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The number of unbounded components in the Poisson Boolean model of continuum percolation in hyperbolic space

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Abstract

We consider the Poisson Boolean model of continuum percolation with balls of fixed radius R in n-dimensional hyperbolic space $\mathbb{H}^n$. Let λ be the intensity of the underlying Poisson process, and let N_C denote the number of unbounded components in the covered region. For the model in any dimension we show that there are intensities such that $N_C = \infty$ a.s. if R is big enough. In $\mathbb{H}^2$ we show a stronger result: for any R there are two intensities λ_c and λ_u where $0 < \lambda_c < \lambda_u < \infty$, such that $N_C = 0$ for $\lambda \in [0, \lambda_c]$, $N_C = \infty$ for $\lambda \in (\lambda_c, \lambda_u)$ and $N_C = 1$ for $\lambda \in [\lambda_u, \infty)$.

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1 Introduction

We begin by describing the fixed radius version of the so called *Poisson Boolean* model in $\mathbb{R}^n$, arguably the most studied continuum percolation model. For a detailed study of this model, we refer to [18]. Let X be a Poisson point process in $\mathbb{R}^n$ with some intensity λ. At each point of X, place a closed ball of radius R. Let C be the union of all balls, and V be the complement of C. The sets V and C will be referred to as the *vacant* and *covered* regions. We say that *percolation occurs* in C (respectively in V) if C (respectively V) contains unbounded (connected) components. For the Poisson Boolean model in $\mathbb{R}^n$, it is known that there is a *critical intensity* $\lambda_c \in (0, \infty)$ such that for $\lambda < \lambda_c$, percolation does not occur in C, and for $\lambda > \lambda_c$, percolation occurs in C. Also, there is a critical intensity $\lambda_c^* \in (0, \infty)$ such that percolation occurs in V if $\lambda < \lambda_c^*$ and percolation does not occur if $\lambda > \lambda_c^*$. Furthermore, if we denote by N_C and N_V the number of unbounded components of C and V respectively, then it is the case that N_C and N_V are both almost sure constants which are either 0 or 1. In $\mathbb{R}^2$ it is also known that $\lambda_c = \lambda_c^*$ and that at λ_c, percolation does not occur in C or V. For $n \geq 3$, Sarkar [21] showed that $\lambda_c < \lambda_c^*$, so that there exists an interval of intensities for which there is an unbounded component in both C and V.

It is possible to consider the Poisson Boolean model in more exotic spaces than $\mathbb{R}^n$, and one might ask if there are spaces for which several unbounded components coexist with positive probability. The main results of this paper is that this is indeed the case for n-dimensional hyperbolic space $\mathbb{H}^n$. We show that there are intensities for which there are almost surely infinitely many unbounded components in the covered region if R is big enough. In $\mathbb{H}^2$ we also show the existence of three distinct phases regarding the number of unbounded components, for any R. It turns out that the main difference between $\mathbb{R}^n$ and $\mathbb{H}^n$ which causes this, is the fact that there is a linear isoperimetric inequality in $\mathbb{H}^n$, which is a consequence of the constant negative curvature of the spaces. In $\mathbb{H}^2$, the linear isoperimetric inequality says that the circumference of a bounded simply connected set is always bigger than the area of the set.

The main result in $\mathbb{H}^2$ is inspired by a theorem due to Benjamini and Schramm. In [6] they show that for a large class of nonamenable planar transitive graphs, there are infinitely many infinite clusters for some parameters in Bernoulli bond percolation. For $\mathbb{H}^2$ we also show that the model does not percolate on λ_c. The discrete analogue of this theorem is due to Benjamini, Lyons, Peres and Schramm and can be found in [4]. It turns out that several techniques from the aforementioned papers are possible to adopt to the continuous setting in $\mathbb{H}^2$.

There is also a discrete analogue to the main result in $\mathbb{H}^n$. In [17], Pak and Smirnova show that for certain Cayley graphs, there is a non-uniqueness phase for the number of unbounded components. In this case, while it is still possible to adopt their main idea to the continuous setting, it is more difficult than for $\mathbb{H}^2$.

The rest of the paper is organized as follows. In section 2 we give a very short review of uniqueness and non-uniqueness results for infinite clusters in Bernoulli percolation on graphs (for a more extensive review, see the survey paper [14]), including the results by Benjamini, Lyons, Peres, Schramm, Pak and Smirnova. In section 3 we review some elementary properties of $\mathbb{H}^n$. In section 4 we introduce the model, and give some basic results. Section 5 is devoted to the proof of the main result in $\mathbb{H}^2$ and section 6 is devoted to the proof of the main theorem for the model in $\mathbb{H}^n$.

2 Non-uniqueness in discrete percolation

Let $G = (V, E)$ be an infinite connected transitive graph with vertex set V and edge set E. In p-Bernoulli bond percolation on G, each edge in E is kept with probability p and deleted with probability $1 - p$, independently of all other edges. All vertices are kept. Let $\mathbf{P}_p$ be the probability measure on the subgraphs of G corresponding to p-Bernoulli percolation. (It is also possible to consider p-Bernoulli site percolation in which it is the vertices that are kept or deleted, and all results we present in this section are valid in this case too.) In this section, ω will denote a random subgraph of G. Connected components of ω will be called *clusters*.

Let I be the event that p-Bernoulli bond percolation contains infinite clusters. One of the most basic facts in the theory of discrete percolation is that there is a critical probability $p_c = p_c(G) \in [0, 1]$ such that $\mathbf{P}_p(I) = 0$ for $p < p_c(G)$ and $\mathbf{P}_p(I) = 1$ for $p > p_c(G)$. What happens on p_c depends on the graph. Above p_c it is known that there is 1 or ∞ infinite clusters for transitive graphs. If we let $p_u = p_u(G)$ be the infimum of the set of $p \in [0, 1]$ such that p-Bernoulli bond percolation has a unique infinite cluster, Schonmann [22] showed for all transitive graphs, one has uniqueness for all $p > p_u$. Thus there are at most three phases for $p \in [0, 1]$ regarding the number of infinite clusters, namely one for which this number is 0, one where the number is ∞ and finally one where uniqueness holds.

A problem which in recent years has attracted much interest is to decide for which graphs $p_c < p_u$. It turns out that whether a graph is *amenable* or not is central in settling this question: For $K \subset V$, the *inner vertex boundary* of K is defined as $\partial_V K := \{y \in K : \exists x \notin K, [x, y] \in E\}$. The *vertex-isoperimetric* constant for G is defined as $\kappa_V(G) := \inf_W \frac{|\partial_V W|}{|W|}$ where the infimum ranges over all finite connected subsets W of V. A bounded degree graph $G = (V, E)$ is said to be *amenable* if $\kappa_V(G) = 0$.

Benjamini and Schramm [7] have made the following general conjecture:

Conjecture 2.1. *If G is transitive, then $p_u > p_c$ if and only if G is nonamenable.*

Of course, one direction of the conjecture is the well-known theorem by Burton and Keane [8] which says that any transitive, amenable graph G has a unique infinite cluster for all $p > p_c$.

The other direction of Conjecture 2.1 has only been partially solved. Here is one such result that will be of particular interest to us, due to Benjamini and Schramm [6]. This can be considered as the discrete analogue to our main theorem in $\mathbb{H}^2$. First, another definition is needed.

