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ALPHA-STABLE BRANCHING AND BETA-COALESCENTS

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Abstract: We determine that the continuous-state branching processes for which the genealogy, suitably time-changed, can be described by an autonomous Markov process are precisely those arising from α-stable branching mechanisms. The random ancestral partition is then a time-changed Λ-coalescent, where Λ is the Beta-distribution with parameters $2-\alpha$ and α, and the time change is given by $Z^{1-\alpha}$, where Z is the total population size. For $\alpha = 2$ (Feller's branching diffusion) and $\Lambda = \delta_0$ (Kingman's coalescent), this is in the spirit of (a non-spatial version of) Perkins' Disintegration Theorem. For $\alpha = 1$ and Λ the uniform distribution on $[0,1]$, this is the duality discovered by Bertoin & Le Gall (2000) between the norming of Neveu's continuous state branching process and the Bolthausen-Sznitman coalescent.

We present two approaches: one, exploiting the 'modified lookdown construction', draws heavily on Donnelly & Kurtz (1999); the other is based on direct calculations with generators.

Key words and phrases: Alpha-stable branching, coalescent, genealogy, lookdown construction.

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1. Introduction and main results

1.1 Introduction

Let $Z = (Z_t)$ be a Feller branching diffusion process satisfying the stochastic differential equation

$$dZ_t = \sqrt{Z_t}\, dW_t, \quad t \geq 0, \tag{1.1}$$

where W is a Wiener process with variance parameter σ^2. Recall (see [16]) that Z arises as a scaling limit as $n \to \infty$ of Galton-Watson processes $(\zeta_k^{(n)})_{k=0,1,...}$ with offspring mean one and offspring variance σ^2, when time is measured in units of n generations and 'mass' is measured in units of n individuals:

$$\big(\frac{1}{n}\zeta^{(n)}_{\lfloor nt \rfloor}\big)_{t\geq 0} \to (Z_t)_{t\geq 0} \quad \text{ in distribution as } n \to \infty. \tag{1.2}$$

Being infinitely divisible, Z_t has a decomposition into a Poissonian superposition of 'clusters':

$$Z_t = \sum_i \chi_i(t), \tag{1.3}$$

where the $\chi_i(t)$ can be thought of as the rescaled sizes of families at time t descended from ancestors at time 0. One way to record who descended from which ancestor at time 0 is to consider a superprocess corresponding to Feller's branching diffusion, i.e. a process (M_t) taking its values in the finite measures μ on $[0,1]$, say, and having generator

$$\mathcal{L}F(\mu) = \frac{\sigma^2}{2}\int_0^1\int_0^1 \mu(da)\delta_a(db)F''(\mu;a,b) \tag{1.4}$$

where $F'(\mu;a) := \frac{\delta F(\mu)}{\delta \mu(a)} := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(F(\mu+\varepsilon\delta_a) - F(\mu))$, and $F''(\mu;a,b) := \delta^2 F(\mu)/\delta\mu(a)\delta\mu(b)$. The total population size process $Z_t := M_t([0,1])$ then is a Feller branching diffusion. For $t > 0$, M_t is a random discrete measure whose atoms $\chi_i(t)\delta_{a_i}$ measure the mass of the offspring at time t descended from an ancestral individual of 'type' a_i at time 0. As long as $Z_t > 0$, the process $R_t := M_t/Z_t$ is well-defined; we will refer to it as the *ratio process*. Generalising a result of [15], Perkins [27] proved that, conditioned on the total population size process Z, the ratio process R_t is a Fleming-Viot process with time inhomogeneous sampling rate. In our setting where there is no spatial motion (i.e. no 'mutation' in the language of genetics) one can therefore think of $(R_t)_{t\geq 0}$ as a time changed Fleming-Viot process (without mutation). Throughout this paper, we shall work in this essentially non-spatial setting in which the Fleming-Viot process simply encodes common ancestry of individuals in the population.

To re-phrase Perkins' result in terms of the generator of R, we put

$$z := \mu([0,1]) \text{ and } \rho := \mu/z,$$

and consider functions $F(\mu)$ of the form

$$F(\mu) = G(\rho) = \int \rho(da_1)...\rho(da_p) f(a_1,...,a_p), \tag{1.5}$$

where $p \in \mathbb{N}$ and $f : [0,1]^p \to \mathbb{R}$ is measurable and bounded. For $a = (a_1,...,a_p) \in [0,1]^p$ and $J \subseteq \{1,...,p\}$, we put

$$a_i^J = a_{\min J} \text{ if } i \in J, \text{ and } a_i^J = a_i \text{ if } i \notin J,\ i = 1,...,p. \tag{1.6}$$

Thinking of a as a sample drawn from ρ, passage from a to a^J means a coalescence of a_i, $i \in J$. Then, Perkins' result implies that, for F as in (1.5),

$$\begin{aligned}(\mathcal{L}F)(\mu) &= z^{-1}\sigma^2 \cdot \sum_{J\subseteq\{1,\ldots,p\},|J|=2} \int \rho(da_1)...\rho(da_p)\left(f(a_1^J,...,a_p^J) - f(a_1,...,a_p)\right) \\ &= z^{-1}\sigma^2 \cdot (\mathcal{F}G)(\rho), \end{aligned} \tag{1.7}$$

where $\mathcal{F}$ is the generator of a standard Fleming-Viot process (without mutation).

It is well-known that there is a duality between the Fleming-Viot process and a coalescent process called Kingman's coalescent, which was introduced in [21]. To define Kingman's coalescent, we first introduce the n-coalescent, which is a continuous-time Markov process taking its values in the set $\mathcal{P}_n$ of partitions of $\{1,\ldots,n\}$. If q denotes the transition rate function for the n-coalescent, then for $\eta, \xi \in \mathcal{P}_n$, we have $q(\eta,\xi) = 1$ if ξ can be obtained by merging two blocks of η and $q(\eta,\xi) = 0$ otherwise. Kingman's coalescent is a continuous-time Markov process whose state space is the set $\mathcal{P}$ of partitions of $\mathbb{N} := \{1,2,\ldots\}$ with the property that for each positive integer n, its restriction to $\{1,\ldots,n\}$ is the n-coalescent.

If normed by the total population size Z_t, the atoms of R_t can be used to define the *random ancestral partition* $\Theta_t \in \mathcal{P}$ using Kingman's paintbox construction [20]. To do this, we define an i.i.d. sequence of random variables $(V_i)_{i=1}^{\infty}$ such that $P(V_i = j) = \chi_j/Z_t$ for all j and then construct Θ_t such that two integers i and j are in the same block if and only if $V_i = V_j$. Since (1.7) says that R_t is a Fleming-Viot process run at speed $\sigma^2 Z_t^{-1}$, the duality between the Fleming-Viot process and Kingman's coalescent implies that if $(\Pi_s)_{s\geq 0}$ is Kingman's coalescent, assumed to be independent of Z, and $T(t) := \int_0^t \sigma^2 Z_s^{-1} ds$, then for each $t \geq 0$ we have

$$\Theta_t \stackrel{d}{=} \Pi_{T(t)}. \tag{1.8}$$

Another way to state (1.8) is through the duality relation

$$\mathbb{E}\left[\int R_t(da_1)...R_t(da_p) f(a_1,...,a_p)\right] = \mathbb{E}\left[\int db_1...db_{|\Pi_{T(t)}|} f_{\Pi_{T(t)}}(b_1,...,b_{|\Pi_{T(t)}|})\right], \tag{1.9}$$

where Π is Kingman's p-coalescent starting at $\pi_0 = \{\{1\},...,\{p\}\}$, and, for any partition $\pi = \{C_1,...,C_q\}$ of $\{1,...,p\}$,

$$f_\pi(b_1,...,b_q) := f(a_1,...,a_p)$$

with $a_i := b_k$ if $i \in C_k$.

Equation (1.8), and equivalently the form of the generator (1.7), have an intuitive interpretation. The random partition $\mathcal{P}_t$ arises through a merging of ancestral lines backwards in time, and any two lines not having merged by time s (backwards from t) coalesce at a rate proportional to the offspring variance σ^2, and inversely proportional to the total population size Z_{t-s}^{-1}, where time is measured in the scale of Kingman's coalescent. This genealogical interpretation can be made precise using the lookdown construction of Donnelly & Kurtz, which we explain later (see [10, 11]).

Kingman's coalescent fits into the family of Λ-coalescents introduced in [28] and [30]. On the other hand, Feller's branching diffusion is a special case of the general continuous state branching processes (CSBP's) initially studied by Jiřina [19], Lamperti [22, 23], and Silverstein [33]. Our goal in this paper is to determine for which continuous-state branching processes the genealogy of the process, suitably time-changed, can be described by an autonomous Markov process. Evidently, this must be some form of coalescent. Although a natural question, noone we spoke to seemed aware of a resolution of

the problem. A detailed analysis of the case in which the CSBP has finite variance is of course well known. Moreover Bertoin & Le Gall [2] showed that the genealogy of a continuous-state branching process studied by Neveu could be described by a coalescent process called the Bolthausen-Sznitman coalescent. Donnelly & Kurtz [11] prepared the ground for a unified treatment, even (in Section 3.1.4) introducing the generalised Fleming-Viot processes rediscovered as duals to Λ-coalescent processes in Bertoin & Le Gall [3] and in Section 5.1 briefly discussing Pitman's Λ-coalescents and time changes that lead to the Kingman coalescent. The long list of authors on this paper arose as a coalescence of three independent groups of workers, all of whom thought it worthwhile to raise the profile of the results of [11] pertaining to the discontinuous CSBP's and at the same time to identify the family of measure-valued branching processes which 'factorise' in this way, namely those arising from a stable branching mechanism.

