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Collision local times, historical stochastic calculus, and competing superprocesses

Steven N. Evans and Edwin A. Perkins

Department of Statistics #3860, University of California at Berkeley
367 Evans Hall, Berkeley CA 94720-3860, USA

Department of Mathematics, University of British Columbia
Vancouver BC V6T 1Z2, CANADA

evans@stat.berkeley.edu   perkins@heron.math.ubc.ca

Abstract: Branching measure–valued diffusion models are investigated that can be regarded as pairs of historical Brownian motions modified by a competitive interaction mechanism under which individuals from each population have their longevity or fertility adversely affected by collisions with individuals from the other population. For 3 or fewer spatial dimensions, such processes are constructed using a new fixed–point technique as the unique solution of a strong equation driven by another pair of more explicitly constructible measure–valued diffusions. This existence and uniqueness is used to establish well–posedness of the related martingale problem and hence the strong Markov property for solutions. Previous work of the authors has shown that in 4 or more dimensions models with the analogous definition do not exist.

The definition of the model and its study require a thorough understanding of random measures called collision local times. These gauge the extent to which two measure–valued processes or an $\mathbb{R}^d$–valued process and a measure–valued process "collide" as time evolves. This study and the substantial amount of related historical stochastic calculus that is developed are germane to several other problems beyond the scope of the paper. Moreover, new results are obtained on the branching particle systems imbedded in historical processes and on the existence and regularity of superprocesses with immigration, and these are also of independent interest.

Keywords: super-process, super-Brownian motion, interaction, local time, historical process, measure-valued Markov branching process, stochastic calculus, martingale measure, random measure

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Contents

1 Introduction

1.1 Background

Consider populations of two different species which migrate, reproduce and compete for the same resources. As a result, when one individual gets close to one or more individuals of the other species its life expectancy decreases or its ability to successfully reproduce is diminished. Continuous space point processes have recently been used by mathematical biologists to model such competing species (see [3], [4] and [28]). One goal of this work was to demonstrate the importance of incorporating realistic spatial structure into population biology models (in contrast to the classical approach that seeks to model total population size directly and usually involves implicit "stirring" assumptions).

Although the biologists are dealing with organisms of a given size and finite interaction range, from a mathematical perspective it is natural to consider a scaling limit in which the interaction becomes purely local and the total population becomes large. In the regime of critical or near critical branching the scaling limit without the inter–species competition is a superprocess (super–Brownian motion if the migration is given by a random walk), and so when competition is present one may expect a scaling limit that is a pair of locally interacting super–Brownian motions. The purpose of this paper is to continue the analysis of such models that was begun in [20].

Super–Brownian models in which the mortality or fertility of individuals is subject to local effects are relatively easy to construct and analyse in one dimension. This is for two reasons. Firstly, one–dimensional super–Brownian motion takes values in the set of measures which are absolutely continuous with respect to Lebesgue measure, and so describing the extent to which two populations collide is easy. Secondly, the interacting model solves a martingale problem that looks like the one for a pair of independent super–Brownian motions except for the addition of tame "drift" terms. The law of such a process can therefore be constructed by using Dawson's Girsanov theorem (see [5]) to produce an absolutely continuous change in the law of a pair of independent super–Brownian motions (see Section 2 of [20]).

Neither of these features is present in higher dimensions: super–Brownian motion takes Lebesgue–singular values and the model we wish to construct has a law that is not absolutely continuous with respect to that of a pair of independent super–Brownian motions (see Theorem 3.11 of [20]). A substantial body of new techniques is therefore needed.

Before moving on to some mathematical preliminaries and the precise definition of the processes that we will study, we mention that there has been considerable recent interest in interacting superprocesses where the interactions affect features of the process other than individual life–times or reproductive efficiencies. For example, [30], [31], [1] and [26] consider models of a single super–Brownian population in which the spatial motion of each individual is affected by the behaviour of the whole population (see, also, [8]). Models for two populations in which the branching rate is subject to local interactions are studied in [10].

1.2 Historical Brownian motion

We begin with some general notation and the definition of historical Brownian motion. The process of interest to us can be thought of as a pair of "competing" historical Brownian motions.

Write C for the space $C(\mathbb{R}_+, \mathbb{R}^d)$ of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^d$. Given $\psi \in C_b^2(\mathbb{R}^{nd}, \mathbb{R})$ ($\equiv$ functions on $\mathbb{R}^{nd}$ with bounded partial derivatives of order 2 or less) and $0 \leq t_1 < \ldots t_n$, let $\bar{\psi} = \bar{\psi}(t_1, \ldots, t_n)(\cdot) : C \to \mathbb{R}$ be given by $\bar{\psi}(y) = \psi(y(t_1), \ldots, y(t_n))$. Put $\bar{y}(t) = (y(t \wedge t_1), \ldots, y(t \wedge t_n))$ and

$$\frac{\bar{\Delta}}{2}\bar{\psi}(t, y) = \frac{1}{2}\sum_{i=1}^{d}\sum_{k=0}^{n-1}\sum_{\ell=0}^{n-1} \mathbf{1}\{t < t_{k+1} \wedge t_{\ell+1}\}\psi_{kd+i,\ell d+i}(\bar{y}(t))$$

(here $\psi_{\alpha\beta}$, $1 \leq \alpha, \beta \leq nd$, are the second order partial derivatives of ψ.)

Set

$$D_S = \{\bar{\psi}(t_1, \ldots, t_n) : 0 \leq t_1 < \ldots < t_n,\ n \in \mathbb{N},\ \psi \in C_K^\infty(\mathbb{R}^{nd})\} \cup \{1\},$$

where $C_K^\infty(\mathbb{R}^{nd})$ is the space of C^∞ functions with compact support. Set $y^t(\cdot) = y(\cdot \wedge t)$ for $y \in C$, put

$$D_1 = \{\phi : \mathbb{R}_+ \times C \to \mathbb{R} \ :\ \phi(t, y) = \phi_1(t)\phi_2(y^t),\ \phi_1 \in C^1(\mathbb{R}_+),\ \phi_2 \in D_S\},$$

and let D_{ST} denote the linear span of D_1. For $\phi \in D_1$ with $\phi(t, y) = \phi_1(t)\phi_2(y^t)$ (where ϕ_1, ϕ_2 are as above) define

$$A\phi(s, y) = \phi_1'(s)\phi_2(y^s) + \phi_1(s)\frac{\bar{\Delta}}{2}\phi_2(s, y), \tag{1.1}$$

and extend A linearly to D_{ST}. Set $S^o = \{(t, y) \in \mathbb{R}_+ \times C : y^t = y\}$. We identify $\phi \in D_{ST}$ and $A\phi$ with their restrictions to S^o.

Given a measurable space $(\Sigma, \mathcal{A})$, let $\mathbf{M}_F(\Sigma)$ denote the space of finite measures on Σ. The notation $b\mathcal{A}$ (respectively, $bp\mathcal{A}$) denotes the set of bounded (respectively, bounded positive) $\mathcal{A}$–measurable functions. We will often use the functional notation $\rho(f)$ to denote $\int f d\rho$ for $\rho \in \mathbf{M}_F(\Sigma)$. When Σ is separable metric space and $\mathcal{A}$ is the corresponding Borel σ–field (which we will denote by $\mathcal{B}(\Sigma)$), we will equip $\mathbf{M}_F(\Sigma)$ with the topology of weak convergence.

Put $\mathbf{M}_F(C)^t = \{\mu \in \mathbf{M}_F(C) : y^t = y,\ \mu-\text{a.e. } y\}$ for $t \geq 0$. Write $\Omega_H[\tau, \infty[$ for the space of continuous, $\mathbf{M}_F(C)$–valued paths, h, on $[\tau, \infty[$ such that $h_t \in \mathbf{M}_F(C)^t$ for all $t \geq \tau$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq \tau}, \mathbb{P})$ be a filtered probability space with the filtration $(\mathcal{F}_t)_{t \geq \tau}$ right–continuous and the σ–field $\mathcal{F}$ universally complete. Suppose that $\tau \in \mathbb{R}_+$ and $\mu \in \mathbf{M}_F(C)^\tau$. We say that H is a *historical Brownian motion* *on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq\tau}, \mathbb{P})$ *started at* (τ, μ) if $(H_t)_{t\geq\tau}$ is a $(\mathcal{F}_t)_{t\geq\tau}$–predictable processes with sample paths almost surely in $\Omega_H[\tau,\infty[$ such that for all $\phi \in D_S$

$$M_t(\phi) = H_t(\phi) - \mu(\phi) - \int_\tau^t H_s(\frac{\bar{\Delta}}{2}\phi_s)\,ds,\ t \geq \tau$$

is a continuous $(\mathcal{F}_t)_{t\geq\tau}$–martingale for which $M_\tau(\phi) = 0$ and

$$\langle M(\phi)\rangle_t = \int_\tau^t \int_C \phi(y)^2 H_s(dy)\,ds.$$

By Theorem 1.3 of [31] this martingale problem uniquely determines the law of H, $\mathbb{Q}^{\tau,\mu}$, on $\Omega_H[\tau,\infty[$.

A simple application of Itô's lemma shows that for $\phi \in D_{ST}$,

$$M_t(\phi) = H_t(\phi_t) - \mu(\phi_\tau) - \int_\tau^t H_s(A\phi_s)\,ds,\ t \geq \tau$$

is a continuous $(\mathcal{F}_t)_{t\geq\tau}$–martingale for which $M_\tau(\phi) = 0$,

$$\langle M(\phi)\rangle_t = \int_\tau^t \int_C \phi(s,y)^2 H_s(dy)\,ds.$$

(Use the fact that $\phi_2(y) = \phi_2(y^t)$, H_t-a.a. y, $\forall t \geq \tau$, a.s., $\forall \phi_2 \in D_S$.)

Let $\Omega_X[\tau,\infty[= C([\tau,\infty[, \mathbf{M}_F(\mathbb{R}^d))$ and define $\Gamma : \Omega_H[\tau,\infty[\to \Omega_X[\tau,\infty[$ by putting

$$\Gamma(h)_t(\phi) = \int \phi(y_t)\,h_t(dy).$$

If $m(A) = \mu(y_\tau \in A)$ and H is as above then $X = \Gamma(H)$ is a super–Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq\tau}, \mathbb{P})$ starting at m at time τ with law on $\Omega_X[\tau,\infty[$ that we will write as $\mathbb{P}^{\tau,m}$. This follows from the martingale problem characterisation of super–Brownian motion (see Ch. 6 of [5]).

1.3 Collision local times

In order describe how the martingale problem for a pair of independent historical Brownian motions has to be modified in order to reflect decreased longevity/fertility due to local collisions, we need to define an object that measures such collisions. Suppose that $(K_t^1)_{t\geq\tau}$ and $(K_t^2)_{t\geq\tau}$ are two predictable, $\mathbf{M}_F(C)$–valued processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq\tau}, \mathbb{P})$ such that almost surely K^i has paths in $\Omega_H[\tau,\infty[$ for $i = 1, 2$. For $\epsilon > 0$ define a continuous, $\mathbf{M}_F(C)$–valued process $(L_t^\epsilon(K^1, K^2))_{t\geq\tau}$ by setting

$$L_t^\epsilon(K^1,K^2)(\phi) = \int_\tau^t \int \left\{\int p_\epsilon(y_1(s) - y_2(s))\,K_s^2(dy_2)\right\} \phi(y_1)\,K_s^1(dy_1)\,ds,$$

where $p_\epsilon(x) = (2\pi\epsilon)^{-d/2}\exp(-|x|^2/2\epsilon)$ for $x \in \mathbb{R}^d$.

We say that K^1 and K^2 have a *field–field collision local time* or *FFCLT* $(L_t(K^1, K^2))_{t\geq\tau}$ if $(L_t(K^1, K^2))_{t\geq\tau}$ is an $(\mathcal{F}_t)_{t\geq\tau}$–predictable, $\mathbf{M}_F(C)$–valued process with sample paths almost surely in $\Omega_H[\tau, \infty[$ such that

$$\lim_{\epsilon\downarrow 0} L_t^\epsilon(K^1, K^2)(\phi) = L_t(K^1, K^2)(\phi)$$

in probability for all $t \geq \tau$ and all bounded continuous functions ϕ on C.

If $(L_t(K^1, K^2))_{t\geq\tau}$ exists, then it is unique up to evanescent sets. Almost surely for all $\tau \leq s < t$ and all $A \in \mathcal{C}$ we have $L_s(K^1, K^2)(A) \leq L_t(K^1, K^2)(A)$, and so there is an almost surely unique Borel random measure on $]\tau, \infty[\times C$ that we will denote as $L(K^1, K^2)$ such that $L(K^1, K^2)(]s, t] \times A) = L_t(K^1, K^2)(A) - L_s(K^1, K^2)(A)$. Adjoin an isolated point Δ to $\mathbf{M}_F(C)$ to form $\mathbf{M}_F^\Delta(C)$ The same notation as above will be used for $\mathbf{M}_F^\Delta(C)$–valued processes, with Δ treated as the 0 measure for purposes of integration.

Remark 1.1 Define continuous $\mathbf{M}_F(\mathbb{R}^d)$–valued processes $(W_t^1)_{t\geq\tau}$ and $(W_t^2)_{t\geq\tau}$ by setting $W^i = \Gamma(K^i)$. The continuous $\mathbf{M}_F(\mathbb{R}^d)$–valued process $(L_t(W^1, W^2))_{t\geq\tau}$ defined by

$$L_t(W^1, W^2)(\phi) = \int_\tau^t \int \phi(y(s))\, L(K^1, K^2)(ds, dy)$$

is the collision local time of W^1 and W^2 defined and studied in [2] and [20]. Note that $L_t(W^1, W^2) = L_t(W^2, W^1)$ (see Section 1 of [2]), but it is certainly not the case that $L_t(K^1, K^2) = L_t(K^2, K^1)$ in general. Note also that we perhaps should write $L(K^1, W^2)$ for $L(K^1, K^2)$, as this object only depends on K^2 through W^2.

A substantial portion of this paper is devoted to investigating the properties of FFCLT's (see Section 3). One of the advances over what was accomplished in [2] and [20] (and the key to the rest of the paper) is the demonstration that, loosely speaking, when (K^1, K^2) are, in a suitable sense, "sub–populations" of a pair of independent historical Brownian motions with $d \leq 3$, then $L(K^1, W^2)(dt)$ can be written as $\int \ell(y, W^2)(dt)\, K_t^1(dy)$, where $\ell(y, W^2)$ is a measure called the *path-field collision local time* (PFCLT) that lives on the set of times t such that $y(t)$ is in the support of W_t^2. In essence, if y is chosen according to Wiener measure, then $\ell(y, W^2)$ is the inhomogeneous additive functional with Revuz measure W_t^2 at time t. Our main tools for studying PFCLT's is a Tanaka–type formula for $\ell(y, W^2)$ in the case when y is chosen according to Wiener measure and W^2 is a suitable sub-population of an independent super–Brownian motion. This material is of independent interest, as FFCLT's and PFCLT's can be expected to occur in any model that modifies historical or super Brownian motions by local interactions. On example is *catalytic super–Brownian motion* for which branching can only occur in the presence of a catalyst that is itself a evolving as an ordinary super–Brownian motion (see, for example, [6], [7], [17] and [23]).

1.4 Smooth measures

The last ingredient we need before describing our interacting historical processes model is a suitable state–space and path–space for the model. As one might imagine, if the initial disposition of the two populations is not sufficiently "dispersed", then the killing mechanism we wish to introduce may be so intense that the process undergoes some sort of initial catastrophe rather than evolving continuously away from its starting state. On the other hand, we can't be too restrictive in our class of initial measures, because in order to have a reasonable strong Markov property we need the process to take values in this class at all times.

Let $\mathbf{M}_{FS}(\mathbb{R}^d) \subset \mathbf{M}_F(\mathbb{R}^d)$ be the set of measures, μ, such that $\int_0^1 r^{1-d} \sup_x \mu(B(x,r))\,dr < \infty$, where $B(x,r)$ is the closed ball of radius r centred at $x \in \mathbb{R}^d$. Write $\mathbf{M}_{FS}(C)^t$ for the subset of $\mathbf{M}_F(C)^t$ consisting of measures μ with the property that $\mu(y(t) \in \cdot) \in \mathbf{M}_{FS}(\mathbb{R}^d)$, that is

$$\int_0^1 r^{1-d} \sup_{x \in \mathbb{R}^d} \mu(\{y : |y(t) - x| \leq r\})\,dr < \infty.$$

Following the pattern of notation used in Appendix A set

$$\begin{aligned} S'_C &= \{(t,\mu) \in \mathbb{R}_+ \times \mathbf{M}_F(C) : \mu \in \mathbf{M}_{FS}(C)^t\}, \\ \hat{S} &= \{(t,\mu^1,\mu^2) \in \mathbb{R}_+ \times \mathbf{M}_F(C) \times \mathbf{M}_F(C) : \mu^i \in \mathbf{M}_{FS}(C)^t,\ i = 1,2\} \end{aligned}$$

and

$$\Omega'_C = \{\omega \in C(\mathbb{R}_+, \mathbf{M}_F^\Delta(C)) : \alpha_C(\omega) < \infty,\, \beta_C(\omega) = \infty\}$$

where

$$\alpha_C = \inf\{t : \omega(t) \neq \Delta\} \text{ and } \beta_C = \inf\{t \geq \alpha_C : (t, \omega(t)) \notin S'_C\} \quad (\inf \emptyset = \infty),$$

and we use the notation $C(\mathbb{R}_+, \mathbf{M}_F^\Delta(C))$ for the subset of the space $D(\mathbb{R}_+, \mathbf{M}_F^\Delta(C))$ of càdlàg $\mathbf{M}_F^\Delta(C)$–valued functions consisting of functions h such that $\Delta \notin \{h(t-), h(t)\}$ implies $h(t-) = h(t)$ for all $t > 0$.

Let $\mathcal{F}'_C$ be the trace of the universally measurable subsets of $C(\mathbb{R}_+, M_F^\Delta(C))$ on Ω'_C

1.5 Competing species martingale problem

Here then is a martingale problem for two interacting historical processes for whom inter–species collisions are unfavourable. Let $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \geq \tau}, \hat{\mathbb{P}})$ be a filtered probability space with the filtration $(\hat{\mathcal{F}}_t)_{t \geq \tau}$ right–continuous and the σ-field $\hat{\mathcal{F}}$ universally complete. Suppose that $r_1, r_2, \tau \in \mathbb{R}_+$ and $\mu^1, \mu^2 \in \mathbf{M}_F(C)^\tau$. Suppressing dependence on (r_1, r_2), we say that the pair $(\hat{H}^1, \hat{H}^2)$ satisfies the martingale problem $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ if:

(i) $(\hat{H}_t^i)_{t \geq 0}$ as sample paths almost surely in Ω'_C, $i = 1, 2$,

(ii) $\hat{H}^i_t = \Delta$, $\forall t < \tau$ a.s., $i = 1, 2$,

(iii) $(\hat{H}^i_t)_{t\geq\tau}$ is $(\hat{\mathcal{F}}_t)_{t\geq\tau}$–predictable, $i = 1, 2$,

(iv) $L(\hat{H}^1, \hat{H}^2)$ and $L(\hat{H}^2, \hat{H}^1)$ exist,

(v) for all $\phi_1, \phi_2 \in D_S$

$$\hat{M}^1_t(\phi_1) = \hat{H}^1_t(\phi_1) - \mu^1(\phi_1) - \int_\tau^t \hat{H}^1_s(\frac{\bar{\Delta}}{2}\phi_1(s))\,ds - r_1 \int_\tau^t \int \phi_1(y)\, L(\hat{H}^1, \hat{H}^2)(ds, dy),\ t \geq \tau,$$

and

$$\hat{M}^2_t(\phi_2) = \hat{H}^2_t(\phi_2) - \mu^2(\phi_2) - \int_\tau^t \hat{H}^2_s(\frac{\bar{\Delta}}{2}\phi_2(s))\,ds - r_2 \int_\tau^t \int \phi_2(y)\, L(\hat{H}^2, \hat{H}^1)(ds, dy),\ t \geq \tau,$$

are continuous $(\hat{\mathcal{F}}_t)_{t\geq\tau}$–martingales for which $\hat{M}^i_\tau(\phi_i) = 0$,

$$\langle \hat{M}^i(\phi_i)\rangle_t = \int_\tau^t \int \phi_i(y)^2\, \hat{H}^i_s(dy)\,ds, \quad \forall t \geq \tau, \text{ a.s.}$$

and $\langle \hat{M}^1(\phi_1), \hat{M}^2(\phi_2)\rangle = 0$.

A simple application of Itô's lemma shows that if $(\hat{H}^1, \hat{H}^2)$ satisfies $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ and $\phi_1, \phi_2 \in D_{ST}$, then

$$\hat{M}^1_t(\phi_1) = \hat{H}^1_t(\phi_1(t)) - \mu^1(\phi_1(\tau)) - \int_\tau^t \hat{H}^1_s(A\phi_1(s))\,ds - r_1 \int_\tau^t \int \phi_1(s, y)\, L(\hat{H}^1, \hat{H}^2)(ds, dy),\ t \geq \tau,$$

and

$$\hat{M}^2_t(\phi_2) = \hat{H}^2_t(\phi_2(t)) - \mu^2(\phi_2(\tau)) - \int_\tau^t \hat{H}^2_s(A\phi_2(s))\,ds - r_2 \int_\tau^t \int \phi_2(s, y)\, L(\hat{H}^2, \hat{H}^1)(ds, dy),\ t \geq \tau,$$

are continuous $(\hat{\mathcal{F}}_t)_{t\geq\tau}$–martingales for which $\hat{M}^i_\tau(\phi_i) = 0$,

$$\langle \hat{M}^i(\phi_i)\rangle_t = \int_\tau^t \int \phi_i(s, y)^2\, \hat{H}^i_s(dy)\,ds$$

and $\langle \hat{M}^1(\phi_1), \hat{M}^2(\phi_2)\rangle = 0$.

Remark 1.2 Suppose that the pair $(\hat{H}^1, \hat{H}^2)$ solves the martingale problem $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ and $\hat{X}^i = \Gamma(\hat{H}^i)$. Set $\nu^i = \hat{X}^i_\tau$, so that $\nu^i(\phi) = \int \phi(y(\tau))\, \mu^i(dy)$. Then for all bounded, continuous functions f_1, f_2 with bounded, continuous first and second order partial derivatives we have

$$\hat{Z}^1_t(f_1) = \hat{X}^1_t(f_1) - \nu^1(f_1) - \int_\tau^t \hat{X}^1_s(\frac{\Delta}{2} f_1)\, ds - r_1 L_t(\hat{X}^1, \hat{X}^2)(f_1),\ t \geq \tau$$

and

$$\hat{Z}^2_t(f_2) = \hat{X}^2_t(f_2) - \nu^2(f_2) - \int_\tau^t \hat{X}^2_s(\frac{\Delta}{2} f_2)\, ds - r_2 L_t(\hat{X}^2, \hat{X}^1)(f_2),\ t \geq \tau$$

are continuous $(\hat{\mathcal{F}}_t)_{t \geq \tau}$–martingales for which $\hat{Z}^i_\tau(f_i) = 0$ and

$$\langle \hat{Z}^i(f_i), \hat{Z}^j(f_j) \rangle_t = \delta_{ij} \int_\tau^t \hat{X}^i_s((f_i)^2)\, ds.$$

Denote this latter martingale problem as $\widehat{MP}_X(\tau, \nu^1, \nu^2)$. Thus $\widehat{MP}_X(\tau, \nu^1, \nu^2)$ is the "two–way killing" martingale problem discussed in [20]. There it was shown that solutions exist if and only if $d \leq 3$, and so throughout this work we will always assume

$$\boxed{d \leq 3.}$$

Roughly speaking, the two populations don't collide enough in higher dimensions to give a non-trivial interaction — even though a non-trivial FFCLT exists for two independent super Brownian motions when $d \leq 5$ (see [2]).

Remark 1.3 For $d \leq 3$ the description of solutions $(\hat{H}^1, \hat{H}^2)$ to $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ developed in this paper can be summarised (very loosely) as follows:

- There is a pair (H^1, H^2) of independent historical Brownian motions such that $\hat{H}^i_t \leq H^i_t$ for all t.
- If we "choose a path $y \in C$ according to H^i_t", then y is a Brownian path stopped at t.
- Any such path is either absent from the support of $\hat{H}^i_t$ or present with the same infinitesimal mass (a little more precisely, the Radon–Nikodym derivative of $\hat{H}^i_t$ against H^i_t is $\{0,1\}$–valued).
- The paths y in the support of H^i_t that are also in the support of $\hat{H}^i_t$ are the ones that have survived being killed off according to the random measure $r_i \ell(y, \hat{X}^j)$ (where $j = 3 - i$).

The non-existence of solutions in $d = 4, 5$ is thus related to the fact that a Brownian motion and an independent super Brownian motion don't collide for $d \geq 4$ and that the natural sequence of approximate PFCLT's fails to converge (rather than converges to 0). It is worth noting that non-existence of solutions for $d = 4, 5$ holds not only for the models considered in this paper, but can also be established for models with other local interactions, such as the one obtained by modifying the above intuitive description so that particles are killed as soon as they encounter the other population (that is, by setting $r_i = \infty$). This will be discussed in a forthcoming paper with Martin Barlow.

Our major result concerning $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ is the following.