Definition 2.2. *Let $G = (V, E)$ be an infinite connected graph and for $W \subset V$ let N_W be the number of infinite clusters of $G \setminus W$. The number $\sup_W N_W$ where the supremum is taken over all finite W is called the* number of ends *of G.*

Theorem 2.3. *Let G be a nonamenable, planar transitive graph with one end. Then $0 < p_c(G) < p_u(G) < 1$ for Bernoulli bond percolation on G.*

Such a general result is not yet available for non-planar graphs. However, below we present a theorem by Pak and Smirnova [17] which proves non-uniqueness for a certain class of Cayley graphs.

Definition 2.4. *Let Γ be a finitely generated group and let $S = \{g_1^{\pm 1}, ..., g_n^{\pm 1}\}$ be a finite symmetric set of generators for Γ. The (right)* Cayley graph *$\Gamma = \Gamma(G, S)$ is the graph with vertex set Γ and $[g, h]$ is an edge in Γ if and only if $g^{-1}h \in S$.*

Let S^k be the multiset of elements of Γ of the type $g_1 g_2 ... g_k$, $g_1, ..., g_k \in S$ and each such element taken with multiplicity equal to the number of ways to write it in this way. Then S^k generates G.

Theorem 2.5. *Suppose $\Gamma = \Gamma(G, S)$ is a nonamenable Cayley-graph and let $\Gamma_k = \Gamma(G, S^k)$. Then for k large enough,*

$$p_c(\Gamma_k) < p_u(\Gamma_k).$$

Theorem 2.5 is the inspiration for our main result in $\mathbb{H}^n$.

3 Hyperbolic space

We consider the unit ball model of n-dimensional hyperbolic space $\mathbb{H}^n$, that is we consider $\mathbb{H}^n$ as the open unit ball in $\mathbb{R}^n$ equipped with the hyperbolic metric. The hyperbolic metric is the metric which to a curve $\gamma = \{\gamma(t)\}_{t=0}^1$ assigns length

$$L(\gamma) = 2 \int_0^1 \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2} dt,$$

and to a set E assigns volume

$$\mu(E) = 2^n \int_E \frac{dx_1 ... dx_n}{(1 - |x|^2)^2}.$$

The *linear isoperimetric inequality* for $\mathbb{H}^2$ says that for all measurable $A \subset \mathbb{H}^2$ with $L(\partial A)$ and $\mu(A)$ well defined,

$$\frac{L(\partial A)}{\mu(A)} \geq 1. \tag{3.1}$$

Denote by $d(x, y)$ the hyperbolic distance between the points x and y. Let $S(x, r) := \{y : d(x, y) \leq r\}$ be the closed hyperbolic ball of radius r centered at x. In what follows, area (resp. length) will always mean hyperbolic area (resp. hyperbolic length). The volume of a ball is given by

$$\mu(S(0, r)) = B(n) \int_0^r \sinh(t)^{n-1}\, dt \tag{3.2}$$

where $B(n) > 0$ is a constant depending only on the dimension. We will make use of the fact that for any $\epsilon \in (0, r)$ there is a constant $K(\epsilon, n) > 0$ independent of r such that

$$\mu(S(0, r) \setminus S(0, r - \epsilon)) \geq K(\epsilon, n)\mu(S(0, r)) \tag{3.3}$$

for all r. For more facts about $\mathbb{H}^n$, we refer to [20].

3.1 Mass transport

Next, we present an essential ingredient to our proofs in $\mathbb{H}^2$, the mass transport principle which is due to Benjamini and Schramm [6]. We denote the group of isometries of $\mathbb{H}^2$ by Isom($\mathbb{H}^2$).

Definition 3.1. *A measure ν on $\mathbb{H}^2 \times \mathbb{H}^2$ is said to be* diagonally invariant *if for all measurable A, $B \subset \mathbb{H}^2$ and $g \in$Isom($\mathbb{H}^2$)*

$$\nu(gA \times gB) = \nu(A \times B).$$

Theorem 3.2. *(*Mass Transport Principle in $\mathbb{H}^2$*) If ν is a positive diagonally invariant measure on $\mathbb{H}^2 \times \mathbb{H}^2$ such that $\nu(A \times \mathbb{H}^2) < \infty$ for some open $A \subset \mathbb{H}^2$, then*

$$\nu(B \times \mathbb{H}^2) = \nu(\mathbb{H}^2 \times B)$$

for all measurable $B \subset \mathbb{H}^2$.

The intuition behind the mass transport principle can be described as follows. One may think of $\nu(A \times B)$ as the amount of mass that goes from A to B. Thus the mass transport principle says that the amount of mass that goes out of A equals the mass that goes into A.

4 The Poisson Boolean model in hyperbolic space

Definition 4.1. *A point process X on $\mathbb{H}^n$ distributed according to the probability measure* **P** *such that for $k \in \mathbb{N}$, $\lambda \geq 0$, and every measurable $A \subset \mathbb{H}^n$ one has*

$$\mathbf{P}[|X(A)| = k] = e^{-\lambda\mu(A)} \frac{(\lambda\mu(A))^k}{k!}$$

is called a Poisson process *with intensity* λ *on* $\mathbb{H}^n$*. Here* $X(A) = X \cap A$ *and* $|\cdot|$ *denotes cardinality.*

In the *Poisson Boolean model* in $\mathbb{H}^n$, at every point of a Poisson process X we place a ball with fixed radius R. More precisely, we let $C = \bigcup_{x \in X} S(x, R)$ and $V = C^c$ and refer to C and V as the covered and vacant regions of $\mathbb{H}^n$ respectively. For $A \subset \mathbb{H}^n$ we let $C[A] := \bigcup_{x \in X(A)} S(x, R)$ and $V[A] := C[A]^c$. For x, $y \in \mathbb{H}^n$ we write $x \leftrightarrow y$ if there is some curve connecting x to y which is completely covered by C. Let $d_C(x, y)$ be the length of the shortest curve connecting x and y lying completely in C if there exists such a curve, otherwise let $d_C(x, y) = \infty$. Similarly, let $d_V(x, y)$ be the length of the shortest curve connecting x and y lying completely in V if there is such a curve, otherwise let $d_V(x, y) = \infty$. The collection of all components of C is denoted by $\mathcal{C}$ and the collection of all components of V is denoted by $\mathcal{V}$. Let N_C denote the number of unbounded components in C and N_V denote the number of unbounded components in V. Next we introduce four *critical intensities* as follows. We let

$$\lambda_c := \inf\{\lambda : N_C > 0 \text{ a.s.}\}, \ \lambda_u = \inf\{\lambda : N_C = 1 \text{ a.s.}\},$$

$$\lambda_c^* = \sup\{\lambda : N_V > 0 \text{ a.s. }\}, \ \lambda_u^* = \sup\{\lambda : N_V = 1 \text{ a.s. }\}.$$

Our main result in $\mathbb{H}^2$ is:

Theorem 4.2. *For the Poisson Boolean model with fixed radius in* $\mathbb{H}^2$

$$0 < \lambda_c < \lambda_u < \infty.$$

Furthermore, with probability 1*,*

$$(N_C, N_V) = \begin{cases} (0,1), & \lambda \in [0, \lambda_c] \\ (\infty, \infty), & \lambda \in (\lambda_c, \lambda_u) \\ (1,0), & \lambda \in [\lambda_u, \infty) \end{cases}$$

The main result in $\mathbb{H}^n$ for any $n \geq 3$ is:

Theorem 4.3. *For the Poisson Boolean model with big enough fixed radius* R *in* $\mathbb{H}^n$*,* $\lambda_c < \lambda_u$*.*

In what follows, we present several quite basic results. The proofs of the following two lemmas, which give the possible values of N_C and N_V are the same as in the $\mathbb{R}^n$ case, see Propositions 3.3 and 4.2 in [18], and are therefore omitted.