1.2 Λ-coalescents and generalised Fleming-Viot processes

Pitman [28] and Sagitov [30] introduced coalescents with multiple collisions, also called Λ-coalescents, which are coalescent processes in which many clusters can merge at once into a single cluster. Here, Λ is a finite measure on $[0,1]$. As with Kingman's coalescent, the Λ-coalescent is a $\mathcal{P}$-valued Markov process whose law can be prescribed by specifying the law of its restriction to $\{1,\dots,n\}$ for all $n \in \mathbb{N}$. If $(\Pi_s)_{s\geq 0}$ is the restriction of a Λ-coalescent to $\{1,\dots,n\}$, then whenever Π_s has p blocks, each transition that involves j of the blocks merging into one happens at rate

$$\beta^{\Lambda}_{p,j} = \int_{[0,1]} y^{j-2}(1-y)^{p-j}\Lambda(dy), \tag{1.10}$$

and these are the only possible transitions. Note that Kingman's coalescent is the special case in which $\Lambda = \delta_0$.

When $\Lambda(\{0\}) = 0$, Pitman showed that the Λ-coalescent can be constructed from a Poisson point process on $[0,\infty)\times\{0,1\}^{\infty}$ with intensity measure $dt \otimes L(d\xi)$. Here L is the measure on $\{0,1\}^{\infty}$ such that $L(A) = \int_{(0,1]} P_y(A) y^{-2}\,\Lambda(dy)$ for all measurable A, where P_y is the law of an infinite sequence of i.i.d. Bernoulli random variables with success probability y. If (t,ξ) is a point of the Poisson process with $\xi = (\xi_1, \xi_2, \dots)$ and $B_1, B_2, \dots$ are the blocks of the coalescent at time $t-$, ordered by their smallest element, then at time t, all of the blocks B_i such that $\xi_i = 1$ merge together, while the other blocks remain unchanged. That is, for each block we flip an independent coin with probability y of heads to determine which blocks participate in the merger. The measure Λ thus governs the rates of the multiple mergers.

We now define the corresponding generalised Fleming-Viot processes. For Λ a finite measure on $[0,1]$, a Λ-Fleming-Viot process takes its values in the probability measures ρ on $[0,1]$ and has generator

$$(\mathcal{R}G)(\rho) = \sum_{J\subseteq\{1,\dots,p\},|J|\geq 2} \beta^{\Lambda}_{p,|J|} \int \rho(da_1)\dots\rho(da_p)(f(a_1^J,\dots a_p^J) - f(a_1,\dots,a_p)), \tag{1.11}$$

where G is a function of the type defined in (1.5) and $\beta^{\Lambda}_{p,j}$ is defined in (1.10). Note that this terminology slightly differs from that in [3]: Bertoin & Le Gall would call this a ν-generalised Fleming-Viot process, with $\nu(dy) = y^{-2}\Lambda(dy)$. As we see in the proof of Theorem 1.1 in Section 3, when $\Lambda(\{0\}) = 0$, the generator can also be written as

$$(\mathcal{R}G)(\rho) = \int_{(0,1]} y^{-2}\Lambda(dy) \int \rho(da)\big(G((1-y)\rho + y\delta_a) - G(\rho)\big).$$

An intuitive way to think about the generator is to consider a Poisson point process on $\mathbb{R}_+ \times (0,1]$ with intensity measure $dt \otimes y^{-2}\Lambda(dy)$ which picks jump times and sizes for (ρ_t). At a jump time t with corresponding jump size y, ρ_{t-} is modified in the following way: pick a according to ρ_{t-}, insert an atom $y\delta_a$, and scale down ρ_{t-} so that the total mass remains equal to one.

As proved in [2], a Λ-Fleming-Viot process is dual to the Λ-coalescent, mirroring the duality between the standard Fleming-Viot process and Kingman's coalescent established in [8].

1.3 Continuous-state branching processes

A continuous-state branching process $(Z_t)_{t\geq 0}$ is a $[0,\infty]$-valued Markov process such that the sum of independent copies of the process started at x and y has the same distribution as the process started at $x+y$. If one excludes processes with the possibility of an instantaneous jump to ∞, the dynamics of a continuous-state branching process (CSBP) are characterised by a triple (σ^2, γ, ν), where σ^2 and γ are nonnegative real numbers, and ν is a measure on $\mathbb{R}_+$ with

$$\int_0^\infty (h^2 \wedge 1)\nu(dh) < \infty. \tag{1.12}$$

The generator of Z is given by

$$\mathcal{L}_Z f(z) \;=\; z\left(\gamma f'(z) + \frac{\sigma^2}{2} f''(z) + \int_{(0,\infty)} (f(z+h) - f(z) - h1_{(0,1]}(h) f'(z))\nu(dh)\right). \tag{1.13}$$

Furthermore Lamperti [22] and Silverstein (Section 4 of [33]) showed that a CSBP with generator (1.13) can be obtained from a Lévy process with no negative jumps whose Laplace exponent is given by $-\Psi(\cdot)$, with

$$\Psi(u) = \gamma u + \frac{\sigma^2}{2}u^2 + \int_0^\infty (e^{-hu} - 1 + hu1_{(0,1]}(h))\,\nu(dh) \tag{1.14}$$

for $u \geq 0$. More precisely, let $(Y_t)_{t\geq 0}$ be a Lévy process such that $Y_0 = Z_0 = s > 0$ and $E[e^{-\lambda Y_t}] = e^{-\lambda s + t\Psi(\lambda)}$. Define $(\tilde{Y}_t)_{t\geq 0}$ to be the process $(Y_t)_{t\geq 0}$ stopped when it hits zero. If $U_t = \inf\{s : \int_0^s \tilde{Y}_u^{-1}\, du = t\}$, then the processes $(Z_t)_{t\geq 0}$ and $(\tilde{Y}_{U(t)})_{t\geq 0}$ have the same law. The function Ψ is called the branching mechanism of the CSBP.

For some triples (σ^2, γ, ν), the process X may explode in finite time or may go extinct. Let $\tau_\infty = \inf\{t : Z_t = \infty\}$ be the explosion time and let $\tau_0 = \inf\{t : Z_t = 0\}$ be the extinction time. Put $\tau := \tau_\infty \wedge \tau_0$. Grey [18] showed that the process is conservative, meaning that $\tau_\infty = \infty$ a.s., if and only if

$$\int_0^\delta \frac{1}{|\Psi(\lambda)|}\, d\lambda = \infty$$

for $\delta > 0$. To give a condition for extinction, let $q = P(\tau_0 < \infty)$ be the extinction probability. Let $m = -\Psi'(0)$, so $E[Z_t] = e^{mt}$ and the process Z is called critical if $m = 0$, subcritical if $m < 0$, and supercritical if $m > 0$. Grey [18] showed that $q > 0$ if and only if, for sufficiently large θ, we have $\Psi(\theta) > 0$ and

$$\int_\theta^\infty \frac{1}{\Psi(\lambda)}\, d\lambda < \infty.$$

Grey also showed that if $q > 0$, then $q < 1$ if and only if $m > 0$.

1.4 Main result

To study the distribution of the random ancestral partition Θ_t for a general CSBP Z, we consider an infinite types model where each ancestor has its own type. This is described by a measure-valued branching process (M_t) taking its values in the finite measures μ on $[0,1]$, and having generator

$$\begin{aligned}\mathcal{L}F(\mu) &= \gamma\int_0^1 \mu(da)F'(\mu;a) + \frac{\sigma^2}{2}\int_0^1\int_0^1 \mu(da)\delta_a(db)F''(\mu;a,b) \\ &+ \int_0^1 \mu(da)\int_{(0,\infty)} \nu(dh)(F(\mu+h\delta_a)-F(\mu)-1_{(0,1]}(h)hF'(\mu;a)),\end{aligned} \tag{1.15}$$

cf. [14] or [7] and see (2.4) below. We write

$$z := \mu([0,1]),\ \rho := \mu/z,\ Z_t := M_t([0,1]),\ R_t := M_t/Z_t.$$

We are now ready to state our main result.

Theorem 1.1. *The ratio process $(R_t)_{0\le t<\tau}$ can be time-changed with an additive functional of the total mass process (Z_t) to obtain a Markov process if and only if*

(i) *$\nu = 0$, or*
(ii) *$\sigma = 0$, and $\nu(dh) = \text{const}\cdot h^{-1-\alpha}dh$ for some $\alpha\in(0,2)$, i.e. Z is an α-stable continuous state branching process.*

In case (i), let $T_t = \int_0^t \sigma^2 Z_s^{-1}\,ds$ and $T^{-1}(t) = \inf\{s : T_s > t\}$. Then the process $(R_{T^{-1}(t)})_{t\ge0}$ is the classical (non-spatial) Fleming-Viot process, dual to Kingman's coalescent. In case (ii), if we let $T_t = \text{const}\cdot\int_0^t Z_s^{1-\alpha}\,ds$ and define $T^{-1}(t)$ as before, then $(R_{T^{-1}(t)})_{t\ge0}$ is the Λ-Fleming-Viot process, where Λ is the Beta$(2-\alpha,\alpha)$ *distribution, i.e. $\Lambda(du) = C_\alpha\, u^{1-\alpha}(1-u)^{\alpha-1}du$.*

If $T_\tau < \infty$ we understand $(R_{T^{-1}(t)})_{t\ge0}$ to be extended for $t\ge T_\tau$ by an independent Λ-Fleming-Viot process started from $R_{T^{-1}(T_\tau-)}$.

In particular, under (ii) we have the following analogue of (1.9),

$$\mathbb{E}\left[\int R_t(da_1)...R_t(da_p)f(a_1,...,a_p)\right] = \mathbb{E}\left[\int db_1...db_{|\Pi_{T(t)}|}f_{\Pi_{T(t)}}(b_1,...,b_{|\Pi_{T(t)}|})\right], \tag{1.16}$$

in which $(\Pi_t)_{t\ge0}$ is now a Beta$(2-\alpha,\alpha)$-coalescent started from $\{\{1\},\dots,\{p\}\}$.

Note that case (i) is a direct consequence of Perkins' result [27]. It is interesting that case (ii) cannot be strengthened to a direct analogue of the Perkins Disintegration Theorem. In the case $\alpha<2$ conditional on the total mass process, the ratio process is not just a time change of an independent generalised Fleming-Viot process: its jump times are now deterministic.

