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Urn-related random walk with drift $\rho x^\alpha / t^\beta$

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Abstract

We study a one-dimensional random walk whose expected drift depends both on time and the position of a particle. We establish a non-trivial phase transition for the recurrence vs. transience of the walk, and show some interesting applications to Friedman's urn, as well as showing the connection with Lamperti's walk with asymptotically zero drift .

Key words: random walks, urn models, martingales.

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1 Introduction

Consider the following stochastic processes X_t which may loosely be described as a random walk on $\mathbb{R}_+$ (or in more generality on $\mathbb{R}$) with the asymptotic drift given by

$$\mu_t := \mathbb{E}\left(X_{t+1} - X_t \,|\, X_t = x\right) \sim \rho \frac{|x|^\alpha}{t^\beta},$$

where ρ, α and β are some fixed constants, and the exact meaning of "$\sim$" will be made precise later. In this paper we establish when this process is recurrent or transient, by finding the whole line of phase transitions in terms of (α, β). We also analyze some critical cases, when the value of ρ becomes important as well. Note that because of symmetry, it is sufficient to consider only these processes on $\mathbb{R}_+$, and from now on we will assume that $X_t \geq 0$ a.s. for all values of t.

The original motivation of this paper is based on an open problem related to Friedman urns, posed in Freedman [4]. In certain regimes of these urns, to the best of our knowledge, it is still unknown whether the number of balls of different colors can overtake each other infinitely many times with a positive probability. We will not describe this problem in more details here, rather we refer the reader directly to Section 6.

Incidentally, the class of stochastic processes we are considering covers simultaneously not only the Friedman urn, but also the walk with an asymptotically zero drift, first probably studied by Lamperti, see [7] and [8]. His one-dimensional walks with drift depending only on the position of the particle naturally arise when proving recurrence of the simple random walk on $\mathbb{Z}^1$ and $\mathbb{Z}^2$ and transience on $\mathbb{Z}^d$, $d \geq 3$. They can be used of course for answering the question of recurrence for a much wider class of models, notably those involving polling systems, for example, see [1] and [9]. It will be not surprising if the model we are considering also covers some other probabilistic models, of which we are unaware at the moment.

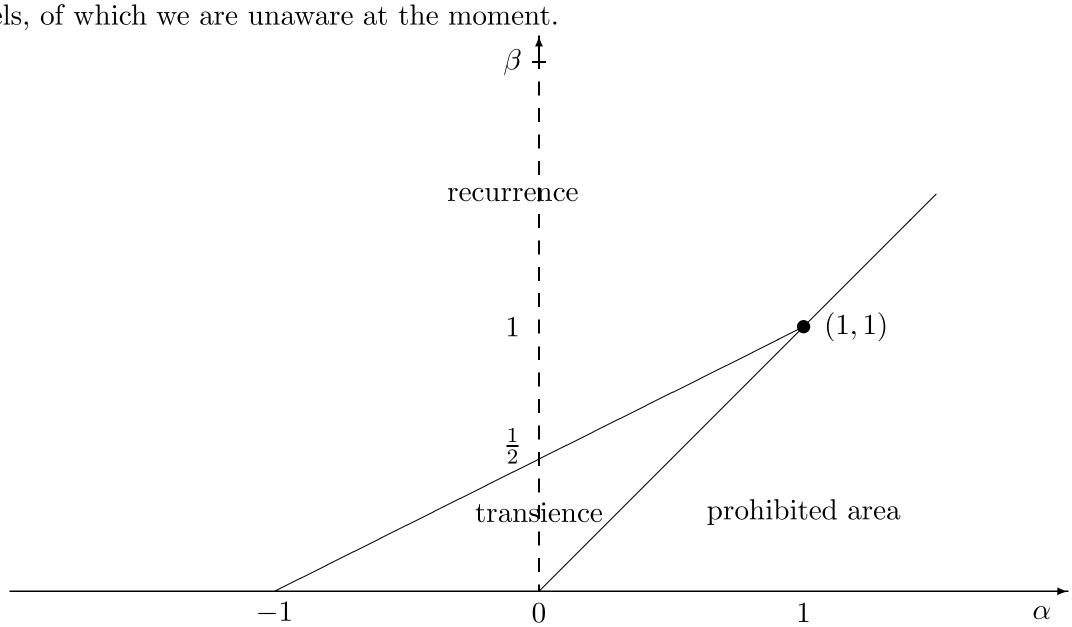

Figure 1: Diagram for (α, β).

In our paper we study the random walk whose drift depends *both on time and the position* of a particle. Throughout the paper we assume that

$$(\alpha,\beta)\in\Upsilon=\{(\alpha,\beta):\ \beta>\alpha \text{ and } \beta\ge 0\}$$

to avoid the situations when the drift becomes unbounded and the borderline cases (the only exception will be $\alpha=\beta=1$). We will show that under some regularity conditions, the walk is transient when (α,β) lie in the following area

$$\mathbf{Trans}=\{(\alpha,\beta):\ 0\le\beta<1,\ 2\beta-1<\alpha<\beta\}\subset\Upsilon$$

and recurrent for (α,β) in

$$\mathbf{Rec}\quad=\quad\Upsilon\setminus\overline{\mathbf{Trans}}=\{(\alpha,\beta):\ \beta\ge 0,\ \alpha<\min(\beta,2\beta-1)\}$$

where $\overline{\mathbf{Trans}}$ denotes the closure of the set **Trans**. In the special critical case $\alpha=\beta=1$ we show that the walk is transient for $\rho>1/2$ and recurrent for $\rho<1/2$. An example of such a walk with $\alpha=\beta=1$ is the process on $\mathbb{Z}_+$ with the following jump distribution:

$$\mathbb{P}(X_{t+1}=n\pm 1\,|\,X_t=n)=\frac{1}{2}\pm\frac{\rho n}{2t}$$

where $t=0,1,2,\dots$. This walk is analyzed in Section 6.

Throughout the paper we will need the following hypothesis. Let X_t be a stochastic process on $\mathbb{R}_+$ with jumps $D_t=X_t-X_{t-1}$ and let $\mathcal{F}_t=\sigma(X_0,X_1,\dots,X_t)$. Let a be some positive constant.

(H1) Uniform boundedness of jumps.
There is a constant $B_1>0$ such that $|D_t|\le B_1$ for all $t\in\mathbb{Z}_+$ a.s.

