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On the Hedging of American Options in Discrete Time Markets with Proportional Transaction Costs

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Abstract

In this note, we consider a general discrete time financial market with proportional transaction costs as in Kabanov and Stricker [4], Kabanov et al. [5], Kabanov et al. [6] and Schachermayer [10]. We provide a dual formulation for the set of initial endowments which allow to super-hedge some American claim. We show that this extends the result of Chalasani and Jha [1] which was obtained in a model with constant transaction costs and risky assets which evolve on a finite dimensional tree. We also provide fairly general conditions under which the expected formulation in terms of stopping times does not work.

Key words : Sum of random convex cones, Transaction costs, American option.

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1 Introduction

We consider a discrete time financial market with proportional transaction costs. These markets have already been widely studied. In particular, a proof of the fundamental theorem of asset pricing was given in Kabanov and Stricker [4] in the case of finite Ω and further developed in Kabanov et al. [5], [6], Rásonyi [8], Schachermayer [10] among others. In these papers, a super-replication theorem is also provided for European contingent claims. The aim of this paper is to extend this theorem to American options. It is well known that, for frictionless markets, the super-replication price of an American claim admits a dual representation in terms of stopping times. However, it is proved in Chalasani and Jha [1] that, in markets with fixed proportional transaction costs and with assets evolving on a finite dimensional tree, this formulation does not hold anymore. In their setting, they show that a correct dual formulation can be obtained if we replace stopping times by *randomized* stopping times, which amounts to work with what they call “approximate martingale node-measures”. In this paper, we provide a new dual formulation for the price of American option which works in the general framework of $\mathcal{C}$-valued processes as introduced in Kabanov et al. [5], and extends the dual formulation of Chalasani and Jha [1].

The rest of the paper is organized as follows. In Section 2, we describe the model and give the dual formulation. The link between our formulation and the one obtained by Chalasani and Jha [1] is explained in Section 3. Section 4 is devoted to counter-examples. The proof of the dual formulation is provided in Section 5.

2 Model and main result

2.1 Problem formulation

Set $\mathbb{T} = \{0, \dots, T\}$ for some $T \in \mathbb{N} \setminus \{0\}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$. In all this paper, inequalities involving random variables have to be understood in the $\mathbb{P}-$a.s. sense. We assume that $\mathcal{F}_T = \mathcal{F}$ and that $\mathcal{F}_0$ is trivial. Given an integer $d \geq 1$, we denote by $\mathcal{K}$ the set of $\mathcal{C}$-valued processes K such that $\mathbb{R}^d_+ \setminus \{0\} \subset \mathrm{int}(K_t)$ for all $t \in \mathbb{T}$.[2]

Following the modelization of Kabanov et al. [6], for a given $K \in \mathcal{K}$ and $x \in \mathbb{R}^d$, we define the process $V^{x,\xi}$ by

$$V_t^{x,\xi} \quad := \quad x + \sum_{s=0}^{t} \xi_s \ , \ t \in \mathbb{T} \ ,$$

[2]Here, we follow Kabanov et al. [6] and say that a sequence of set-valued mappings $(K_t)_{t\in\mathbb{T}}$ is a $\mathcal{C}$-valued process if there is a countable sequence of $\mathbb{R}^d$-valued $\mathbb{F}$-adapted processes $X^n = (X^n_t)_{t\in\mathbb{T}}$ such that, for every $t \in \mathbb{T}$, $\mathbb{P}-$a.s. only a finite but non-zero number of X^n_t is different from zero and $K_t = \mathrm{cone}\{X^n_t \ , \ n \in \mathbb{N}\}$. This means that K_t is the polyhedral cone generated by the $\mathbb{P}-$a.s. finite set $\{X^n_t \ , \ n \in \mathbb{N} \text{ and } X^n_t \neq 0\}$.

where ξ belongs to

$$\mathcal{A}(K) = \{ \xi \in L^0(\mathbb{R}^d;\mathbb{F}) \text{ s.t. } \xi_t \in -K_t \qquad \text{for all } t \in \mathbb{T}\} ,$$

and, for a random set $E \subset \mathbb{R}^d$ $\mathbb{P}-$a.s. and $\mathcal{G} \subset \mathcal{F}$, $L^0(E;\mathbb{F})$ (resp. $L^0(E;\mathcal{G})$) is the collection of $\mathbb{F}$-adapted processes (resp. $\mathcal{G}$-measurable variables) with values in E. The financial interpretation is the following: x is the initial endowment in number of physical units of some financial assets, ξ_t is the amount of physical units of assets which is exchanged at t and $-K_t$ is the set of affordable exchanges given the relative prices of the assets and the level of transaction costs.
Before to go on, we illustrate this modelization through an example (see also Section 3, Kabanov and Stricker [4] and Kabanov et al. [6]).

Example 2.1 *Let us a consider a currency market with d assets whose price process is modelled by the $(0,\infty)^d$-valued $\mathbb{F}$-adapted process S. Let $\mathbb{M}^d_+$ denote the set of d-dimensional square matrices with non-negative entries. Let λ be a $\mathbb{M}^d_+$-valued $\mathbb{F}$-adapted process and consider the $\mathcal{C}$-valued process $(K_t)_{t\in\mathbb{T}}$ defined by*

$$K_t(\omega) = \left\{ x \in \mathbb{R}^d \;:\; \exists a \in \mathbb{M}^d_+ \text{ s.t. } x^i + \sum_{j\neq i\leq d} a^{ji} - a^{ij}\pi^{ij}_t(\omega) \geq 0 \;\; \forall\, i \leq d \right\} ,$$

where $\pi^{ij}_t := (S^j_t/S^i_t)(1+\lambda^{ij}_t)$ for all $i,j \leq d$ and $t \in \mathbb{T}$. In the above formulation, a^{ij} stands for the number of units of asset j obtained by selling $a^{ij}\pi^{ij}_t$ units of assets i. λ^{ij}_t is the coefficient of proportional transaction costs paid in units of asset i for a transfer from asset i to asset j. If $\xi_t \in -K_t$, then we can find some financial transfers $\eta_t = (\eta^{ij}_t)_{i,j\leq d} \in L^0(\mathbb{M}^d_+;\mathcal{F}_t)$ such that

$$\xi^i_t \leq \sum_{j\neq i\leq d} \eta^{ji}_t - \eta^{ij}_t \frac{S^j_t}{S^i_t}(1+\lambda^{ij}_t) , \quad i \leq d \;\; ,$$

i.e. the global change in the portfolio position is the result of single exchanges, η^{ij}_t, between the different financial assets, after possibly throwing away some units of these assets.
The random set K_t denotes the so-called solvency region, i.e. $V_t \in K_t$ means that, up to an immediate transfer ξ_t, V_t can be transformed into a portfolio with no short-position $\tilde{V}_t = V_t + \xi_t \in \mathbb{R}^d_+$. Observe that we can assume, without loss of generality, that

$$(1+\lambda^{ik}_t)(1+\lambda^{kj}_t) \;\; \geq \;\; (1+\lambda^{ij}_t) \quad i,j,k \leq d,\; t \in \mathbb{T} \,. \tag{2.1}$$