In case (ii) when $\alpha = 1$, the CSBP Z is the continuous-state branching process that was studied by Neveu [26]. Also, Λ is the uniform distribution on $[0,1]$, so the Λ-coalescent is the Bolthausen-Sznitman coalescent, which was introduced in [5], so this case corresponds to the result of Bertoin and Le Gall [2]. Bertoin and Le Gall's result was used by Bovier and Kurkova [6] in their study of Derrida's generalised random energy models.

Let $(Z_t^{(1)})_{t\ge0}$ and $(Z_t^{(2)})_{t\ge0}$ be two independent α-stable CSBP's. Equation (1.16) tells us, in particular, that for $\alpha\in[0,1]$, the process $(R_{T^{-1}(t)}([0,a]))_{t\ge0}$, which is a time change of the process $(Z_t^{(1)}/(Z_t^{(1)}+Z_t^{(2)}))_{t\ge0}$, is equal to the dual of the block-counting process introduced in Möhle [25]. These results were generalised by Bertoin and Le Gall in [4].

From (1.14) and the fact that $m = -\Psi'(0)$, we see that in case (i), the branching mechanism of Z is $\Psi(u) = -mu + \frac{\sigma^2}{2}u^2$. In case (ii), when $1 < \alpha < 2$, we have $\Psi(u) = -mu + cu^\alpha$ for some constant $c > 0$. When $\alpha = 1$, we have $\Psi(u) = -du + cu \log u$, where $d \in \mathbb{R}$ is a drift coefficient (see [2]), and when $0 < \alpha < 1$, we have $\Psi(u) = -du - cu^\alpha$ for some constant $c > 0$. Note that d (or m in the case $\alpha > 1$) is the drift of the driving Lévy process; as it can no longer be interpreted as a *mean*, we name it d rather than m in case (ii) when $\alpha \leq 1$. By checking Grey's conditions, it easily follows that Z is conservative except in case (ii) when $0 < \alpha < 1$. Also, $q = 0$ only in case (ii) when $\alpha \leq 1$, and otherwise $q = 1$ only when $m \leq 0$.

Proposition 1.2. *In case (i) of Theorem 1.1, we have $T_\tau = \infty$ a.s. if and only if $m \leq 0$. In case (ii), we have $T_\tau = \infty$ a.s. if and only if either $0 < \alpha \leq 1$ and $d \geq 0$ or $1 < \alpha < 2$ and $m \leq 0$.*

Remark 1.3. *We say a Λ-coalescent $(\Pi_t)_{t\geq 0}$ comes down from infinity if Π_t almost surely has only a finite number of blocks for all $t > 0$. When Λ is the Beta$(2-\alpha, \alpha)$ distribution, it is shown in* [31] *that the Λ-coalescent comes down from infinity if $1 < \alpha < 2$ but not when $0 < \alpha \leq 1$. Combining this observation with Theorem 1.1, we obtain the result that for the α-stable CSBP with $1 < \alpha < 2$, only finitely many individuals at time zero have descendants alive in the population at time $t > 0$. This fact is well-known in the superprocesses literature (see, for example,* [9]*). On the other hand, when $0 < \alpha \leq 1$ and $t > 0$, there are infinitely many individuals at time zero who have descendants alive at time t.*

Proposition 1.2 is now easily understood for $1 < \alpha < 2$. Notice that if $m > 0$ there is a positive probability of more than one infinite line of descent in which case it must be that $T_\tau < \infty$.

In case (ii) of the Theorem, we have

$$(\mathcal{L}F)(\mu) = \text{const} \cdot z^{1-\alpha} \cdot (\mathcal{R}G)(\rho).$$

In view of (1.11) this has the following interpretation. A sample of size p from M_{t+dt} is obtained as follows: first in the time interval $(t, t+dt]$ any j-tuple ($2 \leq j \leq p$) merges with probability $\text{const} \cdot \beta^\Lambda_{p,j} Z_t^{1-\alpha} dt + o(dt)$; the resulting ancestors are then sampled from M_t.

There are three regimes for the Z_t-dependent time change which are qualitatively different:

- In the case $1 < \alpha < 2$ (many small jumps), for large population size z the ratio process R_t runs slower – the law of large numbers starts to take effect. Note that when $m \leq 0$, we can have $T_\tau = \infty$ a.s. even when the CSBP goes extinct a.s. because the process R_t runs quickly when the population size gets small.
- In the case $0 < \alpha < 1$ (many large jumps), for large population size z the process R_t runs quicker - a lot of fluctuations happen. Consequently, we can have $T_\tau = \infty$ a.s. even though $P(\tau_\infty < \infty) = 1$.
- In the case $\alpha = 1$, the speed of the ratio process is independent of the population size and no time change is necessary.

1.5 Heuristics

In order to see why a factorisation of the type considered in this paper will work only in the case of a *stable* branching mechanism, we invite our readers to consider a simplified scenario, where only two 'types' are present. Let X and X' be two independent CSBP's with the same characteristics given by (1.14), and denote by $S_t := X_t + X'_t$, $R_t := X_t/S_t$ the total mass resp. the frequency of the first type.

If at some time t the current total mass is S_{t-} and a new family of size $h > 0$ is created (an event which occurs at rate $S_{t-}\nu(dh)dt$), the *relative* mass of the newborns is $y := h/(S_{t-} + h)$, so

$$\Delta R_t = \begin{cases} y(1-R) & \text{with probability } R_{t-} \text{ and} \\ -yR & \text{with probability } (1-R_{t-})\,. \end{cases}$$

Thus, if we want to find a time-change that eliminates the dependence of the relative jump size y on the current total population size S_{t-}, and hence converts the time-changed R into a Markov process in its own right, the Lévy measure ν must satisfy the factorisation property

$$\forall\, s, y > 0\,,\ s\nu(\{h : h/(h+s) > y\}) = s\nu(\{h : h > sy/(1-y)\}) = g(s)f(y)$$

for some functions f, g. One convinces oneself easily that this forces ν to have algebraic tails (see e.g. the proof of Lemma 1 in Section VIII.8 of [17]), and hence X and X' to be stable branching processes. Details can be found in Lemma 3.5.

2. Genealogies and the lookdown construction

The measure-valued process M_t introduced in Section 1.1 allows us to keep track of which individual, at time zero, is the ancestor of an individual in the population at time t. However, if we wish to trace the genealogy of the population by sampling individuals at time t and following the ancestral lines backwards in time, then we need to know who is the ancestor of a given individual at time t for any time $s < t$.

To extend Theorem 1.1 to a result about the genealogies of CSBP's, we first need to give a precise definition of the genealogies of the CSBP's that arise in Theorem 1.1. Several methods for describing the genealogy of CSBP's have been proposed. Bertoin & Le Gall [2] defined the genealogy using a flow of subordinators $(S^{(s,t)}(a))_{0\le s\le t, a\ge 0}$, where, for fixed s and t, the process $(S^{(s,t)}(a))_{a\ge 0}$ is a subordinator whose law depends only on $t - s$ and we interpret $S^{(s,t)}(a)$ as being the size of the population at time t descended from the first a individuals in the population at time s. Alternatively, Le Gall & Le Jan [24] showed, in the case of a finite first moment, how to describe the genealogy of a CSBP by constructing a 'height process' that determines the continuous analogue of a Galton-Watson tree. See [12] for further developments in this direction.

We choose to define the genealogy of CSBP's by using the lookdown construction of Donnelly & Kurtz [11]. (In that paper it is actually refered to as the 'modified' lookdown construction to distinguish it from the construction of the classical Fleming-Viot superprocess introduced by the same authors in [10]. Here we drop the prefix 'modified'.) We can use this construction to define the genealogy of both continuous-state branching processes and generalised Fleming-Viot processes. This construction allows one to represent a measure-valued process as the empirical measure of a countable system of particles. We now describe a special case of the construction which will be sufficient for the results presented here. We refrain from including a 'Brownian component' (corresponding to case (i) in Thm. 1.1) because that case is well known and cumbersome to incorporate.

Let $n = \sum_i \delta_{(t_i,y_i)}$ be a point configuration on $\mathbb{R}_+ \times (0,1]$ with the property that

$$\sum_{i\,:\,t_i\le t} y_i^2 < \infty \quad \text{for all } t \ge 0. \tag{2.1}$$

We think of each particle being identified by a level $j \in \mathbb{N}$. We equip the levels with types ξ_t^j, $j \in \mathbb{N}$ in some type space E (and we think of $E = [0,1]$ to fit into the previous framework). Initially, we

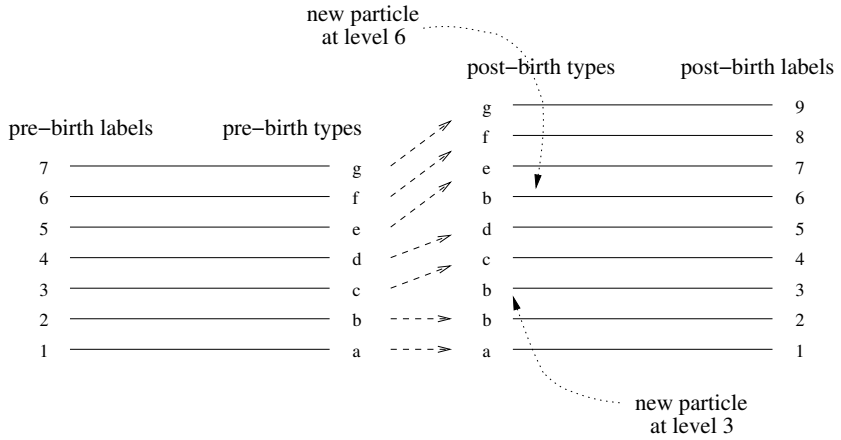

FIGURE 1. Relabelling after a birth event involving levels 2, 3 and 6.

require the types ξ_0^j, $j \in \mathbb{N}$ to be exchangeable and such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \delta_{\xi_0^j} = \frac{\mu}{\mu(E)}$$

for some finite measure μ on E.

In principle, the construction works with any initial distribution of types, not necessarily exchangeable, but then there will be very little to prove about the object obtained. The point is that the construction preserves exchangeability.