Lemma 4.4. N_C *is an almost sure constant which equals* 0*,* 1 *or* ∞*.*

Lemma 4.5. N_V *is an almost sure constant which equals* 0*,* 1 *or* ∞*.*

Next we present some results concerning λ_c and λ_c^*.

Lemma 4.6. *For the Poisson Boolean model with balls of radius R in $\mathbb{H}^n$ it is the case that $\lambda_c(R) > \mu(S(0,2R))^{-1}$.*

The proof is identical to the $\mathbb{R}^n$ case, see Theorem 3.2 in [18].

Proposition 4.7. *Consider the Poisson Boolean model with balls of radius R in $\mathbb{H}^n$. There is $R_0 < \infty$ and a constant $K = K(n) > 0$ independent of R such that for all $R \geq R_0$ we have $\lambda_c(R) \leq K\mu(S(0,2R))^{-1}$.*

Proof. We prove the proposition using a supercritical branching process, the individuals of which are points in $\mathbb{H}^n$. The construction of this branching process is done by randomly distorting a regular tree embedded in the space.

Without loss of generality we assume that there is a ball centered at the origin, and the origin is taken to be the 0'th generation. Let a be such that a six-regular tree with edge length a can be embedded in $\mathbb{H}^2$ in such a way that the angles between edges at each vertex all equal $\pi/3$, and $d(u,v) \geq a$ for all vertices u and v in the tree. Suppose R is so large that $2R - 1 > a$.

Next pick three points x_1, x_2, x_3 on $\partial S(0,2R) \cap \mathbb{H}^2$ such that the angles between the geodesics between the origin and the points is $2\pi/3$. We define a cell associated to x_i as the region in $S(0,2R) \setminus S(0,2R-1)$ which can be reached by a geodesic from the origin which diverts from the geodesic from the origin to x_i by an angle of at most $\pi/6$.

For every cell that contains a Poisson point, we pick one of these uniformly at random, and take these points to be the individuals of the first generation. We continue building the branching process in this manner. Given an individual y in the n:th generation, we consider an arbitrary hyperbolic plane containing y and its parent, and pick two points at distance $2R$ from y in this plane such that the angles between the geodesics from y to these two points and the geodesic from y to its parent are all equal to $2\pi/3$. Then to each of the new points, we associate a cell as before, and check if there are any Poisson points in them. If so, one is picked uniformly at random from each cell, and these points are the children of y.

We now verify that all the cells in which the individuals of the branching process were found are disjoint. By construction, if y is an individual in the branching process, the angles between the geodesics from y to its two possible children and its parent are all in the interval $(\pi/3, \pi)$, and therefore greater than the angles in a six-regular tree. Also, the lengths of these geodesics are in the interval $(2R-1, 2R)$ and therefore larger than a. Thus by the choice of a, if all the individuals were in the same hyperbolic plane, the cells would all be disjoint.

Suppose all individuals are in $\mathbb{H}^2$, with the first individual at the origin. For each child of the origin we may pick two geodesics from the origin to infinity with angle θ less than $\pi/3$ between them that define a sector which contains the child and all of its descendants and no other individuals, and the angle between any of these two geodesics and the geodesic between the origin and the child is $\theta/2$. In the same way, for each child the grandchildren and their corresponding descendants can be divided into sectors with infinite geodesics emanating from the child and so on. Now, such a sector emanating from an individual will contain all the sectors that emanates from descendants in it.

From a sector emanating from an individual, we get a n-dimensional sector by rotating it along the geodesic going through the individual and its corresponding child. Then this n-dimensional sector will contain the corresponding n-dimensional sectors emanating from the child. From this it follows that the cells will always be disjoint.

Now, if the probability that a cell contains a poisson point is greater than 1/2, then the expected number of children to an individual is greater than 1 and so there is a positive probability that the branching process will never die out, which in turn implies that there is an unbounded connected component in the covered region of $\mathbb{H}^n$.

Let B_R denote a cell. By 3.3 there is $K_1 > 0$ independent of R such that $\mu(B_R) \geq K_1\mu(S(0, 2R))$. By the above it follows that

$$\lambda_c(R) \leq \frac{\log 2}{\mu(B_R)} \leq \frac{\log 2}{K_1\mu(S(0, 2R))},$$

completing the proof. $\square$

Lemma 4.8. *For the Poisson Boolean model in $\mathbb{H}^2$, $\lambda_c^* < \infty$.*

Proof. Let Γ be a regular tiling of $\mathbb{H}^2$ into congruent polygons of finite diameter. The polygons of Γ can be identified with the vertices of a planar nonamenable transitive graph $G = (V, E)$. Next, we define a Bernoulli site percolation ω on G. We declare each vertex $v \in V$ to be in ω if and only if its corresponding polygon $\Gamma(v)$ is not completely covered by $C[\Gamma(v)]$. Clearly, the vertices are declared to be in ω or not with the same probability and independently of each other. Now for any v,

$$\lim_{\lambda\to\infty} \mathbf{P}[v \text{ is in } \omega] = 0.$$

Thus, by Theorem 2.3, for λ large enough, there are no infinite clusters in ω. But if there are no infinite clusters in ω, there are no unbounded components of V. Thus $\lambda_c^* < \infty$. $\square$

In $\mathbb{H}^2$, we will need a correlation inequality for *increasing* and *decreasing* events. If ω and ω' are two realizations of a Poisson Boolean model we write $\omega \preceq \omega'$ if any ball present in ω is also present in ω'. An event A measurable with respect to the Poisson process is said to be *increasing* (respectively *decreasing*) if $\omega \preceq \omega'$ implies $1_A(\omega) \leq 1_A(\omega')$ (respectively $1_A(\omega) \geq 1_A(\omega')$).

Theorem 4.9. *(*FKG INEQUALITY*) If A and B are both increasing or both decreasing events measurable with respect to the Poisson process X, then $\mathbf{P}[A \cap B] \geq \mathbf{P}[A]\mathbf{P}[B]$.*

The proof is almost identical to the proof in the $\mathbb{R}^n$ case, see Theorem 2.2 in [18]. In particular, we will use the following simple corollary to Theorem 4.9, the proof of which can be found in [12], which says that if A_1, A_2, ..., A_m are increasing events with the same probability, then

$$\mathbf{P}[A_1] \geq 1 - (1 - \mathbf{P}[\cup_{i=1}^m A_i])^{1/m}.$$

The same holds when A_1, A_2, ..., A_m are decreasing.

For the proof of Theorem 4.2 we need the following lemma, the proof of which is identical to the discrete case, see [14].

Lemma 4.10. *If* $\lim_{d(u,v)\to\infty} \mathbf{P}[u \leftrightarrow v] = 0$ *then there is a.s. not a unique unbounded component in* C.

5 The number of unbounded components in $\mathbb{H}^2$

The aim of this section is to prove Theorem 4.2. We perform the proof in the case $R = 1$ but the arguments are the same for any R. We first determine the possible values of (N_C, N_V) for the model in $\mathbb{H}^2$. The first lemma is an application of the mass transport principle. First, some notation is needed.