Theorem 1.4 *Let $r_1, r_2 \geq 0$ and $(\tau, \mu^1, \mu^2) \in \hat{S}$.*
(a) There is a solution $(\hat{H}^1, \hat{H}^2)$ of $\widehat{MP}_H(\tau, \mu^1, \mu^2)$.
(b) There is a law $\hat{\mathbb{P}}^{\tau,\mu^1,\mu^2}$ on $(\Omega'_C \times \Omega'_C, \mathcal{F}'_C \times \mathcal{F}'_C)$ which is the law of any solution of $\widehat{MP}_H(\tau, \mu^1, \mu^2)$. Moreover, for any $A \in \mathcal{F}'_C \times \mathcal{F}'_C$ the map $(\tau, \mu^1, \mu^2) \mapsto \hat{\mathbb{P}}^{\tau,\mu^1,\mu^2}(A)$ is Borel measurable from $\hat{S}$ to $\mathbb{R}$.
(c) Let $\hat{H} = (\hat{H}^1, \hat{H}^2)$ be a solution of $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ on $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t\geq\tau}, \hat{\mathbb{P}})$ and let $T \geq \tau$ be an a.s. finite $(\hat{\mathcal{F}}_t)_{t\geq\tau}$–stopping time. Then

$$\hat{\mathbb{P}}\left[\phi(\hat{H}_{T+\cdot}) \mid \hat{\mathcal{F}}_T\right](\omega) = \int \phi(\omega'(\cdot + T(\omega)))\, \hat{\mathbb{P}}^{T(\omega),\hat{H}_T(\omega)}(d\omega'), \quad \text{for } \hat{\mathbb{P}}\text{-a.e. } \omega \in \hat{\Omega}$$

for any bounded, measurable function ϕ on $C(\mathbb{R}_+, \mathbf{M}_F(C) \times \mathbf{M}_F(C))$.

Remark 1.5 If $r_1 = 0$ or $r_2 = 0$, then one may easily adapt the methods of [20] (Section 4) to see that $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ is well-posed, and so we will always assume $r_1, r_2 > 0$.

1.6 Approximating systems

Remark 1.3 and the way we make it precise in the course of this paper provides a strong justification for our claim that $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ and $\widehat{MP}_X(\tau, \nu^1, \nu^2)$ realistically capture certain features of two evolving populations subject to purely local inter-species competition. However, additional support for this assertion and a firmer connection with more biologically realistic models would be established if we could show that models satisfying $\widehat{MP}_X(\tau, \nu^1, \nu^2)$ arise as high–density limits of discrete branching particle systems that incorporate mutually disadvantageous encounters between two populations. One such class of models is given by the following family of two–type, long–range contact processes.

Consider two types of particles that occupy sites on the rescaled integer lattice $N^{-\frac{1}{2}}M_N^{-1}\mathbb{Z}^d$, where $N, M_N \in \mathbb{N}$. Several particles of one type can occcupy a site, but particles of different types cannot co-exist at a site. At rate N, a particle of either type at site $x \in N^{-\frac{1}{2}}M_N^{-1}\mathbb{Z}^d$ either dies (with probability $\frac{1}{2}$) or (also with probability $\frac{1}{2}$) it selects a site at random from the set of sites

$$\{y \in N^{-\frac{1}{2}}M_N^{-1}\mathbb{Z}^d : |y_i - x_i| \leq \sqrt{6}N^{-\frac{1}{2}},\ 1 \leq i \leq d\}$$

and produces an offspring there if no particle of the other type is already occupying the site. Define a pair of càdlàg $\mathbf{M}_F(\mathbb{R}^d)$–valued processes $(\hat{X}^{N,1}, \hat{X}^{N,2})$ by

$$\hat{X}_t^{N,i}(A) = N^{-1}\#(\text{particles of type } i \text{ in } A).$$

Increasing M_N lessens the rate at which attempted births fail because of inter–species collisions. Heuristic calculations show that if $d \le 3$, M_N is taken as $\sqrt{6} r^{-\frac{1}{d}} N^{\frac{2}{d}-\frac{1}{2}}$ with $r > 0$ fixed and $\hat{X}_0^{N,i} \to \nu^i$ as $N \to \infty$, $i = 1, 2$, then $(\hat{X}^{N,1}, \hat{X}^{N,2})$ converges in distribution to a solution of $\widehat{MP}_X(0, \nu^1, \nu^2)$ with $r_1 = r_2 = r$. If M_N grows slower than this critical rate, then we expect that the limit exists and solves a martingale problem in which $L(\hat{X}^i, \hat{X}^j)(dt, dx)$ is replaced by a more singular measure living on the times and places of "collisions". In the extreme case when $M_N = 1$ for all N, we expect the limit to be the model discussed above in which collisions of particles with the other population are immediately fatal. On the other hand, if M_N grows faster than the critical rate, then we expect the interaction to disappear and the limit to be just a pair of independent super–Brownian motions.

Moreover, just as super–Brownian motion is "universal" in that many superficially different sequences of branching particle systems converge to it, we expect that a large class of models that incorporate near–critical branching, spatial motion converging to Brownian motion and some form of inter–species competition will converge in the high–density limit to a solution of $\widehat{MP}_X(\tau, \nu^1, \nu^2)$. We leave the investigation of such questions to future work.

There are also other simple, not–so–singular, measure–valued diffusion models that converge to solutions of $\widehat{MP}_X(\tau, \nu^1, \nu^2)$ and $\widehat{MP}_H(\tau, \mu^1, \mu^2)$. For example, let $\widehat{MP}_X^{\epsilon}(\tau, \nu^1, \nu^2)$ and $\widehat{MP}_H^{\epsilon}(\tau, \mu^1, \mu^2)$ denote the analogously defined martingale problems with the field–field collision local times $L(\cdot, \cdot)$ replaced by the approximations $L^{\epsilon}(\cdot, \cdot)$. Dawson's Girsanov theorem shows that $\widehat{MP}_X^{\epsilon}(\tau, \nu^1, \nu^2)$ is well–posed for $\nu^1, \nu^2 \in \mathbf{M}_F(\mathbb{R}^d)$ (see Theorem 2.3 (b) in [20]). We now set $\tau = 0$ and identify $\mathbf{M}_{FS}(\mathbb{R}^d)$ with $\mathbf{M}_{FS}(C)^0$. Let $\hat{\mathbb{Q}}_{\epsilon}^{\nu^1,\nu^2}$ denote the law on $\Omega_X \times \Omega_X$ (where $\Omega_X \equiv \Omega_X[0, \infty[$) of the solution to $\widehat{MP}_X^{\epsilon}(0, \nu^1, \nu^2)$. The same Girsanov argument shows that the law, $\hat{\mathbb{P}}_{\epsilon}^{\nu^1,\nu^2}$ on $\Omega_H \times \Omega_H$ (where $\Omega_H \equiv \Omega_H[0, \infty[$), of any solution to $\widehat{MP}_H^{\epsilon}(0, \nu^1, \nu^2)$ is unique. If $\nu^1, \nu^2 \in \mathbf{M}_{FS}(\mathbb{R}^d)$, then $\{\hat{\mathbb{Q}}_{\epsilon}^{\nu^1,\nu^2} : \epsilon \in [0, 1]\}$ is tight and any subsequential weak limit satisfies $\widehat{MP}_X(0, \nu^1, \nu^2)$ (see Theorem 3.6 of [20]). Theorem 1.4 allows us to strengthen this conclusion. We let $(\hat{H}^1, \hat{H}^2)$ denote the coordinate maps on $\Omega_H \times \Omega_H$ in the following result, which will be proved at the end of Section 7.

Theorem 1.6 *Assume* $\nu^1, \nu^2 \in \mathbf{M}_{FS}(\mathbb{R}^d)$.
(a) $\hat{\mathbb{Q}}_{\epsilon}^{\nu^1,\nu^2}$ *converges weakly to* $\hat{\mathbb{P}}^{0,\nu^1,\nu^2}((\Gamma(\hat{H}^1), \Gamma(\hat{H}^2)) \in \cdot)$ *on* $\Omega_X \times \Omega_X$ *as* $\epsilon \downarrow 0$.
(b) $\hat{\mathbb{P}}_{\epsilon}^{\nu^1,\nu^2}$ *converges weakly to* $\hat{\mathbb{P}}^{0,\nu^1,\nu^2}$ *on* $\Omega_H \times \Omega_H$ *as* $\epsilon \downarrow 0$.

1.7 Overview of the paper

Our overall strategy for proving Theorem 1.4 is to make rigorous sense of the intuition laid out in Remark 1.3.

In Sections 2 and 3 we establish the existence and basic properties of path–field collision local times and field–field collision local times via Tanaka–like semimartingale decompositions. This involves some work on energies and potentials of super-Brownian motion, as one might expect from our picture of the path–field collision local time as an inhomogeneous additive functional of Brownian motion with super-Brownian motion as the Revuz measure.

The most naive idea for making sense of the notion that "a path in the support of H_t^i that is also in the support of $\hat{H}_t^i$ is one that has survived being killing according to a multiple of its path–field collision local time against $\hat{X}^j = \Gamma(\hat{H}^j)$ $(j = 3 - i)$" is to somehow equip each path y in the support of H_t^i with the points of a Poisson process on $[\tau, \infty[$ with intensity $r_i \ell(y, \hat{X}^j)$ and then to kill off paths that receive one or more points. Of course, there is something a little circular in this, because it appears we already need to have $(\hat{H}^1, \hat{H}^2)$ in order to define such Poisson processes.

We proceed a little more obliquely by, loosely speaking, constructing a pair of independent historical Brownian motions (H^1, H^2) in which each path y in the support of H_t^i is equipped with a number of $\mathbb{R}_+ \times [0,1]$–valued marks. Conditional on (H^1, H^2), the marks are laid down according to a Poisson process with intensity $r_i \ell(y, H^j) \otimes m$, where $j = 3 - i$ and m is Lebesgue measure on $[0,1]$. Moreover, the marks inherit the branching structure of (H^1, H^2). For example, if y_1 in the support of $H_{t_1}^i$ and y_2 in the support of $H_{t_2}^i$ are such that that $y_1^s = y_2^s$ for some $s \leq t_1 \wedge t_2$ and the two paths diverge after time s, then the corresponding marks coincide up to time s but are laid down independently (conditional on (H^1, H^2)) thereafter. We call this pair of historical Brownian motions with added marks the *driving process*. With the aid of the driving process, we can define $(\hat{H}^1, \hat{H}^2)$ as the pair such that if we kill a path y in the support of H_t^i at the first time that y has an attached mark (u, z) for which the Radon–Nikodym derivative $(d\ell(y, \hat{X}^j)/d\ell(y, X^j))(u)$ is greater than z, then we recover $\hat{H}_t^i$. We call this implicit definition of $(\hat{H}^1, \hat{H}^2)$ the *strong equation*. We establish pathwise existence and uniqueness of solutions to the strong equation in Section 5 using a fixed–point argument, and show that the unique solution satisfies $\widehat{MP}_H(\tau, \mu^1, \mu^2)$.

The key to proving uniqueness of solutions to $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ (that is, Theorem 1.4(b)) is then to show that given any solution such solution $(\hat{H}^1, \hat{H}^2)$ we can, with extra randomness, build a driving process such that the $(\hat{H}^1, \hat{H}^2)$ is the solution of the strong equation with respect to the driving process. This is carried out in two stages in Sections 6 and 7. Part of the argument depends on working with processes such as $(H_t^{i,\epsilon})_{t \geq \tau + \epsilon}$, where $H_t^{i,\epsilon}(\phi) = \int \phi(y^{t-\epsilon})\, H_t^i(dy)$. The random measure $H_t^{i,\epsilon}$ is atomic, the corresponding set–valued process of supports is a branching particle system and $H^{i,\epsilon}$ converges to H^i as $\epsilon \downarrow 0$. Several results of this sort are in [9], but we need to obtain a number of stronger and more general results. The advantage of working with such approximating, embedded particle systems is that attaching the necessary marks to paths in order to reconstruct the driving processes (or, rather, their approximating particle systems) is reduced to a fairly straightforward exercise involving a finite number of branching paths.

In Section 8 we observe that, as expected, this uniqueness translates into the strong Markov property Theorem 1.4(c), and this in turn gives the existence of strong Markov solutions to $\widehat{MP}_X(\tau, \nu^1, \nu^2)$.

We remark that throughout the paper we are continually dealing with processes of the form $G_t(\phi_t)$, where G is a process such as historical Brownian motion that takes values in the space of finite measures on some space of paths and ϕ_t is defined as the result of performing some sort of Lebesgue or stochastic integration along a path (possibly with a random integrand). We develop a number of new stochastic calculus tools to obtain the the semimartingale decompositions of such processes. This sort of *historical stochastic calculus* has proven to be useful in a number of other contexts such as superprocesses with spatial interations in [30] and [31], nonlinear superprocesses with McKean–Vlasov–like mean field interactions in [29], and explicit stochastic integral representations of functionals of superprocesses in [21]. Also, in Sections 4 and 6 we require new results on the existence and regularity of superprocesses with immigration for quite general spatial motions. These results are presented in Appendix A. The main novelty lies in establishing Hunt and sample-path continuity properties and in deriving semimartingale decompositions.

Unfortunately, we are unable to show uniqueness of solutions for $\widehat{MP}_X(\tau,\nu^1,\nu^2)$. This has been accomplished recently by Leonid Mytnik [27] using a novel and complex duality argument.

The methods developed in this work have proved useful in other settings. For example, they also apply to a related model in which the masses of particles are reduced by inter–species collisions. Rather than state the martingale problem for this model, we go straight to the corresponding strong equation. Here the "driving process" is just a pair (H^1, H^2) of independent historical Brownian motions and the strong equation is

$$\hat{H}_t^i(\phi) = \int \phi(y) \exp(-r_i \ell_t(y, \hat{H}^j))\, H_t^i(dy)$$

for $i = 1, 2$ and $j = 3 - i$. This model will be investigated in a forthcoming paper with Martin Barlow, where it will be shown that a fixed–point technique similar to the one used here suffices to establish existence and uniqueness of solutions. Because of their utility in other contexts, we emphasise the tools for dealing with strong equations and the historical stochastic calculus associated with collision local times as an important feature of this work.

This is a rather large paper with a (necessarily) extensive amount of notation. To assist the reader, we have collected some of the notation together with where it first appears into a table in Appendix D.

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2 Brownian collision local times

Let $(\mathcal{M}_t)_{t\geq\tau}$ denote the universal completion of the canonical right–continuous filtration on $\Omega_X[\tau,\infty[$, where $\tau \geq 0$ is fixed. Write $(\mathcal{C}_t)_{t\geq\tau}$ for the canonical filtration on $C([\tau,\infty[,\mathbb{R}^d)$. Put $\hat{\Omega}_X[\tau,\infty[= C([\tau,\infty[,\mathbb{R}^d) \times \Omega_X[\tau,\infty[$ and $\hat{\mathcal{M}}_t = \mathcal{C}_t \times \mathcal{M}_t$, $t \geq \tau$.

For $\epsilon > 0$, define a continuous $(\hat{\mathcal{M}}_t)_{t\geq 0}$–predictable process on $\hat{\Omega}_X[\tau,\infty[$ by

$$\ell^\epsilon_{\tau,t}(y,\nu) = \int_\tau^t \int p_\epsilon(y(s)-x)\nu_s(dx)\,ds, \quad t \geq \tau.$$

Dependence on τ will usually be suppressed. Note that the expression for $\ell^\epsilon_{\tau,t}(y,\nu)$ still makes sense whenever $\nu : [\tau,\infty[\to \mathbf{M}_F(\mathbb{R}^d)$ is such that $s \mapsto \nu_s(A)$ is Borel measurable for all $A \in \mathcal{B}(\mathbb{R}^d)$ and $\sup_{\tau\leq s\leq t} \nu_s(1) < \infty$, $\forall t > 0$. We will use this generality on occasion in the next section.

Notation 2.1 Let $C_\uparrow$ be the set of nondecreasing continuous functions, $f(t)$, on $\mathbb{R}_+$ such that $f(0) = 0$. If $g \in C_\uparrow$, let

$$\mathbf{M}^g_{FS}(\mathbb{R}^d) = \{\mu \in \mathbf{M}_F(\mathbb{R}^d) : \int_0^\epsilon r^{1-d} \sup_x \mu(B(x,r))\,dr \leq g(\epsilon), \quad \forall \epsilon \in [0,1]\}.$$

Recall from Section 1 that $\mathbf{M}_{FS}(\mathbb{R}^d)$ is the set of measures, μ, such that $\int_0^1 r^{1-d} \sup_x \mu(B(x,r))\,dr < \infty$. Finally let

$$\bar{g}(\nu,\epsilon) = \sup_{\tau'\geq\tau, x\in\mathbb{R}^d} \int_0^\epsilon p_s(x-y)\nu_{\tau'+s}(dy)\,ds$$

and

$$\Omega_{XS}[\tau,\infty[= \{\nu \in \Omega_X[\tau,\infty[: \lim_{\epsilon\downarrow 0} \bar{g}(\nu,\epsilon) = 0 \text{ and } \nu_t = 0 \quad \text{for } t \text{ sufficiently large}\}.$$

Let $W = (T, B_0)$ be a space–time Brownian motion, that is, a diffusion with state–space $\mathbb{R}_+ \times \mathbb{R}^d$ and laws

$$Q^{s,x}(W \in A) = P^x(\{y : (s+\cdot, y(\cdot)) \in A\}),$$

for A a Borel subset of $C(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}^d)$, $s \geq 0$ and $x \in \mathbb{R}^d$, where P^x is Wiener measure starting at x. Let $B(t) = B_0(t - T(0))$, $t \geq T(0)$, be Brownian motion starting at x at time s.

The following result is an easy consequence of Theorem 4.1 and Proposition 4.7 of [20] and their proofs. Recall from Subsection 1.2 that $\mathbb{P}^{\tau,m}$ is the law of super-Brownian motion started at time τ at the measure m.

Theorem 2.2 *(a) For each path ν in $\Omega_{XS}[\tau,\infty[$, there is a $(\mathcal{C}_t)_{t\geq\tau}$–predictable map $\hat{\ell}_t(y,\nu)$ on $[\tau,\infty[\times C([\tau,\infty[,\mathbb{R}^d)$ such that $\hat{\ell}_{\tau+\cdot}(y,\nu) \in C_\uparrow$, $Q^{\tau,x}$-a.s., for each x in $\mathbb{R}^d$ and*

$$\lim_{\epsilon\downarrow 0} \sup_{x\in\mathbb{R}^d} Q^{\tau,x}\left[\sup_{t\geq\tau}\left|\ell^\epsilon_t(B,\nu) - \hat{\ell}_t(B,\nu)\right|^2\right] = 0. \tag{2.1}$$

(b) For each $g \in C_\uparrow$ there is Borel subset, $\Omega^g_{XS}[\tau,\infty[$, of $\Omega_{XS}[\tau,\infty[$, a sequence $\epsilon_n \downarrow 0$, and an $(\hat{\mathcal{M}}_t)_{t\geq\tau}$–predictable process $\ell_t(y,\nu)$on $\hat{\Omega}_X[\tau,\infty[$ satisfying:

(i) $\bigcup_{g\in C_\uparrow} \Omega^g_{XS}[\tau,\infty[= \Omega_{XS}[\tau,\infty[$ *and* $\mathbb{P}^{\tau,m}(\Omega^g_{XS}[\tau,\infty[) = 1$, $\forall m \in \mathbf{M}^g_{FS}(\mathbb{R}^d)$.

(ii) If $\nu \in \Omega^g_{XS}[\tau,\infty[$, *then* $\ell_t(B,\nu) = \hat{\ell}_t(B,\nu)$, $\forall t \geq \tau$, $Q^{\tau,x}$*-a.s.,* $\forall x \in \mathbb{R}^d$ *and so (2.1) holds with* ℓ *in place of* $\hat{\ell}$.

(iii) Let $(B_t)_{t\geq\tau}$ *be a Brownian motion on* $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P}')$ *starting with law* μ *at time* τ, *and let* $\tau \leq T \leq \infty$ *be an* $(\mathcal{F}_t)_{t\geq\tau}$*-stopping time. Then for all* ν *in* $\Omega^g_{XS}[\tau,\infty[$,

$$\lim_{\epsilon\downarrow 0} \mathbb{P}'\left[\sup_{\tau\leq t\leq T,\, t<\infty} \left|\ell^\epsilon_t(B^T,\nu) - \ell_t(B^T,\nu)\right|^2\right] = 0$$

and

$$\lim_{n\to\infty} \sup_{\tau\leq t\leq T,\, t<\infty} \left|\ell^{\epsilon_n}_t(B^T,\nu) - \ell_t(B^T,\nu)\right| = 0 \quad \mathbb{P}'\text{-}a.s.$$

(iv) For all (t,y,ν), $\ell_t(y,\nu) = \ell_t(y^t,\nu^t)$, *where* $\nu^t_s = \nu_s$ *for* $s \leq t$ *and* $\nu^t_s = 0$ *otherwise.*

Proof. Part (a) is immediate from the results cited above.

For (b), one easily can use these arguments to define an increasing sequence of Borel subsets $\{\Omega^{g,n}_{XS}[\tau,\infty[: n \in \mathbb{N}\}$ such that $\Omega^g_{XS}[\tau,\infty[= \bigcup_n \Omega^{g,n}_{XS}[\tau,\infty[$ satisfies (i), and a decreasing sequence of positive numbers $\epsilon_n \downarrow 0$ such that

$$\sup\left\{Q^{\tau,x}\left[\sup_{t\geq\tau}\left|\ell^\epsilon_t(y,\nu) - \ell^\delta_t(y,\nu)\right|^2\right] \;:\; 0<\epsilon,\delta\leq\epsilon_n,\ x\in\mathbb{R}^d,\ \nu\in\Omega^{g,n}_{XS}[\tau,\infty[\right\} < 2^{-n}. \tag{2.2}$$

(An explicit definition of $\Omega^{g,n}_{XS}[\tau,\infty[$ is given below.) Now set

$$\ell_t(y,\nu) = \begin{cases} \lim_{n\to\infty} \ell^{\epsilon_n}_t(y,\nu), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Property (ii) is then clear. We need only comment on the extension to stopped Brownian motions in (iii). Note that

$$\ell^\epsilon_t(B^T,\nu) = \ell^\epsilon_t(B,\nu) \quad \text{and} \quad \ell_t(B^T,\nu) = \ell_t(B,\nu) \quad \text{for all} \quad \tau\leq t\leq T, \quad t<\infty \quad \text{a.s.}$$

Hence the general case follows at once from the $T \equiv \infty$ case which is obvious from (2.2).

□

Remark 2.3 In fact one may take

$$\Omega^g_{XS}[\tau,\infty[= \left\{\nu\in\Omega_{XS}[\tau,\infty[:\ \bar{g}(\nu,\epsilon) \leq N\left(\epsilon^{1/4} + g(c\epsilon^{1/4})\right),\ \forall\epsilon\in[0,1],\ \text{and some } N \text{ in } \mathbb{N}\right\}$$

for some appropriate universal constant c and

$$\Omega^{g,n}_{XS}[\tau,\infty[=\left\{\nu\in\Omega_{XS}[\tau,\infty[:\int_\tau^\infty \nu_s(1)\,ds\le K_n,\quad \bar g(\nu,\epsilon)\le K_n(\epsilon^{1/4}+g(c\epsilon^{1/4})),\ \forall\epsilon\in[0,1]\right\}$$

for a suitable sequence $\{K_n\}$. Note that $\nu^1\in\Omega^g_{XS}[\tau,\infty[$, $\nu^2\in\Omega_X[\tau,\infty[$ and $\nu^2\le\nu^1$ imply $\nu^2\in\Omega^g_{XS}[\tau,\infty[$. In practice we will work with a fixed g and hence a fixed version of $\ell_t(y,\nu)$ given by (2.3). It will, however, be convenient on occasion to use a subsequence of $\{\epsilon_n\}$ in (2.3) to define another version of $\ell_t(y,\nu)$.

We now want to extend $\ell_t(y,\nu)$ to the case when $\nu=X$ is random. This extension in non-trivial. It is not hard to see that if $X=\Gamma(H)$, H is a historical Brownian motion and y "is chosen according to H_T" for some $T\ge\tau$ (see the definition of Campbell measure in the next section), then $(y_t)_{t\ge\tau}$ will be a Brownian motion stopped at time T but $\ell^\epsilon_t(y,X)$ will not converge in any reasonable manner.

Notation 2.4 Put $h_\alpha(x)=\frac12\int_0^\infty e^{-\alpha s/2}p_s(x)\,ds$ where $p_s(x)=(2\pi s)^{-d/2}\exp\{-|x|^2/2s\}$, with $\alpha>0$ if $d\le 2$ and $\alpha\ge 0$ if $d=3$. Set $h_0(x)=1+\log^+(1/|x|)$ if $d=2$ and $h_0(x)=1$ if $d=1$. We abuse the notation and write $h_\alpha(x)=h_\alpha(|x|)$ at times. Then we of course have:

$$\begin{aligned}&h_0(x)=c|x|^{-1},\quad d=3\\&c_1(\alpha)\big(1+\log^+(1/|x|)\big)\le h_\alpha(x)\le c_2(\alpha)\big(1+\log^+(1/|x|),\quad d=2,\\&h_\alpha(x)\le c(\alpha),\quad d=1.\end{aligned}\tag{2.4}$$

Definition 2.5 Fix a filtered probability space $(\Omega,\mathcal F,(\mathcal F_t)_{t\ge\tau},\mathbb P')$. Given an $(\mathcal F_t)_{t\ge\tau}$–stopping time $\tau\le T\le\infty$, an $\mathbb R^d$–valued process $(B_t)_{t\ge\tau}=(B^1_t,\dots,B^d_t)_{t\ge\tau}$ is an $(\mathcal F_t)_{t\ge\tau}$*–Brownian motion stopped at* T if $(B^i_t-B^i_\tau)_{t\ge\tau}$ is a continuous $(\mathcal F_t)_{t\ge\tau}$–martingale for each i and

$$\langle B^i,B^j\rangle_t=\delta_{ij}(t\wedge T-\tau)\quad\forall i,j,\quad t\ge\tau,\quad\text{a.s.}$$

Here, and elsewhere, $\langle M,N\rangle_t\equiv\langle M-M_\tau,N-N_\tau\rangle_t$.

