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Special points of the Brownian net*

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Abstract

The Brownian net, which has recently been introduced by Sun and Swart [SS08], and independently by Newman, Ravishankar and Schertzer [NRS08], generalizes the Brownian web by allowing branching. In this paper, we study the structure of the Brownian net in more detail. In particular, we give an almost sure classification of each point in $\mathbb{R}^2$ according to the configuration of the Brownian net paths entering and leaving the point. Along the way, we establish various other structural properties of the Brownian net.

Key words: Brownian net, Brownian web, branching-coalescing point set.

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1 Introduction and results

1.1 Introduction

The Brownian web, $\mathscr{W}$, is essentially a collection of one-dimensional coalescing Brownian motions starting from every point in space and time $\mathbb{R}^2$. It originated from the work of Arratia [Arr79; Arr81] on the scaling limit of the voter model, and arises naturally as the diffusive scaling limit of the system of one-dimensional coalescing random walks dual to the voter model; see also [FINR04] and [NRS05]. In the language of stochastic flows, the coalescing flow associated with the Brownian web is known as the *Arratia flow*. A detailed analysis of the Brownian web was carried out by Tóth and Werner in [TW98]. More recently, Fontes, Isopi, Newman and Ravishankar [FINR04] introduced a by now standard framework in which the Brownian web is regarded as a random compact set of paths, which (in a suitable topology) becomes a random variable taking values in a Polish space. It is in this framework that the object initially proposed by Arratia in [Arr81] takes on the name *the Brownian web*.

Recently, Sun and Swart [SS08] introduced a generalization of the Brownian web, called *the Brownian net*, $\mathcal{N}$, in which paths not only coalesce, but also branch. From a somewhat different starting point, Newman, Ravishankar and Schertzer independently arrived at the same object. Their alternative construction of the the Brownian net will be published in [NRS08]. The motivation in [SS08] comes from the study of the diffusive scaling limit of one-dimensional branching-coalescing random walks with weak branching, while the motivation in [NRS08] comes from the study of one-dimensional stochastic Ising and Potts models with boundary nucleation. The different constructions of the Brownian net given in [SS08] and [NRS08] complement each other and give different insights into the structure of the Brownian net.

In the Brownian web, at a typical, deterministic point in $\mathbb{R}^2$, there is just a single path leaving the point and no path entering the point. There are, however, random, *special* points, where more than one path leaves, or where paths enter. A full classification of these special points is given in [TW98], see also [FINR06]. The special points of the Brownian web play an important role in the construction of the so-called *marked Brownian web* [FINR06], and also in the construction of the Brownian net in [NRS08]. A proper understanding of the Brownian net thus calls for a similar classification of special points of the Brownian net, which is the main goal of this paper. Along the way, we will establish various properties for the Brownian net $\mathcal{N}$, and the left-right Brownian web $(\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r})$, which is the key intermediate object in the construction of the Brownian net in [SS08].

Several models have been studied recently which have close connections to the Brownian net. One such model is the so-called *dynamical Browian web*, which is a Browian web evolving in time in such a way that at random times, paths switch among outgoing trajectories at points with one incoming, and two outgoing paths. Such a model is similar in spirit to *dynamical percolation*, see e.g. [Hag98]. In [HW07], Howitt and Warren characterized the two-dimensional distributions of the dynamical Brownian web. This leads to two coupled Brownian webs which are similar in spirit to the left-right Brownian web $(\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r})$ in [SS08]. Indeed, there is a close connection between these objects. In [NRS08], the dynamical Brownian web and the Brownian net are constructed in the same framework, and questions of exceptional times (of the former) are investigated. A discrete space-time version of the dynamical Brownian web was studied in [FNRS07].

A second model closely related to the Brownian net is a class of stochastic flows of kernels introduced by Howitt and Warren [HW06]. These stochastic flows are families of random transition kernels, describing a Brownian motion evolving in a random space-time environment. It turns out that these stochastic flows can be constructed through a random switching between outgoing paths in a 'reference' Brownian web, which plays the role of the random environment, similar to the construction of the dynamical Brownian web and the Brownian net in [NRS08]. A subclass of the stochastic flows of kernels of Howitt and Warren turns out to be supported on the Brownian net. This is the subject of the ongoing work [SSS08]. Results established in the present paper, as well as [NRS08], will provide important tools to analyze the Howitt-Warren flows.

Finally, there are close connections between the Brownian net and low temperature scaling limits of one-dimensional stochastic Potts models. In [NRS09], these scaling limits will be constructed with the help of a graphical representation based on a marking of paths in the Brownian net. Their construction uses in an essential way one of the results in the present paper (the local finiteness of relevant separation points proved in Proposition 2.9 below).

In the rest of the introduction, we recall the characterization of the Brownian web and its dual from [FINR04; FINR06], the characterization of the left-right Brownian web and the Brownian net from [SS08], the classification of special points of the Brownian web from [TW98; FINR06], and lastly

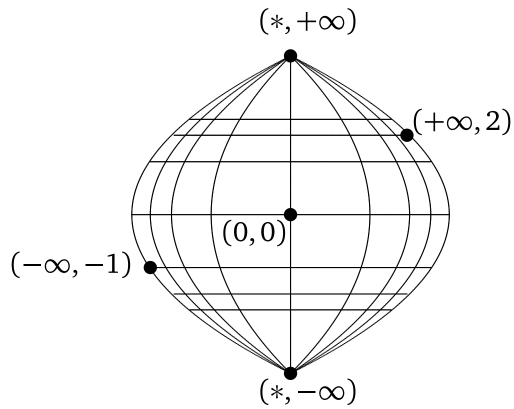


Figure 1: The compactification R^2_c of $\mathbb{R}^2$.

we formulate our main results on the classification of special points for the left-right Brownian web and the Brownian net according to the configuration of paths entering and leaving a point.

1.2 The Brownian web, left-right Brownian web, and Brownian net

Let us first recall from [FINR04] the space of *compact sets of paths* in which the Brownian web and the Brownian net take their values. Let R^2_c denote the completion of the space-time plane $\mathbb{R}^2$ w.r.t. the metric

$$\rho\big((x_1,t_1),(x_2,t_2)\big) = \Big|\tanh(t_1) - \tanh(t_2)\Big| \vee \left|\frac{\tanh(x_1)}{1+|t_1|} - \frac{\tanh(x_2)}{1+|t_2|}\right|. \tag{1.1}$$

As a topological space, R^2_c can be identified with the continuous image of $[-\infty,\infty]^2$ under a map that identifies the line $[-\infty,\infty]\times\{\infty\}$ with a single point $(*,\infty)$, and the line $[-\infty,\infty]\times\{-\infty\}$ with the point $(*,-\infty)$, see Figure 1.

A path π in R^2_c, whose starting time we denote by $\sigma_\pi \in [-\infty,\infty]$, is a mapping $\pi : [\sigma_\pi,\infty] \to [-\infty,\infty]\cup\{*\}$ such that $\pi(\infty)=*$, $\pi(\sigma_\pi)=*$ if $\sigma_\pi=-\infty$, and $t\to(\pi(t),t)$ is a continuous map from $[\sigma_\pi,\infty]$ to (R^2_c,ρ). We then define Π to be the space of all paths in R^2_c with all possible starting times in $[-\infty,\infty]$. Endowed with the metric

$$d(\pi_1,\pi_2) = \Big|\tanh(\sigma_{\pi_1}) - \tanh(\sigma_{\pi_2})\Big| \vee \sup_{t\geq\sigma_{\pi_1}\wedge\sigma_{\pi_2}} \left|\frac{\tanh(\pi_1(t\vee\sigma_{\pi_1}))}{1+|t|} - \frac{\tanh(\pi_2(t\vee\sigma_{\pi_2}))}{1+|t|}\right|, \tag{1.2}$$

(Π,d) is a complete separable metric space. Note that convergence in the metric d can be desrcibed as locally uniform convergence of paths plus convergence of starting times. (The metric d differs slightly from the original choice in [FINR04], which is slightly less natural as explained in the appendix of [SS08].)

We can now define $\mathscr{H}$, the space of compact subsets of (Π,d), equipped with the Hausdorff metric

$$d_{\mathscr{H}}(K_1,K_2) = \sup_{\pi_1\in K_1}\inf_{\pi_2\in K_2} d(\pi_1,\pi_2) \vee \sup_{\pi_2\in K_2}\inf_{\pi_1\in K_1} d(\pi_1,\pi_2). \tag{1.3}$$

The space $(\mathscr{H}, d_{\mathscr{H}})$ is also a complete separable metric space. Let $\mathscr{B}_{\mathscr{H}}$ be the Borel sigma-algebra associated with $d_{\mathscr{H}}$. The Brownian web $\mathscr{W}$ and the Brownian net $\mathscr{N}$ will be $(\mathscr{H}, \mathscr{B}_{\mathscr{H}})$-valued random variables.

In the rest of the paper, for $K \in \mathscr{H}$ and $A \subset R^2_c$, let $K(A)$ denote the set of paths in K with starting points in A. When $A = \{z\}$ for $z \in R^2_c$, we also write $K(z)$ instead of $K(\{z\})$.

We recall from [FINR04] the following characterization of the Brownian web.

Theorem 1.1. [Characterization of the Brownian web]
There exists a $(\mathscr{H}, \mathscr{B}_{\mathscr{H}})$-valued random variable $\mathscr{W}$, called the standard Brownian web, whose distribution is uniquely determined by the following properties:

(a) *For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique path $\pi_z \in \mathscr{W}(z)$.*

(b) *For any finite deterministic set of points $z_1, \ldots, z_k \in \mathbb{R}^2$, the collection $(\pi_{z_1}, \ldots, \pi_{z_k})$ is distributed as coalescing Brownian motions.*

(c) *For any deterministic countable dense subset $\mathscr{D} \subset \mathbb{R}^2$, almost surely, $\mathscr{W}$ is the closure of $\{\pi_z : z \in \mathscr{D}\}$ in (Π, d).*

To each Brownian web $\mathscr{W}$, there is associated a *dual Brownian web* $\hat{\mathscr{W}}$, which is a random set of paths running backward in time [Arr81; TW98; FINR06]. The pair $(\mathscr{W}, \hat{\mathscr{W}})$ is called the *double Brownian web*. By definition, a *backward path* $\hat{\pi}$, with starting time denoted by $\hat{\sigma}_{\hat{\pi}}$, is a function $\hat{\pi} : [-\infty, \hat{\sigma}_{\hat{\pi}}] \to [-\infty, \infty] \cup \{*\}$, such that $t \mapsto (t, \hat{\pi}(t))$ is a continuous map from $[-\infty, \hat{\sigma}_{\hat{\pi}}]$ to R^2_c. We let $(\hat{\Pi}, \hat{d})$ denote the space of backward paths, which is in a natural way isomorphic to (Π, d) through time reversal, and we let $(\hat{\mathscr{H}}, d_{\hat{\mathscr{H}}})$ denote the space of compact subsets of $(\hat{\Pi}, \hat{d})$, equipped with the Hausdorff metric. We say that a dual path $\hat{\pi}$ *crosses* a (forward) path π if there exist $\sigma_\pi \le s < t \le \hat{\sigma}_{\hat{\pi}}$ such that $(\pi(s) - \hat{\pi}(s))(\pi(t) - \hat{\pi}(t)) < 0$. The next theorem follows from [FINR06, Theorem 3.7]; a slightly different construction of the dual Brownian web can be found in [SS08, Theorem 1.9].

Theorem 1.2. [Characterization of the dual Brownian web]
Let $\mathscr{W}$ be the standard Brownian web. Then there exists a $\hat{\mathscr{H}}$-valued random variable $\hat{\mathscr{W}}$, defined on the same probability space as $\mathscr{W}$, called the dual Brownian web, which is almost surely uniquely determined by the following properties:

(a) *For any deterministic $z \in \mathbb{R}^2$, almost surely $\hat{\mathscr{W}}(z)$ consists of a single path $\hat{\pi}_z$, which is the unique path in $\hat{\Pi}(z)$ that does not cross any path in $\mathscr{W}$.*

(b) *For any deterministic countable dense subset $\mathscr{D} \subset \mathbb{R}^2$, almost surely, $\hat{\mathscr{W}}$ is the closure of $\{\hat{\pi}_z : z \in \mathscr{D}\}$ in $(\hat{\Pi}, \hat{d})$.*

It is known that, modulo a time reversal, $\hat{\mathscr{W}}$ is equally distributed with $\mathscr{W}$. Moreover, paths in $\mathscr{W}$ and $\hat{\mathscr{W}}$ interact via Skorohod reflection. (This follows from the results in [STW00], together with the standard discrete aproximation of the double Brownian web.)

We now recall the *left-right Brownian web* $(\mathscr{W}^{\mathrm{l}}, \mathscr{W}^{\mathrm{r}})$, which is the key intermediate object in the construction of the Brownian net in [SS08]. Following [SS08], we call $(l_1, \ldots, l_m; r_1, \ldots, r_n)$ a collection of *left-right coalescing Brownian motions*, if $(l_1, \ldots, l_m)$ is distributed as coalescing Brownian motions each with drift -1, $(r_1, \ldots, r_n)$ is distributed as coalescing Brownian motions each with drift $+1$, paths in $(l_1, \ldots, l_m; r_1, \ldots, r_n)$ evolve independently when they are apart, and the interaction between l_i and r_j when they meet is described by the two-dimensional stochastic differential equation

$$\begin{aligned} \mathrm{d}L_t &= 1_{\{L_t \neq R_t\}} \mathrm{d}B_t^{\mathrm{l}} + 1_{\{L_t = R_t\}} \mathrm{d}B_t^{\mathrm{s}} - \mathrm{d}t, \\ \mathrm{d}R_t &= 1_{\{L_t \neq R_t\}} \mathrm{d}B_t^{\mathrm{r}} + 1_{\{L_t = R_t\}} \mathrm{d}B_t^{\mathrm{s}} + \mathrm{d}t, \end{aligned} \tag{1.4}$$

where $B_t^{\mathrm{l}}, B_t^{\mathrm{r}}, B_t^{\mathrm{s}}$ are independent standard Brownian motions, and (L, R) are subject to the constraint that

$$L_t \leq R_t \text{ for all } t \geq \tau_{L,R}, \tag{1.5}$$

where, for any two paths $\pi, \pi' \in \Pi$, we let

$$\tau_{\pi,\pi'} := \inf\{t > \sigma_\pi \vee \sigma_{\pi'} : \pi(t) = \pi'(t)\} \tag{1.6}$$

denote the first meeting time of π and π', which may be ∞. It can be shown that subject to the condition (1.5), solutions to the SDE (1.4) are unique in distribution [SS08, Proposition 2.1]. (See also [HW07, Proposition 14] for a martingale problem characterization of sticky Brownian motions with drift.) The interaction between left-most and right-most paths is a form of sticky reflection; in particular, if $\{t : L_t = R_t\}$ is nonempty then it is a nowhere dense set with positive Lebesgue measure [SS08, Proposition 3.1].

We cite the following characterization of the left-right Brownian web from [SS08, Theorem 1.5].

Theorem 1.3. [Characterization of the left-right Brownian web]
There exists a $(\mathcal{H}^2, \mathcal{B}_{\mathcal{H}^2})$-valued random variable $(\mathcal{W}^{\mathrm{l}}, \mathcal{W}^{\mathrm{r}})$, called the standard left-right Brownian web, whose distribution is uniquely determined by the following properties:

(a) *For each deterministic $z \in \mathbb{R}^2$, almost surely there are unique paths $l_z \in \mathcal{W}^{\mathrm{l}}$ and $r_z \in \mathcal{W}^{\mathrm{r}}$.*

(b) *For any finite deterministic set of points $z_1, \ldots, z_m, z'_1, \ldots, z'_n \in \mathbb{R}^2$, the collection $(l_{z_1}, \ldots, l_{z_n}; r_{z'_1}, \ldots, r_{z'_n})$ is distributed as left-right coalescing Brownian motions.*

(c) *For any deterministic countable dense subset $\mathcal{D} \subset \mathbb{R}^2$, almost surely $\mathcal{W}^{\mathrm{l}}$ is the closure of $\{l_z : z \in \mathcal{D}\}$ and $\mathcal{W}^{\mathrm{r}}$ is the closure of $\{r_z : z \in \mathcal{D}\}$ in the space (Π, d).*

Comparing Theorems 1.1 and 1.3, we see that $\mathcal{W}^{\mathrm{l}}$ and $\mathcal{W}^{\mathrm{r}}$ are distributed as Brownian webs tilted with drift -1 and $+1$, respectively. Therefore, by Theorem 1.2, the Brownian webs $\mathcal{W}^{\mathrm{l}}$ and $\mathcal{W}^{\mathrm{r}}$ a.s. uniquely determine dual webs $\hat{\mathcal{W}}^{\mathrm{l}}$ and $\hat{\mathcal{W}}^{\mathrm{r}}$, respectively. It turns out that $(\hat{\mathcal{W}}^{\mathrm{l}}, \hat{\mathcal{W}}^{\mathrm{r}})$ is equally distributed with $(\mathcal{W}^{\mathrm{l}}, \mathcal{W}^{\mathrm{r}})$ modulo a rotation by 180°.

Based on the left-right Brownian web, [SS08] gave three equivalent characterizations of the Brownian net, which are called respectively the *hopping, wedge, and mesh characterizations*. We first recall what is meant by hopping, and what are wedges and meshes.

Hopping: Given two paths $\pi_1, \pi_2 \in \Pi$, any $t > \sigma_{\pi_1} \vee \sigma_{\pi_2}$ (note the strict inequality) is called an *intersection time* of π_1 and π_2 if $\pi_1(t) = \pi_2(t)$. By hopping from π_1 to π_2, we mean the construction of a new path by concatenating together the piece of π_1 before and the piece of π_2 after an intersection time. Given the left-right Brownian web $(\mathcal{W}^{\mathrm{l}}, \mathcal{W}^{\mathrm{r}})$, let $H(\mathcal{W}^{\mathrm{l}} \cup \mathcal{W}^{\mathrm{r}})$ denote the set of paths constructed by hopping a finite number of times among paths in $\mathcal{W}^{\mathrm{l}} \cup \mathcal{W}^{\mathrm{r}}$.

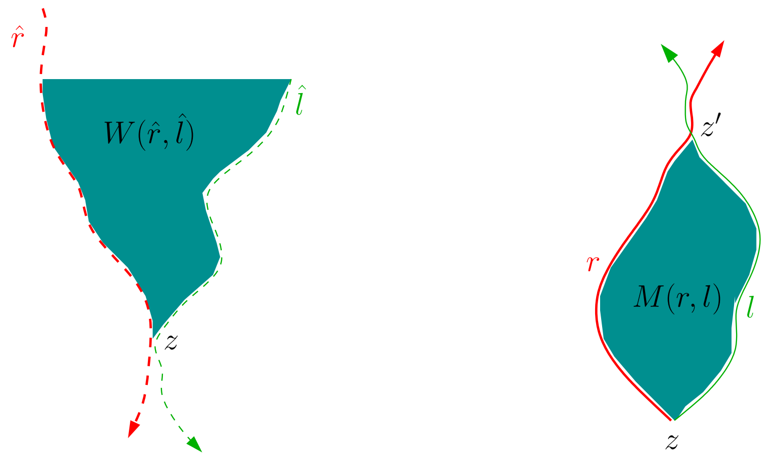


Figure 2: A wedge $W(\hat{r},\hat{l})$ with bottom point z and a mesh $M(r,l)$ with bottom and top points z and z'.

Wedges: Let $(\hat{\mathcal{W}}^{\mathrm{l}},\hat{\mathcal{W}}^{\mathrm{r}})$ be the dual left-right Brownian web almost surely determined by $(\mathcal{W}^{\mathrm{l}},\mathcal{W}^{\mathrm{r}})$. Recall that $\hat{\sigma}_{\hat{\pi}}$ denotes the starting time of a backward path $\hat{\pi}$. Any pair $\hat{l}\in\hat{\mathcal{W}}^{\mathrm{l}}$, $\hat{r}\in\hat{\mathcal{W}}^{\mathrm{r}}$ with $\hat{r}(\hat{\sigma}_{\hat{l}}\wedge\hat{\sigma}_{\hat{r}})<\hat{l}(\hat{\sigma}_{\hat{l}}\wedge\hat{\sigma}_{\hat{r}})$ defines an open set (see Figure 2)

$$W(\hat{r},\hat{l})=\{(x,u)\in\mathbb{R}^2:\hat{\tau}_{\hat{r},\hat{l}}<u<\hat{\sigma}_{\hat{l}}\wedge\hat{\sigma}_{\hat{r}},\ \hat{r}(u)<x<\hat{l}(u)\},\tag{1.7}$$

where, in analogy with (1.6), $\hat{\tau}_{\hat{r},\hat{l}}:=\sup\{t<\hat{\sigma}_{\hat{l}}\wedge\hat{\sigma}_{\hat{r}}:\hat{r}(t)=\hat{l}(t)\}$ denotes the first (backward) hitting time of $\hat{r}$ and $\hat{l}$. Such an open set is called a *wedge* of $(\hat{\mathcal{W}}^{\mathrm{l}},\hat{\mathcal{W}}^{\mathrm{r}})$. If $\hat{\tau}_{\hat{r},\hat{l}}>-\infty$, we call $\tau_{\hat{r},\hat{l}}$ the bottom time, and $(\hat{l}(\hat{\tau}_{\hat{r},\hat{l}}),\tau_{\hat{r},\hat{l}})$ the bottom point of the wedge $W(\hat{r},\hat{l})$.

Meshes: By definition, a *mesh* of $(\mathcal{W}^{\mathrm{l}},\mathcal{W}^{\mathrm{r}})$ is an open set of the form (see Figure 2)

$$M=M(r,l)=\{(x,t)\in\mathbb{R}^2:\sigma_l<t<\tau_{l,r},\ r(t)<x<l(t)\},\tag{1.8}$$

where $l\in\mathcal{W}^{\mathrm{l}}$, $r\in\mathcal{W}^{\mathrm{r}}$ are paths such that $\sigma_l=\sigma_r$, $l(\sigma_l)=r(\sigma_r)$ and $r(s)<l(s)$ on $(\sigma_l,\sigma_l+\epsilon)$ for some $\epsilon>0$. We call $(l(\sigma_l),\sigma_l)$ the bottom point, σ_l the bottom time, $(l(\tau_{l,r}),\tau_{l,r})$ the top point, $\tau_{l,r}$ the top time, r the left (!) boundary, and l the right boundary of M.

Given an open set $A\subset\mathbb{R}^2$ and a path $\pi\in\Pi$, we say π *enters* A if there exist $\sigma_\pi<s<t$ such that $\pi(s)\notin A$ and $\pi(t)\in A$. We say π *enters* A *from outside* if there exists $\sigma_\pi<s<t$ such that $\pi(s)\notin\bar{A}$ and $\pi(t)\in A$. We now recall the following characterization of the Brownian net from [SS08, Theorems 1.3, 1.7, 1.10].

Theorem 1.4. [Characterization of the Brownian net]
There exists a $(\mathcal{H},\mathcal{B}_{\mathcal{H}})$-valued random variable $\mathcal{N}$, the standard Brownian net, whose distribution is uniquely determined by property (a) *and any of the three equivalent properties* (b1)–(b3) *below:*

(a) *There exist $\mathcal{W}^{\mathrm{l}},\mathcal{W}^{\mathrm{r}}\subset\mathcal{N}$ such that $(\mathcal{W}^{\mathrm{l}},\mathcal{W}^{\mathrm{r}})$ is distributed as the left-right Brownian web.*

(b1) *Almost surely, $\mathcal{N}$ is the closure of $H(\mathcal{W}^{\mathrm{l}}\cup\mathcal{W}^{\mathrm{r}})$.*

(b2) *Almost surely,* $\mathcal{N}$ *is the set of paths in* Π *which do not enter any wedge of* $(\hat{\mathcal{W}}^{\rm l}, \hat{\mathcal{W}}^{\rm r})$ *from outside.*

(b3) *Almost surely,* $\mathcal{N}$ *is the set of paths in* Π *which do not enter any mesh of* $(\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r})$.

Remark. Properties (b1)–(b3) in fact imply that the left-right Brownian web $(\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r})$ contained in a Brownian net $\mathcal{N}$ is almost surely uniquely determined by the latter, and for each deterministic $z \in \mathbb{R}^2$, the path in $\mathcal{W}^{\rm l}$, resp. $\mathcal{W}^{\rm r}$, starting from z is just the left-most, resp. right-most, path among all paths in $\mathcal{N}$ starting from z. Since $(\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r})$ uniquely determines a dual left-right Brownian web $(\hat{\mathcal{W}}^{\rm l}, \hat{\mathcal{W}}^{\rm r})$, there exists a dual Brownian net $\hat{\mathcal{N}}$ uniquely determined by and equally distributed with $\mathcal{N}$ (modulo time reversal).

The construction of the Brownian net from the left-right Brownian web can be regarded as an outside-in approach because $\mathcal{W}^{\rm l}$ and $\mathcal{W}^{\rm r}$ are the "outermost" paths among all paths in $\mathcal{N}$. On the other hand, the construction of the Brownian net in [NRS08] can be regarded as an inside-out approach, since they start from a standard Brownian web, which may be viewed as a collection of "innermost" paths, to which new paths are added by allowing branching to the left and right. More precisely, they allow hopping at a set of marked points, which is a Poisson subset of the set of all special points with one incoming and two outgoing paths. One may call this construction the *marking construction* of the Brownian net. In this paper, we will only use the characterizations provided by Theorem 1.4.

1.3 Classification of special points

To classify each point of $\mathbb{R}^2$ according to the local configuration of paths in the Brownian web or net, we first formulate a notion of equivalence among paths entering, resp. leaving, a point, which provides a unified framework. We say that a path $\pi \in \Pi$ *enters* a point $z = (x, t) \in \mathbb{R}^2$ if $\sigma_\pi < t$ and $\pi(t) = x$. We say that π *leaves* z if $\sigma_\pi \leq t$ and $\pi(t) = x$.

Definition 1.5. [Equivalence of paths entering and leaving a point]
We say $\pi_1, \pi_2 \in \Pi$ *are* equivalent paths entering $z = (x, t) \in \mathbb{R}^2$*, denoted by* $\pi_1 \sim^z_{\rm in} \pi_2$*, if* π_1 *and* π_2 *enter* z *and* $\pi_1(t - \varepsilon_n) = \pi_2(t - \varepsilon_n)$ *for a sequence* $\varepsilon_n \downarrow 0$*. We say* π_1, π_2 *are* equivalent paths leaving z*, denoted by* $\pi_1 \sim^z_{\rm out} \pi_2$*, if* π_1 *and* π_2 *leave* z *and* $\pi_1(t + \varepsilon_n) = \pi_2(t + \varepsilon_n)$ *for a sequence* $\varepsilon_n \downarrow 0$.

Note that, on Π, $\sim^z_{\rm in}$ and $\sim^z_{\rm out}$ are not equivalence relations. However, almost surely, they define equivalence relations on the set of all paths in the Brownian web $\mathcal{W}$ entering or leaving z. Due to coalescence, almost surely for all $\pi_1, \pi_2 \in \mathcal{W}$ and $z = (x, t) \in \mathbb{R}^2$,

$$\begin{aligned} \pi_1 \sim^z_{\rm in} \pi_2 \quad &\text{iff} \quad \pi_1 = \pi_2 \text{ on } [t - \varepsilon, \infty) \text{ for some } \varepsilon > 0, \\ \pi_1 \sim^z_{\rm out} \pi_2 \quad &\text{iff} \quad \pi_1 = \pi_2 \text{ on } [t, \infty). \end{aligned} \tag{1.9}$$

Let $m_{\rm in}(z)$, resp. $m_{\rm out}(z)$, denote the number of equivalence classes of paths in $\mathcal{W}$ entering, resp. leaving, z, and let $\hat{m}_{\rm in}(z)$ and $\hat{m}_{\rm out}(z)$ be defined similarly for the dual Brownian web $\hat{\mathcal{W}}$. For the Brownian web, points $z \in \mathbb{R}^2$ are classified according to the value of $(m_{\rm in}(z), m_{\rm out}(z))$. Points of type (1,2) are further divided into types $(1,2)_{\rm l}$ and $(1,2)_{\rm r}$, where the subscript l (resp. r) indicates that the left (resp. right) of the two outgoing paths is the continuation of the (up to equivalence) unique incoming path. Points in the dual Brownian web $\hat{\mathcal{W}}$ are labelled according to their type in the Brownian web obtained by rotating the graph of $\hat{\mathcal{W}}$ in $\mathbb{R}^2$ by 180°. We cite the following result from [TW98, Proposition 2.4] or [FINR06, Theorems 3.11–3.14].

Theorem 1.6. [Classification of special points of the Brownian web]
Let $\mathcal{W}$ be the Brownian web and $\hat{\mathcal{W}}$ its dual. Then almost surely, each $z \in \mathbb{R}^2$ satisfies $m_{\rm out}(z) = \hat{m}_{\rm in}(z)+1$ and $\hat{m}_{\rm out}(z) = m_{\rm in}(z)+1$, and z is of one of the following seven types in $\mathcal{W}/\hat{\mathcal{W}}$: $(0,1)/(0,1)$, $(0,2)/(1,1)$, $(1,1)/(0,2)$, $(0,3)/(2,1)$, $(2,1)/(0,3)$, $(1,2)_{\rm l}/(1,2)_{\rm l}$, and $(1,2)_{\rm r}/(1,2)_{\rm r}$. For each deterministic $t \in \mathbb{R}$, almost surely, each point in $\mathbb{R}\times\{t\}$ is of either type $(0,1)/(0,1)$, $(0,2)/(1,1)$, or $(1,1)/(0,2)$. A deterministic point in $\mathbb{R}^2$ is almost surely of type $(0,1)/(0,1)$.

We do not give a picture to demonstrate Theorem 1.6. However, if in the first row in Figure 3, one replaces each pair consisting of one left-most (green) and right-most (red) path by a single path, and likewise for pairs of dual (dashed) paths, then one obtains a schematic depiction of the 7 types of points in $\mathcal{W}/\hat{\mathcal{W}}$.

We now turn to the problem of classifying the special points of the Brownian net. We start by observing that also in the left-right Brownian web, a.s. for each $z \in \mathbb{R}^2$, the relations $\sim^z_{\rm in}$ and $\sim^z_{\rm out}$ define equivalence relations on the set of paths in $\mathcal{W}^{\rm l} \cup \mathcal{W}^{\rm r}$ entering, resp. leaving z. This follows from (1.9) for the Brownian web and the fact that a.s. for each $l \in \mathcal{W}^{\rm l}$, $r \in \mathcal{W}^{\rm r}$ and $\sigma_l \vee \sigma_r < s < t$ such that $l(s) = r(s)$, one has $l(t) \leq r(t)$ (see Prop. 3.6 (a) of [SS08]). Moreover, by the same facts, the equivalence classes of paths in $\mathcal{W}^{\rm l} \cup \mathcal{W}^{\rm r}$ entering, resp. leaving, z are naturally ordered from left to right.

Our classification of points in the Brownian net $\mathcal{N}$ is mainly based on the equivalence classes of incoming and outgoing paths in $\mathcal{W}^{\rm l} \cup \mathcal{W}^{\rm r}$. To denote the type of a point, we first list the incoming equivalence classes of paths from left to right, and then, separated by a comma, the outgoing equivalence classes of paths from left to right. If an equivalence class contains only paths in $\mathcal{W}^{\rm l}$ resp. $\mathcal{W}^{\rm r}$ we will label it by l, resp. r, while if it contains both paths in $\mathcal{W}^{\rm l}$ and in $\mathcal{W}^{\rm r}$ we will label it by p, standing for pair. For points with (up to equivalence) one incoming and two outgoing paths, a subscript l resp. r means that all incoming paths belong to the left one, resp. right one, of the two outgoing equivalence classes; a subscript s indicates that incoming paths in $\mathcal{W}^{\rm l}$ belong to the left outgoing equivalence class, while incoming paths in $\mathcal{W}^{\rm r}$ belong to the right outgoing equivalence class. If at a point there are no incoming paths in $\mathcal{W}^{\rm l} \cup \mathcal{W}^{\rm r}$, then we denote this by o or n, where o indicates that there are no incoming paths in the net $\mathcal{N}$, while n indicates that there are incoming paths in $\mathcal{N}$ (but none in $\mathcal{W}^{\rm l} \cup \mathcal{W}^{\rm r}$).

Thus, for example, a point is of type $({\rm p},{\rm lp})_{\rm r}$ if at this point there is one equivalence class of incoming paths in $\mathcal{W}^{\rm l} \cup \mathcal{W}^{\rm r}$ and there are two outgoing equivalence classes. The incoming equivalence class is of type p while the outgoing equivalence classes are of type l and p, from left to right. All incoming paths in $\mathcal{W}^{\rm l} \cup \mathcal{W}^{\rm r}$ continue as paths in the outgoing equivalence class of type p.

Points in the dual Brownian net $\hat{\mathcal{N}}$, which is defined in terms of the dual left-right Brownian web $(\hat{\mathcal{W}}^{\rm l}, \hat{\mathcal{W}}^{\rm r})$, are labelled according to their type in the Brownian net and left-right web obtained by rotating the graphs of $\hat{\mathcal{N}}$ and $(\hat{\mathcal{W}}^{\rm l}, \hat{\mathcal{W}}^{\rm r})$ in $\mathbb{R}^2$ by 180°. With the notation introduced above, we can now state our first main result on the classification of points in $\mathbb{R}^2$ for the Brownian net.