(H2) Uniform non-degeneracy on $[a,\infty)$.
There is a constant $B_2>0$ such that whenever $X_{t-1}\ge a$, $\mathbb{E}\,(D_t^2\,|\,\mathcal{F}_{t-1})\ge B_2$ for all $t\in\mathbb{Z}_+$. a.s.

(H3) Uniform boundedness of time to leave [0,a].
The number of steps required for X_t to exit the interval $[0,a]$ starting from any point inside this interval is uniformly stochastically bounded above by some independent random variable $W\ge 0$ with a finite mean $\mu=\mathbb{E}\,W<\infty$, i.e., for all $s\in\mathbb{R}_+$, when $X_s\le a$,

$$\forall x\ge 0\ \ \mathbb{P}(\eta(s)\ge x\,|\,\mathcal{F}_s)\le\mathbb{P}(W\ge x),\text{where } \eta(s)=\inf\{t\ge s:\ X_t>a\}.$$

The rest of the paper is organized as follows. In Section 2 we prove some technical lemmas. In Section 3 we formulate the exact statement about the transience of the process X_t and prove it while in Section 4 we do the same for recurrence. We also study some borderline cases in Section 5, and present an open problem in Section 5.3. Finally, we apply our results to generalized Pólya and Friedman urns in Section 6.

2 Technical facts

First, we will need the following important law of iterated logarithms for martingales.

Lemma 1 (Proposition (2.7) in Freedman (1975))**.** *Suppose that S_n is a martingale adapted to filtration $\mathcal{F}_n$ and $\Delta_n = S_n - S_{n-1}$ are its differences. Let $T_n = \sum_{i=1}^n \mathbb{E}\,(\Delta_i^2 \,|\, \mathcal{F}_{i-1})$, $\sigma_b = \inf\{n : T_n > b\}$, and*

$$L(b) = ess \sup_{\omega} \sup_{n \le \sigma_b(\omega)} |\Delta_n(\omega)|.$$

Suppose $L(b) = o(b/\log\log b)^{1/2}$ as $b \to \infty$. Then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2T_n \log\log T_n}} = 1 \quad a.s. \text{ on } \{T_n \to \infty\}.$$

Lemma 2. *Let X_t, $t = 1, 2, \dots$ be a sequence of random variables adapted to filtration $\mathcal{F}_t$ with differences $D_t = X_t - X_{t-1}$ satisfying (H1), (H2), and (H3) for some $a > 0$. Suppose that on the event $\{X_t \ge a\}$*

$$\mathbb{E}\,(D_{t+1} \,|\, \mathcal{F}_t) \ge 0 \ a.s.$$

Then for any $A > 0$

$$\mathbb{P}(\exists t : \ X_t > A\sqrt{t}) = 1.$$

Note that both the assumptions and conclusion of Lemma 2 are weaker than those of Lemma 1, but it is Lemma 2 what we use further in our paper.

Proof. First, we are going essentially to "freeze" the process X_t whenever it enters the interval $[0, a]$, where it is not a submartingale, until the moment when X_t exits from this interval. Define the function $s(t) : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $s(0) = 0$ and for $t \ge 0$

$$s(t+1) = \begin{cases} t+1, & \text{if } X_t > a \text{ or } X_{t+1} > a \\ s(t), & \text{otherwise.} \end{cases}$$

Let $\tilde{X}_t = X_{s(t)}$. Then $\tilde{X}_n$ is a submartingale satisfying (H1), perhaps with a new constant $\tilde{B}_1 = B_1 + a$. Indeed, when $X_t > a$, $\tilde{X}_t = X_t$ and $\tilde{X}_{t+1} = X_{t+1}$, so $\mathbb{E}\,(\tilde{X}_{t+1} - \tilde{X}_t | \mathcal{F}_t) = \mathbb{E}\,(D_{t+1} | \mathcal{F}_t) \ge 0$. When $X_t < a$ (and so is $\tilde{X}_t < a$), either $X_{t+1} < a$ and then $s(t+1) = s(t)$ implying $\tilde{X}_{t+1} = \tilde{X}_t$, or $X_{t+1} \ge a$ in which case $\tilde{X}_{t+1} = X_{t+1} \ge a > \tilde{X}_t$.

Let

$$\begin{aligned} \tilde{D}_n &= \tilde{X}_n - \tilde{X}_{n-1} \\ Z_n &= \mathbb{E}\,(\tilde{D}_n \,|\, \mathcal{F}_{n-1}) \ge 0 \\ S_n &= X_n - Z_1 - Z_2 - \dots - Z_n. \end{aligned}$$

Then

$$\mathbb{E}\,(S_n - S_{n-1} \,|\, \mathcal{F}_{n-1}) = \mathbb{E}\,(X_n - X_{n-1} - Z_n \,|\, \mathcal{F}_{n-1}) = 0$$

whence S_n is a martingale with differences $\Delta_n := S_n - S_{n-1} = \tilde{D}_n - Z_n$. Note that

$$\mathbb{E}\,(\Delta_n^2 \,|\, \mathcal{F}_{n-1}) = \mathbb{E}\,((S_n - S_{n-1} - Z_{n-1})^2 \,|\, \mathcal{F}_{n-1}) = \mathbb{E}\,(\tilde{D}_n^2 \,|\, \mathcal{F}_{n-1}) - Z_n^2. \tag{1}$$

Let $\eta_0 = 0$ and for $k = 1, 2, \ldots$

$$\begin{aligned} \zeta_k &= \inf\{t \geq \eta_{k-1} : \ X_t \leq a\}, \\ \eta_k &= \inf\{t \geq \zeta_k : \ X_t > a\} \end{aligned}$$

be the consecutive times of entry in and exit from $[0, a]$. Then $\tilde{W}_k := \eta_k - \zeta_k$ are stochastically bounded by i.i.d. random variables $W_1, W_2, \ldots$ with the distribution of W. Therefore

$$\limsup_{m\to\infty} \frac{\sum_{i=1}^m \tilde{W}_i}{m} \leq \mu \text{ a.s.}$$

and consequently the number

$$\begin{aligned} I_n &= \{t \in \{0, 1, \ldots, n-1\} : t \notin [\zeta_k, \eta_k) \text{ for some } k\} \\ &= \{t \in \{0, 1, \ldots, n-1\} : X_t > a\} \end{aligned}$$

of those times which do not belong to some "frozen" interval $[\zeta_k, \eta_k)$ satisfies a.s.

$$|I_n| \geq \frac{n}{2\mu} \tag{2}$$

for n sufficiently large.