Indeed, if this condition is not satisfied for some i,j,k, then it is cheaper to make transfers from i to j by doing a first exchange from i to k and then an other one from k to j, thus leading to an effective transaction cost coefficient equal to $\tilde{\lambda}^{ij}_t = \Big((\lambda^{ik}_t+1)(\lambda^{kj}_t+1)-1\Big) < \lambda^{ij}_t$. Any optimal strategy should then lead to effective transaction costs satisfying (2.1) and we can therefore assume that this property is satisfied by the original ones.

The set of all portfolio processes with initial endowment x is given by

$$A(x;K) \;\; := \;\; \left\{V^{x,\xi},\ \xi \in \mathcal{A}(K)\right\}$$

so that

$$A_t(x;K) \;\; := \;\; \{V_t,\ V \in A(x;K)\}$$

corresponds to the collection of their values at time $t \in \mathbb{T}$.
It is known from the work of Kabanov and Stricker [4], Kabanov et al. [6], Kabanov et al. [5] and Schachermayer [10], see also the references therein, that, under mild no-arbitrage assumptions (see $NA^s(K)$ and $NA^r(K)$ below), the set $A_T(x;K)$ can be written as

$$\left\{g \in L^0(\mathbb{R}^d;\mathcal{F}) \ : \ \mathbb{E}\left[Z_T \cdot g - Z_0 \cdot x\right] \leq 0, \ \ \text{for all } \ Z \in \mathcal{Z}(K),\ (Z \cdot g)^- \in L^1(\mathbb{R};\mathbb{P})\right\}$$

where $\mathcal{Z}(K)$ is the set of $(\mathbb{F},\mathbb{P})$-martingales Z such that

$$Z_t \ \in \ K_t^* \qquad \text{for all } \ t \in \mathbb{T}\ ,$$

and $K_t^*(\omega)$ denotes the positive polar of $K_t(\omega)$, i.e.

$$K_t^*(\omega) \;\; := \;\; \left\{y \in \mathbb{R}^d \ : \ x \cdot y \geq 0\ , \ \ \text{for all } x \in K_t(\omega)\right\}\ .$$

The operator "$\cdot$" denotes the natural scalar product on $\mathbb{R}^d$ and $L^1(E;\mathbb{P})$ (resp. $L^1(E;\mathbb{F},\mathbb{P})$) is the subset of $\mathbb{P}$-integrable elements of $L^0(E;\mathcal{F})$ (resp. $L^0(E;\mathbb{F})$).

In this paper, we are interested in

$$A^s(x;K) \;\; := \;\; \left\{\vartheta \in L^0(\mathbb{R}^d;\mathbb{F}) \ : \ V - \vartheta \in -\mathcal{A}(K) \text{ for some } V \in A(x;K)\right\}\ ,$$

the set of processes which are dominated by a portfolio in the sense of K: $V_t - \vartheta_t \in K_t$, for all $t \in \mathbb{T}$. The relation $V_t - \vartheta_t \in K_t$ means that there is an immediate financial transaction $\xi_t \in -K_t$ such that $V_t + \xi_t = \vartheta_t$. Hence, $A^s(x;K)$ can be interpreted as the set of American claims ϑ, labelled in physical units of the financial assets, which are super-hedgeable when starting with an initial wealth equal to x.
More precisely, our aim is to provide a dual formulation for

$$\Gamma(\vartheta;K) := \left\{x \in \mathbb{R}^d \ : \ \vartheta \in A^s(x;K)\right\}\ , \tag{2.2}$$

the set of initial holdings x that allow to super-hedge ϑ.

2.2 Dual formulation

In analogy with the standard result for markets without transaction cost, one could expect that $\Gamma(\vartheta;K)$ can admit the dual formulation

$$\Theta(\vartheta;K) = \left\{x \in \mathbb{R}^d \ : \ \sup_{\tau \in \mathcal{T}(\mathbb{T})} \mathbb{E}\left[Z_\tau \cdot \vartheta_\tau - Z_0 \cdot x\right] \leq 0\ , \ \ \text{for all } \ Z \in \mathcal{Z}(K)\right\} \tag{2.3}$$

where $\mathcal{T}(\mathbb{T})$ is the set of all $\mathbb{F}$-stopping times with values in $\mathbb{T}$. However, this characterization does not hold true in general, as shown in Section 4. This phenomenon was already pointed out in Chalasani and Jha [1] in a model consisting of one bank account and one risky asset evolving on a finite dimensional tree. In Chalasani and Jha [1], the authors show that a correct dual formulation can be obtained if we replace stopping times by *randomized stopping times*.

In our general framework, this amounts to introduce a new set of dual variables, see Section 3 for an interpretation in terms of *randomized stopping times*. For $\tilde{\mathbb{P}} \sim \mathbb{P}$, the associated set of dual variables, $\mathcal{D}(K, \tilde{\mathbb{P}})$, is defined as the collection of processes $Z \in L^1(\mathbb{R}^d_+; \mathbb{F}, \tilde{\mathbb{P}})$ such that

$$Z_t \in K_t^* \ \text{ and } \ \bar{Z}_t \ := \ \mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{s=t}^{T} Z_s \mid \mathcal{F}_t\right] \in K_t^* \quad \text{for all } \ t \in \mathbb{T} .$$

Example 2.2 *In the model of Example 2.1, we have*

$$K_t^*(\omega) \ = \ \left\{ y \in \mathbb{R}^d_+ \ : \ y^j S^i_t(\omega) \le y^i S^j_t(\omega)(1+\lambda^{ij}_t(\omega)) \ , \ \ i \ne j \le d \right\} .$$