Theorem 1.7. [Classification of special points of the Brownian net]
Let $(\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r})$ be the standard left-right Brownian web, let $(\hat{\mathcal{W}}^{\rm l}, \hat{\mathcal{W}}^{\rm r})$ be its dual, and let $\mathcal{N}$ and $\hat{\mathcal{N}}$ be the associated Brownian net and its dual. Then almost surely, each point in $\mathbb{R}^2$ is of one of the following 20 types in $\mathcal{N}/\hat{\mathcal{N}}$ (see Figure 3):

(1) (o,p)/(o,p), (o,pp)/(p,p), (p,p)/(o,pp), (o,ppp)/(pp,p), (pp,p)/(o,ppp), $({\rm p},{\rm pp})_{\rm l}/({\rm p},{\rm pp})_{\rm l}$, $({\rm p},{\rm pp})_{\rm r}/({\rm p},{\rm pp})_{\rm r}$;

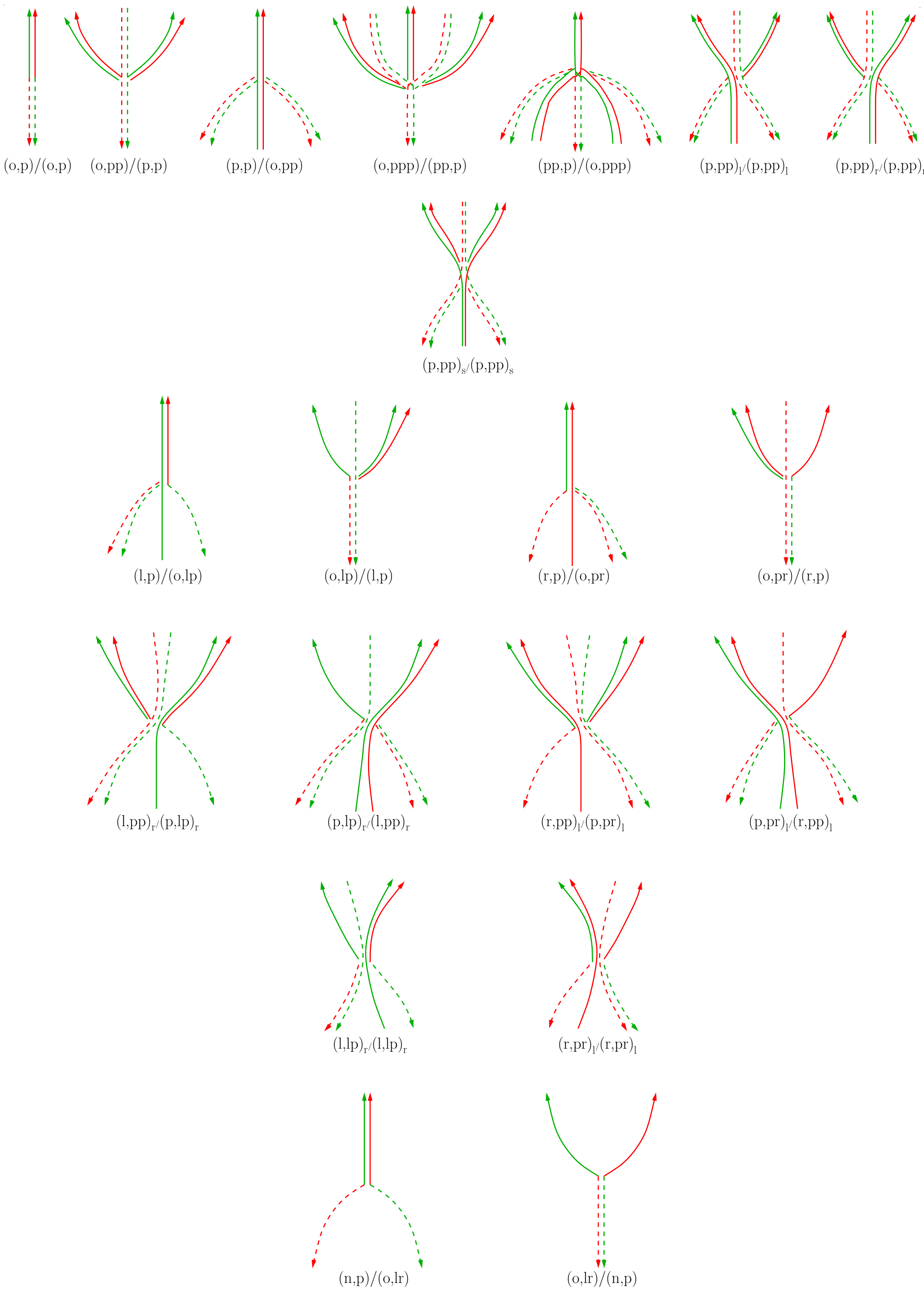


Figure 3: Classification of special points of the Brownian net.

(2) $(\mathrm{p},\mathrm{pp})_\mathrm{s}/(\mathrm{p},\mathrm{pp})_\mathrm{s}$;

(3) (l,p)/(o,lp), (o,lp)/(l,p), (r,p)/(o,pr), (o,pr)/(r,p);

(4) $(\mathrm{l},\mathrm{pp})_\mathrm{r}/(\mathrm{p},\mathrm{lp})_\mathrm{r}$, $(\mathrm{p},\mathrm{lp})_\mathrm{r}/(\mathrm{l},\mathrm{pp})_\mathrm{r}$, $(\mathrm{r},\mathrm{pp})_\mathrm{l}/(\mathrm{p},\mathrm{pr})_\mathrm{l}$, $(\mathrm{p},\mathrm{pr})_\mathrm{l}/(\mathrm{r},\mathrm{pp})_\mathrm{l}$;

(5) $(\mathrm{l},\mathrm{lp})_\mathrm{r}/(\mathrm{l},\mathrm{lp})_\mathrm{r}$, $(\mathrm{r},\mathrm{pr})_\mathrm{l}/(\mathrm{r},\mathrm{pr})_\mathrm{l}$;

(6) (n,p)/(o,lr), (o,lr)/(n,p);

and all of these types occur. For each deterministic time $t \in \mathbb{R}$, almost surely, each point in $\mathbb{R} \times \{t\}$ is of either type (o, p)/(o, p)*,* (o, pp)/(p, p)*, or* (p, p)/(o, pp)*, and all of these types occur. A deterministic point $(x, t) \in \mathbb{R}^2$ is almost surely of type* (o, p)/(o, p)*.*

Remark 1 Our classification is mainly based on the configuration of equivalence classes of paths in the left-right Brownian web $\mathscr{W}^\mathrm{l} \cup \mathscr{W}^\mathrm{r}$ entering and leaving a point. For points of types (o, p) and (n, p), however, we also use information about paths which belong to the Brownian net $\mathscr{N}$ but not to $\mathscr{W}^\mathrm{l} \cup \mathscr{W}^\mathrm{r}$. By distinguishing notationally between these types, we achieve that the type of a point in $\hat{\mathscr{N}}$ is uniquely determined by its type in $\mathscr{N}$. Moreover, by counting n as an incoming path (and counting equivalence classes of types l, r, p as one incoming resp. outgoing path each), we achieve that $m_\mathrm{out}(z) = \hat{m}_\mathrm{in}(z) + 1$ and $\hat{m}_\mathrm{out}(z) = m_\mathrm{in}(z) + 1$, in analogy with the Brownian web.

Remark 2 Modulo symmetry between left and right, and between forward and dual paths, there are only 9 types of points in Theorem 1.7: four types from group (1), and one type each from groups (2)–(6). Group (1) corresponds to the 7 types of points of the Brownian web, except that each equivalence class of paths is now of type p. We call points of type (pp, p) *meeting points*, and points of type $(\mathrm{p},\mathrm{pp})_\mathrm{s}$ (from group (2)) *separation points*. Points in groups (3)–(6) are 'cluster points' that arise as the limit of a nested sequence of excursions between paths in the left-right Brownian web, or its dual (see Proposition 3.11 and Figure 6 below).

1.4 Structure of special points

When contemplating the special points of the Brownian net as depicted in Figure 3, one is struck by the fact that points from groups (3)–(6) (the 'cluster points') look just like certain points from group (1), except that some paths are missing. For example, points of type (l, p)/(o, lp) look like points of type (p, p)/(o, pp), points of type $(\mathrm{l},\mathrm{pp})_\mathrm{r}/(\mathrm{p},\mathrm{lp})_\mathrm{r}$ look like points of $(\mathrm{p},\mathrm{pp})_\mathrm{r}/(\mathrm{p},\mathrm{pp})_\mathrm{r}$, and points of type (n, p)/(o, lp) look like points of type (p, p)/(o, pp). In most cases, when one member of a (dual) left-right pair seems to be missing, the other member of the pair is still present, but for points of group (6), a whole (dual) incoming pair seems to have disappeared. In the present section, we will show how to make these 'missing' paths visible in the form of *reflected left-most* (or *right-most*) *paths*. These reflected left-most (resp. right-most) paths are not elements of $\mathscr{W}^\mathrm{l}$ (resp. $\mathscr{W}^\mathrm{r}$), but they turn out to be countable concatenations of paths in $\mathscr{W}^\mathrm{l}$ (resp. $\mathscr{W}^\mathrm{r}$).

Except for fulfilling the aesthetic role of bringing to light what we feel is missing, these reflected paths also serve the more practical aim of putting limitations on how general paths in the Brownian net $\mathscr{N}$ (and not just the left-right Brownian web $\mathscr{W}^\mathrm{l} \cup \mathscr{W}^\mathrm{r}$) can enter and leave points of various types. Recall that our classicifation theorem (Theorem 1.7) primarily describes how left-most and right-most paths enter and leave points in the plane. In Theorems 1.11 and 1.12 below, we first describe the structure of a special set of reflected left-most and right-most paths near the special points, and then we use this to describe the local structure of the Brownian net at these points. Our results will show that, with the exception of outgoing paths at points of type (o,lr), all Brownian net paths must enter or leave a point squeezed between a pair consisting of one (reflected) left-most and (reflected) right-most path.

We start with some definitions. By definition, we say that a set $\mathcal{K}$ of paths, all starting at the same time t, has a *maximum* (resp. *minimum*) if there exists a path $\pi \in \mathcal{K}$ such that $\pi' \leq \pi$ (resp. $\pi \leq \pi'$) on $[t,\infty)$ for all $\pi' \in \mathcal{K}$. We denote the (necessarily unique) maximum (resp. minimum) of $\mathcal{K}$ by $\max(\mathcal{K})$ (resp. $\min(\mathcal{K})$).

Lemma 1.8. [Reflected left-most and right-most paths]
Almost surely, for each $\hat{\pi} \in \hat{\mathcal{N}}$ and $z=(x,t) \in R^2_{\mathrm{c}}$ such that $t \leq \hat{\sigma}_{\hat{\pi}}$ and $\hat{\pi}(t) \leq x$ (resp. $x \leq \hat{\pi}(t)$), there exists a unique path $l_{z,\hat{\pi}} \in \mathcal{N}$ (resp. $r_{z,\hat{\pi}} \in \mathcal{N}$), defined by

$$\begin{aligned} l_{z,\hat{\pi}} &:= \min\{\pi \in \mathcal{N}(z) : \hat{\pi} \leq \pi \text{ on } [t,\hat{\sigma}_{\hat{\pi}}]\}, \\ r_{z,\hat{\pi}} &:= \max\{\pi \in \mathcal{N}(z) : \pi \leq \hat{\pi} \text{ on } [t,\hat{\sigma}_{\hat{\pi}}]\}. \end{aligned} \tag{1.10}$$

The set

$$\mathcal{F}(l_{z,\hat{\pi}}) := \{s \in (t,\hat{\sigma}_{\hat{\pi}}) : \hat{\pi}(s) = l_{z,\hat{\pi}}(s),\ (\hat{\pi}(s),s) \text{ is of type } (\mathrm{p},\mathrm{pp})_{\mathrm{s}}\} \tag{1.11}$$

is a locally finite subset of $(t,\hat{\sigma}_{\hat{\pi}})$. Let $\mathcal{F}' := \mathcal{F}(l_{z,\hat{\pi}}) \cup \{t,\hat{\sigma}_{\hat{\pi}},\infty\}$ if $\hat{\sigma}_{\hat{\pi}}$ is a cluster point of $\mathcal{F}(l_{z,\hat{\pi}})$ and $\mathcal{F}' := \mathcal{F} \cup \{t,\infty\}$ otherwise. Then, for each $s,u \in \mathcal{F}'$ such that $s<u$ and $(s,u) \cap \mathcal{F}' = \emptyset$, there exists an $l \in \mathcal{W}^{\mathrm{l}}$ such that $l = l_{z,\hat{\pi}}$ on $[s,u]$. If $u < \hat{\sigma}_{\hat{\pi}}$, then $l \leq \hat{\pi}$ on $[u,\hat{\sigma}_{\hat{\pi}}]$ and $\inf\{s' > s : l(s') < \hat{\pi}(s')\} = u$. Analogous statements hold for $r_{z,\hat{\pi}}$.

We call the path $l_{z,\hat{\pi}}$ (resp. $r_{z,\hat{\pi}}$) in (1.10) the *reflected left-most* (resp. *right-most*) *path relative to* $\hat{\pi}$. See Figure 5 below for a picture of a reflected right-most path relative to a dual left-most path. Lemma 1.8 says that such reflected left-most (resp. right-most) paths are well-defined, and are concatenations of countably many left-most (resp. right-most) paths. Indeed, it can be shown that in a certain well-defined way, a reflected left-most path $l_{z,\hat{\pi}}$ always 'turns left' at separation points, except those in the countable set $\mathcal{F}(l_{z,\hat{\pi}})$, where turning left would make it cross $\hat{\pi}$. We call the set $\mathcal{F}(l_{z,\hat{\pi}})$ in (1.11) the set of *reflection times* of $l_{z,\hat{\pi}}$, and define $\mathcal{F}(r_{z,\hat{\pi}})$ analogously. In analogy with (1.10), we also define *reflected dual paths* $\hat{l}_{z,\pi}$ and $\hat{r}_{z,\pi}$ relative to a forward path $\pi \in \mathcal{N}$.

For a given point z in the plane, we now define special classes of reflected left-most and right-most paths, which extend the classes of paths in $\mathcal{W}^{\mathrm{l}}$ and $\mathcal{W}^{\mathrm{r}}$ entering and leaving z.

Definition 1.9. [Extended left-most and right-most paths]
For each $z=(x,t) \in \mathbb{R}^2$, we define

$$\mathcal{W}^{\mathrm{l}}_{\mathrm{in}}(z) := \{l \in \mathcal{W}^{\mathrm{l}} : l \text{ enters } z\} \quad \text{and} \quad \mathcal{W}^{\mathrm{l}}_{\mathrm{out}}(z) := \{l \in \mathcal{W}^{\mathrm{l}} : l \text{ leaves } z\}. \tag{1.12}$$

Similar definitions apply to $\mathcal{W}^{\mathrm{r}}, \hat{\mathcal{W}}^{\mathrm{l}}, \hat{\mathcal{W}}^{\mathrm{r}}, \mathcal{N}$, and $\hat{\mathcal{N}}$. We define

$$\mathcal{E}^{\mathrm{l}}_{\mathrm{in}}(z) := \{l_{z',\hat{r}} \in \mathcal{N}_{\mathrm{in}}(z) : \hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}{}_{\mathrm{out}}(z),\ z'=(x',t'),\ t'<t,\ \hat{r}(t') \leq x'\}, \tag{1.13}$$

and define $\mathcal{E}^{\mathrm{r}}_{\mathrm{in}}(z), \hat{\mathcal{E}}^{\mathrm{l}}{}_{\mathrm{in}}(z), \hat{\mathcal{E}}^{\mathrm{r}}{}_{\mathrm{in}}(z)$ analogously, by symmetry. Finally, we define

$$\mathcal{E}^{\mathrm{l}}_{\mathrm{out}}(z) := \{l_{z,\hat{r}} \in \mathcal{N}_{\mathrm{out}}(z) : \hat{r} \in \hat{\mathcal{E}}^{\mathrm{r}}{}_{\mathrm{in}}(z'),\ z'=(x',t),\ x' \leq x\}, \tag{1.14}$$

and we define $\mathcal{E}^{\mathrm{r}}_{\mathrm{out}}(z), \hat{\mathcal{E}}^{\mathrm{l}}{}_{\mathrm{out}}(z), \hat{\mathcal{E}}^{\mathrm{r}}{}_{\mathrm{out}}(z)$ analogously. We call the elements of $\mathcal{E}^{\mathrm{l}}_{\mathrm{in}}(z)$ and $\mathcal{E}^{\mathrm{l}}_{\mathrm{out}}(z)$ extended left-most paths *entering, resp. leaving z. See Figure 4 for an illustration.*

As we will see in Theorem 1.11 below, the extended left- and right-most paths we have just defined are exactly the 'missing' paths in groups (3)–(6) from Theorem 1.7. Finding the right definition of extended paths that serves this aim turned out to be rather subtle. Note that paths in $\mathscr{E}^{\rm l}_{\rm in}(z)$ and $\mathscr{E}^{\rm r}_{\rm in}(z)$ reflect off paths in $\hat{\mathscr{W}}^{\rm r}{}_{\rm out}(z)$ and $\hat{\mathscr{W}}^{\rm l}{}_{\rm out}(z)$, respectively, but paths in $\mathscr{E}^{\rm l}_{\rm out}(z)$ and $\mathscr{E}^{\rm r}_{\rm out}(z)$ could be reflected off reflected paths. If, instead of (1.14), we would have defined $\mathscr{E}^{\rm l}_{\rm out}(z)$ as the set of all reflected left-most paths that reflect off paths in $\hat{\mathscr{W}}^{\rm r}{}_{\rm in}(z)$ or $\hat{\mathscr{N}}_{\rm in}(z)$, then one can check that for points of type $({\rm o},{\rm lr})$, in the first case we would not have found all 'missing' paths, while in the second case we would obtain too many paths. Likewise, just calling any countable concatenation of left-most paths an extended left-most path would, in view of our aims, yield too many paths.

To formulate our results rigorously, we need one more definition.

Definition 1.10. [Strong equivalence of paths]
We say that two paths $\pi_1,\pi_2\in\Pi$ entering a point $z=(x,t)\in\mathbb{R}^2$ are strongly equivalent*, denoted by $\pi_1 =^z_{\rm in} \pi_2$, if $\pi_1=\pi_2$ on $[t-\varepsilon,t]$ for some $\varepsilon>0$. We say that two paths $\pi_1,\pi_2\in\Pi$ leaving z are* strongly equivalent*, denoted by $\pi_1 =^z_{\rm out} \pi_2$, if $\pi_1=\pi_2$ on $[t,t+\varepsilon]$ for some $\varepsilon>0$. We say that two classes $\mathscr{C},\tilde{\mathscr{C}}$ of paths entering (resp. leaving) z are equal* up to strong equivalence *if for each $\pi\in\mathscr{C}$ there exists a $\tilde\pi\in\tilde{\mathscr{C}}$ such that $\pi =^z_{\rm in} \tilde\pi$ (resp. $\pi =^z_{\rm out} \tilde\pi$), and vice versa.*

Note that by (1.9), a.s. for each $l_1,l_2\in\mathscr{W}^{\rm l}$ and $z\in\mathbb{R}^2$, $l_1 \sim^z_{\rm in} l_2$ implies $l_1 =^z_{\rm in} l_2$ and $l_1 \sim^z_{\rm out} l_2$ implies $l_1 =^z_{\rm out} l_2$. Part (a) of the next theorem shows that our extended left-most and right-most paths have the same property. Parts (b) and (c) say that extended left- and right-most always pair up, and are in fact the 'missing' paths from groups (3)–(6) of Theorem 1.7.

Theorem 1.11. [Extended left-most and right-most paths]
Almost surely, for each $z=(x,t)\in\mathbb{R}^2$:

(a) *If $l_1,l_2\in\mathscr{E}^{\rm l}_{\rm in}(z)$ satisfy $l_1 \sim^z_{\rm in} l_2$, then $l_1 =^z_{\rm in} l_2$. If $l_1,l_2\in\mathscr{E}^{\rm l}_{\rm out}(z)$ satisfy $l_1 \sim^z_{\rm out} l_2$, then $l_1 =^z_{\rm out} l_2$.*

(b) *If z is of type $(\cdot,{\rm p})/({\rm o},\cdot)$, $(\cdot,{\rm pp})/({\rm p},\cdot)$, or $(\cdot,{\rm ppp})/({\rm pp},\cdot)$ in $\mathscr{N}/\hat{\mathscr{N}}$, then, up to strong equivalence, one has $\mathscr{E}^{\rm l}_{\rm out}(z)=\mathscr{W}^{\rm l}_{\rm out}(z)$, $\mathscr{E}^{\rm r}_{\rm out}(z)=\mathscr{W}^{\rm r}_{\rm out}(z)$, $\hat{\mathscr{E}}^{\rm l}{}_{\rm in}(z)=\hat{\mathscr{W}}^{\rm l}{}_{\rm in}(z)$, and $\hat{\mathscr{E}}^{\rm r}{}_{\rm in}(z)=\hat{\mathscr{W}}^{\rm r}{}_{\rm in}(z)$.*

(c) *If z is of type $(\cdot,{\rm pp})/({\rm p},\cdot)$, $(\cdot,{\rm lp})/({\rm l},\cdot)$, $(\cdot,{\rm pr})/({\rm r},\cdot)$, or $(\cdot,{\rm lr})/({\rm n},\cdot)$ in $\mathscr{N}/\hat{\mathscr{N}}$, then up to strong equivalence, one has $\mathscr{E}^{\rm l}_{\rm out}(z)=\{l,l'\}$, $\mathscr{E}^{\rm r}_{\rm out}(z)=\{r,r'\}$, $\hat{\mathscr{E}}^{\rm l}{}_{\rm in}(z)=\{\hat l\}$, and $\hat{\mathscr{E}}^{\rm r}{}_{\rm in}(z)=\{\hat r\}$, where l is the left-most element of $\mathscr{W}^{\rm l}(z)$, r is the right-most element of $\mathscr{W}^{\rm r}(z)$, and $l',r',\hat l,\hat r$ are paths satisfying $l \sim^z_{\rm out} r'$, $\hat r \sim^z_{\rm in} \hat l$, and $l' \sim^z_{\rm out} r$. (See Figure 4.)*

Note that parts (b) and (c) cover all types of points from Theorem 1.7. Indeed, if we are only interested in the structure of $\mathscr{N}_{\rm out}(z)$ and $\hat{\mathscr{N}}_{\rm in}(z)$, then a.s. every point $z\in\mathbb{R}^2$ is of type $(\cdot,{\rm p})/({\rm o},\cdot)$, $(\cdot,{\rm pp})/({\rm p},\cdot)$, $(\cdot,{\rm ppp})/({\rm pp},\cdot)$, $(\cdot,{\rm lp})/({\rm l},\cdot)$, $(\cdot,{\rm pr})/({\rm r},\cdot)$, or $(\cdot,{\rm lr})/({\rm n},\cdot)$.

We finally turn our attention to the way general paths in $\mathscr{N}$ (and not just our extended paths) enter and leave the special points from Theorem 1.7. The next theorem shows that with the exception of outgoing paths at points of type $({\rm o},{\rm lr})$, all Brownian net paths must enter and leave points squeezed between a pair consisting of one extended left-most and one extended right-most path.

Theorem 1.12. [Structure of special points]
Almost surely, for each $z=(x,t)\in\mathbb{R}^2$:

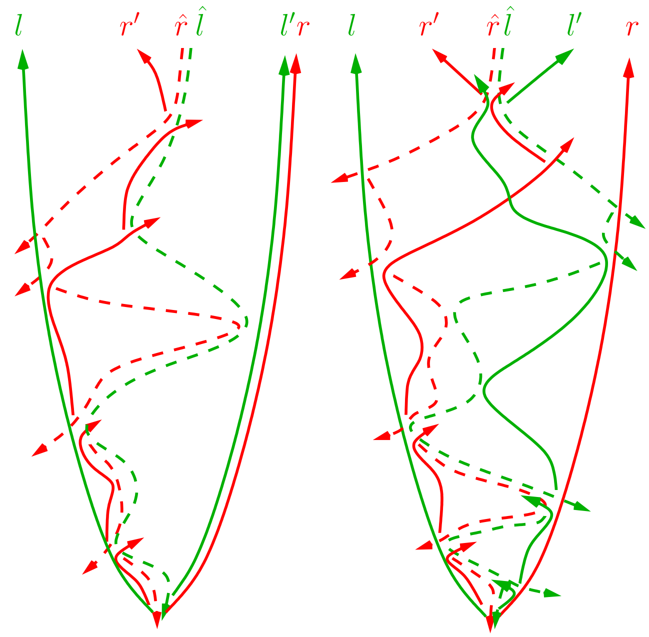


Figure 4: Local structure of outgoing extended left-most and right-most paths and incoming extended dual left-most and right-most paths in points of types $(\cdot, \mathrm{lp})$ and $(\mathrm{o}, \mathrm{lr})$. In the picture on the left, r' and $\hat{r}$ are reflected paths that are not elements of $\mathscr{W}^{\mathrm{r}}$ and $\hat{\mathscr{W}}^{\mathrm{r}}$, respectively. In the picture on the right, only l and r are 'true' left-most and right-most paths, while all other paths are 'missing' paths that are not visible in the schematic picture for points of type $(\mathrm{o}, \mathrm{lr})$ in Figure 3.

(a) *The relation $\sim^z_{\rm in}$ is an equivalence relation on $\mathcal{N}_{\rm in}(z)$. Each equivalence class $\mathscr{C}$ of paths in $\mathcal{N}_{\rm in}(z)$ contains an $l \in \mathcal{E}^{\rm l}_{\rm in}(z)$ and $r \in \mathcal{E}^{\rm r}_{\rm in}(z)$, which are unique up to strong equivalence, and each $\pi \in \mathscr{C}$ satisfies $l \leq \pi \leq r$ on $[t-\varepsilon, t]$ for some $\varepsilon > 0$.*

(b) *If z is not of type $(\rm o, lr)$, then the relation $\sim^z_{\rm out}$ is an equivalence relation on $\mathcal{N}_{\rm out}(z)$. Each equivalence class $\mathscr{C}$ of paths in $\mathcal{N}_{\rm out}(z)$ contains an $l \in \mathcal{E}^{\rm l}_{\rm out}(z)$ and $r \in \mathcal{E}^{\rm r}_{\rm out}(z)$, which are unique up to strong equivalence, and each $\pi \in \mathscr{C}$ satisfies $l \leq \pi \leq r$ on $[t, t+\varepsilon]$ for some $\varepsilon > 0$.*

(c) *If z is of type $(\rm o, lr)$, then there exist $\pi \in \mathcal{N}$ such that $l \sim^z_{\rm out} \pi \sim^z_{\rm out} r$ while $l \not\sim^z_{\rm out} r$, where l and r are the unique outgoing paths at z in $\mathscr{W}^{\rm l}$ and $\mathscr{W}^{\rm r}$, respectively.*

(d) *At points of types with the subscript $\rm l$ (resp. $\rm r$), all incoming paths in $\mathcal{N}$ continue in the left (resp. right) outgoing equivalence class. Except for this restriction, any concatenation of a path in $\mathcal{N}_{\rm in}(z)$ up to time t with a path in $\mathcal{N}_{\rm out}(z)$ after time t is again a path in $\mathcal{N}$.*

1.5 Outline and open problems

In this section, we outline the main structure of our proofs and mention some open problems.

In Section 2 we study separation points, i.e., points of type $(\rm p, pp)_s$ from Theorem 1.7. In a sense, these are the most important points in the Brownian net, since at these points paths in the Brownian net have a choice whether to turn left or right. Also, these are exactly the marked points in [NRS08], and their marking construction shows that the Brownian net is a.s. determined by its set of separation points and an embedded Brownian web.

In order to prepare for our study of separation points, in Section 2.1, we investigate the interaction between forward right-most and dual left-most paths. It turns out that the former are Skorohod reflected off the latter, albeit they may cross the latter from left to right at some random time. In Section 2.2, the results from Section 2.1 are used to prove that crossing points between forward right-most and dual left-most paths are separation points between left-most and right-most paths, and that these points are of type $(\rm p, pp)_s$.

In Section 2.3, we study 'relevant' separation points. By definition, a point $z = (x, t) \in \mathbb{R}^2$ is called an (s, u)-*relevant separation point* for some $-\infty \leq s < u \leq \infty$, if $s < t < u$, there exists a $\pi \in \mathcal{N}$ such that $\sigma_\pi = s$ and $\pi(t) = x$, and there exist incoming $l \in \mathscr{W}^{\rm l}$ and $r \in \mathscr{W}^{\rm r}$ at z such that $l < r$ on (t, u). At (s, u)-relevant separation points, a Brownian net path going from time s to time u has a choice whether to turn left or right that may be relevant for where it ends up at time u. The main result of Section 2.3 says that for deterministic $-\infty < s < t < \infty$, the set of (s, u)-relevant separation points is a.s. a locally finite subset of $\mathbb{R} \times (s, u)$. This fact has several useful consequences. As a first application, in Section 2.4, we prove Lemma 1.8.

In Section 3, we study the image set of the Brownian net started at a fixed time T. Let $\mathcal{N}_T := \{\pi \in \mathcal{N} : \sigma_\pi = T\}$ denote the set of paths in the Brownian net starting at a given time $T \in [-\infty, \infty]$, and let N_T be the *image set* of $\mathcal{N}_T$ in $R^2_{\rm c}$, i.e.,

$$N_T := \{(\pi(t), t) : \pi \in \mathcal{N}_T,\ t \in [T, \infty]\}. \tag{1.15}$$

By [SS08, Prop. 1.13], a.s. for each $T \in [-\infty, \infty]$,

$$\mathcal{N}_T = \{\pi \in \Pi : \sigma_\pi = T,\ \pi \subset N_T\}. \tag{1.16}$$

In view of (1.16), much can be learned about the Brownian net by studying the closed set N_T.

In Section 3.1, it is shown that the connected components of the complement of N_T relative to $\{(x,t) \in R^2_{\rm c} : t \geq T\}$ are meshes of a special sort, called *maximal T-meshes*. In Section 3.2, it is shown that N_T has a local reversibility property that allows one, for example, to deduce properties of meeting points from properties of separation points. Using these facts, in Section 3.3, we give a preliminary classification of points in $\mathbb{R}^2$ based only on the structure of incoming paths in $\mathscr{N}$. In Section 3.4, we use the fractal structure of N_T to prove the existence, announced in [SS08], of random times $t > T$ when $\{x \in \mathbb{R} : (x,t) \in N_T\}$ is not locally finite.

It turns out that to determine the type of a point $z \in \mathbb{R}^2$ in $\mathscr{N}$, according to the classification of Theorem 1.7, except for one trivial ambiguity, it suffices to know the structure of the incoming paths at z in both $\mathscr{N}$ and $\hat{\mathscr{N}}$ according to the preliminary classification from Section 3.3. Therefore, in order to prove Theorem 1.7, we need to know which combinations of types in $\mathscr{N}$ and $\hat{\mathscr{N}}$ are possible according to the latter classification. In particular, proving the existence of points in groups (4) and (5) from Theorem 1.7 depends on showing that there are points where the incoming paths in both $\mathscr{N}$ and $\hat{\mathscr{N}}$ form a nested sequence of excursions. Section 4 contains some excursion theoretic arguments that prepare for this. In particular, in Section 4.1, we prove that there are many excursions between a given left-most path l and dual left-most path $\hat{l}$ on the left of l that are entered by some dual right-most path. In Section 4.2, we prove that there are many points where $\hat{l}$ hits l while at the same time some right-most path makes an excursion away from l.

In Section 5, we finally prove our main results. Section 5.1 contains the proof of Theorem 1.7, while Section 5.2 contains the proofs of Theorems 1.11 and 1.12.

We conclude the paper with two appendices (Appendices A.1 and A.2) containing some facts and proofs that are not used in the main argument but may be of independent interest.

Our investigations leave open a few questions that we believe are important for understanding the full structure of the Brownian net, and which we hope to settle in future work. The first question we would like to mention concerns the image set in (1.15). Fix $-\infty \leq s < u \leq \infty$, and say that a point $z \in \mathbb{R}^2$ is *n-connected* to s and u if $z \in \{(x,t) \in N_s : s < t < u\}$ and one needs to remove at least n points from N_s to disconnect z from $\{(x,t) \in N_s : t = s \text{ or } t = u\}$. Note that the notion of connectedness in N_s is graph theoretic and does not respect the time direction inherent in the Brownian net. Here is a conjecture:

Conjecture 1.13. [3-connected points]
Almost surely for each $-\infty \leq s < u \leq \infty$, the set of 3-connected points to s and u is a locally finite subset of $\mathbb{R} \times (s,u)$, and all 3-connected points are either meeting or separation points.

It is easy to see that all points in $\{(x,t) \in N_s : s < t < u\}$ are 2-connected to s and u. The results in our present paper imply that meeting and separation points are not 4-connected. All (s,u)-relevant separation points are 3-connected, but not all 3-connected separation points are (s,u)-relevant. If Conjecture 1.13 is correct, then the structure of all Brownian net paths going from time s to u can be described by a purely 3-connected, locally finite graph, whose vertices are the 3-connected points.

Other open problems concern the reflected left-most and right-most paths described in Theorem 1.11. Here is another conjecture:

Conjecture 1.14. [Reflection points]
Almost surely for all $z = (x,t) \in \mathbb{R}^2$, if $l, l', \hat{l}, r, r', \hat{r}$ are as in Theorem 1.11 (c), then there exists an $\varepsilon > 0$ such that $\hat{r}(u) = \hat{l}(u)$ for all $u \in \mathscr{F}(\hat{r}) \cap [t, t+\varepsilon]$.

Conjecture 1.14 says that near t, all reflection points of $\hat{r}$ off l lie on $\hat{l}$ (and hence, by symmetry, a similar statement holds for the reflection points of $\hat{l}$ off r). If this is true, then the picture in Figure 4 simplifies a lot. In particular, there exists an $\varepsilon > 0$ such that on $[t, t+\varepsilon]$, the paths are eventually ordered as $l \le r' \le \hat{r} \le \hat{l} \le l' \le r$. Note from Figure 4 that at present, for points of type (o, lr), we cannot even rule out that $l'(u_n) < r'(u_n)$ for a sequence of times $u_n \downarrow t$.

It seems that Conjectures 1.13 and 1.14 cannot be proved with the methods developed in the present paper and [SS08]. Instead, we hope to tackle these problems by calling in the marking construction of the Brownian net developed in [NRS08].

Another open problem is to determine the Hausdorff dimension of the sets of points of each of the types from Theorem 1.7. For the Brownian web, the Hausdorff dimensions of all types of points are known, see [FINR06, Theorem 3.12]. We believe that points from group (1) of Theorem 1.7 have the same Hausdorff dimension as the corresponding points in the Brownian web. Separation points (group (2)) are countable. About the Hausdorff dimensions of points from groups (3)–(6), we know nothing.

1.6 List of global notations

For ease of reference, we collect here some notations that will be used throughout the rest of the paper. In the proofs, some new notation might be derived from the global notations listed here, such as by adding superscripts or subscripts, in which case the objects they encode will be closely related to what the corresponding global notation stands for.

General notation:

$\mathscr{L}(\cdot)$:	law of a random variable.
$\mathscr{D}$:	a deterministic countable dense subset of $\mathbb{R}^2$.
S_ε :	the diffusive scaling map, applied to subsets of $\mathbb{R}^2$, paths, and sets of paths, defined as $S_\varepsilon(x,t) := (\varepsilon x, \varepsilon^2 t)$ for $(x,t) \in \mathbb{R}^2$.
τ :	a stopping time; in Section 4: a time of increase of the reflection process (see (4.84)).
$\tau_{\pi,\pi'}$:	the first meeting time of π and π', defined in (1.6).
(L_t, R_t) :	a pair of diffusions solving the left-right SDE (1.4).
$\sim^z_{\rm in}, \sim^z_{\rm out}$:	equivalence of paths entering, resp. leaving $z \in \mathbb{R}^2$, see Definition 1.5.
$=^z_{\rm in}, =^z_{\rm out}$:	strong equivalence of paths entering, resp. leaving $z \in \mathbb{R}^2$, see Definition 1.10.