The jump times t_i in our point configuration n will correspond to "birth events". Let U_{ij}, $i, j \in \mathbb{N}$, be i.i.d. uniform$([0,1])$. Define for $J \subset \{1,\dots,l\}$ with $|J| \geq 2$,

$$L_J^l(t) := \sum_{i\,:\,t_i \leq t} \prod_{j\in J} 1_{\{U_{ij}\leq y_i\}} \prod_{j\in\{1,\dots,l\}-J} 1_{\{U_{ij}>y_i\}}.$$

$L_J^l(t)$ counts how many times, among the levels in $\{1,\dots,l\}$, exactly those in J were involved in a birth event up to time t. Note that for any configuration $n = \sum \delta_{(t_i,y_i)}$ satisfying (2.1), since $|J| \geq 2$, we have

$$\mathbb{E}[L_J^l(t)] = \sum_{i\,:\,t_i\leq t} y_i^{|J|}(1-y_i)^{l-|J|} \leq \sum_{i\,:\,t_i\leq t} y_i^2 < \infty,$$

so that $L_J^l(t)$ is a.s. finite.

Intuitively, at a jump t_i, each level tosses a uniform coin, and all the levels j with $U_{ij} \leq y_i$ participate in this birth event. Each participating level adopts the type of the smallest level involved. All the other individuals are shifted upwards accordingly, keeping their original order with respect to their levels (see Figure 1). In terms of the sequence $(\xi_t^1, \xi_t^2, \dots)$ this means that if $t = t_i$ is a jump time and

j is the smallest level involved, i.e. $U_{ij} \le y_i$ and $U_{ik} > y_i$ for $k < j$, we put

$$\begin{aligned} \xi_t^k &= \xi_{t-}^k, \quad \text{for } k \le j \\ \xi_t^k &= \xi_{t-}^j, \quad \text{for } k > j \text{ with } U_{ik} \le y_i \\ \xi_t^k &= \xi_{t-}^{k-J_t^k} \quad \text{otherwise,} \end{aligned}$$

where $J_{t_i}^k = \#\{m < k : U_{im} \le y_i\} - 1$. Although the jump times (t_i) may be dense in $\mathbb{R}_+$, assumption (2.1) guarantees that in any finite time interval, the number of 'lookdowns' in which any given pair of type processes ξ^j and ξ^k, $j < k$, is involved is finite. Thus, the processes ξ^j can be defined inductively, details can be found in [11], pp 182. The point of the construction is that for each $t > 0$, $(\xi_t^1, \xi_t^2, \ldots)$ is an exchangeable random vector, so that

$$X_t = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \delta_{\xi_t^j} \tag{2.2}$$

exists almost surely by de Finetti's Theorem.

We now make use of the explicit description of the modified construction to determine the coalescent process embedded in it. Recall the notation L_K^l from above. For each $t \ge 0$ and $k = 1, 2, \ldots$, let $N_k^t(s), 0 \le s \le t$, be the level at time s of the ancestor of the individual at level k at time t. In terms of the L_K^l, for $0 \le s \le t$,

$$\begin{aligned} N_k^t(s) = k &- \sum_{K\subset\{1,\ldots,k\}} \int_{s-}^{t} (N_k^t(u) - \min(K)) 1_{\{N_k^t(u)\in K\}}\, dL_K^k(u) \\ &- \sum_{K\subset\{1,\ldots,k\}} \int_{s-}^{t} (|K \cap \{1,\ldots,N_k^t(u)\}| - 1) 1_{\{N_k^t(u)>\min(K),\, N_k^t(u)\notin K\}}\, dL_K^k(u). \end{aligned} \tag{2.3}$$

Fix $0 \le T$ and, for $t \le T$, define a partition $\mathscr{R}^T(t)$ of $\mathbb{N}$ such that k and l are in the same block of $\mathscr{R}^T(t)$ if and only if $N_k^T(T-t) = N_l^T(T-t)$. Thus, k and l are in the same block if and only if the two levels k and l at time T have the same ancestor at time $T - t$.

We now use this lookdown construction to embed a genealogy into a CSBP (Z_t). We focus on the case in which $\sigma = 0$. As before, let τ be the time of extinction or explosion of Z. Let $(t_i)_{i\in\mathbb{N}}$ be an enumeration of $\{0 \le t < \tau : \Delta Z_t > 0\}$, and put $y_i := \Delta Z_{t_i}/Z_{t_i}$. Note that for any $t < \tau$ we have

$$\sum_{i:t_i\le t} y_i^2 \le (\inf_{0\le r\le t} Z_r)^{-2} \sum_{s\le t} (\Delta Z_s)^2 < \infty.$$

Taking the corresponding $n = \sum \delta_{(t_i,y_i)}$ in the lookdown construction, we obtain partitions $\mathscr{R}^T(t)$ which encode the genealogy of Z. The process $(X_t)_{t\ge 0}$ of (2.2) is then such that

$$\big(Z_t X_t\big)_{0\le t<\tau} \quad \text{is the superprocess with generator (1.15) .} \tag{2.4}$$

The case $\alpha \in (1, 2]$ corresponds to Example 3.1.2 of [11] and then Theorem 3.2 (also of [11]) provides the proof of the claim. For completeness we now check (2.4) for any CSBP with $\sigma = 0$.

Proof of (2.4). First we remark that it suffices to check that

$$F(Z_t X_t) - F(Z_0 X_0) - \int_0^t \mathcal{L}F(Z_s X_s)\, ds \tag{2.5}$$

is a martingale for functions of the type

$$F(\mu) = \psi(|\mu|)\langle \phi, \frac{\mu}{|\mu|}\rangle^m, \quad m \in \mathbb{N}, \psi \in C_c^2(\mathbb{R}_+), \psi(0) = 0, \phi \in B_b([0,1]). \tag{2.6}$$

For such a function, denoting by G_Y the generator of the Lévy process that generates the total mass process via Lamperti's time change, we have

$$\begin{aligned}\mathcal{L}F(\mu) = |\mu| G_Y \psi(|\mu|) \times \langle \phi, \frac{\mu}{|\mu|}\rangle^m + |\mu| \sum_{j=2}^{m} \int_{(0,\infty)} \binom{m}{j} \Big(\frac{h}{|\mu|+h}\Big)^j \Big(\frac{|\mu|}{|\mu|+h}\Big)^{m-j} \\ \times \psi(|\mu|+h)\Big[\langle \phi^j, \frac{\mu}{|\mu|}\rangle \langle \phi, \frac{\mu}{|\mu|}\rangle^{m-j} - \langle \phi, \frac{\mu}{|\mu|}\rangle^m\Big]\nu(dh).\end{aligned} \tag{2.7}$$

To see that this is equivalent to the more familiar 'exponential form' of the martingale problem one can check that the respective linear spans of

$$\begin{aligned}&\big\{(F, \mathcal{L}F) : F(\mu) = \exp(-\langle \phi, \mu\rangle), \phi \in B_b^{++}([0,1])\big\} \quad \text{and} \\ &\big\{(F, \mathcal{L}F) : F(\mu) = \psi(|\mu|)\langle \phi, \frac{\mu}{|\mu|}\rangle^m,\ m \in \mathbb{N}, \psi \in C_c^2(\mathbb{R}_+), \psi(0) = 0, \phi \in B_b([0,1])\big\}\end{aligned}$$

have the same bounded-pointwise closure, where $B_b^{++}([0,1]) = \{\phi \in B_b([0,1]) : \inf \phi > 0\}$, and apply Proposition 3.1, Chapter 4 of [16].

The proof is now straightforward. First we write down the generator A_m for the $(m+1)$-tuple $(Z_t; \xi_t^1, \ldots, \xi_t^m)$ corresponding to the CSBP and the first m levels of the lookdown construction. This can be found in [11]. For the interested reader, in their notation we are taking $Q(t) = Z_t$, $p(q) = q$ (so $P(t) = Q(t)$), $q_1(v) \equiv 0$, $q_2(v, v') = (v' - v)^2$. Still in their notation this implies $U(t) = [P]_t = \sum_{s \le t} (\Delta P_t)^2$, $\beta(v, v') = (v' - v)/v'$, and $\eta(v, dv') = Cv(v' - v)^{-1-\alpha} dv'$. Let $m \in \mathbb{N}$, $x = (x_1, \ldots, x_m)$. For

$$f(v, x_1, \ldots, x_m) = \psi(v) \prod_{i=1}^{m} \phi(x_i)$$

we have

$$\begin{aligned}
A_m f(v,x) &= \prod_{i=1}^m \phi(x_i)\Big[\gamma v\psi'(v) + v\int_{(0,\infty)} \big\{\psi(v+h)-\psi(v)-1_{(0,1]}(h)h\psi'(v)\big\}\,\nu(dh)\Big] \\
&\quad + \sum_{J\subset\{1,\dots,m\}} \int_{\mathbb{R}_+} \Big(\frac{v'-v}{v'}\Big)^{|J|}\Big(1-\frac{v'-v}{v'}\Big)^{m-|J|} \\
&\qquad\qquad \times \psi(v')\Big[\phi(x_{\min J})^{|J|}\prod_{k\in J^c}\phi(x_k) - \prod_{i=1}^m \phi(x_i)\Big]\eta(v,dv') \\
&= \prod_{i=1}^m \phi(x_i)\Big[\gamma v\psi'(v) + v\int_{(0,\infty)} \big\{\psi(v+h)-\psi(v)-1_{(0,1]}(h)h\psi'(v)\big\}\,\nu(dh)\Big] \\
&\quad + \sum_{J\subset\{1,\dots,m\}} \int_{\mathbb{R}_+} \Big(\frac{h}{v+h}\Big)^{|J|}\Big(1-\frac{h}{v+h}\Big)^{m-|J|} \\
&\qquad\qquad \times \psi(v+h)\Big[\phi(x_{\min J})^{|J|}\prod_{k\in J^c}\phi(x_k) - \prod_{i=1}^m \phi(x_i)\Big]v\nu(dh) \\
&= vG_Y\psi(v)\times\prod_{i=1}^m\phi(x_i) + \sum_{\substack{J\subset\{1,\dots,m\},\\ |J|\ge 2}} \int_{(0,\infty)} v\nu(dh)\Big(\frac{h}{v+h}\Big)^{|J|}\Big(1-\frac{h}{v+h}\Big)^{m-|J|} \\
&\qquad\qquad \times \psi(v+h)\Big[\phi(x_{\min J})^{|J|}\prod_{k\in J^c}\phi(x_k) - \prod_{i=1}^m \phi(x_i)\Big].
\end{aligned}$$