The following Tanaka–type representation is the main result of this section.

Theorem 2.6 *Let $(X_t)_{t\ge\tau}$ and $(A_t)_{t\ge\tau}$ be right–continuous, $(\mathcal F_t)_{t\ge\tau}$–adapted, $\mathbf M_F(\mathbb R^d)$–valued processes on $(\Omega,\mathcal F,(\mathcal F_t)_{t\ge\tau},\mathbb P')$. Assume A has non-decreasing paths a.s. and $A_\tau=0$. Let $(B_t)_{t\ge\tau}$ be a (d-dimensional) $(\mathcal F_t)_{t\ge\tau}$–Brownian motion stopped at T. Assume*

(i) For all $\phi\in C^\infty_K(\mathbb R^d)$, the process $M_t(\phi)=X_t(\phi)-X_\tau(\phi)-\int_\tau^t X_s\left(\frac{\Delta\phi}{2}\right)ds-A_t(\phi)$, $t\ge\tau$, is a continuous $(\mathcal F_t)_{t\ge\tau}$–martingale such that $\langle M(\phi)\rangle_t=\int_\tau^t X_s(\phi^2)\,ds$ and $\langle M(\phi),B^j\rangle_t=0$, $1\le j\le d$, $t\ge\tau$, a.s.

(ii) B_τ has law m and $X_\tau = \mu$ a.s., where $\mu \in \mathbf{M}_F(\mathbb{R}^d)$ with $\int \left[\int h_0(x-y)\mu(dx)\right]^2 m(dy) < \infty$.

Then there is an a.s. continuous, non–decreasing, square–integrable $(\mathcal{F}_t)_{t\geq\tau}$–predictable process, $\ell_t(B,X)$, which satisfies

$$\sup_{\tau\leq t\leq u\wedge T} |\ell_t^\epsilon(B,X) - \ell_t(B,X)| \xrightarrow{L^2} 0 \quad \text{as } \epsilon\downarrow 0,\ \forall u > \tau, \tag{2.5}$$

and

$$\begin{aligned}\int h_\alpha(x-B_t)X_t(dx) &= \int h_\alpha(x-B_\tau)\mu(dx) \\ &\quad + \alpha\int_\tau^t\int h_\alpha(x-B_s)X_s(dx)\,ds - \int_\tau^t\int h_\alpha(x-B_s)A(ds,dx) \\ &\quad + \int_\tau^t\int h_\alpha(x-B_s)M(ds,dx) - \int_\tau^t\left(\int \vec{\nabla} h_\alpha(x-B_s)X_s(dx)\right)\cdot dB_s \\ &\quad - \ell_t(B,X), \quad \forall \tau\leq t\leq T,\ a.s.\end{aligned} \tag{2.6}$$

Each of the terms, T_t, in (2.6) satisfies $\mathbb{P}'\left[\sup_{\tau\leq t\leq u\wedge T}|T_t|^2\right] < \infty$, the fourth and fifth terms on the right are continuous martingales, all the terms on the right are a.s. continuous except perhaps the third term which will also be a.s. continuous if $A_t(\mathbb{R}^d)$ is.

Remark 2.7 (a) The martingales $M_t(\phi)$ in hypothesis (i) extend to a family of continuous local martingales, $M_t(\phi) = \int_\tau^t\int \phi(s,\omega,x)M(ds,dx)$, where ϕ is $\mathcal{P}(\mathcal{F}_t)\times\mathcal{B}(\mathbb{R}^d)$ measurable ($\mathcal{P}(\mathcal{F}_t)$ is the predictable σ-field on $[\tau,\infty[\times\Omega)$ and $\langle M(\phi)\rangle_t = \int_\tau^t\int\phi(s,\omega,x)^2X_s(dx)\,ds < \infty$, $\forall t>\tau$, a.s. (see Ch. 2 of [32] and Section 2 of [30]). This extension is used in the Tanaka Formula (2.6).
(b) The construction in Section 5 of [2] will allow us to construct a super-Brownian motion $(X_t^0)_{t\geq\tau}$ such that $X_\tau^0 = \mu$ and $X_t^0 \geq X_t$, $\forall t\geq\tau$, a.s. If M^0 is the orthogonal martingale measure associated with X^0, then this construction also gives

$$\langle M^0(\phi), B^j\rangle_t = 0, \quad \forall t\geq\tau,\ 1\leq j\leq d, \ \text{ a.s., } \forall\phi\in C_K^\infty(\mathbb{R}^d). \tag{2.7}$$

(c) Theorem 2.6 remains valid if the condition on $\langle M(\phi)\rangle$ in hypothesis (i) is weakened to $\langle M(\phi)\rangle_t \leq \gamma\int_\tau^t X_s(\phi^2)\,ds$ $\forall t\geq\tau$ a.s. and all ϕ in $C_K^\infty(\mathbb{R}^d)$ for some $\gamma>0$. We must, however, now assume the existence of X^0 as in (b) and satisfying (2.7). The proof is virtually the same. This extension has proved useful in the two–type model discussed at the end of Section 1 in which inter–species collisions reduce the masses of the colliding particles.

In this work we will usually assume $\mu\in\mathbf{M}_{FS}(\mathbb{R}^d)$ and so the following result shows that hypothesis (ii) will hold.

Corollary 2.8 *If $\mu \in \mathbf{M}_{FS}(\mathbb{R}^d)$, then hypothesis (ii) of Theorem 2.6 is valid for any law m. In particular, the conclusion of Theorem 2.6 will hold whenever hypothesis (i) of that result does.*

Proof. Let $\nu_y([0,r[) = \mu(B(y,r))$. Integration by parts shows that if $y \in \mathbb{R}^d$, then

$$\begin{aligned}\int h_0(x-y)d\mu(x) &= \int_0^\infty h_0(r)\nu_y(dr)\\ &\le \int_0^1 h_0(r)\nu_y(dr) + h_0(1)\mu(\mathbb{R}^d)\\ &\le 2h_0(1)\mu(\mathbb{R}^d) + \int_0^1 cr^{1-d}\nu_y([0,r])dr\\ &= c(\mu) < \infty\end{aligned}$$

because of the hypothesis on μ. The result follows.

$\square$

Proof of Theorem 2.6 By adding an independent Brownian motion to B after time T to obtain a Brownian motion $\tilde{B}$ (not stopped at T!) and setting $\ell_t(B,X) = \ell_{t\wedge T}(\tilde{B},y)$, one easily derives the general case from the $T \equiv \infty$ case. Hence we may set $T \equiv \infty$ in what follows.

Choose $g \in C_K^\infty(\mathbb{R}^d)$ such that $\mathbf{1}\{|x| \le 1\} \le g(x) \le \mathbf{1}\{|x| \le 2\}$ and let $g_n(x) = g(x/n)$. Consider hypothesis (i) with $\phi = g_n$ and use $X_t \le X_t^0$ (see Remark 2.7(b)) and dominated convergence to see that each of the terms in (i), except perhaps $A_t(g_n)$ converges in L^1 as $n \to \infty$. We therefore deduce $A_t(g_n) \xrightarrow{L^1} A_t(1)$, (i) holds for $\phi \equiv 1$ and

$$\mathbb{P}'[A_t(1)] < \infty, \quad \forall t \ge \tau. \tag{2.8}$$

Let $\phi(x,y) = \phi_1(x)\phi_2(y)$ for $\phi_i \in C_K^\infty(\mathbb{R}^d)$. Write $\tilde{\Delta}\phi$ for the $2d$–dimensional Laplacian and $\vec{\nabla}_2\phi \in \mathbb{R}^d$ denotes the partial gradient with respect to the last d variables. Then hypothesis (i), Itô's Lemma and integration by parts give that

$$\begin{aligned}\int \phi(x,B_t)X_t(dx) &= \int \phi(x,B_\tau)\mu(dx) + \int_\tau^t \int \frac{\tilde{\Delta}}{2}\phi(x,B_s)X_s(dx)\,ds\\ &\quad - \int_\tau^t \int \phi(x,B_s)A(ds,dr) + \int_\tau^t \int \phi(x,B_s)M(ds,dx)\\ &\quad + \int_\tau^t \left(\int \vec{\nabla}_2\phi(x,B_s)X_s(dx)\right)\cdot dB_s, \quad \forall t \ge \tau, \text{ a.s.}\end{aligned} \tag{2.9}$$

This remains valid for ϕ in the algebra $\mathcal{A}$ of linear combinations of the above form. Now let $\phi \in C_K^2(\mathbb{R}^{2d})$. By making minor modifications in the last part of the proof of Proposition 5.1.1 of [18]one finds there are $\{\phi_n\}$ in $\mathcal{A}$ such that $\phi_n \to \phi$ and $\tilde{\Delta}\phi_n \to \tilde{\Delta}\phi$ uniformly as $n \to \infty$. Each of the terms in (2.9) with $\phi = \phi_n$ converges in L^1 to the corresponding term with ϕ (use $X \le X^0$ again, and also (2.8) to handle the integrals with respect to A) except perhaps the last. To analyze this term first note that the stochastic integral representation property of super-Brownian motion (Theorem 1.1 of [20]) and ordinary Brownian motion, and (2.7) show that X^0 and B are independent. If $U_K = \inf\{t : X_t^0(1) \ge K\}$ and $\psi_n = \phi - \phi_n$, then for $K > \mu(1)$

$$\begin{aligned}
&\mathbb{P}'\left[\int_\tau^t \mathbf{1}\{s \le U_K\}\left|\int \vec{\nabla}_2\psi_n(x, B_s)X_s(dx)\right|^2 ds\right] \\
&\le K\mathbb{P}'\left[\int_\tau^t \int |\vec{\nabla}_2\psi_n(x, B_s)|^2 X_s^0(dx)\, ds\right] \\
&= KP\left[\int_\tau^t |\vec{\nabla}_2\psi_n(\bar{B}_s)|^2\, ds\right],
\end{aligned} \tag{2.10}$$

where $\bar{B}$ is a $2d$–dimensional Brownian motion starting at time τ with law $\mu \otimes m$ under P (note that μ is not, in general, a probability measure and so we are using the term "law" somewhat loosely). Using Itô's Lemma we see that the right side of (2.10) equals

$$KP\left[\left(\psi_n(\bar{B}_t) - \psi_n(\bar{B}_\tau) - \int_\tau^t \frac{\tilde{\Delta}}{2}\psi_n(\bar{B}_s)\, ds\right)^2\right] \to 0 \quad \text{as} \quad n \to \infty.$$

We now may let $n \to \infty$ in (2.9) (with $\phi = \phi_n$) to see that (2.9) holds for all ϕ in $C_K^2(\mathbb{R}^d)$ (the result is valid for all $t \ge \tau$ by right–continuity). By a truncation argument, as in the derivation of (2.8), one readily sees that (2.9) holds for ϕ in $C_b^2(\mathbb{R}^{2d})$ (that is, for ϕ bounded with bounded continuous partial derivatives of order 2 or less). The bound $X \le X^0$ easily shows that each of the stochastic integrals are L^2 martingales.

Let $\bar{R}_\alpha$ be the α-resolvent for $2d$–dimensional Brownian motion, set $\bar{p}_\epsilon(x, y) = p_\epsilon(x - y)$ and define

$$\bar{h}_\epsilon(x, y) = \bar{R}_\alpha\bar{p}_\epsilon(x, y) = \int_0^\infty e^{-\alpha t}p_{2t+\epsilon}(x - y)dt = \frac{1}{2}\int_\epsilon^\infty e^{-\alpha s/2}p_s(x - y)ds\, e^{\alpha\epsilon/2}. \tag{2.11}$$

Then $\bar{h}_\epsilon \in C_b^2(\mathbb{R}^{2d})$ satisfies $\frac{\tilde{\Delta}}{2}\bar{h}_\epsilon = \alpha\bar{h}_\epsilon - p_\epsilon$. Therefore (2.9) gives

$$\begin{aligned}
\int \bar{h}_\epsilon(x, B_t)X_t(dx) = &\int \bar{h}_\epsilon(x, B_\tau)\mu(dx) + \alpha\int_\tau^t \int \bar{h}_\epsilon(x, B_s)X_s(dx)ds \\
&- \int_\tau^t \int \bar{h}_\epsilon(x, B_s)A(ds, dx) + \int_\tau^t \int \bar{h}_\epsilon(x, B_s)M(ds, dx) \\
&+ \int_\tau^t \int \vec{\nabla}_2\bar{h}_\epsilon(x, B_s)X_s(dx) \cdot dB_s - \ell_t^\epsilon(B, X).
\end{aligned} \tag{2.12}$$

We now derive (2.6) and the existence of $\ell_t(B, X)$ by establishing the convergence of each of the terms in (2.12), except perhaps for the last. Equation (2.11) shows that

$$e^{-\alpha\epsilon/2}\bar{h}_\epsilon(x, y) \uparrow h_\alpha(x - y) \quad \text{as} \quad \epsilon \downarrow 0 \quad \text{and} \quad \bar{h}_\epsilon(x, y) \le ch_\alpha(x - y) \tag{2.13}$$

Monotone convergence shows that the left side of (2.12) and the first three terms on the right side converge to the corresponding integrals with $h_\alpha(x-y)$ in place of $\bar{h}_\epsilon(x,y)$. The limit of the first term in the right is even finite by hypothesis (ii). Before dealing with the finiteness of the other terms, consider the two martingale terms in (2.12).

Lemma 2.9 *For each $t \geq \tau$,*

$$\lim_{\epsilon \downarrow 0} \mathbb{P}'\left[\int_\tau^t \int \left(\bar{h}_\epsilon(x, B_s) - h_\alpha(x - B_s)\right)^2 X_s(dx)\, ds\right] = 0.$$

Proof. The integrand converges pointwise to zero on $\{(s,x) : x \neq B_s\}$ (by (2.13)). This set is of full measure with respect to $\mathbb{P}'[X_s^0(dx)]\, ds$ (recall the independence of X^0 and B) and hence also $\mathbb{P}'[X_s(dx)]\, ds$. Therefore the result will follow from the bound in (2.13) and dominated convergence once we show

$$\mathbb{P}'\left[\int_\tau^t \int h_\alpha(x - B_s)^2 X_s^0(dx)\, ds\right] < \infty. \tag{2.14}$$

Note that if

$$f(u) = \begin{cases} u^{-1/2}, & d = 3, \\ 1 + \log^+(1/u), & d = 2, \\ 1, & d = 1, \end{cases}$$

then

$$\begin{aligned} h_\alpha(x)^2 &= \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{1}\{s \leq t\} e^{-\alpha(s+t)/2} p_s(x) p_t(x) dt ds \\ &\leq c \int_0^\infty \int_0^\infty \mathbf{1}\{s \leq t\} e^{-\alpha(s+t)/2} p_s(x) t^{-d/2} dt ds \\ &= c \int_0^\infty \int_0^\infty \mathbf{1}\{u \leq v\} e^{-\alpha(u+v)} p_{2u}(x) v^{-d/2} dv du \\ &\leq c(\alpha) \int_0^\infty f(u) e^{-\alpha u} p_{2u}(x) du \end{aligned}$$

(where we make the substitutions $s = 2u$ and $t = 2v$ to get the second equality). The expression in (2.14) is therefore bounded by

$$\begin{aligned} &c(\alpha) \int_\tau^t \int_0^\infty f(u) e^{-\alpha u} \iiiint p_{2u}(x-y) p_{s-\tau}(x - x_0) p_{s-\tau}(y - y_0) dx dy\, \mu(dx_0) m(dy_0)\, du\, ds \\ &= c(\alpha) \iint \left[\int_0^{t-\tau} \int_0^\infty f(u) e^{-\alpha(u+w)} p_{2(u+w)}(x_0 - y_0)\, du\, e^{\alpha w} dw\right] \mu(dx_0) m(dy_0). \end{aligned}$$

A simple change of variables shows the term in square brackets is bounded by

$$\begin{aligned}c(\alpha,t)\int_0^\infty \int_{(v-(t-\tau))^+}^{v} & f(u)\,du\,e^{-\alpha v}p_{2v}(x_0-y_0)\,dv \\ &\le c'(\alpha,t)h_\alpha(x_0-y_0).\end{aligned}$$

Now (2.14) follows from the above and hypothesis (ii) of Theorem 2.6 (in fact, a weaker L^1 condition suffices here).

□

Lemma 2.10 *Let $(B_t)_{t\ge 0}$ be a d-dimensional Brownian motion starting at x under P^x. For $\theta>0$ there are constants $c_1=c_1(\theta,d)$ such that for all $x,y,y'\in\mathbb{R}^d$ and $t>0$ the following hold.*
(a) $P^x[|B_t-y|^{-\theta}]\le c_1\left(|y-x|^{-\theta}\wedge t^{-\theta/2}\right)$ for $0<\theta<d$.
(b) $P^x\left[\log^+(1/(B_t-y))\right]\le c_1t^{-\theta/2}$.
(c) $P^x\left[(|B_t-y|^{-\theta}\wedge t^{-\theta/2})(|B_t-y'|^{-\theta}\wedge t^{-\theta/2})\right]\le c_1\left(|y-x|^{-\theta}\wedge t^{-\theta/2}\right)\left(|y'-x|^{-\theta}\wedge t^{-\theta/2}\right)$.

Proof. We may assume $x=0$ and set $P=P^0$.
(a) A simple scaling argument reduces the proof to the $t=1$ case. Then

$$\begin{aligned}P[|B_1-y|^{-\theta}] &= \theta\int_0^\infty r^{-1-\theta}P[|B_1-y|<r]\,dr \\ &\le c\left[\int_0^\infty \mathbf{1}\left\{\frac{|y|}{2}\vee 1\le r\right\}r^{-1-\theta}dr+\int_0^\infty \mathbf{1}\left\{r\le\frac{|y|}{2}\vee 1\right\}r^{-1-\theta}e^{-|y|^2/8}r^d dr\right] \\ &\le c\left[|y|^{-\theta}\wedge 1+e^{-|y|^2/8}(1\vee|y|)^{d-\theta}\right] \\ &\le c\big(|y|^{-\theta}\wedge 1\big).\end{aligned}$$

(b) This is immediate from (a).

(c) Again it suffices to consider $t=1$. Then

$$
\begin{aligned}
&P\left[(|B_1-y|^{-\theta}\wedge 1)(|B_1-y'|^{-\theta}\wedge 1)\right]\\
&=\theta^2\int_1^\infty\int_1^\infty r^{-\theta-1}r'^{-\theta-1}P\left\{|B_1-y|<r,|B_1-y'|<r'\right\}drdr'\\
&\le c\int_1^\infty\int_1^\infty r^{-\theta-1}r'^{-\theta-1}\left[\mathbf{1}\left\{\frac{|y|}{2}\le r,\frac{|y'|}{2}\le r'\right\}\right.\\
&\quad+\mathbf{1}\left\{\frac{|y|}{2}>r,\frac{|y'|}{2}\le r'\right\}e^{-|y|^2/8}r^d+\mathbf{1}\left\{\frac{|y|}{2}\le r,\frac{|y'|}{2}>r'\right\}e^{-|y'|^2/8}r'^d\\
&\quad\left.+\mathbf{1}\left\{\frac{|y|}{2}>r,\frac{|y'|}{2}>r'\right\}\left(e^{-|y|^2/8}r^d\wedge e^{-|y'|^2/8}r'^d\right)\right]drdr'\\
&\le c\left[(|y|^{-\theta}\wedge 1)(|y'|^{-\theta}\wedge 1)+(|y'|^{-\theta}\wedge 1)|y|^de^{-|y|^2/8}+(|y|^{-\theta}\wedge 1)|y'|^de^{-|y'|^2/8}\right.\\
&\quad\left.+\exp\left(-(|y|^2\vee|y'|^2)/8\right)\int_1^\infty\int_1^\infty(r^d\vee r'^d)r^{-\theta-1}r'^{-\theta-1}\mathbf{1}\left\{r<\frac{|y|}{2},r<\frac{|y'|}{2}\right\}drdr'\right]\\
&\le c\left[(|y|^{-\theta}\wedge 1)(|y'|^{-\theta}\wedge 1)+\exp\{-(|y|^2\vee|y'|^2)/8)|y|^d|y'|^d\right]\\
&\le c(|y|^{-\theta}\wedge 1)(|y'|^{-\theta}\wedge 1).
\end{aligned}
$$

□

Lemma 2.11 *For each $t\ge\tau$,*

$$\lim_{\epsilon\downarrow 0}\mathbb{P}'\left[\int_\tau^t\left(\int|\vec\nabla_2\bar h_\epsilon(x,B_s)+\vec\nabla h_\alpha(x-B_s)|X_s(dx)\right)^2ds\right]=0.$$

Proof. Note that

$$
\begin{aligned}
\vec\nabla_2\bar h_\epsilon(x,y)&=-\frac{1}{2}\int_\epsilon^\infty e^{-\alpha s/2}\vec\nabla p_s(x-y)ds\,e^{\alpha\epsilon/2}\\
&\to-\vec\nabla h_\alpha(x-y)\quad\text{as}\quad\epsilon\downarrow 0\quad\text{if}\quad x\ne y.
\end{aligned}
$$

An elementary calculation (use $|\vec\nabla p_s(x)|=|x|s^{-1}p_s(x)$), now shows that

$$|\vec\nabla_2\bar h_\epsilon(x,y)|\le c|x-y|^{1-d}\quad\forall x,y\in\mathbb{R}^d,\epsilon\in]0,1]\quad\text{and some }c>0.$$

Using dominated convergence, as in the proof of Lemma 2.9, we see that it suffices to show that for $d=2$ or $d=3$

$$\mathbb{P}'\left[\int_\tau^t\left(\int|x-B_s|^{1-d}X_s^0(dx)\right)^2ds\right]<\infty.\tag{2.15}$$

If, as above, P^x is Wiener measure starting at x at time 0 and $P^\nu = \int P^x \nu(dx)$, then (see, for example, (A.15) with $L=0$)

$$\begin{aligned}\mathbb{P}\left[\left(\int |x-y|^{1-d} X^0_s(dx)\right)^2\right] &= P^\mu\left[|B_{s-\tau}-y|^{1-d}\right]^2 + \int_0^{s-\tau} P^\mu\left[P^{B_r}\left[|B_{s-\tau-r}-y|^{1-d}\right]^2\right] dr \\ &\le c\left[\int |y-x|^{1-d}\wedge(s-\tau)^{(1-d)/2}\mu(dx)^2 + \int_0^{s-\tau}\int |y-B_r|^{2-2d}\wedge(s-\tau-r)^{1-d} dP^\mu\, dr\right],\end{aligned} \tag{2.16}$$

where Lemma 2.10 is used in the last inequality.

Use this to write (2.15) as $I+II$ where (set $t_0=t-\tau$ and recall X^0 and B are independent)

$$I = \int_0^{t_0} \iint P^m\left[\left(|B_s-x|^{1-d}\wedge s^{(1-d)/2}\right)\left(|B_s-x'|^{1-d}\wedge s^{(1-d)/2}\right)\right] \mu(dx)\,\mu(dx')\,ds$$

and

$$II = \iint \left[\int_0^{t_0}\int_0^s P^0\left[|B_{r+s}+y_0-x_0|^{2-2d}\wedge(s-r)^{1-d}\right] dr\,ds\right] m(dy_0)\,\mu(dx_0).$$

If

$$f(w) = \begin{cases} |w|^{-2}, & d=3, \\ 1+\log^+(1/|w|), & d=2, \end{cases}$$

then a change of variables ($u=s-r, v=s+r$) shows the integral in square brackets in II equals

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^d}\Big\{\int_0^{t_0}\int_0^v p_v\left(w-(y_0-x_0)\right)\left(|w|^2\vee u\right)^{1-d} du\,dv \\ &\quad + \int_{t_0}^{2t_0}\int_0^{2t_0-v} p_v\left(w-(y_0-x_0)\right)\left(|w|^2\vee u\right)^{1-d} du\,dv\Big\}\,dw \\ &\le c(t_0)\int_{\mathbb{R}^d}\int_0^{2t_0} p_v(w-(y_0-x_0))\left[|w|^{4-2d}+f(w)\right] dv\,dw. \\ &\le c(t_0)\int_0^{2t_0} P^{x_0-y_0}\left[f(B_v)\right] dv.\end{aligned}$$

If $d=3$, then Lemma 2.10(a) bounds the above by

$$c(t_0)\int_0^{2t_0} |x_0-y_0|^{-2}\wedge v^{-1}\,dv \le c(t_0)\left(1+\log^+\left(2t_0/|x_0-y_0|^2\right)\right).$$

If $d=2$, then a similar argument using Lemma 2.10(b) bounds the above by $c(t_0)$. The energy condition is now more than enough to show that $II<\infty$.