Paths, Space of Paths:

(R^2_c,ρ) : the compactification of $\mathbb{R}^2$ with the metric ρ, see (1.1).
$z=(x,t)$: point in R^2_c, with position x and time t.
s,t,u,S,T,U : times.
(Π,d) : the space of continuous paths in (R^2_c,ρ) with metric d, see (1.2).
$\Pi(A),\Pi(z)$: the set of paths in Π starting from a set $A\subset R^2_c$ resp. a point $z\in R^2_c$. The same notatation applies to any subset of Π such as $\mathscr{W},\mathscr{W}^{\mathrm{l}},\mathscr{N}$.
π : a path in Π.
σ_π : the starting time of the path π
$\pi(t)$: the position of π at time $t\geq\sigma_\pi$.
$(\hat\Pi,\hat d)$: the space of continuous backward paths in (R^2_c,ρ) with metric $\hat d$, see Theorem 1.2.
$\hat\Pi(A),\hat\Pi(z)$: the set of backward paths in $\hat\Pi$ starting from $A\subset R^2_c$ resp. $z\in R^2_c$.
$\hat\pi$: a path in $\hat\Pi$.
$\hat\sigma_{\hat\pi}$: the starting time of the backward path $\hat\pi$.
$\mathscr{I}(\pi,\hat\pi)$: the set of intersection times of π and $\hat\pi$, see (4.80).
$(\mathscr{H},d_{\mathscr{H}})$: the space of compact subsets of (Π,d) with Hausdorff metric $d_{\mathscr{H}}$, see (1.3).

Brownian webs:

$(\mathscr{W},\hat{\mathscr{W}})$: a double Brownian web consisting of a Brownian web and its dual.
$\pi_z,\hat\pi_z$: the a.s. unique paths in $\mathscr{W}$ resp. $\hat{\mathscr{W}}$ starting from a deterministic point $z\in\mathbb{R}^2$.
$(\mathscr{W}^{\mathrm{l}},\mathscr{W}^{\mathrm{r}})$: the left-right Brownian web, see Theorem 1.3.
$\mathscr{W}_{\mathrm{in}}(z),\mathscr{W}_{\mathrm{out}}(z)$: the set of paths in $\mathscr{W}$ entering, resp. leaving z.
$\mathscr{E}^{\mathrm{l}}_{\mathrm{in}}(z),\mathscr{E}^{\mathrm{l}}_{\mathrm{out}}(z)$: the sets of extended left-most paths entering, resp. leaving z, see Definition 1.9.
$l,r,\hat l,\hat r$: a path in $\mathscr{W}^{\mathrm{l}}$, resp. $\mathscr{W}^{\mathrm{r}}$, $\hat{\mathscr{W}}^{\mathrm{l}}$, $\hat{\mathscr{W}}^{\mathrm{r}}$, often called a left-most, resp. right-most, dual left-most, dual right-most path.
$l_z,r_z,\hat l_z,\hat r_z$: the a.s. unique paths $\mathscr{W}^{\mathrm{l}}(z)$, resp. $\mathscr{W}^{\mathrm{r}}(z)$, $\hat{\mathscr{W}}^{\mathrm{l}}(z)$, $\hat{\mathscr{W}}^{\mathrm{r}}(z)$ starting from a deterministic point $z\in\mathbb{R}^2$.
$l_{z,\hat\pi},r_{z,\hat\pi}$: reflected left-most and right-most paths starting from z, and reflected off $\hat\pi\in\hat{\mathscr{N}}$, see (1.10).
$W(\hat r,\hat l)$: a wedge defined by the paths $\hat r\in\hat{\mathscr{W}}^{\mathrm{r}}$ and $\hat l\in\hat{\mathscr{W}}^{\mathrm{l}}$, see (1.7).
$M(r,l)$: a mesh defined by the paths $r\in\mathscr{W}^{\mathrm{r}}$ and $l\in\mathscr{W}^{\mathrm{l}}$, see (1.8).

Brownian net:

$(\mathscr{N},\hat{\mathscr{N}})$: the Brownian net and the dual Brownian net, see Theorem 1.4.
$\mathscr{N}_T$: the set of paths in $\mathscr{N}$ starting at time $T\in[-\infty,\infty]$.
N_T : the image set of $\mathscr{N}_T$ in $\mathbb{R}^2$, see (1.15).
ξ^K_t : $\{\pi(t):\pi\in\mathscr{N}(K),\ \sigma_\pi\leq t\}$, the set of positions through which there passes a path in the Brownian net starting from the set $K\subset R^2_c$.
$\xi^{(T)}_t$: $\{\pi(t):\pi\in\mathscr{N},\ \sigma_\pi=T\leq t\}$, the same as ξ^K_t, but now for Brownian net paths starting from a given time $T\in[-\infty,\infty]$.
$\hat\xi^K_t$, $\hat\xi^{(T)}_t$: the same as ξ^K_t, $\xi^{(T)}_t$, but defined in terms of the dual Brownian net $\hat{\mathscr{N}}$.
*-mesh : see Definition 3.2

2 Separation points

2.1 Interaction between forward and dual paths

We know from [STW00] that paths in $\mathcal{W}^{\mathrm{l}}$ and $\hat{\mathcal{W}}^{\mathrm{l}}$, resp. $\mathcal{W}^{\mathrm{r}}$ and $\hat{\mathcal{W}}^{\mathrm{r}}$, interact by Skorohod reflection. More precisely, conditioned on $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}$ with deterministic starting point $\hat{z} = (\hat{x}, \hat{t})$, the path $l \in \mathcal{W}^{\mathrm{l}}$ with deterministic starting point $z = (x, t)$, where $t < \hat{t}$, is distributed as a Brownian motion with drift -1 Skorohod reflected off $\hat{l}$. It turns out that conditioned on $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}$, the interaction of $r \in \mathcal{W}^{\mathrm{r}}$ starting at z with $\hat{l}$ is also Skorohod reflection, albeit r may cross $\hat{l}$ at a random time.

Lemma 2.1. [Interaction between dual and forward paths]
Let $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}$, resp. $r \in \mathcal{W}^{\mathrm{r}}$, start from deterministic points $\hat{z} = (\hat{x}, \hat{t})$, resp. $z = (x, t)$, with $t < \hat{t}$. Then almost surely,

(a) *Conditioned on $(\hat{l}(s))_{t \le s \le \hat{t}}$ with $\hat{l}(t) < x$, the path r is given (in distribution) by the unique solution to the Skorohod equation*

$$\begin{aligned} \mathrm{d}r(s) &= \mathrm{d}B(s) + \mathrm{d}s + \mathrm{d}\Delta(s), && t \le s \le \hat{t}, \\ \mathrm{d}r(s) &= \mathrm{d}B(s) + \mathrm{d}s, && \hat{t} \le s, \end{aligned} \tag{2.17}$$

where B is a standard Brownian motion, Δ is a nondecreasing process increasing only when $r(s) = \hat{l}(s)$ (i.e, $\int_t^{\hat{t}} 1_{\{r(s) \neq \hat{l}(s)\}} \mathrm{d}\Delta(s) = 0$), and r is subject to the constraint that $\hat{l}(s) \le r(s)$ for all $t \le s \le \hat{t}$.

(b) *Conditioned on $(\hat{l}(s))_{t \le s \le \hat{t}}$ with $x < \hat{l}(t)$, the path r is given (in distribution) by the unique solution to the Skorohod equation*

$$\begin{aligned} \mathrm{d}r(s) &= \mathrm{d}B(s) + \mathrm{d}s - \mathrm{d}\Delta(s), && t \le s \le \hat{t},\ \Delta(s) < T, \\ \mathrm{d}r(s) &= \mathrm{d}B(s) + \mathrm{d}s + \mathrm{d}\Delta(s), && t \le s \le \hat{t},\ T \le \Delta(s), \\ \mathrm{d}r(s) &= \mathrm{d}B(s) + \mathrm{d}s, && \hat{t} \le s, \end{aligned} \tag{2.18}$$

where B is a standard Brownian motion, Δ is a nondecreasing process increasing only when $r(s) = \hat{l}(s)$, T is an independent mean $1/2$ exponential random variable, and r is subject to the constraints that $r(s) \le \hat{l}(s)$ resp. $\hat{l}(s) \le r(s)$ for all $s \in [t, \hat{t}]$ such that $\Delta(s) < T$ resp. $T \le \Delta(s)$.

The interaction between paths in $\mathcal{W}^{\mathrm{l}}$ and $\hat{\mathcal{W}}^{\mathrm{r}}$ is similar by symmetry. If $z = (\hat{l}(t), t)$, where t is a deterministic time, then exactly two paths $r_1, r_2 \in \mathcal{W}^{\mathrm{r}}$ start from z, where one solves (2.17) and the other solves (2.18). Conditional on $\hat{l}$, the paths r_1 and r_2 evolve independently up to the first time they meet, at which they coalesce.

Remark Lemma 2.1 gives an almost sure construction of a pair of paths $(\hat{l}, r)$ starting from deterministic points $\hat{z}$ and z. By the same argument as in [STW00], we can extend Lemma 2.1 to give an almost sure construction of a finite collection of paths in $(\hat{\mathcal{W}}^{\mathrm{l}}, \mathcal{W}^{\mathrm{r}})$ with deterministic starting points, and the order in which the paths are constructed is irrelevant. (The technical issues involved in consistently defining multiple coalescing-reflecting paths in our case are the same as those in [STW00].) Therefore, by Kolmogorov's extension theorem, Lemma 2.1 may be used to construct $(\hat{\mathcal{W}}^{\mathrm{r}}, \mathcal{W}^{\mathrm{l}})$ restricted to a countable dense set of starting points in $\mathbb{R}^2$. Taking closures and duals, this gives an alternative construction of the double left-right Brownian web $(\mathcal{W}^{\mathrm{l}}, \mathcal{W}^{\mathrm{r}}, \hat{\mathcal{W}}^{\mathrm{l}}, \hat{\mathcal{W}}^{\mathrm{r}})$, and hence of the Brownian net.

Proof of Lemma 2.1 We will prove the following claim. Let $\hat{z} = (\hat{x}, \hat{t})$ and $z = (x, t)$ be deterministic points in $\mathbb{R}^2$ with $t < \hat{t}$ and let $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}$ and $r \in \mathcal{W}^{\mathrm{r}}$ be the a.s. unique dual left-most and forward right-most paths starting from $\hat{z}$ and z, respectively. We will show that it is possible to construct a standard Brownian motion B and a mean $1/2$ exponential random variable T such that $\hat{l}, B$, and T are independent, and such that r is the a.s. unique solution to the equation

$$r(s) = W(s) + \int_t^s (1_{\{\tau \le u\}} - 1_{\{u < \tau\}})\, \mathrm{d}\Delta(u), \qquad s \ge t, \tag{2.19}$$

where $W(s) := x + B(s-t) + (s-t)$, $s \ge t$, is a Brownian motion with drift 1 started at time (x, t), Δ is a nondecreasing process increasing only at times $s \in [t, \hat{t}]$ when $r(s) = \hat{l}(s)$,

$$\tau := \begin{cases} \inf\{s \in [t, \hat{t}] : \Delta(s) \ge T\} & \text{if } x < \hat{l}(t), \\ t & \text{if } \hat{l}(t) \le x, \end{cases} \tag{2.20}$$

and r is subject to the constraints that $r(s) \le \hat{l}(s)$ resp. $\hat{l}(s) \le r(s)$ for all $s \in [t, \hat{t}]$ such that $s < \tau$ resp. $\tau \le s$. Note that (2.19) is a statement about the joint law of $(\hat{l}, r)$, which implies the statements about the conditional law of r given $\hat{l}$ in Lemma 2.1.

Since $\mathbb{P}[\hat{l}(t) = x] = 0$, to show that (2.19)–(2.20) has an a.s. unique solution, we may distinguish the cases $\hat{l}(t) < x$ and $x < \hat{l}(t)$. If $\hat{l}(t) < x$, then (2.19) is a usual Skorohod equation with reflection (see Section 3.6.C of [KS91]) which is known to have the unique solution $r(s) = W(s) + \Delta(s)$, where

$$\Delta(s) = \sup_{t \le u \le s \wedge \hat{t}} \big(\hat{l}(u) - W(u)\big) \vee 0 \qquad (s \ge t). \tag{2.21}$$

If $x < \hat{l}(t)$, then by the same arguments $r(s) = W(s) - \Delta(s)$ $(t \le s \le \tau)$, where

$$\Delta(s) = \sup_{t \le u \le s \wedge \hat{t}} \big(W(u) - \hat{l}(u)\big) \vee 0 \qquad (t \le s \le \tau), \tag{2.22}$$

and

$$\tau := \inf\big\{s \ge t : \sup_{t \le u \le s \wedge \hat{t}} \big(W(u) - \hat{l}(u)\big) \vee 0 = T\big\}, \tag{2.23}$$

which may be infinite. Note that for a.e. path $\hat{l}$, the time τ is a stopping time for W. If $\tau < \infty$, then for $s \ge \tau$ our equation is again a Skorohod equation, with reflection in the other direction, so $r(s) = W^\tau(s) + \Delta^\tau(s)$ where $W^\tau(s) := r(\tau) + W(s) - W(\tau)$ $(s \ge \tau)$, and

$$\Delta^\tau(s) = \sup_{\tau \le u \le s \wedge \hat{t}} \big(\hat{l}(u) - W^\tau(u)\big) \vee 0 \qquad (s \ge \tau). \tag{2.24}$$

To prove that there exist W and T such that r solves (2.19), we follow the approach in [STW00] and use discrete approximation. First, we recall from [SS08] the discrete system of branching-coalescing random walks on $\mathbb{Z}^2_{\mathrm{even}} = \{(x, t) : x, t \in \mathbb{Z}, x + t \text{ is even}\}$ starting from every site of $\mathbb{Z}^2_{\mathrm{even}}$. Here, for $(x, t) \in \mathbb{Z}^2_{\mathrm{even}}$, the walker that is at time t in x jumps at time $t+1$ with probability $\frac{1-\epsilon}{2}$ to $x - 1$, with the same probability to $x + 1$, and with the remaining probability ϵ branches in two walkers situated at $x - 1$ and $x + 1$. Random walks that land on the same position coalesce immediately. Following [SS08], let $\mathscr{U}_\epsilon$ denote the set of branching-coalescing random walk paths on $\mathbb{Z}^2_{\text{even}}$ (linearly interpolated between integer times), and let $\mathscr{U}^{\text{l}}_\epsilon$, resp. $\mathscr{U}^{\text{r}}_\epsilon$, denote the set of left-most, resp. right-most, paths in $\mathscr{U}_\epsilon$ starting from each $z \in \mathbb{Z}^2_{\text{even}}$. There exists a natural dual system of branching-coalescing random walks on $\mathbb{Z}^2_{\text{odd}} = \mathbb{Z}^2 \backslash \mathbb{Z}^2_{\text{even}}$ running backward in time, where $(x,t) \in \mathbb{Z}^2_{\text{even}}$ is a branching point in the forward system if and only if $(x, t+1)$ is a branching point in the backward system, and otherwise the random walk jumping from (x,t) in the forward system and the random walk jumping from $(x,t+1)$ in the backward system are coupled so that they do not cross. Denote the dual collection of branching-coalescing random walk paths by $\hat{\mathscr{U}}_\varepsilon$, and let $\hat{\mathscr{U}}^{\text{l}}_\varepsilon$, resp. $\hat{\mathscr{U}}^{\text{r}}_\varepsilon$, denote the dual set of left-most, resp. right-most, paths in $\hat{\mathscr{U}}_\varepsilon$. Let $S_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ be the diffusive scaling map $S_\varepsilon(x,t) = (\varepsilon x, \varepsilon^2 t)$, and define S_ε applied to a subset of $\mathbb{R}^2$, a path, or a set of paths analogously.

Let ε_n be a sequence satisfying $\varepsilon_n \downarrow 0$. Choose $z_n = (x_n, t_n) \in \mathbb{Z}^2_{\text{even}}$ and $\hat{z}_n = (\hat{x}_n, \hat{t}_n) \in \mathbb{Z}^2_{\text{odd}}$ such that $S_{\epsilon_n}(z_n, \hat{z}_n) \to (z, \hat{z})$ as $n \to \infty$. If we denote by $\hat{l}_n$ the unique path in $\hat{\mathscr{U}}^{\text{l}}_{\varepsilon_n}(\hat{z}_n)$ and by r_n the unique path in $\mathscr{U}^{\text{r}}_{\varepsilon_n}(z_n)$, then by Theorem 5.2 of [SS08], we have

$$\mathscr{L}\left(S_{\varepsilon_n}(\hat{l}_n, r_n)\right) \underset{\varepsilon_n \to 0}{\Longrightarrow} \mathscr{L}(\hat{l}, r), \tag{2.25}$$

where $\mathscr{L}$ denotes law and $\Rightarrow$ denotes weak convergence.

The conditional law of r_n given $\hat{l}_n$ has the following description. Let

$$I_n := \{(y,s) \in \mathbb{Z}^2_{\text{even}} : t \le s < \hat{t},\ \hat{l}(s+1) = y\} \tag{2.26}$$

be the set of points where a forward path can meet $\hat{l}$. Let $I^{\text{l}}_n := \{(y,s) \in I_n : \hat{l}(s+1) < \hat{l}(s)\}$ and $I^{\text{r}}_n := \{(y,s) \in I_n : \hat{l}(s) < \hat{l}(s+1)\}$ be the sets of points where a forward path can meet $\hat{l}$ from the left and right, respectively. Conditional on $\hat{l}_n$, the process $(r_n(s))_{s \ge t_n}$ is a Markov process such that if $(r_n(s), s) \notin I_n$, then $r_n(s+1) = r_n(s)+1$ with probability $(1+\varepsilon_n)/2$, and $r_n(s+1) = r_n(s)-1$ with the remaining probability. If $(r_n(s), s) \in I^{\text{l}}_n$, then $r_n(s+1) = r_n(s)+1$ with probability $2\varepsilon_n/(1+\varepsilon_n)$ and $r_n(s+1) = r_n(s)-1$ with the remaining probability. Here $2\varepsilon_n/(1+\varepsilon_n)$ is the conditional probability that $(r_n(s), s)$ is a branching point of the forward random walks given that $\hat{l}_n(s+1) < \hat{l}_n(s)$. Finally, if $(r_n(s), s) \in I^{\text{r}}_n$, then $r_n(s+1) = r_n(s)+1$ with probability 1.

In view of this, we can construct $\hat{r}_n$ as follows. Independently of $\hat{l}_n$, we choose a random walk $(W_n(s))_{s \ge t_n}$ starting at (x_n, t_n) that at integer times jumps from y to $y-1$ with probability $(1-\varepsilon_n)/2$ and to $y+1$ with probability $(1+\varepsilon_n)/2$. Moreover, we introduce i.i.d. Bernoulli random variables $(X_n(i))_{i \ge 0}$ with $\mathbb{P}[X_n(i) = 1] = 4\varepsilon_n/(1+\varepsilon_n)^2$. We then inductively construct processes Δ_n and r_n, starting at time s in $\Delta_n(s) = 0$ and $r_n(s) = x_n$, by putting

$$\Delta_n(s+1) := \begin{cases} \Delta_n(s)+1 & \text{if } (r_n(s), s) \in I^{\text{l}}_n,\ W_n(s+1) > W_n(s), \text{ and } X_{\Delta_n(s)} = 0, \\ \Delta_n(s)+1 & \text{if } (r_n(s), s) \in I^{\text{r}}_n \text{ and } W_n(s+1) < W_n(s), \\ \Delta_n(s) & \text{otherwise,} \end{cases} \tag{2.27}$$

and

$$r_n(s+1) := \begin{cases} r_n(s) + (W_n(s+1) - W_n(s)) & \text{if } \Delta_n(s+1) = \Delta_n(s), \\ r_n(s) - (W_n(s+1) - W_n(s)) & \text{if } \Delta_n(s+1) > \Delta_n(s). \end{cases} \tag{2.28}$$

This says that r_n evolves as W_n, but is reflected off $\hat{l}_n$ with probability $1 - 4\varepsilon_n/(1+\varepsilon_n)^2$ if it attempts to cross from left to right, and with probability 1 if it attempts to cross from right to left. Note that if $(r_n(s),s)\in I_n^1$, then r_n attempts to cross with probability $(1+\varepsilon_n)/2$, hence the probability that it crosses is $4\varepsilon_n/(1+\varepsilon_n)^2\cdot(1+\varepsilon_n)/2=2\varepsilon_n/(1+\varepsilon_n)$, as required.

Extend $W_n(s),\Delta_n(s)$, and $r_n(s)$ to all real $s\geq t$ by linear interpolation, and set $T_n:=\inf\{i\geq 0:X_i=1\}$. Then

$$r_n(s)=W_n(s)+2\int_t^s(1_{\{\tau_n\leq u\}}-1_{\{u<\tau_n\}})\,\mathrm{d}\Delta_n(u)\qquad(s\geq t),\tag{2.29}$$

where

$$\tau_n:=\begin{cases}\inf\{s\in[t_n,\hat t_n]:\Delta_n(s)\geq T_n\} & \text{if } x_n<\hat l_n(t),\\ t & \text{if } \hat l_n(t)<x_n.\end{cases}\tag{2.30}$$

The process r_n satisfies $r_n(s)\leq\hat l_n(s)-1$ resp. $\hat l_n(s)+1\leq r_n(s)$ for all $s\in[t_n,\hat t_n]$ such that $s\leq\tau_n$ resp. $\tau_n+1\leq s$, and Δ_n increases only for $s\in[t_n,\hat t_n]$ such that $|r_n(s)-\hat l_n(s)|=1$.

By a slight abuse of notation, set $T_{\varepsilon_n}:=\varepsilon_n T_n$, $\tau_{\varepsilon_n}:=\varepsilon_n^2\tau_n$, and let $\hat l_{\varepsilon_n},r_{\varepsilon_n},W_{\varepsilon_n}$, and Δ_{ε_n} be the counterparts of l_n,r_n,W_n, and Δ_n, diffusively rescaled with S_{ε_n}. Then

$$\mathscr{L}(\hat l_{\varepsilon_n},r_{\varepsilon_n})\underset{n\to\infty}{\Longrightarrow}\mathscr{L}(\hat l,r)\quad\text{and}\quad\mathscr{L}(\hat l_{\varepsilon_n},W_{\varepsilon_n},2T_{\varepsilon_n})\underset{n\to\infty}{\Longrightarrow}\mathscr{L}(\hat l,W,T),\tag{2.31}$$

where $\hat l,r$ are the dual and forward path we are interested in, W is a Brownian motion with drift 1, started at (x_n,t_n), T is an exponentially distributed random variable with mean $1/2$, and $\hat l,W,T$ are independent.

It follows from the convergence in (2.31) that the laws of the processes $\tilde\Delta_{\varepsilon_n}(s):=\frac{1}{2}(r_n(s)-W_n(s))$ are tight. By (2.29)–(2.30), for n large enough, one has $\tilde\Delta_{\varepsilon_n}=\Delta_{\varepsilon_n}$ on the event that $\hat l(t)<x$, while on the complementary event, $\tilde\Delta_{\varepsilon_n}$ is $-\Delta_{\varepsilon_n}$ reflected at the level $-T_{\varepsilon_n}$. Using this, it is not hard to see that the processes Δ_{ε_n} are tight. Therefore, going to a subsequence if necessary, by Skorohod's representation theorem (see e.g. Theorem 6.7 in [Bi99]), we can find a coupling such that

$$(\hat l_{\varepsilon_n},r_{\varepsilon_n},W_{\varepsilon_n},2\Delta_{\varepsilon_n},2T_{\varepsilon_n})\underset{n\to\infty}{\longrightarrow}(\hat l,r,W,\Delta,T)\quad\text{a.s.},\tag{2.32}$$

where the paths converge locally uniformly on compacta. Since the Δ_{ε_n} are nondecreasing and independent of T_n, and T has a continuous distribution, one has $\tau_n\to\tau$ a.s. Taking the limit in (2.29)–(2.30), it is not hard to see that r solves the equations (2.19)–(2.20).

If z is of the form $z=(\hat l(t),t)$ for some deterministic t, then the proof is similar, except that now we consider two approximating sequences of right-most paths, one started at $(\hat l_n(t_n)-1,t_n)$ and the other at $(\hat l_n(t_n)+1,t_n)$. ∎

The next lemma is very similar to Lemma 2.1 (see Figure 5).

Lemma 2.2. [Sequence of paths crossing a dual path]
Let $\hat z=(\hat x,\hat t)\in\mathbb{R}^2$ and $t<\hat t$ be a deterministic point and time. Let $\hat B,B_i$ $(i\geq 0)$ be independent, standard Brownian motions, and let $(T_i)_{i\geq 0}$ be independent mean $1/2$ exponential random variables. Set $\hat l(\hat t-s):=\hat x+\hat B_s+s$ $(s\geq 0)$. Set $\tau_0:=t$ and define inductively paths $(r_i(s))_{s\geq\tau_i}$ $(i=0,\ldots,M)$ starting at $r_i(\tau_i)=\hat l(\tau_i)$ by the unique solutions to the Skorohod equation

$$\begin{aligned}\mathrm{d}r_i(s)&=\mathrm{d}\tilde B_i(s)+\mathrm{d}s-\mathrm{d}\Delta_i(s), && \tau_i\leq s\leq\tau_{i+1},\\ \mathrm{d}r_i(s)&=\mathrm{d}\tilde B_i(s)+\mathrm{d}s+\mathrm{d}\Delta_i(s), && \tau_{i+1}\leq s\leq\hat t,\\ \mathrm{d}r_i(s)&=\mathrm{d}\tilde B_i(s)+\mathrm{d}s, && \hat t\leq s,\end{aligned}\tag{2.33}$$

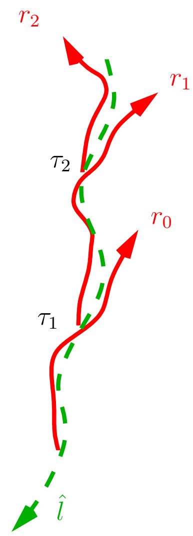


Figure 5: A sequence of paths in $\mathscr{W}^{\mathrm{r}}$ crossing $\hat{l} \in \hat{\mathscr{W}}^{\mathrm{l}}$.

where Δ_i is a nondecreasing process increasing only when $r_i(s) = \hat{l}(s)$, the process r_i is subject to the constraints that $r_i(s) \leq \hat{l}(s)$ on $[\tau_i, \tau_{i+1}]$ and $\hat{l}(s) \leq r_i(s)$ on $[\tau_{i+1}, \hat{t}]$, and $\tau_{i+1} := \inf\{s \in (\tau_i, \hat{t}) : \Delta_i(s) > T_i\}$ $(i = 0, \dots, M-1)$. The induction terminates at $M := \inf\{i \geq 0 : \Delta_i(\hat{t}) \leq T_i\}$, and we set $\tau_{M+1} := \hat{t}$. The Brownian motions $\tilde{B}_i$ in (2.33) are inductively defined as $\tilde{B}_0 := B_0$, and, for $i = 1, \dots, M$,

$$\tilde{B}_i(s) := \begin{cases} B_i(s) & \tau_i \leq s \leq \sigma_i, \\ \tilde{B}_{i-1}(s) & \sigma_i \leq s, \end{cases} \tag{2.34}$$

where $\sigma_i := \inf\{s \geq \tau_i : r_i(s) = r_{i-1}(s)\}$.

Then $M < \infty$ a.s. and we can couple $\{\hat{l}, r_0, \dots, r_M\}$ to a left-right Brownian web and its dual in such a way that $\hat{l} \in \hat{\mathscr{W}}^{\mathrm{l}}$ and $r_i \in \mathscr{W}^{\mathrm{r}}$ for $i = 0, \dots, M$.

Proof This follows by discrete approximation in the same way as in the proof of Lemma 2.1. Note that (2.34) ensures that $r_0, \dots, r_M$ coalesce when they meet. To see that $M < \infty$ a.s., define $(r'(s))_{s \geq t}$ by $r' := r_i$ on $[\tau_i, \tau_{i+1}]$ and $r' := r_M$ on $[\hat{t}, \infty)$, and set $B(s) := \int_t^s \mathrm{d}B(u)$ and $\Delta(s) := \int_t^s \mathrm{d}\Delta(u)$ where $\mathrm{d}B := \mathrm{d}\tilde{B}_i$ and $\mathrm{d}\Delta := \mathrm{d}\Delta_i$ on $[\tau_i, \tau_{i+1}]$ and $\mathrm{d}B := \mathrm{d}\tilde{B}_M$ on $[\hat{t}, \infty)$. Then B is a Brownian motion, Δ is a nondecreasing process increasing only when $r'(s) = \hat{l}(s)$, and r' solves

$$\begin{aligned} \mathrm{d}r'(s) &= \mathrm{d}B(s) + \mathrm{d}s - \mathrm{d}\Delta(s), && t \leq s \leq \hat{t}, \\ \mathrm{d}r'(s) &= \mathrm{d}B(s) + \mathrm{d}s, && \hat{t} \leq s, \end{aligned} \tag{2.35}$$

subject to the constraint that $r'(s) \leq \hat{l}(s)$ for all $t \leq s \leq \hat{t}$. In particular, we have $\Delta(\hat{t}) = \sup_{s \in [t, \hat{t}]}(B(s) - \hat{l}(s) + \hat{l}(t)) < \infty$ and the times $\tau_1, \dots, \tau_M$ are created by a Poisson point process on $[t, \hat{t}]$ with intensity measure $2\,\mathrm{d}\Delta$. ∎

2.2 Structure of separation points

In this section we apply the results from the previous section to study the structure of separation points. We start with some definitions. First, we recall the definition of intersection points from [SS08], and formally define meeting and separation points.

Definition 2.3. [Intersection, meeting, and separation points]
We call $z = (x, t) \in \mathbb{R}^2$ an intersection point of $\pi_1, \pi_2 \in \Pi$ if $\sigma_{\pi_1}, \sigma_{\pi_2} < t$ and $\pi_1(t) = \pi_2(t) = x$. If furthermore $\pi_1 \neq \pi_2$ on $(t - \varepsilon, t)$ (resp. $(t, t + \varepsilon)$) for some $\varepsilon > 0$, then we call z a meeting (resp. separation) point of π_1 and π_2. Intersection, meeting, and separation points of dual paths are defined analogously.

Crossing points of two forward paths $\pi_1, \pi_2 \in \Pi$ have been defined in [SS08]. Below, we define crossing points of a forward path π and a dual path $\hat{\pi}$.

Definition 2.4. [Crossing points]
We say that a path $\pi \in \Pi$ crosses a path $\hat{\pi} \in \hat{\Pi}$ from left to right at time t if there exist $\sigma_\pi \leq t_- < t < t_+ \leq \hat{\sigma}_{\hat{\pi}}$ such that $\pi(t_-) < \hat{\pi}(t_-)$, $\hat{\pi}(t_+) < \pi(t_+)$, and $t = \inf\{s \in (t_-, t_+) : \hat{\pi}(s) < \pi(s)\} = \sup\{s \in (t_-, t_+) : \pi(s) < \hat{\pi}(s)\}$. Crossing from right to left is defined analogously. We call $z = (x, t) \in \mathbb{R}^2$ a crossing point of $\pi \in \Pi$ and $\hat{\pi} \in \hat{\Pi}$ if $\pi(t) = x = \hat{\pi}(t)$ and π crosses $\hat{\pi}$ either from left to right or from right to left at time t.

A disadvantage of the way we have defined crossing is that it is possible to find paths $\pi \in \Pi$ and $\hat{\pi} \in \hat{\Pi}$ with $\sigma_\pi < \hat{\sigma}_{\hat{\pi}}$, $\pi(\sigma_\pi) < \hat{\pi}(\sigma_\pi)$, and $\hat{\pi}(\hat{\sigma}_{\hat{\pi}}) < \pi(\hat{\sigma}_\pi)$, such that π crosses $\hat{\pi}$ from left to right at no time in $(\sigma_\pi, \hat{\sigma}_{\hat{\pi}})$. The next lemma shows, however, that such pathologies do not happen for left-most and dual right-most paths.

Lemma 2.5. [Crossing times]
Almost surely, for each $r \in \mathscr{W}^{\mathrm{r}}$ and $\hat{l} \in \hat{\mathscr{W}}^{\mathrm{l}}$ such that $\sigma_r < \hat{\sigma}_{\hat{l}}$, $r(\sigma_r) < \hat{l}(\sigma_r)$, and $\hat{l}(\hat{\sigma}_{\hat{l}}) < r(\hat{\sigma}_{\hat{l}})$, there exists a unique $\sigma_r < \tau < \hat{\sigma}_{\hat{l}}$ such that r crosses $\hat{l}$ from left to right at time τ. Moreover, one has $r \leq \hat{l}$ on $[\sigma_r, \tau]$, $\hat{l} \leq r$ on $[\tau, \hat{\sigma}_{\hat{l}}]$, and there exist $\varepsilon_n, \varepsilon'_n \downarrow 0$ such that $r(\tau - \varepsilon_n) = \hat{l}(\tau - \varepsilon_n)$ and $r(\tau + \varepsilon'_n) = \hat{l}(\tau + \varepsilon'_n)$. Analogous statements hold for left-most paths crossing dual right-most paths from right to left.