So for test functions $F(q,\rho) := \psi(q)\langle\phi,\rho\rangle^m$ the generator $\mathbb{A}$ of the pair (Q,X), consisting of the driving total mass process and the empirical measure process, obtained from the lookdown construction is

$$\begin{aligned}
\mathbb{A}F(q,\rho) &= \big\langle A_m f(q,\cdot), \rho^{\otimes m}\big\rangle \\
&= qG_Y\psi(q)\times\langle\phi,\rho\rangle^m \\
&\quad + \sum_{\substack{J\subset\{1,\dots,m\},\\ |J|\ge 2}} \int_{(0,\infty)} q\nu(dh)\Big(\frac{h}{q+h}\Big)^{|J|}\Big(1-\frac{h}{q+h}\Big)^{m-|J|} \\
&\qquad\qquad \times \psi(q+h)\Big[\big\langle\phi^{|J|},\rho\big\rangle\langle\phi,\rho\rangle^{m-|J|} - \langle\phi,\rho\rangle^m\Big],
\end{aligned}$$

(see Theorem 4.1 and equation (4.4) in [11]). Now substituting $q = |\mu|$, $\rho = \mu/|\mu|$, we see that $\mathbb{A}F(|\mu|,\mu/|\mu|) = \mathcal{L}F(\mu)$ as required. □

While the classical duality in Theorem 1.1 is only a statement about distributions, there is also a pathwise version, given by the lookdown construction, which relates both processes on the same probability space.

Theorem 2.1. *Assume that an α-stable branching superprocess (M_t) has been obtained as above from an α-stable continuous-state branching process (Z_t) and the lookdown construction. Define T_t from Z as in Theorem 1.1. Assume that either $0<\alpha\le 1$ and $d\ge 0$ or $1<\alpha<2$ and $m\le 0$, so that $T_\tau=\infty$ a.s. Fix $t>0$, and for $0\le s\le t$, let $\Pi_s = \mathscr{R}^{T^{-1}(t)}(T^{-1}(t)-T^{-1}(t-s))$. Then, the $\mathcal{P}$-valued process $(\Pi_s)_{0\le s\le t}$ is a* Beta$(2-\alpha,\alpha)$*-coalescent.*

Remark 2.2. *An analogous result holds in the general case. If $T_\tau < \infty$ we can augment the point process $\sum_{s \le T_\tau} \Delta\tilde{Z}_s/\tilde{Z}_s$ that is used in the lookdown construction with an independent auxiliary Poisson point process in a similar way to the proof of Lemma 3.7 and thus produce a* $\mathrm{Beta}(2-\alpha, \alpha)$*-coalescent that lives for all time. However, the auxiliary part no longer has anything to do with the genealogy of the given realisation of the measure-valued branching process.*

Remark 2.3. *We refer to* [32] *for another route leading, at least for $\alpha \in (1, 2]$, from α-stable branching to Beta-coalescents, this time via a particle approximation obtained from a Galton-Watson branching process by holding the population size fixed by randomly sampling N offspring in each generation to survive.*

3. Proofs

Our first step will be to consider the time change in Theorem 1.1 and to prove Proposition 1.2, which gives necessary and sufficient conditions to have $T_\tau = \infty$ a.s. Define the continuous-state branching process Z_t and the time-change T_t as in Theorem 1.1. Let $(Y_t)_{t\ge 0}$ be a Lévy process with Laplace exponent $-\Psi$ such that $Y_0 > 0$. We write $\{\mathcal{F}_t : t \ge 0\}$ for its canonical filtration. For all $a \in \mathbb{R}$, let $\zeta(a) = \inf\{t : Y_t = a\}$. Then, as discussed previously, we may assume that $Z_t = Y_{U(t)}$, where $U_t = \inf\{s : \int_0^s Y_u^{-1}\, du = t\}$, for $t < \int_0^{\zeta(0)} Y_u^{-1}\, du$. For $t \ge \int_0^{\zeta(0)} Y_u^{-1}\, du$, we have $Z_t = 0$ if $\zeta(0) < \infty$ and $Z_t = \infty$ if $\zeta(0) = \infty$. Recall that $\tau = \tau_0 \wedge \tau_\infty$, where $\tau_\infty = \inf\{t : Z_t = \infty\}$, $\tau_0 = \inf\{t : Z_t = 0\}$ are the explosion resp. extinction times of Z. The following lemma shows how we can combine the two time changes and express the condition that $T_\tau = \infty$ in terms of the Lévy process Y. For this result, and the rest of this section, case (i) of Theorem 1.1 corresponds to setting $\alpha = 2$.

Lemma 3.1. *We have $T_\tau = \infty$ a.s. if and only if $\int_0^{\zeta(0)} Y_t^{-\alpha}\, dt = \infty$ a.s.*

Proof. For $0 \le t \le \zeta(0)$, define $K(t) = \int_0^t Y_u^{-1}\, du$ and note that $Z_{K(t)} = Y_t$. Therefore,

$$\int_0^{\zeta(0)} Y_t^{-\alpha}\, dt = \int_0^{\zeta(0)} Y_t^{1-\alpha} Y_t^{-1}\, dt = \int_0^{\zeta(0)} Z_{K(t)}^{1-\alpha} Y_t^{-1}\, dt.$$

Note that if $\tau = \tau_0 < \infty$ then $\tau_0 = K(\zeta(0))$, and if $\tau = \tau_\infty < \infty$ then $\tau_\infty = K(\zeta(0))$. Also, if $\tau = \infty$ then $K(\zeta(0)) = \infty$, so we have $K(\zeta(0)) = \tau$. Since the function $t \mapsto K(t)$, defined on $(0, \zeta(0))$, is almost surely absolutely continuous with derivative $K'(t) = Y_t^{-1}$, we make the change of variables $s = K(t)$ to obtain

$$\int_0^{\zeta(0)} Y_t^{-\alpha}\, dt = \int_0^{\tau} Z_s^{1-\alpha}\, ds. \tag{3.1}$$

The lemma is now immediate from the definition of T_τ. □

Lemma 3.2. *Let $(W_t)_{t\ge 0}$ be a stable Lévy process having Laplace exponent $\Phi(u) = \mathrm{sgn}(1-\alpha) c u^\alpha$, where $\alpha \in (0, 1) \cup (1, 2]$ and $c > 0$. Let r and x be positive real numbers. Then*

$$P(|W_t - W_0| \le rx \text{ for } 0 \le t \le x^\alpha) = \eta_{\alpha,r},$$

where $\eta_{\alpha,r}$ is a positive constant which does not depend on x.

Proof. For $t \ge 0$, let $\tilde{W}_t = W_t - W_0$. From the form of Φ, it is straightforward to verify the scaling property, which says that for any $k > 0$, the processes $(\tilde{W}_t)_{t\ge 0}$ and $(k^{-1/\alpha}\tilde{W}_{kt})_{t\ge 0}$ have the same law. Therefore by taking $k = x^{-\alpha}$, we get

$$P(|\tilde{W}_t| < rx \text{ for } 0 \le t \le x^\alpha) = P(|\tilde{W}_t| < r \text{ for } 0 \le t \le 1). \tag{3.2}$$

It remains to show that the probability on the right-hand side of (3.2), which we call $\eta_{\alpha,r}$, is positive. For $0 < \alpha < 1$, $M_1 = \sup_{0\le s\le 1} |W_s|$ has a stable density which is strictly positive on $(0,\infty)$. For $1 < \alpha < 2$, exact asymptotics for $\eta_{\alpha,r}$ as $r \to 0$ are given e.g. in Proposition 3, Chapter VIII of [1]. □

We now combine Lemma 3.1 with scaling properties of Lévy processes to prove Proposition 1.2. We present the proof in the form of two lemmas.

Lemma 3.3. *We have $T_\tau = \infty$ a.s. if either $\alpha = 1$, $0 < \alpha < 1$ and $d \ge 0$, or $1 < \alpha \le 2$ and $m \le 0$.*

Proof. First, note that if $\alpha = 1$, then time is changed only by a constant factor. Therefore, since Z neither explodes nor goes extinct, we have $T_\tau = \infty$ a.s.

We next consider $1 < \alpha \le 2$ and $m \le 0$. In this case, $\zeta(b) < \infty$ and we have $Y_{\zeta(b)} = b$ for all $b < Y_0$ a.s. because Y has no negative jumps (see e.g. p. 188 of [1]). For $n \in \mathbb{N}$, write $x_n = Y_0 2^{-n}$ and let A_n be the event that $\frac{1}{2}x_n < Y_t < \frac{3}{2}x_n$ for all $t \in [\zeta(x_n), \zeta(x_n) + x_n^\alpha]$. If A_n occurs, then $\zeta(x_{n+1}) \ge \zeta(x_n) + x_n^\alpha$, and so

$$\int_{\zeta(x_n)}^{\zeta(x_{n+1})} Y_t^{-\alpha}\, dt \ge \int_{\zeta(x_n)}^{\zeta(x_n)+x_n^\alpha} Y_t^{-\alpha}\, dt \ge x_n^\alpha [(3/2)x_n]^{-\alpha} = (2/3)^\alpha.$$

Therefore, if infinitely many of the A_n occur a.s., then $\int_0^{\zeta(0)} Y_t^{-\alpha}\, dt = \infty$ a.s., which by Lemma 3.1 implies that $T_\tau = \infty$ a.s. It thus suffices to show that infinitely many of the A_n occur a.s. By the strong Markov property, the events A_n are independent, so by the Borel-Cantelli Lemma, it suffices to show that $\sum_{n=1}^\infty P(A_n) = \infty$.