Next, since $\tilde{D}_n$'s are bounded, we have $|\Delta_n| \leq |\tilde{D}_n| + |\mathbb{E}\,(\tilde{D}_n \,|\, \mathcal{F}_{n-1})| \leq 2(B_1 + a)$. Therefore, $L(b) \leq 2(B_1 + a)$ and the conditions of Lemma 1 are met. First, suppose that

$$T_n = \sum_{i=1}^n \mathbb{E}\,(\Delta_i^2 \,|\, \mathcal{F}_{i-1}) \to \infty,$$

then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2T_n \log\log T_n}} = 1 \quad \text{a.s.}$$

Therefore, for infinitely many n's we would have

$$S_n \geq \sqrt{T_n \log\log T_n}.$$

Using (1), this results in

$$\begin{aligned} X_n &= \sum_{i=1}^n Z_i + S_n \geq \sum_{i=1}^n Z_i + \sqrt{\sum_{i=1}^n (\mathbb{E}\,(\tilde{D}_i^2 \,|\, \mathcal{F}_{i-1}) - Z_i^2) \log\log T_n} \\ &\geq \sum_{i\in 1+I_n}^n Z_i + \sqrt{\sum_{i\in 1+I_n} (\mathbb{E}\,(D_i^2 \,|\, \mathcal{F}_{i-1}) - Z_i^2) \log\log T_n} \end{aligned} \tag{3}$$

since $i - 1 \in I_n$ implies $X_{i-1} > a$ and consequently $\tilde{D}_i = D_i$ (note that each term in the sums above is non-negative). Let $0 \leq k \leq |I_n|$ be the number of those Z_i's, $i \in I_n$ such that $Z_i < \sqrt{B_2/2}$. Then (3) together with $\mathbb{E}\,(D_i^2 \,|\, \mathcal{F}_{i-1}) \geq B_2$ yields

$$X_n \geq (|I_n| - k)\sqrt{\frac{B_2}{2}} + \sqrt{\frac{kB_2 \log\log T_n}{2}} \geq \sqrt{\frac{nB_2 \log\log T_n}{2\mu}}$$

for n large enough, taking into account the fact that $T_n \le B_1^2 n$ and inequality (2). This implies the statement of Lemma 2, since we assumed $T_n \to \infty$.

On the other hand, on the complementary event $\sum_{i=1}^{\infty} \mathbb{E}\,(\Delta_i^2 \,|\, \mathcal{F}_{i-1}) < \infty$, by e.g. Theorem in Chapter 12 in Williams (1991) S_n converges a.s. to a finite quantity S_∞, and we obviously must also have $\mathbb{E}\,(\tilde{D}_n^2 \,|\, \mathcal{F}_{n-1}) - Z_n^2 \to 0$ yielding

$$\liminf_{i\to\infty,\ i:\ X_{i-1}>a} Z_i \ge \sqrt{B_2}.$$

Combining this with (2), we obtain

$$\liminf_{n\to\infty} \frac{X_n}{n} = \liminf_{n\to\infty} \frac{S_n + Z_1 + Z_2 + \cdots + Z_n}{n} \ge \frac{\sqrt{B_2}}{2\mu}$$

which is even a stronger statement than we need to prove. ∎

Lemma 3. *Fix $a > 0$, $c > 0$, $\gamma \in (0,1)$, and consider a Markov process X_t, $t = 0, 1, 2, \dots$ on $\mathbb{R}_+$ with jumps $D_t = X_t - X_{t-1}$, for which the hypotheses (H1) and (H2) hold. Suppose that for some large $n > 0$ the process starts at $X_0 \in (a, \gamma n]$, and that on the event $\{a \le X_t \le n\}$*

$$\mathbb{E}\,(D_t \,|\, \mathcal{F}_{t-1}) \le \frac{c}{n}.$$

Let

$$\tau = \inf\{t:\ X_t < a \text{ or } X_t > n\}.$$

be the time to exit $[a, n]$. Then

- *(i) $\tau < \infty$ a.s.;*
- *(ii) $\mathbb{P}(X_\tau < a) \ge \nu = \nu(\gamma, c, B_2) > 0$ uniformly in n.*

Proof. First, let us show that the process X_t must exit $[a, n]$ in a finite time. Since $|D_t| \le B_1$, by Markov inequality for non-negative random variables for any $\varepsilon > 0$ we have

$$\mathbb{P}(B_1^2 - D_t^2 \ge (1-\varepsilon^2) B_1^2 \,|\, \mathcal{F}_{t-1}) \le \frac{\mathbb{E}\,(B_1^2 - D_t^2 \,|\, \mathcal{F}_{t-1})}{(1-\varepsilon^2) B_1^2} \le (1-\varepsilon^2)^{-1} \left(1 - \frac{B_2}{B_1^2}\right) \tag{4}$$

Hence for a sufficiently small $\varepsilon > 0$ the RHS of (4) can be made smaller than 1, whence there is a $\delta > 0$ such that

$$\mathbb{P}(D_t^2 \ge (\varepsilon B_1)^2 \,|\, \mathcal{F}_{t-1}) > 2\delta.$$

In turn, this implies that at least one of the probabilities $\mathbb{P}(D_t \ge \varepsilon B_1 \,|\, \mathcal{F}_{t-1})$ or $\mathbb{P}(D_t \le -\varepsilon B_1 \,|\, \mathcal{F}_{t-1})$ is larger than δ. Hence from any starting point the walk can exit $[a, n]$ in at most $n/(\varepsilon B_1)$ steps with probability at least $\delta^{n/(\varepsilon B_1)}$, yielding that $\varepsilon B_1 \tau / n$ is stochastically bounded by a geometric random variable with parameter $\delta^{n/(\varepsilon B_1)}$, which is not only finite but also has all finite moments.