Proof By [SS08, Lemma 3.4 (b)] it suffices to prove the statements for paths with deterministic starting points. Set

$$\begin{aligned} \tau &:= \sup\{s \in (\sigma_r, \hat{\sigma}_{\hat{l}}) : r(s) < \hat{l}(s)\}, \\ \tau' &:= \inf\{s \in (\sigma_r, \hat{\sigma}_{\hat{l}}) : \hat{l}(s) < r(s)\}. \end{aligned} \tag{2.36}$$

Lemma 2.1 and the properties of Skorohod reflection imply that $\tau = \tau'$, hence r crosses $\hat{l}$ from left to right at the unique crossing time τ. To see that $r(\tau - \varepsilon_n) = \hat{l}(\tau - \varepsilon_n)$ and for some $\varepsilon_n \downarrow 0$, we consider, in Lemma 2.1 (b), the unique solution $(\tilde{r}, \tilde{\Delta})$ to the Skorohod equation

$$\begin{aligned} \mathrm{d}\tilde{r}(s) &= \mathrm{d}B(s) + \mathrm{d}s - \mathrm{d}\tilde{\Delta}(s), \qquad && t \leq s \leq \hat{t}, \\ \mathrm{d}\tilde{r}(s) &= \mathrm{d}B(s) + \mathrm{d}s, && \hat{t} \leq s. \end{aligned} \tag{2.37}$$

Then $(\tilde{r}(s), \tilde{\Delta}(s)) = (r(s), \Delta(s))$ for all $s \leq \tau$ and $\tilde{\Delta}$ is independent of T. It follows that the set $\{s \in [t, \hat{t}] : \tilde{\Delta}(s) = T\}$, if it is nonempty, a.s. consists of one point, hence $\tilde{\Delta}(s) < T$ for all $s < \tau$, which implies our claim. By symmetry between forward and dual, and between left and right paths, we also have $r(\tau+\varepsilon'_n)=\hat{l}(\tau+\varepsilon'_n)$ for some $\varepsilon'_n \downarrow 0$. ∎

The next proposition says that the sets of crossing points of $(\hat{\mathcal{W}}^{\rm l},\mathcal{W}^{\rm r})$ and $(\mathcal{W}^{\rm l},\hat{\mathcal{W}}^{\rm r})$, of separation points of $(\mathcal{W}^{\rm l},\mathcal{W}^{\rm r})$ and $(\hat{\mathcal{W}}^{\rm l},\hat{\mathcal{W}}^{\rm r})$, and of points of type $({\rm p},{\rm pp})_{\rm s}/({\rm p},{\rm pp})_{\rm s}$ in $\mathcal{N}/\hat{\mathcal{N}}$ as defined in Section 1.3, all coincide.

Proposition 2.6. [Separation points]
Almost surely for each $z\in\mathbb{R}^2$, the following statements are equivalent:

(i) *z is a crossing point of some $l\in\mathcal{W}^{\rm l}$ and $\hat{r}\in\hat{\mathcal{W}}^{\rm r}$,*

(ii) *z is a crossing point of some $\hat{l}\in\hat{\mathcal{W}}^{\rm l}$ and $r\in\mathcal{W}^{\rm r}$,*

(iii) *z is a separation point of some $l\in\mathcal{W}^{\rm l}$ and $r\in\mathcal{W}^{\rm r}$,*

(iv) *z is a separation point of some $\hat{l}\in\hat{\mathcal{W}}^{\rm l}$ and $\hat{r}\in\hat{\mathcal{W}}^{\rm r}$,*

(v) *z is of type $({\rm p},{\rm pp})_{\rm s}$ in $\mathcal{N}$.*

(vi) *z is of type $({\rm p},{\rm pp})_{\rm s}$ in $\hat{\mathcal{N}}$.*

Moreover, the set $\{z\in\mathbb{R}^2 : z$ is of type $({\rm p},{\rm pp})_{\rm s}$ in $\mathcal{N}\}$ is a.s. countable.

Proof We will prove the implications (i)⇒(iii)⇒(v)⇒(ii). By symmetry between forward and dual paths, this then implies that (ii)⇒(iv)⇒(vi)⇒(i), hence all conditions are equivalent.

To prove (i)⇒(iii), let $z=(x,t)$ be a crossing point of $\hat{r}\in\hat{\mathcal{W}}^{\rm r}$ and $l\in\mathcal{W}^{\rm l}$. By [SS08, Lemma 3.4 (b)], without loss of generality, we can assume that $\hat{r}$ and l start from deterministic points with $\sigma_l<t<\hat{\sigma}_{\hat{r}}$. By Lemma 2.2 and the fact that paths in $\hat{\mathcal{W}}^{\rm r}$ cannot cross, there exists an $\hat{r}'\in\hat{\mathcal{W}}^{\rm r}(z)$ and $\varepsilon>0$ such that $l\leq\hat{r}'$ on $[t-\varepsilon,t]$. By Lemma 2.5, we can find $s\in(t-\varepsilon,t)$ such that $l(s)<\hat{r}'(s)$. Now any path $r\in\mathcal{W}^{\rm r}$ started at a point (y,s) with $l(s)<x<\hat{r}'(s)$ is confined between l and $\hat{r}'$, hence passes through z. Since r cannot cross $\hat{r}$ we have $\hat{r}\leq r$ on $[t,\hat{\sigma}_{\hat{r}}]$. Since l and r spend positive Lebesgue time together whenever they meet by [SS08, Prop. 3.6 (c)], while r and $\hat{r}$ spend zero Lebesgue time together by [SS08, Prop. 3.2 (d)], z must be a separation point of l and r.

To prove (iii)⇒(v), let $z=(x,t)$ be a separation point of $l\in\mathcal{W}^{\rm l}$ and $r\in\mathcal{W}^{\rm r}$ so that $l(s)<r(s)$ on $(t,t+\varepsilon]$ for some $\varepsilon>0$. Without loss of generality, we can assume that l and r start from deterministic points with $\sigma_l,\sigma_r<t$. Choose $\hat{t}\in(t,t+\varepsilon]\cap\mathcal{T}$ where $\mathcal{T}\subset\mathbb{R}$ is some fixed, deterministic countable dense set. By Lemma 2.2, there exist unique $\hat{t}>\tau_1>\cdots>\tau_M>\sigma_l$, $\tau_0:=\hat{t}$, $\tau_{M+1}:=-\infty$, with $0\leq M<\infty$, and $\hat{r}_0\in\hat{\mathcal{W}}^{\rm r}(l(\tau_0),\tau_0),\ldots,\hat{r}_M\in\mathcal{W}^{\rm r}(l(\tau_M),\tau_M)$, such that $l\leq\hat{r}_i$ on $[\tau_{i+1}\vee\sigma_l,\tau_i]$ for each $0\leq i\leq M$, and τ_{i+1} is the crossing time of $\hat{r}_i$ and l for each $0\leq i<M$. (See Figure 5, turned upside down.) Note that all the paths $\hat{r}_i$ are confined to the left of r because paths in $\mathcal{W}^{\rm r}$ and $\hat{\mathcal{W}}^{\rm r}$ cannot cross. Since $M<\infty$, one of the paths, say $\hat{r}=\hat{r}_i$, must pass through the separation point z. Since by Proposition 2.2 (b) of [SS08], l and r spend positive Lebesgue time together on $[t-\varepsilon',t]$ for all $\varepsilon'>0$, and by Proposition 3.1 (d) of [SS08], $\hat{r}$ and r spend zero Lebesgue time together, $\hat{r}$ must cross l at z. Similarly, there exists a $\hat{l}\in\hat{\mathcal{W}}^{\rm l}$ starting at $(r(s'),s')$ for some $s'\in(t,t+\varepsilon]$ such that $\hat{l}\leq r$ on $[t,s']$ and $\hat{l}$ crosses r in the point z. Again because paths in $\mathcal{W}$ and $\hat{\mathcal{W}}$ spend zero Lebesgue time together, z must be a separation point of $\hat{r}$ and $\hat{l}$.

For each point (y,s) with $l(s) < y < \hat{r}(s)$ we can find a path $r \in \mathcal{W}^{\mathrm{r}}(y,s)$ that is confined between l and $\hat{r}$, so using the fact that $\mathcal{W}^{\mathrm{r}}$ is closed we see that there exists a path $r' \in \mathcal{W}^{\mathrm{r}}(z)$ that is confined between l and $\hat{r}$. By Lemma 2.5, there exist $\varepsilon_n \downarrow 0$ such that $l(t+\varepsilon_n) = \hat{r}(t+\varepsilon_n)$, so $l \sim^{z}_{\mathrm{out}} r'$. Similarly, there exists $l' \in \mathcal{W}^{\mathrm{l}}$ starting from z which is confined between $\hat{l}$ and r and satisfies $l' \sim^{z}_{\mathrm{out}} r$. The point z must therefore be of type $(1,2)_{\mathrm{l}}$ in $\mathcal{W}^{\mathrm{l}}$ and of type $(1,2)_{\mathrm{r}}$ in $\mathcal{W}^{\mathrm{r}}$, and hence of type $(\mathrm{p},\mathrm{pp})_{\mathrm{s}}$ in $\mathcal{N}$.

To prove (v)⇒(ii), let $l \in \mathcal{W}^{\mathrm{l}}$ and $r \in \mathcal{W}^{\mathrm{r}}$ be the left-most and right-most paths separating at z and let $r' \in \mathcal{W}^{\mathrm{r}}(z)$ and $l' \in \mathcal{W}^{\mathrm{l}}(z)$ be the right-most and left-most paths such that $l \sim^{z}_{\mathrm{out}} r'$ and $l' \sim^{z}_{\mathrm{out}} r$. Then, by [SS08, Prop. 3.2 (c) and Prop. 3.6 (d)], any $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}$ started in a point $z' = (x',t')$ with $t < t'$ and $r'(t') < x' < l'(t')$ is contained between r' and l', hence must pass through z. Since $\hat{l}$ cannot cross l and, by [SS08, Prop. 3.2 (d)], spends zero Lebesgue time with l, while by [SS08, Prop. 3.6 (c)], l and r spend positive Lebesgue time in $[t-\varepsilon,t]$ for any $\varepsilon > 0$, the path $\hat{l}$ must cross r in z.

The fact that the set of separation points is countable, finally, follows from the fact that by [SS08, Lemma 3.4 (b)], each separation point between some $l \in \mathcal{W}^{\mathrm{l}}$ and $r \in \mathcal{W}^{\mathrm{r}}$ is the separation point of some $l' \in \mathcal{W}^{\mathrm{l}}(\mathcal{D})$ and $r' \in \mathcal{W}^{\mathrm{r}}(\mathcal{D})$, where $\mathcal{D}$ is a fixed, deterministic, countable dense subset of $\mathbb{R}^2$. ∎

2.3 Relevant separation points

In a sense, separation points are the most important points in the Brownian net, since these are the points where paths have a choice whether to turn left or right. In the present section, we prove that for deterministic times $s < u$, there are a.s. only locally finitely many separation points at which paths in $\mathcal{N}$ starting at time s have to make a choice that is relevant for determining where they end up at time u.

We start with a useful lemma.

Lemma 2.7. [Incoming net paths]
Almost surely for each $-\infty \leq s < u < \infty$ and $-\infty < x_- \leq x_+ < \infty$:

(a) *For each $\pi \in \mathcal{N}$ such that $\sigma_\pi = s$ and $x_- \leq \pi(u) \leq x_+$ there exist $\hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}(x_-,u)$ and $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}(x_+,u)$ such that $\hat{r} \leq \pi \leq \hat{l}$ and $\hat{r} < \hat{l}$ on (s,u).*

(b) *If there exist $\hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}(x_-,u)$ and $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}(x_+,u)$ such that $\hat{r} < \hat{l}$ on (s,u), then there exists a $\pi \in \mathcal{N}$ such that $\sigma_\pi = s$ and $\hat{r} \leq \pi \leq \hat{l}$ on (s,u).*

Proof Part (b) follows from the steering argument used in [SS08, Lemma 4.7]. To prove part (a), choose $x_-^{(n)} \uparrow x_-$, $x_-^{(n)} \downarrow x_-$, $\hat{r}_n \in \hat{\mathcal{W}}^{\mathrm{r}}(x_-^{(n)},u)$ and $\hat{l}_n \in \hat{\mathcal{W}}^{\mathrm{l}}(x_+^{(n)},u)$. Since paths in the Brownian net do not enter wedges from outside (see Theorem 1.4 (b2)), one has $\hat{r}_n \leq \pi \leq \hat{l}_n$ and $\hat{r}_n < \hat{l}_n$ on (s,u). By monotonicity, $\hat{r}_n \uparrow \hat{r}$ and $\hat{l}_n \downarrow \hat{l}$ for some $\hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}(x_-,u)$ and $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}(x_+,u)$. The claim now follows from the nature of convergence of paths in the Brownian web (see [SS08, Lemma 3.4 (a)]). ∎

The next lemma introduces our objects of interest.

Lemma 2.8. [Relevant separation points]
Almost surely, for each $-\infty \leq s < u \leq \infty$ and $z = (x,t) \in \mathbb{R}^2$ with $s < t < u$, the following statements are equivalent:

(i) *There exists a* $\pi \in \mathcal{N}$ *starting at time* s *such that* $\pi(t) = x$ *and* z *is the separation point of some* $l \in \mathcal{W}^{\mathrm{l}}$ *and* $r \in \mathcal{W}^{\mathrm{r}}$ *with* $l < r$ *on* (t, u)*.*

(ii) *There exists a* $\hat{\pi} \in \mathcal{N}$ *starting at time* u *such that* $\hat{\pi}(t) = x$ *and* z *is the separation point of some* $\hat{l} \in \mathcal{W}^{\mathrm{l}}$ *and* $\hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}$ *with* $\hat{r} < \hat{l}$ *on* (s, t)*.*

Proof By symmetry, it suffices to prove (i)⇒(ii). If z satisfies (i), then by Lemma 2.7, there exists a $\hat{\pi} \in \mathcal{N}$ starting at time u such that $\hat{\pi}(t) = x$, and there exist $\hat{l} \in \mathcal{W}^{\mathrm{l}}(z)$, $\hat{r} \in \mathcal{W}^{\mathrm{r}}(z)$ such that $\hat{r} < \hat{l}$ on (s, t). Since z is the separation point of some $l \in \mathcal{W}^{\mathrm{l}}$ and $r \in \mathcal{W}^{\mathrm{r}}$, by Proposition 2.6, z is also the separation point of some $\hat{l}' \in \mathcal{W}^{\mathrm{l}}$ and $\hat{r}' \in \mathcal{W}^{\mathrm{r}}$. Again by Proposition 2.6, z is of type $(\mathrm{p}, \mathrm{pp})_{\mathrm{s}}$, hence we must have $\hat{l}' = \hat{l}$ and $\hat{r}' = \hat{r}$ on $[-\infty, t]$. ∎

If z satisfies the equivalent conditions from Lemma 2.8, then, in line with the definition given in Section 1.5, we say that z is an (s, u)*-relevant separation point*. We will prove that for deterministic $S < U$, the set of (S, U)-relevant separation points is a.s. locally finite. Let $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \mathrm{d}y$ denote the distribution function of the standard normal distribution and set

$$\Psi(t) := \frac{e^{-t}}{\sqrt{\pi t}} + 2\Phi(\sqrt{2t}) \qquad (0 < t \leq \infty). \tag{2.38}$$

For any $-\infty \leq S < U \leq \infty$, we write

$$R_{S,U} := \{z \in \mathbb{R}^2 : z \text{ is an } (S, U)\text{-relevant separation point}\}. \tag{2.39}$$

Below, $|A|$ denotes the cardinality of a set A.

Proposition 2.9. [Density of relevant separation points]
For deterministic $-\infty \leq S \leq s < u \leq U \leq \infty$ *and* $-\infty < a < b < \infty$*,*

$$\mathbb{E}\big[\big|R_{S,U} \cap ([a, b] \times [s, u])\big|\big] = 2(b - a) \int_s^u \Psi(t - S)\Psi(U - t)\mathrm{d}t. \tag{2.40}$$

In particular, if $-\infty < S < U < \infty$*, then* $R_{S,U}$ *is a.s. a locally finite subset of* $\mathbb{R} \times [S, U]$*.*

Proof It suffices to prove the statement for deterministic $S < s < u < U$; the general statement then follows by approximation. Set

$$E_s = \{\pi(s) : \pi \in \mathcal{N},\ \sigma_\pi = S\} \quad \text{and} \quad F_u = \{\hat{\pi}(u) : \hat{\pi} \in \hat{\mathcal{N}},\ \hat{\sigma}_{\hat{\pi}} = U\}. \tag{2.41}$$

By [SS08, Prop. 1.12], E_s and F_u are spatially homogeneous (in law), locally finite point sets on $\mathbb{R}$ with densities $\Psi(s - S)$ and $\Psi(U - u)$, respectively.

Since the restrictions of the Brownian net to $\mathbb{R} \times [S, s]$ and $\mathbb{R} \times [s, u]$ are independent (which follows from the discrete approximation in [SS08, Thm. 1.1]), at each point $x \in E_s$ there start a.s. unique paths $l_{(x,s)} \in \mathcal{W}^{\mathrm{l}}(x, s)$ and $r_{(x,s)} \in \mathcal{W}^{\mathrm{r}}(x, s)$. Likewise, for each $y \in F_u$ there start a.s. unique $\hat{r}_{(y,u)} \in \hat{\mathcal{W}}^{\mathrm{r}}(y, u)$ and $\hat{l}_{(y,u)} \in \hat{\mathcal{W}}^{\mathrm{l}}(y, u)$. Let

$$\begin{aligned} Q_{s,u} &:= \{(x, y) : x \in E_s,\ y \in F_u,\ l_{(x,s)}(u) < y < r_{(x,s)}(u)\} \\ &= \{(x, y) : x \in E_s,\ y \in F_u,\ \hat{r}_{(y,u)}(s) < x < \hat{l}_{(y,u)}(s)\}, \end{aligned} \tag{2.42}$$

where the equality follows from the fact that paths in $\mathscr{W}^1$ and $\hat{\mathscr{W}}^1$ a.s. do not cross. Note that if $z\in R\times(s,u)$ is an (S,U)-relevant separation point and $\pi,\hat\pi$ are as in Lemma 2.8, then $(\pi(s),\hat\pi(u))\in Q_{s,u}$. Conversely, if $(x,y)\in Q_{s,u}$, then the point $z=(w,\tau)$ defined by $\tau:=\sup\{t\in(s,u):l_{(x,s)}(t)=r_{(x,s)}(t)\}$ and $w:=l_{(x,s)}(\tau)$ is an (S,U)-relevant separation point.

Conditional on E_s, for each $x\in E_s$, the paths $l_{(x,s)}$ and $r(x,s)$ are Brownian motions with drift -1 and $+1$, respectively, hence $\mathbb{E}[r(x,s)(u)-l_{(x,s)}(u)]=2(u-s)$. Since the restrictions of the Brownian net to $\mathbb{R}\times[S,s]$, $\mathbb{R}\times[s,u]$, and $\mathbb{R}\times[u,U]$ are independent, and since the densities of E_s and F_u are $\Psi(s-S)$ and $\Psi(U-u)$, respectively, it follows that for each $a<b$,

$$\mathbb{E}\big[\big|(x,y)\in Q_{s,u}:x\in(a,b)\}|\big]=2(b-a)(u-s)\Psi(s-S)\Psi(U-u). \tag{2.43}$$

For $n\geq 1$, set $D_n:=\{S+k(U-S)/n:0\leq k\leq n-1\}$, and for $t\in(S,U)$ write $\lfloor t\rfloor_n:=\sup\{t'\in D_n:t'\leq t\}$. By our previous remarks and the equicontinuity of the Brownian net, for each $z=(x,t)\in R_{S,U}$ there exist $(x_n,y_n)\in Q_{\lfloor t\rfloor_n,\lfloor t\rfloor_n+1/n}$ such that $x_n\to x$ and $y_n\to x$. It follows that for any $S<s<u<U$ and $a<b$,

$$\Big|R_{S,U}\cap((a,b)\times(s,u))\Big|\leq\liminf_{n\to\infty}\Big|\{(x,y)\in Q_{t,t+1/n}:t\in D_n\cap(s,u),\ x\in(a,b)\}|, \tag{2.44}$$

and therefore, by Fatou,

$$\begin{aligned}\mathbb{E}\big[\big|R_{S,U}\cap((a,b)\times(s,u))\big|\big]&\leq\lim_{n\to\infty}\sum_{t\in D_n\cap(s,u)}\mathbb{E}\big[\big|\{(x,y)\in Q_{t,t+1/n}:x\in(a,b)\}|\big]\\&=2(b-a)\int_s^u\Psi(t-S)\Psi(U-t)\mathrm{d}t,\end{aligned} \tag{2.45}$$

where in the last step we have used (2.43) and Riemann sum approximation. This proves the inequality $\leq$ in (2.40). In particular, our argument shows that the set $R_{S,T}\cap((a,b)\times(S,T))$ is a.s. finite.

By our previous remarks, each point $(x,y)\in Q_{t,t+1/n}$ gives rise to an (S,U)-relevant separation point $z\in\mathbb{R}\times(t,t+1/n)$. To get the complementary inequality in (2.40), we have to deal with the difficulty that a given z may correspond to more than one $(x,y)\in Q_{t,t+1/n}$. For $\delta>0$, set

$$E_t^\delta:=\big\{x\in E_t:E_t\cap(x-\delta,x+\delta)=\{x\}\big\}. \tag{2.46}$$

and define F_t^δ similarly. By Lemma 2.10 below, for each $\varepsilon>0$, we can find $\delta>0$ such that

$$\begin{aligned}\mathbb{E}[|E_t^\delta\cap(a,b)|]&\geq(1-\varepsilon)(b-a)\Psi(t-S),\\\mathbb{E}[|F_t^\delta\cap(a,b)|]&\geq(1-\varepsilon)(b-a)\Psi(U-t)\end{aligned} \tag{2.47}$$

for all $t\in[s,u]$. Arguing as before, we find that

$$\mathbb{E}\big[\big|\{(x,y)\in Q_{t,t'}:x\in(a,b)\cap E_t^\delta\}|\big]\geq 2(1-\varepsilon)(b-a)(t'-t)\Psi(t-S)\Psi(U-t') \tag{2.48}$$

for all $s\leq t\leq t'\leq u$. Similarly, by symmetry between forward and dual paths,

$$\begin{aligned}&\mathbb{E}\big[\big|\{(x,y)\in Q_{t,t'}:x\in(a,b),\ y\in F_{t'}^\delta\}|\big]\\&=\mathbb{E}\big[\big|\{(x,y)\in Q_{t,t'}:y\in(a,b)\cap F_{t'}^\delta\}|\big]\geq 2(1-\varepsilon)(b-a)(t'-t)\Psi(t-S)\Psi(U-t').\end{aligned} \tag{2.49}$$

Set

$$Q^{\delta,K}_{t,t'} := \{(x,y) \in Q_{t,t'} : x \in E^\delta_t,\ y \in F^\delta_{t'},\ x-K \le l_{(x,t)} \le r_{(x,t)} \le x+K \text{ on } [t,t']\}, \tag{2.50}$$

where $l_{(x,t)}$ and $r_{(x,t)}$ are the a.s. unique left-most and right-most paths starting from (x,t). Then, by (2.48) and (2.49), for each $\varepsilon > 0$ we can choose $\delta > 0$ and $K > 0$ such that

$$\mathbb{E}\big[\big|\{(x,y) \in Q^{\delta,K}_{t,t'} : x \in (a,b)\}|\big] \ge 2(1-3\varepsilon)(b-a)(t'-t)\Psi(t-S)\Psi(U-t'). \tag{2.51}$$

By the equicontinuity of the net,

$$\begin{aligned}&\Big|R_{S,U} \cap ((a,b)\times(s,u))\Big| \\ &\quad \ge \limsup_{n\to\infty} \Big|\{(x,y) \in Q^{\delta,K}_{t,t+1/n} : t \in D_n \cap (s,u),\ x \in (a,b)\}\Big|.\end{aligned} \tag{2.52}$$

Since the random variables on the right-hand side of (2.52) are bounded from above by the integrable random variable

$$\big((b-a)/\delta + 1\big)\Big|R_{S,U} \cap \big((a-K,b+K)\times(s,u)\big)\Big|, \tag{2.53}$$

we can take expectations on both sides of (2.52) and let $\varepsilon \to 0$, to get the lower bound in (2.40).

The final statement of the proposition follows by observing that the integral on the right-hand side of (2.40) is finite if $-\infty < S = s < u = U < \infty$. ∎

To complete the proof of Proposition 2.9, we need the following lemma.

Lemma 2.10. [Uniform finiteness]
For $0 \le t \le \infty$, set $\xi^{(-t)}_0 := \{\pi(0) : \pi \in \mathcal{N},\ \sigma_\pi = -t\}$ and $\xi^{(-t)}_{0,\delta} := \{x \in \xi^{(-t)}_0 : \xi^{(-t)}_0 \cap (x-\delta, x+\delta) = \{x\}\}$. Then, for each compact set $K \subset (0,\infty]$ and $-\infty < a < b < \infty$, one has

$$\lim_{\delta\downarrow 0} \sup_{t\in K} \mathbb{E}\big[|(\xi^{(-t)}_0 \backslash \xi^{(-t)}_{0,\delta}) \cap (a,b)|\big] = 0. \tag{2.54}$$

Proof Set $F_t := |\xi^{(-t)}_0 \cap (a,b)|$, $F^\delta_t := |\xi^{(-t)}_{0,\delta} \cap (a,b)|$, and $f_\delta(t) := \mathbb{E}[F_t - F^\delta_t]$. By [SS08, Prop. 1.12], $\mathbb{E}[F_t] < \infty$ a.s. for all $t \in (0,\infty]$. Since $F^\delta_t \uparrow F_t$ as $\delta \downarrow 0$, it follows that $f_\delta(t) = \mathbb{E}[F_t - F^\delta_t] \downarrow 0$ as $\delta \downarrow 0$, for each $t \in (0,\infty]$. Since $F_s - F^\delta_s \downarrow F_t - F^\delta_t$ as $s \uparrow t$, the f_δ are continuous functions on $(0,\infty]$ decreasing to zero, hence $\lim_{\delta\downarrow 0}\sup_{t\in K} f_\delta(t) = 0$ for each compact $K \subset (0,\infty]$. ∎

The following simple consequence of Proposition 2.9 will often be useful.

Lemma 2.11. [Local finiteness of relevant separation points]
Almost surely, for each $-\infty \le s < u \le \infty$, the set $R_{s,u}$ of all (s,u)-relevant separation points is a locally finite subset of $\mathbb{R} \times (s,u)$.

Proof Let $\mathscr{T}$ be a deterministic countable dense subset of $\mathbb{R}$. Then, by Proposition 2.9, $R_{s,u}$ is a locally finite subset of $\mathbb{R} \times [s,u]$ for each $s < u$, $s,u \in \mathscr{T}$. For general $s < u$, we can choose $s_n, u_n \in \mathscr{T}$ such that $s_n \downarrow s$ and $u_n \uparrow u$. Then $R_{s_n,u_n} \uparrow R_{s,u}$, hence $R_{s,u}$ is locally finite. ∎

2.4 Reflected paths

In this section, we prove Lemma 1.8. We start with two preparatory lemmas.

Lemma 2.12. [Forward and dual paths spend zero time together]
Almost surely, for any $s < u$, one has

$$\int_s^u 1_{\{\xi_t^{(s)} \cap \hat{\xi}_t^{(u)} \neq \emptyset\}} \mathrm{d}t = 0, \tag{2.55}$$

where, for $t \in (s,u)$,

$$\begin{aligned} \xi_t^{(s)} &:= \{\pi(t) : \pi \in \mathcal{N},\ \sigma_\pi = s\}, \\ \hat{\xi}_t^{(u)} &:= \{\hat{\pi}(t) : \hat{\pi} \in \hat{\mathcal{N}},\ \hat{\sigma}_{\hat{\pi}} = u\}. \end{aligned} \tag{2.56}$$

In particular, one has $\int_s^u 1_{\{\pi(t)=\hat{\pi}(t)\}} \mathrm{d}t = 0$ for any $\pi \in \mathcal{N}$ and $\hat{\pi} \in \hat{\mathcal{N}}$ with $\sigma_\pi = s$ and $\hat{\sigma}_{\hat{\pi}} = u$.

Proof It suffices to prove the statement for deterministic times. In that case, the expectation of the quantity in (2.55) is given by

$$\int_s^u \mathbb{P}[\xi_t^{(s)} \cap \hat{\xi}_t^{(u)} \neq \emptyset] \mathrm{d}t. \tag{2.57}$$

By Proposition 1.12 of [SS08], $\xi_t^{(s)}$ and $\hat{\xi}_t^{(u)}$ are stationary point processes with finite intensity. By the independence of $\mathcal{N}|_{(-\infty,t]}$ and $\hat{\mathcal{N}}|_{[t,\infty)}$, $\xi_t^{(s)}$ and $\hat{\xi}_t^{(u)}$ are independent. It follows that $\mathbb{P}[\xi_t^{(s)} \cap \hat{\xi}_t^{(u)} \neq \emptyset] = 0$ for each $t \in (s,u)$. Fubini's Theorem then implies (2.55) almost surely. ∎

Part (b) of the next lemma is similar to Lemma 2.5.

Lemma 2.13. [Crossing of dual net paths]

(a) *Almost surely, for each $r \in \mathcal{W}^{\mathrm{r}}$ and $\hat{\pi} \in \hat{\mathcal{N}}$ such that $\sigma_r < \hat{\sigma}_{\hat{\pi}}$ and $\hat{\pi}(\sigma_r) < r(\sigma_r)$, one has $\hat{\pi} \leq r$ on $[\sigma_r, \hat{\sigma}_{\hat{\pi}}]$.*

(b) *Almost surely, for each $r \in \mathcal{W}^{\mathrm{r}}$ and $\hat{\pi} \in \hat{\mathcal{N}}$ such that $\sigma_r < \hat{\sigma}_{\hat{\pi}} < \infty$ and $r(\sigma_r) < \hat{\pi}(\sigma_r)$, if $S := \{s \in (\sigma_r, \hat{\sigma}_{\hat{\pi}}) : r(s) = \hat{\pi}(s),\ (r(s), s) \text{ is of type } (\mathrm{p}, \mathrm{pp})_{\mathrm{s}}\} = \emptyset$, then $r \leq \hat{\pi}$ on $[\sigma_r, \hat{\sigma}_{\hat{\pi}}]$, while otherwise, r crosses $\hat{\pi}$ from left to right at time $\tau := \inf(S)$, and one has $r \leq \hat{\pi}$ on $[\sigma_r, \tau]$ and $\hat{\pi} \leq r$ on $[\tau, \hat{\sigma}_{\hat{\pi}}]$.*

Analogous statements hold for left-most paths crossing dual Brownian net paths from right to left.

Proof To prove part (a), imagine that $r(t) < \hat{\pi}(t)$ for some $\sigma_r < t \leq \hat{\sigma}_{\hat{\pi}}$. Let $\hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}$ be the left-most element of $\hat{\mathcal{N}}(\hat{\pi}(t), t)$. Then, by [SS08, Prop. 3.2 (c)], one has $r \leq \hat{r} \leq \hat{\pi}$ on $[\sigma_r, t]$, contradicting our assumption that $\hat{\pi}(\sigma_r) < r(\sigma_r)$.

To prove part (b), set

$$\begin{aligned} t &:= \sup\{s \in (\sigma_r, \hat{\sigma}_{\hat{\pi}}) : r(s) < \hat{\pi}(s)\}, \\ t' &:= \inf\{s \in (\sigma_r, \hat{\sigma}_{\hat{\pi}}) : \hat{\pi}(s) < r(s)\}. \end{aligned} \tag{2.58}$$

Then $t < t'$ by what we have just proved and therefore $t = t'$ by Lemma 2.12, hence t is the unique crossing point of r and $\hat{\pi}$, $r \leq \hat{\pi}$ on $[\sigma_r, t]$, and $\hat{\pi} \leq r$ on $[t, \hat{\sigma}_{\hat{\pi}}]$.

To see that $(r(t),t)$ is a separation point, by Proposition 2.6, it suffices to show that there exists some $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}$ that crosses r at time t. By [SS08, Lemma 3.4 (b)], we can without loss of generality assume that r starts from a deterministic point. Choose $\hat{t} > t$ with $\hat{\pi}(\hat{t}) < r(\hat{t})$ and $\hat{t} \in \mathcal{T}$, where $\mathcal{T} \subset \mathbb{R}$ is some fixed, deterministic countable dense set. By Lemma 2.2, there exist unique $\hat{t} > \tau_1 > \cdots > \tau_M > \sigma_r$, $\tau_0 := \hat{t}$, $\tau_{M+1} := -\infty$, with $0 \leq M < \infty$, and $\hat{l}_0 \in \hat{\mathcal{W}}^{\mathrm{l}}(r(\tau_0),\tau_0),\ldots,\hat{l}_M \in \hat{\mathcal{W}}^{\mathrm{l}}(r(\tau_M),\tau_M)$, such that $\hat{l}_i \leq r$ on $[\tau_{i+1} \vee \sigma_r, \tau_i]$ for each $0 \leq i \leq M$, and τ_{i+1} is the crossing time of $\hat{l}_i$ and r for each $0 \leq i < M$. We claim that $\tau_i = t$ for some $i = 1,\ldots,M$.

By [SS08, Prop. 1.8], $\hat{\pi} \leq \hat{l}_0$ on $(-\infty,\hat{t}]$, so $M \geq 1$ and $\tau_1 \geq t$. If $\tau_1 = t$ we are done. Otherwise, we claim that $\hat{\pi} \leq \hat{l}_1$ on $(-\infty,\tau_1]$. To see this, assume that $\hat{l}_1(s) < \hat{\pi}(s)$ for some $s < \tau_1$. Then we can start a left-most path l between $\hat{l}_1$ and $\hat{\pi}$. By what we have proved in part (a) and [SS08, Prop. 3.2 (c)], l is contained by $\hat{l}_1$ and $\hat{\pi}$, hence l and r form a wedge of $(\mathcal{W}^{\mathrm{l}},\mathcal{W}^{\mathrm{r}})$ which by the characterization of $\hat{\mathcal{N}}$ using wedges (Theorem 1.4 (b2)) cannot be entered by $\hat{\pi}$, which yields a contradiction. This shows that $\hat{\pi} \leq \hat{l}_1$ on $(-\infty,\tau_1]$. It follows that $M \geq 2$ and $\tau_2 \geq t$. Continuing this process, using the finiteness of M, we see that $\tau_i = t$ for some $i = 1,\ldots,M$.

