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# Convergence of values in optimal stopping and convergence of optimal stopping times

François COQUET
CREST-ENSAI
Campus de Ker Lann
35170 Bruz, France
and LMAH
Université du Havre, France
E-mail: fcoquet@ensai.fr

Sandrine TOLDO
IRMAR
Antenne de Bretagne de l'ENS Cachan
Campus de Ker Lann
35170 Bruz, France
E-mail: sandrine.toldo@bretagne.ens-cachan.fr

**Abstract**

Under the hypothesis of convergence in probability of a sequence of càdlàg processes $(X^n)_n$ to a càdlàg process $X$, we are interested in the convergence of corresponding values in optimal stopping and also in the convergence of optimal stopping times. We give results under hypothesis of inclusion of filtrations or convergence of filtrations.

**Key words:** Values in optimal stopping, Convergence of stochastic processes, Convergence of filtrations, Optimal stopping times, Convergence of stopping times.

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# 1 Introduction

Let us consider a càdlàg process $X$. Denote by $\mathcal{F}^X$ its natural filtration and by $\mathcal{F}$ the right-continuous associated filtration ($\forall t, \mathcal{F}_t = \mathcal{F}^X_{t^+}$). Denote also by $\mathcal{T}_T$ the set of $\mathcal{F}-$stopping times bounded by $T$.
Let $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ a bounded continuous function. We define the value in optimal stopping of horizon $T$ for the process $X$ by:

$$\Gamma(T) = \sup_{\tau \in \mathcal{T}_T} \mathbb{E}[\gamma(\tau, X_\tau)].$$

We call a stopping time $\tau$ *optimal* whenever $\mathbb{E}[\gamma(\tau, X_\tau)] = \Gamma(T)$.

**Remark 1** As it is noticed in Lamberton and Pagès (1990), the value of $\Gamma(T)$ only depends on the law of $X$.

Throughout this paper, we will deal with the problem of stability of values in optimal stopping, and of optimal stopping times, under approximations of the process $X$. To be more precise, let us consider a sequence $(X^n)_n$ of processes which converges in probability to a limit process $X$. For all $n$, we denote by $\mathcal{F}^n$ the natural filtration of $X^n$ and by $\mathcal{T}^n_T$ the set of $\mathcal{F}^n-$stopping times bounded by $T$. Then, we define the values in optimal stopping $\Gamma_n(T)$ by $\Gamma_n(T) = \sup_{\tau \in \mathcal{T}^n_T} \mathbb{E}[\gamma(\tau, X^n_\tau)]$. The main aims of this paper are first to give conditions under which $(\Gamma_n(T))_n$ converges to $\Gamma(T)$, and second, when it is possible to find a sequence $(\tau_n)$ of optimal stopping times w.r.t. the $X_n$'s, to give further conditions under which the sequence $(\tau_n)$ converges to an optimal stopping time w.r.t $X$.

In his unpublished manuscript (Aldous, 1981), Aldous proved that if $X$ is quasi-left continuous and if extended convergence (in law) of $((X^n, \mathcal{F}^n))_n$ to $(X, \mathcal{F})$ holds, then $(\Gamma_n(T))_n$ converges to $\Gamma(T)$. In their paper (Lamberton and Pagès, 1990), Lamberton and Pagès obtained the same result under the hypothesis of weak extended convergence of $((X^n, \mathcal{F}^n))_n$ to $(X, \mathcal{F})$, quasi-left continuity of the $X^n$'s and Aldous' criterion of tightness for $(X^n)_n$.
Another way to study this problem is to consider the Snell envelopes associated to the processes. We recall that the Snell envelope of a process $Y$ is the smallest supermartingale larger than $Y$ (see e.g. (El Karoui, 1979)). The value in optimal stopping can be written as the value at 0 of a Snell envelope, as it is used for example in (Mulinacci and Pratelli, 1998), where a result of convergence of Snell envelopes for the Meyer-Zheng topology is proved.

Section 2 is devoted to convergence of values in optimal stopping. The main difficulty is to prove that $\Gamma(T) \geqslant \limsup \Gamma_n(T)$ and both papers (Aldous, 1981) and (Lamberton and Pagès, 1990) need weak extended convergence to prove it. We prove that this inequality actually holds whenever filtrations $\mathcal{F}^n$ are included into the limiting filtration $\mathcal{F}$, or when convergence of filtrations holds.
The main idea in our proof of the inequality $\Gamma(T) \geqslant \limsup \Gamma_n(T)$ is the following. We build a sequence $(\tau^n)$ of $\mathcal{F}^n-$stopping times bounded by $T$. Then, we extract a convergent subsequence of $(\tau^n)$ to a random variable $\tau$ and, at the same time, we compare $\mathbb{E}[\gamma(\tau, X_\tau)]$ and $\Gamma(T)$. This is carried out through two methods.

First, we enlarge the space of stopping times, by considering the randomized stopping times and the topology introduced in (Baxter and Chacon, 1977). Baxter and Chacon have shown that the space of randomized stopping times with respect to a right-continuous filtration with the associated topology is compact. We use this method in subsection 2.3 when holds the hypothesis of inclusion of filtrations $\mathcal{F}^n \subset \mathcal{F}$ (which means that $\forall t \in [0,T], \mathcal{F}_t^n \subset \mathcal{F}_t$). We point out that this assumption is simpler and easier to check than the extended convergence used in (Aldous, 1981) and (Lamberton and Pagès, 1990) or our own alternate hypothesis of convergence of filtrations.
However, when inclusion of filtrations does not hold, we follow an idea already used, in a slightly different way, in (Aldous, 1981) and in (Lamberton and Pagès, 1990), that is to enlarge the filtration $\mathcal{F}$ associated to the limiting process $X$. In subsection 2.4, we enlarge (as little as possible) the limiting filtration so that the limit $\tau^*$ of a convergent subsequence of the randomized $(\mathcal{F}^n)$ stopping times associated to the $(\tau^n)_n$ is a randomized stopping time for this enlarged filtration and we assume that convergence of filtrations (but not necessarily extended convergence) holds. In doing so, we do not need to introduce the prediction process which Aldous needed to define extended convergence. We also point out that convergence of processes joined to convergence of filtrations does not always imply extended convergence (see (Mémin, 2003) for a counter example). So the result given in this subsection is somewhat different from those of (Aldous, 1981) and (Lamberton and Pagès, 1990).

When convergence of values in optimal stopping holds, it is natural to wonder wether the associated optimal stopping times (when existing) do converge. Here again, the main problem is that, in general, the limit of a sequence of stopping times is not a stopping time. It may happen that the limit in law of a sequence of $\mathcal{F}-$stopping times is not the law of a $\mathcal{F}-$stopping time (see the example in (Baxter and Chacon, 1977)). In section 3, we shall give conditions, including again convergence of filtrations, under which the limit in probability of a sequence of $(\mathcal{F}^n)-$stopping times is a $\mathcal{F}-$stopping time (and not only a stopping time for a larger filtration). This caracterization will allow us to deduce a result of convergence of optimal stopping times when the limit process $X$ has independent increments.

Finally, in section 4, we give applications of the previous results to discretizations and also to financial models.

In what follows, we are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We fix a positive real $T$. Unless otherwise specified, every $\sigma$-field is supposed to be included in $\mathcal{A}$, every process will be indexed by $[0,T]$ and taking values in $\mathbb{R}$ and every filtration will be indexed by $[0,T]$. $\mathbb{D} = \mathbb{D}([0,T])$ denotes the space of càdlàg functions from $[0,T]$ to $\mathbb{R}$. We endow $\mathbb{D}$ with the Skorokhod topology.
For technical background about Skorokhod topology, the reader may refer to (Billingsley, 1999) or (Jacod and Shiryaev, 2002).

## 2 Convergence of values in optimal stopping

### 2.1 Statement of the results

The notion of convergence of filtrations has been defined in (Hoover, 1991) and, in a slightly different way, in (Coquet, Mémin and Słomiński, 2001). Here, we use the definition taken from the latter paper:

**Definition 2** *We say that $(\mathcal{F}^n = (\mathcal{F}^n_t)_{t\in[0,T]})_n$ converges weakly to $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ if for every $A \in \mathcal{F}_T$, $(\mathbb{E}[1_A|\mathcal{F}^n_\cdot])_n$ converges in probability to $\mathbb{E}[1_A|\mathcal{F}_\cdot]$ for the Skorokhod topology. We denote $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.*

In the papers (Aldous, 1978) and (Aldous, 1989), Aldous has introduced the following Criterion for tightness :

$$\forall \varepsilon > 0, \lim_{\delta\downarrow 0} \limsup_{n\to+\infty} \sup_{\sigma,\nu\in\mathcal{T}^n_T, \sigma\leqslant\nu\leqslant\sigma+\delta} \mathbb{P}[|X^n_\sigma - X^n_\nu| \geqslant \varepsilon] = 0 \tag{1}$$

where $\mathcal{T}^n_T$ is the set of the stopping times for the natural filtration of the process $X^n$, bounded by $T$. This Criterion is a standard tool for functional limit theorems when the limit is quasi-left continuous. It happens to be at the heart of the following theorem, whose proof is the main purpose of this section.