To prove the second claim of the lemma, first we establish the following elementary inequality. Fix a $k \ge 1$ and consider the function $g(x) = (1-x)^k - 1 + kx - k(k-1)x^2/4$. Since $g(0) = 0$,

$g'(0)=0$, and $g''(x)=k(k-1)((1-x)^{k-2}-1/2)\geq 0$ for $|x|\leq 1/(2k)$, we have $g(x)\geq 0$ on this interval. Consequently,

$$(1-x)^k-1\geq -kx+\frac{k(k-1)x^2}{4} \text{ for } x\in\left[-\frac{1}{2k},\frac{1}{2k}\right]. \tag{5}$$

Now let $Z_t=2n-X_t$ and $Y_t=Z_t^k$ for some $k\geq 1$ to be chosen later. Suppose that $n>2kB_1$. Then, on the event $\{X_t\in[a,n]\}$ we have $Z_t\in[n,2n]$ yielding $|D_{t+1}/Z_t|\leq B_1/n\leq 1/(2k)$ and thus by (5) we have

$$\begin{aligned}
\mathbb{E}\left(Y_{t+1}-Y_t\,|\,\mathcal{F}_t\right) &= Y_t\mathbb{E}\left(\left(1-\frac{D_{t+1}}{Z_t}\right)^k-1\,|\,\mathcal{F}_t\right)\\
&\geq kY_t\left[-\frac{\mathbb{E}\left(D_{t+1}\,|\,\mathcal{F}_t\right)}{Z_t}+\frac{(k-1)\mathbb{E}\left(D_{t+1}^2\,|\,\mathcal{F}_t\right)}{4Z_t^2}\right]\\
&\geq kY_t\left[-\frac{c}{nZ_t}+\frac{(k-1)B_2}{4Z_t^2}\right]\geq kY_t\left[-\frac{c}{n^2}+\frac{(k-1)B_2}{16n^2}\right]>0,
\end{aligned}$$

once $k>1+16c/B_2$.

Hence $Y_{t\wedge\tau}$ is a non-negative submartingale. By the optional stopping theorem,

$$\mathbb{E}\left(Y_\tau\right)\geq Y_0\geq[(2-\gamma)n]^k.$$

On the other hand,

$$\mathbb{E}\left(Y_\tau\right)=\mathbb{E}\left(Y_\tau;\ X_\tau<a\right)+\mathbb{E}\left(Y_\tau;\ X_\tau>n\right)\leq(2n)^k\mathbb{P}(X_\tau<a)+n^k(1-\mathbb{P}(X_\tau<a)),$$

yielding

$$\mathbb{P}(X_\tau<a)\geq\frac{(2-\gamma)^k-1}{2^k-1}=:\nu>0.$$

$\blacksquare$

Lemma 4. *Suppose that X_t, $t=0,1,\dots$ is a submartingale satisfying (H1). Then for any $x>0$*

$$\mathbb{P}\left(\inf_{0\leq t\leq hx^2}X_t<X_0-bx\right)\leq c(h,b,B_1)=\frac{4hB_1^2}{b^2}.$$

Proof. Let $Z_n=\mathbb{E}\left(X_{n+1}-X_n\,|\,\mathcal{F}_n\right)\geq 0$. Then

$$S_t=X_0-(X_t-Z_1-Z_2-\cdots-Z_t)=(X_0-X_t)+Z_1+\cdots+Z_t\geq X_0-X_t$$

is a square-integrable martingale with $S_0=0$, since $|S_n|\leq|X_0|+2nB_1$. Moreover, since

$$\begin{aligned}
\mathbb{E}\left((S_t-S_{t-1})^2\,|\,\mathcal{F}_{t-1}\right) &= \mathbb{E}\left((X_t-X_{t-1}-Z_t)^2\,|\,\mathcal{F}_{t-1}\right)\\
&= \mathbb{E}\left((X_t-X_{t-1})^2\,|\,\mathcal{F}_{t-1}\right)-Z_t^2\leq B_1^2
\end{aligned}$$

we have

$$A_n:=\sum_{t=1}^n\mathbb{E}\left((S_t-S_{t-1})^2\,|\,\mathcal{F}_{t-1}\right)\leq nB_1^2.$$

By Doob's maximum ^{2} inequality (see Durrett, pp. 254–255),

$$\mathbb{E}\left(\sup_{0\le m\le n}|S_m|^2\right)\le 4\mathbb{E}\,S_n^2=4A_n\le 4nB_1^2.$$

Consequently, by Chebyshev's inequality

$$\begin{aligned}\mathbb{P}\left(\inf_{0\le t\le hx^2}X_t<X_0-bx\right) &= \mathbb{P}\left(\sup_{0\le t\le hx^2}X_0-X_t>bx\right)\\ &\le \mathbb{P}\left(\sup_{0\le t\le hx^2}|S_t|>bx\right)<\frac{4(hx^2)B_1^2}{b^2x^2}=\frac{4hB_1^2}{b^2}.\end{aligned}$$

■

3 Transience

Theorem 1. *Consider a Markov process X_t, $t=0,1,2,\dots$ on $\mathbb{R}_+$ with increments $D_t=X_t-X_{t-1}$ which satisfies (H1), (H2), and (H3) for some $a>0$. Suppose that for t sufficiently large on the event $\{X_t\ge a\}$ we have either*

(i) for some $\rho>1/2$

$$\mathbb{E}\left(D_{t+1}\,|\,\mathcal{F}_t\right)\ge\frac{\rho X_t}{t},$$

or

(ii) for some $\rho>0$ and $(\alpha,\beta)\in$ **Trans**

$$\mathbb{E}\left(D_{t+1}\,|\,\mathcal{F}_t\right)\ge\frac{\rho X_t^\alpha}{t^\beta}.$$

Then X_t is transient in the sense that for any starting point $X_0=x$ we have

$$\mathbb{P}(\lim_{t\to\infty}X_t=\infty)=1.$$

Proof. Consider $Y_t=t/X_t^2$. Then

$$\begin{aligned}Y_{t+1}-Y_t &= \frac{t+1}{(X_t+D_{t+1})^2}-\frac{t}{X_t^2}=\frac{t+1}{X_t^2}\left[\frac{1}{(1+D_{t+1}/X_t)^2}-\frac{1}{1+1/t}\right]\\ &\le \frac{t+1}{X_t^2}\left[\frac{1}{t}-2\frac{D_{t+1}}{X_t}+3\frac{D_{t+1}^2}{X_t^2}+O\left(\frac{D_{t+1}}{X_t}\right)^3\right]\end{aligned}$$

yielding

$$\mathbb{E}\left(Y_{t+1}-Y_t\,|\,\mathcal{F}_t\right)\le\frac{1+1/t}{X_t^2}Q_t \tag{6}$$

where

$$Q_t=1-2\rho\frac{t^{1-\beta}}{X_t^{1-\alpha}}+3B_1^2\frac{t}{X_t^2}+O(X_t^{-3}).$$

Consider two cases:

(i) $\alpha=\beta=1$, then $Q_t = 1-2\rho+3B_1^2\frac{t}{X_t^2}+O(X_t^{-3})$;

(ii) $(\alpha,\beta)\in$ **Trans**.