Conversely, if $r(s) = \hat{\pi}(s)$ for some $s \in (\sigma_r, \hat{\sigma}_{\hat{\pi}})$ such that $z := (r(s),s)$ is of type $(\mathrm{p},\mathrm{pp})_{\mathrm{s}}$, then by Lemma 2.7 (b), the path $\hat{\pi}$ is contained between the left-most and right-most paths starting at z that are *not* continuations of incoming left-most and right-most paths. It follows that any incoming right-most path r at z must satisfy $\hat{\pi} \leq r$ on $[s,\hat{\sigma}_{\hat{\pi}}]$. This shows that t is the first separation point that r meets on $\hat{\pi}$. ∎

Proof of Lemma 1.8 We will prove the statement for reflected right-most paths; the statement for left-most paths then follows by symmetry. It follows from the image set property (see formula (1.16) or [SS08, Prop. 1.13]) and the local equicontinuity of the Brownian net that $r_{z,\hat{\pi}}(s) := \sup\{\pi(s) : \pi \in \mathcal{N}(z),\ \pi \leq \hat{\pi} \text{ on } [t,\hat{\sigma}_{\hat{\pi}}]\}$ $(s \geq t)$ defines a path $r_{z,\hat{\pi}} \in \mathcal{N}$. Put

$$\begin{aligned}\mathcal{F} := \big\{s \in (t,\hat{\sigma}_{\hat{\pi}}) : \exists r \in \mathcal{W}^{\mathrm{r}} \text{ such that } r \text{ crosses } \hat{\pi} \text{ at } (\hat{\pi}(s),s) \text{ and } \exists \pi \in \mathcal{N}(z) \\ \text{such that } \pi(s) = \hat{\pi}(s), \text{ and } \pi \leq \hat{\pi} \text{ on } [t,s]\big\}.\end{aligned} \tag{2.59}$$

By Lemma 2.13, $(\hat{\pi}(s),s)$ is a $(t,\hat{\sigma}_{\hat{\pi}})$-relevant separation point for each $s \in \mathcal{F}$, so by Lemma 2.11, $\mathcal{F}$ is a locally finite subset of $(t,\hat{\sigma}_{\hat{\pi}})$.

We claim that

$$\begin{aligned}\forall s \in \mathcal{F} \text{ and } \pi \in \mathcal{N}(z) \text{ s.t. } \pi(s) = \hat{\pi}(s) \text{ and } \pi \leq \hat{\pi} \text{ on } [t,s] \\ \exists \pi' \in \mathcal{N}(z) \text{ s.t. } \pi' = \pi \text{ on } [t,s] \text{ and } \pi \leq \hat{\pi} \text{ on } [t,\hat{\sigma}_{\hat{\pi}}].\end{aligned} \tag{2.60}$$

To prove this, set $\pi_0 := \pi$ and $s_0 := s$ and observe that $(\pi_0(s_0),s_0)$ is a separation point where some right-most path r crosses $\hat{\pi}$. Let r' be the outgoing right-most path at $(\pi_0(s_0),s_0)$ that is not equivalent to r. By Lemma 2.7 (b), $r' \leq \hat{\pi}$ on $[s_0,s_0+\varepsilon]$ for some $\varepsilon > 0$. Let π_1 be the concatenation of π_0 on $[t,s_0]$ with r' on $[s_0,\infty]$. Since the Brownian net is closed under hopping [SS08, Prop. 1.4], using the structure of separation points, it is not hard to see that $\pi_1 \in \mathcal{N}$. (Indeed, we may hop from π_0 onto the left-most path entering $(\pi_0(s_0),s_0)$ at time s and then hop onto r at some time $s+\varepsilon$ and let $\varepsilon \downarrow 0$, using the closedness of $\mathcal{N}$.) The definition of $\mathcal{F}$ and Lemma 2.13 (b) imply that either $\pi_1 \leq \hat{\pi}$ on $[t,\hat{\sigma}_{\hat{\pi}}]$ or there exists some $s_1 \in \mathcal{F}$, $s_1 > s_0$ such that r' crosses $\hat{\pi}$ at s_1. In that case, we can continue our construction, leading to a sequence of paths π_n and times $s_n \in \mathcal{F}$ such that $\pi_n \leq \hat{\pi}$ on $[t,s_n]$. Since $\mathcal{F}$ is locally finite, either this process terminates after a finite number of steps, or $s_n \uparrow \hat{\sigma}_{\hat{\pi}}$. In the latter case, using the compactness of $\mathcal{N}$, any subsequential limit of the π_n gives the desired path π'.

We next claim $\mathscr{F} \subset \mathscr{F}(r_{z,\hat\pi})$, where the latter is defined as in (1.11). Indeed, if $s \in \mathscr{F}$, then $(\hat\pi(s),s)$ is of type $(\mathrm{p},\mathrm{pp})_{\mathrm{s}}$ by Lemma 2.13 (b). Moreover, we can find some $\pi \in \mathscr{N}(z)$ such that $\pi(s) = \hat\pi(s)$ and $\pi \leq \hat\pi$ on $[t,s]$. By (2.60), we can modify π on $[s,\hat\sigma_{\hat\pi}]$ so that it stays on the left of $\hat\pi$, hence $\pi \leq r_{z,\hat\pi}$ by the maximality of the latter, which implies that $r_{z,\hat\pi}(s) = \hat\pi(s)$ hence $s \in \mathscr{F}(r_{z,\hat\pi})$.

Set $\mathscr{F}' := \mathscr{F} \cup \{t, \hat\sigma_{\hat\pi}, \infty\}$ if $\hat\sigma_{\hat\pi}$ is a cluster point of $\mathscr{F}$ and $\mathscr{F}' := \mathscr{F} \cup \{t,\infty\}$ otherwise. Let $s,u \in \mathscr{F}^-$ satisfy $s < u$ and $(s,u) \cap \mathscr{F}' = \emptyset$. Let $\mathscr{T}$ be some fixed, deterministic countable dense subset of $\mathbb{R}$. By Lemma 2.12 we can choose $t_n \in \mathscr{T}$ such that $t_n \downarrow s$ and $r_{z,\hat\pi}(t_n) < \hat\pi(t_n)$. Choose $r_n \in \mathscr{W}^{\mathrm{r}}(r_{z,\hat\pi}(t_n),t_n)$ such that $r_{z,\hat\pi} \leq r_n$ on $[t_n,\infty]$. By [SS08, Lemma 8.3], the concatenation of $r_{z,\hat\pi}$ on $[t,t_n]$ with r_n on $[t_n,\infty]$ is a path in $\mathscr{N}$, hence r_n can cross $\hat\pi$ only at times in $\mathscr{F}$, and therefore $r_n \leq \hat\pi$ on $[t_n,u]$. Using the compactness of $\mathscr{W}^{\mathrm{r}}$, let $r \in \mathscr{W}^{\mathrm{r}}$ be any subsequential limit of the r_n. Then $r_{z,\hat\pi} \leq r$ on $[s,\infty]$, $r \leq \hat\pi$ on $[s,u]$, and, since $\mathscr{N}$ is closed, the concatenation of $r_{z,\hat\pi}$ on $[t,s]$ and r on $[s,\infty]$ is a path in $\mathscr{N}$.

If $u < \hat\sigma_{\hat\pi}$, then $r_{z,\hat\pi}(u) \leq r(u) \leq \hat\pi(u)$ while $r_{z,\hat\pi}(u) = \hat\pi(u)$ by the fact that $\mathscr{F} \subset \mathscr{F}(r_{z,\hat\pi})$. In this case, since $(\hat\pi(u),u)$ is a separation point, by Lemma 2.13 (b), the path r crosses $\hat\pi$ at u, $\hat\pi \leq r$ on $[u,\hat\sigma_{\hat\pi}]$, and $\inf\{s' > s : \hat\pi(s') < r(s')\} = u$. Let π denote the concatenation of $r_{z,\hat\pi}$ on $[t,s]$ and r on $[s,\infty]$. By (2.60), if $u < \hat\sigma_{\hat\pi}$, then we can modify π on $[u,\infty]$ such that it stays on the left of $\hat\pi$. Therefore, whether $u < \hat\sigma_{\hat\pi}$ or not, by the maximality of $r_{z,\hat\pi}$, we have $r_{z,\hat\pi} = r$ on $[s,u]$.

To complete our proof we must show that $\mathscr{F} \supset \mathscr{F}(r_{z,\hat\pi})$. To see this, observe that if $s \in \mathscr{F}(r_{z,\hat\pi})$, then, since $r_{z,\hat\pi}$ is a concatenation of right-most paths, by Lemma 2.13 (b), some right-most path crosses $\hat\pi$ at s, hence $s \in \mathscr{F}$. ∎

3 Incoming paths and the image set

3.1 Maximal T-meshes

Let $\mathscr{N}_T := \{\pi \in \mathscr{N} : \sigma_\pi = T\}$ denote the set of paths in the Brownian net starting at a given time $T \in [-\infty,\infty]$ and let N_T be its image set, defined in (1.15). We call N_T the *image set of the Brownian net started at time T*. In the present section, we will identify the connected components of the complement of N_T relative to $\{(x,t) \in R^2_{\mathrm{c}} : t \geq T\}$.

The next lemma is just a simple observation.

Lemma 3.1. [Dual paths exit meshes through the bottom point]
If $M(r,l)$ is a mesh with bottom point $z = (x,t)$ and $\hat\pi \in \hat{\mathscr{N}}$ starts in $M(r,l)$, then $r(s) \leq \hat\pi(s) \leq l(s)$ for all $s \in [t,\hat\sigma_{\hat\pi}]$.

Proof Immediate from Lemma 2.13 (a). ∎

We will need a concept that is slightly stronger than that of a mesh.

Definition 3.2. [$*$-meshes]
A mesh $M(r,l)$ with bottom point $z = (x,t)$ is called a $*$-mesh if there exist $\hat r \in \hat{\mathscr{W}}^{\mathrm{r}}$ and $\hat l \in \hat{\mathscr{W}}^{\mathrm{l}}$ with $\hat r \sim^z_{\mathrm{in}} \hat l$, such that r is the right-most element of $\mathscr{W}^{\mathrm{r}}(z)$ that passes on the left of $\hat r$ and l is the left-most element of $\mathscr{W}^{\mathrm{l}}(z)$ that passes on the right of $\hat l$.

Lemma 3.3. [Characterization of $*$-meshes]
Almost surely for all $z = (x, t) \in \mathbb{R}^2$ and $\hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}, \hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}$ such that $\hat{r} \sim^{z}_{\mathrm{in}} \hat{l}$, the set

$$M_z(\hat{r}, \hat{l}) := \{z' \in \mathbb{R}^2 : \forall \hat{\pi} \in \hat{\mathcal{N}}(z')\ \exists \varepsilon > 0 \text{ s.t. } \hat{r} \leq \hat{\pi} \leq \hat{l} \text{ on } [t, t + \varepsilon]\} \tag{3.61}$$

is a $*$-mesh with bottom point z. Conversely, each $*$-mesh is of this form.

Remark A simpler characterization of $*$-meshes (but one that is harder to prove) is given in Lemma 3.13 below. Once Theorem 1.7 is proved, it will turn out that if a mesh is not a $*$-mesh, then its bottom point must be of type $(\mathrm{o}, \mathrm{ppp})$. (See Figure 3.)

Proof of Lemma 3.3 Let $z = (x, t) \in \mathbb{R}^2$ and $\hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}, \hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}$ satisfy $\hat{r} \sim^{z}_{\mathrm{in}} \hat{l}$. Let r be the right-most element of $\mathcal{W}^{\mathrm{r}}(z)$ that passes on the left of $\hat{r}$ and let l be the left-most element of $\mathcal{W}^{\mathrm{l}}(z)$ that passes on the right of $\hat{l}$. Then, obviously, $M(r, l)$ is a $*$-mesh, and each $*$-mesh is of this form. We claim that $M(r, l) = M_z(\hat{r}, \hat{l})$.

To see that $M(r, l) \supset M_z(\hat{r}, \hat{l})$, note that if $z' = (x', t') \notin M(r, l)$ and $t' > t$, then either there exists an $\hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}(z')$ that stays on the left of r, or there exists an $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}(z')$ that stays on the right of l; in either case, $z' \notin M_z(\hat{r}, \hat{l})$.

To see that $M(r, l) \subset M_z(\hat{r}, \hat{l})$, assume that $z' = (x', t') \in M(r, l)$. Since each path in $\hat{\mathcal{N}}(z')$ is contained by the left-most and right-most dual paths starting in z', it suffices to show that each $\hat{r}' \in \hat{\mathcal{W}}^{\mathrm{r}}(z')$ satisfies $\hat{r} \leq \hat{r}'$ on $[t, t + \varepsilon]$ for some $\varepsilon > 0$ and each $\hat{l}' \in \hat{\mathcal{W}}^{\mathrm{l}}(z')$ satisfies $\hat{l}' \leq \hat{l}$ on $[t, t + \varepsilon']$ for some $\varepsilon' > 0$. By symmetry, it suffices to prove the statement for $\hat{r}'$. So imagine that $\hat{r}' < \hat{r}$ on (t, t'). Then there exists an $r' \in \mathcal{W}^{\mathrm{r}}(z)$ that stays between $\hat{r}'$ and $\hat{r}$, contradicting the fact that r is the right-most element of $\mathcal{W}^{\mathrm{r}}(z)$ that passes on the left of $\hat{r}$. ∎

Definition 3.4. [Maximal T-meshes]
For a given $T \in [-\infty, \infty)$, we call a mesh $M(r, l)$ with bottom point $z = (x, t)$ a T-mesh if $M(r, l)$ is a $*$-mesh and $t \geq T$, and a maximal T-mesh if it is not contained in any other T-mesh. A maximal T-mesh with $T = -\infty$ is called a mesh of the backbone of the Brownian net.

Proposition 3.5. [Properties of maximal T-meshes]
Almost surely for each $T \in [-\infty, \infty)$:

(a) *A set of the form (3.61), with $z = (x, t)$, is a maximal T-mesh if and only if $t = T$ or if $t > T$ and $\hat{r} < \hat{l}$ on (T, t).*

(b) *The maximal T-meshes are mutually disjoint, and their union is the set $\{(x, t) \in R^2_{\mathrm{c}} : t \geq T\} \backslash N_T$, where N_T is defined in (1.15).*

Proof Let $M(r, l)$ and $M(r', l')$ be $*$-meshes with bottom points $z = (x, t)$ and $z' = (x', t')$, respectively, with $t \geq t' \geq T$. Then either $M(r, l)$ and $M(r', l')$ are disjoint, or there exists a $z'' \in M(r, l) \cap M(r', l')$. In the latter case, by Lemma 3.3, any $\hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}(z'')$ and $\hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}(z'')$ must pass through z and z' (in this order) and we have $M(r, l) = M_z(\hat{r}, \hat{l})$ and $M(r', l') = M_{z'}(\hat{r}, \hat{l})$. It follows that $M_z(\hat{r}, \hat{l}) \subset M_{z'}(\hat{r}, \hat{l})$, where the inclusion is strict if and only if $t > t'$. Moreover, a set of the form (3.61), with $z = (x, t)$, is a maximal T-mesh if and only if there exists no $z' = (x', t)$ with $t' \in [T, t)$ such that $\hat{r}$ and $\hat{l}$ are equivalent incoming paths at z'. This proves part (a).

We have just proved that T-meshes are either disjoint or one is contained in the other, so maximal T-meshes must be mutually disjoint. Let $O_T := \{(x,t) \in R^2_{\rm c} : t \geq T\} \backslash N_T$. It is easy to see that $O_T \subset \mathbb{R} \times (T, \infty)$. Consider a point $z = (x,t) \in \mathbb{R}^2$ with $t > T$. If $z \notin O_T$, then by Lemma 2.7 there exist $\hat r \in \hat{\mathcal{W}}^{\rm r}(z)$ and $\hat l \in \hat{\mathcal{W}}^{\rm l}(z)$ with the property that there does not exist a $z' = (x', t')$ with $t' \geq T$ such that $\hat r \sim^{z'}_{\rm in} \hat l$, hence by Lemma 3.3, z is not contained in any T-mesh. On the other hand, if $z \in O_T$, then by Lemma 2.7 (b) and the nature of convergence of paths in the Brownian web (see [SS08, Lemma 3.4 (a)]), there exist $\hat r \in \hat{\mathcal{W}}^{\rm r}$ and $\hat l \in \hat{\mathcal{W}}^{\rm l}$ starting from points (x_-, t) and (x_+, t), respectively, with $x_- < x < x_+$, such that $\hat r(s) = \hat l(s)$ for some $s \in (T, t)$. Now, setting $u := \inf\{s \in (T,t) : \hat r(s) = \hat l(s)\}$ and $z' := (\hat r(u), u)$, by Lemma 3.3, $M_{z'}(\hat r, \hat l)$ is a maximal T-mesh that contains z. ∎

3.2 Reversibility

Recall from (1.15) the definition of the image set N_T of the Brownian net started at time T. It follows from [SS08, Prop. 1.15] that the law of $N_{-\infty}$ is symmetric with respect to time reversal. In the present section, we extend this property to $T > -\infty$ by showing that locally on $\mathbb{R} \times (T, \infty)$, the law of N_T is absolutely continuous with respect to its time-reversed counterpart. This is a useful property, since it allows us to conclude that certain properties that hold a.s. in the forward picture also hold a.s. in the time-reversed picture. For example, meeting and separation points have a similar structure, related by time reversal. (Note that this form of time-reversal is different from, and should not be confused with, the dual Brownian net.)

We write $\mu \ll \nu$ when a measure μ is absolutely continuous with respect to another measure ν, and $\mu \sim \nu$ if μ and ν are equivalent, i.e., $\mu \ll \nu$ and $\nu \ll \mu$.

Proposition 3.6. [Local reversibility]
Let $-\infty < S, T < \infty$ and let N_T be the image set of the Brownian net started at time T. Define $\mathcal{R}_S : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathcal{R}_S(x,t) := (x, S-t)$. Let $K \subset \mathbb{R}^2$ be a compact set such that $K, \mathcal{R}_S(K) \subset \mathbb{R} \times (T, \infty)$. Then

$$\mathbb{P}[N_T \cap \mathcal{R}_S(K) \in \cdot\,] \sim \mathbb{P}[\mathcal{R}_S(N_T \cap K) \in \cdot\,]. \tag{3.62}$$

Proof By the reversibility of the backbone, it suffices to prove that

$$\mathbb{P}[N_T \cap K \in \cdot\,] \sim \mathbb{P}[N_{-\infty} \cap K \in \cdot\,]. \tag{3.63}$$

Choose some $T < s < \min\{t : (x,t) \in K\}$. By [SS08, Prop. 1.12], the set

$$\hat\xi^K_s := \{\hat\pi(s) : \hat\pi \in \hat{\mathcal{N}}(K)\} \tag{3.64}$$

is a.s. a finite subset of $\mathbb{R}$, say

$$\hat\xi^K_s = \{X_1, \ldots, X_M\} \quad \text{with} \quad X_1 < \cdots < X_M. \tag{3.65}$$

For $U = T, -\infty$, let us write

$$\xi^{(U)}_s := \{\pi(s) : \pi \in \mathcal{N},\ \sigma_\pi = U\}. \tag{3.66}$$

By Lemma 2.7 and the fact that s is deterministic, for any $z = (x,t) \in K$, one has $z \in N_U$ if and only if there exist $\hat r \in \hat{\mathcal{W}}^{\rm r}(z)$ and $\hat l \in \hat{\mathcal{W}}^{\rm l}(z)$ such that $\hat r < \hat l$ on $[s,t)$ and $\xi^{(U)}_s \cap (\hat r(s), \hat l(s)) \neq \emptyset$. Thus, we can write

$$N_U \cap K = \bigcup_{i \in I_U} N_i, \tag{3.67}$$

where

$$N_i := \{z = (x,t) \in K : \exists \hat{r} \in \hat{\mathcal{W}}^{\mathrm{r}}(z),\ \hat{l} \in \hat{\mathcal{W}}^{\mathrm{l}}(z) \\ \text{s.t. } \hat{r} < \hat{l} \text{ on } [s,t) \text{ and } \hat{r}(s) \le X_i,\ \hat{l}(s) \ge X_{i+1}\} \tag{3.68}$$

and

$$I_U := \{i : 1 \le i \le M-1,\ \xi_s^{(U)} \cap (X_i, X_{i+1}) \neq \emptyset\}. \tag{3.69}$$

It follows that

$$\mathbb{P}[N_U \cap K \in \cdot\,] = \mathbb{E}\Big[\mathbb{P}\Big[\bigcup_{i \in I_U} N_i \in \cdot \,\Big|\, \{X_1, \ldots, X_M\}\Big]\Big], \tag{3.70}$$

where

$$\begin{aligned}&\mathbb{P}\Big[\bigcup_{i \in I_U} N_i \in \cdot \,\Big|\, \{X_1, \ldots, X_M\}\Big] \\ &\quad = \sum_{\mathcal{I} \subset \{1,\ldots,M-1\}} \mathbb{P}\Big[\bigcup_{i \in \mathcal{I}} N_i \in \cdot \,\Big|\, \{X_1, \ldots, X_M\}\Big]\mathbb{P}\Big[I_U = \mathcal{I} \,\Big|\, \{X_1, \ldots, X_M\}\Big].\end{aligned} \tag{3.71}$$

Here the sum ranges over all subsets $\mathcal{I}$ of $\{1, \ldots, M-1\}$. The statement of the proposition now follows from (3.70) and (3.71) by observing that

$$\mathbb{P}\Big[I_U = \mathcal{I} \,\Big|\, \{X_1, \ldots, X_M\}\Big] > 0 \quad \text{a.s.} \tag{3.72}$$

for all $\mathcal{I} \subset \{1, \ldots, M-1\}$ and $U = T, -\infty$. ∎

3.3 Classification according to incoming paths

In this section, we give a preliminary classification of points in the Brownianb net that is entirely based on incoming paths. Note that if there is an incoming path $\pi \in \mathcal{N}$ at a point $z = (x,t)$, then $z \in N_T$ for some $T < t$, where N_T is the image set of the Brownian net started at time T, defined in (1.15). Therefore in this section, our main task is to classify the special points of N_T.

For a given $T \in [-\infty, \infty)$, let us say that a point $z = (x,t) \in N_T \cap (\mathbb{R} \times (T, \infty))$ is *isolated from the left* if

$$\sup\{x' \in \mathbb{R} : (x',t) \in N_T,\ x' < x\} < x. \tag{3.73}$$

Points that are isolated from the right are defined similarly, with the supremum replaced by an infimum and both inequality signs reversed.

Lemma 3.7. [Isolated points]
A point $z = (x,t) \in N_T \cap (\mathbb{R} \times (T, \infty))$ is isolated from the left if and only if there exists an incoming path $l \in \mathcal{W}^{\mathrm{l}}$ at z. An analogous statement holds if z is isolated from the right.

Proof By Proposition 3.5 (b), if $z = (x,t)$ is isolated from the left, then there exists a maximal T-mesh $M(r,l)$ with left and right boundary r and l, bottom time strictly smaller than t, and top time strictly larger than t, such that $l(t) = x$. Conversely, if there exists an incoming path $l \in \mathcal{W}^{\mathrm{l}}$ at z, then by [SS08, Lemma 6.5], there exists a mesh $M(r,l)$ with bottom time in (T,t) and top time in (t, ∞), such that $l(t) = x$. Therefore, by the characterization of the Brownian net with meshes (see Theorem 1.4 (b3)), z is isolated from the left. ∎

We now give a classification of points in $\mathbb{R}^2$ based on incoming paths in the Brownian net. Recall Definition 2.3 of intersection, meeting, and separation points.

Definition 3.8. [Classification by incoming paths]
We say that a point $z = (x, t) \in \mathbb{R}^2$ is of type

(C_o) *if there is no incoming $\pi \in \mathcal{N}$ at z,*

(C_n) *if there is an incoming $\pi \in \mathcal{N}$ at z, but there is no incoming $\pi \in \mathcal{W}^l \cup \mathcal{W}^r$ at z,*

(C_l) *if there is an incoming $l \in \mathcal{W}^l$ at z, but there is no incoming $r \in \mathcal{W}^r$ at z,*

(C_r) *if there is an incoming $r \in \mathcal{W}^r$ at z, but there is no incoming $l \in \mathcal{W}^l$ at z,*

(C_s) *if z is a separation point of some $l \in \mathcal{W}^l$ and $r \in \mathcal{W}^r$,*

(C_m) *if z is a meeting point of some $l \in \mathcal{W}^l$ and $r \in \mathcal{W}^r$,*

(C_p) *if there is an incoming $l \in \mathcal{W}^l$ at z and an incoming $r \in \mathcal{W}^r$ at z, and z is not of type* (C_s) *or* (C_m).

Note that by Lemma 2.7, for any $T < t$ such that $z \in N_T$, points of types (C_l), (C_r), and (C_n) are either not isolated from the left, or not isolated from the right, or both. In view of this, we call these points *cluster points*. Points of the types (C_l) and (C_r) are called *one-sided cluster points* and points of type (C_n) *two-sided cluster points*. Proposition 3.11 below shows that, among other things, cluster points are the limits of nested sequences of excursions between left-most and right-most paths.

The main result of this section is the following.

Lemma 3.9. [Classification by incoming paths]

(a) *Almost surely, each point in $\mathbb{R}^2$ is of exactly one of the types* (C_o), (C_n), (C_l), (C_r), (C_s), (C_m), *and* (C_p), *and each of these types occurs in $\mathbb{R}^2$.*

(b) *For deterministic $t \in \mathbb{R}$, a.s. each point in $\mathbb{R} \times \{t\}$ is of type* (C_o) *or* (C_p), *and both these types occur.*

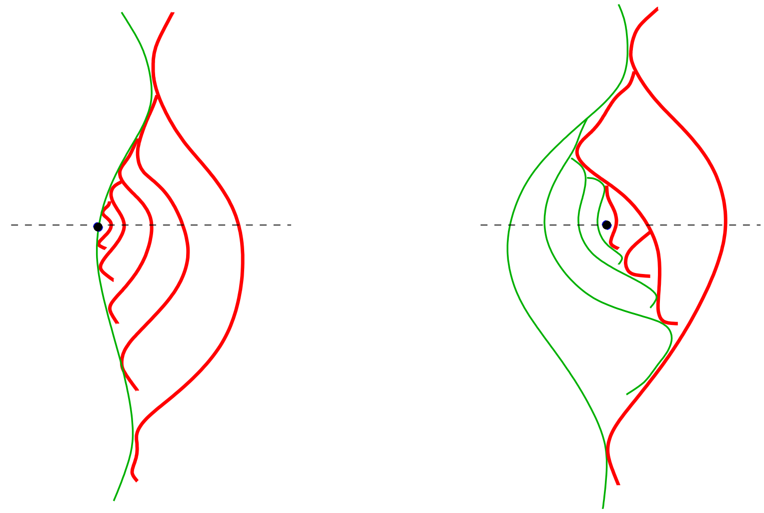
(c) *Each deterministic $z \in \mathbb{R}^2$ is a.s. of type* (C_o).

Proof of Lemma 3.9 (b) and (c) If $t \in \mathbb{R}$ is deterministic, and $T < t$, then by [SS08, Prop 1.12], the set $N_T \cap (\mathbb{R} \times \{t\})$ is locally finite. In particular, if there is an incoming path $\pi \in \mathcal{N}$ at a point $z \in \mathbb{R} \times \{t\}$, then z is isolated from the left and right. Since each meeting or separation point of some $l \in \mathcal{W}^l$ and $r \in \mathcal{W}^r$ is the meeting or separation point of some left-most and right-most path chosen from a fixed, deterministic countable dense set, and since paths started at deterministic starting points a.s. do not meet or separate at deterministic times, z must be of type (C_p).

If $z = (x, t) \in \mathbb{R}^2$ is deterministic, then by [SS08, Prop 1.12], for each $n \geq 1$, a.s. there is no path $\pi \in \mathcal{N}$ with $\sigma_\pi = t - 1/n$ and $\pi(t) = x$, hence z must be of type (C_o). ∎

Before proving Lemma 3.9 (a), we first establish some basic properties for each type of points in Definition 3.8. We start with a definition and a lemma.

By definition, we say that two paths $\pi, \pi' \in \Pi$ make an *excursion* from each other *on a time interval* (s, u) if $\sigma_\pi, \sigma_{\pi'} < s$ (note the strict inequality), $\pi(s) = \pi'(s)$, $\pi \neq \pi'$ on (s, u), and $\pi(u) = \pi'(u)$. The next lemma says that excursions between left-most and right-most paths are rather numerous.



(C$_\mathrm{l}$) (C$_\mathrm{n}$)

Figure 6: Nested excursions around cluster points.

Lemma 3.10. [Excursions along left-most paths]
Almost surely, for each $l \in \mathscr{W}^{\mathrm{l}}$ and for each open set O such that $l \cap O \neq \emptyset$, there exists an $r \in \mathscr{W}^{\mathrm{r}}$ such that r makes an excursion from l during a time interval (s, u), and $\{(x,t) : t \in [s,u], x \in [l(t), r(t)]\} \subset O$.

Proof Choose $s < u$ such that $\{l(t) : t \in [s,u]\} \subset O$ and choose some t from a fixed, deterministic countable dense subset of $\mathbb{R}$ such that $t \in (s,u)$. By Lemma 3.9 (b), there exists a unique incoming path $r \in \mathscr{W}^{\mathrm{r}}$ at $(l(t), t)$, and r does not separate from l at time t. Since by [SS08, Prop. 3.6 (b)], the set $\{v : r(v) = l(v)\}$ is nowhere dense, we can find $u_1 > s_1 > u_2 > s_2 > \cdots$ such that $u_n \downarrow t$ and r makes an excursion from l during the time interval (s_n, u_n) for each $n \geq 1$. By the local equicontinuity of the Brownian net, we can choose n large enough such that $\{(x,t) : t \in [s_n, u_n], x \in [l(t), r(t)]\} \subset O$. ∎

Proposition 3.11. [Structure of points with incoming paths]

(a) *If $z = (x,t)$ is of type (C$_\mathrm{l}$), then there exist $l \in \mathscr{W}^{\mathrm{l}}$ and $r_n \in \mathscr{W}^{\mathrm{r}}$ ($n \geq 1$), such that $l(t) = x < r_n(t)$, each r_n makes an excursion away from l on a time interval $(s_n, u_n) \ni t$, $[s_n, u_n] \subset (s_{n-1}, u_{n-1})$, $s_n \uparrow t$, $u_n \downarrow t$, and $r_n(t) \downarrow x$. (See Figure 6). By symmetry, an analogous statement holds for points of type (C$_\mathrm{r}$).*

(b) *If $z = (x,t)$ is of type (C$_\mathrm{n}$), then there exist $l_1 \in \mathscr{W}^{\mathrm{l}}, r_2 \in \mathscr{W}^{\mathrm{r}}, l_3 \in \mathscr{W}^{\mathrm{l}}, \ldots$, such that $l_{2n+1}(t) < x < r_{2n}(t)$, each path ($l_n$ for n odd, r_n for n even) in the sequence makes an excursion away from the previous path on a time interval $(s_n, u_n) \ni t$, $[s_n, u_n] \subset (s_{n-1}, u_{n-1}]$, $s_n \uparrow t$, $u_n \downarrow t$, $l_{2n+1}(t) \uparrow x$, and $r_{2n}(t) \downarrow x$ (the monotonicity here need not be strict).*

(c) *If $z = (x,t)$ is of type (C$_\mathrm{s}$), then for each $T < t$ with $z \in N_T$, there exist maximal T-meshes $M(r,l)$ and $M(r',l')$ with bottom times strictly smaller than t and top times strictly larger than t, and a maximal T-mesh $M(r'',l'')$ with bottom point z, such that $l \sim^z_{\mathrm{in}} r'$, $l \sim^z_{\mathrm{out}} r''$, and $l'' \sim^z_{\mathrm{out}} r'$.*

(d) *If $z = (x,t)$ is of type (C_m), then for each $T < t$ with $z \in N_T$, there exist maximal T-meshes $M(r,l)$ and $M(r',l')$ with bottom times strictly smaller than t and top times strictly larger than t, and a maximal T-mesh $M(r'',l'')$ with top point z, such that $l \sim^z_{\rm in} r''$, $l'' \sim^z_{\rm in} r'$, and $l \sim^z_{\rm out} r'$.*

(e) *If $z = (x,t)$ is of type (C_p), then for each $T < t$ with $z \in N_T$, there exist maximal T-meshes $M(r,l)$ and $M(r',l')$ with bottom times strictly smaller than t and top times strictly larger than t, such that $l \sim^z_{\rm in} r'$ and $l \sim^z_{\rm out} r'$.*

Proof (a): Let l be an incoming left-most path at z and choose $T < t$ such that $l \subset N_T$. Choose t_n from some fixed, deterministic countable dense subset of $\mathbb{R}$ such that $t_n \uparrow t$. By Lemma 3.9 (b), for each n there is a unique incoming path $r_n \in \mathscr{W}^{\rm r}$ at $(l(t_n),t_n)$. By assumption, r_n does not pass through z, hence r_n makes an excursion away from l on a time interval (s_n,u_n) with $t_n \le s_n < t < u_n \le \infty$. Clearly $s_n \uparrow t$ and $u_n \downarrow u_\infty$ for some $u_\infty \ge t$. We claim that $u_\infty = t$. Indeed, if $u_\infty > t$, then $(l(s_n),s_n)$ $(n \ge 1)$ are (s_1,u_∞)-relevant separation points, hence the latter are not locally finite on $\mathbb{R} \times (s_1,u_\infty)$, contradicting Lemma 2.11. By going to a subsequence if necessary, we can asssure that $s_{n-1} < s_n$ and $u_n < u_{n-1}$. The fact that $r_n(t) \downarrow x$ follows from the local equicontinuity of the Brownian net.

(b): Let $\pi \in \mathscr{N}$ be an incoming path at z and choose $T < t$ such that $\pi \subset N_T$. By Lemma 2.7 (a), there exist $\hat r \in \hat{\mathscr{W}}^{\rm r}(z)$ and $\hat l \in \hat{\mathscr{W}}^{\rm l}(z)$ such that $\hat r < \hat l$ on (T,t) and $\hat r \le \pi \le \hat l$ on $[T,t]$. Choose an arbitrary $l_1 \in \mathscr{W}^{\rm l}$ starting from $z_1 = (x_1,s_1)$ with $s_1 \in (T,t)$ and $x_1 \in (\hat r(s_1),\hat l(s_1))$. Since there is no incoming left-most path at z, and since l_1 cannot cross $\hat l$, the path l_1 must cross $\hat r$ at some time $s_2 < t$. By Proposition 2.6, there exists $r_2 \in \mathscr{W}^{\rm r}$ such that $(l_1(s_2),s_2)$ is a separation point of l_1 and r_2, and r_2 lies on the right of $\hat r$. Since there is no incoming right-most path at z, and since r_2 cannot cross $\hat r$, the path r_2 must cross $\hat l$ at some time $s_3 < t$, at which a path $l_3 \in \mathscr{W}^{\rm l}$ separates from r_2, and so on. Repeating this procedure gives a sequence of paths $l_1 \in \mathscr{W}^{\rm l}, r_2 \in \mathscr{W}^{\rm r}, l_3 \in \mathscr{W}^{\rm l},\ldots$ such that the n-th path in the sequence separates from the $(n-1)$-th path at a time $s_n \in (s_{n-1},t)$, and we have $l_1(t) \le l_3(t) \le \cdots < x < \cdots \le r_4(t) \le r_2(t)$. By the a.s. local equicontinuity of $\mathscr{W}^{\rm l} \cup \mathscr{W}^{\rm r}$ and $\hat{\mathscr{W}}^{\rm l} \cup \hat{\mathscr{W}}^{\rm r}$, it is clear that $s_n \uparrow t$, $l_{2n+1}(t) \uparrow x$ and $r_{2n}(t) \downarrow x$. Let $u_n := \inf\{s \in (s_n,\infty) : l_{n-1} = r_n\}$ if n is even and $u_n := \inf\{s \in (s_n,\infty) : r_{n-1} = l_n\}$ if n is odd. Then $u_n \downarrow u_\infty \ge t$. The same argument as in the proof of part (a) shows that $u_\infty = t$.