Definition 5.1. *If* H *is a random subset of* $\mathbb{H}^2$ *which is measurable with respect to the Poisson process, we say that the distribution of* H *is* $\mathrm{Isom}(\mathbb{H}^2)$-invariant *if* gH *has the same distribution as* H *for all* $g \in \mathrm{Isom}(\mathbb{H}^2)$.

In our applications, H will typically be a union of components from C or V or something similar.

Lemma 5.2. *Suppose* H *is a random subset of* $\mathbb{H}^2$ *which is measurable with respect to the Poisson process, such that its distribution is Isom(*$\mathbb{H}^2$*)-invariant. Also suppose that if* B *is a bounded subset of* $\mathbb{H}^2$*, then* $L(B \cap \partial H) < \infty$ *a.s. and* B *intersects only finitely many components of* H *a.s. If* H *contains only finite components a.s., then for any measurable* $A \subset \mathbb{H}^2$

$$\mathbf{E}[\mu(A \cap H)] \leq \mathbf{E}[L(A \cap \partial H)].$$

Before the proof we describe the intuition behind it: we place mass of unit density in all of $\mathbb{H}^2$. Then, if h is a component of H, the mass inside h is transported to the boundary of h. Then we use the mass transport principle: the expected amount of mass transported out of a subset A equals the expected amount of mass transported into it. Finally we combine this with the isoperimetric inequality (3.1).

Proof. For A, $B \subset \mathbb{H}^2$, let

$$\eta(A \times B,\, H) := \sum_h \frac{\mu(B \cap h)\, L(A \cap \partial h)}{L(\partial h)}.$$

and let $\nu(A \times B) := \mathbf{E}[\eta(A \times B,\, H)]$. (Note that only components h that intersect both A and B give a non-zero contribution to the sum above.) Since the distribution of H is Isom($\mathbb{H}^2$)-invariant, we get for each $g \in$Isom($\mathbb{H}^2$)

$$\begin{aligned}\nu(gA \times gB) &= \mathbf{E}[\eta(gA \times gB,\, H)] = \mathbf{E}[\eta(gA \times gB,\, gH)] \\ &= \mathbf{E}[\eta(A \times B,\, H)] = \nu(A \times B).\end{aligned}$$

Thus, ν is a diagonally invariant positive measure on $\mathbb{H}^2 \times \mathbb{H}^2$. We have $\nu(\mathbb{H}^2 \times A) = \mathbf{E}\left[\mu(A \cap H)\right]$ and

$$\nu(A \times \mathbb{H}^2) = \mathbf{E}\left[\sum_h \frac{\mu(h)\, L(A \cap \partial h)}{L(\partial h)}\right] \leq \mathbf{E}[L(A \cap \partial H)]$$

where the last inequality follows from the linear isoperimetric inequality. Hence, the claim follows by Theorem 3.2. □

In the following lemmas, we exclude certain combinations of N_C and N_V. The first lemma can be considered as a continuous analogue to Lemma 3.3 in [6].

Lemma 5.3. *If H is a union of components from $\mathcal{C}$ and $\mathcal{V}$ such that the distribution of H is Isom($\mathbb{H}^2$)-invariant, then H and/or H^c contains unbounded components almost surely.*

Proof. Suppose H and $D := H^c$ contains only finite components, and let in this proof $\mathcal{H}_0$ and $\mathcal{D}_0$ be the collections of the components of H and D respectively. Then every element h of $\mathcal{H}_0$ is surrounded by a unique element h' of $\mathcal{D}_0$, which in turn is surrounded by a unique element h'' of $\mathcal{H}_0$. In the same way, every element d of $\mathcal{D}_0$ is surrounded by a unique element d' of $\mathcal{H}_0$ which in turn is surrounded by a unique element d'' of $\mathcal{D}_0$. Inductively, for $j \in \mathbb{N}$, let $\mathcal{H}_{j+1} := \{h'' \,:\, h \in \mathcal{H}_j\}$ and $\mathcal{D}_{j+1} := \{d'' \,:\, d \in \mathcal{D}_j\}$. Next, for $r \in \mathbb{N}$, let

$$A_r := \bigcup_{j=0}^{r} (\{h \in \mathcal{H}_0 \,:\, \sup\{i \,:\, h \in \mathcal{H}_i\} = j\} \cup \{d \in \mathcal{D}_0 \,:\, \sup\{i \,:\, d \in \mathcal{D}_i\} = j\}).$$

In words, $\mathcal{H}_j$ and $\mathcal{D}_j$ define layers of components from H and D. Thus A_r is the union of all layers of components from H and D that have at most r layers inside of them. Now let B be some ball in $\mathbb{H}^2$. Note that $L(B \cap \partial A_r) \leq L(B \cap \partial C)$ and $\mathbf{E}[L(B \cap \partial C)] < \infty$. Also, almost surely, there is some random r_0 such that B will be completely covered by A_r for all $r \geq r_0$. Thus the dominated convergence theorem gives

$$\lim_{r\to\infty} \mathbf{E}[\mu(B \cap A_r)] = \mu(B) \text{ and } \lim_{r\to\infty} \mathbf{E}[L(B \cap \partial A_r)] = 0.$$

Since the distribution of A_r is Isom($\mathbb{H}^2$)-invariant we get by Lemma 5.2 that there is $r_1 < \infty$ such that for $r \geq r_1$,

$$\mathbf{P}[A_r \text{ has unbounded components}] > 0.$$

But by construction, for any r it is the case that A_r has only finite components. Hence the initial assumption is false. □

Lemma 5.4. *The cases* $(N_C, N_V) = (\infty, 1)$ *and* $(N_C, N_V) = (1, \infty)$ *have probability* 0.

Proof. Suppose $N_C = \infty$. First we show that it is possible to pick $r > 0$ such that the event

$$A(x,r) := \{S(x,r) \text{ intersects at least 2 disjoint unbounded components of } C[S(x,r)^c]\}$$

has positive probability for $x \in \mathbb{H}^2$. Suppose $S(x, r_0)$ intersects an unbounded component of C for some $r_0 > 0$. Then if $S(x, r_0)$ does not intersect some unbounded component of $C[S(x, r_0)^c]$, there must be some ball centered in $S(x, r_0 + 2)\backslash S(x, r_0 + 1)$ being part of an unbounded component of $C[S(x, r_0 + 1)^c]$, which is to say that $S(x, r_0 + 1)$ intersects an unbounded component of $C[S(x, r_0 + 1)^c]$. Clearly we can find $\tilde{r}$ such that

$$B(x, \tilde{r}) := \{S(x, \tilde{r}) \text{ intersects at least 3 disjoint unbounded components of } C\}.$$

By the above discussion it follows that $\mathbf{P}[A(x, \tilde{r}) \cup A(x, \tilde{r} + 1)] > 0$, which proves the existence of r such that $A(x, r)$ has positive probability. Pick such an r and let $E(x, r) := \{S(x, r) \subset C[S(x, r)]\}$. E has positive probability and is independent of A so $A \cap E$ has positive probability. By planarity, on $A \cap E$, V contains at least 2 unbounded components. So with positive probability, $N_V > 1$. By Lemma 4.5, $N_V = \infty$ a.s. This finishes the first part of the proof. Now instead suppose $N_V = \infty$ and pick $r > 0$ such that

$$A(x,r) := \{S(x,r) \text{ intersects at least two unbounded components of } V\}$$

has positive probability. Let

$$B(x,r) := \{C[S(x, r + 1)^c] \text{ contains at least 2 unbounded components}\}.$$

On A, $C\backslash S(x, r)$ contains at least two unbounded components, which in turn implies that B occurs. Since $\mathbf{P}[A] > 0$ this gives $\mathbf{P}[B] > 0$. Since B is independent of $F(x, r) := \{|X(S(x, r + 1))| = 0\}$ which has positive probability, $\mathbf{P}[B \cap F] > 0$. On $B \cap F$, C contains at least two unbounded components. By Lemma 4.4 we get $N_C = \infty$ a.s. $\square$

The proof of the next lemma is very similar to the discrete case, see Lemma 11.12 in [12], but is included for the convenience of the reader.