In the first case, Q_t and hence (6) are negative as long as $Y_t = t/X_t^2 \le r$ for some positive constant $r<(2\rho-1)/3B_1^2$. (Note that this would imply $X_t \ge \sqrt{t/r}\ge a$ for large enough t). Fix an arbitrary small $\varepsilon>0$ and suppose that for some time s we have $Y_s = s/X_s^2 \le \varepsilon r$. Let

$$\tau=\tau(s)=\inf\{t>s:\ Y_t\ge r\}.$$

Then $Y_{t\wedge\tau}$ is a non-negative supermartingale, hence it a.s. converges to some random limit $Y_\infty=\lim_{t\to\infty}Y_t$. By Fatou's lemma, $\mathbb{E}\,Y_\infty\le Y_s\le\varepsilon r$. On the other hand,

$$\mathbb{E}\,Y_\infty=\mathbb{E}\,(Y_\infty;\ \tau<\infty)+\mathbb{E}\,(Y_\infty;\ \tau=\infty)\ge r\mathbb{P}(\tau<\infty)$$

hence $\mathbb{P}(\tau<\infty)\le\varepsilon$.

Finally, to show that for any $\varepsilon>0$ with probability 1 there is an s such that $s/X_s^2\le\varepsilon r$ we apply Lemma 2. Consequently, $\mathbb{P}(\tau(s)=\infty \text{ for some } s)=1$ yielding $\limsup_{t\to\infty}t/X_t^2\le r$ a.s., and thus $\mathbb{P}(X_t\to\infty)=1$.

Now consider case (ii) and observe that $0\le\beta<1$ and $1-\alpha>0$. Suppose that

$$X_t^{2-2\delta}\le t\le X_t^2 \text{ for some } \delta\in\left(0,\frac{1+\alpha-2\beta}{2(1-\beta)}\right).$$

Then

$$Q_t \;=\; 1-2\rho\frac{t^{1-\beta}}{X_t^{1-\alpha}}\left(1-\frac{3B_1^2}{2\rho}\frac{t^\beta}{X_t^{1+\alpha}}\right)+O(X_t^{-3}).$$

Since $t\le X_t^2$, and $2\beta<\alpha+1$,

$$\frac{t^\beta}{X_t^{1+\alpha}}\le\frac{X_t^{2\beta}}{X_t^{1+\alpha}}=\frac{1}{X_t^{1+\alpha-2\beta}}=o(1),$$

therefore

$$Q(t)\;\le\;1-2\rho\frac{X^{2(1-\beta)(1-\delta)}}{X_t^{1-\alpha}}(1-o(1))+O(X_t^{-3})=1-2\rho X^{2(1-\beta)(1-\delta)-(1-\alpha)}(1-o(1))<0$$

since $2(1-\beta)(1-\delta)-(1-\alpha)>0$ due to the choice of δ. Therefore, on the event $\{X_t^{2-2\delta}\le t\le X_t^2\}$, Y_t is a supermartingale by inequality (6).

Define the following areas:

$$\begin{aligned} M &= \{s,x\ge0:\ x^{2-\delta}>s\},\\ R &= \{s,x\ge0:\ x^{2-2\delta}>s\},\\ L &= \{s,x\ge0:\ x^2<s\}.\end{aligned}$$

By Lemma 2, there will be infinitely many times s for which $s \le X_s^2/2$, so that $(s, X_s) \notin L$. Fix such an s and let

$$\tau = \tau(s) = \inf\{t > s : \ (t, X_t) \in L \cup M\}.$$

Then $Y^{(*)} := Y_{t\wedge\tau(s)}$ is a bounded supermartingale which a.s. converges to $Y_\infty^{(*)}$; we have $\mathbb{E}\, Y_\infty^{(*)} \le 1/2$ and as before obtain that on the event $\{\tau < \infty\}$, $\mathbb{P}(Y_\tau \in L) \le 1/2$ independently of s. Therefore, either $\tau(s) = \infty$ for some s which implies transience immediately, or by Borel-Cantelli lemma there will be infinitely many times s for which $(s, X_s) \in M$. From now assume that the latter is the case.

Consider the sequence of stopping times when (t, X_t) crosses the curve $t = X_t^{2-\delta}$, then reaches either area L or area R before crossing this curve again. Rigorously, suppose that for some $t = \sigma_0$ we have $(t, X_t) \in M$ and it has just entered area M. Set

$$\eta_0 = \inf\{t > \sigma_0 : \ (t, X_t) \in L \cup R\}.$$

Then for $k \ge 0$ let

$$\begin{aligned} \sigma_{k+1} &= \begin{cases} \inf\{t > \eta_k : \ (t, Y_t) \in M\}, & \text{if } (\eta_k, X_{\eta_k}) \in L \\ \inf\{t > \eta_k : \ (t, Y_t) \notin M\}, & \text{if } (\eta_k, X_{\eta_k}) \in R. \end{cases} \\ \eta_{k+1} &= \inf\{t > \sigma_{k+1} : \ (t, Y_t) \in L \cup R\}. \end{aligned}$$

Thus we have

$$\sigma_0 < \eta_0 < \sigma_1 < \eta_1 < \sigma_2 < \eta_2 \ldots$$

Of course, it could happen that one of these stopping times is infinity and hence all the remaining ones equal infinity as well; however this would imply that $(t, X_t) \notin L$ for all large t, which in turn implies transience (recall that we have assumed that we visit the area M infinitely often). Therefore, let us assume from now on that all η_k's and σ_k's are finite.