Turning to I, we may use Lemma 2.10(c) to bound it by

$$2c_1 \iiint \left[\int_0^{t_0} \left(|y-x| \vee \sqrt{s}\right)^{1-d} \left(|y-x'| \vee \sqrt{s}\right)^{1-d} ds\right] \mathbf{1}\{|y-x| \leq |y-x'|\}\, m(dy)\, \mu(dx)\, \mu(dx').$$

Consider only $d=3$ as $d=2$ is even easier. In this case the above equals

$$\begin{aligned}
&c(t_0) \iiint \left[2|x'-y|^{-2} + 2|x'-y|^{-2} \log^+\left(\frac{|x'-y|}{|x-y|}\right)\right] \mathbf{1}\{|x-y| \leq |x'-y|\} m(dy)\, \mu(dx)\, \mu(dx') \\
&\quad \leq c(t_0) \iiint |x'-y|^{-1}|x-y|^{-1} \left[1 + \frac{|x-y|}{|x'-y|} \log^+ \frac{|x'-y|}{|x-y|}\right] \\
&\qquad \mathbf{1}\{|x-y| \leq |x'-y|\}\, m(dy)\, \mu(dx)\, \mu(dx') \\
&\quad \leq c(t_0) \int \left[\int h_0(|x-y|)\, \mu(dx)\right]^2 m(dy)
\end{aligned}$$

which is finite by hypothesis (ii) of Theorem 2.6. This shows I is finite and hence proves (2.15) and completes the proof.

$\square$

Lemma 2.12 *(a)* $\lim_{\epsilon \downarrow 0} \mathbb{P}' \left[\int_\tau^t \left[\int \left|\bar{h}_\epsilon(x, B_s) - h_\alpha(x - B_s)\right| X_s(dx)\right]^2 ds\right] = 0$, $\forall t > \tau$.
(b) $\lim_{\epsilon \downarrow 0} \mathbb{P}' \left[\int_\tau^t \int |\bar{h}_\epsilon(x, B_s) - h_\alpha(x - B_s)| A(ds, dx)^2\right] = 0$, $\forall t > \tau$.
(c) $\lim_{\epsilon \downarrow 0} \mathbb{P}' \left[\sup_{\tau \leq t \leq u} \left|\int \bar{h}_\epsilon(x, B_t) - h_\alpha(x - B_t) X_t(dx)\right|^2\right] = 0$, $\forall u > \tau$.

Proof. (a) As in the proof of Lemma 2.9 (see (2.14)), it suffices to prove

$$\mathbb{P}' \left[\int_\tau^t \int h_\alpha(x - B_s) X_s^0(dx)^2\, ds\right] < \infty.$$

This is immediate from (2.4) and (2.15) (the case $d=1$ being trivial).
(b) The decomposition (2.12) shows that

$$\begin{aligned}
\int_\tau^t \int \bar{h}_\epsilon(x, B_s) A(ds, dx) \leq &\int \bar{h}_\epsilon(x, B_\tau) \mu(dx) + \alpha \int_\tau^t \int \bar{h}_\epsilon(x, B_s) X_s(dx)\, ds \\
&+ \int_\tau^t \bar{h}_\epsilon(x, B_s) M(ds, dx) + \int_\tau^t \int \vec{\nabla}_2 \bar{h}_\epsilon(x, B_s) X_s(dx) \cdot dB_s.
\end{aligned}$$

Lemmas 2.9, 2.11, part (a) and hypothesis (ii) of Theorem 2.6 (and (2.13)) show that the right side of the above converges in L^2 to the corresponding expression with $h_\alpha(x - B_s)$ in place of $\bar{h}_\epsilon(x, B_s)$. Use

(2.13) and Fatou's Lemma to see that $\mathbb{P}'\left[\int_\tau^t \int h_\alpha(x-B_s)A(ds,dx)^2\right] < \infty$. This shows that $A(ds,dx)$ does not charge $\{(s,x) : B_s = x\}$ a.s. and therefore (by (2.13))

$$\lim_{\epsilon\downarrow 0} \bar{h}_\epsilon(x,B_s) - h_\alpha(x-B_s) = 0, \quad A(ds,dx) - \text{a.e.}, \ \mathbb{P}' - \text{a.s.}$$

(Implicit in the above is $d > 1$, but this last assertion is trivial if $d = 1$.) The result now follows from (2.13), the above square–integrability and dominated convergence.

(c) Argue as in the proof of Lemma 5.6(b) of [2] (see the last displayed equation in the proof) to see that for $u > \tau$ fixed

$$\lim_{\delta\downarrow 0} \sup_{\tau\le t\le u} \int h_\alpha(x-B_t)\mathbf{1}\{|x-B_t|\le\delta\}X_t^0(dx) = 0, \quad \text{a.s.} \tag{2.17}$$

We have

$$\begin{aligned}\sup_{\tau\le t\le u}\left|\int \bar{h}_\epsilon(x,B_t) - h_\alpha(x-B_t)X_t(dx)\right| &\le \left(\sup_{\tau\le t\le u} X_t(1)\right)\sup\{|\bar{h}_\epsilon(x,y)-h_\alpha(x-y)| : |x-y|>\delta\} \\ &\quad + c\sup_{\tau\le t\le u}\int h_\alpha(x-B_t)1(|x-B_t|\le\delta)X_t^0(dx) \quad (\text{by } (2.13)).\end{aligned}$$

Use (2.17) and the uniform convergence of $\bar{h}_\epsilon(x,y)$ to $h_\alpha(x-y)$ on $\{(x,y) : |x-y| > \delta\}$ as $\epsilon\downarrow 0$ (see Lemma 5.3 of [2]) to see that the left side of the above approaches 0 a.s. as $\epsilon\downarrow 0$. If we could show

$$\mathbb{P}'\left[\sup_{\tau\le t\le u}\int h_\alpha(x-B_t)X_t(dx)^2\right] < \infty, \tag{2.18}$$

then the result would follow from the above, (2.13) and dominated convergence. To prove (2.18) use (2.12) to see that

$$\begin{aligned}\sup_{\tau\le t\le u}\int \bar{h}_\epsilon(x,B_t)X_t(dx) &\le \int \bar{h}_\epsilon(x,B_\tau)\,\mu(dx) + \alpha\int_\tau^t\int \bar{h}_\epsilon(x,B_s)X_s(dx)\,ds \\ &\quad + \sup_{\tau\le t\le u}\left|\int_\tau^t\int \bar{h}_\epsilon(x,B_s)M(ds\,dx)\right| + \left|\int_\tau^t\int \vec{\nabla}_2\bar{h}_\epsilon(x,B_s)X_s(dx)\cdot dB_s\right|.\end{aligned}$$

The right side converges in L^2 to the corresponding expression with $h_\alpha(x-B_s)$ in place of $\bar{h}_\epsilon(x,B_s)$ by hypothesis (ii) of Theorem 2.6, part (a), Lemmas 2.9 and 2.11, and Doob's maximal L^2 inequality. An elementary argument using Fatou's Lemma now gives (2.18) and completes the proof.

□

Proof of Theorem 2.6 (continued) Return now to (2.12). Lemmas 2.9, 2.11, 2.12 and hypothesis (ii) of Theorem 2.6 (with (2.13)) show that if T_t^ϵ denotes the left side of (2.12) or any of the first 5 terms on the right side and T_t is the corresponding term with $h_\alpha(x-y)$ in place of $\bar{h}_\epsilon(x,y)$ (the initial condition is independent of t), then for any $u>\tau$, $\sup_{\tau\le t\le u}|T_t^\epsilon - T_t|\xrightarrow{L^2}0$ as $\epsilon\downarrow 0$. Therefore there is an a.s. continuous, non–decreasing, square–integrable $(\mathcal{F}_t)_{t\ge\tau}$–predictable process $(\ell_t(B,X))_{t\ge\tau}$ such that (2.5) and (2.6) hold with $T\equiv\infty$. If $A_t(\mathbb{R}^d)$ is a.s. continuous then so is $\int_\tau^t\int \bar{h}_\epsilon(x,B_s)A(ds,dx)$ and hence the same is true of $\int_\tau^t\int h_\alpha(x-B_s)A(ds,dx)$ by Lemma 2.12(b). The remaining assertions in the Theorem are now obvious.

□

Remark 2.13 There is some possible confusion over the notation $\ell_t(y,\nu)$ in Theorem 2.2(b) and the notation $\ell_t(B,X)(\omega)$ in Theorem 2.6. We claim that given a process X and a time T as in Theorem 2.6 we may construct $\ell_t(y,\nu)$ as in Theorem 2.2 (and more specifically (2.3)) so that

$$\ell_t(B,X)(\omega)=\ell_t(y,\nu)\mid_{(y,\nu)=(B_\cdot(\omega),X_\cdot(\omega))},\quad \forall\tau\le t\le T,\ \mathbb{P}'\text{-a.s.} \tag{2.19}$$

Given $\{\epsilon_n\}$ as in (2.3) we may (by (2.5)) choose a subsequence $\{\epsilon_{n_k}\}$ such that

$$\ell_t(B,X)=\lim_{k\to\infty}\ell_t^{\epsilon_{n_k}}(B,X),\quad \forall\tau\le t\le T,\ \mathbb{P}'\text{-a.s.}$$

Now replace $\{\epsilon_n\}$ with $\{\epsilon_{n_k}\}$ in (2.3) to define $\ell_t(y,\nu)$ so that (2.19) holds. Clearly this argument can accommodate countably many $\{X_n\}$ in (2.19).

3 Historical collision local times

3.1 Path–field collision local times

Let $(\Omega,\mathcal{H},(\mathcal{H}_t)_{t\ge\tau},\mathbb{P})$ be a filtered probability space such that $(\mathcal{H}_t)_{t\ge\tau}$ is right–continuous and the σ–field $\mathcal{H}$ is universally complete. Let (H^1,H^2) be a pair of independent historical Brownian motions starting at (τ,μ^1) and (τ,μ^2), respectively, defined on $(\Omega,\mathcal{H},(\mathcal{H}_t)_{t\ge\tau},\mathbb{P})$ and with corresponding martingale measures (M^1,M^2).

Definition 3.1 Write $\mathcal{M}(H^1,H^2)$ for the collection of pairs of predictable, $\mathbf{M}_F(C)$–valued processes (G^1,G^2) with values in $\Omega_H[\tau,\infty[$ such that there are nondecreasing, predictable, $\mathbf{M}_F(C)$–valued processes (A^1,A^2), null at τ, and with sample paths almost surely in $\Omega_H[\tau,\infty[$ such that: $G_t^i\le H_t^i$, $i=1,2$, for all $t\ge\tau$, and for all $\phi^1,\phi^2\in D_{ST}$ we have that

$$N_t^1(\phi^1)=G_t^1(\phi_t^1)-\mu^1(\phi_\tau^1)-\int_\tau^t G_s(A\phi_s^1)\,ds-\int_\tau^t\int\phi^1(s,y)\,A^1(ds,dy)$$

and

$$N_t^2(\phi^2)=G_t^2(\phi_t^2)-\mu^2(\phi_\tau^2)-\int_\tau^t G_s(A\phi_s^2)\,ds-\int_\tau^t\int\phi^2(s,y)\,A^2(ds,dy)$$

are continuous $(\mathcal{H}_t)_{t\geq\tau}$–martingales null at τ for which

$$\langle N^i(\phi^i)\rangle_t = \int_\tau^t \int \phi_s^i(y)^2\, G_s^i(dy)ds$$

and $\langle N^i(\phi^i), N^j(\phi^j)\rangle = \langle M^i(\phi^i), N^j(\phi^j)\rangle = 0$ for $i \neq j$.

Let $\mathcal{C}$ denote the Borel σ-field on C and, as in Section 2, write $(\mathcal{C}_t)_{t\geq 0}$ for the canonical filtration. Using the extension procedure in Section 2 of [32], we can construct orthogonal martingale measures N^i (defined on appropriate $(\mathcal{C}_t \times \mathcal{H}_t)_{t\geq\tau}$–predictable integrands) that have the following properties. If γ^i is $(\mathcal{C}_t \times \mathcal{H}_t)_{t\geq\tau}$–predictable and $\int_\tau^t \int \gamma^i(s,y)^2\, G_s^i(dy)ds < \infty$ for all $t \geq \tau$, $\mathbb{P}$-a.s. (respectively, $\mathbb{P}[\int_\tau^t \int \gamma^i(s,y)^2\, G_s^i(dy)ds] < \infty$ for all $t \geq \tau$), then $\int_\tau^t \int \gamma^i(s,y)\, dN^i(s,y)$ is a continuous local martingale (respectively, a continuous square - integrable martingale). We have that $N_t^i(\phi^i) = \int_\tau^t \phi^i(s,y)\, dN^i(ds,dy)$. Moreover, $\langle \int_\tau^\cdot \int \gamma^i(s,y)\, dN^i(s,y)\rangle_t = \int_\tau^t \int \gamma^i(s,y)^2\, G_s^i(dy)ds$ and $\langle \int_\tau^\cdot \int \gamma^1(s,y)\, dN^1(s,y), \int_\tau^\cdot \int \gamma^2(s,y)\, dN^2(s,y)\rangle = 0$. The analogous extensions of course also hold for M^i and $\langle \int_\tau^\cdot \int \gamma^i(s,y)\, dN^i(s,y), \int_\tau^\cdot \int \gamma^j(s,y)\, dM^j(s,y)\rangle = 0$ for $i \neq j$.

For the rest of this section we will assume that (H^1, H^2) is a pair of independent historical Brownian motions starting at (τ, μ^1) and (τ, μ^2), respectively, with $\mu^i \in \mathbf{M}_{FS}(C)^\tau$, $i = 1, 2$, and $(G^1, G^2) \in \mathcal{M}(H^1, H^2)$.

We first extend the results in Section 2 concerning $\ell_t(y, \nu)$ and $\ell_t(y, X)$ to the setting where y is chosen according to H^i instead of being a fixed Brownian path.

Definition 3.2 Given a bounded $(\mathcal{H}_t)_{t\geq\tau}$–stopping time $T \geq \tau$, *the normalised Campbell measure associated with* H^i is the probability measure $\bar{\mathbb{P}}_T^{H^i}$ on on $(C \times \Omega, \mathcal{C} \times \mathcal{H}) \equiv (\hat{\Omega}, \hat{\mathcal{H}})$ given by

$$\int \gamma(y,\omega)\, \bar{\mathbb{P}}_T^{H^i}(dy, d\omega) = \int \left[\int \gamma(y,\omega)\, H_T^i(dy)\right] \mathbb{P}(d\omega)/\mu^i(1).$$

Let $\hat{\mathcal{H}}_t = \mathcal{C}_t \times \mathcal{H}_t$ and $\hat{\mathcal{H}}_t^*$ denotes its universal completion.

Definition 3.3 Say that $\Lambda \subset [\tau, \infty[\times \hat{\Omega}$ is H^i–evanescent if $\Lambda \subset \Lambda_1$ where Λ_1 is $(\hat{\mathcal{H}}_t^*)_{t\geq\tau}$–predictable and

$$\sup_{\tau\leq u\leq t} 1_{\Lambda_1}(u, y, \omega) = 0, \quad H_t^i\text{-a.a. } y,\ \forall t \geq \tau,\ \mathbb{P}' - a.s.$$

Say that a property holds H^i-a.e. iff it holds off an H^i-evanescent set.

Definition 3.4 Let $(X_t)_{t\geq\tau}$ be an $(\mathcal{H}_t)_{t\geq\tau}$–optional, $\mathbf{M}_F(\mathbb{R}^d)$–valued process on $\hat{\Omega}$ such that

$$\sup_{\tau\leq s\leq t} X_s(1) < \infty, \quad \forall t \geq \tau, \text{ a.s.}$$

A *path–field collision local time* (or PFCLT) for H^i with respect to X is an $(\hat{\mathcal{H}}_t)_{t\geq\tau}$–predictable process $(t,y,\omega)\to\ell_t(y,X)(\omega)$ such that for any bounded $(\mathcal{H}_t)_{t\geq\tau}$–stopping time $T\geq\tau$,

$$\sup_{\tau\leq t\leq T}|\ell_t^\epsilon(y,X(\omega))-\ell_t(y,X)(\omega)|\to 0 \text{ in } \bar{\mathbb{P}}_T^{H^i}\text{-probability as } \epsilon\downarrow 0.$$

If $(G_t)_{t\geq\tau}$ is an $\mathbf{M}_F(C)$–valued process and $X=\Gamma(G)$ is as above, we write $\ell_t(y,G)$ (and $\ell_t^\epsilon(y,G)$) for $\ell_t(y,X)$ (and $\ell_t^\epsilon(y,X)$) and call the former the PFCLT for H^i with respect to G (if it exists).

Remark 3.5 A simple application of the section theorem shows that the PFCLT for H^i with respect to X is unique up to H^i-evanescent sets. To see this note that if ℓ_t' is another PFCLT then

$$\{(t,\omega): t\geq\tau, H_t^i(\sup_{\tau\leq s\leq t}|\ell_s(y,X)-\ell_s'(y,X)|>0)>0\}$$

is $(\mathcal{H}_t)_{t\geq\tau}$–predictable by Corollary 3.6 of [30] (use the fact that if f is bounded and $(\hat{H}_t^*)_{t\geq\tau}$–predictable, then so is f^*, where $f_t^*=\sup_{\tau\leq s\leq t}f_s=\sup_{\tau\leq s<t}f_s\vee f_t$) and therefore is evanescent (in the usual sense) by the above definition and the section theorem. A slightly more involved application of the section theorem shows that $\ell_\cdot(y,X)|_{[\tau,t]}$ is non-decreasing and continuous H_t^i-a.a. y, $\forall t\geq\tau$, a.s. (that is, H^i-a.e.).

Lemma 3.6 *Let $T\geq\tau$ be a bounded $(\mathcal{H}_t)_{t\geq\tau}$–stopping time and set $X=\Gamma(G^2)$. Under $\bar{\mathbb{P}}_T^{H^1}$, $B_t(y,\omega)=y_t$ is a $(\hat{\mathcal{H}}_t)_{t\geq\tau}$–Brownian motion stopped at T and (B,X) satisfies the hypotheses of Theorem 2.6 on $(\hat{\Omega},\hat{\mathcal{H}},(\hat{\mathcal{H}}_t)_{t\geq\tau},\bar{\mathbb{P}}_T^{H^1})$.*

Proof. Theorem 2.6 of [30] gives the first assertion about B. To prove that X satisfies hypothesis (i), fix t_0 large and $\phi\in C_K^\infty(\mathbb{R}^d)$, and set $\bar{\phi}(y)=\phi(y(t_0))$. If $\nu(A)=\mu^2(\{y:y_\tau\in A\})$, then the definition of $\mathcal{M}(H^1,H^2)$ shows that for $\tau\leq t\leq t_0$,

$$X_t(\phi)=\nu(\phi)+\int_\tau^t X_s(\frac{\Delta\phi}{2})ds+N_t^2(\bar{\phi})-A_t(\phi),$$

where $A_t(\phi)=\int_\tau^t\int\phi(y(s))A^2(ds,dy)$ and $N_t^2(\bar{\phi})\equiv M_t(\phi)$ is an $(\mathcal{H}_t)_{t\geq\tau}$–martingale under $\mathbb{P}$ satisfying

$$\langle M(\phi)\rangle_t=\int_\tau^t\int\phi(y(t_0))^2G_s^2(dy)ds=\int_\tau^t X_s(\phi^2)ds,\ \ \tau\leq t\leq t_0.$$

If $\tau\leq s<t\leq t_0$, $A\in\mathcal{H}_s$ and f is bounded and $\mathcal{C}_s$-measurable, then

$$\begin{aligned}\mu^1(1)\bar{\mathbb{P}}_T^{H^1}\left[(M_t(\phi)-M_s(\phi))1_Af(y)\right]&=\mathbb{P}\left[H_T^1(f)(M_t(\phi)-M_s(\phi))1_A\right]\\&=\mathbb{P}\left[(H_{s\wedge T}^1(f)+\int_{s\wedge T}^T\int f(y)M^1(ds,dy))(N_t^2(\bar{\phi})-N_s^2(\bar{\phi}))1_A\right]\\&\qquad\text{(for example, by Theorem 2.6 of [31])}\\&=0,\end{aligned}$$

the last because N^2 and M^1 are orthogonal. This shows $M_t(\phi)$ is an $(\hat{\mathcal{H}}_t)_{t\geq\tau}$–martingale under $\bar{\mathbb{P}}_T^{H^1}$.

If $\hat{B}_t = B_t - B_\tau$ and $A_s = A \cap \{T > s\}$, then

$$
\begin{aligned}
&\bar{\mathbb{P}}_T^{H^1}\left[\left(\hat{B}^i_t M_t(\phi) - \hat{B}^i_s M_s(\phi)\right) f(y) 1_A\right] \\
&\quad = \mathbb{P}\left[\left(H^1_T\left(\hat{B}^i_t f\right) M_t(\phi) - H^1_T\left(\hat{B}^i_s f\right) M_s(\phi)\right) 1_A\right] \\
&\quad = \mathbb{P}\left[H^1_{T\wedge t}\left(\left(\hat{B}^i_{t\wedge T} - \hat{B}^i_{s\wedge T}\right) f\right) M_t(\phi) 1_{A_s}\right] + \mathbb{P}\left[H^1_{T\wedge t}\left(\hat{B}^i_{s\wedge T} f\right)(M_t(\phi) - M_s(\phi)) 1_A\right] \\
&\qquad\qquad \text{(by Theorem 2.6 } (K_2) \text{ of [31] and a truncation argument)} \\
&\quad = \mathbb{P}\left[\int_s^{T\wedge t}\int (y^i_u - y^i_s) f(y) M^1(du,dy) \bar{N}^2_t(\bar{\phi}) 1_{A_s}\right] \\
&\qquad + \mathbb{P}\left[(H^1_{s\wedge T}(\hat{B}^i_{s\wedge T} f) + \int_s^{t\wedge T}\int \hat{B}^i_{s\wedge T} f(y) M^1(du,dy))\left(N^2_t(\bar{\phi}) - N^2_s(\bar{\phi})\right) 1_A\right] \\
&\qquad\qquad ((K_2) \text{ and } (K_4) \text{ of Theorem 2.6 in [31]}) \\
&\quad = 0
\end{aligned}
$$

by the orthogonality of M^1 and N^2. Therefore $\langle M(\phi), B^i\rangle_t = 0$ and hypothesis (i) of Theorem 2.6 holds on $(\hat{\Omega}, \hat{\mathcal{H}}, \hat{\mathcal{H}}_t, \bar{\mathbb{P}}_T^{H^1})$. Hypothesis (ii) of Theorem 2.6 is valid by Corollary 2.8 and our hypothesis on μ^1.

□

Theorem 3.7 *Let $i, j \in \{1, 2\}$ be distinct.*
(a) The PFCLT for H^i with respect to G^j exists. Moreover by replacing $\{\epsilon_n\}$ (in (2.3) and Theorem 2.2(b)) with an appropriate subsequence we may assume that $\ell_t(y, G^j)(\omega) \equiv \ell_t(y,\nu)|_{\nu=G^j(\omega)}$, $\forall t \geq \tau$, $y \in C$ and

$$
\lim_{n\to\infty}\sup_{\tau\leq t\leq u}|\ell^{\epsilon_n}_t(y, G^j) - \ell_t(y, G^j)| = 0, \quad H^i_u\text{-a.e. } y,\ \forall u \geq \tau,\ a.s. \tag{3.1}
$$

(b) If $\mathcal{T}_N$ is the set of $(\mathcal{H}_t)$–stopping times bounded by N, then

$$
\begin{aligned}
&\sup_{T\in\mathcal{T}_N}\mathbb{P}\left[H^i_T\left(\sup_{\tau\leq t\leq T}|\ell^\epsilon_t(y, G^j) - \ell_t(y, G^j)|^2\right)\right] \\
&\qquad = \mathbb{P}\left[H^i_N\left(\sup_{\tau\leq t\leq N}|\ell^\epsilon_t(y, G^j) - \ell_t(y, G^j)|^2\right)\right] \to 0 \text{ as } \epsilon\downarrow 0.
\end{aligned}
$$

(c) For any $N > \tau$,

$$
\begin{aligned}
&\sup_{\tau\leq t\leq N} H^i_t\left(\sup_{\tau\leq u\leq t}|\ell^\epsilon_u(\cdot, G^j) - \ell_u(\cdot, G^j)|^2\right) \to 0 \text{ in } \mathbb{P}\text{–probability as } \epsilon\downarrow 0, \\
&\sup_{\tau\leq t\leq N} H^i_t(\sup_{\tau\leq u\leq t}|\ell^\epsilon_u(\cdot, G^j) - \ell_u(\cdot, G^j)|) \to 0 \text{ in } L^2 \text{ as } \epsilon\downarrow 0,
\end{aligned}
$$

and

$$\sup_{\tau\le t\le N}\mathbb{P}\left[\int \ell_t(y,G^j)^2 H_t^i(dy)\right]=\mathbb{P}\left[\int \ell_N(y,G^j)^2 H_N^i(dy)\right]<\infty.$$

Proof. If ϵ and k are positive, let

$$R_t^{\epsilon,k}=\int \sup_{\tau\le u\le t}|\ell_u^\epsilon(y,G^j)-\ell_u(y,G^j)|^2\wedge k\ H_t^i(dy).$$

Observe that $R_t^{\epsilon,k}$ is a non-negative a.s. continuous $(\mathcal{H}_t)_{t\ge\tau}$–submartingale by Theorem 2.23 of [31] (Remark 3.5 shows the hypotheses of that result are in force). By the weak L^1 inequality we have

$$\begin{aligned}\mathbb{P}\left\{\sup_{\tau\le t\le N}R_t^{\epsilon,k}>\delta\right\}&\le\delta^{-1}\mu^i(1)\bar{\mathbb{P}}_N^{H^i}\left[\sup_{\tau\le u\le N}|\ell_u^\epsilon(y,G^j)-\ell_u(y,G^j)|^2\right]\\&\to 0\text{ as }\epsilon\downarrow 0\end{aligned}\tag{3.2}$$

by Lemma 3.6 and Theorem 2.6. By replacing $\{\epsilon_n\}$ in (2.3) with an appropriate subsequence we may assume (let $k\uparrow\infty$ in (3.2) and use an obvious notation)

$$\mathbb{P}\left\{\sup_{\tau\le t\le n}R_t^{\epsilon_n,\infty}>2^{-n}\right\}\le 2^{-n-1}.$$

The Borel-Cantelli lemma implies that $\mathbb{P}$-a.s. for large n,

$$\sup_{\tau\le t\le n}\int\sup_{\tau\le u\le t}|\ell_u^{\epsilon_n}(y,G^j)-\ell_u(y,G^j)|^2H_t^i(dy)\le 2^{-n}.$$

A further application of Borel-Cantelli shows that (3.1) holds when we define $\ell_t(y,G^j)(\omega)=\ell_t(y,\nu)|_{\nu=G^j(\omega)}$. Let $k\uparrow\infty$ in (3.2) and use Fatou's lemma to see that

$$\begin{aligned}&\mathbb{P}\left\{\sup_{\tau\le t\le N}H_t^i\left(\sup_{\tau\le u\le t}|\ell_u^\epsilon(\cdot,G^j)-\ell_u(\cdot,G^j)|^2\right)>\delta\right\}\\&\le\lim_{k\to\infty}\mathbb{P}\left\{\sup_{\tau\le t\le N}R_t^{\epsilon,k}>\delta\right\}\\&\to 0\text{ as }\epsilon\downarrow 0\quad\text{by (3.2)}.\end{aligned}$$

This proves the first assertion in (c), and the rest of (a) follows at once. The submartingale property mentioned above shows that for $T\in\mathcal{T}_N$,

$$\mathbb{P}\left[H_T^i\left(\sup_{\tau\le t\le T}|\ell_t^\epsilon(y,G^j)-\ell_t(y,G^j)|^2\wedge k\right)\right]\le\mathbb{P}\left[H_N^i\left(\sup_{\tau\le t\le N}|\ell_t^\epsilon(y,G^j)-\ell_t(y,G^j)|^2\wedge k\right)\right],$$

and so letting $k\to\infty$ we get the equality in (b). Applying (3.2) now completes the proof of (b).