To prepare for parts (c)–(e), we note that if there exist incoming paths $l \in \mathscr{W}^{\rm l}$ and $r \in \mathscr{W}^{\rm r}$ at a point $z \in \mathbb{R}^2$ and $T < t$ is such that $z \in N_T$, then by Lemma 3.7, z is isolated from the left and from the right, hence by Proposition 3.5 (b), there exist maximal T-meshes $M(r,l)$ and $M(r',l')$ with bottom times strictly smaller than t and top times strictly larger than t, such that $l(t) = x = r'(t)$. We now prove (c)–(e).

(c): If z is a separation point of some left-most and right-most path, then by Proposition 2.6, z must be a separation point of the paths l and r' mentioned above, and there exist paths $r'' \in \mathscr{W}^{\rm r}(z)$ and $l'' \in \mathscr{W}^{\rm l}(z)$ such that $l \sim^z_{\rm out} r''$, and $l'' \sim^z_{\rm out} r'$, and z is the bottom point of the mesh $M(r'',l'')$. Note that $M(r'',l'')$ is a maximal T-mesh by Definition 3.4.

(d): If z is a meeting point of some left-most and right-most path, then z must be a meeting point of the paths l and r' mentioned above. This means that z is a point where two maximal T-meshes meet. Recall from Proposition 3.5 that the maximal T-meshes are the connected components of the open set $(\mathbb{R} \times (T,\infty)) \backslash N_T$. Thus, reversing time for N_T in some compact environment of z makes z into a separation point of two maximal T-meshes. Therefore, by local reversibility (Proposition 3.6) and what we have just proved about points of type (C_s), there exists a maximal T-mesh $M(r'',l'')$ with top point z, such that $l \sim^z_{\rm in} r''$, $l'' \sim^z_{\rm in} r'$, and $l \sim^z_{\rm out} r'$.

(e): If z is not a separation or meeting point of any left-most and right-most path, then the paths l and r' mentioned above must satisfy $l \sim^z_{\rm in} r'$ and $l \sim^z_{\rm out} r'$. ∎

Proof of Lemma 3.9 (a) The statement that each point in $\mathbb{R}^2$ belongs to exactly one of the types $(\mathrm{C_o})$, $(\mathrm{C_n})$, $(\mathrm{C_l})$, $(\mathrm{C_r})$, $(\mathrm{C_s})$, $(\mathrm{C_m})$, and $(\mathrm{C_p})$ is entirely self-evident, except that we have to show that a point cannot at the same time be a separation point of some $l \in \mathscr{W}^{\rm l}$ and $r \in \mathscr{W}^{\rm r}$, and a meeting point of some (possibly different) $l' \in \mathscr{W}^{\rm l}$ and $r' \in \mathscr{W}^{\rm r}$. This however follows from Proposition 2.6.

It follows from parts (b) and (c) of the lemma (which have already been proved) that points of the types $(\mathrm{C_o})$ and $(\mathrm{C_p})$ occur. Obviously, points of types $(\mathrm{C_s})$ and $(\mathrm{C_m})$ occur as well. To prove the existence of cluster points, it suffices to establish the existence of nested sequences of excursions, which follows from Lemma 3.10. ∎

Proposition 3.11 yields a useful consequence.

Lemma 3.12. [Separation and meeting points]
If $z \in \mathbb{R}^2$ is a separation (resp. meeting) point of two paths $\pi, \pi' \in \mathcal{N}$, then z is of type $(\mathrm{p},\mathrm{pp})_{\rm s}$ (resp. (pp,p)).

Proof We start by showing that if $z \in \mathbb{R}^2$ is a separation (resp. meeting) point of two paths $\pi, \pi' \in \mathcal{N}$, then z is of type $(\mathrm{C_s})$ (resp. $(\mathrm{C_m})$). Thus, we need to show that a.s. for any $z \in \mathbb{R}^2$, if $\pi, \pi' \in \mathcal{N}_T$ are incoming paths at z, one has $\pi \sim^z_{\rm in} \pi'$ if z is not a meeting point, and $\pi \sim^z_{\rm in} \pi'$ if z is not a separation point.

If z is not a cluster point, these statements follow from the configuration of maximal T-meshes around z as described in Proposition 3.11 (c)–(e).

If z is a two-sided cluster point, then any path $\pi \in \mathcal{N}$ must pass through the top points of the nested excursions around z, hence all paths in $\mathcal{N}(z)$ are equivalent as outgoing paths at z. If z is a one-sided cluster point, then by [SS08, Prop. 1.8], $l \leq \pi$ on $[t, \infty)$ for all incoming net paths at z, hence all incoming net paths must pass through the top points of the nested excursions and therefore be equivalent as outgoing paths.

Our previous argument shows that at a cluster point $z = (x,t)$, for any $T < t$ such that $z \in N_T$, all paths $\pi \in \Pi$ such that $\pi(t) = x$ and $\pi \subset N_T$ are equivalent as outgoing paths at z. By local reversibility (Proposition 3.6), it follows that all paths $\pi \in \Pi$ such that $\pi(t) = x$ and $\pi \subset N_T$ are equivalent as ingoing paths at z. (Note that reversing time in N_T does not change the fact that z is a cluster point.)

This completes the proof that if $z \in \mathbb{R}^2$ is a separation (resp. meeting) point of two paths $\pi, \pi' \in \mathcal{N}$, then z is of type $(\mathrm{C_s})$ (resp. $(\mathrm{C_m})$). If z is of type $(\mathrm{C_s})$, then by Proposition 2.6, z is of type $(\mathrm{p},\mathrm{pp})_{\rm s}$. If z is of type $(\mathrm{C_m})$, then by Proposition 3.11 (d), there are exactly two incoming left-right pairs at z, and there is at least one outgoing left-right pair at z. Therefore, z must be of type $(2,1)$ in both $\mathscr{W}^{\rm l}$ and $\mathscr{W}^{\rm r}$, hence there are no other outgoing paths in $\mathscr{W}^{\rm l} \cup \mathscr{W}^{\rm r}$ at z, so z is of type (pp,p). ∎

Remark Lemma 3.12 shows in particular that any meeting point of two paths $\pi, \pi' \in \mathscr{W}^{\rm l} \cup \mathscr{W}^{\rm r}$ is of type (pp,p). This fact can be proved by more elementary methods as well. Consider the Markov process (L, R, L') given by the unique weak solutions to the SDE

$$\begin{aligned} \mathrm{d}L_t &= 1_{\{L_t < R_t\}} \mathrm{d}B^{\rm l}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\rm s}_t - \mathrm{d}t, \\ \mathrm{d}R_t &= 1_{\{L_t < R_t\}} \mathrm{d}B^{\rm r}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\rm s}_t + \mathrm{d}t, \\ \mathrm{d}L'_t &= \mathrm{d}B^{\rm l'}_t - \mathrm{d}t, \end{aligned} \tag{3.74}$$

where $B^{\mathrm{l}}, B^{\mathrm{r}}, B^{\mathrm{s}}$, and $B^{\mathrm{l}'}$ are independent Brownian motions, and we require that $L_t \leq R_t$ for all $t \geq 0$. Set

$$\tau = \inf\{t \geq 0 : R_t = L'_t\}. \tag{3.75}$$

Then the claim follows from the fact that the process started in $(L_0, R_0, L'_0) = (0, 0, \varepsilon)$ satisfies

$$\lim_{\varepsilon \to 0} \mathbb{P}^{(0,0,\varepsilon)}\big[L_\tau = R_\tau\big] = 1, \tag{3.76}$$

which can be shown by a submartingale argument. Since this proof is of interest on its own, we give it in Appendix A.2.

The next lemma is a simple consequence of Lemma 3.12.

Lemma 3.13. [Characterization of $*$-meshes]
A mesh $M(r,l)$ with bottom point $z = (x,t)$ is a $*$-mesh if and only if there exists no $\pi \in \mathscr{W}^{\mathrm{l}}(z) \cup \mathscr{W}^{\mathrm{r}}(z)$, $\pi \neq l, r$, such that $r \leq \pi \leq l$ on $[t, t+\varepsilon]$ for some $\varepsilon > 0$.

Proof If $M(r,l)$ is a mesh with bottom point $z = (x,t)$ and there exists some $l \neq l' \in \mathscr{W}^{\mathrm{l}}(z)$ such that $r \leq l' \leq l$ on $[t, t+\varepsilon]$ for some $\varepsilon > 0$, then $r < l'$ on $[t, t+\varepsilon]$ by [SS08, Prop. 3.6 (a)], hence by Lemma 3.1 we can find $\hat{r} \in \hat{\mathscr{W}}^{\mathrm{r}}$ and $\hat{l} \in \hat{\mathscr{W}}^{\mathrm{l}}$ such that $r \leq \hat{r} \leq \hat{l} \leq l'$ on $[t, t+\varepsilon]$, which implies that $M(r,l)$ is not a $*$-mesh. By symmetry, the same is true if there exists some $r \neq r' \in \mathscr{W}^{\mathrm{r}}(z)$ such that $r \leq r' \leq l$ on $[t, t+\varepsilon]$.

For any mesh $M(r,l)$, by Lemma 3.1, we can find $\hat{r} \in \hat{\mathscr{W}}^{\mathrm{r}}$ and $\hat{l} \in \hat{\mathscr{W}}^{\mathrm{l}}$ such that $r \leq \hat{r} \leq \hat{l} \leq l'$ on $[t, t+\varepsilon]$ for some $\varepsilon > 0$. We claim that $\hat{r} \sim^{z}_{\mathrm{in}} \hat{l}$ if $M(r,l)$ satisfies the assumptions of Lemma 3.13; from this, it then follows that $M(r,l)$ is a $*$-mesh. To prove our claim, assume that $\hat{r} \not\sim^{z}_{\mathrm{in}} \hat{l}$. Then, by Lemma 3.12, there exist $\hat{l}' \in \hat{\mathscr{W}}^{\mathrm{l}}$ and $\hat{r}' \in \hat{\mathscr{W}}^{\mathrm{r}}$ such that $\hat{r} \leq \hat{l}' < \hat{r}' \leq \hat{l}$ on $[t, t+\varepsilon']$ for some $\varepsilon' > 0$. By [SS08, Prop 3.6 (d)], this implies that there exist $l \in \mathscr{W}^{\mathrm{l}}(z)$ such that $\hat{l}' \leq l \leq \hat{r}'$ on $[t, t+\varepsilon']$, hence $M(r,l)$ does not satisfy the assumptions of Lemma 3.13. ∎

3.4 Special times

For any closed set $K \subset \mathbb{R}^2$, set

$$\xi^K_t := \{\pi(t) : \pi \in \mathscr{N}(K),\ \sigma_\pi \leq t\}. \tag{3.77}$$

It has been proved in [SS08, Theorem 1.11] that for any closed $A \subset \mathbb{R}$, the process $(\xi^{A\times\{0\}}_t)_{t\geq 0}$ is a Markov process taking values in the space of closed subsets of $\mathbb{R}$. It was shown in [SS08, Prop 1.12] that $\xi^{A\times\{0\}}_t$ is a.s. a locally finite point set for each deterministic $t > 0$. It was claimed without proof there that there exists a dense set of times $t > 0$ such that $\xi^{A\times\{0\}}_t$ is not locally finite. Indeed, with the help of Lemma 3.10, we can prove the following result.

Proposition 3.14. [No isolated points]
Almost surely, there exists a dense set $\mathscr{T} \subset (0,\infty)$ such that for each $t \in \mathscr{T}$ and for each closed $A \subset \mathbb{R}$, the set $\xi^{A\times\{0\}}_t$ contains no isolated points.

Proof We claim that it suffices to prove the statement for $A=\mathbb{R}$. To see this, suppose that $x\in\xi_t^{A\times\{0\}}$ is not isolated from the left in $\xi_t^{\mathbb{R}\times\{0\}}$ for some $t>0$. It follows from the characterization of the Brownian net using meshes (see Theorem 1.4 (b3)) that the pointwise infimum $\pi:=\inf\{\pi'\in\mathcal{N}(A\times\{0\}):\pi'(t)=x\}$ defines a path $\pi\in\mathcal{N}(A\times\{0\})$. Let l be the left-most element of $\mathcal{W}^{\rm l}(\pi(0),0)$. By Lemma 3.7, there is no incoming left-most path at (x,t), hence we must have $l(t)<x$. Since x is not isolated from the left in $\xi_t^{\mathbb{R}\times\{0\}}$, there are $\pi_n\in\mathcal{N}$ starting at time 0 such that $\pi_n(t)\in(l(t),x)$ for each n, and $\pi_n(t)\uparrow x$. Now each π_n must cross either l or π, so by the fact that the Brownian net is closed under hopping ([SS08, Prop. 1.4]), x is not isolated from the left in $\xi_t^{A\times\{0\}}$.

To prove the proposition for $A=\mathbb{R}$, we claim that for each $0<s<u$ and for each $n\geq 1$, we can find $s\leq s'<u'\leq u$ such that

$$(x-\tfrac{1}{n},x+\tfrac{1}{n})\cap\xi_t^{\mathbb{R}\times\{0\}}\neq\{x\}\quad\forall t\in(s',u'),\ x\in(-n,n)\cap\xi_t^{\mathbb{R}\times\{0\}}. \tag{3.78}$$

To show this, we proceed as follows. If (3.78) holds for $s'=s$ and $u'=u$ we are done. Otherwise, we can find some $t_1\in(s,u)$ and $x_1\in(-n,n)\cap\xi_{t_1}^{\mathbb{R}\times\{0\}}$ such that $(x_1-\frac{1}{n},x_1+\frac{1}{n})\cap\xi_{t_1}^{\mathbb{R}\times\{0\}}=\{x_1\}$. In particular, x_1 is an isolated point so there is an incoming $l_1\in\mathcal{W}^{\rm l}$ at (x_1,t_1), hence by Lemma 3.10 we can find an $r_1\in\mathcal{W}^{\rm r}$ such that r_1 makes an excursion from l_1 during a time interval (s_1,u_1) with $s<s_1<u_1<u$, with the additional property that $x_1-\frac{1}{2n}\leq l_1<r_1\leq x_1+\frac{1}{2n}$ on (s_1,u_1). Now either we are done, or there exists some $t_2\in(s_1,u_1)$ and $x_2\in(-n,n)\cap\xi_{t_2}^{\mathbb{R}\times\{0\}}$ such that $(x_2-\frac{1}{n},x_2+\frac{1}{n})\cap\xi_{t_2}^{\mathbb{R}\times\{0\}}=\{x_2\}$. In this case, we can find $l_2\in\mathcal{W}^{\rm l}$ and $r_2\in\mathcal{W}^{\rm r}$ making an excursion during a time interval (s_2,u_2), with the property that l_2 and r_2 stay in $[x_2-\frac{1}{2n},x_2+\frac{1}{2n}]$. We iterate this process if necessary. Since x_{m+1} must be at least a distance $\frac{1}{2n}$ from each of the points $x_1,\ldots,x_m$, this process terminates after a finite number of steps, proving our claim.

By what we have just proved, for any $0<s<u$, we can find $s\leq s_1\leq s_2\leq\cdots\leq u_2\leq u_1\leq u$ such that

$$(x-\tfrac{1}{n},x+\tfrac{1}{n})\cap\xi_t^{\mathbb{R}\times\{0\}}\neq\{x\}\quad\forall n\geq 1,\ t\in(s_n,u_n),\ x\in(-n,n)\cap\xi_t^{\mathbb{R}\times\{0\}}. \tag{3.79}$$

Necessarily $\bigcap_n(s_n,u_n)=\{t\}$ for some $t\in\mathbb{R}$, and we conclude that $\xi_t^{\mathbb{R}\times\{0\}}$ contains no isolated points. ∎

4 Excursions

4.1 Excursions between forward and dual paths

Excursions between left-most and right-most paths have already been studied briefly in Section 3.3. In this section, we study them in more detail. In particular, in order to prove the existence of points from groups (4) and (5) of Theorem 1.7, we will need to prove that, for a given left-most path l and a dual left-most path $\hat{l}$ that hits l from the left, there exist nested sequences of excursions of right-most paths away from l, such that each excursion interval contains an intersection point of l and $\hat{l}$. As a first step towards proving this, we will study excursions between l and $\hat{l}$.

By definition, we say that a forward path $\pi\in\Pi$ and a backward path $\hat{\pi}\in\hat{\Pi}$ make an *excursion* from each other *on a time interval* (s_-,s_+) if $\sigma_\pi<s_-<s_+<\hat{\sigma}_{\hat{\pi}}$ (note the strict inequalities),

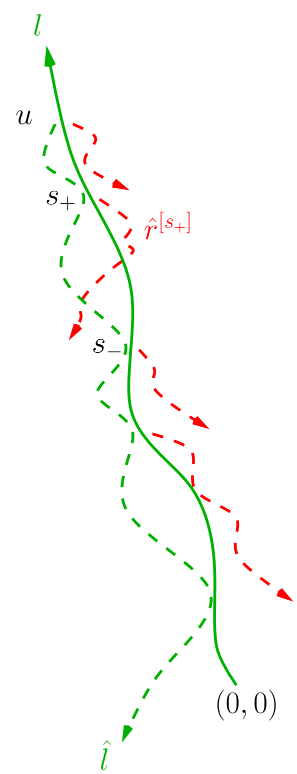


Figure 7: Excursions between a left-most and dual left-most path, with dual right-most paths starting at the top of each excursion.

$\pi(s_-)=\hat\pi(s_-)$, $\pi\neq\pi'$ on (s_-,s_+), and $\pi(s_+)=\hat\pi(s_+)$. We write

$$\begin{aligned}\mathscr{I}(\pi,\hat\pi)&:=\big\{t\in(\sigma_\pi,\hat\sigma_{\hat\pi}):\pi(t)=\hat\pi(t)\big\},\\ \mathscr{I}_+(\pi,\hat\pi)&:=\big\{s_+:\pi\text{ and }\hat\pi\text{ make an excursion}\\&\qquad\text{from each other on }(s_-,s_+)\text{ for some }\sigma_\pi<s_-<s_+<\hat\sigma_{\hat\pi}\big\}.\end{aligned}\tag{4.80}$$

Our next proposition is the main result of this section. Below, for given $s_+\in\mathscr{I}_+(l,\hat l)$, it is understood that s_- denotes the unique time such that l and $\hat l$ make an excursion from each other on (s_-,s_+). See Figure 7.

Proposition 4.1. [Excursions entered by a dual path]
Almost surely, for each $l\in\mathscr{W}^{\rm l}$ and $\hat l\in\hat{\mathscr{W}}^{\rm l}$ such that $\sigma_l<\hat\sigma_{\hat l}$ and $\hat l\leq l$ on $[\sigma_l,\hat\sigma_{\hat l}]$, there starts at each $s_+\in\mathscr{I}_+(l,\hat l)$ a unique $\hat r^{[s_+]}\in\hat{\mathscr{W}}^{\rm r}(l(s_+),s_+)$ such that $l\leq\hat r^{[s_+]}$ on $[s_+-\varepsilon,s_+]$ for some $\varepsilon>0$. Moreover, the set

$$\mathscr{I}'_+(l,\hat l):=\big\{s_+\in\mathscr{I}_+(l,\hat l):\hat r^{[s_+]}\text{ crosses }l\text{ at some time in }(s_-,s_+)\big\}\tag{4.81}$$

is a dense subset of $\mathscr{I}(l,\hat l)$. The same statements hold when $\hat l$ is replaced by a path $\hat r\in\hat{\mathscr{W}}^{\rm r}$.

The proof of Proposition 4.1 is somewhat long and depends on excursion theory. We start by studying the a.s. unique left-most path started at the origin. Let $l\in\mathscr{W}^{\rm l}(0,0)$ be that path and fix some deterministic $u>0$. By the structure of special points of the Brownian web (see, e.g., [SS08, Lemma 3.3 (b)]), $\hat{\mathscr{W}}^{\rm l}(l(u),u)$ contains a.s. two paths, one on each side of l. Let $\hat l$ be the one on the left of l. Set

$$X_t:=\tfrac{1}{\sqrt2}\big(l(u-t)-\hat l(u-t)\big)\qquad(t\in[0,u]).\tag{4.82}$$

Our first lemma, the proof of which can be found below, says that X is standard Brownian motion reflected at the origin.

Lemma 4.2. [Reflected Brownian motion]
There exists a standard Brownian motion $B=(B_t)_{t\geq0}$ such that

$$X_t=B_t+\Psi_t\qquad(t\in[0,u])\quad\text{where}\quad\Psi_t:=-\inf_{s\in[0,t]}B_s\qquad(t\geq0).\tag{4.83}$$

Extend X_t to all $t\geq0$ by (4.83), and put

$$\begin{aligned}S_\tau&:=\inf\{t>0:\Psi_t>\tau\}\qquad(\tau\geq0),\\ \mathscr{T}&:=\{\tau\geq0:S_{\tau-}<S_\tau\}.\end{aligned}\tag{4.84}$$

Then the intervals of the form $(S_{\tau-},S_\tau)$ with $\tau\in\mathscr{T}$ are precisely the intervals during which X makes an excursion away from 0. Define a random point set N on $(0,\infty)^2$ by

$$N:=\{(h_\tau,\tau):\tau\in\mathscr{T}\}\quad\text{where}\quad h_\tau:=S_\tau-S_{\tau-}.\tag{4.85}$$

The following fact is well-known.

Lemma 4.3. [Poisson set of excursions]
The set N is a Poisson point process with intensity measure $\nu(\mathrm{d}x)\mathrm{d}\tau$, where

$$\nu(\mathrm{d}h)=\frac{\mathrm{d}h}{\sqrt{2\pi h^3}}.\tag{4.86}$$

Proof It follows from Brownian scaling that $(S_\tau)_{\tau \geq 0}$ is a stable subordinator with exponent $1/2$, and this implies that $\nu(\mathrm{d}h) = ch^{-3/2}\mathrm{d}h$ for some $c > 0$. The precise formula (4.86) can be found in [KS91, Sect. 6.2.D]. ∎

To explain the main idea of the proof of Proposition 4.1, we formulate one more lemma, which will be proved later.

Lemma 4.4. [Dual paths at top points of excursions]
Almost surely, at each $s_+ \in \mathscr{I}_+(l, \hat{l})$ there starts a unique $\hat{r}^{[s_+]} \in \hat{\mathscr{W}}^{\mathrm{r}}(l(s_+), s_+)$ such that $l \leq \hat{r}^{[s_+]}$ on $[s_+ - \varepsilon, s_+]$ for some $\varepsilon > 0$.

Set

$$\mathscr{T}_u := \{\tau \in \mathscr{T} : S_\tau < u\} = \mathscr{T} \cap (0, \Psi_u). \tag{4.87}$$

and observe that

$$\mathscr{I}_+(l, \hat{l}) = \{U_\tau : \tau \in \mathscr{T}_u\} \quad \text{where} \quad U_\tau := u - S_{\tau-} \qquad (\tau \in \mathscr{T}). \tag{4.88}$$

For $d \geq 1$ and $h > 0$, let $\mathscr{E}^d_h$ be the space of all continuous functions $f : [0, h] \to \mathbb{R}^d$ such that $f(0) = 0$, and set $\mathscr{E}^d := \{(f, h) : h > 0,\ f \in \mathscr{E}^d_h\}$. Using Lemma 4.4, for each $\tau \in \mathscr{T}_u$, we define a random function $(\tilde{R}^\tau, L^\tau, \hat{L}^\tau) \in \mathscr{E}^3_{h_\tau}$ by

$$\begin{aligned} \tilde{R}^\tau_t &:= l(U_\tau) - \big(\hat{r}^{[U_\tau]}(U_\tau - t) \vee \hat{l}(U_\tau - t)\big), \\ L^\tau_t &:= l(U_\tau) - l(U_\tau - t), \qquad\qquad 0 \leq t \leq h_\tau, \\ \hat{L}^\tau_t &:= l(U_\tau) - \hat{l}(U_\tau - t), \end{aligned} \tag{4.89}$$

where U_τ is as in (4.88). Note that modulo translation and time reversal, the triple $(\tilde{R}^\tau, L^\tau, \hat{L}^\tau)$ is just $(\hat{r}^{[U_\tau]}, l, \hat{l})$ during the time interval $[U_\tau - h_\tau, U_\tau]$ when $\hat{l}$ and l make an excursion away from each other, and $\hat{r}^{[U_\tau]}$ coalesces with $\hat{l}$ upon first hitting $\hat{l}$. Let N^3_u be the random subset of $\mathscr{E}^3 \times (0, \Psi_u)$ defined by

$$N^3_u := \big\{(\tilde{R}^\tau, L^\tau, \hat{L}^\tau, h_\tau, \tau) : \tau \in \mathscr{T}_u\big\}. \tag{4.90}$$

We will show that N^3_u can be extended to a Poisson point process N^3 on $\mathscr{E}^3 \times (0, \infty)$. Proposition 4.1 will then be established by showing that N^3_u contains infinitely many points $(\tilde{R}^\tau, L^\tau, \hat{L}^\tau, h_\tau, \tau)$ with the property that $\tilde{R}^\tau$ crosses L^τ before h_τ. (Note that since we are only interested in the time when $\hat{r}^{[U_\tau]}$ crosses l, there is no need to follow $\hat{r}^{[U_\tau]}$ after it meets $\hat{l}$. This is why we have defined $\tilde{R}^\tau_t$ in such a way that it coalesces with $\hat{L}^\tau_t$.)

Proof of Lemma 4.2 Set

$$\begin{aligned} L_t &:= l(u) - l(u - t), \\ \hat{L}_t &:= l(u) - \hat{l}(u - t) \end{aligned} \qquad (t \in [0, u]). \tag{4.91}$$

We know from [STW00] (see also Lemma 2.1 and [SS08, formula (6.17)]) that conditioned on l, the dual path $\hat{l}$ is distributed as a Brownian motion with drift -1, Skorohod reflected off l. Therefore, on $[0, u]$, the paths L and $\hat{L}$ are distributed as solutions to the SDE

$$\begin{aligned} \mathrm{d}L_t &= \mathrm{d}B^{\mathrm{l}}_t - \mathrm{d}t, \\ \mathrm{d}\hat{L}_t &= \mathrm{d}B^{\hat{\mathrm{l}}}_t - \mathrm{d}t + \mathrm{d}\Phi_t, \end{aligned} \tag{4.92}$$

where B^{l}_t and $B^{\hat{\mathrm{l}}}_t$ are independent, standard Brownian motions, Φ_t is a nondecreasing process, increasing only when $L_t = \hat{L}_t$, and one has $L_t \le \hat{L}_t$ for all $t \in [0,u]$. Extending our probability space if necessary, we may extend solutions of (4.92) so that they are defined for all $t \ge 0$. Set

$$\begin{aligned} B^-_t &:= \tfrac{1}{\sqrt{2}}\big(B^{\hat{\mathrm{l}}}_t - B^{\mathrm{l}}_t\big), & B^+_t &:= \tfrac{1}{\sqrt{2}}\big(B^{\hat{\mathrm{l}}}_t + B^{\mathrm{l}}_t\big), \\ X_t &:= \tfrac{1}{\sqrt{2}}\big(\hat{L}_t - L_t\big), & Y_t &:= \tfrac{1}{\sqrt{2}}\big(\hat{L}_t + L_t\big). \end{aligned} \tag{4.93}$$

Then B^-_t and B^+_t are independent standard Brownian motions,

$$\begin{aligned} X_t &= B^-_t + \Psi_t, \\ Y_t &= B^+_t - \sqrt{2}t + \Psi_t, \end{aligned} \tag{4.94}$$

where $X_t \ge 0$ and $\Psi_t := \frac{1}{\sqrt{2}}\Phi_t$ increases only when $X_t = 0$. In particular, setting $B := B^-$ and noting that X in (4.94) solves a Skorohod equation, the claims in Lemma 4.2 then follow. ∎

For each $\tau \in \mathscr{T}$, we define $X^\tau \in \mathscr{E}_{h_\tau}$ by

$$X^\tau(t) := X_{S_{\tau-}+t} \qquad (t \in [0,h_\tau]), \tag{4.95}$$

and we define a point process N^1 on $\mathscr{E}^1 \times [0,\infty)$ by

$$N^1 := \big\{(X^\tau, h_\tau, \tau) : \tau \in \mathscr{T}\big\}. \tag{4.96}$$

The following facts are well-known.

Lemma 4.5. [Excursions of reflected Brownian motion]
There exists a σ-finite measure μ^1 on $\mathscr{E}^1$ such that N^1 is a Poisson point process on $\mathscr{E}^1 \times [0,\infty)$ with intensity measure $\mu^1(\mathrm{d}(f,h))\mathrm{d}\tau$. The measure μ^1 may be written as

$$\mu^1(\mathrm{d}(f,h)) = \mu^1_h(\mathrm{d}f)\nu(\mathrm{d}h), \tag{4.97}$$

where ν is the measure in (4.86) and the μ^1_h are probability measures on $\mathscr{E}^1_h$ $(h>0)$. There exists a random function $F : [0,1] \to [0,\infty)$ with $F(0) = 0 = F(1)$ and $F > 0$ on $(0,1)$, such that

$$\mu^1_h = \mathbb{P}[F_h \in \cdot\,] \quad \textit{where} \quad F_h(t) := \sqrt{h}F(t/h) \qquad (t \in [0,h]). \tag{4.98}$$

Proof The existence of the excursion measure μ^1 follows from general excursion theory, see [Ber96, Chapter IV] or [RW94, Chapter VI.8]; a precise description of μ^1 for reflected Brownian motion can be found in [RW94, Sect. VI.55]. Since ν is the marginal of the measure μ^1, it has to be the measure in (4.86). Formula (4.98) is a result of Brownian scaling. ∎

Let $(L,\hat{L})$ be the solution to (4.92), extended to all $t \ge 0$. For each $\tau \in \mathscr{T}$, we define $(L^\tau, \hat{L}^\tau) \in \mathscr{E}^2_{h_\tau}$ by

$$\begin{aligned} L^\tau(t) &:= L_{S_{\tau-}+t} - L_{S_{\tau-}}, \\ \hat{L}^\tau(t) &:= \hat{L}_{S_{\tau-}+t} - L_{S_{\tau-}}, \end{aligned} \qquad t \in [0,h_\tau], \tag{4.99}$$

and we define a point process N^2 on $\mathscr{E}^2 \times [0,\infty)$ by

$$N^2 := \big\{(L^\tau, \hat{L}^\tau, h_\tau, \tau) : \tau \in \mathscr{T}\big\}. \tag{4.100}$$

Lemma 4.6. [Excursions between a left and dual left path]
The set N^2 is a Poisson point process on $\mathscr{E}^2 \times [0,\infty)$ with intensity $\mu_h^2(\mathrm{d}f)\nu(\mathrm{d}h)$, where ν is the measure in (4.86) and the μ_h^2 are probability measures on $\mathscr{E}_h^2$ $(h > 0)$ given by

$$\mu_h^2 = \mathbb{P}[(F_h^-, F_h^+) \in \cdot\,] \quad \textit{with} \quad F_h^{\pm}(t) := B_t - t \pm \tfrac{1}{\sqrt{2}} F_h(t) \quad (t \in [0,h]), \tag{4.101}$$

where F_h is a random variable as in (4.98) and B a Brownian motion independent of F_h.

Proof This follows from the fact that, by (4.93) and (4.94),

$$\begin{aligned} L_t &= \tfrac{1}{\sqrt{2}}\big(Y_t - X_t\big) = B_t^+ - t + \tfrac{1}{\sqrt{2}}\Psi_t - \tfrac{1}{\sqrt{2}}X_t, \\ \hat{L}_t &= \tfrac{1}{\sqrt{2}}\big(Y_t + X_t\big) = B_t^+ - t + \tfrac{1}{\sqrt{2}}\Psi_t + \tfrac{1}{\sqrt{2}}X_t, \end{aligned} \tag{4.102}$$

where B^+ is a standard Brownian motion independent of X and Ψ. Note that restrictions of B^+ to disjoint excursion intervals are independent and that, since Ψ increases only at times t when $L_t = \hat{L}_t$, it drops out of the formulas for L^τ and $\hat{L}^\tau$. ∎

Remark General excursion theory tells us how a strong Markov process can be constructed by piecing together its excursions from a singleton. In our situation, however, we are interested in excursions of the process $(L_t, \hat{L}_t)$ from the set $\{(x,x) : x \in \mathbb{R}\}$, which is not a singleton. Formula (4.102) shows that apart from motion during the excursions, the process $(L_t, \hat{L}_t)$ also moves along the diagonal at times when $L_t = \hat{L}_t$, even though such times have zero Lebesgue measure. (Indeed, it is possible to reconstruct $(L_t, \hat{L}_t)$ from its excursions and local time in $\{(x,x) : x \in \mathbb{R}\}$, but we do not pursue this here.)

Together with Lemma 4.6, the next lemma implies that the point process N_u^3 on $\mathscr{E}^3 \times (0, \Psi_u)$ defined in (4.90) is a Poisson point process, as claimed.

Lemma 4.7. [Distribution of crossing times]
The paths $(\tilde{R}^\tau)_{\tau \in \mathscr{T}_u}$ are conditionally independent given l and $\hat{l}$, and their conditional law up to coalescence with $\hat{L}^\tau$ is given by the solution to the Skorohod equation

$$\begin{aligned} \mathrm{d}\tilde{R}_t^\tau &= \mathrm{d}B_t + \mathrm{d}t - \mathrm{d}\Delta_t, \qquad && \Delta_t < T, \\ \mathrm{d}\tilde{R}_t^\tau &= \mathrm{d}B_t + \mathrm{d}t + \mathrm{d}\Delta_t, \qquad && T \le \Delta_t, \end{aligned} \qquad 0 \le t \le h_\tau \wedge \inf\{s \ge 0 : \tilde{R}_s^\tau = \hat{L}_s^\tau\}, \tag{4.103}$$

where B is a standard Brownian motion, Δ is a nondecreasing process increasing only when $\tilde{R}_t^\tau = L_t^\tau$, T is an independent mean $1/2$ exponential random variable, and $\tilde{R}^\tau$ is subject to the constraints that $\tilde{R}_t^\tau \le L_t^\tau$ resp. $L_t^\tau \le \tilde{R}_t^\tau$ when $\Delta_t < T$ resp. $T \le \Delta_t$.