Before giving the theorem, we recall the links proved by Aldous between quasi-left continuous process and his Criterion :

**Proposition 3** *Let $(X^n)$ and $X$ be càdlàg processes.*

1. *If $X^n \xrightarrow{\mathcal{L}} X$ and if Aldous's Criterion for tightness (1) is filled, then $X$ is quasi-left continuous.*
2. *If $(X^n, \mathcal{F}^n) \to (X, \mathcal{F})$ and if $X$ is quasi-left continuous, then Aldous's Criterion for tightness (1) is filled.*

In the second part of the previous Proposition, there is an assumption of extended convergence. This convergence has been introduced in (Aldous, 1981) with prediction processes. Then, Aldous proved a characterization of this convergence using conditionnal expectation. Here, we use this characterization as a definition :

**Definition 4** *Let $(X^n)$ and $X$ be càdlàg processes and their right continuous natural filtrations $(\mathcal{F}^n)$ and $\mathcal{F}$. We have extended convergence of $(X^n, \mathcal{F}^n)$ to $(X, \mathcal{F})$ if for every $k \in \mathbb{N}$, for every right-continuous bounded functions $\phi_1, \ldots, \phi_k : \mathbb{D} \to \mathbb{R}$, we have*

$$(X^n, \mathbb{E}[\phi_1(X^n)|\mathcal{F}^n], \ldots, \mathbb{E}[\phi_k(X^n)|\mathcal{F}^n]) \xrightarrow{\mathcal{L}} (X, \mathbb{E}[\phi_1(X)|\mathcal{F}], \ldots, \mathbb{E}[\phi_k(X)|\mathcal{F}])$$

*for the Skorokhod topology. We denote $(X^n, \mathcal{F}^n) \to (X, \mathcal{F})$.*

Now, let us give our main result.

**Theorem 5** *Let us consider a càdlàg process $X$ and a sequence $(X^n)_n$ of càdlàg processes. Let $\mathcal{F}$ be the right-continuous filtration associated to the natural filtration of $X$ and $(\mathcal{F}^n)_n$ the natural filtrations of the processes $(X^n)_n$. We assume that :*
*- $X^n \xrightarrow{\mathbb{P}} X$,*
*- Aldous' Criterion for tightness (1) is filled,*
*- either for all $n$, $\mathcal{F}^n \subset \mathcal{F}$, either $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.*
*Then, $\Gamma_n(T) \xrightarrow[n\to\infty]{} \Gamma(T)$.*

The proof of Theorem 5 will be carried out through two steps in next subsections:
- Step 1: show that $\Gamma(T) \leqslant \liminf \Gamma_n(T)$ in subsection 2.2,
- Step 2: show that $\Gamma(T) \geqslant \limsup \Gamma_n(T)$ in subsections 2.3 and 2.4.

Let us give at once an extension of Theorem 5 which will prove useful for the application to finance in Section 4.

**Corollary 6** *Let $(\gamma^n)_n$ be a sequence of continuous bounded functions on $[0,T]\times\mathbb{R}$ which uniformly converges to a continuous bounded function $\gamma$. Let $X$ be a càdlàg process and $(X^n)_n$ a sequence of càdlàg processes. Let $\mathcal{F}$ be the right-continuous filtration of the process $X$ and $\mathcal{F}^n$ the natural filtration of $X^n$. We assume that :*
*- $X^n \xrightarrow{\mathbb{P}} X$,*
*- Aldous' Criterion for tightness (1) is filled,*
*- either for all $n$, $\mathcal{F}^n \subset \mathcal{F}$, either $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.*
*We consider the values in optimal stopping defined by:*

$$\Gamma(T) = \sup_{\tau\in\mathcal{T}_T} \mathbb{E}[\gamma(\tau, X_\tau)] \quad \textit{and} \quad \Gamma_n(T) = \sup_{\tau^n\in\mathcal{T}_T^n} \mathbb{E}[\gamma^n(\tau^n, X^n_{\tau^n})].$$

*Then $\Gamma_n(T) \xrightarrow[n\to\infty]{} \Gamma(T)$.*

## 2.2 Proof of the inequality $\Gamma(T) \leqslant \liminf \Gamma_n(T)$

In this section, we give a lower semi-continuity result. The hypotheses are not the weakest possible, but will be sufficient to prove Theorem 5.

**Theorem 7** *Let us consider a càdlàg process $X$, the right-continuous filtration $\mathcal{F}$ associated to the natural filtration of $X$, a sequence of càdlàg processes $(X^n)_n$ and their natural filtrations $(\mathcal{F}^n)_n$. We suppose that $X^n \xrightarrow{\mathbb{P}} X$. Then $\Gamma(T) \leqslant \liminf \Gamma_n(T)$.*

Proof
We only give here the sketch of the proof, which is not very different from those in (Lamberton and Pagès, 1990) and (Aldous, 1981).

To begin with, we can prove that, if $\tau$ is a $\mathcal{F}^X-$stopping time bounded by $T$ and taking values in a discrete set $\{t_i\}_{i\in I}$ such that $\mathbb{P}[\Delta X_{t_i} \neq 0] = 0$, $\forall i$, and if we define $\tau^n$ by $\tau^n(\omega) = \min\{t_i : i \in \{j : \mathbb{E}[1_{A_j}|\mathcal{F}^n_{t_j}](\omega) > 1/2\}\}$, $\forall\omega$, where $A_i = \{\tau = t_i\}$, then, $(\tau^n)$ is a sequence of $(\mathcal{T}^n_T)$ such that $(\tau^n, X^n_{\tau^n}) \xrightarrow{\mathbb{P}} (\tau, X_\tau)$.

Let us then consider a finite subdivision $\pi$ of $[0,T]$ such that no fixed time of discontinuity of $X$ belongs to $\pi$. We denote by $\mathcal{T}_T^\pi$ the set of $\mathcal{F}$ stopping times taking values in $\pi$, and we define:

$$\Gamma^\pi(T) = \sup_{\tau\in\mathcal{T}_T^\pi} \mathbb{E}[\gamma(\tau, X_\tau)].$$

Applying the previous result to stopping times belonging to $\mathcal{T}_T^\pi$ shows that $\Gamma^\pi(T) \leqslant \liminf \Gamma_n(T)$.
At last, using an increasing sequence $(\pi^k)_k$ of subdivisions such that $|\pi^k| \xrightarrow[k\to+\infty]{} 0$ and such that $\mathbb{P}[\Delta X_s \neq 0] = 0\ \forall s \in \pi_k$, standard computations prove that $\Gamma^{\pi^k}(T) \xrightarrow[k\to+\infty]{} \Gamma(T)$, and Theorem 7 follows. □

## 2.3 Proof of the inequality $\Gamma(T) \geqslant \limsup \Gamma_n(T)$ if for every $n$, $\mathcal{F}^n \subset \mathcal{F}$

### 2.3.1 Randomized stopping times

The notion of randomized stopping times has been introduced in (Baxter and Chacon, 1977) and this notion has been used in (Meyer, 1978) under the french name "temps d'arrêt flous".

We are given a filtration $\mathcal{F}$. Let us denote by $\mathcal{B}$ the Borel $\sigma$-field on $[0,1]$. Then, we define the filtration $\mathcal{G}$ on $\Omega\times[0,1]$ such that $\forall t$, $\mathcal{G}_t = \mathcal{F}_t \times \mathcal{B}$. A map $\tau : \Omega\times[0,1] \to [0,+\infty]$ is called a randomized $\mathcal{F}-$stopping time if $\tau$ is a $\mathcal{G}-$stopping time. We denote by $\mathcal{T}^*$ the set of randomized stopping times and by $\mathcal{T}_T^*$ the set of randomized stopping times bounded by $T$. $\mathcal{T}$ is included in $\mathcal{T}^*$ and the application $\tau \mapsto \tau^*$, where $\tau^*(\omega,t) = \tau(\omega)$ for every $\omega$ and every $t$, maps $\mathcal{T}$ into $\mathcal{T}^*$. In the same way, $\mathcal{T}_T$ is included in $\mathcal{T}_T^*$.
On the space $\Omega\times[0,1]$, we build the probability measure $\mathbb{P}\otimes\mu$ where $\mu$ is Lebesgue's measure on $[0,1]$. In their paper (Baxter and Chacon, 1977), Baxter and Chacon define the convergence of randomized stopping times by the following:

$$\tau^{*,n} \xrightarrow{BC} \tau^* \ \text{ iff } \ \forall f \in \mathcal{C}_b([0,\infty]), \forall Y \in L^1(\Omega,\mathcal{F},\mathbb{P}), \mathbb{E}[Y f(\tau^{*,n})] \to \mathbb{E}[Y f(\tau^*)],$$

where $\mathcal{C}_b([0,\infty])$ is the set of bounded continuous functions on $[0,\infty]$.
This kind of convergence is a particular case of "stable convergence" as introduced in (Renyi, 1963) and studied in (Jacod and Mémin, 1981).

The main point for us here is, as it is shown in (Baxter and Chacon, 1977, Theorem 1.5), that the set of randomized stopping times for a right-continuous filtration is compact for Baxter and Chacon's topology (which is not true for the set of ordinary stopping times).

The following Proposition will be the main argument in the proof of Theorem 13 below.

**Proposition 8** *Let us consider a sequence of filtrations $(\mathcal{F}^n)$ and a right-continuous filtration $\mathcal{F}$ such that $\forall n$, $\mathcal{F}^n \subset \mathcal{F}$. Let $(\tau^n)_n$ be a sequence of $(\mathcal{T}_T^n)_n$. Then, there exists a randomized $\mathcal{F}-$stopping time $\tau^*$ and a subsequence $(\tau^{\varphi(n)})_n$ such that $\tau^{*,\varphi(n)} \xrightarrow{BC} \tau^*$ where for every $n$, $\tau^{*,n}(\omega,t) = \tau^n(\omega)$ $\forall\omega$, $\forall t$.*

Proof

For every $n$, $\mathcal{F}^n \subset \mathcal{F}$, $(\tau^n)_n$ is a sequence of $\mathcal{F}-$stopping times so, by definition, $(\tau^{*,n})_n$ is a sequence of randomized $\mathcal{F}-$stopping times. According to (Baxter and Chacon, 1977, Theorem 1.5), we can find a randomized $\mathcal{F}-$stopping time $\tau^*$ and a subsequence $(\tau^{\varphi(n)})_n$ such that $\tau^{*,\varphi(n)} \xrightarrow{BC} \tau^*$. □

Now, we define $X_{\tau^*}$ by $X_{\tau^*}(\omega, v) = X_{\tau^*(\omega,v)}(\omega)$, for every $(\omega, v) \in \Omega \times [0, 1]$. Then, we can prove the following Lemma:

**Lemma 9** *Let* $\Gamma^*(T) = \sup\limits_{\tau^* \in \mathcal{T}_T^*} \mathbb{E}[\gamma(\tau^*, X_{\tau^*})]$. *Then* $\Gamma^*(T) = \Gamma(T)$.