By the strong Markov property, $P(A_n) = P(|Y_t - Y_0| \le x_n/2 \text{ for all } 0 \le t \le x_n^\alpha)$. Define a process $(W_t)_{t\ge 0}$ by $W_t = Y_t - mt$. Then W is a Lévy process with Laplace exponent $-cu^\alpha$ for some $c > 0$. Let B_n be the event that $|W_t - W_0| \le x_n/4$ for all $0 \le t \le x_n^\alpha$, and note that $P(B_n) = \eta_{\alpha,1/4}$ by Lemma 3.2. If B_n occurs and $mx_n^\alpha \le x_n/4$, then $|Y_t - Y_0| \le x_n/2$ for all $0 \le t \le x_n^\alpha$. Note that $mx_n^\alpha \le x_n/4$ if and only if $4m \le x_n^{1-\alpha}$, which is true for sufficiently large n. It follows that $P(A_n) \ge P(B_n) = \eta_{\alpha,1/4}$ for sufficiently large n, which implies that $\sum_{n=1}^\infty P(A_n) = \infty$.

Next, suppose $0 < \alpha < 1$ and $d \ge 0$. In this case, the process Y is a stable subordinator with nonnegative drift added, so Y is nondecreasing and $\lim_{t\to\infty} Y_t = \infty$ a.s. Define a sequence of stopping times $(S_n)_{n=0}^\infty$ by $S_0 = 0$ and $S_n = \inf\{t : Y_t \ge 2Y_{S_{n-1}}\}$ for $n \ge 1$. Note that for all n, we have $S_n < \infty$ a.s. Let A_n be the event that $S_{n+1} - S_n \ge Y_{S_n}^\alpha$. If A_n occurs, then

$$\int_{S_n}^{S_{n+1}} Y_t^{-\alpha}\, dt \ge (S_{n+1} - S_n)(2Y_{S_n})^{-\alpha} \ge 2^{-\alpha}.$$

Thus, once again it suffices to show that infinitely many of the A_n occur a.s. Let $\mathcal{F}_{S_n}$ be the σ-field generated by the stopped process $Y_{\cdot\wedge S_n}$. Note that $A_n \in \mathcal{F}_{S_{n+1}}$, so by the conditional Borel-Cantelli Lemma (see Section 4.3 of [13]), it suffices to show that $\sum_{n=1}^\infty P(A_n|\mathcal{F}_{S_n}) = \infty$ a.s.

For $b > 0$, let $p(b) = P(Y_t - Y_0 \le b \text{ for all } 0 \le t \le b^\alpha)$. By the strong Markov property, $P(A_n|\mathcal{F}_{S_n}) = p(Y_{S_n})$. Let $W_t = Y_t - dt$, and let $q(b) = P(W_t - W_0 \le b/2 \text{ for all } 0 \le t \le b^\alpha)$. If $W_t - W_0 \le b/2$ for all $0 \le t \le b^\alpha$ and $db^\alpha \le b/2$, then $Y_t - Y_0 \le b$ for all $0 \le t \le b^\alpha$. We have $db^\alpha \le b/2$ if and only if $2d \le b^{1-\alpha}$, which is true for sufficiently large b. It follows that $p(b) \ge q(b)$ for sufficiently large b. Since $Y_{S_n} \ge 2^n Y_0$, we have $p(Y_{S_n}) \ge q(Y_{S_n})$ for sufficiently large n. However, $q(b) = \eta_{\alpha,1/2}$ for all b by Lemma 3.2, so $\sum_{n=1}^\infty P(A_n|\mathcal{F}_{S_n}) = \infty$ a.s. □

Lemma 3.4. *If $1 < \alpha \le 2$ and $m > 0$, or if $0 < \alpha < 1$ and $d < 0$, then $P(T_\tau < \infty) > 0$.*

Proof. First, suppose $1 < \alpha \leq 2$ and $m > 0$. Now the event $\{\zeta(0) = \infty\}$ has positive probability, and as then Y grows approximately linearly with $\inf_t Y_t > 0$, (3.1) will be finite. Indeed, let $W_t = Y_t - mt - Y_0$ for all $t \geq 0$, so $(W_t)_{t\geq 0}$ is a stable Lévy process with Laplace exponent $\Psi(u) = -cu^\alpha$ for some $c > 0$. We have $\limsup_{t\to\infty} t^{-1}|W_t| = 0$ a.s. (see, for example, p. 222 of [1]). Therefore, if $0 < a < m$, there exists M such that $M < \infty$ a.s. and $Y_t = W_t + mt + Y_0 \geq at$ for all $t \geq M$. Thus, the process Y_t drifts to ∞, in the terminology of p. 167 of [1], and since $Y_0 > 0$, we have $P(Y_t > Y_0/2 \text{ for all } t \geq 0) > 0$. On the event that $Y_t > Y_0/2$ for all $t \geq 0$, we have

$$\int_0^\infty Y_t^{-\alpha}\, dt \leq \int_0^M (Y_0/2)^{-\alpha}\, dt + \int_M^\infty (at)^{-\alpha}\, dt < \infty.$$

It follows that $P(T_\tau < \infty) > 0$.

Next, suppose $0 < \alpha < 1$ and $d < 0$. Now the event $\{\zeta(0) < \infty\}$ has positive probability, and on this event, $Y_t - 0$ will look approximately like $\text{const} \cdot (\zeta(0) - t)$ for t near $\zeta(0)-$, so that (3.1) will be finite.

More formally, for all $t \geq 0$, let $I_t = \inf\{Y_s : 0 \leq s \leq t\}$ be the infimum process, and for $x \leq Y_0$, let $S_x = \inf\{t \geq 0 : I_t < x\}$. Since $d < 0$, the process $(Y_t)_{t\geq 0}$ is a Lévy process with no negative jumps that is not a subordinator. Therefore, we can apply Theorem 1 on p. 189 of [1] to $-Y$ to see that the process $(S_{Y_0-t})_{t\geq 0}$ is a subordinator, killed at an independent exponential time κ. If $\kappa > Y_0$ then $S_0 < \infty$, which means $Y_t < 0$ for some t and therefore $\zeta(0) < \infty$. Since κ has an exponential distribution, it follows that $P(\zeta(0) < \infty) > 0$. Furthermore, we have $S_0 = \zeta(0)$ almost surely on the event $\{\zeta(0) < \infty\}$.

Using the time-reversal property of subordinators, we see that, conditional on the event $\{\zeta(0) < \infty\}$, the process $(\tilde{S}_t)_{0\leq t\leq Y_0}$ defined by $\tilde{S}_t = \zeta(0) - S_t$ is a subordinator. It follows (see Proposition 8 on p. 84 of [1]) that $\lim_{t\downarrow 0} t^{-1}\tilde{S}_t = \beta$, where β is the drift coefficient for the subordinator $\tilde{S}$. It follows that for any $\beta' > \beta$ there exists $\epsilon > 0$ such that $\tilde{S}_t \leq \beta' t$ for all $0 \leq t \leq \epsilon$. Therefore, there exists $B < \infty$ such that $\tilde{S}_t \leq Bt$, and so $S_t \geq \zeta(0) - Bt$, for all $0 \leq t \leq Y_0$. Thus, if $0 \leq t \leq \zeta(0)$, then $S_{(\zeta(0)-t)/B} \geq t$. It follows that if $0 \leq t \leq \zeta(0)$, then $I_t \geq (\zeta(0) - t)/B$. Hence, on the event $\{\zeta(0) < \infty\}$, we have

$$\int_0^{\zeta(0)} Y_t^{-\alpha}\, dt \leq \int_0^{\zeta(0)} I_t^{-\alpha}\, dt \leq \int_0^{\zeta(0)} [(\zeta(0) - t)/B]^{-\alpha}\, dt = B^\alpha \int_0^{\zeta(0)} s^{-\alpha}\, ds,$$

which is finite because $0 < \alpha < 1$. Thus, $\int_0^{\zeta(0)} Y_t^{-\alpha}\, dt < \infty$ a.s. on $\{\zeta_0 < \infty\}$, which by Lemma 3.1 implies that $P(T_\tau < \infty) > 0$, as claimed. □

We now show rigorously that the generator $\mathcal{L}$ of our measure-valued process $(M_t)_{t\geq 0}$ applied to functions of the form (1.5) factorises precisely under the conditions of Theorem 1.1.

Lemma 3.5. *Let ν be a measure on $(0, \infty)$ satisfying (1.12), and for $z > 0$ let $\lambda_z = \phi_z(\nu)$ be the image of ν under the mapping given by*

$$\phi_z : h \mapsto r := \frac{h}{z + h}. \tag{3.3}$$

There exists a measure λ on $\mathbb{R}_+$ and a measurable mapping $\mathfrak{s} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lambda_z = \mathfrak{s}(z)\lambda \tag{3.4}$$

if and only if, for some $\alpha \in (0, 2)$,

$$\nu(dh) = \text{const} \cdot h^{-1-\alpha} dh.$$

In this case, $\mathfrak{s}(z) = \text{const} \cdot z^{-\alpha}$, and $\lambda(dr) = \text{const} \cdot r^{-2}\text{Beta}(2 - \alpha, \alpha)(dr)$.