Lemma 5.5. *The case* $(N_C, N_V) = (1, 1)$ *has probability* 0.

Proof. Assume $(N_C, N_V) = (1, 1)$ a.s. Fix $x \in \mathbb{H}^2$. Denote by $A^u_C(k)$ (respectively $A^d_C(k)$, $A^r_C(k)$, $A^l_C(k)$) the event that the uppermost (respectively lowermost, rightmost, leftmost) quarter of

$\partial S(x,k)$ intersects an unbounded component of $C \backslash S(x,k)$. Clearly, these events are increasing. Since $N_C = 1$ a.s.,

$$\lim_{k\to\infty} \mathbf{P}[A_C^u(k) \cup A_C^d(k) \cup A_C^r(k) \cup A_C^l(k)] = 1.$$

Hence by the corollary to the FKG-inequality, $\lim_{k\to\infty} \mathbf{P}[A_C^t(k)] = 1$ for $t \in \{u,\, d,\, r,\, l\}$. Now let $A_V^u(k)$ (respectively $A_V^l(k)$, $A_V^r(k)$, $A_V^l(k)$) be the event that the uppermost (respectively lowermost, rightmost, leftmost) quarter of $\partial S(x,k)$ intersects an unbounded component of $V \backslash S(x,k)$. Since these events are decreasing, we get in the same way as above that $\lim_{k\to\infty} \mathbf{P}[A_V^t(k)] = 1$ for $t \in \{u,\, d,\, r,\, l\}$. Thus we may pick k_0 so big that $\mathbf{P}[A_C^t(k_0)] > 7/8$ and $\mathbf{P}[A_V^t(k_0)] > 7/8$ for $t \in \{u,\, d,\, r,\, l\}$. Let

$$A := A_C^u(k_0) \cap A_C^d(k_0) \cap A_V^l(k_0) \cap A_V^r(k_0).$$

Bonferroni's inequality implies $\mathbf{P}[A] > 1/2$. On A, $C \backslash S(x,k_0)$ contains two disjoint unbounded components. Since $N_C = 1$ a.s., these two components must almost surely on A be connected. The existence of such a connection implies that there are at least two unbounded components of V, an event with probability 0. This gives $\mathbf{P}[A] = 0$, a contradiction. ☐

Proposition 5.6. *Almost surely,* $(N_C, N_V) \in \{(1,0),\, (0,1),\, (\infty,\infty)\}$.

Proof. By Lemmas 4.4 and 4.5, each of N_C and N_V is in $\{0,\, 1,\, \infty\}$. Lemma 5.3 with $H \equiv C$ rules out the case $(0,0)$. Hence Lemmas 5.4 and 5.5 imply that it remains only to rule out the cases $(0,\infty)$ and $(\infty,0)$. But since every two unbounded components of C must be separated by some unbounded component of V, $(\infty,0)$ is impossible. In the same way, $(0,\infty)$ is impossible. ☐

5.1 The situation at λ_c and λ_c^*

It turns out that to prove the main theorem, it is necessary to investigate what happens regarding N_C and N_V at the intensities λ_c and λ_c^*. Our proofs are inspired by the proof of Theorem 1.1 in [4], which says that critical Bernoulli bond and site percolation on nonamenable Cayley graphs does not contain infinite clusters.

Theorem 5.7. *At* λ_c*,* $N_C = 0$ *a.s.*

Proof. We begin with ruling out the possibility of a unique unbounded component of C at λ_c. Suppose $\lambda = \lambda_c$ and that $N_C = 1$ a.s. Denote the unique unbounded component of C by U. By Proposition 5.6, V contains only finite components a.s. Let $\epsilon > 0$ be small and remove each point in X with probability ϵ and denote by X_ϵ the remaining points. Furthermore, let $C_\epsilon = \cup_{x\in X_\epsilon} S(x,1)$. Since X_ϵ is a Poisson process with intensity $\lambda_c(1-\epsilon)$ it follows that C_ϵ will contain only bounded components a.s. Let $\mathcal{C}_\epsilon$ be the collection of all components of C_ϵ. We will now construct H_ϵ as a union of elements from $\mathcal{C}_\epsilon$ and $\mathcal{V}$ such that the distribution of H_ϵ will be $\text{Isom}(\mathbb{H}^2)$-invariant. For each $z \in \mathbb{H}^2$ we let $U_\epsilon(z)$ be the union of the components of $U \cap C_\epsilon$ being closest to z. We let each h from $\mathcal{C}_\epsilon \cup \mathcal{V}$ be in H_ϵ if and only if $\sup_{z\in h} d(z, U) < 1/\epsilon$ and $U_\epsilon(x) = U_\epsilon(y)$ for all x, $y \in h$. We want to show that for ϵ small enough, H_ϵ contains unbounded components with positive probability. Let B be some ball. It is clear that $L(B \cap \partial H_\epsilon) \to 0$ a.s. and also that $\mu(B \cap H_\epsilon) \to \mu(B)$ a.s. when $\epsilon \to 0$. Also $L(B \cap \partial H_\epsilon) \leq L(B \cap (\partial C_\epsilon \cup \partial C))$ and $\mathbf{E}[L(B \cap (\partial C_\epsilon \cup \partial C))] \leq K < \infty$ for some constant K independent of ϵ. By the dominated convergence theorem, we have

$$\lim_{\epsilon \to 0} \mathbf{E}[\mu(B \cap H_\epsilon)] = \mu(B) \text{ and } \lim_{\epsilon \to 0} \mathbf{E}[L(B \cap \partial H_\epsilon)] = 0.$$

Therefore we get by Lemma 5.2 that H_ϵ contains unbounded components with positive probability when ϵ is small enough. Suppose $h_1, h_2, ...$ is an infinite sequence of distinct elements from $\mathcal{C}_\epsilon \cup \mathcal{V}$ such that they constitute an unbounded component of H_ϵ. Then $U_\epsilon(x) = U_\epsilon(y)$ for all x, y in this component. Hence $U \cap C_\epsilon$ contains an unbounded component (this particular conclusion could not have been made without the condition $\sup_{z\in h} d(z, U) < 1/\epsilon$ in the definition of $U_\epsilon(z)$). Therefore we conclude that the existence of an unbounded component in H_ϵ implies the existence of an unbounded component in C_ϵ. Hence C_ϵ contains an unbounded component with positive probability, a contradiction.

We move on to rule out the case of infinitely many unbounded components of C at λ_c. Assume $N_C = \infty$ a.s. at λ_c. As in the proof of Lemma 5.4, we choose r such that for $x \in \mathbb{H}^2$ the event

$$A(x, r) := \\ \{S(x, r) \text{ intersects at least 3 disjoint unbounded components of } C[S(x, r)^c]\}$$

has positive probability. Let $B(x, r) := \{S(x, r) \subset C[S(x, r)]\}$ for $x \in \mathbb{H}^2$. Since A and B are independent, it follows that $A \cap B$ has positive probability. On $A \cap B$, x is contained in an unbounded component U of C. Furthermore, $U \backslash S(x, r + 1)$ contains at least three disjoint unbounded components. Now let Y be a Poisson process independent of X with some positive intensity. We call a point $y \in \mathbb{H}^2$ a *encounter point* if

- $y \in Y$;
- $A(y, r) \cap B(y, r)$ occurs;
- $S(y, 2(r + 1)) \cap Y = \{y\}$.