It follows that $\mathcal{D}(K, \tilde{\mathbb{P}})$ is the collection of processes $Z \in L^1(\mathbb{R}^d_+; \mathbb{F}, \tilde{\mathbb{P}})$ such that

$$Z^j_t S^i_t \le Z^i_t S^j_t (1+\lambda^{ij}_t) \ \text{ and } \ \bar{Z}^j_t S^i_t \le \bar{Z}^i_t S^j_t (1+\lambda^{ij}_t) \ \ \forall\, i,j \le d \ , \ t \in \mathbb{T} .$$

In the following, we shall say that a subset B of $L^0(\mathbb{R}^d; \mathbb{F})$ is closed in measure if it is closed in probability when identified as a subset of $L^0(\mathbb{R}^{d\times(T+1)}; \mathcal{F})$, i.e.

$$v^n \in B \text{ and } \forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}\left[\sum_{t\in\mathbb{T}} \|v^n_t - v_t\| > \varepsilon\right] = 0 \ \Longrightarrow \ v \in B .$$

We then have the following characterization of $A^s(K) := A^s(0; K)$.

Theorem 2.1 *Assume that $A^s(K)$ is closed in measure and that the no-arbitrage condition*

$$NA(K) \ : \quad A_T(0;K) \cap L^0(\mathbb{R}^d_+; \mathcal{F}) \ = \ \{0\}$$

holds. Then, the following assertions are equivalent :

(i) $\vartheta \in A^s(K)$

(ii) for all $\tilde{\mathbb{P}} \sim \mathbb{P}$ and $Z \in \mathcal{D}(K, \tilde{\mathbb{P}})$ such that $(\vartheta \cdot Z)^- \in L^1(\mathbb{R}; \mathbb{F}, \tilde{\mathbb{P}})$ we have

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t=0}^{T} \vartheta_t \cdot Z_t\right] \ \le \ 0 .$$

(iii) for some $\tilde{\mathbb{P}} \sim \mathbb{P}$ we have

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t=0}^{T} \vartheta_t \cdot Z_t\right] \quad \leq \quad 0$$

for all $Z \in \mathcal{D}(K, \tilde{\mathbb{P}})$ such that $(\vartheta \cdot Z)^- \in L^1(\mathbb{R}; \mathbb{F}, \tilde{\mathbb{P}})$.

Since $A^s(K) = A^s(0; K) = A^s(x; K) - x$, this immediately provides a dual formulation for $\Gamma(\vartheta; K)$.

Corollary 2.1 *Let the conditions of Theorem 2.1 hold. Then, for all $\vartheta \in L^0(\mathbb{R}^d; \mathbb{F})$,*

$$\Gamma(\vartheta; K) = \left\{ x \in \mathbb{R}^d \ : \ \mathbb{E}\left[\sum_{t=0}^{T} \vartheta_t \cdot Z_t\right] \leq \bar{Z}_0 \cdot x \ \ \forall\, Z \in \mathcal{D}(K; \mathbb{P}),\ (Z \cdot \vartheta)^- \in L^1(\mathbb{R}; \mathbb{F}, \mathbb{P}) \right\}.$$

Remark 2.1 The integrability condition on $(Z \cdot \vartheta)^-$ is trivially satisfied if there is some $\mathbb{R}^d$-valued constant c such that $\vartheta_t + c \in K_t$ for all $t \in \mathbb{T}$, i.e. the *liquidation value* of ϑ is uniformly bounded from below. Indeed, in that case $Z_t \cdot (\vartheta_t + c) \geq 0$ for all $Z_t \in L^0(K_t^*; \mathcal{F})$.

Following the approach of Kabanov et al. [5] and Kabanov et al. [6] the closure property of $A^s(0; K)$ can be obtained under the general assumption

$$\xi \in \mathcal{A}(K) \text{ and } \sum_{t \in \mathbb{T}} \xi_t = 0 \implies \xi_t \in K_t^0 \quad \text{for all } t \in \mathbb{T} \tag{2.4}$$

where $K^0 = (K_t^0)_{t \in \mathbb{T}}$ is defined by $K_t^0 = K_t \cap (-K_t)$ for $t \in \mathbb{T}$.

Proposition 2.1 *Assume that* (2.4) *holds, then $A^s(K)$ is closed in measure.*

Remark 2.2 1. In the case of efficient frictions, i.e. $K_t^0 = \{0\},\ \ \forall t \in \mathbb{T}$, it is shown in Kabanov et al. [5] that the assumption (2.4) is a consequence of the *strict no-arbitrage* property

$$NA^s(K): \ A_t(0; K) \cap L^0(K_t; \mathcal{F}_t) \quad \subset \quad L^0(K_t^0; \mathcal{F}_t) \ \text{ for all } \ t \in \mathbb{T}\,.$$

The financial interpretation of the assumption $K_t^0 = \{0\}$ is that there is no couple of assets which can be exchanged freely, without paying transaction costs. In the model of Example 2.1, it is easily checked that it is equivalent to $\lambda_t^{ij} + \lambda_t^{ji} > 0$ for all $t \in \mathbb{T}$, because of (2.1).

2. In the case where K_t^0 may not be trivial, (2.4) holds under the *robust no-arbitrage* condition introduced by Schachermayer [10] and further studied by Kabanov et al. [6],

$$NA^r(K): \ NA(\tilde{K}) \text{ holds for some } \tilde{K} \in \mathcal{K} \text{ which dominates } K,$$

where $\tilde{K}$ dominates K if $K_t \setminus K_t^0 \ \subset \ \mathrm{ri}(\tilde{K}_t) \quad$ for all $\ t \in \mathbb{T}$.

In finance, this means that there is a model with strictly bigger transaction costs (in the directions where they are not equal to zero) in which there is still no-arbitrage in the sense of NA.

3. It is shown in Penner [7] that the condition $K_t^0 = \{0\}$ in 1. can be replaced by the weaker one: $L^0(K_t^0; \mathcal{F}_{t-1}) \subset L^0(K_{t-1}^0; \mathcal{F}_{t-1})$ for all $1 \leq t \leq T$. See also Rásonyi [8].