For the second assertion in (c) define $M_t^{\epsilon,k}$ in the same manner as $R_t^{\epsilon,k}$ but with no square on the integrand. Apply Doob's strong L^2 inequality to the non-negative submartingales $M^{\epsilon,k}$ and let $k \uparrow \infty$ as usual to see that

$$M_t^\epsilon = \int \sup_{\tau \le u \le t} |\ell_u^\epsilon(y, G^j) - \ell_u(y, G^j)| H_t^i(dy)$$

satisfies

$$\begin{aligned}\mathbb{P}\Big[\sup_{\tau \le t \le N} (M_t^\epsilon)^2\Big] \le c\mathbb{P}\left[(M_N^\epsilon)^2\right] &\le c\mu^i(1)\mathbb{P}\left[H_N^i(\sup_{\tau \le t \le N} |\ell_t^\epsilon(y, G^j) - \ell_t(y, G^j)|^2)\right] \\ &\to 0 \text{ as } \epsilon \downarrow 0 \text{ by (b)}.\end{aligned}$$

It remains only to prove the last assertion of (c). The first equality follows from the submartingale property of $t \to \int \ell_t(y, G^j)^2 \wedge k\ H_t^i(dy)$ and then letting $k \to \infty$. The finiteness of the integral is immediate from the square–integrability of the terms in (2.6) (in Theorem 2.6) and Lemma 3.6.

□

Remark 3.8 If $\{(G^{1,k}, G^{2,k}) : k \in \mathbb{N}\}$ is a sequence in $\mathcal{M}(H^1, H^2)$, then a diagonalization argument shows we may choose $\{\epsilon_n\}$ so that the above result holds for each $G^{j,k}$.

3.2 Field–field collision local times

In this section we wish to establish the existence of FFCLT's for pairs $(G^1, G^2) \in \mathcal{M}(H^1, H^2)$. Here and throughout the rest of the paper, we encounter processes of the form $H_t^i(\phi_t)$, where $\phi_t(y, \omega)$ is, in turn, the result of some sort of Lebesgue or stochastic integration along the path y. It will be necessary for us to have semimartingale decompositions of such processes. The following is our first example.

Remark 3.9 If γ is a bounded, $(\hat{\mathcal{H}}_t^*)_{t \ge \tau}$–predictable function, Theorem 3.8 of [30] shows that $\mathbb{P}$-a.s. for all $t \ge \tau$,

$$\int \left\{ \int_\tau^t \gamma(s, y) ds \right\} H_t^i(dy) = \int_\tau^t \int \left\{ \int_\tau^s \gamma(u, y) du \right\} M^i(ds, dy) + \int_\tau^t \int \gamma(s, y) H_s^i(dy) ds, \tag{3.3}$$

and each of the terms is $\mathbb{P}$-a.s. continuous. The proof relies on the fact (Lemma 3.5 of [30]) that if ψ is bounded and $\mathcal{C}_u$–measurable ($u \ge \tau$), then

$$H_t^i(\psi) = H_u^i(\psi) + \int_\tau^t \int 1(s > u)\psi(y^s) M^i(ds, dy), \quad \forall t \ge u \text{ a.s.} \tag{3.4}$$

Consequently, (3.3) will hold whenever H^i is a continuous measure-valued process satisfying (3.4), M^i is an orthogonal martingale measure with the usual square function, and H^i satisfies a mild integrability condition (for example, if the total mass process is dominated by that of super-Brownian motion). One such extension is given by Theorem 2.8 of [31]. (Note this result also shows that (3.3) may fail in general for G^i if $A^i \neq 0$.) In the future we will therefore use (3.3) for other processes satisfying (3.4) without detailed justification. The argument is as in [30].

Theorem 3.10 *(a) For $i \neq j$, the FFCLT $L(H^i, G^j)$ exists and, moreover, for all bounded and continuous $\phi : C \to \mathbb{R}$ and $u \geq \tau$,*

$$\lim_{\epsilon \downarrow 0} \mathbb{P}\left[\sup_{\tau \leq t \leq u} \left|L_t^\epsilon(H^i, G^j)(\phi) - L_t(H^i, G^j)(\phi)\right|^2\right] = 0.$$

(b) For each bounded $(\hat{\mathcal{H}}_t^*)$–optional function γ we have that $\mathbb{P}$-a.s. for $i \neq j$ and all $t \geq \tau$,

$$\begin{aligned} \int \left\{\int_\tau^t \gamma(s,y)\ell(y,G^j)(ds)\right\} H_t^i(dy) = &\int_\tau^t \int \left\{\int_\tau^s \gamma(u,y)\ell(y,G^j)(du)\right\} M^i(ds,dy) \\ &+ \int_\tau^t \int \gamma(s,y) L(H^i,G^j)(ds,dy), \end{aligned} \tag{3.5}$$

where the stochastic integral is a continuous, square–integrable martingale and each term is square–integrable.

Proof. Part (a) may be proved by adapting the Tanaka formula proof of Theorem 5.9 of [2] (see Theorem 3.13(a) below). We give a different argument using Theorem 3.7 because it simultaneously handles (b) and will also handle situations in which the arguments of [2] do not apply. We have avoided some shortcuts peculiar to the present setting to give a proof which is readily modified to cover some subsequent scenarios (for example, in the proof of Lemma 5.3 below). Consider first the case where $\gamma(s,y,\omega) = \mathbf{1}_{]T_1,T_2]}(s)\gamma_1(y,\omega)$ where $\tau \leq T_1 \leq T_2 \leq \infty$ are $(\hat{\mathcal{H}}_t)_{t\geq\tau}$–stopping times and $\gamma_1 \in b\hat{\mathcal{H}}_{T_1}$ (note that $\gamma \equiv 1$ is possible). Then by (3.3) we have

$$\begin{aligned} \int \left\{\int_\tau^t \gamma(s,y)\ell^\epsilon(y,G^j)(ds)\right\} H_t^i(dy) = &\int_\tau^t \int \left\{\int_\tau^s \gamma(u,y)\ell^\epsilon(y,G^j)(du)\right\} M^i(ds,dy) \\ &+ \int_\tau^t \int \gamma(s,y) L^\epsilon(H^i,G^j)(ds,dy), \ \forall t \geq \tau, \ \mathbb{P}-\text{a.s.} \end{aligned} \tag{3.6}$$

For the left side of (3.6) equation we have (with $i \neq j$)

$$
\begin{aligned}
&\sup_{\tau \leq t \leq N} \left| \iint_\tau^t \gamma(s,y)[\ell^\epsilon(y,G^j)(ds) - \ell(y,G^j)(ds)]H_t^i(dy) \right| \\
&\qquad \leq \|\gamma_1\|_\infty \sup_{\tau \leq t \leq N} \sum_{m=1}^{2} \int |\ell^\epsilon(y,G^j)(T_m(y,\omega) \wedge t) - \ell(y,G^j)(T_m(y,\omega) \wedge t)| H_t^i(dy) \\
&\qquad \leq 2\|\gamma_1\|_\infty \sup_{\tau \leq t \leq N} \int \sup_{\tau \leq u \leq t} |\ell_u^\epsilon(y,G^j) - \ell_u(y,G^j)| H_t^i(dy) \\
&\qquad \to 0 \text{ in } L^2 \text{ as } \epsilon \downarrow 0,
\end{aligned}
$$

the last by Theorem 3.7(c).

For the martingale terms in (3.6) we have

$$
\begin{aligned}
&\mathbb{P}\left[\int_\tau^t \int \left(\int_\tau^s \gamma(u,y)\ell^\epsilon(y,G^j)(du) - \int_\tau^s \gamma(u,y)\ell(y,G^j)(du)\right)^2 H_s^i(dy)ds\right] \\
&\leq \|\gamma_1\|_\infty^2 \mathbb{P}\left[\int_\tau^t \int \left(\ell_{s\wedge T_2}^\epsilon(y,G^j) - \ell_{s\wedge T_2}(y,G^j) - \ell_{s\wedge T_1}^\epsilon(y,G^j) + \ell_{s\wedge T_1}(y,G^j)\right)^2 H_s^i(dy)ds\right] \\
&\to 0 \text{ as } \epsilon \downarrow 0 \text{ (Theorem 3.7(b))}.
\end{aligned}
$$

It follows from (3.6) and the above that there is an a.s. non-decreasing, continuous, square–integrable $(\mathcal{H}_t)$–predictable process, $(L_t(\gamma))_{t \geq \tau}$, such that

$$
\sup_{\tau \leq t \leq N} \left| \int_\tau^t \int \gamma(s,y) L^\epsilon(H^i,G^j)(ds,dy) - L_t(\gamma) \right| \to 0 \text{ in } L^2 \quad \forall N \in \mathbb{N},
$$

and

$$
\begin{aligned}
\int \left\{\int_\tau^t \gamma(s,y)\ell(y,G^j)(ds)\right\} H_t^i(dy) = \int_\tau^t \int \left\{\int_\tau^s \gamma(u,y)\ell(y,G^j)(du)\right\} M^i(ds,dy) \\
+ L_t(\gamma).
\end{aligned}
\tag{3.7}
$$

The same conclusions hold if γ is a finite linear combination of the γ's considered above (write $\gamma \in \mathcal{L}$). If γ is bounded, $(\hat{\mathcal{H}}_t)$–optional and has left limits, then there are $\{\gamma_n\}$ in $\mathcal{L}$ and $(\hat{\mathcal{H}}_t)$–stopping times $\tau \leq T_n \uparrow \infty$ such that $\|\gamma^{T_n} - \gamma_n\|_\infty \to 0$, and so $\sup_n \|\gamma_n\|_\infty < \infty$ (see [11] IV.64(c)). Now let $\phi : C \to \mathbb{R}$ be bounded and continuous and set $\gamma(s,y,\omega) = \phi(y^s)$. Choose $\{T_n\}$ and $\{\gamma_n\}$ as above and define $\tilde{\gamma}_n(s,y,\omega) = \mathbf{1}_{[T_n(y),\infty[}(s)$. Then, since $y = y^s$ for H_s^i-a.a. y, $\forall s \geq \tau$. a.s., we have w.p.1 $\forall t \geq \tau$,

$$
\begin{aligned}
L_t^\epsilon(H^i,G^j)(|\phi - \gamma_n|) &= L_t^\epsilon(H^i,G^j)(|\gamma - \gamma_n|) \\
&\leq L_t^\epsilon(H^i,G^j)(|\gamma - \gamma^{T_n}|) + \|\gamma^{T_n} - \gamma_n\|_\infty L_t^\epsilon(H^i,G^j)(1) \\
&\leq \|\phi\|_\infty L_t^\epsilon(H^i,G^j)(\tilde{\gamma}_n) + \|\gamma^{T_n} - \gamma_n\|_\infty L_t^\epsilon(H^i,G^j)(1) \\
&\to \|\phi\|_\infty L_t(\tilde{\gamma}_n) + \|\gamma^{T_n} - \gamma_n\|_\infty L_t(1) \text{ in } L^2 \text{ as } \epsilon \downarrow 0 \text{ by the above.}
\end{aligned}
\tag{3.8}
$$

Use (3.7) to see that

$$\begin{aligned}\|L_t(\tilde{\gamma}_n)\|_2 \leq & \mathbb{P}\Big[\int (\ell_t(y,G^j) - \ell_{t\wedge T_n}(y,G^j))\, H_t^i(dy)^2\Big]^{1/2} \\ & + \mathbb{P}\Big[\int_\tau^t \int (\ell_s(y,G^j) - \ell_{s\wedge T_n}(y,G^j))^2\, H_s^i(dy)ds\Big]^{1/2} \\ & \to 0 \text{ as } n\to\infty\end{aligned}$$

by dominated convergence, since each of the terms in (3.7) with $\gamma \equiv 1$ is in L^2. Therefore (3.8) approaches 0 in L^2 as $n \to \infty$. So, by first choosing $n = n_0$ large and then ϵ, ϵ' small, we see from the L^2–convergence of $L^\epsilon(H^i, G^j)(\gamma_n)$ as $\epsilon \downarrow 0$ that

$$\sup_{\tau\leq t\leq N} |L_t^\epsilon(H^i,G^j)(\phi) - L_t^{\epsilon'}(H^i,G^j)(\phi)| \to 0 \text{ in } L^2 \text{ as } \epsilon, \epsilon' \downarrow 0.$$

The existence of the FFCLT $L_t(H^i, G^j)$ satisfying (a) is now easy (see, for example, the proof of Theorem 5.9 in [2]). If $\gamma(s,y,\omega) = \gamma_1(y)\gamma_2(\omega)1_{[u,v[}(s)$ for continuous $\gamma_1 \in b\mathcal{C}_u$ and $\gamma_2 \in b\mathcal{H}_u$, then by its definition $L_t(\gamma) = \int_\tau^t \int \gamma(s,y)L(H^i,G^j)(ds,dy)$ and so (3.7) gives (3.5) in this case. A standard monotone class argument now implies (3.5) for bounded $(\hat{\mathcal{H}}_t)_{t\geq\tau}$–predictable γ and hence for bounded $(\hat{\mathcal{H}}_t)_{t\geq\tau}$–optional γ by IV.66 of [11]. An easy modification of the proof of Theorem 3.8 in [30] allows one to handle $(\hat{\mathcal{H}}_t^*)_{t\geq\tau}$–optional integrands.

□

We now wish to establish the existence of the FFCLT $L(G^1,G^2)$ for $(G^1,G^2) \in \mathcal{M}(H^1,H^2)$ and show that, in an appropriate sense, $L(G^1,G^2)$ is continuous in (G^1,G^2). We will do this using the Tanaka formula approach of [2].

We first need to dispose of some formalities concerning the the generator of the space–time process associated with the Brownian path process and extend the semimartingale decomposition appearing in Definition 3.1 to cover a more general class of integrands than D_{ST}. The notation we set up will also be useful in Section 6.

We will use the construction in Appendix A in the following special settings. Put $E = C$, $E^\partial = E \cup \{\partial\}$ (with ∂ added as a discrete point), $S^o = \{(t,y) \in \mathbb{R}_+ \times E : y^t = y\}$ (as in Section 1), $E_t = \{y \in C : y^t = y\} = C^t$,

$$\Omega^o = \{\omega \in C(\mathbb{R}_+, E^\partial) : \alpha^o(\omega) < \infty,\ \beta^o(\omega) = \infty\},$$

where

$$\alpha^o(\omega) = \inf\{t : \omega(t) \neq \partial\} \text{ and } \beta^o(\omega) = \inf\{t \geq \alpha^o(\omega) : \omega(t) \notin E_t\} \quad (\inf\emptyset = \infty).$$

Let $\mathcal{F}^o$ be the trace of the universally measurable subsets of $C(\mathbb{R}_+, E^\partial)$ on Ω^o. (Recall that, with a slight abuse of the usual notation, $C(\mathbb{R}_+, E^\partial)$ denotes the subspace of $D(\mathbb{R}_+, E^\partial)$ consisting of functions, f, such that $f(t) = f(t-)$ if $\partial \notin \{f(t), f(t-)\}$.)

Definition 3.11 If $(s, y) \in S^o$ and $\omega \in C$ satisfies $\omega(0) = y(s)$ let

$$(y/s/\omega)(u) = \begin{cases} y(u), & \text{if } u \leq s, \\ \omega(u-s), & \text{if } u > s; \end{cases}$$

and define $\bar{w}(s, y, \omega) \in \Omega^o$ by

$$\bar{w}(s, y, \omega)(t) = \begin{cases} \partial, & \text{if } t < s, \\ (y/s/\omega)^t & \text{if } t \geq s. \end{cases}$$

Recall that P^x is Wiener measure on C starting at $x \in \mathbb{R}^d$. Define $P^{s,y}$ on $(\Omega^o, \mathcal{F}^o)$ for $(s, y) \in S^o$ by

$$P^{s,y}(A) = P^{y(s)}\{\omega : \bar{w}(s, y, \omega)(\cdot) \in A\}.$$

Thus $P^{s,y}$ is the law of the Brownian path–process starting at time s with the path y.

Put $S^2 = \{(s, y_1, y_2) \in \mathbb{R}_+ \times C \times C : y_i^s = y_i,\ i = 1, 2\}$. Define a Markov semigroup $P_t^2 : b\mathcal{B}(S^2) \to b\mathcal{B}(S^2)$, $t \geq 0$, by

$$P_t^2\phi(s, y_1, y_2) = \int \phi(s+t, \omega_1(s+t), \omega_2(s+t))\, P^{s,y_1} \otimes P^{s,y_2}(d\omega_1, d\omega_2).$$

Thus $(P_t^2)_{t\geq 0}$ is the semigroup of the space–time process associated with a pair of independent Brownian path–processes.

The weak generator, $\mathcal{A}^2$, associated with this semigroup is the set of $(\phi, \psi) \in b\mathcal{B}(S^2) \times b\mathcal{B}(S^2)$ such that $t \mapsto P_t^2\psi(s, y_1, y_2)$ is right–continuous for each $(s, y_1, y_2) \in S^2$ and $P_t^2\phi = \phi + \int_0^t P_r^2\psi dr$ pointwise on S^2. Write $D(\mathcal{A}^2)$ for the set of ϕ such that $(\phi, \psi) \in \mathcal{A}^2$ for some ψ and set $\psi = \mathcal{A}^2\phi$ (clearly, ψ is unique).

Let D_o^2 denote the set of functions $\phi \in C(S^2)$ of the form

$$\phi(s, y_1, y_2) = \phi_0(s) \prod_{\ell=1}^{2} \prod_{i=0}^{n} \phi_{i,\ell}(y_\ell(t_i \wedge s)),$$

for some $\phi_0 \in C_K^\infty(\mathbb{R}_+)$, $\phi_{i,\ell} \in C_K^\infty(\mathbb{R}^d)$ and $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = \infty$. Observe for such a ϕ that, if we write $(B(t))_{t\geq 0}$ for the coordinate variables on C, then

$$
\begin{aligned}
&P_t^2\phi(s, y_1, y_2) \\
&\quad = \phi_0(s) \sum_{i=0}^n \sum_{j=1}^n \mathbf{1}\left\{t_i \vee (t_j - t) \leq s < t_{i+1} \wedge (t_{j+1} - t)\right\} \left[\prod_{\ell=1}^2 \left[\prod_{k=0}^i \phi_{k,\ell}(y_\ell(t_k))\right]\right] \\
&\quad\quad \times \prod_{\ell=1}^2 P^{y_\ell(s)} \left[\prod_{k=i+1}^j \phi_{k,\ell}(B(t_k - s)) \prod_{k=j+1}^n \phi_{k,\ell}(B(t))\right] \\
&\quad = \sum_{0\leq i\leq j\leq n} \mathbf{1}\left\{t_i \vee (t_j - t) \leq s < t_{i+1} \wedge (t_{j+1} - t)\right\} \prod_{\ell=1}^2 \left(\prod_{k=0}^i \phi_{k,\ell}(y_\ell(t_k))\right) \tilde{\phi}_{i,j}(s, y_\ell(s)),
\end{aligned}
$$

say (with the convention $\tilde{\phi}_{n,n} = 1$).

Put $I_{i,j} = [t_i \vee (t_j - 1), t_{i+1} \wedge (t_{j+1} - t)[$. For $0 \leq i < n$ the functions $\tilde{\phi}_{i,j}$ belong to the class $C_0^\infty(I_{i,j} \times \mathbb{R}^d)$ of infinitely differentiable functions which vanish at ∞ together with all their derivatives. Let D^2 be the set of $\phi \in C_b(S^2)$ such that

$$
\phi(s, y_1, y_2) = \sum_{m=0}^M \mathbf{1}\{u_m \leq s < u_{m+1}\} \left[\prod_{\ell=1}^2 \prod_{k=0}^m \phi_{k,\ell}(y_\ell(u_k))\right] \tilde{\phi}_m(s, y_1(s), y_2(s)) \tag{3.9}
$$

with $\phi_{k,\ell} \in C_K^\infty(\mathbb{R}^d) \cap \{1\}$, $0 = u_0 < \ldots < u_{M+1} < u_{M+1} \leq \infty$, $\tilde{\phi}_m \in C_0^\infty([u_m, u_{m+1}[\times \mathbb{R}^{2d})$ for $0 \leq m < M$, and $\tilde{\phi}_m = 1$. For $\phi \in D^2$ with the above representation, Itô's lemma shows that $\phi \in D(\mathcal{A}^2)$ and

$$
\mathcal{A}^2\phi(s, y_1, y_2) = \sum_{m=0}^M \mathbf{1}\{u_m \leq s < u_{m+1}\} \left[\prod_{\ell=1}^2 \prod_{k=0}^m \phi_{k,\ell}(y_\ell(u_k))\right] \left[\frac{\partial \tilde{\phi}_m}{\partial s} + \frac{\tilde{\Delta}}{2}\tilde{\phi}_m\right] (s, y_1(s), y_2(s)),
$$

where, as in Section 2, $\tilde{\Delta}$ is the $2d$–dimensional Laplacian.

We have seen that P_t^2 maps D_o^2 into $D^2 \subset D(\mathcal{A}^2)$ for each $t \geq 0$. Let $\langle D^2\rangle$ denote the linear span of D^2. A minor modification of the proof of Proposition 1.3.3 of [18] establishes a bounded–pointwise version of that result and enables us to conclude that if $\phi \in D(\mathcal{A}^2)$, then there exists $\phi_n \in \langle D^2\rangle$, $n \in \mathbb{N}$, such that $\phi_n \to \phi$ and $\mathcal{A}^2\phi_n \to \mathcal{A}^2\phi$ bounded–pointwise as $n \to \infty$.

Lemma 3.12 *If $\phi \in D(\mathcal{A}^2)$, then*

$$
\begin{aligned}
G_t^1 \otimes G_t^2(\phi) = \mu^1 \otimes \mu^2(\phi_\tau) &+ \int_\tau^t \iint \phi(s, y_1, y_2)\ \left[G_s^1(dy_1)dN^2(s, y_2) + G_s^2(dy_2)dN^1(s, y_1)\right] \\
&- \int_\tau^t \iint \phi(s, y_1, y_2)\ \left[G_s^1(dy_1)A^2(ds, dy_2) + G_s^2(dy_2)A^1(ds, dy_1)\right] \\
&+ \int_\tau^t \iint \mathcal{A}^2\phi(s, y_1, y_2)\, G_s^1(dy_1)G_s^2(dy_2)ds, \quad \forall t \geq \tau,\ a.s.
\end{aligned}
$$

Proof. By the above and dominated convergence, it suffices to prove the statement of the lemma for $\phi \in D^2$. Fix such a ϕ with the representation (3.9). We will check that the statement holds for $t \in [u_m, u_{m+1}[$ for successive m. For this it further suffices to establish the statement of the lemma with integrals over $[\tau, t]$ replaced by integrals over $[u_m, t]$ and the term $\mu^1 \otimes \mu^2(\phi_\tau)$ replaced by $G^1_{u_m} \otimes G^2_{u_m}(\phi_{u_m})$.