We will prove Lemmas 4.4 and 4.7 in one stroke.

Proof of Lemmas 4.4 and 4.7 We start by showing that almost surely, at each $s_+ \in \mathscr{I}_+(l, \hat{l})$ there starts at most one $\hat{r}^{[s_+]} \in \hat{\mathscr{W}}^{\mathrm{r}}(l(s_+), s_+)$ such that $l \le \hat{r}^{[s_+]}$ on $[s_+ - \varepsilon, s_+]$ for some $\varepsilon > 0$. Indeed, since dual right-most paths cannot cross l from left to right [SS08, Prop. 3.6 (d)], if there is more than one dual right-most path starting at $(l(s_+), s_+)$ on the right of l, then there start at least three dual right-most paths at this point. It follows that $(l(s_+), s_+)$ is a meeting point of $\mathscr{W}^{\mathrm{r}}$ and hence, by Lemma 3.12, also a meeting point of $\mathscr{W}^{\mathrm{l}}$. This contradicts the existence of an incoming dual left-most path (see Theorem 1.6).

To prove the other statements we use discrete approximation (compare the proof of Lemma 2.1). As in [SS08], we consider systems of branching-coalescing random walks on $\mathbb{Z}^2_{\rm even}$ with branching probabilities $\varepsilon_n \to 0$. Diffusively rescaling space and time as $(x,t) \mapsto (\varepsilon_n x, \varepsilon_n^2 t)$ then yields the Brownian net in the limit. In the discrete system, we consider the left-most path l_n starting at the origin, we choose $u_n \in \mathbb{N}$ such that $\varepsilon_n u_n \to u$ and consider the dual left-most path $\hat{l}_n$ started at time u_n at distance one to the left of l_n. For each $0 \leq i \leq u_n$, we let $i^+ := \inf\{j \geq i : l_n(j) - \hat{l}_n(j) = 1\}$ and we let $\hat{r}_n(i)$ denote the position at time i of the dual right-most path started at time i^+ at distance one on the right of l_n. Then $\hat{r}_n$ is the concatenation of dual right-most paths, started anew immediately on the right of l_n each time $\hat{l}_n$ is at distance one from l_n. In analogy with the definition of $\tilde{R}^\tau$ in (4.89), we set $\tilde{r}_n(i) := \hat{l}_n(i) \vee \hat{r}_n(i)$.

Now $(l_n, \hat{l}_n)$, diffusively rescaled, converges in distribution to $(l, \hat{l})$ where l is the left-most path in the Brownian net starting at the origin and $\hat{l}$ is the a.s. unique dual left-most path starting at $(l(u), u)$ that lies on the left of l. Moreover, the set of times when $\hat{l}_n$ is at distance one from l_n, diffusively rescaled, converges to the set $\{s \in (0,u] : \hat{l}(s) = l(s)\}$. (This follows from the fact that the reflection local time of $\hat{l}_n$ off l_n converges to its continuum analogue, and the latter increases whenever $\hat{l}(s) = l(s)$.)

In the diffusive scaling limit, the path $\tilde{r}_n$ converges to a path $\tilde{r}$ such that $t \mapsto \tilde{r}(u-t)$ is right-continuous and is set back to $l(u-t)$ each time $\hat{l}(u-t)$ meets $l(u-t)$. Between these times, in the same way as in the proof of Lemma 2.1, we see that the conditional law of $\tilde{r}$ given l and $\hat{l}$ is as described in Lemma 4.7.

Since $\tilde{r}_n$ is the concatenation of dual right-most paths, we see that at each time $s_+ \in \mathscr{I}_+(l, \hat{l})$ there starts at least one dual right-most path that lies on the right of l on $[s_+ - \varepsilon, s_+]$ for some $\varepsilon > 0$, completing the proof of Lemma 4.4. ∎

Proof of Proposition 4.1 We first prove the claims for the a.s. unique paths $l \in \mathscr{W}^{\rm l}(0,0)$ and $\hat{l} \in \hat{\mathscr{W}}^{\rm l}(l(u), u)$ such that $\hat{l}$ lies on the left of l. For each $\tau \in \mathscr{T}_u$, set

$$C_\tau := h_\tau \wedge \inf\{t \in [0, h_\tau] : L^\tau(t) < \tilde{R}^\tau(t)\}. \tag{4.104}$$

Let N be the Poisson point process in (4.85), let N_u denote the restriction of N to $(0,\infty) \times (0, \Psi_u)$, and set

$$N'_u := \{(h_\tau, \tau) : \tau \in \mathscr{T}_u,\ C_\tau < h_\tau\}. \tag{4.105}$$

Then N'_u is a thinning of N_u, obtained by independently keeping a point $(h_\tau, \tau) \in N_u$ with probability $\rho(h_\tau)$, where $\rho : (0,\infty) \to (0,\infty)$ is some function. Indeed, by Lemmas 4.6 and 4.7, $\rho(h)$ has the following description. Pick a random variable F_h as in Lemma 4.5, two standard Brownian motions B, B', and a mean $1/2$ exponential random variable T, independent of each other. Set

$$L(t) := B_t - t - \tfrac{1}{\sqrt{2}} F_h \qquad (t \in [0,h]). \tag{4.106}$$

Let (R', Δ) be the solution to the Skorohod equation

$$\mathrm{d}R'_t = \mathrm{d}B'_t + \mathrm{d}t - \mathrm{d}\Delta_t \qquad (0 \leq t \leq h), \tag{4.107}$$

reflected to the left off L, and set

$$C := h \wedge \inf\{t \in [0,h] : \Delta_t \geq T\}. \tag{4.108}$$

Then

$$\rho(h) = \mathbb{P}[C < h] = \mathbb{E}[1 - e^{-2\Delta_h}] \geq \mathbb{E}[1 - e^{-2\Delta'_h}] =: \rho'(h), \tag{4.109}$$

where

$$\begin{aligned}\Delta_h &= \sup_{t\in[0,h]} \big(B'_t + t - L_t\big) = \sup_{t\in[0,h]} \big(B'_t - B_t + 2t + \tfrac{1}{\sqrt{2}} F_h(t)\big) \\ &\geq \sup_{t\in[0,h]} \big(B'_t - B_t + \tfrac{1}{\sqrt{2}} F_h(t)\big) =: \Delta'_h.\end{aligned} \tag{4.110}$$

It follows from Brownian scaling (see (4.98)) that

$$\rho'(h) = \mathbb{E}[1 - e^{-2\sqrt{h}\Delta'_1}], \tag{4.111}$$

hence, since $h^{-1/2}(1 - e^{-2\sqrt{h}\Delta'_1}) \uparrow 2\Delta'_h$ as $h \downarrow 0$,

$$\lim_{h\to 0} h^{-1/2} \rho'(h) = 2\mathbb{E}[\Delta'_1] > 0. \tag{4.112}$$

By (4.86), it follows that, for some $c > 0$,

$$\int_{0+} \nu(\mathrm{d}h)\rho(h) \geq c \int_{0+} h^{-3/2} h^{1/2} \mathrm{d}h = \infty, \tag{4.113}$$

i.e., the intensity measure of the thinned Poisson point process is not integrable, hence the set $\mathscr{T}'_u := \{\tau : (h_\tau, \tau) \in N'_u\}$ is a dense subset of $(0, \Psi_u)$, hence $\mathscr{I}'_+(l, \hat{l})$ is dense in $\mathscr{I}(l, \hat{l})$. This completes the proof for the special paths $l \in \mathscr{W}^{\mathrm{l}}(0,0)$ and $\hat{l} \in \hat{\mathscr{W}}^{\mathrm{l}}(l(u), u)$ such that $\hat{l}$ lies on the left of l.

To prove the same statement for $l \in \mathscr{W}^{\mathrm{l}}(0,0)$ and $\hat{r} \in \hat{\mathscr{W}}^{\mathrm{r}}(l(u), u)$ where $\hat{r}$ lies on the left of l, first note that because u is deterministic, Lemma 2.1 implies the existence of such an $\hat{r}$. Set

$$\hat{R}_t := l(u) - \hat{r}(u - t) \qquad (t \in [0, u]), \tag{4.114}$$

and let L be as in (4.91). Then, by Lemma 2.1, L and $\hat{R}$ are distributed as solutions to the SDE (compare (4.92))

$$\begin{aligned}\mathrm{d}L_t &= \mathrm{d}B^{\mathrm{l}}_t - \mathrm{d}t, \\ \mathrm{d}\hat{R}_t &= \mathrm{d}B^{\hat{\mathrm{r}}}_t + \mathrm{d}t + \mathrm{d}\Phi_t,\end{aligned} \tag{4.115}$$

where B^{l}_t and $B^{\hat{\mathrm{r}}}_t$ are independent, standard Brownian motions, Φ_t is a nondecreasing process, increasing only when $L_t = \hat{R}_t$, and $L_t \leq \hat{R}_t$ for all $t \in [0, u]$. By Girsanov, solutions of (4.115) are equivalent in law to solutions of (4.92), hence we can reduce this case to the case of a dual left-most path. In particular, by what we have already proved, almost surely for each $s_+ \in \mathscr{I}_+(l, \hat{r})$ there exists a unique $\hat{r}^{[s_+]} \in \hat{\mathscr{W}}^{\mathrm{r}}(l(s_+), s_+)$ that lies on the right of l, and the set

$$\mathscr{I}'_+(l, \hat{r}) := \big\{s_+ \in \mathscr{I}_+(l, \hat{r}) : \hat{r}^{[s_+]} \text{ crosses } l \text{ at some time in } (s_-, s_+)\big\} \tag{4.116}$$

is a dense subset of $\mathscr{I}(l, \hat{r})$.

By translation invariance, $\mathscr{I}'_+(l, \hat{l})$ is dense in $\mathscr{I}(l, \hat{l})$ for each $l \in \mathscr{W}^{\mathrm{l}}$ started from a point $z = (x, t) \in \mathbb{Q}^2$ and $\hat{l} \in \hat{\mathscr{W}}^{\mathrm{l}}(l(u), u)$ that lies on the left of l. By [SS08, Lemma 3.4 (b)], we can generalize this to arbitrary $l \in \mathscr{W}^{\mathrm{l}}$. Since any dual left-most path that hits l from the left at a time s must have coalesced with some left-most path started in $(l(u), u)$ for some $u \in \mathbb{Q}$ with $u > s$ and lying on the left of l, we can generalize our statement to arbitrary $l \in \mathscr{W}^{\mathrm{l}}$ and $\hat{l} \in \hat{\mathscr{W}}^{\mathrm{l}}$. The argument for dual right-most paths is the same. ∎

4.2 Excursions around hitting points

With the help of Proposition 4.1, we can prove the following result. Recall the definition of $\mathscr{I}(\pi,\hat{\pi})$ from (4.80).

Proposition 4.8. [Excursions around intersection points]
Almost surely, for each $l \in \mathscr{W}^{\mathrm{l}}$ and $\hat{l} \in \hat{\mathscr{W}}^{\mathrm{l}}$ such that $\sigma_l < \hat{\sigma}_{\hat{l}}$ and $\hat{l} \leq l$ on $[\sigma_l, \hat{\sigma}_{\hat{l}}]$, the set

$$\begin{aligned}\mathscr{I}''(l,\hat{l}) := \{t \in \mathscr{I}(l,\hat{l}) : \exists r \in \mathscr{W}^{\mathrm{r}} \text{ s.t. } r \text{ makes an excursion from } l \\ \text{during a time interval } (s,u) \text{ with } \sigma_l < s < t < u < \hat{\sigma}_{\hat{l}}\}\end{aligned} \tag{4.117}$$

is a dense subset of $\mathscr{I}(l,\hat{l})$. The same statements hold when $\hat{l}$ is replaced by a path $\hat{r} \in \hat{\mathscr{W}}^{\mathrm{r}}$.

Proof Since the set $\mathscr{I}'_+(l,\hat{l})$ from (4.81) is dense in $\mathscr{I}(l,\hat{l})$, for each $t \in \mathscr{I}(l,\hat{l})\backslash\mathscr{I}_+(l,\hat{l})$ we can find $s_+^{(n)} \in \mathscr{I}'_+(l,\hat{l})$ such that $s_+^{(n)} \uparrow t$. Our claim will follow provided we show that infinitely many of the $s_+^{(n)}$ are in $\mathscr{I}''(l,\hat{l})$. It suffices to show that at least one $s_+^{(n)}$ is in $\mathscr{I}''(l,\hat{l})$; then the same argument applied to the sequence started after $s_+^{(n)}$ gives the existence of another such point, and so on, ad infinitum. So imagine that $s_+^{(n)} \notin \mathscr{I}''(l,\hat{l})$ for all n. Let $z_n = (l(c_n), c_n)$, with $c_n \in (s_-^{(n)}, s_+^{(n)})$, be the point where $\hat{r}^{[t_n]}$ crosses l. By Proposition 2.6, there exists an incoming path $r_n \in \mathscr{W}^{\mathrm{r}}$ at z_n such that r_n separates from l in z_n. Set $\tau_n := \inf\{s > c_n : l(s) = r_n(s)\}$. If $\tau_n < s_+^{(n)}$ then l and r_n form a wedge of $(\mathscr{W}^{\mathrm{l}}, \mathscr{W}^{\mathrm{r}})$ which cannot be entered by $\hat{r}^{[t_n]}$, leading to a contradiction. If $\tau_n = s_+^{(n)}$, then $s_+^{(n)}$ is a meeting point of l and r_n, hence by Lemma 3.12, $(l(s_+^{(n)}), s_+^{(n)})$ is of type $(2,1)/(0,3)$ in $\mathscr{W}^{\mathrm{l}}$, which contradicts the existence of the dual incoming path $\hat{l}$. Finally, we cannot have $\tau_n \in (s_+^{(n)}, \hat{\sigma}_{\hat{l}})$ because of our assumption that $s_+^{(n)} \notin \mathscr{I}''(l,\hat{l})$, so we conclude that $\tau_n \geq \sigma_{\hat{l}}$ for all n. It follows that the z_n are $(\sigma_l, \hat{\sigma}_{\hat{l}})$-relevant separation points, contradicting Lemma 2.11. ∎

5 Structure of special points

5.1 Classification of special points

Recall the preliminary classification of points in the Brownian net given in Section 3.3, which is based only on the structure of incoming paths. In this section, we turn our attention to the more detailed classification from Theorem 1.7, which also uses information about outgoing paths that are not continuations of incoming paths. We start with a preliminary lemma.

Lemma 5.1. [No incoming paths]
Almost surely for each $z = (x,t) \in \mathbb{R}^2$, there is no incoming path $\pi \in \mathscr{N}$ at z if and only if $\hat{\mathscr{W}}^{\mathrm{r}}(z)$ and $\hat{\mathscr{W}}^{\mathrm{l}}(z)$ each contain a single path, $\hat{r}$ and $\hat{l}$, say, and $\hat{r} \sim_{\mathrm{out}}^{z} \hat{l}$.

Proof Let $\hat{r}$ be the left-most element of $\hat{\mathscr{W}}^{\mathrm{r}}(z)$ and let $\hat{l}$ be the righ-most element of $\hat{\mathscr{W}}^{\mathrm{l}}(z)$. Then $\hat{r} \leq \hat{l}$, and by Lemma 2.7, there is an incoming path $\pi \in \mathscr{N}$ at z if and only if $\hat{r} < \hat{l}$ on $(t-\varepsilon, t)$ for some $\varepsilon > 0$. ∎

Theorem 1.7 follows from the following result.

Theorem 5.2. [Classification of points in the Brownian net]
Let $\mathcal{N}$ be the standard Brownian net and let $\hat{\mathcal{N}}$ be its dual. Then, using the classification of Definition 3.8, almost surely, each point in $\mathbb{R}^2$ is of one of the following 19 types in $\mathcal{N}/\hat{\mathcal{N}}$:

(1) $(\mathrm{C_o})/(\mathrm{C_o})$, $(\mathrm{C_o})/(\mathrm{C_p})$, $(\mathrm{C_p})/(\mathrm{C_o})$, $(\mathrm{C_o})/(\mathrm{C_m})$, $(\mathrm{C_m})/(\mathrm{C_o})$, $(\mathrm{C_p})/(\mathrm{C_p})$;

(2) $(\mathrm{C_s})/(\mathrm{C_s})$;

(3) $(\mathrm{C_l})/(\mathrm{C_o})$, $(\mathrm{C_o})/(\mathrm{C_l})$, $(\mathrm{C_r})/(\mathrm{C_o})$, $(\mathrm{C_o})/(\mathrm{C_r})$;

(4) $(\mathrm{C_l})/(\mathrm{C_p})$, $(\mathrm{C_p})/(\mathrm{C_l})$, $(\mathrm{C_r})/(\mathrm{C_p})$, $(\mathrm{C_p})/(\mathrm{C_r})$;

(5) $(\mathrm{C_l})/(\mathrm{C_l})$, $(\mathrm{C_r})/(\mathrm{C_r})$;

(6) $(\mathrm{C_n})/(\mathrm{C_o})$, $(\mathrm{C_o})/(\mathrm{C_n})$;

and all of these types occur. Moreover, these points correspond to the types listed in Theorem 1.7 (in the same order), where points of type $(\mathrm{C_p})/(\mathrm{C_p})$ are either of type $(\mathrm{p},\mathrm{pp})_\mathrm{l}/(\mathrm{p},\mathrm{pp})_\mathrm{l}$ or $(\mathrm{p},\mathrm{pp})_\mathrm{r}/(\mathrm{p},\mathrm{pp})_\mathrm{r}$. For each deterministic time $t\in\mathbb{R}$, almost surely, each point in $\mathbb{R}\times\{t\}$ is of either type $(\mathrm{C_o})/(\mathrm{C_o})$, $(\mathrm{C_o})/(\mathrm{C_p})$, or $(\mathrm{C_p})/(\mathrm{C_o})$, and all of these types occur. A deterministic point $(x,t)\in\mathbb{R}^2$ is almost surely of type $(\mathrm{C_o})/(\mathrm{C_o})$.

For clarity, we split the proof into three lemmas.

Lemma 5.3. [Forward and dual types]
Almost surely, for all $z\in\mathbb{R}^2$:

(a) *If z is of type $(\mathrm{C_s})$ in $\mathcal{N}$, then z is of type $(\mathrm{C_s})$ in $\hat{\mathcal{N}}$.*

(b) *If z is of type $(\mathrm{C_m})$ in $\mathcal{N}$, then z is of type $(\mathrm{C_o})$ in $\hat{\mathcal{N}}$.*

(c) *If z is of type $(\mathrm{C_n})$ in $\mathcal{N}$, then z is of type $(\mathrm{C_o})$ in $\hat{\mathcal{N}}$.*

(d) *If z is of type $(\mathrm{C_l})$ in $\mathcal{N}$, then z is not of type $(\mathrm{C_r})$ in $\hat{\mathcal{N}}$.*

In particular, each point in $\mathbb{R}^2$ is of one of the 19 types listed in Theorem 5.2.

Lemma 5.4. [Existence of types]
Almost surely, all types of points listed in Theorem 5.2 occur. For each deterministic time $t\in\mathbb{R}$, almost surely, each point in $\mathbb{R}\times\{t\}$ is of either type $(\mathrm{C_o})/(\mathrm{C_o})$, $(\mathrm{C_o})/(\mathrm{C_p})$, or $(\mathrm{C_p})/(\mathrm{C_o})$, and all of these types occur. A deterministic point $(x,t)\in\mathbb{R}^2$ is almost surely of type $(\mathrm{C_o})/(\mathrm{C_o})$.

Lemma 5.5. [Structure of points]
Almost surely, with respect to $\mathcal{N}/\hat{\mathcal{N}}$ and for all $z\in\mathbb{R}^2$:

(a) *If z is of type $(\mathrm{C_o})/(\mathrm{C_o})$, then z is of type $(\mathrm{o},\mathrm{p})/(\mathrm{o},\mathrm{p})$.*

(b) *If z is of type $(\mathrm{C_p})/(\mathrm{C_o})$, then z is of type $(\mathrm{p},\mathrm{p})/(\mathrm{o},\mathrm{pp})$.*

(c) *If z is of type $(\mathrm{C_m})/(\mathrm{C_o})$, then z is of type $(\mathrm{pp},\mathrm{p})/(\mathrm{o},\mathrm{ppp})$.*

(d) *If z is of type $(\mathrm{C_p})/(\mathrm{C_p})$, then z is of type $(\mathrm{p},\mathrm{pp})_\mathrm{l}/(\mathrm{p},\mathrm{pp})_\mathrm{l}$ or $(\mathrm{p},\mathrm{pp})_\mathrm{r}/(\mathrm{p},\mathrm{pp})_\mathrm{r}$.*

(e) *If z is of type* $(\mathrm{C_s})/(\mathrm{C_s})$*, then z is of type* $(\mathrm{p},\mathrm{pp})_\mathrm{s}/(\mathrm{p},\mathrm{pp})_\mathrm{s}$.

(f) *If z is of type* $(\mathrm{C_l})/(\mathrm{C_o})$*, then z is of type* $(\mathrm{l},\mathrm{p})/(\mathrm{o},\mathrm{lp})$.

(g) *If z is of type* $(\mathrm{C_l})/(\mathrm{C_p})$*, then z is of type* $(\mathrm{l},\mathrm{pp})_\mathrm{r}/(\mathrm{p},\mathrm{lp})_\mathrm{r}$.

(h) *If z is of type* $(\mathrm{C_l})/(\mathrm{C_l})$*, then z is of type* $(\mathrm{l},\mathrm{lp})_\mathrm{r}/(\mathrm{l},\mathrm{lp})_\mathrm{r}$.

(i) *If z is of type* $(\mathrm{C_n})/(\mathrm{C_o})$*, then z is of type* $(\mathrm{n},\mathrm{p})/(\mathrm{o},\mathrm{lr})$.

Analogous statements hold for the remaining types in Theorem 5.2 by left-right and forward-backward symmetry.

Proof of Lemma 5.3 Part (a) follows from Proposition 2.6. If z is of type $(\mathrm{C_m})$ in $\mathscr{N}$, then by Lemma 3.12, there is a single outgoing left-right pair at z, hence by Lemma 5.1, z is of type $(\mathrm{C_o})$ in $\hat{\mathscr{N}}$, proving part (b). If z is of type $(\mathrm{C_n})$ in $\mathscr{N}$, then each outgoing Brownian net path at z must pass through the top points of the nested excursions around z as described in Proposition 3.11 (a). Hence also in this case, there is a single outgoing left-right pair at z, so by Lemma 5.1, z is of type $(\mathrm{C_o})$ in $\hat{\mathscr{N}}$, proving part (c). To prove part (d), suppose that z is of type $(\mathrm{C_l})$ and that $\hat{r} \in \hat{\mathscr{W}}^{\mathrm{r}}$ enters z. By Proposition 3.11 (a), there exist right-most paths r_n making a sequence of nested excursions away from l. Since each r_n forms with l a wedge of $(\mathscr{W}^{\mathrm{l}},\mathscr{W}^{\mathrm{r}})$ that cannot be entered by $\hat{r}$, the latter must satisfy $\hat{r} \leq l$ on $(t, t+\varepsilon)$ for some $\varepsilon > 0$. Since $\hat{r}$ reflects off l, we can find some $u > t$ such that $\hat{r}(u) < l(u)$. Now any path in $\hat{\mathscr{W}}^{\mathrm{l}}$ started in $(\hat{r}(u), l(u)) \times \{u\}$ must enter z, hence z is not of type $(\mathrm{C_r})$ in $\hat{\mathscr{N}}$. It is easy to check that (a)–(d) rules out all but the 19 types listed in Theorem 5.2. ∎

Proof of Lemma 5.4 It follows from Lemma 3.9 that each deterministic $z \in \mathbb{R}^2$ is of type $(\mathrm{C_o})/(\mathrm{C_o})$, and that for each deterministic $t \in \mathbb{R}$, a.s. each point in $\mathbb{R} \times \{t\}$ is of one of the types $(\mathrm{C_o})/(\mathrm{C_o})$, $(\mathrm{C_p})/(\mathrm{C_o})$, $(\mathrm{C_o})/(\mathrm{C_p})$, and $(\mathrm{C_p})/(\mathrm{C_p})$. By the structure of the Brownian web (see, e.g., [SS08, Lemma 3.3]), points of type $(\mathrm{C_p})/(\mathrm{C_p})$ do not occur at deterministic times. Since points of type $(\mathrm{C_p})$ do occur at deterministic times both in $\mathscr{N}$ and in $\mathscr{N}$, we conclude that a.s. each point in $\mathbb{R} \times \{t\}$ is of one of the types $(\mathrm{C_o})/(\mathrm{C_o})$, $(\mathrm{C_p})/(\mathrm{C_o})$, and $(\mathrm{C_o})/(\mathrm{C_p})$, and all of these types occur.

To prove the existence of all 19 types of points listed in Theorem 5.2, by symmetry between left and right and between forward and dual paths, it suffices to prove the existence of the 9 types of points listed in Lemma 5.5. We have just established the existence of points of types $(\mathrm{C_o})/(\mathrm{C_o})$ and $(\mathrm{C_p})/(\mathrm{C_o})$. The existence of points of types $(\mathrm{C_s})/(\mathrm{C_s})$, $(\mathrm{C_m})/(\mathrm{C_o})$, and $(\mathrm{C_n})/(\mathrm{C_o})$ follows from Lemma 5.3 and the existence of points of types $(\mathrm{C_s})$, $(\mathrm{C_m})$, and $(\mathrm{C_n})$ in $\mathscr{N}$. Hence, we are left with the task of establishing the existence of points of types $(\mathrm{C_p})/(\mathrm{C_p})$, $(\mathrm{C_l})/(\mathrm{C_o})$, $(\mathrm{C_l})/(\mathrm{C_p})$, and $(\mathrm{C_l})/(\mathrm{C_l})$.

By the structure of the Brownian web, there exist $l \in \mathscr{W}^{\mathrm{l}}$ and $\hat{l} \in \hat{\mathscr{W}}^{\mathrm{l}}$ such that $l \leq \hat{l}$ on $(\sigma_l, \hat{\sigma}_{\hat{l}})$ and the set $\mathscr{I}(l,\hat{l})$ defined in (4.80) is not empty. Since $\mathscr{I}(l,\hat{l})$ is uncountable and since the set of all crossing points in $\mathbb{R}^2$ is countable, there exist lots of points $z = (x,t) \in \mathscr{I}(l,\hat{l})$ such that no path in $\hat{\mathscr{W}}^{\mathrm{r}}$ crosses l and no path in $\mathscr{W}^{\mathrm{r}}$ crosses $\hat{l}$ at z. For such points, we can find $r \in \mathscr{W}^{\mathrm{r}}$ and $\hat{r} \in \hat{\mathscr{W}}^{\mathrm{r}}$ such that $l \leq r \leq \hat{r} \leq \hat{l}$ on $[t-\varepsilon, t+\varepsilon]$ for some $\varepsilon > 0$. Of all types of points listed in Theorem 5.2, only points of type $(\mathrm{C_p})/(\mathrm{C_p})$ have incoming paths in $\mathscr{W}^{\mathrm{l}}$, $\mathscr{W}^{\mathrm{r}}$, $\hat{\mathscr{W}}^{\mathrm{r}}$, and $\hat{\mathscr{W}}^{\mathrm{l}}$, hence, by Lemma 5.3, z must be of this type.

We are left with the task of establishing the existence of points of types $(C_l)/(C_o)$, $(C_l)/(C_p)$, and $(C_l)/(C_l)$. By Lemma 3.7 and Proposition 3.11 (b), a point z is of type (C_l) in $\mathscr{N}$ if and only if there is an incoming path $l \in \mathscr{W}^l$ at z and there are right-most paths r_n making a nested sequence of excursions from l, as described in Proposition 3.11 (b). We need to show that we can choose these nested excursions in such a way that z is of type (C_o), (C_p), or (C_l) in $\hat{\mathscr{N}}$.

Fix a path $l \in \mathscr{W}^l$ and let $\{\hat{l}_n\}_{n\in\mathbb{N}}$ be paths in $\hat{\mathscr{W}}^l$ starting from a deterministic countable dense subset of $\mathbb{R}^2$. Since $\hat{l}_1$ is reflected off l, using Lemma 3.10, we can find a path $r_1 \in \mathscr{W}^r$ such that r_1 makes an excursion from l during an interval (s_1, u_1) with $l_1 \neq l$ on $[s_1, u_1]$. By the same arguments, we can inductively find paths $r_n \in \mathscr{W}^r$ such that r_n makes an excursion from l during an interval (s_n, u_n) with $l_n \neq l$ on $[s_n, u_n] \subset (s_{n-1}, u_{n-1})$. We can choose the r_n such that $\bigcap_n [s_n, u_n] = \{t\}$ for some $t \in \mathbb{R}$. Then setting $z := (l(t), t)$ yields a point such that z is of type (C_l) in $\mathscr{N}$ and no path in $\hat{\mathscr{W}}^l$ enters z, hence (by Lemma 5.3 (d)), z must be of type $(C_l)/(C_o)$.

To construct points of type $(C_l)/(C_p)$, we fix paths $l \in \mathscr{W}^l$ and $\hat{r} \in \hat{\mathscr{W}}^r$ with $\hat{r} \leq l$ on $[\sigma_l, \hat{\sigma}_{\hat{r}}]$ such that the set $\mathscr{I}(l, \hat{r})$ defined in (4.80) is not empty. By Proposition 4.8 and Lemma 3.10, we can inductively find paths $r_n \in \mathscr{W}^r$ such that r_n makes an excursion from l during an interval (s_n, u_n) with $[s_n, u_n] \subset (s_{n-1}, u_{n-1})$ and $(s_n, u_n) \cap \mathscr{I}(l, \hat{r}) \neq \emptyset$. Choosing $\{t\} = \bigcap_n [s_n, u_n]$ and setting $z := (l(t), t)$ then yields a point of type (C_l) in $\mathscr{N}$ such that $\hat{r}$ enters z, hence (by Lemma 5.3 (d)), z must be of type $(C_l)/(C_p)$.

Finally, to construct points of type $(C_l)/(C_l)$, we fix paths $l \in \mathscr{W}^l$ and $\hat{l} \in \hat{\mathscr{W}}^r$ with $\hat{l} \leq l$ on $[\sigma_l, \hat{\sigma}_{\hat{l}}]$ such that the set $\mathscr{I}(l, \hat{l})$ defined in (4.80) is not empty. By Proposition 4.8 and Lemma 3.10, we can inductively find intervals (s_n, u_n) with $[s_n, u_n] \subset (s_{n-1}, u_{n-1})$ and $(s_n, u_n) \cap \mathscr{I}(l, \hat{l}) \neq \emptyset$, and paths $r_{2n} \in \mathscr{W}^r$ and $\hat{r}_{2n+1} \in \hat{\mathscr{W}}^r$ such that r_{2n} makes an excursion from l during (s_{2n}, u_{2n}) and $\hat{r}_{2n+1}$ makes an excursion from $\hat{l}$ during (s_{2n+1}, u_{2n+1}). Choosing $\{t\} = \bigcap_n [s_n, u_n]$ and setting $z := (l(t), t)$ then yields a point of type $(C_l)/(C_l)$. ∎

Proof of Lemma 5.5 Part (e) about separation points has already been proved in Proposition 2.6. It follows from Lemma 3.12 that at points of all types except (C_m), all incoming paths in $\mathscr{N}$ are equivalent. Therefore, points of type (C_o) are of type $(o, \cdot)$, points of type (C_n) are of type $(n, \cdot)$, points of type (C_l) are of type $(l, \cdot)$, and points of types (C_p) are of type $(p, \cdot)$, while by Lemma 3.12, points of type (C_m) are of type (pp, p). We next show that, the configuration of incoming paths in $\mathscr{N}$ at z determines the configuration of outgoing paths in $\hat{\mathscr{N}}$ at z:

(o) If z is of type (C_o) in $\mathscr{N}$, then z is of type $(\cdot, p)$ in $\hat{\mathscr{N}}$.

(p) If z is of type (C_p) in $\mathscr{N}$, then z is of type $(\cdot, pp)$ in $\hat{\mathscr{N}}$.

(m) If z is of type (C_m) in $\mathscr{N}$, then z is of type $(\cdot, ppp)$ in $\hat{\mathscr{N}}$.

(l) If z is of type (C_l) in $\mathscr{N}$, then z is of type $(\cdot, lp)$ in $\hat{\mathscr{N}}$.

(n) If z is of type (C_n) in $\mathscr{N}$, then z is of type $(\cdot, lr)$ in $\hat{\mathscr{N}}$.

Statement (o) follows from Lemma 5.1.

To prove statement (p), we observe that if $z = (x, t)$ is of type (C_p), then by Proposition 3.11 (e), for each $T < t$ with $z \in N_T$, there exist maximal T-meshes $M_T^- = M(r', l)$ and $M_T^+ = M(r, l')$ with bottom times strictly smaller than t and top times strictly larger than t, such that $l \sim_{\rm in}^z r$ and $l \sim_{\rm out}^z r$.

Therefore, by the structure of the Brownian web and ordering of paths, there exist $\hat{l}^-, \hat{l}^+ \in \hat{\mathscr{W}}^{\rm l}(z)$ and $\hat{r}^-, \hat{r}^+ \in \hat{\mathscr{W}}^{\rm r}(z)$ such that $r' \le \hat{r}^- \le \hat{l}^- \le l \le r \le \hat{r}^+ \le \hat{l}^+ \le l'$ on $[t-\varepsilon, t]$, for some $\varepsilon > 0$. By Lemma 3.1, the paths $\hat{l}^-$ and $\hat{r}^-$ both pass through the bottom point of M^-_T. Since T can be chosen arbitrarily close to t, we must have $\hat{l}^- \sim^z_{\rm out} \hat{r}^-$, and similarly $\hat{l}^+ \sim^z_{\rm out} \hat{r}^+$.

The proof of statement (m) is similar, where in this case we use the three maximal T-meshes from Proposition 3.11 (d).

If $z = (x, t)$ is of type $(\mathrm{C_l})$, then by the structure of the Brownian web and ordering of paths, there exist $\hat{l}^-, \hat{l}^+ \in \hat{\mathscr{W}}^{\rm l}(z)$ such that $\hat{l}^- \le l \le \hat{l}^+$, and there exists a unique $\hat{r} \in \hat{\mathscr{W}}^{\rm r}(z)$ with $\hat{r} \le \hat{l}^-$. By Lemma 3.7, z is isolated from the left in N_T for any $T < t$ with $z \in N_T$, hence by Proposition 3.5 (b), there exists a maximal T-mesh $M(r, l)$ with bottom time strictly smaller than t and top time strictly larger than t, such that $l(t) = x$. Now any path in $\hat{\mathscr{N}}$ on the left of l must exit $M(r, l)$ through its bottom point, hence the same argument as before shows that $\hat{l}^+ \sim^z_{\rm out} \hat{r}$, proving statement (l).