Proof

- $\mathcal{T}_T$ is included into $\mathcal{T}_T^*$, hence $\Gamma(T) \leqslant \Gamma^*(T)$.
- Let $\tau^* \in \mathcal{T}_T^*$. We consider, for every $v$, $\tau_v(\omega) = \tau^*(\omega, v), \forall \omega$.

For every $v \in [0, 1]$, for every $t \in [0, T]$,

$$\{\omega : \tau_v(\omega) \leqslant t\} \times \{v\} = \{(\omega, x) : \tau^*(\omega, x) \leqslant t\} \cap (\Omega \times \{v\}).$$

But, $\{(\omega, x) : \tau^*(\omega, x) \leqslant t\} \in \mathcal{F}_t \times \mathcal{B}$ because $\tau^*$ is a randomized $\mathcal{F}-$stopping time and $\Omega \times \{v\} \in \mathcal{F}_t \times \mathcal{B}$. So, $\{\omega : \tau_v(\omega) \leqslant t\} \times \{v\} \in \mathcal{F}_t \times \mathcal{B}$. Consequently,

$$\{\omega : \tau_v(\omega) \leqslant t\} \in \mathcal{F}_t.$$

Hence, for every $v$, $\tau_v$ is a $\mathcal{F}-$stopping time bounded by $T$. We have:

$$\begin{aligned}
\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] &= \int_\Omega \int_0^1 \gamma(\tau^*(\omega, v), X_{\tau^*(\omega,v)}(\omega)) d\mathbb{P}(\omega) dv \\
&= \int_0^1 \left( \int_\Omega \gamma(\tau^*(\omega, v), X_{\tau_v(\omega)}(\omega)) d\mathbb{P}(\omega) \right) dv \\
&= \int_0^1 \mathbb{E}[\gamma(\tau_v, X_{\tau_v})] dv \\
&\leqslant \Gamma(T) \ \text{ because, for every } v,\ \tau_v \in \mathcal{T}_T.
\end{aligned}$$

Taking the sup over $\tau^*$ in $\mathcal{T}_T^*$, we get $\Gamma^*(T) \leqslant \Gamma(T)$.

Lemma 9 is proved. □

The following proposition will also be useful:

**Proposition 10** *Let us consider a sequence* $(X^n)_n$ *of càdlàg adapted processes that converges in law to a càdlàg process X. Let* $(\tau^n)_n$ *be a sequence of stopping times such that the associated sequence* $(\tau^{*,n})_n$ *of randomized stopping times* ($\tau^{*,n}(\omega, t) = \tau^n(\omega)\ \forall \omega,\ \forall t$) *converges in law to a random variable V. We suppose that* $(\tau^{*,n}, X^n) \xrightarrow{\mathcal{L}} (V, X)$ *and that Aldous' Criterion (1) is filled. Then* $(\tau^{*,n}, X^n_{\tau^{*,n}}) \xrightarrow{\mathcal{L}} (V, X_V)$.

Proof

The proof that follows the lines of the proof of (Aldous, 1981, Corollary 16.23) is given in two steps.

Step 1 : $\mathbb{P}[\Delta X_V \neq 0] = 0$.
According to Skorokhod Representation Theorem, we can suppose that $(\tau^{*,n}, X^n) \xrightarrow{a.s.} (V, X)$. Let $(\Lambda^n)$ be a sequence of change of times associated to the almost sure convergence of $(X^n)$ to $X$. We have $\sup_t |\Lambda^n(t) - t| \xrightarrow{a.s.} 0$ and $\sup_t |X^n_{\Lambda^n(t)} - X_t| \xrightarrow{a.s.} 0$. Then,

$$\begin{aligned} |X^n_{\tau^{*,n}} - X_V| &\leqslant |X^n_{\tau^{*,n}} - X_{(\Lambda^n)^{-1}(\tau^{*,n})}| + |X_{(\Lambda^n)^{-1}(\tau^{*,n})} - X_V| \\ &\leqslant \sup_t |X^n_{\Lambda^n(t)} - X_t| + |X_{(\Lambda^n)^{-1}(\tau^{*,n})} - X_V| \\ &\xrightarrow{a.s.} 0 \end{aligned}$$

by choice of $(\Lambda^n)$ and because $\mathbb{P}[\Delta X_V \neq 0] = 0$. We have proved that $X^n_{\tau^{*,n}} \xrightarrow{a.s.} X_V$, so $(\tau^{*,n}, X^n_{\tau^{*,n}}) \xrightarrow{a.s.} (V, X_V)$. Hence, $(\tau^{*,n}, X^n_{\tau^{*,n}}) \xrightarrow{\mathcal{L}} (V, X_V)$.

Step 2 : $\mathbb{P}[\Delta X_V \neq 0] > 0$.
We can find a sequence $(\delta_k)_k$ that decreases to 0 and such that for every $k$, $\mathbb{P}[\Delta X_{V+\delta_k} \neq 0] = 0$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a uniformly continuous function.

$$\begin{aligned} |\mathbb{E}[f(\tau^{*,n}, X^n_{\tau^{*,n}}) - f(V, X_V)]| \leqslant\ & |\mathbb{E}[f(\tau^{*,n}, X^n_{\tau^{*,n}}) - f(\tau^{*,n} + \delta_k, X^n_{\tau^{*,n}+\delta_k})]| \\ & + |\mathbb{E}[f(\tau^{*,n} + \delta_k, X^n_{\tau^{*,n}+\delta_k}) - f(V + \delta_k, X_{V+\delta_k})]| \\ & + |\mathbb{E}[f(V + \delta_k, X_{V+\delta_k}) - f(V, X_V)]|. \end{aligned}$$

We have :
- $\forall k$, $\limsup_{n\to+\infty} \mathbb{E}[f(\tau^{*,n} + \delta_k, X^n_{\tau^{*,n}+\delta_k}) - f(V + \delta_k, X_{V+\delta_k})] = 0$ thanks to Step 1, so we have the convergence : $\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{E}[f(\tau^{*,n} + \delta_k, X^n_{\tau^{*,n}+\delta_k}) - f(V + \delta_k, X_{V+\delta_k})] = 0$.
- $\lim_{k\to+\infty} \mathbb{E}[f(V + \delta_k, X_{V+\delta_k}) - f(V, X_V)] = 0$ as $X_{V+\delta_k} \xrightarrow[k\to+\infty]{p.s.} X_V$ bacause $X$ is right-continuous. So, $\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{E}[f(V + \delta_k, X_{V+\delta_k}) - f(V, X_V)] = 0$.
- $\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{E}[f(\tau^{*,n}, X^n_{\tau^{*,n}}) - f(\tau^{*,n} + \delta_k, X^n_{\tau^{*,n}+\delta_k})] = 0$ according to Aldous' Criterion for tighness and because $(\delta_k)_k$ decreases to 0.
Proposition 10 is proved. □

**Remark 11** We point out that, in this proposition, Aldous' Criterion is filled for genuine -not randomized- stopping times.

**Proposition 12** *Let us consider a sequence $(X^n)_n$ of càdlàg adated processes converging in probability to a càdlàg process X. Let $(\tau^{*,n})_n$ be a sequence of randomized stopping times converging to the randomized stopping time $\tau$ under Baxter and Chacon's topology.*
*Then $(X^n, \tau^{*,n}) \xrightarrow{\mathcal{L}} (X, \tau^*)$.*

Proof
- As $(X^n)_n$ and $(\tau^{*,n})_n$ are tight, $((X^n, \tau^{*,n}))_n$ is tight for the product topology.
- We are now going to identify the limit throught finite-dimensional convergence.

Let $k \in \mathbb{N}$ and $t_1 < \ldots < t_k$ such that for every $i$, $\mathbb{P}[\Delta X_{t_i} \neq 0] = 0$. We are going to show that $(X^n_{t_1}, \ldots, X^n_{t_k}, \tau^{*,n}) \xrightarrow{\mathcal{L}} (X_{t_1}, \ldots, X_{t_k}, \tau^*)$.
Let $f : \mathbb{R}^k \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be bounded continuous functions.

$$\begin{aligned}
&|\mathbb{E}[f(X^n_{t_1}, \ldots, X^n_{t_k})g(\tau^{*,n})] - \mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^*)]| \\
\leqslant\ & |\mathbb{E}[(f(X^n_{t_1}, \ldots, X^n_{t_k}) - f(X_{t_1}, \ldots, X_{t_k}))g(\tau^{*,n})]| \\
&\quad + |\mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^{*,n})] - \mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^*)]| \\
\leqslant\ & \|g\|_\infty \mathbb{E}[|f(X^n_{t_1}, \ldots, X^n_{t_k}) - f(X_{t_1}, \ldots, X_{t_k})|] \\
&\quad + |\mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^{*,n})] - \mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^*)]|
\end{aligned}$$