If $m = M$, then $\tilde{\phi}_m = 1$ and the revised claim is clear from Definition 3.1. Assume $m < M$. If $\tilde{\phi}_m(s, x_1, x_2) = \tilde{\phi}^0_m(s) \prod_{\ell=1}^2 \tilde{\phi}^\ell_m(x_\ell)$ for $\tilde{\phi}^0_m \in C^\infty_0(\mathbb{R}_+)$ and $\tilde{\phi}^\ell_m \in C^\infty_K(\mathbb{R}^d)$, then the revised claim in this special case is clear from Definition 3.1 and Itô's lemma. A routine truncation argument (see, for example, the proof of Proposition 5.1.1 of [18]) shows that the revised claim is still valid for $\tilde{\phi}_m$ of this special product form, but with $\tilde{\phi}^\ell_m \in C^\infty_0(\mathbb{R}^d)$. An application of Theorem 1.3.3 of [18] gives that the subset of $C_0(\mathbb{R}_+ \times \mathbb{R}^{2d})$ consisting of the linear span of this latter class of functions is a core for the generator of space–time Brownian motion, and hence the revised claim holds for all $\phi \in D^2$.

□

Theorem 3.13 *(a) The FFCLT $L(G^1, G^2)$ exists. For all $\phi \in C_b(C)$ and $u \geq \tau$*

$$\lim_{\epsilon \downarrow 0} \sup_{(G^1, G^2) \in \mathcal{M}(H^1, H^2)} \mathbb{P}\left[\sup_{\tau \leq t \leq u} |L^\epsilon_t(G^1, G^2)(\phi) - L_t(G^1, G^2)(\phi)| \wedge 1\right] = 0.$$

(b) Suppose that $\{(G^{1,k}, G^{2,k})\}_{k \in \mathbb{N} \cup \{\infty\}} \subset \mathcal{M}(H^1, H^2)$ is such that $\mathbb{P} - a.s.$ for all $t \geq \tau$ and $i = 1, 2$, $\lim_{k \to \infty} G^{i,k}_t = G^{i,\infty}_t$ in $\mathbf{M}_F(C)$. Then for all $t \geq \tau$ and $\phi \in C_b(C)$,

$$\lim_{k \to \infty} \mathbb{P}\left[\sup_{\tau \leq s \leq t} |L_s(G^{1,k}, G^{2,k})(\phi) - L_s(G^{1,\infty}, G^{2,\infty})(\phi)|\right] = 0.$$

Proof. (a) We modify the proof of the $\mathbf{M}_F(\mathbb{R}^d)$–valued version appearing as Theorem 5.10 in [2]. For $\phi \in b\mathcal{B}(S^2)$, write $R^2_\lambda \phi = \int_0^\infty e^{-\lambda t} P^2_t \phi, dt$, $\lambda > 0$ for the resolvent associated with P^2_t. Standard arguments show that if $\phi \in b\mathcal{B}(S^2)$ and $t \mapsto P^2_t \phi(s, y_1, y_2)$ is right–continuous for each $(s, y_1, y_2) \in S^2$, then $R^2_\lambda \phi \in D(\mathcal{A}^2)$ and $\mathcal{A}^2 R^2_\lambda \phi = \lambda R^2_\lambda \phi - \phi$. Given $\psi \in C_b(C)$, set $\psi_\epsilon(s, y_1, y_2) = p_\epsilon(y_1(s) - y_2(s))\psi((y^s_1 + y^s_2)/2)$, $\epsilon > 0$, and put $G_{\lambda,\epsilon}\psi = R^2_\lambda \psi_\epsilon$. We may take $\lambda = 0$ here if $d = 3$.

Apply Lemma 3.12 with $\phi = G_{\lambda,\epsilon}\psi$ and let $\epsilon \downarrow 0$ to prove that $L_t(G^1, G^2)$ exists and to derive a Tanaka formula for it. The argument is a minor alteration of the proof of Theorem 5.10 of [2]. As in the proof of that result, we conclude that for all $\phi \in C_b(C)$ and $u \geq \tau$,

$$\lim_{\epsilon \downarrow 0} \sup_{(G^1, G^2) \in \mathcal{M}(H^1, H^2)} \mathbb{P}\left[\sup_{\tau \leq t \leq u} \left|\tilde{L}^\epsilon_t(G^1, G^2)(\phi) - L_t(G^1, G^2)(\phi)\right| \wedge 1\right] = 0,$$

where

$$\tilde{L}^\epsilon_t(G^1, G^2)(\phi) = \int_\tau^t \int \left\{\int p_\epsilon(y_1(s) - y_2(s))\, G^2_s(dy_2)\right\} \phi((y_1 + y_2)/2)\, G^1_s(dy_1)\, ds.$$

Proceed as in Lemma 3.4 of [20] to replace $\tilde{L}^\epsilon$ by L^ϵ.
(b) If $\epsilon > 0$, then, since $G_s^{1,k} \otimes G_s^{2,k} \to G_s^{1,\infty} \otimes G_s^{2,\infty}$ for each s a.s., one has for ϕ as in the theorem

$$L_t^\epsilon(G^{1,k}, G^{2,k})(\phi) \to L_t^\epsilon(G^{1,\infty}, G^{2,\infty})(\phi), \qquad \forall t \geq \tau, \text{ a.s. as } k \uparrow \infty.$$

For $\tau \leq s < t$,

$$\begin{aligned} &|L_t^\epsilon(G^{1,k}, G^{2,k})(\phi) - L_s^\epsilon(G^{1,k}, G^{2,k})(\phi)| \\ &\quad \leq \|\phi\|_\infty |L_t^\epsilon(H^1, H^2)(1) - L_s^\epsilon(H^1, H^2)(1)|, \quad \forall k \in \mathbb{N} \cup \{\infty\}, \end{aligned}$$

and so by Arzela–Ascoli we have

$$\lim_{k\to\infty} \mathbb{P}\left[\sup_{\tau \leq s \leq t} |L_s^\epsilon(G^{1,k}, G^{2,k})(\phi) - L_s^\epsilon(G^{1,\infty}, G^{2,\infty})(\phi)| \wedge 1\right] = 0, \quad \forall t > \tau, \ \mathbb{P}\text{-a.s.} \tag{3.10}$$

Part (a) shows that

$$\lim_{\epsilon \downarrow 0} \sup_{k \in \mathbb{N}\cup\{\infty\}} \mathbb{P}\left[\sup_{\tau \leq s \leq t} |L_s^\epsilon(G^{1,k}, G^{2,k})(\phi) - L_s(G^{1,k}, G^{2,k})(\phi)| \wedge 1\right] = 0. \tag{3.11}$$

By first choosing ϵ so that the expression in (3.11) is small and then using (3.10) we prove that

$$\lim_{k\to\infty} \mathbb{P}\left[\sup_{\tau \leq s \leq t} |L_s(G^{1,k}, G^{2,k})(\phi) - L_s(G^{1,\infty}, G^{2,\infty})(\phi)| \wedge 1\right] = 0.$$

Finally, we can drop the truncation by 1 as the rest of the integrand is bounded by $2\|\phi\|_\infty L_t(H^1, H^2)(1) \in L^2$ (the latter, for example, by Theorem 3.10(a)).

□

3.3 Continuity of PFCLT's and Radon–Nikodym derivatives

Theorem 3.14 *Assume* $\{(G^{1,k}, G^{2,k}) : k \in \mathbb{N} \cup \{\infty\}\} \subset \mathcal{M}(H^1, H^2)$, $G_t^{1,k} \downarrow G_t^{1,\infty}$ *and* $G_t^{2,k} \uparrow G_t^{2,\infty}$, $\forall t \geq \tau$, *a.s. as* $k \to \infty$. *Then*

$$\ell_t(y, G^{1,k}) \downarrow \ell_t(y, G^{1,\infty}), \quad \forall \tau \leq t \leq u, \quad H_u^2\text{-a.a. } y, \quad \forall u \geq \tau, \ \mathbb{P}\text{-a.s.}$$

and

$$\ell_t(y, G^{2,k}) \uparrow \ell_t(y, G^{2,\infty}), \quad \forall \tau \leq t \leq u, \quad H_u^1\text{-a.a. } y, \quad \forall u \geq \tau, \ \mathbb{P}\text{-a.s.}$$

Proof. By (3.1), Remark 3.8, and the obvious monotonicity of $\ell_t^\epsilon(y, G^{2,k})$ in k, if $\ell_t^\infty(y,\omega)$ is defined to be $\lim_{k\to\infty}(\ell_t(y, G^{2,k})(\omega)$, if the limit exists, and 0, otherwise, then

$$\ell_t(y, G^{2,k}) \uparrow \ell_t^\infty(y), \quad \forall \tau \leq t \leq u, \quad H_u^2\text{-a.a. } y, \quad \forall u \geq \tau, \quad \mathbb{P}\text{-a.s.}$$

Equation (3.1) and Remark 3.8 also show that

$$\ell_t(y, G^{2,k}) - \ell_s(y, G^{2,k}) \leq \ell_t(y, G^{2,\infty}) - \ell_s(y, G^{2,\infty}) \ \ \forall \tau \leq s \leq t \leq u \ H_u^2\text{-a.a. } y \ \ \forall u \geq \tau \text{ a.s.}$$

These observations imply that ℓ^∞ also satisfies

$$\ell_t^\infty(y) - \ell_s^\infty(y) \leq \ell_t(y, G^{2,\infty}) - \ell_s(y, G^{2,\infty}), \ \ \forall \tau \leq s \leq t \leq u, \ \ H_u^2\text{-a.a. } y, \ \ \forall u \geq \tau, \text{ a.s.} \tag{3.12}$$

Theorem 3.10(b) shows that if $T \geq \tau$ is a bounded $(\mathcal{H}_t)_{t \geq \tau}$–stopping time then

$$\begin{aligned}\mathbb{P}\left[\int \ell_T(y, G^{2,\infty}) - \ell_T(y, G^{2,k})\, H_T^1(dy)\right] =& \mathbb{P}\left[L_T(H^1, G^{2,\infty})(1) - L_T(H^1, G^{2,k})(1)\right] \\ & \to 0 \text{ as } k \to \infty \text{ (by Theorem 3.13(b).}\end{aligned}$$

This shows that $\ell_T(y, G^{2,\infty}) = \ell_T^\infty(y)$ for H_T^1-a.e. y, $\mathbb{P}$-a.s. and so, by the section theorem,

$$\ell_u(y, G^{2,\infty}) = \ell_u^\infty(y), \quad H_u^1\text{-a.a. } y, \quad \forall u \geq \tau, \quad \mathbb{P}\text{-a.s.}$$

This, together with (3.12), shows that

$$\ell_t(y, G^{2,\infty}) = \ell_t^\infty(y), \quad \forall \tau \leq t \leq u, \ H_u^1\text{-a.e. } y, \ \forall u \geq \tau, \ \mathbb{P}\text{-a.s.}$$

A similar argument for $\ell_t(y, G^{1,\infty})$ completes the proof.

□

Lemma 3.15 *Assume $(\tilde{G}_t^2)_{t \geq \tau}$ is an optional $\mathbf{M}_F(C)$–valued process such that $\tilde{G}_t^2 \leq H_t^2$ $\forall t \geq \tau$ and the PFCLT, $\ell(y, \tilde{G}^2)$, for H^1 with respect to $\tilde{G}^2$ exists. There exists an $(\hat{\mathcal{H}}_t)_{t \geq \tau}$–predictable function $\lambda :]\tau, \infty[\times C \times \Omega \to [0,1]$ such that for all bounded $(\hat{\mathcal{H}}_t)_{t \geq \tau}$–predictable functions ϕ and for all bounded $(\mathcal{H}_t)_{t \geq \tau}$–stopping times $T \geq \tau$, we have*

$$\int_{]\tau,T]} \phi(s,y)\lambda(s,y)\, d\ell_s(y, H^2) = \int_{]\tau,T]} \phi(s,y)\, d\ell_s(y, \tilde{G}^2), \quad \bar{\mathbb{P}}_T^{H^1}\text{-a.s.}$$

Proof. Observe that for each bounded $(\mathcal{H}_t)_{t \geq \tau}$–stopping times $T \geq \tau$ that for $\bar{\mathbb{P}}_T^{H^1}$ - a.e. (y, ω) the measure $d\ell_s(y, \tilde{G}^2)(\omega)$ restricted to $]\tau, T(\omega)]$ is dominated by the measure $d\ell_s(y, H^2)(\omega)$ restricted to $]\tau, T(\omega)]$. Therefore, if we set

$$\lambda(s,y)(\omega) = \limsup_{k \to \infty} \frac{\ell_s(y, \tilde{G}^2) - \ell_{s-k^{-1}}(y, \tilde{G}^2)}{\ell_s(y, H^2) - \ell_{s-k^{-1}}(y, H^2)} \wedge 1,$$

then λ is $(\hat{\mathcal{H}}_t)_{t \geq \tau}$–predictable and for any bounded Borel function f on $]\tau, \infty[$

$$\int_{]\tau,T]} f(s)\lambda(s,y)\, d\ell_s(y, H^2) = \int_{]\tau,T]} f(s)\, d\ell_s(y, \tilde{G}^2), \quad \bar{\mathbb{P}}_T^{H^1} - \text{a.s.}$$

It follows that if ϕ is of the form $\phi(s,y,\omega) = \sum_{k=1}^n \xi_k(y,\omega)\mathbf{1}_{]u_k,v_k]}(s)$ with $\tau \le u_1 \le v_1 \le \dots u_n \le v_n$ and $\xi_k \in b(\hat{\mathcal{H}}_{u_k})$, then

$$\int_{]\tau,T]} \phi(s,y)\lambda(s,y)d\ell_s(y,H^2) = \int_{]\tau,T]} \phi(s,y)d\ell_s(y,\tilde{G}^2)$$

$\bar{\mathbb{P}}_T^{H^1}$-a.s., and the result in general follows by a monotone class argument.

□

3.4 Smoothness of FFCLT's

Lemma 3.16 *There is a $(\mathcal{H}_t)_{t\ge\tau}$–predictable, $\mathbf{M}_F(C)$–valued process $(K_t(G^1,G^2))_{t\ge\tau}$ such that*

$$L(G^1,G^2)(A\times B) = \int_A K_t(G^1,G^2)(B)\,dt$$

for all $A \in \mathcal{B}([\tau,\infty[)$ and $B \in \mathcal{C}$.

Proof. The proof is standard once one establishes that $\mathbb{P}$-a.s. the measure $A \mapsto L(H^1,H^2)(A\times C)$ is absolutely continuous with respect to Lebesgue measure. This, however, follows from Corollary 4.5 of [19].

□

Lemma 3.17 *(a) $\mathbb{P}$-a.s. for all $t \ge \tau$ the random measure $L(G^1,G^2)\otimes G_t^1$ assigns no mass to the set*

$$\{((s_1,y_1),y_2)\;:\;\tau \le s_1 \le t,\; y_1 = y_2^{s_1}\}$$

.

(b) $\mathbb{P}$-a.s. the random measure $L(G^1,G^2)\otimes L(G^1,G^2)$ assigns no mass to the set

$$\{((s_1,y_1),(s_2,y_2))\;:\;\tau \le s_1 \le s_2,\; y_1 = y_2^{s_1}\}.$$

Proof. Since $G^i \le H^i$, $i=1,2$, it suffices in both parts to consider the special case $(G^1,G^2) = (H^1,H^2)$.
(a) Let $(K_t(H^1,H^2))_{t\ge\tau}$ be the process guaranteed by Lemma 3.16. For $a > 0$ put

$$K_t^a(H^1,H^2) = \begin{cases} K_t(H^1,H^2), & \text{if } K_t(H^1,H^2)(C) \le a, \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

By (3.3),

$$
\begin{aligned}
&\iint_\tau^t \int \mathbf{1}\{y_1 = y_2^s\}\, K_s^a(H^1,H^2)(dy_1)\, ds\, H_t^1(dy_2) \\
&\quad \overset{m}{=} \int_\tau^t \iint \mathbf{1}\{y_1 = y_2^s\}\, K_s^a(H^1,H^2)(dy_1)\, H_s^1(dy_2)\, ds \\
&\quad = \int_\tau^t \iint \mathbf{1}\{y_1 = y_2\}\, H_s^1(dy_2)\, K_s^a(H^1,H^2)(dy_1)\, ds \\
&\quad = 0,
\end{aligned}
$$

where $=^m$ means that the two sides differ by a continuous martingale that is null at τ, and the last equality follows because $\mathbb{P}$-a.s. H_t^1 is diffuse for all $t > \tau$ by Proposition 4.1.8 of [9]. The leftmost member is therefore $\mathbb{P}$-a.s. 0 for all $t \geq \tau$. Letting $a \to \infty$ and applying monotone convergence establishes the result.

b) Fix $a > 0$ and put $T = \inf\{t \geq \tau : L_t(H^1,H^2)(C) \geq a\}$. By Theorem 3.10(b) and part (a) we have

$$
\begin{aligned}
&\int_\tau^{t\wedge T} \iint_\tau^{s_2 \wedge T} \int \mathbf{1}\{y_1 = y_2^{s_1}\}\, L(H^1,H^2)(ds_1, dy_1)\, L(H^1,H^2)(ds_2, dy_2) \\
&\quad \overset{m}{=} \iint_\tau^{t\wedge T} \int_\tau^{s_2\wedge T} \int \mathbf{1}\{y_1 = y_2^{s_1}\}\, L(H^1,H^2)(ds_1, dy_1)\, \ell(y_2, H^2)(ds_2)\, H_{t\wedge T}^1(dy_2) \\
&\quad = 0.
\end{aligned}
$$

Letting $a \to \infty$ and applying monotone convergence establishes the result.

□

4 Driving processes

4.1 Marked historical Brownian motion

As explained in Section 1, the "driving process" in our strong equation approach to building a solution of $\widehat{MP}_H(\tau,\mu^1,\mu^2)$ will be, in essence, a pair of independent historical Brownian motions (H^1,H^2) in which each path in the support of H_t^i is equipped with a number of $\mathbb{R}_+ \times [0,1]$–valued marks. Conditional on (H^1,H^2), the marks are laid down according to a Poisson process with intensity $r_i \ell(y, H^j) \otimes m$, where $j = 3 - i$ and m is Lebesgue measure on $[0,1]$. Moreover, the marks inherit the branching structure of (H^1,H^2).

As a first step in construction such a marked pair, we consider the simpler problem of marking a single historical process, where each path is marked according to a Poisson with intensity $\ell(y,\nu) \otimes m$ for some deterministic ν.

Recall from Section 2 that under $Q^{s,x}$, $W = (T, B_0)$ is a space–time Brownian motion and $B(t) = B_0(t - T(0))$ for $t \geq T(0)$ is a Brownian motion starting at x at time $s \geq 0$. Fix ν in $\Omega_{XS}[\tau,\infty[$. For

$s \geq \tau$ let $\ell_{s,t}(y,\nu) = \ell_t(y,\nu)$ be as in Theorem 2.2(b) but with τ replaced by s, where we have chosen $g \in C_\uparrow$ satisfying $\nu \in \Omega^g_{XS}[\tau,\infty[$ (so that $(\nu_u)_{u\geq s} \in \Omega^g_{XS}[s,\infty[)$ and then an appropriate version of ℓ given by (2.3). We write $\ell_s(y,\nu)(dt)$ for the obvious measure on $\mathcal{B}([s,\infty[)$ where y is a (possibly stopped) Brownian path starting at time s. Dependence on s will be supressed if there is no ambiguity.

An argument very similar to the proof of Proposition 4.2 of [20] shows that

$$\begin{aligned}(t;(s,x,z);(ds',dx',dz')) \mapsto\ & Q^{s,x}(W_t \in (ds',dx') \\ & \times \{\exp(-\ell_{s,t+s}(B,\nu))\delta_z(dz') + (1-\exp(-\ell_{s,t+s}(B,\nu)))m(dz')\}\end{aligned}$$

is a time-homogeneous Feller transition function on $[\tau,\infty[\times\mathbb{R}^d\times[0,1]$. Let (T,B_0,F) be a Feller process with this transition function and write $R^{s,x,z}_\nu$, $(s,x,z) \in [\tau,\infty[\times\mathbb{R}^d\times[0,1]$, for the corresponding family of laws on $D(\mathbb{R}_+,[\tau,\infty[\times\mathbb{R}^d\times[0,1])$. Thus, under $R^{s,x,z}_\nu$ the pair (T,B_0) evolves just as under $Q^{s,x}$. Conditional on (T,B_0), the process F is a jump–hold process that makes jumps at rate $\ell_s(B,\nu)(s+dt)$ at time t. The successive places to which F moves when making such jumps are i.i.d. with common distribution m.

Definition 4.1 Let $\mathbf{M}_\#$ denote the set of measures, n, on $\mathbb{R}_+\times[0,1]$ with the following properties:

(i) $n(A) \in \mathbb{N}\cup\{\infty\}$ for all $A \in \mathcal{B}(\mathbb{R}_+\times[0,1])$;

(ii) $n([0,k]\times[0,1]) < \infty$, for all $k \in \mathbb{N}$;

(iii) $n(\{t\}\times[0,1]) \in \{0,1\}$, for all $t \geq 0$.

Note that if we equip $\mathbf{M}_\#$ with the trace of the vague topology on locally finite measures on $\mathbb{R}_+\times[0,1]$, then it becomes a metrisable Lusin space. For $n \in \mathbf{M}_\#$ and $t \geq 0$ put $n^t = n(\cdot\cap[0,t]\times[0,1])$.

We will use the construction in Appendix A (with L=0) in the following setting. Put $E^\# = C\times\mathbf{M}_\#$,

$$S^\# = \{(t,y,n) \in \mathbb{R}_+\times E^\# : (y^t,n^t) = (y,n)\}, \quad E^\#_t = \{(y,n) \in E^\# : (y^t,n^t) = (y,n)\}$$

and

$$\Omega^\# = \{\omega \in C(\mathbb{R}_+,(E^\#)^\partial) : \alpha^\#(\omega) < \infty,\ \beta^\#(\omega) = \infty\},$$

where

$$\alpha^\#(\omega) = \inf\{t : \omega(t) \neq \partial\}$$

and

$$\beta^\#(\omega) = \inf\{t \geq \alpha^\#(\omega) : \omega(t) \notin E^\#_t\}.$$

Let $\mathcal{F}^{\#}$ be the trace of the universally measurable subsets of $C(\mathbb{R}_+, (E^{\#})^{\partial})$ on $\Omega^{\#}$.

If $(s, y, n) \in S^{\#}$ and $(\omega, \eta) \in C \times \mathbf{M}_{\#}$ satisfies $\omega(0) = y(s)$ let $((y, n)/s/(\omega, \eta)) = (y', n')$, where

$$y'(u) = \begin{cases} y(u), & \text{if } u \le s, \\ \omega(u - s), & \text{if } u > s, \end{cases}$$

and

$$n'(A) = n(\{(t, z) : (t, z) \in A,\ 0 \le t < s\}) + \eta(\{(t, z) : (s + t, z) \in A, t \ge 0\}).$$

For $t \ge 0$ define $(y, n)^t = (y^t, n^t)$ and

$$\breve{w}(s, (y, n), (\omega, \eta))(t) = \begin{cases} \partial, & \text{if } t < s, \\ ((y, n)/s/(\omega, \eta))^t, & \text{if } t \ge s. \end{cases}$$

Let $N : D([\tau, \infty[, [0, 1]) \to \mathbf{M}_{\#}$ be given by

$$N(f) = \begin{cases} 0, & \text{if } \operatorname{card}\{s \le t : f(s) \ne f(s-)\} = \infty \text{ for some } t \ge \tau, \\ \sum\{\delta_{(s, f(s))} : f(s) - f(s-) \ne 0\}, & \text{otherwise.} \end{cases}$$

For $(s, y, n) \in S^{\#}$, define a probability measure $P_\nu^{s,y,n}$ on $(\Omega^{\#}, \mathcal{F}^{\#})$ by setting

$$P_\nu^{s,y,n}(A) = R_\nu^{s,y(s),0}(\breve{w}(s, (y, n), (B_0(\cdot), N(F(\cdot)))) \in A).$$

Write $\breve{W}_t$, $t \ge 0$, for the coordinate variables on $\Omega^{\#}$. If $\omega \in \Omega^{\#}$ is such that $\breve{W}_t(\omega) \in C \times \mathbf{M}_{\#}$, put $(\breve{B}_t(\omega), \breve{N}_t(\omega)) = \breve{W}_t(\omega)$. It will be a convenient abuse of notation to set $\breve{B}_t(\omega) = \breve{N}_t(\omega) = \partial = \breve{W}_t(\omega)$ otherwise. Let $\mathcal{F}^{\#}_{[s,t+]}$ denote the universal completion of $\bigcap_n \sigma\{\breve{W}_r : s \le t \le t + n^{-1}\}$ for $0 \le s \le t$. It is not hard to see by an argument similar to the proof of Theorem 2.2.1 of [9] that, in the nomenclature of Appendix A, $\breve{W} = (\Omega^{\#}, \mathcal{F}^{\#}, \mathcal{F}^{\#}_{[s,t+]}, \breve{W}_t, P_\nu^{s,y,n})$ is the canonical realisation of an inhomogeneous Hunt process (IHP) with càdlàg paths in $E_t^{\#} \subset E^{\#}$.