For $t \ge \sigma_k$, $k \ge 0$, consider a supermartingale $Y^{(k)} = Y_{t\wedge\eta_k}$. Since the jumps of X_t are bounded, $X_{\sigma_k} = \sigma_k^{1/(2-\delta)} + O(1)$ and $\mathbb{E}\, Y_{\eta_k} \le Y_{\sigma_k} = X_{\sigma_k}^{-\delta}(1 + o(1))$ and as before, we obtain that

$$\mathbb{P}\left((\eta_k, X_{\eta_k}) \in L \,|\, \mathcal{F}_{\sigma_k}\right) \le X_{\sigma_k}^{-\delta}(1 + o(1)) = \frac{1 + o(1)}{\sigma_k^{\delta/(2-\delta)}}. \tag{7}$$

On the other hand, starting at (σ_k, X_{σ_k}) it takes a lot of time for (t, X_t) to reach L, and also if $(\eta_k, X_{\eta_k}) \in R$ it takes a lot of time to exit M, since the walk has to go against the drift. More precisely,

$$\sigma_{k+1} - \sigma_k \ge (\eta_k - \sigma_k) 1_{(\eta_k, X_{\eta_k}) \in L} + (\sigma_{k+1} - \eta_k) 1_{(\eta_k, X_{\eta_k}) \in R}. \tag{8}$$

Set $x := X_{\sigma_k}$,

$$h = \frac{1}{2(2B_1 + 1)^2} < \frac{1}{8B_1^2},$$

and observe that since $2hx^2 - x^{2-\delta} > hx^2$

$$\left\{ \inf_{0 \le i \le hx^2} X_{\sigma_k + i} \ge x\sqrt{2h} \right\} \subseteq \{(\sigma_k + i, X_{\sigma_k + i}) \notin L \text{ for all } 0 \le i \le hx^2\}. \tag{9}$$

By Lemma 4, the probability of the LHS of (9) is larger than

$$1 - \frac{4hB_1^2}{(1-\sqrt{2h})^2} = \frac{1}{2}.$$

Similarly, when $(\eta_k, X_{\eta_k}) \in R$ set $y := X_{\eta_k} > x^{\frac{2-\delta}{2-2\delta}}$. Since

$$(y-x)^{2-\delta} - x^{2-\delta} > x^{\frac{(2-\delta)^2}{2-2\delta}}(1+o(1)) - x^{2-\delta} = x^{2+\frac{\delta^2}{2-2\delta}}(1+o(1)) \gg x^2$$

we have

$$\left\{ \inf_{0 \le i \le x^2/(8B_1^2)} X_{\eta_k+i} \ge y - x \right\} \subseteq \left\{ (\eta_k + i, X_{\eta_k+i}) \in M \text{ for all } 0 \le i \le \frac{x^2}{8B_1^2} \right\}. \tag{10}$$

By Lemma 4 the probability of the LHS of (10) is also less than $1/2$. Therefore, since $a < 1/(8B_1^2)$, from (8) we obtain

$$\mathbb{P}\left(\sigma_{k+1} - \sigma_k \ge aX_{\sigma_k}^2\right) > \frac{1}{2}.$$

On the other hand, provided σ_k is large enough,

$$aX_{\sigma_k}^2 = a\sigma_k^{\frac{2}{2-\delta}}(1+o(1)) > 3\sigma_k$$

yielding that for large k for some $C_1 > 0$

$$\sigma_k \ge C_1 4^{k/2} = C_1 2^k.$$

Consequently, the probability in (7) is bounded by

$$\frac{1+o(1)}{(C_1 2^k)^{\frac{\delta}{2-\delta}}},$$

which is summable over k. By the Borel-Cantelli lemma only finitely many events $\{(\eta_k, X_{\eta_k}) \in L\}$ occur, or, equivalently, for large times $(t, X_t) \notin L$. This yields transience. ∎

4 Recurrence

Theorem 2. *Consider a Markov process X_t, $t = 0, 1, 2, \ldots$ on $\mathbb{R}_+$ with increments $D_t = X_t - X_{t-1}$, satisfying (H1) and (H2) for some $a > 0$. Suppose that on the event $\{X_t \ge a\}$ either*

(i) for some $\rho < 1/2$

$$\mathbb{E}\left(D_{t+1} \mid \mathcal{F}_t\right) \le \frac{\rho X_t}{t},$$

or

*(ii) for some $\rho > 0$ and $(\alpha, \beta) \in$ **Rec***

$$\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) \le \frac{\rho X_t^\alpha}{t^\beta}.$$

Then X_t is "recurrent" in the sense that for any starting point $X_0 = x$ we have

$$\mathbb{P}(\exists\, t \ge 0 \text{ such that } X_t < a) = 1.$$

Hence also $\mathbb{P}(X_t < a$ infinitely often$) = 1$.

Proof. Consider $Y_t = X_t^2/t \ge 0$ and assume $X_t \ge a$. Then

$$Y_{t+1} - Y_t = \frac{(X_t + D_{t+1})^2}{t+1} - \frac{X_t^2}{t} = \frac{2tX_tD_{t+1} - X_t^2 + tD_{t+1}^2}{t(t+1)}$$

whence

$$\begin{aligned}(t+1)\mathbb{E}\left(Y_{t+1} - Y_t \,|\, \mathcal{F}_t\right) &= \mathbb{E}\left(D_{t+1}^2 \,|\, \mathcal{F}_t\right) + [2X_t\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) - X_t^2/t] \\ &\le B_1^2 + (2\rho\kappa_t - 1)\, X_t^2/t \le B_1^2 - (1 - 2\rho\kappa_t)\, Y_t \qquad (11)\end{aligned}$$

where

$$\kappa_t = \frac{X_t^{\alpha-1}}{t^{\beta-1}}.$$

Consider the following three cases:

(i) $\alpha = \beta = 1$, then $\kappa_t = 1$;

(ii-a) $\beta > \alpha \ge 1$, then since $X_t \le B_1 t$, $\kappa_t \le B_1^\alpha/t^{\beta-\alpha} \to 0$ as $t \to \infty$;

(ii-b) $\alpha < 1$, $\beta > (\alpha+1)/2$, then whenever $Y_t = X_t^2/t \ge r$ for some fixed positive constant r we have

$$\kappa_t = \frac{1}{X_t^{1-\alpha} t^{\beta-1}} \le \frac{r^{(\alpha-1)/2}}{t^{(1-\alpha)/2} t^{\beta-1}} = \frac{r^{(\alpha-1)/2}}{t^{\beta-(\alpha+1)/2}} \to 0 \text{ as } t \to \infty.$$