But, $X^n \xrightarrow{\mathbb{P}} X$ and for every $i$, $\mathbb{P}[\Delta X_{t_i} \neq 0] = 0$ so $(X^n_{t_1}, \ldots, X^n_{t_k}) \xrightarrow{\mathbb{P}} (X_{t_1}, \ldots, X_{t_k})$. Moreover, $f$ is bounded continuous, so

$$\mathbb{E}[|f(X^n_{t_1}, \ldots, X^n_{t_k}) - f(X_{t_1}, \ldots, X_{t_k})|] \xrightarrow[n\to+\infty]{} 0.$$

On the other hand, by definition of Baxter and Chacon's convergence,

$$\mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^{*,n})] - \mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^*)] \xrightarrow[n\to+\infty]{} 0.$$

Then,

$$\mathbb{E}[f(X^n_{t_1}, \ldots, X^n_{t_k})g(\tau^{*,n})] - \mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^*)] \xrightarrow[n\to+\infty]{} 0.$$

Using a density argument, we can expand the previous result to continuous and bounded fonctions from $\mathbb{R}^{k+1}$ to $\mathbb{R}$. More precisely, for every $\varphi : \mathbb{R}^{k+1} \to \mathbb{R}$ continuous and bounded, we have:

$$\mathbb{E}[\varphi(X^n_{t_1}, \ldots, X^n_{t_k}, \tau^{*,n})] \xrightarrow[n\to+\infty]{} \mathbb{E}[\varphi(X_{t_1}, \ldots, X_{t_k}, \tau^*)].$$

It follows that $(X^n_{t_1}, \ldots, X^n_{t_k}, \tau^{*,n}) \xrightarrow{\mathcal{L}} (X_{t_1}, \ldots, X_{t_k}, \tau^*)$. At last, the tightness of the sequence $((X^n, \tau^{*,n}))_n$ and the finite-dimensional convergence on a dense set to $(X, \tau^*)$ imply $(X^n, \tau^{*,n}) \xrightarrow{\mathcal{L}} (X, \tau^*)$. $\square$

### 2.3.2 Application to the proof of the inequality $\limsup \Gamma_n(T) \leqslant \Gamma(T)$

We can now prove our first result about convergence of optimal values.

**Theorem 13** *Let us consider a càdlàg process $X$, its right-continuous filtration $\mathcal{F}$, a sequence $(X^n)_n$ of càdlàg processes and their natural filtrations $(\mathcal{F}^n)_n$. We suppose that :*
*- $X^n \xrightarrow{\mathbb{P}} X$,*
*- Aldous' Criterion for tightness (1) is filled,*
*- $\forall n$, $\mathcal{F}^n \subset \mathcal{F}$.*
*Then $\limsup \Gamma_n(T) \leqslant \Gamma(T)$.*

Proof

There exists a subsequence $(\Gamma_{\varphi(n)}(T))_n$ which converges to $\limsup \Gamma_n(T)$.

Let us fix $\varepsilon > 0$. We can find a sequence $(\tau^{\varphi(n)})_n$ of $(\mathcal{T}_T^{\varphi(n)})_n$ such that

$$\forall n,\ \mathbb{E}[\gamma(\tau^{\varphi(n)}, X^{\varphi(n)}_{\tau^{\varphi(n)}})] \geqslant \Gamma_{\varphi(n)}(T) - \varepsilon.$$

We consider the sequence $(\tau^{*,\varphi(n)})_n$ of randomized stopping times associated to $(\tau^{\varphi(n)})_n$: for every $n$, $\tau^{*,\varphi(n)}(\omega, t) = \tau^{\varphi(n)}(\omega)$, $\forall \omega$, $\forall t$. $\mathcal{F}^{\varphi(n)} \subset \mathcal{F}$ and $(\tau^{\varphi(n)})$ is a sequence of $\mathcal{F}^{\varphi(n)}-$stopping times bounded by $T$, so using Proposition 8, there exists a randomized $\mathcal{F}-$stopping time $\tau^*$ and a subsequence $(\tau^{\varphi\circ\psi(n)})$ such that $\tau^{*,\varphi\circ\psi(n)} \xrightarrow{BC} \tau^*$. Moreover $X^{\varphi\circ\psi(n)} \xrightarrow{\mathbb{P}} X$, so by Proposition 12, $(X^{\varphi\circ\psi(n)}, \tau^{*,\varphi\circ\psi(n)}) \xrightarrow{\mathcal{L}} (X, \tau^*)$. Then, using Proposition 10, we have: $(\tau^{*,\varphi\circ\psi(n)}, X^{\varphi\circ\psi(n)}_{\tau^{*,\varphi\circ\psi(n)}}) \xrightarrow{\mathcal{L}} (\tau^*, X_{\tau^*})$. Since $\gamma$ is continuous and bounded, we deduce:

$$\mathbb{E}[\gamma(\tau^{*,\varphi\circ\psi(n)}, X^{\varphi\circ\psi(n)}_{\tau^{*,\varphi\circ\psi(n)}})] \to \mathbb{E}[\gamma(\tau^*, X_{\tau^*})].$$

But, $\mathbb{E}[\gamma(\tau^{*,\varphi\circ\psi(n)}, X^{\varphi\circ\psi(n)}_{\tau^{*,\varphi\circ\psi(n)}})] = \mathbb{E}[\gamma(\tau^{\varphi\circ\psi(n)}, X^{\varphi\circ\psi(n)}_{\tau^{\varphi\circ\psi(n)}})]$ by definition of $(\tau^{*,n})$, and by construction of $\varphi$, $\mathbb{E}[\gamma(\tau^{\varphi\circ\psi(n)}, X^{\varphi\circ\psi(n)}_{\tau^{\varphi\circ\psi(n)}})] \geqslant \Gamma_{\varphi\circ\psi(n)}(T) - \varepsilon$. So,

$$\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \geqslant \lim \Gamma_{\varphi\circ\psi(n)}(T) - \varepsilon = \limsup \Gamma_n(T) - \varepsilon.$$

We hence have proved that for any $\varepsilon > 0$ we can find a randomized stopping time $\tau^*$ such that $\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \geqslant \limsup \Gamma_n(T) - \varepsilon$.

As by definition $\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \leqslant \Gamma^*(T)$ and $\varepsilon$ is arbitrary, it follows that $\Gamma^*(T) \geqslant \limsup \Gamma_n(T)$.

At last, recall that $\Gamma^*(T) = \Gamma(T)$ by Lemma 9 to conclude that $\Gamma(T) \geqslant \limsup \Gamma_n(T)$. $\square$

**Remark 14** We were able to prove the previous theorem, because we knew something about the nature of the limit of the subsequence of stopping times thanks to Proposition 8. If we remove the hypothesis of inclusion of filtrations $\mathcal{F}^n \subset \mathcal{F}, \forall n$, the limit of the subsequence needs no longer be a randomized $\mathcal{F}-$stopping time, and we cannot always compare $\mathbb{E}[\gamma(\tau^*, X_{\tau^*})]$ to $\Gamma^*(T)$.

However, the result of Theorem 13 remains true under other settings, as we shall prove in next subsection.

## 2.4 Proof of the inequality $\limsup \Gamma_n(T) \leqslant \Gamma(T)$ if $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$

**Theorem 15** *Let us consider a sequence of càdlàg processes $(X^n)_n$, their natural filtrations $(\mathcal{F}^n)_n$, a càdlàg process $X$ and its right-continuous natural filtration $\mathcal{F}$. We suppose that*
*- $X^n \xrightarrow{\mathbb{P}} X$,*
*- Aldous' Criterion for tightness (1) is filled,*
*- $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.*
*Then $\limsup \Gamma_n(T) \leqslant \Gamma(T)$.*

Proof

Our proof is more or less scheduled as the second part of the proof in (Aldous, 1981, Theorem

17.2). The main difference is that we do not need extended convergence in our theorem: instead, we use convergence of filtrations.

We can find a subsequence $(\Gamma_{\varphi(n)}(T))_n$ converging to $\limsup \Gamma_n(T)$.

Let us take $\varepsilon > 0$. There exists a sequence $(\tau^{\varphi(n)})_n$ of $(\mathcal{T}_T^{\varphi(n)})_n$ such that

$$\forall n, \mathbb{E}[\gamma(\tau^{\varphi(n)}, X^{\varphi(n)}_{\tau^{\varphi(n)}})] \geqslant \Gamma_{\varphi(n)}(T) - \varepsilon.$$

Let us consider the sequence $(\tau^{*,\varphi(n)})_n$ of associated randomized $\mathcal{F}^{\varphi(n)}-$stopping times like in 2.3.1. Taking the filtration $\mathcal{H} = (\bigvee_n \mathcal{F}^n) \vee \mathcal{F}$, $(\tau^{*,\varphi(n)})$ is a bounded sequence of randomized $\mathcal{H}-$stopping times. Then, using (Baxter and Chacon, 1977, Theorem 1.5), we can find a further subsequence (still denoted $\varphi$) and a randomized $\mathcal{H}-$stopping time $\tau^*$ ($\tau^*$ is not a priori a randomized $\mathcal{F}-$stopping time) such that

$$\tau^{*,\varphi(n)} \xrightarrow{BC} \tau^*.$$

Using Proposition 12, we obtain $(X^{\varphi(n)}, \tau^{*,\varphi(n)}) \xrightarrow{\mathcal{L}} (X, \tau^*)$. Then, Proposition 10 gives the convergence $(\tau^{\varphi(n)}, X^{\varphi(n)}_{\tau^{\varphi(n)}}) \xrightarrow{\mathcal{L}} (\tau^*, X_{\tau^*})$. So, $\mathbb{E}[\gamma(\tau^{\varphi(n)}, X^{\varphi(n)}_{\tau^{\varphi(n)}})] \xrightarrow[n\to+\infty]{} \mathbb{E}[\gamma(\tau^*, X_{\tau^*})]$. On the other hand, $\mathbb{E}[\gamma(\tau^{\varphi(n)}, X^{\varphi(n)}_{\tau^{\varphi(n)}})] \geqslant \Gamma_{\varphi(n)}(T) - \varepsilon$. So, letting $n$ go to infinity leads to

$$\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \geqslant \limsup \Gamma_n(T) - \varepsilon. \tag{2}$$

Our next step will be to prove the following

**Lemma 16**

$$\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \leqslant \Gamma^*(T).$$

PROOF

Let us consider the smaller right-continuous filtration $\mathcal{G}$ such that $X$ is $\mathcal{G}-$adapted and $\tau^*$ is a randomized $\mathcal{G}-$stopping time. It is clear that $\mathcal{F} \subset \mathcal{G}$. For every $t$, we have

$$\mathcal{G}_t \times \mathcal{B} = \bigcap_{s>t} \sigma(A \times B, \{\tau^* \leqslant u\}, A \in \mathcal{F}_s, u \leqslant s, B \in \mathcal{B}).$$

We consider the set $\tilde{\mathcal{T}}_T$ of randomized $\mathcal{G}-$stopping times bounded by $T$ and we define $\tilde{\Gamma}(T) = \sup_{\tilde{\tau}\in\tilde{\mathcal{T}}_T} \mathbb{E}[\gamma(\tilde{\tau}, X_{\tilde{\tau}})]$.