The third condition above means that if y_1 and y_2 are two encounter points, then $S(y_1, r + 1)$ and $S(y_2, r + 1)$ are disjoint sets. By the above, it is clear that given $y \in Y$, the probability that y is an encounter point is positive.

We now move on to show that if y is an encounter point and U is the unbounded component of C containing y, then each of the disjoint unbounded components of $U \backslash S(y, r+1)$ contains a further encounter point.

Let $m(s,t) = 1$ if t is the unique encounter point closest to s in C, and $m(s,t) = 0$ otherwise. Then let for measurable sets A, $B \subset \mathbb{H}^2$

$$\eta(A \times B) = \sum_{s \in Y(A)} \sum_{t \in Y(B)} m(s,t)$$

and

$$\nu(A \times B) = \mathbf{E}[\eta(A \times B)].$$

Clearly, ν is a positive diagonally invariant measure on $\mathbb{H}^2 \times \mathbb{H}^2$. Suppose A is some ball in $\mathbb{H}^2$. Since $\sum_{t \in Y} m(s,t) \leq 1$ we get $\nu(A \times \mathbb{H}^2) \leq \mathbf{E}[|Y(A)|] < \infty$. On the other hand, if y is an encounter point lying in A and with positive probability there is no encounter point in some of the unbounded components of $U \backslash S(y, r+1)$ we get $\sum_{s \in Y} \sum_{t \in Y(A)} m(s,t) = \infty$ with positive probability, so $\nu(\mathbb{H}^2 \times A) = \infty$, which contradicts Theorem 3.2.

The proof now continues with the construction of a forest F, that is a graph without loops or cycles. Denote the set of encounter points by T, which is a.s. infinite by the above. We let each $t \in T$ represent a vertex $v(t)$ in F. For a given $t \in T$, let $U(t)$ be the unbounded component of C containing t. Then let k be the number of unbounded components of $U(t) \backslash S(t, r+1)$ and denote these unbounded components by C_1, C_2,..., C_k. For $i = 1$, 2, ..., k put an edge between $v(t)$ and the vertex corresponding to the encounter point in C_i which is closest to t in C (this encounter point is unique by the nature of the Poisson process).

Next, we verify that F constructed as above is indeed a forest. If v is a vertex in F, denote by $t(v)$ the encounter point corresponding to it. Suppose v_0, v_1, ..., $v_n = v_0$ is a cycle of length ≥ 3, and that $d_C(t(v_0), t(v_1)) < d_C(t(v_1), t(v_2))$. Then by the construction of F it follows that $d_C(t(v_1), t(v_2)) < d_C(t(v_2), t(v_3)) < \ldots < d_C(t(v_{n-1}), t(v_0)) < d_C(t(v_0), t(v_1))$ which is impossible. Thus we must have that $d_C(t(v_i), t(v_{i+1}))$ is the same for all $i \in \{0, 1, .., n-1\}$. The assumption $d_C(t(v_0), t(v_1)) > d_C(t(v_1), t(v_2))$ obviously leads to the same conclusion. But if $y \in Y$, the probability that there are two other points in Y on the same distance in C to y is 0. Hence, cycles exist with probability 0, and therefore F is almost surely a forest.

Now define a bond percolation $F_\epsilon \subset F$: Define C_ϵ in the same way as above. Let each edge in F be in F_ϵ if and only if both encounter points corresponding to its end-vertices are in the same component of C_ϵ. Since C_ϵ contains only bounded components, F_ϵ contains only finite connected components.

For any vertex v in F we let $K(v)$ denote the connected component of v in F_ϵ and let $\partial_F K(v)$ denote the inner vertex boundary of $K(v)$ in F. Since the degree of each vertex in F is at least 3, and F is a forest, it follows that at least half of the vertices in $K(v)$ are also in $\partial_F K(v)$. Thus we conclude

$$\mathbf{P}[x \in T,\, v(x) \in \partial_F K(v(x)) | x \in Y] \geq \frac{1}{2}\mathbf{P}[x \in T | x \in Y].$$

The right-hand side of the above is positive and independent of ϵ. But the left-hand side tends to 0 as ϵ tends to 0, since when ϵ is small, it is unlikely that an edge in F is not in F_ϵ. This is a contradiction. □

By Proposition 5.6, if $N_C = 0$ a.s., then $N_V = 1$ a.s. Thus we have an immediate corollary to Theorem 5.7.

Corollary 5.8. *At λ_c, $N_V = 1$ a.s.*

Next, we show the corresponding results for λ_c^*. Obviously, the nature of V is quite different from that of C, but still the proof of Theorem 5.9 below differs only in details to that of Theorem 5.7. We include it for the convenience of the reader.

Theorem 5.9. *At λ_c^*, $N_V = 0$ a.s.*

Proof. Suppose $N_V = 1$ a.s. at λ_c^* and denote the unbounded component of V by U. Then C contains only finite components a.s. by Proposition 5.6. Let $\epsilon > 0$ and let Z be a Poisson process independent of X with intensity ϵ. Let $C_\epsilon := \cup_{x \in X \cup Z} S(x,1)$ and $V_\epsilon := C_\epsilon^c$. Since $X \cup Z$ is a Poisson process with intensity $\lambda_c^* + \epsilon$ it follows that C_ϵ has a unique unbounded component a.s. and hence V_ϵ contains only bounded components a.s. Let $\mathcal{V}_\epsilon$ be the collection of all components of V_ϵ. Define H_ϵ in the following way: For each $z \in \mathbb{H}^2$ we let $U_\epsilon(z)$ be the union of the components of $U \cap V_\epsilon$ being closest to z. We let each $h \in \mathcal{C} \cup \mathcal{V}_\epsilon$ be in H_ϵ if and only if $\sup_{z \in h} d(z, U) < 1/\epsilon$ and $U_\epsilon(x) = U_\epsilon(y)$ for all x, $y \in h$. As in the proof of Theorem 5.7, for $\epsilon > 0$ small enough, H_ϵ contains an unbounded component with positive probability, and therefore V_ϵ contains an unbounded component with positive probability, a contradiction.

Now suppose that $N_V = \infty$ a.s. at λ_c^*. Then also $N_C = \infty$ by Proposition 5.6. Therefore, for $x \in \mathbb{H}^2$, we can choose $r > 1$ big such that the intersection of the two independent events

$$A(x,r) := \{S(x,r) \text{ intersects at least 3 disjoint unbounded components of } C[S(x,r)^c]\}$$

and $B(x,r) := \{|X(S(x,r))| = 0\}$ has positive probability. Next, suppose that Y is a Poisson process independent of X with some positive intensity. Now we redefine what an encounter point is: call $y \in \mathbb{H}^2$ an encounter point if

- $y \in Y$;
- $A(y,r) \cap B(y,r)$ occurs;

- $S(y, 2r) \cap Y = \{y\}$.