Define $\mathbf{M}_F(C \times \mathbf{M}_{\#})^t$ as the set of those μ in $\mathbf{M}_F(C \times \mathbf{M}_{\#})$ for which $(y, n)^t = (y, n)$ μ-a.e. and, in the notation of Appendix A (with E replaced by $E^{\#}$, S^o replaced by $S^{\#}$, and so on), set

$$S' = \{(t, \breve{\mu}) \in \mathbb{R}_+ \times \mathbf{M}_F(C \times \mathbf{M}_{\#}) : \breve{\mu} \in \mathbf{M}_F(C \times \mathbf{M}_{\#})^t\}$$

and

$$\Omega' = \{\omega \in C(\mathbb{R}_+, \mathbf{M}_F^{\Delta}(C \times \mathbf{M}_{\#})) : \alpha'(\omega) < \infty,\ \beta'(\omega) = \infty\},$$

where

$$\alpha'(\omega) = \inf\{t : \omega(t) \ne \Delta\} \quad (\inf \emptyset = \infty)$$

and

$$\beta'(\omega) = \inf\{t \geq \alpha'(\omega) : (t, \omega(t)) \notin S'\}.$$

Let $\mathcal{F}'$, $\mathcal{F}'_{[s,t+]}$ and $\mathcal{F}'_t \equiv \mathcal{F}_{[0,t+]}$ be σ–fields defined on Ω' as in Appendix A. Following Theorem A.1, given $(\tau, \breve{\mu}) \in S'$, let $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}$ denote the probability measure on $(\Omega', \mathcal{F}')$ that is the law of the $\breve{W}$ superprocess (with zero immigration) starting at $(\tau, \breve{\mu})$. Write $\breve{H}_t$ for the coordinate maps on Ω'. Let $\breve{M}_\nu$ be the orthogonal martingale measure constructed from the martingales appearing in the semimartingale description of $\breve{H}$ under $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}$.

On $E^\# = C \times \mathbf{M}_\#$, let $\breve{\mathcal{D}}$ denote the Borel σ-field and let $\breve{\mathcal{D}}_t$ denote the σ-field generated by the map $(y, n) \mapsto (y^t, n^t)$. Set $\hat{\mathcal{F}}_t = \breve{\mathcal{D}}_t \times \mathcal{F}'_t$, and write $\hat{\mathcal{F}}_t^*$ for the universal completion of $\hat{\mathcal{F}}_t$.

For the next three results we fix $(\tau, \breve{\mu}) \in S'$.

Lemma 4.2 *Suppose that $u \geq \tau$ and $\phi \in b\breve{\mathcal{D}}_u$. Put $Z_t = \mathbf{1}\{t > u\}(\breve{H}_t(\phi) - \breve{H}_u(\phi))$, $t \geq \tau$. Then $(Z_t)_{t\geq\tau}$ is a continuous $(\mathcal{F}'_t)_{t\geq\tau}$–martingale under $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}$. In fact, $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}$-a.s. for all $t \geq \tau$,*

$$Z_t = \int_\tau^t \mathbf{1}\{s \geq u\}\phi(y, n)\, d\breve{M}_\nu(s, y, n).$$

Proof. We have $\phi(y, n) = \phi(y^u, n^u)$ by the same argument as Theorem IV.96.c of [11]. Define $\tilde{\phi} \in b\mathcal{B}(S^\#)$ by

$$\tilde{\phi}(s, y, n) = \begin{cases} \phi(y, n) = \phi(y^u, n^u), & \text{if } s \geq u, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\tilde{\phi}$ is finely continuous for the space–time process associated with $\breve{W}$. As we remark in Appendix A (just before Theorem A.1), we can use the resolvent for the space–time process associated with $\breve{W}$ to write $\tilde{\phi}$ as the bounded–pointwise limit of a sequence of functions $\{\phi^k\}_{k\in\mathbb{N}}$ such that the pair (ϕ^k, ψ^k) belongs to the weak generator considered in Appendix A for some ψ^k. Moreover, it is clear from this construction that $\psi^k(s, y, n) = 0$ for $s \geq u$. Thus $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}$-a.s. for all $t \geq \tau$,

$$\begin{aligned}
Z_t &= \lim_k \mathbf{1}\{t > u\}(\breve{H}_t(\phi_s^k) - \breve{H}_u(\phi_s^k)) \\
&= \lim_k \int_\tau^t \mathbf{1}\{s \geq u\}\phi^k(s, y, n)\, d\breve{M}_\nu(s, y, n) \qquad \text{by Theorem A.3} \\
&= \int_\tau^t \mathbf{1}\{s \geq u\}\phi(y, n)\, d\breve{M}_\nu(s, y, n).
\end{aligned}$$

□

Definition 4.3 For $\epsilon > 0$ define a continuous, $\mathbf{M}_F(C \times \mathbf{M}_\#))$–valued process $(L^\epsilon_t(\breve{H}, \nu))_{t \geq \tau}$ by setting

$$L^\epsilon_t(\breve{H}, \nu)(\phi) = \int_\tau^t \int \left(\int p_\epsilon(y(s) - x)\, \nu_s(dx) \right) \phi(y, n)\, \breve{H}_s(d(y, n))\, ds.$$

Using the same nomenclature as in Section 3 for a similar concept, we say that $\breve{H}$ and ν have a *field–field collision local time* $(L(\breve{H}, \nu))_{t \geq \tau}$ if $(L(\breve{H}, \nu))_{t \geq \tau}$ is a predictable, continuous, $\mathbf{M}_F(C \times \mathbf{M}_\#)$–valued process such that

$$\lim_{\epsilon \downarrow 0} L^\epsilon_t(\breve{H}, \nu)(\phi) = L_t(\breve{H}, \nu)(\phi)$$

in $\mathbb{Q}^{\tau, \breve{\mu}}_\nu$-probability for all $t \geq \tau$ and all bounded continuous functions ϕ on $C \times \mathbf{M}_\#$. In this case there is an almost surely unique Borel random measure on $]\tau, \infty[\times C \times \mathbf{M}_\#$ that we will also denote by $L(\breve{H}, \nu)$ such that $L(\breve{H}, \nu)(]s, t] \times A) = L_t(\breve{H}, \nu)(A) - L_s(\breve{H}, \nu)(A)$ for $\tau \leq s \leq t$.

Theorem 4.4 *Under $\mathbb{Q}^{\tau, \breve{\mu}}_\nu$ the collision local time, $L(\breve{H}, \nu)$, exists and in fact*

$$\lim_{\epsilon \downarrow 0} \mathbb{Q}^{\tau, \breve{\mu}}_\nu \left[\sup_{\tau \leq t \leq u} |L^\epsilon_t(\breve{H}, \nu)(\phi) - L_t(\breve{H}, \nu)(\phi)|^2 \right] = 0$$

for all bounded, continuous $\phi : C \times \mathbf{M}_\# \to \mathbb{R}$ and $u \geq \tau$. If $\gamma(s, y, n, \omega)$ is a bounded $(\hat{\mathcal{F}}^*_t)_{t \geq \tau}$–optional process, then

$$\begin{aligned}\iint_\tau^t \gamma(s, y, n) \ell(y, \nu)(ds)\, \breve{H}_t(d(y, n)) = {} & \int_\tau^t \int \left[\int_\tau^s \gamma(u, y, n) \ell(y, \nu)(du) \right] dM_\nu(s, y, n) \\ & + \int_\tau^t \int \gamma(s, y, n) L(\breve{H}, \nu)(ds, dy, dn).\end{aligned}$$

Each term is square–integrable and a.s. continuous, and the stochastic integral is a $(\mathcal{F}'_t)_{t \geq \tau}$–martingale.

Proof. This is a minor modification of the proof of Theorem 3.10. One needs an analogue of Theorem 3.7 with ν in place of the random G^j and this follows in the same way using Theorem 2.2 in place of Theorem 2.6.

□

Theorem 4.5 *Suppose that $\phi \in D_S$ with $\psi = \frac{\bar{\Delta}}{2}\phi$ and $\xi : \mathbf{M}_\# \mapsto \mathbb{R}$ is of the form $\xi(n) = \zeta(n(A_1), \ldots, n(A_a))$ for some $A_1, \ldots, A_a \in \mathcal{B}(\mathbb{R}_+ \times [0, 1])$ and bounded, continuous $\zeta : \mathbb{R}^a \to \mathbb{R}$.*

Then $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}$-a.s. we have for all $t \geq \tau$

$$\begin{aligned}
\int \phi(y)\xi(n)\,\breve{H}_t(d(y,n)) &= \int \phi(y)\xi(n)\,\breve{\mu}(d(y,n)) \\
&+ \int_\tau^t \int \phi(y)\xi(n)\,d\breve{M}_\nu(s,y,n) + \int_\tau^t \int \psi(s,y)\xi(n)\,\breve{H}_s(d(y,n))\,ds \\
&+ \int_\tau^t \iint_{[0,1]} \phi(y)\{\xi(n^{s-}+\delta_{s,z}) - \xi(n^{s-})\}\,m(dz)\,L(\breve{H},\nu)(d(y,n,s)).
\end{aligned}$$

Proof. It follows from the definition of $(\breve{B},\breve{N})$ that under each measure $P_\nu^{s,y,n}$ the process $\phi(\breve{B}_t) - \int_s^t \psi(u,\breve{B}_u)\,du$, $t \geq s$, is a continuous martingale that is uniformly bounded on finite intervals and

$$\xi(\breve{N}_t) - \int_{]s,t]}\int_{[0,1]} \{\xi(\breve{N}_{u-}+\delta_{u,z}) - \xi(\breve{N}_{u-})\}\,m(dz)\,\ell_s(\breve{B}_t,\nu)(du),\ t \geq s$$

is a martingale with locally finite variation. By stochastic calculus,

$$\begin{aligned}
\phi(\breve{B}_t)\xi(\breve{N}_t) &- \int_{]s,t]}\int_{[0,1]} \phi(\breve{B}_u)\left\{\xi(\breve{N}_{u-}+\delta_{u,z}) - \xi(\breve{N}_{u-})\right\}\,m(dz)\,\ell_s(\breve{B}_t,\nu)(du) \\
&- \int_s^t \psi(u,\breve{B}_u)\xi(\breve{N}_{u-})\,du, \quad t \geq s,
\end{aligned}$$

is a martingale.

For $k \in \mathbb{N}$ set

$$U_k(s,y) = \inf\{t \geq s : \ell_{s,t}(y,\nu) \geq k\}.$$

Then,

$$\begin{aligned}
&\phi(\breve{B}_{t\wedge U_k(s,\breve{B}_t)})\xi(\breve{N}_{t\wedge U_k(s,\breve{B}_t)}) \\
&\quad - \int_s^{t\wedge U_k(s,\breve{B}_t)} \int_{[0,1]} \phi(\breve{B}_u)\left\{\xi(\breve{N}_{u-}+\delta_{u,z}) - \xi(\breve{N}_{u-})\right\}\,m(dz)\,\ell_s(\breve{B}_{t\wedge U_k(s,\breve{B}_t)},\nu)(du) \\
&\quad - \int_s^{t\wedge U_k(s,\breve{B}_t)} \psi(u,\breve{B}_u)\xi(\breve{N}_{u-})\,du, \quad t \geq s,
\end{aligned}$$

is a bounded martingale. Note that the second term is a bounded, measurable function of $(\breve{B}_t,\breve{N}_t)$. It follows from Theorem A.3 that under $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}$,

$$\int \phi(y(\cdot \wedge U_k(\tau,y))\xi(n(\cdot \cap [0,U_k(\tau,y)])) \,\breve{H}_t(d(y,n)) = \int \phi(y)\xi(n)\,\breve{\mu}(d(y,n))$$
$$+ \int \left[\int_\tau^{t\wedge U_k(s,y)} \int_{[0,1]} \phi(y^s)\left\{\xi(n^{s-}+\delta_{s,z}) - \xi(n^{s-})\right\} m(dz)\,\ell_\tau(y,\nu)(ds)\right] \breve{H}_t(d(y,n))$$
$$+ \int_\tau^{t\wedge U_k(\tau,y)} \int \left[\phi(y)\xi(n) - \int_\tau^s \int_{[0,1]} \phi(y^u)\left\{\xi(n^{u-}+\delta_{u,z}) - \xi(n^{u-})\right\}\right.$$
$$\left.\times\, m(dz)\,\ell_\tau(y,\nu)(du)\right] d\breve{M}_\nu(s,y,n)$$
$$+ \int_\tau^t \int \psi(s, y(\cdot \wedge U_k(\tau,y))\xi(n(\cdot \cap [0,U_k(\tau,y)])\,\breve{H}_s(d(y,n))\,ds.$$

Now apply Theorem 4.4 to the second term on the right hand side in the above and then let $k \to \infty$ and use the square–integrability from this result to apply dominated convergence and hence complete the proof.

□

Corollary 4.6 *Almost surely under $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}$, for all $t > \tau$, $\breve{H}_t(\{(y,n) : n \neq n^{t-}\}) = 0$.*

Proof. Fix $u > \tau$. For $p \in \mathbb{N}$ let $\tau = t_0^p \leq \ldots \leq t_{K(p)}^p = u$ be such that $\lim_p \sup_k t_k^p - t_{k-1}^p = 0$. Put $\xi_k^p(n) = \mathbf{1}\{n(]t_{k-1}^p, t_k^p] \times [0,1]) > 0\}$. By writing the indicator function of $]0,\infty[$ as a bounded–pointwise limit of continuous functions, we can conclude from Theorem 4.5 that

$$\sum_k \mathbf{1}\{t \in]t_{k-1}^p, t_k^p]\} \int \xi_k^p(n)\,\breve{H}_t(d(y,n))$$
$$= \sum_k \mathbf{1}\{t \in]t_{k-1}^p, t_k^p]\}\Big\{\int \xi_k^p(n)\,\breve{\mu}(d(y,n)) + \int_\tau^t \int \xi_k^p(n)\,d\breve{M}_\nu(s,y,n)$$
$$+ \int_\tau^t \iint_{[0,1]} \left\{\xi_k^p(n^{s-}+\delta_{s,z}) - \xi_k^p(n^{s-})\right\} m(dz)\,L(\breve{H},\nu)(d(s,y,n))\Big\}.$$

By construction, $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}$-a.s., for all $t \geq \tau$, for $\breve{H}_t$-a.e. (y,n) we have $(y,n) = (y^t, n^t)$. Thus

$$\sum_k \mathbf{1}\{t \in]t_{k-1}^p, t_k^p]\} \int \mathbf{1}\left\{n(]t_{k-1}^p, t] \times [0,1]) > 0\right\} \breve{H}_t(d(y,n))$$
$$- \sum_k \mathbf{1}\{t \in]t_{k-1}^p, t_k^p]\} \int_{]t_{k-1}^p, t]} \iint_{[0,1]} \left\{\xi_k^p(n^{s-}+\delta_{s,z}) - \xi_k^p(n^{s-})\right\} m(dz)\,L(\breve{H},\nu)(d(s,y,n)) \quad (4.1)$$
$$= \sum_k \mathbf{1}\{t \in]t_{k-1}^p, t_k^p]\} \int_{]t_{k-1}^p, t]} \int \mathbf{1}\left\{n(]t_{k-1}^p, s] \times [0,1]) > 0\right\} d\breve{M}_\nu(s,y,n).$$

Let us first observe that, upon taking expectations of both sides in this equality and then letting $p \to \infty$, we can conclude from the continuity of $L(\breve{H}, \nu)$ and dominated convergence that $\breve{H}_t(\{(y,n) : n \neq n^{t-}\}) = 0$ almost surely for a fixed $t \in [\tau, u]$.

Now, by Doob's L^2 maximal inequality, we have

$$\begin{aligned}
&\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}\left[\sup_{\tau < t \le u}\left\{\sum_k \mathbf{1}\{t \in]t_{k-1}^p, t_k^p]\} \int_{]t_{k-1}^p, t]} \int \mathbf{1}\{n(]t_{k-1}^p, s] \times [0,1]) > 0\}\, d\breve{M}_\nu(s,y,n)\right\}^2\right] \\
&\le \sum_k \breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}\left[\sup_{t_{k-1}^p < t \le t_k^p}\left\{\int_{]t_{k-1}^p, t]} \int \mathbf{1}\{n(]t_{k-1}^p, s] \times [0,1]) > 0\}\, d\breve{M}_\nu(s,y,n)\right\}^2\right] \\
&\le 4 \sum_k \breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}\left[\int_{]t_{k-1}^p, t_k^p]} \int \mathbf{1}\{n(]t_{k-1}^p, s] \times [0,1]) > 0\}\, \breve{H}_s(d(y,n))\, ds\right] \\
&= 4 \breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}\left[\int_{]\tau,u]} \int \sum_k \mathbf{1}\{s \in]t_{k-1}^p, t_k^p]\} \mathbf{1}\{n(]t_{k-1}^p, s] \times [0,1]) > 0\}\, \breve{H}_s(d(y,n))\, ds\right] \\
&\to 0
\end{aligned}$$

as $p \to \infty$ by Fubini's theorem, the bounded convergence theorem, and what what we have concluded above for a fixed time. We can again appeal to (4.1) and the continuity of $L(\breve{H}, \nu)$ to establish the result.

□

4.2 Construction of the driving process

Our aim in this section is to construct the *driving processes* described informally in Subsection 1.7.

Abusing the α', β' notation slightly, define Ω'_H as Ω' but with C in place of $C \times \mathbf{M}_\#$. Similarly, define $(\mathcal{F}_t^H)_{t \ge 0}$ as the universal completion of the canonical right–continuous filtration on Ω'_H, and let H be the canonical coordinate process on Ω'_H. We remark that the definition of Ω'_H is very similar to that of Ω'_C in Section 1. The difference is that we do not require that the paths on Ω'_H take values in $\mathbf{M}_{FS}(C)$ after the birth time α'. We may, and shall, consider the law $\mathbb{Q}^{\tau,\mu}$ of historical Brownian motion as a law on Ω'_H (as well as $\Omega_H[\tau, \infty[$) by setting $\alpha' \equiv \tau$, $\mathbb{Q}^{\tau,\mu}$-a.s.

Let $\pi : \mathbf{M}_F(C \times \mathbf{M}_\#) \to \mathbf{M}_F(C)$ and $\Pi : \Omega' \to \Omega'_H$ be the natural projection maps. It is clear from the original definition that the law of $\Pi(\breve{H})$ under $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}$ is the law $\mathbb{Q}^{\tau,\pi(\breve{\mu})}$ of historical Brownian motion started at $(\tau, \pi(\breve{\mu}))$.

As in Proposition 4.3 of [20], for each $A \in \mathcal{F}'$ the map $(\tau', \breve{\mu}', \nu') \mapsto \breve{\mathbb{Q}}_{\nu'}^{\tau',\breve{\mu}'}(A)$ is Borel from $S = \{(\tau', \breve{\mu}', \nu') : (\tau', \breve{\mu}') \in S',\ \nu' \in \Omega_{XS}[\tau', \infty[\}$ to $\mathbb{R}$. Now apply Proposition C.1 of Appendix C with S as above, $T = \Omega'$, $E = \Omega'_H$, and $Z = \Pi$, to obtain a jointly measurable version of the conditional probabilities $(\tau, \breve{\mu}, \nu, h) \mapsto \breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}(\cdot \mid \Pi = h)$. Write this version simply as $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}(\cdot \mid h)$.

Let $(\check{H}^1, \check{H}^2)$ be the coordinate processes on $\Omega' \times \Omega'$, let $(\mathcal{F}_t'')_{t \geq 0}$ be the universal completion of the canonical right–continuous filtration on this product space, and let (H^1, H^2) be the coordinate processes on $\Omega_H' \times \Omega_H'$. We suppose for the rest of this section that $\tau \geq 0$, $\check{\mu}^1, \check{\mu}^2 \in \mathbf{M}_F(C \times \mathbf{M}_\#)^\tau$ with $\mu^i = \pi(\check{\mu}^i) \in \mathbf{M}_{FS}(C)^\tau$ and $r_1, r_2 > 0$ are fixed parameters. Dependence on (r_1, r_2) will be suppressed in our notation. Then $\mathbb{Q}^{\tau,\mu^i} \circ \Gamma^{-1}$ is the law of a super-Brownian motion and so, by Proposition 4.7 of [20], $\mathbb{Q}^{\tau,\mu^i}\{h : \Gamma(h) \in \Omega_{XS}[\tau, \infty[\} = 1$. Therefore, we may define a Borel probability measure $\check{\mathbb{P}}^{\tau,\check{\mu}^1,\check{\mu}^2}$ on $\Omega' \times \Omega'$ by setting

$$\check{\mathbb{P}}^{\tau,\check{\mu}^1,\check{\mu}^2}(A) = \int \left[\check{\mathbb{Q}}^{\tau,\check{\mu}^1}_{r_1\Gamma(h^2)}(\cdot \mid h^1) \otimes \check{\mathbb{Q}}^{\tau,\check{\mu}^2}_{r_2\Gamma(h^1)}(\cdot \mid h^2)\right](A)\,[\mathbb{Q}^{\tau,\mu^1} \otimes \mathbb{Q}^{\tau,\mu^2}](d(h^1, h^2)),$$

where $\Gamma(h^i)_t \equiv \Gamma(h^i)_\tau$ for $t \in [0, \tau]$. The pair $(\check{H}^1, \check{H}^2)$ under $\check{\mathbb{P}}^{\tau,\check{\mu}^1,\check{\mu}^2}$ is the pair of driving processes described in Subsection 1.7. Observe that

$$\check{\mathbb{P}}^{\tau,\check{\mu}^1,\check{\mu}^2} \circ (\Pi, \Pi)^{-1} = \mathbb{Q}^{\tau,\mu^1} \otimes \mathbb{Q}^{\tau,\mu^2},$$

so that the pair $(\Pi \circ \check{H}^1, \Pi \circ \check{H}^2)$ is indeed a pair of independent historical Brownian motions.

We need to define a suitable notion of a field–field collision local time for the pair $(\check{H}^1, \check{H}^2)$.

Definition 4.7 For $\epsilon > 0$ define a continuous $\mathbf{M}_F(C \times \mathbf{M}_\#)$–valued process $(L_t^\epsilon(\check{H}^1, \check{H}^2))_{t \geq \tau}$ by setting

$$L_t^\epsilon(\check{H}^1, \check{H}^2)(\phi) = \int_\tau^t \int \left(\int p_\epsilon(y_1(s) - y_2(s))\, \check{H}_s^2(d(y_2, n_2)) \right) \phi(y_1, n_1)\, \check{H}_s^1(d(y_1, n_1))\, ds.$$

We say that $\check{H}^1$ and $\check{H}^2$ have a *field–field collision local time* $(L_t(\check{H}^1, \check{H}^2))_{t \geq \tau}$ if $(L_t(\check{H}^1, \check{H}^2))_{t \geq \tau}$ is a predictable $\mathbf{M}_F(C \times \mathbf{M}_\#)$–valued process with almost surely continuous sample paths such that

$$\lim_{\epsilon \downarrow 0} L_t^\epsilon(\check{H}^1, \check{H}^2)(\phi) = L_t(\check{H}^1, \check{H}^2)(\phi)$$

in $\check{\mathbb{P}}^{\tau,\check{\mu}^1,\check{\mu}^2}$-probability for all $t \geq \tau$ and all bounded continuous functions ϕ on $C \times \mathbf{M}_\#$.

In Theorem 4.9 below we will establish a semimartingale decomposition for $(\check{H}_1, \check{H}_2)$ under $\check{\mathbb{P}}^{\tau,\check{\mu}^1,\check{\mu}^2}$ and show that the martingales appearing in that decomposition extend to orthogonal martingale measures defined on the following class of integrands.

Definition 4.8 A process $(\gamma(s, y, n))_{s \geq \tau}$ is τ–admissible (or just admissible if there is no ambiguity) if

$$\int_\tau^t \int \gamma(s, y, n)^2\, (\check{H}_s^1 + \check{H}_s^2)(dy, dn)\, ds < \infty, \quad \forall t \geq \tau, \quad \check{\mathbb{P}}^{\tau,\check{\mu}^1,\check{\mu}^2} - \text{a.s.}$$

and there exists a $(\check{\mathcal{D}}_t \times \mathcal{F}_t'')_{t \geq \tau}$–predictable process $(\gamma'(s, y, n))_{s \geq \tau}$ such that

$$\int_\tau^t \int \left[\gamma(s, y, n) - \gamma'(s, y, n)\right]^2\, (\check{H}_s^1 + \check{H}_s^2)(dy, dn)\, ds = 0, \quad \forall t \geq \tau, \quad \check{\mathbb{P}}^{\tau,\check{\mu}^1,\check{\mu}^2} - \text{a.s.}$$

Theorem 4.9 *Under $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$ there exist orthogonal martingale measures $\breve{M}^1$ and $\breve{M}^2$ defined on admissible integrands such that*

$$\left\langle \int_\tau^\cdot \int \gamma(s,y,n)\, d\breve{M}^i(s,y,n) \right\rangle_t = \int_\tau^t \int \gamma(s,y,n)^2\, \breve{H}_s^i(d(y,n))\, ds$$

and

$$\left\langle \int_\tau^\cdot \int \gamma^{(1)}(s,y,n)\, d\breve{M}^1(s,y,n), \int_\tau^\cdot \int \gamma^{(2)}(s,y,n)\, d\breve{M}^2(s,y,n) \right\rangle_t = 0.$$

Suppose that $\phi \in D_S$ with $\psi = \frac{\bar{\Delta}}{2}\phi$, and $\xi : \mathbf{M}_\# \mapsto \mathbb{R}$ is of the form $\xi(n) = \zeta(n(A_1), \ldots, n(A_a))$ for some $A_1, \ldots, A_a \in \mathcal{B}(\mathbb{R}_+ \times [0,1])$ and bounded continuous, $\zeta : \mathbb{R}^a \to \mathbb{R}$. Then $L(\breve{H}^i, \breve{H}^j)(i \neq j)$ exists and

$$\begin{aligned}
&\int \phi(y)\xi(n)\, \breve{H}_t^i(d(y,n)) \\
&= \int \phi(y)\xi(n)\, \breve{\mu}^i(d(y,n)) + \int_\tau^t \int \phi(y)\xi(n)\, d\breve{M}^i(s,y,n) + \int_\tau^t \int \psi(s,y)\xi(n)\, \breve{H}_s^i(d(y,n))\, ds \\
&\quad + r_i \int_\tau^t \iint_{[0,1]} \phi(y) \left\{ \xi(n^{s-} + \delta_{s,z}) - \xi(n^{s-}) \right\}\, m(dz)\, L(\breve{H}^i, \breve{H}^j)(d(s,y,n)),
\end{aligned}$$

where the stochastic integral is a continuous, square–integrable martingale. In particular, $\Pi(\breve{H}_1)$ and $\Pi(\breve{H}_2)$ are independent historical Brownian motions on $(\Omega'^2, \mathcal{F}'^2, (\mathcal{F}_t'')_{t \geq \tau}, \breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2})$. Moreover if $f : C \times \mathbf{M}_\# \mapsto \mathbb{R}$ is bounded and continuous then

$$\sup_{\tau \leq t \leq u} |L_t^\epsilon(\breve{H}^1, \breve{H}^2)(f) - L_t(\breve{H}^1, \breve{H}^2)(f)| \to 0 \quad \text{in } L^2(\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}) \qquad \forall u \geq \tau.$$

First we need some preliminary results.

