$d(x,\mathcal{V}) = |x| \le \frac{\langle x,\nu\rangle}{\zeta} \le \frac{(1+\gamma)\langle z,\nu\rangle}{\zeta} \le \frac{(1+\gamma)s_1\rho_z}{\rho_*\zeta},$$

which shows that property 2 is satisfied with $\nu \doteq (1+\gamma)s_1/\rho_*\zeta < \infty$. Furthermore, property 3 is a simple consequence of the fact that since $Q_C^+ \subset N_1(0)$ for every $C \in \mathcal{P}(\mathcal{U})$ and $\rho_z < r_z$, we have $x \in Q_z^+ \subset N_{r_z}(z)$.

Lastly, note that since $\beta \in (0,1)$, we have

$$\cup_{k\in\mathbb{N}} H[s_1(1-\beta)^k, s_1(1-\beta)^{k-1}(1+\beta)] \cap \partial G = H(0, s_1(1+\beta)] \cap \partial G,$$

which is contained in $[\cup_{z\in S} Q_z^-] \cap \partial G$ by (6.94). Recalling that $\mathcal{V} = \{0\}$ and using (6.90) we then obtain

$$\mathcal{U} \cap N_r(\mathcal{V}) = [\partial G \setminus \{0\}] \cap N_r(0) \subset \partial G \cap H(0,s_1] \subset \partial G \cap [\cup_{z\in S} Q_z^-].$$

Also, the fact that $\rho_C < \kappa$ and the definition (6.91) of κ yield

$$[\cup_{z\in S} Q_z^+] \cap G \subset N_R(0) = N_R(\mathcal{V}).$$

This proves properties 4 and 5. To complete the proof of the lemma, it only remains to establish the claim.

Proof of Claim. For $j = 1,\ldots,J-1$, let $\mathcal{P}_j(\mathcal{U})$ be the collection of subsets corresponding to j-dimensional faces in ∂G:

$$\mathcal{P}_j(\mathcal{U}) \doteq \{C \in \mathcal{P}(\mathcal{U}) : \dim[F_C] = j\},$$

where $\dim[A]$ represents the dimension of the affine hull of A. Note that $\mathcal{U} = \cup_{j=1}^{J-1} \cup_{C\in\mathcal{P}_j(\mathcal{U})} F_C$. In order to establish the first property of the claim it therefore suffices to show that for every $j = 1,\ldots,J-1$, there exist $\beta_j \in (0,1)$ and a finite set of points $S_1^j \subset H_{s_1} \cap [\cup_{C\in\mathcal{P}_j(\mathcal{U})} F_C]$ and associated scalars $\rho_z \in (0,\kappa)$, $z \in S_1^j$, such that

$$H[s_1(1-\beta_j), s_1(1+\beta_j)] \cap \left[\cup_{C\in\mathcal{P}_j(\mathcal{U})} F_C\right] \subset \left[\cup_{z\in\cup_{i=1}^j S_1^i} Q_z^-\right] \cap \left[\cup_{C\in\mathcal{P}_j(\mathcal{U})} F_C\right] \tag{6.98}$$

where, as usual, $Q_z^- \doteq z + \rho_z Q_{I(z)}^-$. Indeed then setting $\beta \doteq \min_j \beta_j$ and $S_1 = \cup_{j=1}^J S_1^j$, the union of (6.98) over $j = 1,\ldots,J-1$ yields property 1 of the claim.

We shall prove (6.98) by induction on j. As explained below, it is easy to see that (6.98) holds when $j = 1$. For each $C \in \mathcal{P}_1(\mathcal{U})$, note that $H_{s_1} \cap F_C$ is equal to a point. Let $S_1^1 \doteq \{H_{s_1} \cap F_C : C \in \mathcal{P}(\mathcal{U})\}$ be the finite collection of these points as C varies over $\mathcal{P}(\mathcal{U})$. Now define