(Note that (ii-a) and (ii-b) together cover the set **Rec**.) Set $r = B_1^2/(1-2\rho) > 0$ in the first case, and set $r = 2B_1^2$ otherwise. Then for t sufficiently large from (11) we have

$$\mathbb{E}\left(Y_{t+1} - Y_t \,|\, \mathcal{F}_t\right) \le 0. \qquad (12)$$

Let $s \ge 0$, and set $\tau(s) = \inf\{t \ge s : \; Y_t \le r\}$. Equation (12) yields that $Y_{t\wedge\tau(s)}$ is a supermartingale, therefore a.s. there is a limit $Y_\infty = \lim_{t\to\infty} Y_{t\wedge\tau(s)}$. This implies that either $\tau(s) < \infty$ for infinitely many $s \in \mathbb{Z}_+$, or that there is a (random) S such that $\tau(S) = \infty$. In both cases we conclude that there is a possibly random value Z such that $X_t^2 \le Zt$ for infinitely many times $t_k \in \mathbb{Z}_+$, $k = 1, 2, \dots$.

First, suppose that $\alpha \ge 0$. Then for a fixed t_k define a process $X_t' = X_{t+t_k}$. Set $n = 2X_{t_k}$ and $\gamma = 1/2$, and observe that the process X_t' satisfies the conditions of Lemma 3 with some $c = c(2\beta - \alpha, \rho, Z) > 0$. Indeed, when $X_t \le n$, the drift of X_t is at most of order $n^\alpha/t^\beta \sim$

$1/n^{2\beta-\alpha} \le 1/n$ since $2\beta - \alpha \ge 1$ and $\alpha \ge 0$. Hence, there is a constant $\nu > 0$, independent of t_k, such that

$$\mathbb{P}(X_t,\ t \ge t_k,\ \text{reaches } [0,a] \text{ before } [n,\infty) \,|\, \mathcal{F}_{t_k}) \ge \nu.$$

Therefore, by the second Borel–Cantelli Lemma (Durrett, p. 240) $\{X_t \le a\}$ for infinitely many t's.

Now suppose that $\alpha < 0$. Consider $W_t = X_t^{1-\nu}$ for some $0 < \nu < 1$. Then

$$\begin{aligned}
\mathbb{E}\left(W_{t+1} - W_t \,|\, \mathcal{F}_t\right) &= X_t^{1-\nu}\mathbb{E}\left((1 + D_{t+1}/X_t)^{1-\nu} - 1 \,|\, \mathcal{F}_t\right)\\
&= (1-\nu)X_t^{1-\nu}\mathbb{E}\left(\frac{D_{t+1}}{X_t} - \frac{\nu}{2}\frac{D_{t+1}^2}{X_t^2} + O(X_t^{-3}) \,|\, \mathcal{F}_t\right)\\
&\le (1-\nu)X_t^{-1-\nu}\left(\frac{X_t^{1+\alpha}}{t^\beta} - \frac{\nu B_2}{2} + O(X_t^{-3})\right) \qquad (13)
\end{aligned}$$

Let $n = n_k = \sqrt{Zt_k}$, so that $X_{t_k} \le n$. Since $2\beta > 1 + \alpha$, we can fix an $\zeta > 1$ such that

$$2\beta > \zeta(1+\alpha).$$

Consider the process W_t for $t \in [t_k, \eta]$ where

$$\eta = \eta_k := \inf\{t \ge t_k :\ X_t \le a \text{ or } X_t \ge n^\zeta\}.$$

Then $\eta < \infty$ a.s. from the same argument as in part (i) of Lemma 3.
Moreover, for $t \in [t_k, \eta]$

$$\frac{X_t^{1+\alpha}}{t^\beta} \le \frac{(n^\zeta)^{1+\alpha}}{(n^2/Z)^\beta} = \frac{Z^\beta}{n^{2\beta-\zeta(1+\alpha)}} \to 0 \text{ as } k \to \infty.$$

since $t_k \to \infty$ and hence $n_k \to \infty$. Therefore the RHS of (13) is negative and thus $W_{t\wedge\eta}$ is a supermartingale. Consequently, by the optional stopping theorem

$$n^{1-\nu} \ge X_{t_k}^{1-\nu} = W_{t_k} \ge \mathbb{E}\left(W_\eta \,|\, \mathcal{F}_{t_k}\right) \ge (1-p)(n^\zeta)^{1-\nu}$$

where $p = p_k = \mathbb{P}(X_\eta \le a \,|\, \mathcal{F}_{t_k})$. This implies

$$p_k \ge 1 - \frac{1}{n_k^{(\zeta-1)(1-\nu)}} \to 1 \text{ as } k \to \infty.$$

finishing the proof of the theorem. ■

5 Special cases

5.1 Case $\alpha = \beta \ge 0$

Since we can always rescale the process X_t by a positive constant, in this section we assume that $B_1 = 1$. Then, in turn, it is also reasonable to restrict our attention only to the case $\rho \le 1$, since if the jumps of X_t can be indeed close to 1 with a positive probability, we might have $X \approx t$, and the drift of order $\rho(X/t)^\beta$ with $\rho > 1$ would imply that the drift is in fact larger than $1 = B_1$ leading to a contradiction, so the model would not be properly defined.

Theorem 3 ($\alpha = \beta < 1$)**.** *Consider a Markov process X_t, $t = 0, 1, 2, \dots$ on $\mathbb{R}_+$ with increments $D_t = X_t - X_{t-1}$, satisfying (H1), (H2), and (H3) for some $a > 0$. Suppose that for some $\beta < 1$ and $\rho \in (0, 1]$ on the event $\{X_t \geq a\}$*

$$\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) \geq \rho \left(\frac{X_t}{t}\right)^{\beta},$$

Then X_t is transient.

Proof. The proof is identical to the proof of Theorem 1, case (ii). ∎

Theorem 4 ($\alpha = \beta > 1$)**.** *Consider a Markov process X_t, $t = 0, 1, 2, \dots$ on $\mathbb{R}_+$ with increments $D_t = X_t - X_{t-1}$, satisfying (H1) and (H2) for some $a > 0$. Suppose that for some $\beta > 1$ and $\rho < 1$ on the event $\{X_t \geq a\}$ the process X_t satisfies*

$$\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) \leq \rho \left(\frac{X_t}{t}\right)^{\beta}.$$

Then X_t is recurrent.