By definition of $\mathcal{G}$, $\tau^* \in \tilde{\mathcal{T}}_T$ so

$$\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \leqslant \tilde{\Gamma}(T). \tag{3}$$

In order to prove Lemma 16, we will use the following Lemma, which is an adaptation of (Lamberton and Pagès, 1990, Proposition 3.5) to our enlargement of filtration:

**Lemma 17** *If $\mathcal{G}_t \times \mathcal{B}$ and $\mathcal{F}_T \times \mathcal{B}$ are conditionally independent given $\mathcal{F}_t \times \mathcal{B}$ for every $t \in [0, T]$, then $\tilde{\Gamma}(T) = \Gamma^*(T)$.*

Proof

The proof is the same as the proof of (Lamberton and Pagès, 1990, Proposition 3.5) with $(\mathcal{F}_t \times \mathcal{B})_{t\in[0,T]}$ and $(\mathcal{G}_t \times \mathcal{B})_{t\in[0,T]}$ instead of $\mathcal{F}^Y$ and $\mathcal{F}$ and with a process $X^*$ such that for every $\omega$, for every $v \in [0,1]$, for every $t \in [0,T]$, $X^*_t(\omega, v) = X_t(\omega)$ instead of the process $Y$. □

Back to Lemma 16, we have to prove the conditional independence required in Lemma 17 which, according to (Brémaud and Yor, 1978, Theorem 3), is equivalent to the following assumption:

$$\forall t \in [0,T], \forall Z \in L^1(\mathcal{F}_T \times \mathcal{B}), \mathbb{E}[Z|\mathcal{F}_t \times \mathcal{B}] = \mathbb{E}[Z|\mathcal{G}_t \times \mathcal{B}]. \tag{4}$$

The main part of what is left in this subsection is devoted to show that the assumptions of Theorem 15 do imply (4), therefore fulfilling the assumptions needed to make Lemma 17 work. Note that in order to prove (4) (Aldous, 1981) and in (Lamberton and Pagès, 1990) use extended convergence, which needs not hold under the hypothesis of Theorem 15 (see (Mémin, 2003) for a counter-example).

Without loss of generality, we suppose from now on that $\tau^{*,n} \xrightarrow{BC} \tau^*$ instead of $\tau^{*,\varphi(n)} \xrightarrow{BC} \tau^*$. Moreover, as $X^n \xrightarrow{\mathbb{P}} X$ and Aldous' Criterion for tightness (1) is filled, using the results of (Aldous, 1981), $X$ is quasi-left continuous.

- As $\mathcal{F} \subset \mathcal{G}$, $\forall t \in [0,T]$, $\forall Z \in L^1(\mathcal{F}_T \times \mathcal{B})$, $\mathbb{E}_{\mathbb{P}\otimes\mu}[Z|\mathcal{F}_t \times \mathcal{B}]$ is $\mathcal{G}_t \times \mathcal{B}-$measurable.
- We shall show that $\forall t \in [0,T], \forall Z \in L^1(\mathcal{F}_T \times \mathcal{B}), \forall C \in \mathcal{G}_t \times \mathcal{B}$,

$$\mathbb{E}_{\mathbb{P}\otimes\mu}[\mathbb{E}_{\mathbb{P}\otimes\mu}[Z|\mathcal{F}_t \times \mathcal{B}]1_C] = \mathbb{E}_{\mathbb{P}\otimes\mu}[Z1_C].$$

Let $t \in [0,T]$ and $\varepsilon > 0$ be fixed, and take $Z \in L^1(\mathcal{F}_T \times \mathcal{B})$. By definition of $\mathcal{G}_t \times \mathcal{B}$, it suffices to prove that for every $A \in \mathcal{F}_t$, for every $s \leqslant t$ and for every $B \in \mathcal{B}$,

$$\begin{aligned}
&\int\int_{\Omega\times[0,1]} Z(\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d\mathbb{P}(\omega)dv \\
&\quad = \int\int_{\Omega\times[0,1]} \mathbb{E}_{\mathbb{P}\otimes\mu}[Z|\mathcal{F}_t \times \mathcal{B}](\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d\mathbb{P}(\omega)dv.
\end{aligned} \tag{5}$$

We first prove that (5) holds for $Z = 1_{A_1\times A_2}$, $A_1 \in \mathcal{F}_T$, $A_2 \in \mathcal{B}$.
We can find $l \in \mathbb{N}$, $s_1 < \ldots < s_l$ and a continuous bounded function $f$ such that

$$\mathbb{E}_{\mathbb{P}}[|1_{A_1} - f(X_{s_1}, \ldots, X_{s_l})|] \leqslant \varepsilon. \tag{6}$$

Then

$$\int\int |1_{A_1\times A_2}(\omega,v) - f(X_{s_1}(\omega), \ldots, X_{s_l}(\omega))1_{A_2}(v)|d\mathbb{P}(\omega)dv \leqslant \varepsilon.$$

Let us fix $A \in \mathcal{F}_t$. We can find $k \in \mathbb{N}$, $t_1 < \ldots < t_k \leqslant t$ and $H : \mathbb{R}^k \to \mathbb{R}$ bounded continuous such that

$$\mathbb{E}_{\mathbb{P}}[|1_A - H(X_{t_1}, \ldots, X_{t_k})|] \leqslant \varepsilon. \tag{7}$$

Let $u > t$ such that $\mathbb{P}[\Delta\mathbb{E}[f(X_{s_1}, \dots, X_{s_l})|\mathcal{F}_u] \neq 0] = 0$ and $\mathbb{P}[\tau^* = u] = 0$.
Fix $s \leqslant t$. We can find a bounded continuous function $G$ such that

$$\mathbb{E}_{\mathbb{P}\otimes\mu}[|1_{\{\tau^* \leqslant s\}} - G(\tau^* \wedge u)|] \leqslant \varepsilon. \tag{8}$$

$B \in \mathcal{B}$ and the set of continuous functions is dense into $L^1(\mu)$, so there exists $g : \mathbb{R} \to \mathbb{R}$ bounded continuous such that

$$\int |1_B(v) - g(v)|dv \leqslant \varepsilon. \tag{9}$$

We are going to show that

$$\begin{aligned} &\int\int \mathbb{E}_{\mathbb{P}\otimes\mu}[f(X_{s_1}, \dots, X_{s_l})1_{A_2}|\mathcal{F}_u \otimes \mathcal{B}](\omega, v)H(X_{t_1}(\omega), \dots, X_{t_k}(\omega)) \\ &\qquad G(\tau^*(\omega, v) \wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega, v) \\ = &\int\int f(X_{s_1}(\omega), \dots, X_{s_l}(\omega))1_{A_2}(v)H(X_{t_1}(\omega), \dots, X_{t_k}(\omega)) \\ &\qquad G(\tau^*(\omega, v) \wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega, v). \end{aligned}$$

$X^n \xrightarrow{\mathbb{P}} X$ and $f$ is a bounded continuous function, so that

$$f(X^n_{s_1}, \dots, X^n_{s_l}) \xrightarrow{L^1} f(X_{s_1}, \dots, X_{s_l}).$$

Moreover, $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ so using (Coquet, Mémin and Słomiński, 2001, Remark 2),

$$\mathbb{E}_{\mathbb{P}}[f(X^n_{s_1}, \dots, X^n_{s_l})|\mathcal{F}^n_{\cdot}] \xrightarrow{\mathbb{P}} \mathbb{E}_{\mathbb{P}}[f(X_{s_1}, \dots, X_{s_l})|\mathcal{F}_{\cdot}].$$

Since $\mathbb{P}[\Delta\mathbb{E}[f(X_{s_1}, \dots, X_{s_l})|\mathcal{F}_u] \neq 0] = 0$, we have

$$\mathbb{E}_{\mathbb{P}}[f(X^n_{s_1}, \dots, X^n_{s_l})|\mathcal{F}^n_u] \xrightarrow{\mathbb{P}} \mathbb{E}_{\mathbb{P}}[f(X_{s_1}, \dots, X_{s_l})|\mathcal{F}_u]$$

and since $f$ is bounded,

$$\mathbb{E}_{\mathbb{P}}[f(X^n_{s_1}, \dots, X^n_{s_l})|\mathcal{F}^n_u] \xrightarrow{L^1} \mathbb{E}_{\mathbb{P}}[f(X_{s_1}, \dots, X_{s_l})|\mathcal{F}_u]. \tag{10}$$