3 Interpretation in terms of "approximate martingale node-measures" and "randomized stopping times"

In this section, we show that the dual formulation of Corollary 2.1 is indeed an extension of the result of Chalasani and Jha [1]. To this purpose, we consider the example treated in the above paper. It corresponds to a financial market with one non-risky asset S^1 and one risky asset S^2. For sake of simplicity, we restrict ourselves to this 2-dimensional case, although it should be clear that the above arguments can be easily extended to the multivariate setting. We also assume that the interest rate associated to the non-risky asset is equal to zero and normalize $S^1 \equiv 1$, otherwise all quantities have to be divided by S^1 which amounts to considering S^1 as a numéraire and working with discounted values. Here, S^2 is a $(0, \infty)$-valued $\mathbb{F}$-adapted process. Given μ and λ in $(0, 1)$, the model considered in Chalasani and Jha [1] corresponds to the $\mathcal{C}$-valued process $(K_t)_{t \in \mathbb{T}}$ defined by

$$K_t(\omega) = \left\{x \in \mathbb{R}^2 \ : \ x^1 + S_t^2(\omega) \left[(1-\mu)x^2 \mathbb{1}_{x^2 \geq 0} + (1+\lambda)x^2 \mathbb{1}_{x^2 < 0}\right] \geq 0\right\} ,$$

so that

$$K_t^*(\omega) = \left\{y \in \mathbb{R}^2_+ \ : \ S_t^2(\omega)(1-\mu)y^1 \leq y^2 \leq S_t^2(\omega)(1+\lambda)y^1\right\} .$$

It follows that $\mathcal{D}(K, \mathbb{P})$ is the collection of processes $Z \in L^1(\mathbb{R}^2_+; \mathbb{F}, \mathbb{P})$ such that

$$S_t^2(1-\mu)Z_t^1 \leq Z_t^2 \leq S_t^2(1+\lambda)Z_t^1 \ \text{ and } \ S_t^2(1-\mu)\bar{Z}_t^1 \leq \bar{Z}_t^2 \leq S_t^2(1+\lambda)\bar{Z}_t^1 \quad (3.1)$$

for all $t \in \mathbb{T}$.

3.1 "Approximate martingale node-measures"

We first provide an alternative dual formulation in terms of what Chalasani and Jha [1] call "approximate martingale node-measures". Although we are not considering a finite probability space, we keep the term "node measure" used in the above paper for ease of comparison.

Given Z in $\mathcal{D}(K, \mathbb{P})$, let us define $\hat{Z}^1 = Z^1/S^1 = Z^1$ and $\hat{Z}^2 = Z^2/S^2$ so that (3.1) can be written equivalently in

$$(1-\mu)\hat{Z}_t^1 \leq \hat{Z}_t^2 \leq (1+\lambda)\hat{Z}_t^1 \ \text{ and } \ S_t^2(1-\mu)E_t^{q^Z}[\mathbf{1}] \leq E_t^{q^Z}[\chi^Z S^2] \leq E_t^{q^Z}[\mathbf{1}]S_t^2(1+\lambda) \quad (3.2)$$

where $\mathbf{1}$ denotes the constant process equal to 1, and, for a bounded from below process α

$$E_t^{q^Z}[\alpha] := \mathbb{E}\left[\sum_{t\le s\le T} q_s^Z \alpha_s \mid \mathcal{F}_t\right] \text{ with } \chi_t^Z := \hat{Z}_t^2/\hat{Z}_t^1 \text{ and } q_t^Z := \hat{Z}_t^1/\mathbb{E}\left[\sum_{s\in\mathbb{T}} \hat{Z}_s^1\right],$$

for all $t \in \mathbb{T}$. Here, we use the convention $0/0 = 0$. Hence, any element Z in $\mathcal{D}(K,\mathbb{P})$ is such that (χ^Z, q^Z) belongs to the set $\mathcal{Q}(K,\mathbb{P})$ of elements (χ, q) of $L^1(\mathbb{R}_+^2;\mathbb{F},\mathbb{P})$ such that, for all $t \in \mathbb{T}$,

$$\mathbb{E}\left[\sum_{t\in\mathbb{T}} q_t\right] = 1,\ \ 1-\mu \le \chi_t \le 1+\lambda \text{ and } S_t^2(1-\mu)E_t^q[\mathbf{1}] \le E_t^q[\chi S^2] \le E_t^q[\mathbf{1}]S_t^2(1+\lambda)\ , \tag{3.3}$$

where $E_t^q[\cdot]$ is defined as above with q in place of q^Z.

The set $\mathcal{Q}(K,\mathbb{P})$ coincides with the set of *approximate martingale node-measure* defined in Chalasani and Jha [1] (see Definition 6.3). More precisely, for $(\chi, q) \in \mathcal{Q}(K,\mathbb{P})$, the *node-measure* is defined by the map

$$(A,B) \in \mathcal{F}\times\mathcal{P}(\mathbb{T}) \quad\mapsto\quad \mathbb{E}\left[\mathbb{1}_A \sum_{t\in\mathbb{T}} q_t \mathbb{1}_{t\in B}\right]$$

where $\mathcal{P}(\mathbb{T})$ is the collection of subsets of $\mathbb{T}$. The term χ does not appear in the formulation of the above paper because the authors do no take into account the transaction costs that are possibly paid when the option is exercised and the hedging portfolio is liquidated.

Conversely, given $(\chi, q) \in \mathcal{Q}(K,\mathbb{P})$ it is clear that we can find $Z \in \mathcal{D}(K,\mathbb{P})$ such that $(\chi^Z, q^Z) = (\chi, q)$. It follows from Corollary 2.1 that for $\vartheta \in L^0(\mathbb{R}^2;\mathbb{F},\mathbb{P})$ such that ϑ^1 and ϑ^2 are uniformly bounded from below, we have

$$\begin{aligned} h(\vartheta;K) &:= \inf\{x^1 \in \mathbb{R} \ :\ (x^1,0) \in \Gamma(\vartheta;K)\} \\ &= \sup_{Z\in\mathcal{D}(K,\mathbb{P})} E_0^{q^Z}[\vartheta^1 + \chi^Z \vartheta^2 S^2] \\ &= \sup_{(\chi,q)\in\mathcal{Q}(K,\mathbb{P})} E_0^q[\vartheta^1 + \chi\vartheta^2 S^2]\ . \end{aligned} \tag{3.4}$$

This extends the dual formulation in terms of *node-measures* obtained by Chalasani and Jha [1], see Theorem 9.1, to the general discrete time case where we take into account the transaction costs that are possibly paid when the option is exercised. Since ϑ corresponds in our framework to a claim labelled in units of the assets, the corresponding amounts are given by the process $(\vartheta^1, \vartheta^2 S^2)$.