Proof. Fix $\zeta \in (\rho, 1)$ and consider $Y_t = X_t / t^{\zeta}$. Then, calculating as before, we obtain

$$\mathbb{E}\left(Y_{t+1} - Y_t \,|\, \mathcal{F}_t\right) \leq \frac{X_t}{t(t+1)^{\zeta}} \left(\rho (X_t/t)^{\beta-1} - \zeta\right).$$

Since $\limsup X_t/t \leq B_1 = 1$ and $\rho < \zeta$, for large t this is negative and hence Y_t is a non-negative supermartingale converging almost sure. On the other hand, $\zeta < 1$, thus implying

$$\lim_{t\to\infty} \frac{X_t}{t} = \lim_{t\to\infty} \frac{Y_t}{t^{1-\zeta}} = 0 \text{ a.s.}$$

and consequently since $\beta > 1$ for some sufficiently large t we have $(X_t/t)^{\beta-1} < 1/4$. Therefore, for large t,

$$\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) \leq \rho \left(\frac{X_t}{t}\right)^{\beta-1} \times \frac{X_t}{t} \leq \frac{1}{4} \frac{X_t}{t},$$

and hence X_t is recurrent by Theorem 2. ∎

The following statement immediately follows from Theorems 1 and 2.

Corollary 1 ($\alpha = \beta = 1$)**.** *Suppose that X_t is a process satisfying (H1), (H2), and (H3) for some $a > 0$.*

(i) If for some $\rho < 1/2$

$$\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) \leq \frac{\rho X_t}{t} \quad \text{when } X_t \geq a$$

then X_t is recurrent.

(ii) If for some $\rho > 1/2$

$$\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) \geq \frac{\rho X_t}{t} \quad \text{when } X_t \geq a$$

then X_t is transient.

5.2 Case $\alpha \le 0$, $\beta = 0$

In this case, the drift is of order ρ/X_t^ν where $\nu = -\alpha \ge 0$. This is the situation resolved by Lamperti [7] and [8].

Theorem 5 ($\alpha = -1$, $\beta = 0$)**.** *Suppose that X_t is a process satisfying (H1) and (H2) for some $a > 0$. Then, when $X_t \ge a$,*

(i) if for some $\rho \le 1/2$

$$\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) \le \rho \frac{\mathbb{E}\left(D_{t+1}^2 \,|\, \mathcal{F}_t\right)}{X_t}$$

then X_t is recurrent;

(ii) if for some $\rho > 1/2$

$$\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) \ge \rho \frac{\mathbb{E}\left(D_{t+1}^2 \,|\, \mathcal{F}_t\right)}{X_t}$$

then X_t is transient.

Corollary 2 ($\alpha \in (-\infty, -1) \cup (-1, 0)$, $\beta = 0$)**.** *Suppose that X_t is a process satisfying (H1) and (H2) for some $a > 0$. Then, when $X_t \ge a$,*

(i) if for some $\nu > 1$

$$\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) \le \frac{\rho}{X_t^\nu}$$

then X_t is recurrent;

(ii) if for some $\nu < 1$

$$\mathbb{E}\left(D_{t+1} \,|\, \mathcal{F}_t\right) \ge \frac{\rho}{X_t^\nu}$$

then X_t is transient.

5.3 Case $2\beta - \alpha = 1$, $-1 \le \alpha \le 1$: open problem

Two cases $\alpha = \beta = 1$ and $\alpha = -1$, $\beta = 0$ are already covered. It is also straightforward that when $\alpha = 0$, $\beta = 1/2$ by the law of iterated logarithm the process is recurrent for any ρ.

Cases $-1 < \alpha < 0$, $\beta = \frac{1}{2}(\alpha + 1)$ and $0 < \alpha < 1$, $\beta = \frac{1}{2}(\alpha + 1)$: unfortunately, we cannot find a general sensible criteria to separate recurrence and transience here, and leave this as an **open problem**.

6 Application to urn models

Fix a constant $\sigma > 0$. Consider a Friedman-type urn process (W_n, B_n), with the following properties. We choose a white ball with probability $W_n/(W_n + B_n)$ and a black ball with a complementary probability; whenever we draw a white (black resp.) ball, we add a random quantity A of white (black resp.) balls and $\sigma - A$ black (white resp.) balls; For simplicity, suppose $0 \le A \le \sigma$ a.s. A special case when A is not random is considered in Freedman (1964). Following his notations, let $\alpha = \mathbb{E}\, A$, $\beta = \sigma - \alpha$, and $\rho = (\alpha - \beta)/\sigma = (\alpha - \beta)/(\alpha + \beta)$. Also assume that $\alpha > \beta > 0$.

It turns out that this urn can be coupled with a random walk described above. Indeed, for $t = 0, 1, 2, \dots$ set $X_t = |W_t - B_t|/(\beta - \alpha) \in \mathbb{Z}_+ \subset \mathbb{R}_+$. Without much loss of generality assume that the process starts at time $(W_0 + B_0)/\sigma \in \mathbb{Z}$, then $t = (W_t + B_t)/\sigma \in \mathbb{Z}$.

Consequently, once $X_t \neq 0$,

$$\mathbb{E}\,(X_{t+1} - X_t \,|\, \mathcal{F}_t) = \frac{1}{2}\left(1 + \frac{(\beta-\alpha)X_t}{\sigma t}\right)(+1) + \frac{1}{2}\left(1 - \frac{(\beta-\alpha)X_t}{\sigma t}\right)(-1) = \frac{\rho X_t}{t}.$$

Corollary 3.3 in Friedman (1965), states that when $\rho > 1/2$, $W_n - B_n = W_0 - B_0$ (equivalently, $X_n = 0$) occurs for finitely many n with a positive probability, and after the Corollary Friedman says that he does not know whether this event has, in fact, probability 1. On the other hand, our Theorem 2 answers this question positively – indeed, a.s. there will be finitely many times when the difference between the number of white and black balls in the urn equals a particular constant.

See Janson [6] and Pemantle [10] for more on urn models.

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