Using that $H$, $G$, and $f$ are continuous and bounded, we can show that:

$$\begin{aligned} &\int\int f(X^n_{s_1}(\omega), \dots, X^n_{s_l}(\omega))1_{A_2}(v)H(X^n_{t_1}(\omega), \dots, X^n_{t_k}(\omega)) \\ &\qquad G(\tau^{*,n}(\omega, v) \wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega, v) \\ \xrightarrow[n\to+\infty]{} &\int\int f(X_{s_1}(\omega), \dots, X_{s_l}(\omega))1_{A_2}(v)H(X_{t_1}(\omega), \dots, X_{t_k}(\omega)) \\ &\qquad G(\tau^*(\omega, v) \wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega, v). \end{aligned} \tag{11}$$

On the other hand, $\mathbb{E}[f(X_{s_1}, \dots, X_{s_l})1_{A_2}|\mathcal{F}_u \times \mathcal{B}] = \mathbb{E}[f(X_{s_1}, \dots, X_{s_l})|\mathcal{F}_u]1_{A_2}$.
Using again that $H$, $G$ and $f$ are continuous and bounded and the convergence (10), we have:

$$\begin{aligned} &\int\int \mathbb{E}[f(X^n_{s_1}, \dots, X^n_{s_l})1_{A_2}|\mathcal{F}^n_u \times \mathcal{B}](\omega, v)H(X^n_{t_1}(\omega), \dots, X^n_{t_k}(\omega)) \\ &\qquad G(\tau^{*,n} \wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega, v) \\ \xrightarrow[n\to+\infty]{} &\int\int \mathbb{E}[f(X_{s_1}, \dots, X_{s_l})1_{A_2}|\mathcal{F}_u \times \mathcal{B}](\omega, v)H(X_{t_1}(\omega), \dots, X_{t_k}(\omega)) \\ &\qquad G(\tau^*(\omega, v) \wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega, v). \end{aligned} \tag{12}$$

But, $H(X^n_{t_1},\dots,X^n_{t_k})$ is $\mathcal{F}^n_u\times\mathcal{B}-$measurable and $G(\tau^n\wedge u)$ and both $g(U)$, where $\forall\omega\in\Omega$, $\forall v\in[0,1]$, $U(\omega,v)=v$, are also $\mathcal{F}^n_u\times\mathcal{B}-$measurable, by continuity of $G$ and $g$. It follows that

$$\begin{aligned}
&\mathbb{E}[\mathbb{E}[f(X^n_{s_1},\dots,X^n_{s_l})1_{A_2}|\mathcal{F}^n_u\times\mathcal{B}]H(X^n_{t_1},\dots,X^n_{t_k})G(\tau^n\wedge u)g(U)]\\
&\quad= \mathbb{E}[\mathbb{E}[f(X^n_{s_1},\dots,X^n_{s_l})1_{A_2}H(X^n_{t_1},\dots,X^n_{t_k})G(\tau^n\wedge u)g(U)|\mathcal{F}^n_u\times\mathcal{B}]]\\
&\quad= \mathbb{E}[f(X^n_{s_1},\dots,X^n_{s_l})1_{A_2}H(X^n_{t_1},\dots,X^n_{t_k})G(\tau^n\wedge u)g(U)]
\end{aligned}$$

Identifying limits in (11) and (12), we obtain:

$$\begin{aligned}
&\int\int \mathbb{E}[f(X_{s_1},\dots,X_{s_l})1_{A_2}|\mathcal{F}_u\times\mathcal{B}](\omega,v)H(X_{t_1}(\omega),\dots,X_{t_k}(\omega))\\
&\qquad G(\tau^*(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v)\\
&= \int\int f(X_{s_1}(\omega),\dots,X_{s_l}(\omega))1_{A_2}(v)H(X_{t_1}(\omega),\dots,X_{t_k}(\omega))\\
&\qquad G(\tau^*(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v).
\end{aligned}\tag{13}$$

Then, using the approximations (6), (7), (8), (9) and the fact that $\mathbb{E}[f(X_{s_1},\dots,X_{s_l})|\mathcal{F}_{\cdot}]$ is a càdlàg process, we can deduce from (13) the equality (5):

$$\begin{aligned}
&\int\int \mathbb{E}[Z|\mathcal{F}_t\times\mathcal{B}](\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d(\mathbb{P}\otimes\mu)(\omega,v)\\
&\quad= \int\int Z(\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d(\mathbb{P}\otimes\mu)(\omega,v),
\end{aligned}$$

for every $t\in[0,T]$, for every $Z=1_{A_1\times A_2}$, $A_1\in\mathcal{F}_T$, $A_2\in\mathcal{B}$, for every $A\in\mathcal{F}_t$, for every $s\leqslant t$, for every $B\in\mathcal{B}$.

It follows through a monotone class argument, linearity and density that (5) holds whenever $Z$ is $\mathcal{F}_T\times\mathcal{B}-$measurable and integrable.
Hence, for every $t\in[0,T]$, for every $Z\in L^1(\mathcal{F}_T\times\mathcal{B})$, for every $C\in\mathcal{G}_t\times\mathcal{B}$ (by definition of $\mathcal{G}_t\times\mathcal{B}$),

$$\mathbb{E}_{\mathbb{P}\otimes\mu}[\mathbb{E}_{\mathbb{P}\otimes\mu}[Z|\mathcal{F}_t\times\mathcal{B}]1_C]=\mathbb{E}_{\mathbb{P}\otimes\mu}[Z1_C].$$

We hence have checked (4), therefore the assumption of Lemma 17 is filled, and we readily deduce Lemma 16 from (3). □

Recall now inequality (2): from the definition of $\tau^*$ and Lemma 17, whose assumption is filled as we just have shown, it follows that

$$\begin{aligned}
\limsup\Gamma_n(T)-\varepsilon &\leqslant \mathbb{E}[\gamma(\tau^*,X_{\tau^*})]\\
&\leqslant \tilde{\Gamma}(T)=\Gamma^*(T).
\end{aligned}$$

As such a randomized stopping time $\tau^*$ exists for arbitrary $\varepsilon>0$, we conclude that

$$\limsup\Gamma_n(T)\leqslant\Gamma^*(T)$$

and Lemma 9 shows now that $\Gamma^*(T)=\Gamma(T)$. Theorem 15 is proved. □

To sum up this section, under the hypothesis of Theorem 5, we have proved the inequality $\Gamma(T) \leqslant \liminf \Gamma_n(T)$ in Theorem 7. Then, we have shown that $\Gamma(T) \geqslant \limsup \Gamma_n(T)$ when inclusion of filtrations $\mathcal{F}^n \subset \mathcal{F}$ (in Theorem 13) or convergence of filtrations $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ (in Theorem 15) hold, provided that Aldous' Criterion for tightness (1) is filled by the sequence $(X^n)$. At last, Theorem 5 is proved.

# 3 Convergence of optimal stopping times

**Definition 18** *$\tau$ is an optimal stopping time for $X$ if $\tau$ is a $\mathcal{F}-$stopping time bounded by $T$ such that $\mathbb{E}[\gamma(\tau, X_\tau)] = \Gamma(T)$.*

Some results of existence of optimal stopping time are given for instance in (Shiryaev, 1978) in the case of Markov processes.

Now, let $(X^n)_n$ be a sequence of càdlàg processes that converges in probability to a càdlàg process $X$. Let $(\mathcal{F}^n)_n$ be the natural filtrations of processes $(X^n)_n$ and $\mathcal{F}$ the right-continuous filtration of $X$. We suppose again that Aldous' Criterion for tightness (1) is filled and that we have the convergence of values in optimal stopping: $\Gamma_n(T) \to \Gamma(T)$ (see Section 1 for the notations).

We consider, if it exists, a sequence $(\tau^n_{op})_n$ of optimal stopping times associated to the $(X^n)$. $(\tau^n_{op})_n$ is tight so we can find a subsequence which converges in law to a random variable $\tau$. There are at least two problems to solve. First, is $\tau$ a $\mathcal{F}-$stopping time or (at least) is the law of $\tau$ the law of a $\mathcal{F}-$stopping time ? Then, if the answer is positive, is $\tau$ optimal for $X$, i.e. have we $\mathbb{E}[\gamma(\tau, X_\tau)] = \Gamma(T)$ ?

It is not difficult to answer the second question as next result shows:

**Lemma 19** *We suppose that $\Gamma_n(T) \xrightarrow[n\to+\infty]{} \Gamma(T)$ and that Aldous' Criterion for tightness (1) is filled. Let $(\tau^n_{op})_n$ be a sequence of optimal stopping times associated to $(X^n)_n$. Assume that $\tau$ is a stopping time such that, along some subsequence $\varphi$, $(X^{\varphi(n)}, \tau^{\varphi(n)}_{op}) \xrightarrow{\mathcal{L}} (X, \tau)$. Then $\tau$ is an optimal $\mathcal{F}-$stopping time.*

Proof

$(X^{\varphi(n)}, \tau^{\varphi(n)}_{op}) \xrightarrow{\mathcal{L}} (X, \tau)$ so according to Proposition 10, $(\tau^{\varphi(n)}_{op}, X^{\varphi(n)}_{\tau^{\varphi(n)}_{op}}) \xrightarrow{\mathcal{L}} (\tau, X_\tau)$. $\gamma$ is bounded and continuous, so $\mathbb{E}[\gamma(\tau^{\varphi(n)}_{op}, X^{\varphi(n)}_{\tau^{\varphi(n)}_{op}})] \xrightarrow[n\to+\infty]{} \mathbb{E}[\gamma(\tau, X_\tau)]$. $(\tau^{\varphi(n)}_{op})$ is a sequence of optimal $(\mathcal{F}^{\varphi(n)})-$stopping times, so for every $n$, $\mathbb{E}[\gamma(\tau^{\varphi(n)}_{op}, X^{\varphi(n)}_{\tau^{\varphi(n)}_{op}})] = \Gamma_{\varphi(n)}(T)$ where $\Gamma_{\varphi(n)}(T)$ is the value in optimal stopping for $X^{\varphi(n)}$. Moreover, $\Gamma_{\varphi(n)}(T) \xrightarrow[n\to+\infty]{} \Gamma(T)$. So, by unicity of the limit, $\mathbb{E}[\gamma(\tau, X_\tau)] = \Gamma(T)$. Finally, $\tau$ is an optimal $\mathcal{F}-$stopping time. □

Now, it remains to find a criterion to determine wether the limit of a sequence $(\tau^n)$ of $(\mathcal{F}^n)-$stopping times is a $\mathcal{F}-$stopping time. Next proposition gives such a criterion involving convergence of filtrations, and which will prove useful in the applications of Section 4.