3.2 "Randomized stopping times"

In Theorem 9.1 of Chalasani and Jha [1] one can also find an equivalent formulation in terms of *randomized stopping times*. A *randomized stopping time* X is a non negative $\mathbb{F}$-adapted process such that $\sum_{t\in\mathbb{T}} X_t = 1$. We denote by $\mathcal{X}$ the set of

all *randomized stopping times*. Observe that for a stopping time τ, the process defined by $X := (\mathbb{1}_{\tau=t})_{t\in\mathbb{T}}$ belongs to $\mathcal{X}$. Chalasani and Jha [1] show that there is a one to one correspondence between node measures and pairs $(X,\mathbb{Q})$ where X is a *randomized stopping time* and $\mathbb{Q}$ is a $\mathbb{P}$-absolutly continuous probability measure (see Theorem 5.4 in [1]). This result can be easily extended to our framework as shown below.
Given an adapted process χ such that $(1-\mu) \leq \chi_t \leq (1+\lambda)$ for all $t \in \mathbb{T}$, the $\mathbb{P}$-equivalent measure $\mathbb{Q}$ is called a (χ, X)-*approximate martingale measure* if for all $t \in \mathbb{T}$

$$S_t^2(1-\mu)X_t^+ \leq \mathbb{E}^{\mathbb{Q}}\left[\sum_{t\leq s\leq T} X_s\chi_s S_s^2 \mid \mathcal{F}_t\right] \leq X_t^+ S_t^2(1+\lambda) \tag{3.5}$$

where $X_t^+ := \mathbb{E}^{\mathbb{Q}}\left[\sum_{t\leq s\leq T} X_s \mid \mathcal{F}_t\right]$ for $t\in\mathbb{T}$. Observe that $X_t^+ = \sum_{t\leq s\leq T} X_s$ since X is $\mathbb{F}$-adapted and $\sum_{t\in\mathbb{T}} X_t = 1$. Denoting by $\mathcal{P}(X;K,\mathbb{P})$ the associated set of pairs $(\chi,\mathbb{Q})$ such that the above inequalities hold, we then obtain as in Chalasani and Jha [1] that

$$h(\vartheta;K) \quad = \quad \sup_{X\in\mathcal{X}} \sup_{(\chi,\mathbb{Q})\in\mathcal{P}(X;K,\mathbb{P})} \mathbb{E}^{\mathbb{Q}}\left[\sum_{t\in\mathbb{T}} X_t\left(\vartheta_t^1 + \chi_t\vartheta_t^2 S_t^2\right)\right] =: h(\mathcal{X})\ . \tag{3.6}$$

Here again, the χ is added to the formulation of Chalasani and Jha [1] to take into account the transaction costs that are possibly paid when the option is exercised.
To obtain the last equality first, we argue in two steps :
1. First, take $(\chi,q)\in\mathcal{Q}(K,\mathbb{P})$ and define

$$N_t = \mathbb{E}\left[\sum_{t\leq s\leq T} q_s \mid \mathcal{F}_t\right] \quad \text{and} \quad D_t = \mathbb{E}\left[\sum_{t\leq s\leq T} q_s \mid \mathcal{F}_{t-1}\right]$$

for all $0\leq t\leq T$ with the convention $\mathcal{F}_{-1}=\mathcal{F}_0$. Then, let the $\mathbb{F}$-adapted processes H and X be defined inductively by $(H_0,X_0)=(1,q_0)$, $H_{t+1}=H_tN_{t+1}/D_{t+1}$ and $X_t = q_t/H_t$ for all $1\leq t\leq T$. Then, H is a martingale starting from 1 and we can define the associated probability measure $\mathbb{Q}$ by $d\mathbb{Q}/d\mathbb{P}=H_T$. Moreover, one easily checks by using an inductive argument and the identity $D_{t+1}=N_t-q_t$ that, for all $1\leq k\leq T$, one has $\sum_{j=0}^k X_{T-j} = N_{T-k}/H_{T-k}$. For $k=T$, this shows that $\sum_{t=0}^T X_t = N_0/H_0 = 1$, recall the left hand-side of (3.3). Hence, X is a randomized stopping time. Rewritting q in terms of (H,X) in (3.3)-(3.4), one obtains (3.5) and the expectation entering in the definition of $h(\mathcal{X})$. This shows that $h(\mathcal{X})\geq h(\vartheta;K)$.
2. Conversely, observe that for $X\in\mathcal{X}(K,\mathbb{P})$ and $(\chi,\mathbb{Q})\in\mathcal{P}(X;K,\mathbb{P})$, then (χ,q) $\in\mathcal{Q}(K,\mathbb{P})$ with q defined by $q_t := X_tH_t/\mathbb{E}\left[\sum_{s\in\mathbb{T}} X_sH_s\right]$ and $H_t := \mathbb{E}\left[d\mathbb{Q}/d\mathbb{P}\mid\mathcal{F}_t\right]$ for $t\in\mathbb{T}$. In view of (3.4), this shows that $h(\mathcal{X})\leq h(\vartheta;K)$.

4 Counter examples

In this section, we first show that the duality relation

$$\mathbf{D}(K): \ \Gamma(\vartheta; K) = \Theta(\vartheta; K) \ \text{ for all } \ \vartheta \in L^0(\mathbb{R}^d; \mathbb{F})$$

does not hold for a large class of $\mathcal{C}$-valued process $K \in \mathcal{K}$ (recall the definitions of Γ and Θ in equations (2.2) and (2.3)). For $x \in \mathbb{R}^d$, let us define

$$c_t(x) \ := \ \min\{c \in \mathbb{R} \ : \ c\mathbf{1}_1 - x \in K_t\} \ . \tag{4.1}$$

In financial terms, $c_t(x)$ is the minimal amount, in terms of the first asset, necessary to dominate x in the sense of K_t at time t. If the first asset is interpreted as a numeraire, it corresponds to the *constitution value* of x in terms of this numeraire. Here, $\mathbf{1}_1$ stands for the $\mathbb{R}^d$ vector $(1, 0, \ldots, 0)$.

Proposition 4.1 *If there exists $x \in \mathbb{R}^d$ such that*

(i) $y - c_0(x)\mathbf{1}_1 \in K_0^0 \Rightarrow y - x \in K_0^0$ or $\mathbb{P}[y - x \in K_1] < 1$

(ii) $x - c_0(x)\mathbf{1}_1 \notin K_0$.

Then, there exists ϑ such that $\Theta(\vartheta; K) \neq \Gamma(\vartheta; K)$, i.e. $\mathbf{D}(K)$ is not satisfied.