$$\rho^{(1)} \doteq \inf_{z\in S_1^1} r_z > 0, \tag{6.99}$$

and for each $z \in S_1^1$, set $\rho_z \doteq \rho^{(1)}$ and let Q_z^- be defined as above. By assumption, $0 \in (Q_C)^\circ$ and so $Q_z^- \cap F_C$ is a line segment containing z in its relative interior and for each $z \in S_1^1$, there exists a neighbourhood of z that is contained in Q_z^-. Hence there exists $\beta_1 \in (0,1)$ and $\varepsilon_1 > 0$ such that for every $C \in \mathcal{P}_1(\mathcal{U})$,

$$H[s_1(1-\beta_1), s_1(1+\beta_1)] \cap F_C \subset Q_z^- \cap F_C$$

(where $z = H_{s_1} \cap F_C$) and

$$H[s_1(1-\beta_1), s_1(1+\beta_1)] \cap N_{\varepsilon_1}(F_C) \subset Q_z^-. \tag{6.100}$$

Taking the union of the penultimate relation over $C \in \mathcal{P}(\mathcal{U})$ and $z \in S_1^1$ yields (6.98) for $j = 1$. Now suppose there exists $k \leq J-1$ such that for all $j \leq k-1$, there exist $\beta_j \in (0,1), \varepsilon_j > 0$ and a finite set of points $S_1^j \subset H_{s_1} \cap [\cup_{C \in \mathcal{P}_j(\mathcal{U})} F_C]$ such that (6.98) holds and for every $j \leq k-1$, the relation

$$H[s_1(1-\beta_j), s_1(1+\beta_j)] \cap \left[\cup_{C \in \mathcal{P}_j(\mathcal{U})} N_{\varepsilon_j}(F_C)\right] \subset \cup_{z \in \cup_{1 \leq j \leq k-1} S_1^j} Q_z^- \tag{6.101}$$

is satisfied. We will show that then (6.98) also holds for $j = k$ and (6.101) holds with $k-1$ replaced by k. Indeed, the argument in this case is analogous to the case $j = 1$. The difference now is that it is no longer true that the intersection of the k-dimensional faces with H_{s_1} are mutually disjoint and therefore the direct analog of (6.99) does not hold. Indeed, the intersections of multiple k-dimensional faces yield lower-dimensional faces. But by the second induction assumption (6.101), we have already covered a neighbourhood of the intersection of H_{s_1} with these lower-dimensional faces. Thus once these neighbourhoods are excised from the k-dimensional faces, we once again obtain disjoint sets and an inequality analogous to (6.99) will be true.. To make this reasoning precise, fix $C \in \mathcal{P}_k(\mathcal{U})$. By the definition of r_z, it follows that if

$$z \in \tilde{F}_C \doteq [F_C \cap H_{s_1}] \setminus [\cup_{1 \leq j \leq k-1} \cup_{\tilde{C} \in \mathcal{P}_j(\mathcal{U})} N_{\varepsilon_j}(F_{\tilde{C}})],$$

then $r_z > \min_{1 \leq j < k} \varepsilon_j$. Thus choose $0 < \rho^{(k)} < \min_{1 \leq j < k} \varepsilon_j \wedge \kappa$ (note that this is the analogue of (6.99) that we wanted). Since $\cup_{C \in \mathcal{P}_k(\mathcal{U})} \tilde{F}_C$ is bounded and $0 \in (Q_C^-)^\circ$ for every C, it follows that there exist $\beta_k \in (0,1)$, $\varepsilon_k > 0$ and a finite number of points $S_1^k = \cup_{C \in \mathcal{P}_k(\mathcal{U})} S_1^{k,C}$ with $S_1^{k,C} \subset \tilde{F}_C$ such that if $\rho_z \doteq \rho^{(k)}$ and (as before) $Q_z^- = z + \rho_z Q_{I(z)}^-$ for $z \in S_1^k$, then (6.98) is satisfied with $j = k$ and (6.101) holds with $k-1$ replaced by k. By induction, this shows that (6.98) holds for all $j = 1, \ldots, J-1$, and the proof of the first property of the claim is complete. The proof of the second property is analogous and therefore omitted. ■

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