By the above discussion,

$$\mathbf{P}[y \text{ is an encounter point} \mid y \in Y] > 0.$$

If y is a encounter point, y is contained in an unbounded component U of V and $U \backslash S(y, r)$ contains at least 3 disjoint unbounded components. Again we construct a forest F using the encounter points and define a bond percolation $F_\epsilon \subset F$. Let V_ϵ be defined as above. Each edge of F is declared to be in F_ϵ if and only if both its end-vertices are in the same component of V_ϵ. The proof is now finished in the same way as Theorem 5.7. □

Again, Proposition 5.6 immediately implies the following corollary:

Corollary 5.10. *At λ_c^*, $N_C = 1$ a.s.*

5.2 Proof of Theorem 4.2

Here we combine the results from the previous sections to prove our main theorem in $\mathbb{H}^2$.

Proof of Theorem 4.2. If $\lambda < \lambda_u$ then Proposition 5.6 implies $N_V > 0$ a.s. giving $\lambda \leq \lambda_c^*$. If $\lambda > \lambda_u$ the same proposition gives $N_V = 0$ a.s. giving $\lambda \geq \lambda_c^*$. Thus

$$\lambda_u = \lambda_c^*. \tag{5.1}$$

By Theorem 5.7 $N_C = 0$ a.s. at λ_c, so $N_V > 0$ a.s. at λ_c by Proposition 5.6. Thus by Theorem 5.9

$$\lambda_c < \lambda_c^*. \tag{5.2}$$

Hence the desired conclusion follows by (5.1), (5.2) and Lemma 4.8. □

6 The number of unbounded components in $\mathbb{H}^n$

This section is devoted to the proof of Theorem 4.3.

First part of proof of Theorem 4.3. In view of Lemma 4.10, it is enough to show that $\mathbf{P}[u \leftrightarrow v] \to 0$ as $d(u, v) \to \infty$ for some intensity above λ_c. We use a duplication trick. Let X_1 and X_2 be two independent copies of the Poisson Boolean model. If we for some $\epsilon > 0$ can find points u and v on an arbitrarily large distance from each other such that u is connected to v in X_1 with probability at least ϵ, then the event

$$B(u, v) := \{u \text{ is connected to } v \text{ in both } X_1 \text{ and } X_2\}$$

has probability at least ϵ^2. So it is enough to show that $\mathbf{P}[B(u,v)] \to 0$ as $d(u,v) \to \infty$ at some intensity above λ_c.

Fix points u and v and suppose $d(u,v) = d$. Let $k = \lceil d/(2R) \rceil$. That is, k is the smallest number of balls of radius R needed to connect the points u and v. Thus, for $B(u,v)$ to occur, there must be at least one sequence of at least k distinct connected balls in X_1, such that the first ball contains u and the last ball contains v, and at least one such sequence of balls in X_2. This in turn obviously implies that there is at least one sequence of at least k connected balls in X_1 such that the first ball contains u, and the last ball intersects the first ball of a sequence of at least k connected balls in X_2, where the last ball in this sequence contains u. In this sequence of at least $2k$ balls, the center of the first ball is at distance at most $2R$ from the center of the last ball.

Let $l \geq 2k$. Next we estimate the expected number of sequences of balls as above of length l. Denote this number by $N(l)$. Now, if we consider sequences of balls as above of length l, without the condition that the last ball contains u, then the expected number of such sequences is easily seen to be bounded by $\lambda^l \mu(S(0,2R))^l$ (as for example in the proof of Theorem 3.2 in [18]). Let $P_R(l)$ be the probability that the center of the last ball in such a sequence is at most at distance $2R$ from the center of the first ball. Then $N(l) \leq \lambda^l \mu(S(0,2R))^l P_R(l)$.

Now

$$\mathbf{P}[B(u,v)] \leq \sum_{l=2k}^{\infty} N(l) \leq \sum_{l=2k}^{\infty} \left(\lambda \mu(S(0,2R))\right)^l P_R(l).$$

We will now estimate the terms in the sum above.

Lemma 6.1. *Suppose $X_0, X_1, ...X_k$ is a sequence of distinct points in a Poisson point process in $\mathbb{H}^n$ such that $d(X_i, X_{i+1}) < 2R$ for $i = 0, 1, ..., k-1$. Then there is a sequence of i.i.d. random variables Y_1, $Y_2, ...$ with positive mean such that*

$$\mathbf{P}[d(X_0, X_k) \leq 2R] \leq \mathbf{P}[\sum_{i=1}^{k-1} Y_i \leq 2R].$$

In other words, $P_R(k) \leq \mathbf{P}[\sum_{i=1}^{k-1} Y_i \leq 2R]$.

The distribution of Y_i will be defined in the proof.

First part of proof of Lemma 6.1. Note that given the point X_i, the distribution of the point X_{i+1} is the uniform distribution on $S(X_i, 2R)$. Put $d_i := d(X_i, X_{i+1})$. Then $d_0, d_1...$ is a sequence of independent random variables with density

$$\frac{d}{dr}\frac{\mu(S(0,r))}{\mu(S(0,2R))} = \frac{\sinh(r)^{n-1}}{\int_0^{2R} \sinh(t)^{n-1}\,dt} \text{ for } r \in [0, 2R]. \tag{6.1}$$

Next we write

$$\mathbf{P}[d(X_0, X_k) < 2R] = \mathbf{P}\left[\sum_{i=0}^{k-1}(d(X_0, X_{i+1}) - d(X_0, X_i)) < 2R\right]. \tag{6.2}$$

The terms in the sum 6.2 are neither independent nor identically distributed. However, we will see that the sum is always larger than a sum of i.i.d. random random variables with positive mean. Suppose without loss of generality that X_0 is at the origin. Let γ_i be the geodesic between 0 and X_i and let φ_i be the geodesic between X_i and X_{i+1} for $i \geq 1$. Let θ_i be the angle between γ_i and φ_i for $i \geq 1$ and let $\theta_0 = \pi$. Then $\theta_1, \theta_2, ...$ is a sequence of independent random variables, uniformly distributed on $[0, \pi]$. Since the geodesics γ_i, γ_{i+1} and φ_i lie in the same hyperbolic plane, we can express $d(0, X_{i+1})$ in terms of $d(0, X_i)$, $d(X_i, X_{i+1})$ and θ_i using the first law of cosines for triangles in hyperbolic space (see [20], Theorem 3.5.3), which gives that

$$\begin{aligned} d(0, X_{i+1}) - d(0, X_i) = \cosh^{-1}\Big(&\cosh(d_i)\cosh(d(0, X_i)) \\ &- \sinh(d_i)\sinh(d(0, X_i))\cos(\theta_i)\Big) - d(0, X_i). \end{aligned} \tag{6.3}$$

Next we prove a lemma that states that the random variable above dominates a random variable which is independent of $d(0, X_i)$. Put

$$f(x, y, \theta) := \cosh^{-1}(\cosh(x)\cosh(y) - \sinh(x)\sinh(y)\cos(\theta)) - y.$$

Lemma 6.2. *For fixed x and θ, the function $f(x, y, \theta)$ is strictly decreasing in y and $g(x, \theta) := \lim_{y\to\infty} f(x, y, \theta) = \log(\cosh(x) - \sinh(x)\cos(\theta))$.*