The proof is postponed to the end of the section.

Remark 4.1 Condition (ii) means that there are directions with efficient frictions at time 0. Condition (i) has the following interpretation. If a portfolio y is equivalent to the *constitution value* of x then it dominates x in the sense of K_0. However, since x and y have the same constitution value, $c_0(x) = c_0(y)$, it can not be *too large*. In particular, if it is not equivalent to x, then it can not dominate x component by component. In that case, we assume that there is randomness enough so that the probability that y still dominates x at time 1 is less than 1.

Remark 4.2 1. If $K_0^0 = \{0\}$ and $x \neq c_0(x)\mathbf{1}_1$ then (ii) holds since by definition we already have $c_0(x)\mathbf{1}_1 - x \in K_0$. If we also assume that $\mathbb{P}[c_1(x) > c_0(x)] > 0$ then (i) is satisfied too.

Example 4.1 *1. Efficient frictions:* consider the following cones

$$K_t \ = \ \left\{(x^1, x^2) \in \mathbb{R}^2 \ : x^1 + (1+\lambda_t)x^2 \geq 0 \ , \ x^1 + (1-\mu_t)x^2 \geq 0\right\},$$

where $t \in \mathbb{T} := \{0, 1\}$, $\lambda_0 < \lambda_1$ and $\mu_0, \mu_1 \in (0, 1)$. Observe that $K_0^0 = \{0\}$. For $x = (0, 1)$, $c_0(x) = 1 + \lambda_0 < c_1(x) = 1 + \lambda_1$. Then, the conditions of the remark above hold so that $\mathbf{D}(K)$ is not true.

2. Partial frictions: consider the preceding case where we add an asset which has no transaction cost with the first one, i.e.

$$K_t \ = \ \left\{(x^1, x^2, x^3) \in \mathbb{R}^3 \ : x^1 + (1+\lambda_t)x^2 + x^3 \geq 0 \ , \ x^1 + (1-\mu_t)x^2 + x^3 \geq 0\right\}.$$

We put $x = (0, 1, 0)$ so that assumption (ii) holds. We now check (i). It is clear that if $y - c_0(x)\mathbf{1}_1 \in K_0^0$ then $y - x \notin K_0^0$. Observe that $y = (y^1, 0, y^3)$ with $y^1 + y^3 = c_0(x)$, so $y^1 + (-1)(1+\lambda_1) + y^3 < 0$ which implies that $y - x \notin K_1$.

On the contrary, we can also show that $\mathbf{D}(K)$ does not *only hold* in the case where $K_t = K_t^0 + \mathbb{R}^d_+$, i.e. there is no transaction costs.

Proposition 4.2 *There exists $(\Omega, \mathcal{F}, \mathbb{P})$ and $K \in \mathcal{K}$ such that $NA(K)$ holds, $K_t^0 = \{0\}$ for all t, and such that for all $\vartheta \in L^0(\mathbb{R}^d; \mathbb{F})$ we have $\Theta(\vartheta; K) = \Gamma(\vartheta; K)$.*

Proof. We take Ω trivial, i.e. $|\Omega| = 1$ with $\mathcal{F}_0 = \mathcal{F}_T = \{\Omega, \emptyset\}$, and put $K = K_0$ constant. Then, $x \in \Theta(\vartheta; K)$ reads $\sup\limits_{Z_t \in K_t^*} Z_t \cdot (\vartheta_t - x) \leq 0$, i.e. $x - \vartheta \in K_t$ for all $t \in \mathbb{T}$. □

This example shows that, for $\mathbf{D}(K)$ to be wrong, we need not only to have non zero transaction costs but also enough randomness in the direction where transaction costs are positive.

Proof of Proposition 4.1: Let x be such that $(i)-(ii)$ are satisfied. We consider the asset ϑ defined by $\vartheta_t = c_0(x)\mathbf{1}_1 \mathbb{1}_{\{t=0\}} + x\mathbb{1}_{\{t>0\}}$. By the martingale property of Z,

$$
\begin{aligned}
\sup_{\tau \in \mathcal{T}(\mathbb{T})} \mathbb{E}\left[Z_\tau \cdot \vartheta_\tau - Z_0 \cdot (c_0(x)\mathbf{1}_1)\right] &= \sup_{\tau \in \mathcal{T}(\mathbb{T})} \mathbb{E}\left[Z_\tau \cdot (x - c_0(x)\mathbf{1}_1)\mathbb{1}_{\{\tau>0\}}\right] \\
&= \max\{0\ ;\ Z_0 \cdot (x - c_0(x)\mathbf{1}_1)\}
\end{aligned}
$$

which is non positive by (4.1). Hence, $c_0(x)\mathbf{1}_1 \in \Theta(\vartheta; K)$. If $\mathbf{D}(K)$ holds, then there exists a portfolio $V \in A(c_0(x)\mathbf{1}_1; K)$ such that $V_0 - c_0(x)\mathbf{1}_1 \in K_0$ and therefore $V_0 - c_0(x)\mathbf{1}_1 \in K_0^0$. By (i) there are two cases. If $V_0 - x \in K_0^0$, then $x - c_0(x)\mathbf{1}_1 \in K_0^0 \subset K_0$ which is a contradiction to (ii). If $\mathbb{P}[V_0 - x \in K_1] < 1$, we can not have $V_1 - x = V_0 + \xi_1 - x \in K_1$ with $\xi_1 \in -K_1$. □

5 Proofs

In this section, we first provide the proof of Theorem 2.1. It follows from standard arguments based on the Hahn-Banach separation theorem. For ease of notations, we simply write $A(K)$ and $A^s(K)$ in place of $A(0; K)$ and $A^s(0; K)$. We denote by $\mathcal{L}^0$ the set of $\mathbb{F}$-adapted processes with values in $\mathbb{R}^d$ and by $\mathcal{L}^1(\tilde{\mathbb{P}})$ (resp. $\mathcal{L}^\infty$) the subset of these elements which are $\tilde{\mathbb{P}}$-integrable, $\tilde{\mathbb{P}} \sim \mathbb{P}$, (resp. bounded). Observe that $\mathcal{L}^0$ (resp. $\mathcal{L}^\infty$) can be identified as a subset of $L^0(\mathbb{R}^{d\times(T+1)}; \mathcal{F})$ (resp. $L^\infty(\mathbb{R}^{d\times(T+1)}; \mathcal{F})$, the set of bounded elements of $L^0(\mathbb{R}^{d\times(T+1)}; \mathcal{F})$).