Proof. The "if" direction follows by inspection; hence it suffices to prove the "only if" direction. For $c > 0$, write ψ_c for the mapping $h \mapsto c \cdot h$. Evidently, for all z, c,

$$\phi_z = \phi_{cz} \circ \psi_c \tag{3.5}$$

Hence, using (3.4) and (3.5),

$$\mathfrak{s}(z)\lambda = \phi_{cz}(\psi_c(\nu)). \tag{3.6}$$

On the other hand, again by (3.4),

$$\phi_{cz}(\nu) = \mathfrak{s}(cz)\lambda \tag{3.7}$$

Inverting (3.7),

$$\nu = \mathfrak{s}(cz)\phi_{cz}^{-1}(\lambda), \tag{3.8}$$

and inverting (3.6) and using (3.8),

$$\psi_c(\nu) = \frac{\mathfrak{s}(z)}{\mathfrak{s}(cz)}\nu. \tag{3.9}$$

Choosing $z = 1$ and putting $\tilde{\mathfrak{s}}(\frac{1}{c}) = \mathfrak{s}(1)/\mathfrak{s}(c)$, we obtain for the tail probabilities

$$\ell(h) := \nu([h, \infty))$$

the relation

$$\ell\big(\frac{h}{c}\big) = \psi_c(\nu)([h, \infty)) = \tilde{\mathfrak{s}}\big(\frac{1}{c}\big)\nu([h, \infty)) = \tilde{\mathfrak{s}}\big(\frac{1}{c}\big)\ell(h).$$

From this it follows readily that $\ell(h) = \text{const} \cdot h^{-\alpha}$, where $\alpha \in (0, 2)$ by assumption (1.12). Finally, it is straightforward to check that λ is of the claimed form. □

Proof of Theorem 1.1. Recall $z = |\mu|$, $\rho = \mu/z$. We claim that for functions $F(\mu) = G(\rho)$ of the form (1.5)

$$(\mathcal{L}F)(\mu) = z^{-1}\sigma^2 \cdot (\mathcal{F}G)(\rho) + z \cdot (\mathcal{R}_{\nu,z}G)(\rho), \tag{3.10}$$

where $(\mathcal{F}G)(\rho)$ is as in (1.7), and

$$(\mathcal{R}_{\nu,z}G)(\rho) = \int_0^1 \rho(da) \int_{(0,\infty)} \nu(dh) \left(G\left(\frac{\mu + h\delta_a}{z + h}\right) - G(\rho)\right). \tag{3.11}$$

Indeed, we may assume without loss of generality that the f appearing in (1.5) is symmetric (since the r.h.s. of (1.5) does not change if f is replaced by its symmetrised version). Since any continuous symmetric function from $[0, 1]^p$ to $\mathbb{R}$ can be uniformly approximated by functions of the form $f(a_1, ..., a_p) = \prod_{i=1}^p \phi(a_i)$, it suffices to consider functions F of the form $F(\mu) = H(\frac{1}{z}\langle\phi, \mu\rangle) = H(\langle\phi, \rho\rangle)$ with $H : \mathbb{R} \to \mathbb{R}$ differentiable and ϕ bounded. Since

$$F'(\mu; a) = H'\big(\langle\phi, \rho\rangle\big)\frac{1}{z}(\phi(a) - \langle\phi, \rho\rangle),$$

and consequently $\int \mu(da)F'(\mu; a) = 0$, we see that in the case $\sigma = 0$ (1.15) turns into (3.10) and (3.11). The general form of our claim follows by combining this with (1.4) and (1.7).

Defining λ_z as the image of the measure ν under the mapping (3.3), we can re-write equation (3.11) as

$$(\mathcal{R}_{\nu,z}G)(\rho) = \int \rho(da) \int_{(0,1)} \lambda_z(dr)(G((1 - r)\rho + r\delta_a) - G(\rho)).$$

By Lemma 3.5, λ_z factorises in the desired form if and only if, for some $\alpha \in (0, 2)$,

$$(\mathcal{R}_{\nu,z}G)(\rho) = \text{const} \cdot z^{-\alpha}(\mathcal{R}G)(\rho)$$

where

$$(\mathcal{R}G)(\rho) = \int \rho(da) \int_{(0,1)} \frac{1}{r^2} \Lambda(dr)(G((1-r)\rho + r\delta_a) - G(\rho)) \tag{3.12}$$

and

$$\Lambda := \mathrm{Beta}(2-\alpha, \alpha).$$

It remains to check that $\mathcal{R}G$ given by (3.12) is indeed the generator of a generalised Fleming-Viot process, i.e. $\mathcal{R}G$ is of the form (1.11) for functions G of the form (1.5).

For this purpose, let $A_1, ..., A_p$ be i.i.d. with distribution ρ, and let, for $r \in (0,1)$, $\mathfrak{J} \subseteq \{1, ..., p\}$ be the random success times of a coin tossing with success probability r and independent of $(A_1, ..., A_p)$. Using the notation introduced in (1.6) it is readily checked that

$$\begin{aligned}\int_{(0,1)} \rho(da)(G((1-r)\rho + r\delta_a) - G(\rho)) &= \mathbf{E}[f(A_1^{\mathfrak{J}}, ..., A_p^{\mathfrak{J}}) - f(A_1, ..., A_p)] \\ &= \sum_{J \subseteq \{1,...,p\}} r^{|J|}(1-r)^{p-|J|} \int \rho(da_1)...\rho(da_p)(f(a_1^J, ...a_p^J) - f(a_1, ..., a_p)).\end{aligned}$$

Noting that the subsets J with $|J| \in \{0, 1\}$ do not contribute to the sum, we obtain

$$\begin{aligned}(\mathcal{R}G)(\rho) &= \int_{(0,1)} \Lambda(dr) \frac{1}{r^2} \sum_{J \subseteq \{1,...,p\}, |J| \geq 2} r^{|J|}(1-r)^{p-|J|} \\ &\quad \int \rho(da_1)...\rho(da_p)(f(a_1^J, ..., a_p^J) - f(a_1, ..., a_p)) \qquad (3.13) \\ &= \sum_{J \subseteq \{1,...,p\}, |J| \geq 2} \beta^{\Lambda}_{p,|J|} \int \rho(da_1)...\rho(da_p)(f(a_1^J, ...a_p^J) - f(a_1, ..., a_p)),\end{aligned}$$

where $\beta^{\Lambda}_{p,j}$ is defined in (1.10).

Thus, $\mathcal{R}$ is precisely of the form appearing in [3] p. 280, and the second summand on the right hand side of (3.10) is const $\cdot\, z^{1-\alpha}(\mathcal{R}G)(\rho)$. □

It remains to prove Theorem 2.1. We achieve this by two lemmas. The first says that both the Λ-Fleming-Viot process and the Λ-coalescent can be obtained through a realisation-wise lookdown construction from a time-homogeneous Poisson point process on $[0, \infty) \times (0, 1]$. The second explains how the required time-homogeneous Poisson point process can be recovered from an α-stable CSBP.

Lemma 3.6. *Let Λ be a finite measure on $(0, 1]$ and let $N = \sum_i \delta_{(t_i, y_i)}$ be a Poisson point process on $[0, \infty) \times (0, 1]$ with intensity measure $dt \otimes y^{-2}\Lambda(dy)$. Note that (2.1) holds almost surely, so we can define the measures X_t and the partitions $\mathscr{R}^T(t)$ via the lookdown construction in Section 2.*

(i) *The process $(X_t)_{t \geq 0}$ is the Λ-Fleming-Viot process.*
(ii) *For fixed $T > 0$, the process $(\mathscr{R}^T(t))_{0 \leq t \leq T}$ is a Λ-coalescent run for time T.*

Proof. Part (i) is a direct consequence of Section 4 of [11]. In their notation, we choose

$$Q(t) := \sum_{i: t_i \leq t} y_i^2, \quad P(t) \equiv 1, \quad U(t) = Q(t), \quad t \geq 0,$$

$p(v) \equiv 1$, $q_1(v) \equiv 0$, $q_2(v, v') = v' - v$. The process Q is Markov. In fact it is a driftless subordinator with Lévy measure given by

$$\ell(A) = \int_{[0,1]} 1_A(r^2) \frac{1}{r^2} \Lambda(dr) \quad \text{for all } A \subset [0,1] \text{ Borel.} \tag{3.14}$$

The transition kernel $\eta(v, dv')$ of Q is given by $\ell(v + dv')$. We let the type space $E = [0,1]$ and the motion operator B be the 0-operator. Consider, for $m \in \mathbb{N}$, test functions of the type

$$f(v', x_1, \ldots, x_m) = \psi(v') \prod_{i=1}^{m} \phi(x_i).$$

Then, using $x = (x_1, \ldots, x_m)$ the $(m+1)$-tuple consisting of the process Q and the first m levels of the lookdown construction has generator (equation (4.2) in [11])

$$\begin{aligned} A_m f(v, x) &= Gf(v, x) + \sum_{J \subset \{1,\ldots,m\}} \int_{\mathbb{R}^+} (\sqrt{v' - v})^{|J|} (1 - \sqrt{v' - v})^{m - |J|} \\ &\qquad \times \left(f(v', \theta_J(x)) - f(v', x) \right) \eta(v, dv') \\ &= Gf(v, x) + \sum_{J \subset \{1,\ldots,m\}} \int_{[0,1]} y^{|J|} (1 - y)^{m - |J|} \\ &\qquad \times \left(f(v + y^2, x^J) - f(v + y^2, x) \right) \frac{1}{y^2} \Lambda(dy), \end{aligned} \tag{3.15}$$

where G is the generator of Q and x^J is as defined in (1.6). Note that the martingale problem for A_m is well-posed due to the boundedness of

$$\int_{\mathbb{R}^+} (\sqrt{v' - v})^2 \, \eta(v, dv') = \int_{[0,1]} y^2 \frac{1}{y^2} \Lambda(dy) = \Lambda([0,1]) < \infty.$$

Let X_t be the empirical process obtained from the lookdown construction by

$$X_t = \lim_{k \to \infty} \frac{1}{k} \sum_{k} \xi_t^k.$$

By [11], Theorem 4.1, we have that for the pair (Q_t, X_t) and f as above,

$$\langle f(Q_t, \cdot), X_t^m \rangle - \int_0^t \langle A_m f(Q_s, \cdot), X_s^m \rangle \, ds \tag{3.16}$$

is a martingale with respect to the canonical filtration of (Q, X).

Note that for f as above with $\psi \equiv 1$, we have

$$\begin{aligned} &\langle A_m f(Q_t, \cdot), X_t^m \rangle \\ &\quad = \iint_{E^m} \sum_{J \subset \{1,\ldots,m\}} \int_{[0,1]} y^{|J|} (1 - y)^{m - |J|} \\ &\qquad \times \left[\phi(x_{\min J})^j \prod_{k \in J^c} \phi(x_k) - \prod_{i=1}^{m} \phi(x_i) \right] \frac{1}{y^2} \Lambda(dy) \, X_t^m(dx_1, \ldots, dx_m). \end{aligned} \tag{3.17}$$

Observe that only terms involving J with $|J| \geq 2$ contribute to the above integral, so that we can rewrite, by slight abuse of notation abbreviating $f(Q_t, \cdot) = f(\cdot)$,

$$\begin{aligned}&\langle A_m f(\cdot), X_t^m \rangle \\ &\quad = \sum_{J:|J|\geq 2} \beta^{\Lambda}_{m,|J|} \iint_{E^m} \left[f(x^J) - f(x)\right] X_t^m(dx_1, \ldots, dx_m), \end{aligned} \tag{3.18}$$

where x^J is defined as in (1.6). This is the generator in the martingale problem for the generalised Fleming-Viot process stated in Bertoin & Le Gall [3], which is well-posed (by duality with the Λ-coalescent).