**Proposition 20** *We suppose $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$. Let $(\tau^n)_n$ be a sequence of $(\mathcal{F}^n)-$stopping times that converges in probability to a $\mathcal{F}_T-$measurable random variable $\tau$. Then $\tau$ is a $\mathcal{F}-$stopping time.*

Proof
$\tau^n \xrightarrow{\mathbb{P}} \tau$ so $1_{\{\tau^n \leqslant .\}} \xrightarrow{\mathbb{P}} 1_{\{\tau \leqslant .\}}$ for the Skorokhod topology.
We fix $t$ such that $\mathbb{P}[\tau = t] = 0$. Then, $1_{\{\tau^n \leqslant t\}} \xrightarrow{\mathbb{P}} 1_{\{\tau \leqslant t\}}$. The sequence $(1_{\{\tau^n \leqslant t\}})_n$ is uniformly integrable, so $1_{\{\tau^n \leqslant t\}} \xrightarrow{L^1} 1_{\{\tau \leqslant t\}}$. $\tau$ is $\mathcal{F}_T-$measurable, so $1_{\{\tau \leqslant t\}}$ is $\mathcal{F}_T-$measurable. As $1_{\{\tau^n \leqslant t\}} \xrightarrow{L^1} 1_{\{\tau \leqslant t\}}$, $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ and $1_{\{\tau \leqslant t\}}$ is $\mathcal{F}_T-$measurable, according to (Coquet, Mémin and Słomiński, 2001, Remark 2), we have:

$$\mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_.] \xrightarrow{\mathbb{P}} \mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_.].$$

Let us prove that $\mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_t] \xrightarrow{\mathbb{P}} \mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_t]$.
Fix $\eta > 0$ and $\varepsilon > 0$.
$\mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_.]$ is a càdlàg process, so we can find $s \in ]t, T]$ satisfying $\mathbb{P}[\Delta \mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_s] \neq 0] = 0$ and such that

$$\mathbb{P}[|\mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_s] - \mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_t]| \geqslant \eta/3] \leqslant \varepsilon/3.$$

Then, we have $\mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_s] \xrightarrow{\mathbb{P}} \mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_s]$ and we can find $n_0$ such that for every $n \geqslant n_0$,

$$\mathbb{P}[|\mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_s] - \mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_s]| \geqslant \eta/3] \leqslant \varepsilon/3.$$

On the other hand,

$$\mathbb{P}[|\mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_t] - \mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_s] \geqslant \eta/3] = 0$$

because $\{\tau^n \leqslant t\} \in \mathcal{F}^n_t$ as $(\tau^n)_n$ is a sequence of $(\mathcal{F}^n)-$stopping times, and $\{\tau^n \leqslant t\} \in \mathcal{F}^n_s$ since $s \geqslant t$.
Finally, for every $n \geqslant n_0$,

$$\begin{aligned}
&\mathbb{P}[|\mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_t] - \mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_t]| \geqslant \eta] \\
&\leqslant \mathbb{P}[|\mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_t] - \mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_s] \geqslant \eta/3] + \mathbb{P}[|\mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_s] - \mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_s]| \geqslant \eta/3] \\
&\qquad + \mathbb{P}[|\mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_s] - \mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_t]| \geqslant \eta/3] \\
&\leqslant \varepsilon.
\end{aligned}$$

Hence,

$$\mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_t] \xrightarrow{\mathbb{P}} \mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_t].$$

But, $(\tau^n)_n$ is a sequence of $(\mathcal{F}^n)-$stopping times, so $\forall n$, $\mathbb{E}[1_{\{\tau^n \leqslant t\}}|\mathcal{F}^n_t] = 1_{\{\tau^n \leqslant t\}}$. Moreover, $1_{\{\tau^n \leqslant t\}} \xrightarrow{\mathbb{P}} 1_{\{\tau \leqslant t\}}$. By unicity of the limit, $\mathbb{E}[1_{\{\tau \leqslant t\}}|\mathcal{F}_t] = 1_{\{\tau \leqslant t\}}$ *a.s.* Then, for every $t$ such that $\mathbb{P}[\tau = t] = 0$, $\{\tau \leqslant t\} \in \mathcal{F}_t$.

Next, the right continuity of $\mathcal{F}$ implies that for every $t$, $\{\tau \leqslant t\} \in \mathcal{F}_t$. Finally $\tau$ is a $\mathcal{F}-$stopping time. □

**Remark 21** A sufficient condition to get the $\mathcal{F}_T-$measurability of the limit may be the inclusion of terminal $\sigma-$fields $\mathcal{F}^n_T \subset \mathcal{F}_T, \forall n$. Indeed, under this hypothesis, $(\tau^n)$ is a sequence of $\mathcal{F}_T-$measurable variables. Hence the limit is also $\mathcal{F}_T-$measurable.

**Remark 22** Even if convergence of filtrations $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ holds, the limit of a sequence of $(\mathcal{F}^n)-$stopping times is not a priori $\mathcal{F}_T-$measurable. For example, if $\mathcal{F}$ is the trivial filtration, the assumption $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ is always true. However, the limit of a sequence of $(\mathcal{F}^n)-$stopping times may not be a constant, so it is not always $\mathcal{F}_T-$measurable.

Proposition 20 and Lemma 19 allow us to give a result of convergence of optimal stopping times when the processes $X^n$ have independent increments.

**Theorem 23** *Let $(X^n)$ be a sequence of càdlàg processes which converges in law to a quasi-left continuous process $X$. We suppose that the processes $(X^n)$ have independent increments. Let $(\mathcal{F}^n)$ be the natural filtrations of processes $(X^n)$ and $\mathcal{F}$ be the right-continuous filtration associated to the process $X$. Let $(\tau^n)$ be a sequence of optimal $(\mathcal{F}^n)-$stopping times. If $(X,\tau)$ is the limit in law of a subsequence of $((X^n,\tau^n))_n$ and if $\tau$ is $\mathcal{F}_T-$measurable, then $\tau$ is an optimal stopping time for $X$.*

Proof
$((X^n,\tau^n))$ is tight because $(\tau^n)$ is bounded and $(X^n)$ is convergent. So we can extract a subsequence $((X^{\varphi(n)},\tau^{\varphi(n)}))$ which converges in law to $(X,\tau)$.
Let us prove that $\tau$ is a $\mathcal{F}-$stopping time.
Using the Skorokhod representation theorem, we can find a probability space $(\tilde{\Omega},\tilde{\mathcal{G}},\tilde{\mathbb{P}})$ on which are defined $(\tilde{X}^{\varphi(n)},\tilde{\tau}^{\varphi(n)}) \sim (X^{\varphi(n)},\tau^{\varphi(n)})$ and $(\tilde{X},\tilde{\tau}) \sim (X,\tau)$ such that $(\tilde{X}^{\varphi(n)},\tilde{\tau}^{\varphi(n)}) \xrightarrow{a.s} (\tilde{X},\tilde{\tau})$ in $(\tilde{\Omega},\tilde{\mathcal{G}},\tilde{\mathbb{P}})$. As $(X^n)$ are processes with independent increments, $(\tilde{X}^{\varphi(n)})$ also are. Using (Mémin, 2003, Proposition 3), we have the extended convergence

$$(\tilde{X}^{\varphi(n)},\mathcal{F}^{\tilde{X}^{\varphi(n)}}) \xrightarrow{\mathbb{P}} (\tilde{X},\mathcal{F}^{\tilde{X}})$$

where $\mathcal{F}^{\tilde{X}^{\varphi(n)}}$ (resp. $\mathcal{F}^{\tilde{X}}$) is the natural filtration of the process $\tilde{X}^{\varphi(n)}$ (resp. $\tilde{X}$).
On the other hand, $\tau$ is $\mathcal{F}_T-$measurable. So, we can find a measurable function $f$ such that $\tau = f(X)$. $(\tilde{X},\tilde{\tau}) \sim (X,\tau)$ and $(X,\tau) = (X,f(X))$, so $\tilde{\tau} = f(\tilde{X})$ *a.s.* Hence, $\tilde{\tau}$ is $\mathcal{F}^{\tilde{X}}_T-$measurable.
Moreover, $\tilde{\tau}^{\varphi(n)} \xrightarrow{a.s.} \tilde{\tau}$ by construction and, as $(\tau^{\varphi(n)})$ is a sequence of $(\mathcal{F}^{\varphi(n)})-$stopping times, $(\tilde{\tau}^{\varphi(n)})$ is a sequence of $(\mathcal{F}^{\tilde{X}^{\varphi(n)}})-$stopping times.
Then, using Proposition 20, $\tilde{\tau}$ is a $\mathcal{F}^{\tilde{X}}-$stopping time.
Next, using Proposition 3, Aldous' Criterion for tightness is filled because $\tilde{X}$ is quasi-left continuous and $(\tilde{X}^{\varphi(n)},\mathcal{F}^{\tilde{X}^{\varphi(n)}}) \xrightarrow{\mathcal{L}} (\tilde{X},\mathcal{F}^{\tilde{X}})$. Moreover, $\Gamma^{\tilde{X}^{\varphi(n)}}(T) \to \Gamma^{\tilde{X}}(T)$ according to Theorem 5.
Then, according to Lemma 19, $\tilde{\tau}$ is an optimal $\mathcal{F}^{\tilde{X}}-$stopping time.
Finally, $\tau$ is an optimal $\mathcal{F}-$stopping time. □