Proposition 5.1 *Let the conditions of Theorem 2.1 hold. Then, for all $\tilde{\mathbb{P}} \sim \mathbb{P}$, there is some $Z \in \mathcal{D}(K; \tilde{\mathbb{P}}) \cap \mathcal{L}^\infty$ such that*

$$
\sup_{\vartheta \in A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}})} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t \in \mathbb{T}} Z_t \cdot \vartheta_t\right] \leq 0 .
$$

Proof. Since $A^s(K)\cap\mathcal{L}^1(\tilde{\mathbb{P}})$ is closed in $L^1(\mathbb{R}^{d\times(T+1)};\mathcal{F},\tilde{\mathbb{P}})$ (when identified with a subset of $L^1(\mathbb{R}^{d\times(T+1)};\mathcal{F},\tilde{\mathbb{P}})$) and convex, it follows from the Hahn-Banach separation theorem, $NA(K)$ and the fact that $A^s(K)\cap\mathcal{L}^1(\tilde{\mathbb{P}})$ is a cone which contains $-\mathcal{L}^\infty$, that there some $\eta=(\eta_t)_{t\in\mathbb{T}}\in L^\infty(\mathbb{R}_+^{d\times(T+1)};\mathcal{F})$ such that

$$\sup_{\vartheta\in A^s(K)\cap\mathcal{L}^1(\tilde{\mathbb{P}})}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t\in\mathbb{T}}\eta_t\cdot\vartheta_t\right]\ \le\ 0\,. \tag{5.1}$$

By possibly replacing η_t by $\mathbb{E}\left[\eta_t\mid\mathcal{F}_t\right]$, we can assume that η is $\mathbb{F}$-adapted. Fix some arbitrary $\xi\in\mathcal{A}(K)\cap\mathcal{L}^\infty$, so that $V^{0,\xi}\in A^s(K)\cap\mathcal{L}^1(\tilde{\mathbb{P}})$. Since

$$\sum_{t\in\mathbb{T}}\eta_t\cdot V_t^{0,\xi}\ =\ \sum_{t\in\mathbb{T}}\xi_t\cdot\left(\sum_{s=t}^{T}\eta_s\right)$$

we deduce from the above inequality that

$$\sup_{\xi\in\mathcal{A}(K)\cap\mathcal{L}^\infty}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t\in\mathbb{T}}\bar{\eta}_t\cdot\xi_t\right]\ \le\ 0\,,$$

where we defined

$$\bar{\eta}_t\ :=\ \mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{s=t}^{T}\eta_s\mid\mathcal{F}_t\right]\quad t\in\mathbb{T}\,.$$

This shows that $\bar{\eta}_t\in K_t^*$ for all $t\in\mathbb{T}$. For an arbitrary bounded element ξ_t in $L^0(K_t;\mathcal{F}_t)$, the process $V_s^{0,\xi}=-\mathbb{1}_{\{s=t\}}\xi_t$, $s\in\mathbb{T}$, belongs to $A^s(K)$. In view of (5.1), this implies that $\eta_t\in K_t^*$. □

Proposition 5.2 *Let the conditions of Theorem 2.1 hold. Fix $\tilde{\mathbb{P}}\sim\mathbb{P}$ and $\vartheta\in\mathcal{L}^1(\tilde{\mathbb{P}})$. If*

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t=0}^{T}\vartheta_t\cdot Z_t\right]\ \le\ 0$$

for all $Z\in\mathcal{D}(K,\tilde{\mathbb{P}})$ such that $\vartheta\cdot Z\in\mathcal{L}^1(\tilde{\mathbb{P}})$, then $\vartheta\in A^s(K)$.

Proof. Since $A^s(K)\cap\mathcal{L}^1(\tilde{\mathbb{P}})$ is closed and convex, if $\vartheta\notin A^s(K)$, we can find some $\eta=(\eta_t)_{t\in\mathbb{T}}\in L^\infty(\mathbb{R}^{d\times(T+1)};\mathcal{F})$ such that

$$\sup_{\tilde{\vartheta}\in A^s(K)\cap\mathcal{L}^1(\tilde{\mathbb{P}})}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t=0}^{T}\tilde{\vartheta}_t\cdot\eta_t\right]\ <\ \mathbb{E}^{\tilde{\mathbb{P}}}\left[\sum_{t=0}^{T}\vartheta_t\cdot\eta_t\right]\,.$$

By the same arguments as in the proof of Proposition 5.1, we can assume that η is $\mathbb{F}$-adapted and show that $\eta_t\in K_t^*$ and $\bar{\eta}_t\in K_t^*$ for all $t\in\mathbb{T}$. Hence, $\eta\in\mathcal{D}(K,\tilde{\mathbb{P}})$ which leads to a contradiction. □

Proof of Theorem 2.1 1. In view Proposition 5.2, the implication (ii) $\Rightarrow$ (i) is obtained by considering $\tilde{\mathbb{P}}$ with density with respect to $\mathbb{P}$ defined by $H/\mathbb{E}\left[H\right]$ with $H:=\exp(-\sum_{t\in\mathbb{T}}\|\vartheta_t\|)$.

2. It is clear that (ii) implies (iii). For the reverse implication, let $\bar{\mathbb{P}}$ be the probability measure satisfying (iii). Let $\tilde{\mathbb{P}} \sim \mathbb{P}$, $Z \in \mathcal{D}(K, \tilde{\mathbb{P}})$ and ϑ be such that $(\vartheta \cdot Z)^- \in L^1(\mathbb{R}; \mathbb{F}, \tilde{\mathbb{P}})$. Set $\tilde{H}_t := \mathbb{E}\left[d\tilde{\mathbb{P}}/d\bar{\mathbb{P}} \mid \mathcal{F}_t\right]$. Then, $\tilde{H}Z \in \mathcal{D}(K, \bar{\mathbb{P}})$ and $(\vartheta \cdot (\tilde{H}Z))^- \in L^1(\mathbb{R}; \mathbb{F}, \bar{\mathbb{P}})$. We can then apply the inequality of (iii) to $\tilde{H}Z$ and ϑ under $\bar{\mathbb{P}}$. This implies: $\mathbb{E}^{\tilde{\mathbb{P}}}[\sum_{t\in\mathbb{T}} \vartheta_t \cdot Z_t] \leq 0$.