Part (ii) is immediate from the construction. □

The final lemma of this section identifies the distribution of the process of relative jump sizes of our time-changed CSBP and thus, combined with Lemma 3.6, completes the proof of Theorem 2.1.

Lemma 3.7. *Assume that either $0 < \alpha \leq 1$ and $d \geq 0$ or $1 < \alpha < 2$ and $m \leq 0$. Then, writing $\tilde{Z}_t := Z_{T^{-1}(t)}$,*

$$\sum_{t\,:\,\Delta\tilde{Z}_t>0} \delta_{(t,\Delta\tilde{Z}_t/\tilde{Z}_t)}$$

is a Poisson point process on $[0,\infty) \times (0,1)$ with intensity measure $dt \otimes y^{-2}\Lambda(dy)$, and Λ is the Beta$(2-\alpha, \alpha)$ *distribution.*

In the complementary case, there exists a Poisson point process $\bar{N} = \sum_i \delta_{(t_i,y_i)}$ with the same intensity measure and a random time η such that

$$\left(\sum_{i\,:\,t_i\leq\eta} \delta_{(t_i,y_i)} \,,\ \eta \right) \stackrel{d}{=} \left(\sum_{t\leq T_\tau\,:\,\Delta\tilde{Z}_t>0} \delta_{(t,\Delta\tilde{Z}_t/\tilde{Z}_t)} \,,\ T_\tau \right).$$

Proof. As in the preamble to this section, Z can be expressed as a time change of a Lévy process Y. Evidently, $\tilde{Z}$ is also a time change of Y. Indeed, writing $B_t = \int_0^t Y_s^{-\alpha}\, ds$ for $t < \zeta(0)$ and $B_t = B_{\zeta(0)-}$ for $t \geq \zeta(0)$, we have that $\tilde{Z}_t = Y_{B^{-1}(t)}$ with the stopping times $B^{-1}(s) := \inf\{t \geq 0 : B_t > s\}$. Observe that under the given conditions, from Lemma 3.1 we have $B_{\zeta(0)-} = \infty$, so that $B^{-1}(s)$ is defined for all s. In this case it therefore suffices to check that the random measure

$$\sum_{t\,:\,\Delta Y_{B^{-1}(t)}>0} \delta_{(t,\Delta Y_{B^{-1}(t)}/Y_{B^{-1}(t)})}$$

is a Poisson point process on $\mathbb{R}_+ \times (0,1)$ with intensity measure $dt \otimes r^{-2}\text{Beta}(2-\alpha,\alpha)(dr)$. To this end, let $U_t := \sum_{s\leq t\wedge\zeta(0)} (\Delta Y_s/Y_s)^2$, $\tilde{U}_s := U_{B^{-1}(s)}$. It is enough to show that $\tilde{U}$ is a subordinator (without drift) with Lévy measure given by (3.14) with $\Lambda = \text{Beta}(2-\alpha,\alpha)$, so that the square roots of its jumps form the required Poisson point process. Fix a continuously differentiable function f with compact support and let

$$H(s, \Delta Y_s) := 1_{\{\zeta(0)>s\}}\big(f(U_{s-} + (\Delta Y_s/(\Delta Y_s + Y_{s-}))^2) - f(U_{s-})\big),$$

so $f(U_t) = f(U_0) + \sum_{s\le t} H(s,\Delta Y_s)$. Put

$$\begin{aligned} M_t^f &:= f(U_t) - \int_0^t \int_{(0,\infty)} \big(f(U_{s-} + (h/(h+Y_{s-}))^2) - f(U_{s-})\big) h^{-1-\alpha} dh ds \\ &= f(U_t) - \int_0^t \int_{(0,1)} \big(f(U_{s-}+u^2) - f(U_{s-})\big) \frac{1}{u^2} u^{1-\alpha}(1-u)^{\alpha-1} du\, Y_{s-}^{-\alpha} ds \end{aligned}$$

(we have substituted $u = h/(Y_{s-}+h)$ in the second line). We now show that M_t^f is a uniformly integrable $(\mathcal{F}_t)$-martingale. Let K be such that $f(u) = 0$ for $u \ge K$, and let L_f denote the Lipschitz constant of f.

$$\begin{aligned} \mathbb{E}\Big[\int_0^t \int_{(0,\infty)} |H(s,y)|\nu(dy)ds\Big] &\le L_f \mathbb{E}\Big[\int_0^{t\wedge\zeta(0)} \int_{(0,\infty)} 1_{\{U_s\le K\}} \big(h/(h+Y_s)\big)^2 h^{-1-\alpha} dh\, ds\Big] \\ &= L_f \frac{\Gamma(2-\alpha)\Gamma(\alpha)}{\Gamma(2)} \mathbb{E}\Big[\int_0^{t\wedge\zeta(0)} 1_{\{U_s\le K\}} Y_s^{-\alpha} ds\Big]. \end{aligned} \tag{3.19}$$

Let us assume for the moment that

$$\mathbb{E}\Big[\int_0^{\zeta(0)} 1_{\{U_s\le K\}} Y_s^{-\alpha} ds\Big] < \infty \quad \text{for all } K > 0. \tag{3.20}$$

Given this, (3.19) together with standard results on Poisson point processes (see e.g. [29], Cor. XII.1.11) implies that M^f is a martingale w.r.t. $(\mathcal{F}_t)$. Note that for $t > 0$

$$|M_t^f| \le ||f||_\infty + L_f \frac{\Gamma(2-\alpha)\Gamma(\alpha)}{\Gamma(2)} \int_0^{\zeta(0)} 1_{\{U_s\le K\}} Y_s^{-\alpha} ds,$$

and that the right hand side has finite expectation by (3.20), so that $(M_t^f)_{t\ge 0}$ is uniformly integrable. Using $Y_{s-}^{-\alpha} ds = dB_s$, uniform integrability and the Optional Stopping Theorem applied to the stopping times $B^{-1}(s)$, we see that

$$f(\tilde{U}_s) - \int_0^s \int_{(0,1)} \big(f(\tilde{U}_{s-}+u^2) - f(\tilde{U}_{s-})\big) \frac{1}{u^2} u^{1-\alpha}(1-u)^{\alpha-1} du ds$$

is a martingale with respect to the new filtration $\{\tilde{\mathcal{F}}_s := \mathcal{F}_{B^{-1}(s)}\}$. The corresponding well-posed martingale problem is solved by the subordinator with Lévy measure ℓ defined in (3.14) with $\Lambda = \text{Beta}(2-\alpha,\alpha)$, see Thm. 3.4 in Chapter 8 of [16].

In the complementary case, when $B_{\zeta(0)-} < \infty$, provided we can check (3.20), we can apply Lemma 5.16, Chapter 4 of [16] to find a version of $\tilde{U}$ that lives for all time.

It remains to check (3.20). The proof is reminiscent of that of Lemma 3.3. Here is a sketch (in the strictly stable case):
Let $\tau_{(a,b)^c} := \inf\{t > 0 : Y_t \notin (a,b)\}$. Note that by scaling we have for all $x > 0$

$$\mathbb{E}_x\Big[\int_0^{\tau_{(\frac12 x, \frac32 x)^c}} Y_s^{-\alpha}\, ds\Big] = \mathbb{E}_1\Big[\int_0^{\tau_{(\frac12,\frac32)^c}} Y_s^{-\alpha}\, ds\Big] \le 2^\alpha \mathbb{E}_1\big[\tau_{(\frac12,\frac32)^c}\big] < \infty,$$

$$q := \mathbb{P}_x\big(\exists\, s \le \tau_{(\frac12 x,\frac32 x)^c} : \Delta Y_s > \tfrac14 x\big) = \mathbb{P}_1\big(\exists\, s \le \tau_{(\frac12,\frac32)^c} : \Delta Y_s > \tfrac14\big) > 0.$$

Define a sequence of stopping times via $T_0 := 0$, $T_n := \inf\{t > T_{n-1} : Y_t \notin (\frac{1}{2}Y_{T_{n-1}}, \frac{3}{2}Y_{T_{n-1}})\}$. Put $A_n := \{\exists s \in (T_{n-1}, T_n) : \Delta Y_s \geq \frac{1}{4}Y_{T_{n-1}}\}$. Note that $T_n \nearrow \zeta(0)$, and that the sequence $1_{A_1}, 1_{A_2}, \ldots$ is i.i.d. by the strong Markov property. Furthermore $A_n \subset \{U_{T_n} - U_{T_{n-1}} \geq 1/36\}$. Thus

$$\begin{aligned}\int_0^{T\wedge\zeta(0)} 1_{\{U_s\leq K\}} Y_s^{-\alpha} ds &= \sum_{n=1}^{\infty} \int_{T_{n-1}}^{T_n} 1_{\{U_s\leq K\}} Y_s^{-\alpha} ds \\ &\leq \sum_{n=1}^{\infty} 1_{\left\{\sum_{j=1}^{n-1} 1_{A_j} \leq \lfloor 36K \rfloor\right\}} \int_{T_{n-1}}^{T_n} Y_s^{-\alpha}\, ds,\end{aligned}$$

and the expectation of the right hand side is (in an obvious notation)

$$\mathbb{E}_1\Big[\int_0^{\tau_{(\frac{1}{2},\frac{3}{2})^c}} Y_s^{-\alpha}\, ds\Big] \times \sum_{n=1}^{\infty} \mathrm{Bin}_{n-1,q}(\{0,1,\ldots,\lfloor 36K\rfloor\}) < \infty.$$

□

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