# 4 Applications

## 4.1 Application to discretizations

**Proposition 24** *Let us consider a quasi-left continuous process $X$ with independent increments. Let $(\pi^n = \{t^n_1,\dots t^n_{k^n}\})_n$ be an increasing sequence of subdivisions of $[0,T]$ with mesh*

*going to 0* ($|\pi^n| \xrightarrow[n\to+\infty]{} 0$). *We define the sequence of discretized processes* $(X^n)_n$ *by* $\forall n$, $\forall t$, $X_t^n = \sum_{i=1}^{k^n-1} X_{t_i^n} 1_{t_i^n \leqslant t < t_{i+1}^n}$.
*Let us denote by* $\mathcal{F}$ *the right-continuous natural filtration of* $X$ *and by* $(\mathcal{F}^n)_n$ *the natural filtrations of the* $(X^n)_n$.
*Then, using the notations of Section 1,* $\Gamma_n(T) \xrightarrow[n\to+\infty]{} \Gamma(T)$. *Moreover, if* $(\tau^n)$ *is a sequence of optimal stopping times associated to the processes* $(X^n)$ *and if* $(X, \tau)$ *is the limit in law of a subsequence of* $(X^n, \tau^n)$*, then* $\tau$ *is an optimal stopping time for* $X$.

Proof
$X^n \xrightarrow[n\to+\infty]{} X$ *a.s.* then in probability, for every $n$ $\mathcal{F}^n \subset \mathcal{F}$ by definition of $X^n$. Moreover, $X$ is quasi-left continuous and $(X^n)$ is a sequence of discretized processes, so we can easily check that Aldous' Criterion is filled. So, using Theorem 5, $\Gamma_n(T) \xrightarrow[n\to+\infty]{} \Gamma(T)$.
On the other hand, for every $n$, $\mathcal{F}_T^n \subset \mathcal{F}_T$. So $\tau$ is $\mathcal{F}_T-$measurable. Then, according to Theorem 23, $\tau$ is an optimal stopping time for $X$. □

## 4.2 Application to financial models

### 4.2.1 The models

The convergence of properly normalized Cox-Ross-Rubinstein models to a Black-Scholes model is a standard in financial mathematics. By convergence, it is usually meant here convergence of option prices. We are going to apply our results to prove that a sequence of so-called rational times of exercise for an american put in a Cox-Ross-Rubinstein converge, under the same normalization, to a rational time of exercise for an american put in the Black-Scholes model.

We just recall here the classical notation for both models.

The Black-Scholes model on an interval $[0, T]$ consists in a market with one non-risky asset of price $S_t^0 = S_0^0 e^{rt}$ at time $t$, $r$ denoting the instant interest rate, and a risky asset whose price is governed by the following stochastic differential equation:

$$S_t = S_t(\mu dt + \sigma dB_t) \tag{14}$$

where $\mu$ and $\sigma$ are positive reals and $(B_t)$ is a standart Brownian motion. We denote by $\mathbb{P}^*$ the risk-neutral probability, under which the actualized price of the risky asset is a martingale, and by $(\mathcal{F}_t)_{t\leq T}$ the filtration generated by the Brownian motion $B$.

If we are given an american put option with maturity $T$ and strike price $K$, then its optimal value is defined as

$$\Gamma^S(T) = \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{P}^*}[e^{-r\tau}(K - S_\tau)^+],$$

where $\mathcal{T}_T$ is the set of $\mathcal{F}-$stopping times bounded by $T$, and the expectation is taken under $\mathbb{P}^*$.
A rational exercise time is then a stopping time $\tau^0$ such that

$$\mathbb{E}_{\mathbb{P}^*}[e^{-r\tau^0}(K - S_{\tau^0})^+] = \Gamma^S(T).$$

We now build a sequence of random walks approaching $B$, following the construction of (Knight, 1962). We refer to (Itô and McKean, 1974) for explicit details. We only need to know here that Knight has built an array $(Y_i^n)$ such that, for every $n$, $(Y_i^n)_i$ is a sequence of $\mathcal{F}_T-$mesurable independent Bernoulli variables, such that $\mathbb{P}[Y_i^n = 1] = \mathbb{P}[Y_i^n = -1] = 1/2$, and for which, if we put $B_t^n = \sqrt{\frac{T}{n}} \sum_{i=1}^{[\frac{nt}{T}]} Y_i^n$, holds the following convergence:

$$\mathbb{P}\left[\lim_{n\uparrow+\infty} \sup_{t\in[0,T]} |B_t^n - B_t| = 0\right] = 1. \tag{15}$$

The last step is to build the Cox-Ross-Rubinstein models based upon the array $(Y_i^n)$ in such a way that holds the convergence of binomial prices for the risky assets (denoted by $S_n$) to $S$ (the reader will find the appropriate normalizations, e.g. in (Lamberton and Lapeyre, 1997) or (Shiryaev, 1999) or any textbook on mathematical finance).
For each $n$, the maximal expectation of profit for the associated Cox-Ross-Rubinstein model is given by:

$$\Gamma^{S^n}(T) = \sup_{\tau^n \in \mathcal{T}_T^n} \mathbb{E}_{\mathbb{P}^{*,n}}[(1 + rT/n)^{-([\tau^n n/T])}(S_{\tau^n}^n - K)^+]$$

where $(\mathcal{F}^n)$ denotes the (piecewise constant) filtration generated by the price process $(S_i^n)_i$ (which is also the filtration generated by the process $B^n$), $\mathcal{T}_T^n$ is the set of $\mathcal{F}^n-$stopping times bounded by $T$, and $\mathbb{P}^{*,n}$ is the equivalent probability making the actualised price process a $\mathcal{F}^n-$martingale.

### 4.2.2 Convergence of values in optimal stopping

Having used Knight's construction ensures us that

$$(B^n, S^n) \xrightarrow{a.s.} (B, S)$$

and as the $B^n$'s are processes with independent increments and $S^n$ and $S$ are bijective functions of $B^n$ and $B$, (Mémin, 2003, Proposition 3) gives the extended convergence: $(S^n, \mathcal{F}^{S^n}) \xrightarrow{\mathbb{P}} (S, \mathcal{F}^S)$. Thanks to Theorem 6, whose hypothesis is clearly fulfilled, we deduce then

$$\Gamma^{S^n}(T) \xrightarrow[n\to+\infty]{} \Gamma^S(T) \tag{16}$$

Remark at last that (16) holds regardless of the specific construction of the prelimit Cox-Ross-Rubinstein models. Indeed, according to Remark 1, the value in optimal stopping only depends on the law of the underlying process hence every Cox-Ross-Rubinstein model with the same law as $S$ (and any Black-Scholes limiting model) would perfectly fit, provided that the correct normalizations are performed in order to have the convergence (in law) of the price processes.
To sum up, we have just proved the following result:

**Proposition 25** *When approximating a Black-Scholes model by a sequence of Cox-Ross-Rubinstein models, we have convergence of the associated sequence of values in optimal stopping:*

$$\Gamma^{S^n}(T) \xrightarrow[n\to+\infty]{} \Gamma^S(T).$$

This convergence is already well known (see (Mulinacci and Pratelli, 1998), (Lamberton , 1993) or (Amin and Khanna , 1994) for example). What is new in this paper is the result of convergence of optimal stopping times proved in the next section.

#### 4.2.3 Convergence of optimal stopping times

In the same framework as above, let us end with the study of the convergence of optimal stopping times.

$S^n$ is a Markov process so there exists a sequence $(\tau_{op}^n)$ of optimal $\mathcal{F}^n-$stopping times. The sequence $(B, B^n, S^n, \tau_{op}^n)$ is tight, and up to some subsequence, we can assume that

$$(B, B^n, S^n, \tau_{op}^n) \xrightarrow{\mathcal{L}} (B, B, S, \tau)$$

for some random variable $\tau$.

By Skorokhod's representation lemma, we can assume that this convergence holds almost surely (preserving the fact that al the processes are functions of the Brownian motion $B$).

As in previous subsection, convergence of filtrations holds, moreover our specific construction (Knight's one) ensures that for every $n$, $\tau^n$ is $B_T^n-$measurable, hence $\mathcal{F}_T-$measurable. From Proposition 20, we deduce that $\tau$ is a $\mathcal{F}-$stopping time.

As moreover, $\Gamma^{S^n}(T) \to \Gamma^S(T)$, Lemma 19 now says that $\tau$ is indeed an optimal $\mathcal{F}-$stopping time.

We have just proved the following result:

**Proposition 26** *When approximating a Black-Scholes model by a sequence of Cox-Ross-Rubinstein models based on Knight's construction, if a subsequence of $((B^n, S^n, \tau_{op}^n))_n$ -where $(\tau_{op}^n)_n$ is a sequence of optimal stopping times for the prelimit models- converges in law to $(B, S, \tau)$, then $\tau$ is an optimal stopping time for Black-Scholes model.*

**Remark 27** We stress once more on the fact that, whereas Proposition 25 remains true for every Cox-Ross-Rubinstein approximation of a Black-Scholes models, the proof of Proposition 26 rely upon the fact that $\tau$ is actually a stopping time for the natural filtration associated filtration of Black and Scholes model (and not for a larger one like in the existing papers), for which we need Knight's construction of the prelimit models.

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