Finally, if $z = (x, t)$ is of type $(\mathrm{C_n})$, then by the structure of the Brownian web, $\hat{\mathscr{W}}^{\rm r}(z)$ and $\hat{\mathscr{W}}^{\rm l}(z)$ each contain a unique path, say $\hat{r}$ and $\hat{l}$. By Lemma 5.1, $\hat{r} \not\sim^z_{\rm out} \hat{l}$, so by ordering of paths, $\hat{r} < \hat{l}$ on $(t - \varepsilon, t)$ for some $\varepsilon > 0$. This proves statement (n).

To complete our proof, we must determine how incoming paths continue for points of the types $(\mathrm{C_p})/(\mathrm{C_p})$, $(\mathrm{C_l})/(\mathrm{C_p})$, and $(\mathrm{C_l})/(\mathrm{C_l})$.

Points of type $(\mathrm{C_p})/(\mathrm{C_p})$ must be either of type $(\mathrm{p,pp})_{\rm l}/(\mathrm{p,pp})_{\rm l}$ or $(\mathrm{p,pp})_{\rm r}/(\mathrm{p,pp})_{\rm r}$. Hence, points of at least one of these types must occur, so by symmetry between left and right, both types must occur.

If $l \in \mathscr{W}^{\rm l}$ and $\hat{l} \in \hat{\mathscr{W}}^{\rm l}$ are incoming paths at a point z and $l \le \hat{l}$, then it has been shown in the proof of Lemma 5.4 that z must either be of type $(\mathrm{C_p})/(\mathrm{C_p})$ or of type $(\mathrm{C_s})/(\mathrm{C_s})$. It follows that at points of type $(\mathrm{C_l})/(\mathrm{C_p})$ and $(\mathrm{C_l})/(\mathrm{C_l})$, the incoming path $l \in \mathscr{W}^{\rm l}$ continues on the right of the incoming path $\hat{l} \in \hat{\mathscr{W}}^{\rm l}$. ∎

5.2 Structure of special points

In this section, we prove Theorems 1.11 and 1.12. We start with a preparatory lemma.

Lemma 5.6. [Excursions between reflected paths]
Let $\hat{\pi}_1, \hat{\pi}_2 \in \hat{\mathscr{N}}$, $z_i = (x_i, t_i) \in \mathbb{R}^2$ with $t_i < \hat{\sigma}_{\hat{\pi}_i}$ and $x_i \le \hat{\pi}_i(t_i)$, and let $r_i = r_{z_i, \hat{\pi}_i}$ $(i = 1, 2)$ be reflected right-most paths off $\hat{\pi}_1$ and $\hat{\pi}_2$. Assume that r_1 makes an excursion from r_2 during a time interval (s, u) with $u < \hat{\sigma}_{\hat{\pi}_1}, \hat{\sigma}_{\hat{\pi}_2}$. Then there exists a $t \in (s, u]$ such that $\hat{\pi}_1$ separates from $\hat{\pi}_2$ at time t.

Proof Without loss of generality we may assume that $r_1 < r_2$ on (s, u). Since r_1 separates from r_2 at time s, by the structure of reflected paths (Lemma 1.8), we must have $s \in \mathscr{F}(r_1)$ but $s \notin \mathscr{F}(r_2)$, and hence $\hat{\pi}_1(s) = r_1(s) = r_2(s) < \hat{\pi}_2(s)$. By the structure of separation points and Lemma 2.7 (a), we have $r_1 \le \hat{\pi}_1 \le r_2$ on $[s, s + \varepsilon]$ for some $\varepsilon > 0$. Set $\tau := \inf\{t > u : r_2(t) < \hat{\pi}_1(t)\}$. Then $\tau \le u$ since r_1 does not cross $\hat{\pi}_1$ and spends zero Lebesgue time with $\hat{\pi}_1$ (see Lemma 2.12). By Lemma 2.13 (a) and the structure of reflected paths, we have $\tau \in \mathscr{F}(r_2)$, hence $\hat{\pi}_1(\tau) = \hat{\pi}_2(\tau)$. It follows that $\hat{\pi}_1$ and $\hat{\pi}_2$ separate at some time $t \in (s, \tau]$. ∎

Proof of Theorem 1.11 (a) To prove part (a), assume that $l_i = l_{z_i, \hat{r}_i} \in \mathscr{E}^{\rm l}_{\rm in}(z)$ $(i = 1, 2)$ satisfy

$l_1 \sim^z_{\rm in} l_2$ but $l_1 \not\approx^z_{\rm in} l_2$. Then l_1 and l_2 make excursions away from each other on a sequence of intervals (s_k, u_k) with $u_k \uparrow t$. By Lemma 5.6, it follows that $\hat r_i \in \hat{\mathscr{W}}^{\rm r}{}_{\rm out}(z)$ $(i = 1, 2)$ separate at a sequence of times $t_k \uparrow t$, which contradicts (1.9). If $l_i = l_{z,\hat r_i} \in \mathscr{E}^{\rm l}_{\rm out}(z)$ $(i = 1, 2)$ satisfy $l_1 \sim^z_{\rm out} l_2$ but $l_1 \not\approx^z_{\rm out} l_2$, then the same argument shows that the paths $\hat r_i \in \hat{\mathscr{E}}^{\rm r}{}_{\rm in}(z'_i)$ with $z'_i = (x'_i, t)$ and $x'_i \leq x$ separate at a sequence of times $t_k \downarrow t$. It follows that $z'_1 = z'_2 =: z'$, $r_1 \sim^{z'}_{\rm in} r_2$, but $r_1 \not\approx^{z'}_{\rm in} r_2$, contradicting what we have just proved. ∎

Before we continue we prove one more lemma.

Lemma 5.7. [Containment between extremal paths]
Almost surely, for each $z = (x, t) \in \mathbb{R}^2$, if l is the left-most element of $\mathscr{W}^{\rm l}(z)$ and r is the right-most element of $\mathscr{W}^{\rm r}$, then every $\pi \in \mathscr{N}(z)$ satisfies $l \leq \pi \leq r$ on $[t, \infty)$ and every $\hat\pi \in \hat{\mathscr{N}}_{\rm in}(z)$ satisfies $l \leq \hat\pi \leq r$ on $[t, \hat\sigma_{\hat\pi}]$.

Proof By symmetry, it suffices to prove the statements for l. The first statement then follows by approximation of l with left-most paths starting on the left of x, using the fact that paths in $\mathscr{N}$ cannot paths in $\mathscr{W}^{\rm l}$ from right to left [SS08, Prop. 1.8]. The second statement follows from Lemma 2.7 (a). ∎

Proof of Theorems 1.11 (b)–(c) and 1.12 (a)–(c). Since these are statements about incoming paths only, or about outgoing paths only, it suffices to consider the following three cases: **Case I** points of type $(\cdot, \rm p)/(o, \cdot)$, **Case II** points of types $(\cdot, \rm pp)/(p, \cdot)$ and $(\cdot, \rm ppp)/(pp, \cdot)$, and **Case III** points of types $(\cdot, \rm lp)/(l, \cdot)$, $(\cdot, \rm pr)/(r, \cdot)$, or $(\cdot, \rm lr)/(n, \cdot)$. In each of these cases, when we prove Theorem 1.12 (a), we actually prove the analogue for $\hat{\mathscr{N}}_{\rm in}(z)$.

Case I In this case, by Lemma 5.7, all paths in $\mathscr{N}_{\rm out}(z)$ are contained between the equivalent paths l and r. Note that this applies in particular to paths in $\mathscr{E}^{\rm l}_{\rm out}(z)$ and $\mathscr{E}^{\rm r}_{\rm out}(z)$, so by Theorem 1.11 (a), $\mathscr{E}^{\rm l}_{\rm out}(z) = \mathscr{W}^{\rm l}_{\rm out}(z)$ and $\mathscr{E}^{\rm l}_{\rm out}(z) = \mathscr{W}^{\rm l}_{\rm out}(z)$ up to strong equivalence. By Lemma 2.7 (a), $\hat{\mathscr{N}}_{\rm in}(z) = \emptyset$.

Case II We start with points of type $(\cdot, \rm pp)/(p, \cdot)$. Write $\mathscr{W}^{\rm l}(z) = \{l, l'\}$ and $\mathscr{W}^{\rm r} = \{r, r'\}$ where $l \sim^z_{\rm out} r'$ and $l' \sim^z_{\rm out} r$. By Lemma 3.13, r' and l' form a maximal t-mesh, hence, by Proposition 3.5 (b) and Lemma 5.7, all paths in $\mathscr{N}_{\rm out}(z)$ are either contained between l and r' or between l' and r. By Theorem 1.11 (a), this implies that $\mathscr{E}^{\rm l}_{\rm out}(z) = \{l, l'\}$ and $\mathscr{E}^{\rm r}_{\rm out}(z) = \{r, r'\}$ up to strong equivalence. To prove the statements about dual incoming paths, we note that by Lemma 5.5, points of type $(\cdot, \rm pp)/(p, \cdot)$ are of type $(\rm C_p)$ or $(\rm C_s)$ in $\hat{\mathscr{N}}$. Therefore, by Proposition 3.11 (c) and (e), there are unique paths $\hat r \in \hat{\mathscr{W}}^{\rm r}{}_{\rm in}(z)$ and $\hat l \in \hat{\mathscr{W}}^{\rm l}{}_{\rm in}(z)$ and all paths in $\hat{\mathscr{N}}_{\rm in}(z)$ are eventually contained between the equivalent paths $\hat r$ and $\hat l$. By Theorem 1.11 (a), $\hat{\mathscr{E}}^{\rm l}{}_{\rm in}(z) = \{\hat l\}$ and $\hat{\mathscr{E}}^{\rm r}{}_{\rm in}(z) = \{\hat r\}$ up to strong equivalence.

For points of type $(\cdot, \rm ppp)/(pp, \cdot)$ the argument is similar, except that there are now two maximal t-meshes with bottom point z, and for the dual incoming paths we must use Proposition 3.11 (d).

Case III Our proof in this case actually also works for points of type $(\cdot, \rm pp)/(p, \cdot)$, although for these points, the argument given in Case II is simpler.

Let l denote the left-most element of $\mathscr{W}^{\rm l}(z)$ and let r denote the right-most element of $\mathscr{W}^{\rm r}(z)$. By Lemma 2.7 (a), any path $\pi \in \hat{\mathscr{N}}_{\rm in}(z)$ must stay in $V := \{(x_+, t_+) : t_+ > t,\ l(t_+) \leq x_+ \leq r(t_+),\ l < r \text{ on } (t, t_+)\}$. Choose any $z_+ \in V$ and define (by Lemma 1.8) reflected paths by $\hat r := \hat r_{z_+, l} = \min\{\hat\pi' \in \mathscr{N}(z_+) : l \leq \hat\pi' \text{ on } [t, t_+]\}$ and $\hat l := \hat l_{z_+, r}$. By Lemma 3.12, all paths in $\hat{\mathscr{N}}_{\rm in}(z)$ are equivalent, hence $\hat r \sim^z_{\rm in} \hat l$ and, by Theorem 1.11 (a), $\hat{\mathscr{E}}^{\rm l}{}_{\rm in}(z) = \{\hat l\}$ and $\hat{\mathscr{E}}^{\rm r}{}_{\rm in}(z) = \{\hat r\}$ up to strong equivalence. Since our construction does not depend on the point z_+, each path $\hat{\pi} \in \hat{\mathcal{N}}_{\rm in}(z)$ satisfies $\hat{r} \leq \hat{\pi} \leq \hat{l}$ on $[t, t+\varepsilon]$ for some $\varepsilon > 0$.

Now fix some $z_+ = (x_+, t_+) \in V$ and put $\hat{r} := \hat{r}_{z_+, l}$ and $\hat{l} := \hat{l}_{z_+, r}$. By what we have just proved we can choose z_+ in such a way that $\hat{l} \in \hat{\mathcal{W}}^{\rm l}{}_{\rm in}(z)$ if the latter set is nonempty and $\hat{r} \in \hat{\mathcal{W}}^{\rm l}{}_{\rm in}(z)$ if that set is nonempty. In any case $\hat{\mathcal{E}}^{\rm l}{}_{\rm in}(z) = \{\hat{l}\}$ and $\hat{\mathcal{E}}^{\rm r}{}_{\rm in}(z) = \{\hat{r}\}$ up to strong equivalence. Define (by Lemma 1.8) reflected paths $r' := r_{z,\hat{l}}$ and $l' := l_{z,\hat{r}}$. Since any path in $\hat{l}' \in \hat{\mathcal{E}}^{\rm l}{}_{\rm in}$ is strongly equivalent to $\hat{l}$, each reflected path of the form $r_{z,\hat{l}'}$ is strongly equivalent to r'. On the other hand, it is easy to see that each reflected path of the form $r_{z,\hat{l}'}$ with $\hat{l}' \in \mathcal{E}^{\rm l}_{\rm in}(z')$, $z' = (x', t')$, $x < x'$ is strongly equivalent to r. It follows that, up to strong equivalence, $\mathcal{E}^{\rm r}_{\rm out}(z) = \{r, r'\}$, and by a symmetric argument $\mathcal{E}^{\rm l}_{\rm out}(z) = \{l, l'\}$.

Since $\hat{r} \sim^{z}_{\rm in} \hat{l}$, since $\hat{r}$ and $\hat{l}$ spend positive Lebesgue time together whenever they meet while r' and $\hat{l}$ spend zero Lebesgue time together, we must have $r'(t_n) < \hat{r}(t_n)$ for a sequence of times $t_n \downarrow t$. Since r' can cross $\hat{r}'$ only at times in $\mathcal{F}(\hat{r})$, we must have $l(t'_n) = r'(t'_n)$ for a sequence of times $t'_n \downarrow t$, i.e., $l \sim^{z}_{\rm out} r'$. A symmetric argument shows that $l' \sim^{z}_{\rm out} r$. It follows from the definition of reflected paths that any $\pi \in \mathcal{N}(z)$ such that $\pi \leq \hat{l}$ on $[t, t_+]$ must satisfy $l \leq \pi \leq r'$ on $[t, t_+]$. Since all paths in $\hat{\mathcal{E}}^{\rm l}{}_{\rm in}(z)$ are strongly equivalent, this shows that if $\pi \in \mathcal{N}(z)$ satisfies $\pi \leq \hat{l}'$ on $[t, t+\varepsilon]$ for some $\hat{l}' \in \mathcal{E}^{\rm l}_{\rm in}(z)$ and $\varepsilon > 0$, then $l \leq \pi \leq r'$ on $[t, t+\varepsilon']$ for some $\varepsilon' > 0$. A similar conclusion holds if $\hat{r}' \leq \pi$ on $[t, t+\varepsilon]$ for some $\hat{r}' \in \mathcal{E}^{\rm r}_{\rm in}(z)$ and $\varepsilon > 0$.

By Lemma 2.13 (a), a path $\pi \in \mathcal{N}(z)$ can cross $\hat{l}$ from right to left only at times in $\mathcal{F}(\hat{l})$ and π can cross $\hat{r}$ from left to right only at times in $\mathcal{F}(\hat{r})$. In particular, if $\hat{l} \in \hat{\mathcal{W}}^{\rm l}$, then any path $\pi \in \mathcal{N}(z)$ that enters the region to the right of $\hat{l}$ must stay there till time t_+. It follows that for points of type $(\cdot, {\rm lp})/({\rm l}, \cdot)$ or $(\cdot, {\rm pp})/({\rm p}, \cdot)$, any $\pi \in \mathcal{N}(z)$ must either satisfy $\hat{l} \leq \pi$ on $[t, t_+]$ or $\pi \leq \hat{l}$ on $[t, t+\varepsilon]$ for some $\varepsilon > 0$. By what we have just proved, this implies that either $l' \leq \pi \leq r$ on $[t, t_+]$ or $l \leq \pi \leq r'$ on $[t, t+\varepsilon']$ for some $\varepsilon' > 0$. By symmetry, a similar argument applies to points of type $(\cdot, {\rm pr})/({\rm r}, \cdot)$.

This completes the proof of Theorems 1.11 (b)–(c) and 1.12 (a)–(b). To prove Theorem 1.12 (c), we must show that at points of type $({\rm o}, {\rm lr})/({\rm n}, {\rm p})$ there exist paths π that infinitely often cross over between l and r in any time interval $[t, t+\varepsilon]$ with $\varepsilon > 0$. Since $r' \notin \mathcal{W}^{\rm r}_{\rm out}(z)$, we have $\mathcal{F}(r') = \{t_n : n \geq 0\}$ for some $t_n \downarrow t$. Let $r_n \in \mathcal{W}^{\rm r}$ be the right-most path such that $r' = r_n$ on $[t_{n+1}, t_n]$ and set $\tau_n := \inf\{u > t_n : r_n(u) = r(u)\}$. By the compactness of $\mathcal{W}^{\rm r}$ we have $r_n \uparrow r'$ for some $r' \in \mathcal{W}^{\rm r}(z)$ and hence, since the latter set has only one element, $r_n \uparrow r$. This shows that $\tau_n \downarrow t$. By a symmetry, there exist times $t'_n, \tau'_n \downarrow t$ and $l_n \in \mathcal{W}^{\rm l}$ such that $l_n = l'$ on $[t'_{n+1}, t'_n]$ and $l_n(\tau'_n) = l(\tau'_n)$. Using these facts and the hopping construction of the Brownian net, it is easy to construct the desired path $\pi \in \mathcal{N}(z)$ such that $l \sim^{z}_{\rm out} \pi \sim^{z}_{\rm out} r$. ∎

Proof of Theorem 1.12 (d) At all points of types with the subscript l, there is a dual path $\hat{r}$ that, by Lemma 2.13 (a), cannot be crossed by any path $\pi \in \mathcal{N}$ entering z (as we have have just seen it must) between the unique incoming paths $l \in \mathcal{W}^{\rm l}_{\rm in}(z)$ and $r \in \mathcal{W}^{\rm r}_{\rm in}(z)$, and hence prevents π from leaving z in the right outgoing equivalence class. By symmetry, an analogue statement holds for points of types with the subscript r.

In all other cases where $\mathcal{N}_{\rm in}(z)$ is not empty, either the point is of type $({\rm p}, {\rm pp})_{\rm s}$ or there is a single outgoing equivalence class. In all these cases, using the fact that the Brownian net is closed under hopping, it is easy to see that any concatenation of a path in $\mathcal{N}_{\rm in}(z)$ up to time t with a path in

$\mathcal{N}_{\text{out}}(z)$ after time t is again a path in $\mathcal{N}$. ∎

A Extra material

A.1 Some simple lemmas

Lemma A.1. [Incoming paths]
Let $(\mathcal{W}^{\text{l}}, \mathcal{W}^{\text{r}}, \hat{\mathcal{W}}^{\text{l}}, \hat{\mathcal{W}}^{\text{r}})$ be the standard left-right Brownian web and its dual. For $z \in \mathbb{R}^2$, Let $m_{\text{in}}^{\text{lr}}(z)$ and $m_{\text{out}}^{\text{lr}}(z)$ denote respectively the number of equivalence classes of paths in $(\mathcal{W}^{\text{l}}, \mathcal{W}^{\text{r}})$ entering and leaving z. Let $\hat{m}_{\text{in}}^{\text{lr}}(z)$ and $\hat{m}_{\text{out}}^{\text{lr}}(z)$ be defined similarly for $(\hat{\mathcal{W}}^{\text{l}}, \hat{\mathcal{W}}^{\text{r}})$. Then

(a) *For each deterministic $z \in \mathbb{R}^2$, almost surely $m_{\text{in}}^{\text{lr}}(z) + \hat{m}_{\text{in}}^{\text{lr}}(z) = 0$.*

(b) *For each deterministic $t \in \mathbb{R}$, almost surely $m_{\text{in}}^{\text{lr}}(x,t) + \hat{m}_{\text{in}}^{\text{lr}}(x,t) \leq 1$ for all $x \in \mathbb{R}$.*

(c) *Almost surely, $m_{\text{in}}^{\text{lr}}(z) + \hat{m}_{\text{in}}^{\text{lr}}(z) \leq 2$ for all $z \in \mathbb{R}^2$.*

Proof By Lemma 3.4 (b) of [SS08], it suffices to consider only paths in $(\mathcal{W}^{\text{l}}, \mathcal{W}^{\text{r}}, \hat{\mathcal{W}}^{\text{l}}, \hat{\mathcal{W}}^{\text{r}})$ starting from a deterministic countable dense set $\mathcal{D} \subset \mathbb{R}^2$, which we denote by

$$(\mathcal{W}^{\text{l}}(\mathcal{D}), \mathcal{W}^{\text{r}}(\mathcal{D}), \hat{\mathcal{W}}^{\text{l}}(\mathcal{D}), \hat{\mathcal{W}}^{\text{r}}(\mathcal{D})).$$

Part (a) follows from the fact that a Brownian motion (with constant drift) almost surely does not hit a deterministic space-time point.

Part (b) follows from the fact, paths in $\mathcal{W}^{\text{l}}(\mathcal{D}) \cup \mathcal{W}^{\text{r}}(\mathcal{D})$, resp. $\hat{\mathcal{W}}^{\text{l}}(\mathcal{D}) \cup \hat{\mathcal{W}}^{\text{r}}(\mathcal{D})$, evolve independently before they meet, hence almost surely, no two paths in $\mathcal{W}^{\text{l}}(\mathcal{D}) \cup \mathcal{W}^{\text{r}}(\mathcal{D})$, resp. $\hat{\mathcal{W}}^{\text{l}}(\mathcal{D}) \cup \hat{\mathcal{W}}^{\text{r}}(\mathcal{D})$, can first meet at a deterministic time. Also note that, $(\mathcal{W}^{\text{l}}(\mathcal{D}), \mathcal{W}^{\text{r}}(\mathcal{D}))|_{(-\infty,t)}$, resp. $(\mathcal{W}^{\text{l}}(\mathcal{D}), \mathcal{W}^{\text{r}}(\mathcal{D}))|_{(t,\infty)}$, the restriction of paths in $(\mathcal{W}^{\text{l}}(\mathcal{D}), \mathcal{W}^{\text{r}}(\mathcal{D}))$ to the time interval $(-\infty, t)$, resp. (t, ∞), are independent. Hence it follows that $(\mathcal{W}^{\text{l}}(\mathcal{D}), \mathcal{W}^{\text{r}}(\mathcal{D}))|_{(-\infty,t)}$ and $(\hat{\mathcal{W}}^{\text{l}}(\mathcal{D}), \hat{\mathcal{W}}^{\text{r}}(\mathcal{D}))|_{(t,\infty)}$ are also independent, and almost surely, no path in $\mathcal{W}^{\text{l}}(\mathcal{D}) \cup \mathcal{W}^{\text{r}}(\mathcal{D})$ starting before time t can be at the same position at time t as some path in $\hat{\mathcal{W}}^{\text{l}}(\mathcal{D}) \cup \hat{\mathcal{W}}^{\text{r}}(\mathcal{D})$ starting after time t.

The proof of (c) is similar. By symmetry, it suffices to show that: (1) the probability of $\pi_1, \pi_2, \pi_3 \in \mathcal{W}^{\text{l}}(\mathcal{D}) \cup \mathcal{W}^{\text{r}}(\mathcal{D})$ first meeting at the same space-time point is zero; (2) the probability of $\pi_1, \pi_2 \in \mathcal{W}^{\text{l}}(\mathcal{D}) \cup \mathcal{W}^{\text{r}}(\mathcal{D})$ first meeting at a point (x,t), and $\hat{\pi}(t) = x$ for some $\hat{\pi} \in \hat{\mathcal{W}}^{\text{l}}(\mathcal{D}) \cup \hat{\mathcal{W}}^{\text{r}}(\mathcal{D})$ starting after time t is zero. Claim (1) holds because π_1, π_2 and π_3 evolve independently before they meet. For claim (2), note that conditioned on $T = \inf\{t > \sigma_{\pi_1} \vee \sigma_{\pi_2} : \pi_1(t) = \pi_2(t)\}$, the distribution of $(\mathcal{W}^{\text{l}}(\mathcal{D} \cap ((T,\infty) \times \mathbb{R})), \mathcal{W}^{\text{r}}(\mathcal{D} \cap ((T,\infty) \times \mathbb{R})))$ is independent of $(\pi_1, \pi_2)|_{(-\infty,T)}$, which is a consequence of the strong Markov property of a collection of left-right coalescing Brownian motions. Hence $(\hat{\mathcal{W}}^{\text{l}}(\mathcal{D}), \hat{\mathcal{W}}^{\text{r}}(\mathcal{D}))|_{(T,\infty)}$ is also independent of $(\pi_1, \pi_2)|_{(-\infty,T)}$ conditioned on T. The claim then follows. ∎

Lemma A.2. [Intersection points of $\mathcal{W}^{\text{l}}$ and $\mathcal{W}^{\text{r}}$]
For $l \in \mathcal{W}^{\text{l}}$ and $r \in \mathcal{W}^{\text{r}}$ with deterministic starting points z_{l}, resp. z_{r}, let $T = \inf\{s \geq \sigma_l \vee \sigma_r : l(s) = r(s)\}$, $I = \{s > T : l(s) = r(s)\}$, and let μ_I denote the measure on $\mathbb{R}$ defined by $\mu_I(A) = \ell(I \cap A)$, where ℓ is the Lebesgue measure. Then

(a) *Almost surely on the event* $T < \infty$, $\lim_{t\downarrow 0} t^{-1}\mu_I([T, T+t]) = 1$.

(b) $\mathbb{P}(l(T+t) = r(T+t) \,|\, T < \infty) \uparrow 1$ *as* $t \downarrow 0$.

Proof By the strong Markov property of the unique weak solution of (1.4) (see Proposition 2.1 of [SS08]), it suffices to verify (a) and (b) for the solution of (1.4) with initial condition $L_0 = R_0 = 0$. Let $W_\tau = \sqrt{2}B_\tau + 2\tau$, where B_τ is a standard Brownian motion. Let

$$X_\tau = W_\tau + R_\tau \quad \text{with} \quad R_\tau = -\inf_{0\le s\le\tau} W_s \tag{A.118}$$

denote W_τ Skorohod reflected at 0. Then recall from the proof of Proposition 2.1 and Lemma 2.2 in [SS08] that, the process $D_t = R_t - L_t$ is a time change of X_τ, converting the local time of X at 0 into real time. More precisely, D_t is equally distributed with X_{T_t}, where the inverse of T is defined by $T_\tau^{-1} = \tau + \frac{1}{2}R_\tau$. In particular, $\int_0^t 1_{\{X_{T_s}=0\}} ds = \frac{1}{2}R_{T_t}$. Therefore to show part (a), it suffices to show that, almost surely

$$\lim_{t\downarrow 0}\frac{R_{T_t}}{2t} = \lim_{\tau\downarrow 0}\frac{1}{1+\frac{2\tau}{R_\tau}} = 1. \tag{A.119}$$

By Girsanov's theorem, it suffices to show that

$$\lim_{\tau\downarrow 0}\frac{\tau}{R'_\tau} = 0 \quad \text{a.s.}, \tag{A.120}$$

where $R'_\tau = -\sqrt{2}\inf_{0\le s\le\tau} B_s$ for a standard Brownian motion B_s. This can be proved by a straightforward Borel-Cantelli argument, which we leave to the reader.

To prove part (b), it suffices to show the monotonicity of $\mathbb{P}[L_s = R_s]$ in s, because by (a), we have $\lim_{t\downarrow 0}\frac{1}{t}\int_0^t \mathbb{P}[L_s = R_s]ds = \lim_{t\downarrow 0}\mathbb{E}\big[\frac{1}{t}\int_0^t 1_{\{L_s=R_s\}}ds\big] = 1$. Note that by the same arguments as in the proof of Proposition 16 in [SS08], $D_t = R_t - L_t$ is the unique nonnegative weak solution of the SDE

$$dD_t = 1_{\{D_t>0\}}\sqrt{2}\,dB_t + 2dt, \tag{A.121}$$

which defines a time homogeneous strong Markov process with continuous paths in $[0,\infty)$. By coupling, it is clear that any such Markov process preserves the stochastic order, i.e., if D_s and D'_s are two solutions of (A.121) with initial laws $\mathscr{L}(D_0) \le \mathscr{L}(D'_0)$, then $\mathscr{L}(D_t) \le \mathscr{L}(D'_t)$ for all $t \ge 0$, where $\le$ denotes stochastic order. In particular, if $D_0 = 0$, then $\mathscr{L}(D_s) \le \mathscr{L}(D_t)$ for all $s \le t$, since $\delta_0 \le \mathscr{L}(D_{t-s})$. Therefore $\mathbb{P}(D_s = 0) \ge \mathbb{P}(D_t = 0)$ for all $s \le t$, which is the desired monotonicity. ∎

A.2 Meeting points

In this appendix, we give an alternative proof of the fact that any meeting point of two paths $\pi, \pi' \in \mathscr{W}^{\mathrm{l}} \cup \mathscr{W}^{\mathrm{r}}$ is of type (pp, p). Recall that any pair $L \in \mathscr{W}^{\mathrm{l}}$ and $R \in \mathscr{W}^{\mathrm{r}}$ with deterministic starting points solve the SDE (1.4). Consider the SDE (3.74). Let us change variables and denote $X_t = L'_t - L_t$, $Y_t = R_t - L_t$, so

$$\begin{aligned} \mathrm{d}X_t &= 1_{\{Y_t\,>\,0\}}(\mathrm{d}B^{\mathrm{l}'}_t - \mathrm{d}B^{\mathrm{l}}_t) + 1_{\{Y_t\,=\,0\}}(\mathrm{d}B^{\mathrm{l}'}_t - \mathrm{d}B^{\mathrm{s}}_t), \\ \mathrm{d}Y_t &= 1_{\{Y_t\,>\,0\}}(\mathrm{d}B^{\mathrm{r}}_t - \mathrm{d}B^{\mathrm{l}}_t) + 2\mathrm{d}t, \end{aligned} \tag{A.122}$$

with the constraint that $Y_t \geq 0$. In terms of the process (X,Y), the stopping time τ from (3.75) becomes $\tau = \inf\{t \geq 0 : X_t = Y_t\}$, and (3.76) becomes

$$\lim_{\varepsilon \to 0} \mathbb{P}^{\varepsilon,0}[Y_\tau = 0] = 1. \tag{A.123}$$

The generator of the process (X,Y) is

$$G = \frac{\partial^2}{\partial x^2} + 1_{\{y>0\}} \frac{\partial^2}{\partial y^2} + 1_{\{y>0\}} \frac{\partial^2}{\partial x \partial y} + 2\frac{\partial}{\partial y}. \tag{A.124}$$

Define a function $g(x,y)$ on the domain $0 \leq y < x$ by

$$g(x,y) = \frac{y}{x}, \tag{A.125}$$

so that

$$Gg(x,y) = \frac{2y}{x^3} - 1_{\{y>0\}} \frac{1}{x^2} + \frac{2}{x} = 1_{\{y>0\}} \frac{2y-x}{x^3} + \frac{2}{x}. \tag{A.126}$$

Now define the domain $\Delta = \{(x,y) : 0 \leq y < x/2,\ 0 < x < 1\}$. Then the first term on the right-hand side of (A.126) is nonpositive on Δ. We compensate the second term on the right-hand side of (A.126) by adding another function to g. Namely, let $f = 8\sqrt{x}$, then $Gf(x,y) = -2x^{-3/2}$. Hence

$$G(g+f) \leq 0 \quad \text{on} \quad \Delta, \tag{A.127}$$

which means that $(g+f)(X_t, Y_t)$ is a local supermartingale before it exits Δ. Let $\partial\Delta$ denote the boundary of Δ. Observe that

$$\begin{aligned} &g + f \geq 0 \quad \text{on} \quad \Delta, \\ &g + f \geq \tfrac{1}{2} \quad \text{on} \quad \partial\Delta \backslash \{(0,0)\}, \\ &\lim_{\varepsilon \to 0} (g+f)(\varepsilon, 0) = 0. \end{aligned} \tag{A.128}$$

It follows that as $\varepsilon \downarrow 0$, the probability that the process (X,Y) started at $(\varepsilon,0)$ exits Δ from $\partial\Delta\backslash\{(0,0)\}$ tends to zero. Consequently, the process must exit from $(0,0)$, and (A.123) and (3.76) then follow.

Now we prove that any meeting point of two paths $\pi, \pi' \in \mathcal{W}^{\mathrm{l}} \cup \mathcal{W}^{\mathrm{r}}$ is of type (pp,p). By Lemma 3.4 (b) of [SS08], it suffices to show this for paths in $(\mathcal{W}^{\mathrm{l}}, \mathcal{W}^{\mathrm{r}})$ starting from a deterministic countable dense subset $\mathcal{D} \subset \mathbb{R}^2$. Without loss of generality, assume $l_1, l_2 \in \mathcal{W}^{\mathrm{l}}$ start respectively from $z_1 = (x_1, t_1), z_2 = (x_2, t_2) \in \mathcal{D}$ with $t_1 \leq t_2$ and $l_1(t_2) < l_2(t_2)$. Let $\tau_\varepsilon = \inf\{s \geq t_2 : l_2(s) - l_1(s) \leq \varepsilon\}$. Then for any $\delta > 0$, we can find $z_1' \in \mathcal{D}$ and $r_1 \in \mathcal{W}^{\mathrm{r}}(z_1')$ with z_1' sufficiently close to $(l_1(\tau_\varepsilon), \tau_\varepsilon)$, such that with probability at least $1-\delta$, l_1 and r_1 will meet, at which time l_1 and l_2 still have not met and are at most 2ε distance apart. Since $\varepsilon, \delta > 0$ can be arbitrary, (3.76) implies that almost surely, the first meeting point z of l_1 and l_2 is also a meeting point of l_2 and some $r_1 \in \mathcal{W}^{\mathrm{r}}(\mathcal{D})$ bounded between l_1 and l_2. In fact, by Lemma A.1 (c), we have $l_1 \sim^z_{\mathrm{in}} r_1$. Repeating the argument for r_1 and l_2, there must exist $r_2 \in \mathcal{W}^{\mathrm{r}}(\mathcal{D})$ which enters z and $r_2 \sim^z_{\mathrm{in}} l_2$. The case of meeting points of $l \in \mathcal{W}^{\mathrm{l}}(\mathcal{D})$ and $r \in \mathcal{W}^{\mathrm{r}}(\mathcal{D})$ is similar. ∎

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