Proof. For simplicity write $a = a(x) := \cosh(x)$ and $b = b(x, \theta) := \sinh(x)\cos(\theta)$. Then by rewriting

$$f(x, y, \theta) = \log\left(\frac{a\cosh(y) - b\sinh(y) + \sqrt{(a\cosh(y) - b\sinh(y))^2 - 1}}{\exp(y)}\right) \tag{6.4}$$

we get by easy calculations that the limit as $y \to \infty$ is as desired. It remains to show that $f_y'(x, y, \theta) < 0$ for all x, y and θ. We have that

$$f_y'(x, y, \theta) = -1 + \frac{-b\cosh(y) + a\sinh(y)}{\sqrt{-1 + a\cosh(y) - b\sinh(y)}\sqrt{1 + a\cosh(y) - b\sinh(y)}} \tag{6.5}$$

which is less than 0 if

$$\sqrt{-1 + a\cosh(y) - b\sinh(y)}\sqrt{1 + a\cosh(y) - b\sinh(y)} > a\sinh(y) - b\cosh(y) \tag{6.6}$$

If the right hand side in 6.6 is negative then we are done, otherwise, taking squares and simplifying gives that the inequality 6.6 is equivalent to the simpler inequality

$$a^2 - b^2 > 1$$

which holds since $a^2 - b^2 = \cosh^2(x) - \sinh^2(x)\cos^2(\theta) > \cosh^2(x) - \sinh^2(x) = 1$, completing the proof of the lemma. $\square$

Second part of proof of Lemma 6.1. Letting $Y_i := g(d_i, \theta_i)$ we have (since $Y_0 > 0$),

$$\mathbf{P}[d(X_0, X_k) < 2R] \leq \mathbf{P}\left[\sum_{i=0}^{k-1} Y_i < 2R\right] \leq \mathbf{P}\left[\sum_{i=1}^{k-1} Y_i < 2R\right] \tag{6.7}$$

where g is as in Lemma 6.2, which concludes the proof. $\square$

We now want to bound the probability in Lemma 6.1, and for this we have the following technical lemma, which in a slightly different form than below is due to Patrik Albin.

Lemma 6.3. *Let Y_i be defined as above. There is a function $h(R, \epsilon)$ such that for any $\epsilon \in (0, 1)$ we have $h(R, \epsilon) \sim Ae^{-R(1-\epsilon)}$ as $R \to \infty$ for some constant $A = A(\epsilon) \in (0, \infty)$ independent of R and such that for any $R > 0$,*

$$\mathbf{P}\left[\sum_{i=1}^{k} Y_i < 2R\right] \leq h(R, \epsilon)^k e^R. \tag{6.8}$$

Proof. Let K be the complete elliptic integral of the first kind (see [11], pp. 313-314). Then we have

$$\begin{aligned}
\mathbf{E}[e^{-Y_1/2}|d_1] &= \mathbf{E}\left[\frac{1}{\sqrt{\cosh(d_1) - \sinh(d_1)\cos(\theta_1)}}\middle| d_1\right] \\
&= \mathbf{E}\left[\frac{e^{-d_1/2}}{\sqrt{1 - \cos(\theta_1/2)^2(1 - e^{-2d_1})}}\middle| d_1\right] \\
&= \frac{2e^{-d_1/2}K(\sqrt{1 - e^{-2d_1}})}{\pi}.
\end{aligned}$$

Using the relation $K(x) = \pi\ {}_2F_1(1/2, 1/2, 1, x)/2$ where ${}_2F_1$ is the hypergeometric function (see [11], Equation 13.8.5), we have

$$\mathbf{E}[e^{-Y_1/2}|d_1] = e^{-d_1/2}\,{}_2F_1(1/2, 1/2, 1, 1 - e^{-2d_1}).$$

Since ${}_2F_1(1/2, 1/2, 1, \cdot)$ is continuous on $\{z \in \mathbb{C} : |z| \leq \rho\}$ for any $\rho \in (0, 1)$, this gives

$$\mathbf{E}[e^{-Y_1/2}|d_1] \leq A_1 e^{-d_1/2} \text{ for } d_1 \leq x_0, \tag{6.9}$$

for some constant $A_1(x_0) > 0$, for any $x_0 > 0$. Large values of d_1 makes the argument of ${}_2F_1(1/2, 1/2, 1, 1-e^{-2d_1})$ approach the radius of convergence 1 of ${}_2F_1(1/2, 1/2, 1, \cdot)$ so we perform the quadratic transformation

$$
{}_2F_1(a, b, 2b, x) = (1-x)^{-a/2} {}_2F_1\left(a, 2b-a, b+1/2, -\frac{(1-\sqrt{1-x})^2}{4\sqrt{1-x}}\right),
$$

(see [10], Equation 2.11.30), giving

$$
\mathbf{E}[e^{-Y_1/2} | d_1] = {}_2F_1\left(1/2, 1/2, 1, -e^{d_1}(1-e^{-d_1})^2/4\right).
$$

By the asymptotic behaviour of the hypergeometric function (here the analytic continuation of the hypergeometric function is used), we have

$$
|{}_2F_1(1/2, 1/2, 1, x)| \sim A_2 \frac{\log |x|}{\sqrt{|x|}}
$$

as $|x| \to \infty$ (see [10], Equation 2.3.2.9), for some constant $A_2 > 0$. Combining this with 6.9 we get

$$
\mathbf{E}[e^{-Y_1/2} | d_1] \leq A_3(1+d_1)e^{-d_1/2} \leq A_4 e^{-(1-\epsilon)d_1/2}
$$

for $d_1 > 0$, for any $\epsilon \in (0,1)$, for some constants $A_3 > 0$ and $A_4(\epsilon) > 0$. Thus

$$
\begin{aligned}
\mathbf{E}[e^{-Y_1/2}] &\leq \mathbf{E}[A_4 e^{-d_1(1-\epsilon)/2}] \\
&= A_4 \frac{\int_0^{2R} \sinh(t)^{n-1} e^{-t(1-\epsilon)/2}\, dt}{\int_0^{2R} \sinh(t)^{n-1}\, dt}
\end{aligned}
$$

Clearly $h(R,\epsilon) := A_4 \int_0^{2R} \sinh(t)^{n-1} e^{-t(1-\epsilon)/2}\, dt \big/ \int_0^{2R} \sinh(t)^{n-1}\, dt \sim A e^{-R(1-\epsilon)}$ as $R \to \infty$ for some constant $A \in (0, \infty)$. Finally we get using Markov's inequality that

$$
\begin{aligned}
\mathbf{P}\left[\sum_{i=1}^{k} Y_i < 2R\right] &= \mathbf{P}\left[e^{-\frac{1}{2}\sum_{i=1}^{k} Y_i} > e^{-R}\right] \\
&\leq e^R \mathbf{E}\left[e^{-\frac{1}{2}\sum_{i=1}^{k} Y_i}\right] \\
&= e^R \mathbf{E}\left[e^{-Y_1/2}\right]^k \\
&\leq h(R,\epsilon)^k e^R
\end{aligned}
$$

completing the proof. $\square$

Second part of proof of Theorem 4.3. By the estimates in Proposition 4.7 and Lemma 6.3 we get that

$$
\sum_{l=2k}^{\infty} (\lambda_c(R)\mu(S(0,2R)))^l P_R(l) \leq e^R \sum_{l=2k}^{\infty} K^l h(R,\epsilon)^{l-1}
$$

for any $\epsilon \in (0,1)$ and some constant $K \in (0,\infty)$. Thus if we take R big enough, the sum goes to 0 as $k \to \infty$. This is also the case if we replace λ_c with $t\lambda_c$ for some $t > 1$, proving that there are intensities above λ_c for which there are infinitely many unbounded connected components in the covered region of $\mathbb{H}^n$ for R big enough. □

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