3. The last implication (i) $\Rightarrow$ (ii) is trivial. Indeed, recall that, for $\xi \in \mathcal{A}(K)$,

$$\mathbb{E}\left[\sum_{t\in\mathbb{T}} Z_t \cdot V_t^{0,\xi}\right] \;=\; \mathbb{E}\left[\sum_{t\in\mathbb{T}} \bar{Z}_t \cdot \xi_t\right] .$$

Since $\bar{Z}_t \in L^0(K_t^*; \mathcal{F}_t)$ and $\xi_t \in L^0(-K_t; \mathcal{F}_t)$, the last term is non-positive. Moreover, $V_t^{0,\xi} - \vartheta_t \in L^0(K_t; \mathcal{F}_t)$ implies $Z_t \cdot V_t^{0,\xi} \geq Z_t \cdot \vartheta_t$. □

We now provide the proof of Proposition 2.1. The following Lemma can be found in Kabanov and Stricker [3].

Lemma 5.1 *Set $\mathcal{G} \subset \mathcal{F}$ and E be a closed subset of $\mathbb{R}^d$. Let $(\eta^n)_{n\geq 1}$ be a sequence in $L^0(E; \mathcal{G})$. Set $\tilde{\Omega} := \{\liminf_{n\to\infty} \|\eta^n\| < \infty\}$. Then, there is an increasing sequence of random variables $(\tau(n))_{n\geq 1}$ in $L^0(\mathbb{N}; \mathcal{G})$ such that $\tau(n) \to \infty$ and , for each $\omega \in \tilde{\Omega}$, $\eta^{\tau(n)}(\omega)$ converges to some $\eta^*(\omega)$ with $\eta^* \in L^0(E; \mathcal{G})$.*

Proof of Proposition 2.1. We use an inductive argument. For $t \in \mathbb{T}$, we denote by Σ_t the set of processes $\vartheta \in \mathcal{L}^0$ such that

$$\exists\, \xi \in \mathcal{A}(K) \text{ s.t. } \sum_{s=t}^{\tau} \xi_s - \vartheta_\tau \;\in\; K_\tau \quad \text{for all } \; t \leq \tau \leq T .$$

Clearly, Σ_T is closed in measure. Assume that Σ_{t+1} is closed and let ϑ^n be a sequence in Σ_t such that $\vartheta_s^n \to \vartheta_s$ for $t \leq s \leq T$. Let $\xi^n \in \mathcal{A}(K)$ be such that

$$\sum_{s=t}^{\tau} \xi_s^n - \vartheta_\tau^n \;\in\; K_\tau \quad \text{for all } \; t \leq \tau \leq T .$$

Set $\tilde{\Omega} := \{\liminf_{n\to\infty} \|\xi_t^n\| < \infty\}$. Since $\tilde{\Omega} \in \mathcal{F}_t$, we can work separately on $\tilde{\Omega}$ and $\tilde{\Omega}^c$.

1. If $\mathbb{P}\left[\tilde{\Omega}\right] = 1$, after possibly passing to a random sequence (see Lemma 5.1), we can assume that ξ_t^n converges $\mathbb{P}$ − a.s. to some $\xi_t \in L^0(-K_t; \mathcal{F}_t)$. Since

$$\sum_{s=t+1}^{\tau} \xi_s^n - (\vartheta_\tau^n - \xi_t^n) \;\in\; K_\tau \quad \text{for all } \; t+1 \leq \tau \leq T ,$$

and Σ_{t+1} is closed, we can find some $\tilde{\xi} \in \mathcal{A}(K)$ such that

$$\sum_{s=t+1}^{\tau} \tilde{\xi}_s - (\vartheta_\tau - \xi_t) \;\in\; K_\tau \quad \text{for all } \; t+1 \leq \tau \leq T .$$

This shows that $\vartheta \in \Sigma_t$.
2. If $\mathbb{P}\left[\tilde{\Omega}\right] < 1$, then we can assume without loss of generality that $\mathbb{P}\left[\tilde{\Omega}\right] = 0$. Following line by line the proof of Lemma 2 in Kabanov et al. [6] and using the K_s's closure property, we can find some $\hat{\xi} \in \mathcal{A}(K)$ with $\|\hat{\xi}_t\| = 1$ such that

$$\kappa_\tau := \sum_{s=t}^{\tau} \hat{\xi}_s \quad \in \quad K_\tau \quad \text{for all } \; t \le \tau \le T \; .$$

Since $0 = \sum_{s=t}^{\tau} \hat{\xi}_s - \kappa_\tau = \sum_{s=t}^{\tau-1} \hat{\xi}_s + (\hat{\xi}_\tau - \kappa_\tau)$, and $\hat{\xi}_\tau, -\kappa_\tau \in -K_\tau$, we deduce from (2.4) that $\hat{\xi}_\tau - \kappa_\tau \in K^0_\tau$. Therefore,

$$\hat{\xi}_\tau \in K^0_\tau \;\text{ and }\; \kappa_\tau = \sum_{s=t}^{\tau} \hat{\xi}_s \quad \in \quad K^0_\tau \quad \text{for all } \; t \le \tau \le T \; . \tag{5.2}$$

Since $\|\hat{\xi}_t\| = 1$, there is a partition of $\tilde{\Omega}$ into disjoint subsets $\Gamma_i \in \mathcal{F}_t$ such that $\Gamma_i \subset \{(\hat{\xi}_t)^i \neq 0\}$ for $i = 1, \ldots, d$. We then define

$$\check{\xi}^n_s \quad = \quad \sum_{i=1}^{d} \left(\xi^n_s - \beta^{n,i}_t \hat{\xi}_s \right) \mathbb{1}_{\Gamma_i} \quad s \in \mathbb{T}$$

with $\beta^{n,i}_t = (\xi^n_t)^i / (\hat{\xi}_t)^i$ on Γ_i, $i = 1, \ldots, d$. In view of (5.2) and definition of ξ^n, we have

$$\sum_{s=t}^{\tau} \check{\xi}^n_s - \vartheta^n_\tau \quad \in \quad K_\tau \quad \text{for all } \; t \le \tau \le T \; ,$$

since $K_\tau - K^0_\tau \subset K_\tau, \;\; \tau \in \mathbb{T}$. We can then proceed as in Kabanov et al. [6] and obtain the required result by repeating the above argument with $(\check{\xi}^n)_{n\ge 1}$ instead of $(\xi^n)_{n\ge 1}$ and by iterating this procedure a finite number of times. $\square$

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