Lemma 4.10 *Suppose that $\breve{\mu} \in \mathbf{M}_F(C \times \mathbf{M}_\#)^\tau$, $\mu = \pi(\breve{\mu})$, $\nu \in \Omega_{XS}[\tau, \infty[$, and $A \in \mathcal{F}_s'$ for some $s \geq \tau$.*
(a) $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}(A) = \breve{\mathbb{Q}}_{\nu^s}^{\tau,\breve{\mu}}(A)$.
(b) $\breve{\mathbb{Q}}_\nu^{\tau,\breve{\mu}}(A \mid h) = \breve{\mathbb{Q}}_{\nu^s}^{\tau,\breve{\mu}}(A \mid h)$ for $\mathbb{Q}^{\tau,\mu}$ -a.e. h.

Proof. (a) This is an easy exercise using the associated non-linear equation (A.1) for the log–Laplace transform and the fact that $P_\nu^{s',y,n} = P_{\nu^s}^{s',y,n}$ on the σ-field generated by the maps $(y,n) \mapsto (y^t, n^t)$ for $s' \leq t \leq s$.
(b) Suppose that $B_1 \in \mathcal{F}_s^H$ and $B_2 \in \sigma(\{H_t : t \geq s\})$. Put $B = B_1 \cap B_2$. Then, by part (a) and the

Markov property of $\breve{H}$,

$$\begin{aligned}\int \breve{\mathbb{Q}}^{\tau,\breve{\mu}}_{\nu^s}(A \mid h)\mathbf{1}_B(h)\,\mathbb{Q}^{\tau,\mu}(dh) &= \int \breve{\mathbb{Q}}^{\tau,\breve{\mu}}_{\nu^s}(A \mid h)\mathbf{1}_B(h)\,\breve{\mathbb{Q}}^{\tau,\breve{\mu}}_{\nu^s}\circ\Pi^{-1}(dh)\\
&= \breve{\mathbb{Q}}^{\tau,\breve{\mu}}_{\nu^s}\left[\mathbf{1}_A(\breve{H})\mathbf{1}_{B_1}(\Pi(\breve{H}))\mathbf{1}_{B_2}(\Pi(\breve{H}))\right]\\
&= \breve{\mathbb{Q}}^{\tau,\breve{\mu}}_{\nu^s}\left[\mathbf{1}_A(\breve{H})\mathbf{1}_{B_1}(\Pi(\breve{H}))\breve{\mathbb{Q}}^{s,\breve{H}_s}_{\nu^s}\{\breve{h}\in\Omega' : \Pi(\breve{h})\in B_2\}\right]\\
&= \breve{\mathbb{Q}}^{\tau,\breve{\mu}}_{\nu^s}\left[\mathbf{1}_A(\breve{H})\mathbf{1}_{B_1}(\Pi(\breve{H}))\mathbb{Q}^{s,\pi(\breve{H}_s)}(B_2)\right]\\
&= \breve{\mathbb{Q}}^{\tau,\breve{\mu}}_{\nu}\left[\mathbf{1}_A(\breve{H})\mathbf{1}_{B_1}(\Pi(\breve{H}))\mathbb{Q}^{s,\pi(\breve{H}_s)}(B_2)\right]\\
&= \int \breve{\mathbb{Q}}^{\tau,\breve{\mu}}_{\nu}(A \mid h)\mathbf{1}_B(h)\,\mathbb{Q}^{\tau,\mu}(dh),\end{aligned}$$

reversing the above steps. A monotone class argument shows that the equality of the two extreme members of the preceeding chain extends to all $B\in\mathcal{F}^H_\infty$.

□

Lemma 4.11 *For $\theta^i : \mathbb{R}_+\times C\times\mathbf{M}_\# \to \mathbb{R}$ of the form $\theta^i(t,y,n) = \mathbf{1}\{t>u^i\}\breve{\theta}^i(y,n)$ where $u^i\geq\tau$ and $\breve{\theta}^i \in b\breve{\mathcal{D}}_{u^i}$, set*

$$\breve{M}^i_t(\theta^i) = \mathbf{1}\{t>u^i\}\left[\breve{H}^i_t(\breve{\theta}^i) - \breve{H}^i_{u^i}(\breve{\theta}^i)\right].$$

Then $\{\breve{M}^i_t(\theta^i)\}_{t\geq\tau}$ is a continuous $(\mathcal{F}''_t)_{t\geq\tau}$–martingale under $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$ with quadratic variation

$$\langle\breve{M}^i(\theta^i)\rangle_t = \int_\tau^t \breve{H}^i_s((\theta^i_s)^2)\,ds.$$

Moreover,

$$\langle\breve{M}^1(\theta^1),\breve{M}^2(\theta^2)\rangle_t = 0.$$

Proof. Consider $\tau\leq s\leq t$ and $U^i\in b\mathcal{F}'_s$ for $i=1,2$. We have

$$\begin{aligned}&\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\left[(\breve{M}^1_t(\theta^1)-\breve{M}^1_s(\theta^1))U^1(\breve{H}^1)U^2(\breve{H}^2)\right]\\
&= \mathbb{Q}^{\tau,\mu^1}\otimes\mathbb{Q}^{\tau,\mu^2}\left[\breve{\mathbb{Q}}^{\tau,\breve{\mu}^1}_{r_1\Gamma(H^2)}\left[(\breve{M}^1_t(\theta^1)-\breve{M}^1_s(\theta^1))U^1(\breve{H}^1)\mid H^1\right]\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{r_2\Gamma(H^1)}\left[U^2(\breve{H}^2)\mid H^2\right]\right]\\
&= \mathbb{Q}^{\tau,\mu^1}\otimes\mathbb{Q}^{\tau,\mu^2}\left[\breve{\mathbb{Q}}^{\tau,\breve{\mu}^1}_{r_1\Gamma(H^2)}\left[(\breve{M}^1_t(\theta^1)-\breve{M}^1_s(\theta^1))U^1(\breve{H}^1)\mid H^1\right]\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{r_2\Gamma(H^1)^s}\left[U^2(\breve{H}^2)\mid H^2\right]\right]\\
&= \mathbb{Q}^{\tau,\mu^1}\otimes\mathbb{Q}^{\tau,\mu^2}\left[\mathbb{Q}^{\tau,\mu^1}\otimes\mathbb{Q}^{\tau,\mu^2}\left[\breve{\mathbb{Q}}^{\tau,\breve{\mu}^1}_{\Gamma(H^2)}\left[(\breve{M}^1_t(\theta^1)-\breve{M}^1_s(\theta^1))U^1(\breve{H}^1)\mid H^1\right]\mid\mathcal{F}^H_s\times\mathcal{F}^H_\infty\right]\right.\\
&\qquad\left.\times\,\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\Gamma(H^1)^s}\left[U^2(\breve{H}^2)\mid H^2\right]\right],\end{aligned}$$

where the second equality follows from Lemma 4.10.

Suppose that $g \in \mathcal{F}_\infty^H \times \mathcal{F}_\infty^H$. A monotone class argument starting with g of the form $g(h^1, h^2) = g(h^1)g(h^2)$ shows that

$$\mathbb{Q}^{\tau,\mu^1} \otimes \mathbb{Q}^{\tau,\mu^2}\left[g(H^1, H^2) \mid \mathcal{F}_s^H \times \mathcal{F}_\infty^H\right](h^1, h^2) = \mathbb{Q}^{\tau,\mu^1}\left[g(H^1, h^2) \mid \mathcal{F}_s^H\right](h^1)$$

for $\mathbb{Q}^{\tau,\mu^1} \otimes \mathbb{Q}^{\tau,\mu^2}$-a.e. (h^1, h^2). We may therefore continue the last chain of equalities to get

$$\mathbb{Q}^{\tau,\mu^1} \otimes \mathbb{Q}^{\tau,\mu^2}\left[\mathbb{Q}^{\tau,\mu^1}\left[\breve{\mathbb{Q}}^{\tau,\breve{\mu}^1}_{\Gamma(H^2)}\left[(\breve{M}_t^1(\theta^1) - \breve{M}_s^1(\theta^1))U^1(\breve{H}^1) \mid H^1\right] \mid \mathcal{F}_s^H\right] \breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\Gamma(H^1)^s}\left[U^2(\breve{H}^2) \mid H^2\right]\right].$$

Recall from Proposition 4.7 of [20] that $\Gamma(H^2) \in \Omega_{XS}[\tau, \infty[$, $\mathbb{Q}^{\tau,\mu^2}$-a.s. It therefore certainly suffices to show that if $\breve{\mu} \in \mathbf{M}_F(C \times \mathbf{M}_\#)^\tau$, $\mu = \pi(\breve{\mu})$, $\nu \in \Omega_{XS}[\tau, \infty[$, and

$$\breve{M}_t(\theta^i) = \mathbf{1}\{t > u^i\}\left[\breve{H}_t(\breve{\theta}^i) - \breve{H}_{u^i}(\breve{\theta}^i)\right],$$

then

$$\mathbb{Q}^{\tau,\mu}\left[\breve{\mathbb{Q}}^{\tau,\breve{\mu}}_\nu\left[(\breve{M}_t(\theta^1) - \breve{M}_s(\theta^1))U^1(\breve{H}) \mid H\right] \mid \mathcal{F}_s^H\right] = 0$$

$\mathbb{Q}^{\tau,\mu}$-a.s.

Write $(\mathcal{G}_t^H)_{t\geq 0}$ for the filtration on Ω' generated by the process $\Pi(\breve{H})$. Recall (by Proposition 4.7 of [20]) that for each fixed $\nu \in \Omega_{XS}[\tau, \infty[$, $\mathbb{Q}^{\tau,\mu} = \breve{\mathbb{Q}}^{\tau,\breve{\mu}}_\nu \circ \Pi^{-1}$; and, in fact, for $U^* \in b\mathcal{F}_\infty^H$, $V^* \in b\mathcal{F}_s^H$,

$$\mathbb{Q}^{\tau,\mu}\left[\mathbb{Q}^{\tau,\mu}\left[U^*(H) \mid \mathcal{F}_s^H\right] V^*(H)\right] = \breve{\mathbb{Q}}^{\tau,\breve{\mu}}_\nu\left[\breve{\mathbb{Q}}^{\tau,\breve{\mu}}_\nu\left[U^*(\Pi(\breve{H})) \mid \mathcal{G}_s^H\right] V^*(\Pi(\breve{H}))\right].$$

It therefore further suffices to show that

$$\breve{\mathbb{Q}}^{\tau,\breve{\mu}}_\nu\left[\breve{\mathbb{Q}}^{\tau,\breve{\mu}}_\nu\left[(\breve{M}_t(\theta^1) - \breve{M}_s(\theta^1))U^1(\breve{H}) \mid \Pi(\breve{H})\right] \mid \mathcal{G}_s^H\right] = 0 \tag{4.2}$$

$\breve{\mathbb{Q}}^{\tau,\breve{\mu}}_\nu$-a.s. Now the left-hand side in (4.2) is just

$$\breve{\mathbb{Q}}^{\tau,\breve{\mu}}_\nu\left[(\breve{M}_t(\theta^1) - \breve{M}_s(\theta^1))U^1(\breve{H}) \mid \mathcal{G}_s^H\right],$$

which is 0, $\breve{\mathbb{Q}}^{\tau,\breve{\mu}}_\nu$-a.s. from Lemma 4.2 and the fact that $\mathcal{G}_s^H \subset \mathcal{F}'_s$. This completes the proof that $\breve{M}^1(\theta^1)$ (and, similarly, $\breve{M}^2(\theta^2)$) is a martingale.

The claim regarding quadratic variations is immediate from Lemma 4.2, Fubini's theorem, and the construction of $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$ (keeping in mind, of course, the fact that the quadratic variation can be obtained by an almost sure sample path construction).

It remains to establish the claim regarding covariations. Given what we have already established, it suffices to show that, with U^1, U^2 defined as above,

$$\begin{aligned}
&\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\left[(\breve{M}_t^1(\theta^1)-\breve{M}_s^1(\theta^1))(\breve{M}_t^2(\theta^2)-\breve{M}_s^2(\theta^2))U^1(\breve{H}^1)U^2(\breve{H}^2)\right]\\
&\equiv \mathbb{Q}^{\tau,\mu^1}\otimes\mathbb{Q}^{\tau,\mu^2}\Big[\breve{\mathbb{Q}}^{\tau,\breve{\mu}^1}_{r_1\Gamma(H^2)}\left[(\breve{M}_t^1(\theta^1)-\breve{M}_s^1(\theta^1))U^1(\breve{H}^1)\mid H^1\right]\\
&\quad\times\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{r_2\Gamma(H^1)}\left[(\breve{M}_t^2(\theta^2)-\breve{M}_s^2(\theta^2))U^2(\breve{H}^2)\mid H^2\right]\Big]\\
&=0.
\end{aligned}$$

This will follow from (4.2) if we can show that

$$\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{r_2\Gamma(H^1)}\left[(\breve{M}_t^2(\theta^2)-\breve{M}_s^2(\theta^2))U^2(\breve{H}^2)\mid H^2\right]=\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{r_2\Gamma(H^1)^s}\left[(\breve{M}_t^2(\theta^2)-\breve{M}_s^2(\theta^2))U^2(\breve{H}^2)\mid H^2\right]$$

$\mathbb{Q}^{\tau,\mu^1}\otimes\mathbb{Q}^{\tau,\mu^2}$-a.s.

By Fubini's theorem and a monotone class argument, it therefore suffices to show for all $\nu\in\Omega_{XS}[\tau,\infty[$ and $B\in\mathcal{F}_\infty^H$ that

$$\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\nu}\left[(\breve{M}_t(\theta^2)-\breve{M}_s(\theta^2))U^2(\breve{H})\mathbf{1}_B(\Pi(\breve{H}))\right]=\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\nu^s}\left[(\breve{M}_t(\theta^2)-\breve{M}_s(\theta^2))U^2(\breve{H})\mathbf{1}_B(\Pi(\breve{H}))\right].$$

From the definition of $\breve{M}_t(\theta^2)$ (and by considering $s\le u^i$ and $s>u^i$ separately), it suffices in turn to show that for $v\ge u\ge\tau$, $\breve{\theta}\in b\breve{\mathcal{D}}_u$, $U\in b\mathcal{F}'_u$, and $C\in\mathcal{F}_\infty^H$ of the form $C=C_1\cap C_2$ with $C_1\in\mathcal{F}_u^H$ and $C_2\in\sigma(\{H_t:t\ge u\})$ that

$$\begin{aligned}
&\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\nu}\left[(\breve{H}_v(\breve{\theta})-\breve{H}_u(\breve{\theta}))U(\breve{H})\mathbf{1}_C(\Pi(\breve{H}))\right]\\
&\quad=\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\nu^u}\left[(\breve{H}_v(\breve{\theta})-\breve{H}_u(\breve{\theta}))U(\breve{H})\mathbf{1}_C(\Pi(\breve{H}))\right].
\end{aligned}$$

By the Markov property,

$$\begin{aligned}
&\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\nu^u}\left[(\breve{H}_v(\breve{\theta})-\breve{H}_u(\breve{\theta}))U(\breve{H})\mathbf{1}_C(\Pi(\breve{H}))\right]\\
&\quad=\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\nu^u}\left[U(\breve{H})\mathbf{1}_{C_1}(\Pi(\breve{H}))\breve{\mathbb{Q}}^{u,\breve{H}_u}_{\nu^u}\left[(\breve{H}_v(\breve{\theta})-\breve{H}_u(\breve{\theta}))\mathbf{1}_{C_2}(\Pi(\breve{H}))\right]\right].
\end{aligned}$$

By the Markov property and Lemma 4.10,

$$\begin{aligned}
&\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\nu}\left[(\breve{H}_v(\breve{\theta})-\breve{H}_u(\breve{\theta}))U(\breve{H})\mathbf{1}_C(\Pi(\breve{H}))\right]\\
&\quad=\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\nu}\left[U(\breve{H})\mathbf{1}_{C_1}(\Pi(\breve{H}))\int\{\breve{h}_v(\breve{\theta})-\breve{h}_u(\breve{\theta})\}\mathbf{1}_{C_2}(\Pi(\breve{h}))\breve{\mathbb{Q}}^{u,\breve{H}_u}_{\nu}(d\breve{h})\right]\\
&\quad=\breve{\mathbb{Q}}^{\tau,\breve{\mu}^2}_{\nu^u}\left[U(\breve{H})\mathbf{1}_{C_1}(\Pi(\breve{H}))\int\{\breve{h}_v(\breve{\theta})-\breve{h}_u(\breve{\theta})\}\mathbf{1}_{C_2}(\Pi(\breve{h}))\breve{\mathbb{Q}}^{u,\breve{H}_u}_{\nu}(d\breve{h})\right].
\end{aligned}$$

It is therefore enough to consider the case $u = \tau$ and establish

$$\breve{\mathbb{Q}}_{\nu}^{\tau,\breve{\mu}^2}\left[(\breve{H}_v(\breve{\theta}) - \breve{H}_\tau(\breve{\theta}))\mathbf{1}_{C_2}(\Pi(\breve{H}))\right] = \breve{\mathbb{Q}}_{\nu^\tau}^{\tau,\breve{\mu}^2}\left[(\breve{H}_v(\breve{\theta}) - \breve{H}_\tau(\breve{\theta}))\mathbf{1}_{C_2}(\Pi(\breve{H}))\right].$$

It is clear from the definition of $\breve{W}$ that for $r \geq \tau$, $P_{\nu}^{r,y,n}$ and $P_{\nu^\tau}^{r,y,n}$ agree on $\sigma(\{\breve{B}_t : t \geq \tau\})$. Using the Markov property, it is straightforward to derive an integral equation that characterises the joint log Laplace transform of $(\breve{H}_v(\breve{\theta}), H_{t_1}(\psi_1), \ldots, H_{t_n}(\psi_n))$ for $\tau \leq t_1 \leq \ldots \leq t_n$ and $\psi_1, \ldots, \psi_n \in b\mathcal{C}$ that generalises (A.1) in Appendix A (cf. Theorem 1.2 of [14]). It is clear from this equation that the transforms are equal under $\breve{\mathbb{Q}}_{\nu}^{\tau,\breve{\mu}^2}$ and $\breve{\mathbb{Q}}_{\nu^\tau}^{\tau,\breve{\mu}^2}$, and this leads to the equality we require.

□

The following result is immediate from Corollary 4.6 and the construction of $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$.

Lemma 4.12 *Almost surely under $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$, for all $t > \tau$, $\breve{H}_t^i(\{(y,n) : n \neq n^{t-}\}) = 0$.*

Proof of Theorem 4.9 Consider γ of the form

$$\gamma(t,y,n) = \mathbf{1}\{u_1 < t \leq u_2\}\kappa_1(y,n)\chi_1 + \ldots \mathbf{1}\{u_\ell < t \leq u_{\ell+1}\}\kappa_\ell(y,n)\chi_\ell,$$

where $\tau \leq u_1 \leq u_2 \leq \ldots \leq u_{\ell+1}$, $\kappa_k \in b\breve{\mathcal{D}}_{u_k}$, and $\chi_k \in b\mathcal{F}''_{u_k}$ for $k = 1, \ldots, \ell$. Put

$$\breve{M}_t^i(\gamma) = \sum_{k=1}^{\ell}\left[\breve{H}_{t\wedge u_{k+1}}^i(\kappa_k) - \breve{H}_{t\wedge u_k}^i(\kappa_k)\right]\chi_k.$$

It follows from Lemma 4.11 that $\breve{M}_t^i(\gamma)$ is a continuous martingale such that

$$\langle \breve{M}^i(\gamma)\rangle_t = \int_\tau^t \breve{H}_s^i(\gamma_s^2)\,ds.$$

Moreover, if $\breve{M}_t^1(\gamma^{(1)})$ and $\breve{M}_t^2(\gamma^{(2)})$ are two martingales of this form, then they are orthogonal.

It is straightforward to apply the extension procedure in Section 2 of [32] to construct two orthogonal martingale measures $\breve{M}^1$ and $\breve{M}^2$ under $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$ defined on admissible integrands and with the stated quadratic variation and covariation properties such that for γ as above

$$\breve{M}_t^i(\gamma) = \int_\tau^t \int \gamma(s,y,n)\,d\breve{M}^i(s,y,n).$$

We next want to relate the orthogonal martingale measures $\breve{M}^i$ that we have just constructed to the orthogonal martingale measures $\breve{M}_\nu$ considered in Subsection 4.1.

Say that a function $\theta' : \mathbb{R}_+ \times C \times \mathbf{M}_\# \to \mathbb{R}$ is a simple $(\breve{D}_t)_{t \geq \tau}$–predictable function if it is of the form

$$\theta'(t, y, n) = \mathbf{1}\{u_1 < t \leq u_2\}\theta_1(y, n) + \ldots \mathbf{1}\{u_\ell < t \leq u_{\ell+1}\}\theta_\ell(y, n)$$

for $u_1 \leq u_2 \leq \ldots \leq u_{\ell+1}$ and $\breve{\theta}_k \in b\breve{\mathcal{D}}_{u_k}$, $k = 1, \ldots, \ell$. Suppose that θ is a $\mathcal{B}([\tau, \infty[) \times \breve{\mathcal{D}}$–measurable function such that

$$\breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}\left[\int_\tau^t \int \theta^2(s, y, n)\, \breve{H}_s^i(d(y, n))\, ds\right] < \infty,$$

$\forall t \geq \tau$. By Lemma 4.12, $\mathbb{P}^{\tau, \breve{\mu}^1, \breve{\mu}^2}$–a.s. for all $t > \tau$, $\theta(t, y, n) = \theta(t, y^t, n^{t-})$ for $\breve{H}_t^i$–a.e. (y, n). In particular, θ is an admissible integrand (when considered as a function on $\mathbb{R}_+ \times C \times \mathbf{M}_\# \times \Omega' \times \Omega'$). Therefore, we can find a sequence $\{\theta_k\}_{k=1}^\infty$ of simple $(\breve{\mathcal{D}}_t)_{t \geq \tau}$–predictable functions such that

$$\breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}\left[\int_\tau^k \int \left[\theta_k(s, y, n) - \theta(s, y, n)\right]^2\, \breve{H}_s^1(d(y, n))\, ds\right] < 2^{-2k}, \quad \forall k \in \mathbb{N},$$

and hence,

$$\lim_{k \to \infty} \sup_{\tau \leq s \leq k} \left\{\int_\tau^s \int \theta_k(u, y, n)\, d\breve{M}^1(u, y, n) - \int_\tau^s \int \theta(u, y, n)\, d\breve{M}^1(u, y, n)\right\}^2 = 0,$$

both $\breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}$–a.s. and in $L^1(\breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2})$.

By construction of $\breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}$, it follows from Borel-Cantelli that for $\mathbb{Q}^{\tau, \mu^2}$-a.e. h^2 (say for h^2 not in a $\mathbb{Q}^{\tau, \mu^2}$–null set N)

$$\breve{\mathbb{Q}}^{\tau, \breve{\mu}^1}_{r_1\Gamma(h^2)}\left[\int_\tau^t \int \theta^2(s, y, n)\, \breve{H}_s(d(y, n))\, ds\right] < \infty, \quad \forall t \geq \tau,$$

and

$$\breve{\mathbb{Q}}^{\tau, \breve{\mu}^1}_{r_1\Gamma(h^2)}\left[\int_\tau^k \int \left[\theta_k(s, y, n) - \theta(s, y, n)\right]^2\, \breve{H}_s(d(y, n))\, ds\right] < 2^{-k} \text{ for } k > k_0(h^2), \tag{4.3}$$

Consequently, for $\breve{h}^2 \in \Omega'$ such that $\Pi(\breve{h}^2) = h^2 \notin N$ and for $\breve{\mathbb{Q}}^{\tau, \breve{\mu}^1}_{r_1\Gamma(h^2)}$- a.a. $\breve{h}^1$,

$$\lim_{k \to \infty} \sup_{\tau \leq t \leq k} \left|\int_\tau^t \int \theta_k(s, y, n)\, d\breve{M}^1(s, y, n)(\breve{h}^1, \breve{h}^2) - \tilde{M}_t(\breve{h}^1)\right| = 0$$

for some continuous process $\tilde{M}$ on Ω'. Thus

$$\int_\tau^t \int \theta(s, y, n)\, d\breve{M}^1(s, y, n) = \tilde{M}_t(\breve{H}^1), \qquad \forall t \geq \tau,\ \breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}\text{-a.s.} \tag{4.4}$$

By Lemma 4.2 (and possibly enlarging N),

$$\int_\tau^t \int \theta_k(s,y,n)\, d\breve{M}^1(s,y,n)(\breve{h}^1,\breve{h}^2) = \int_\tau^t \int \theta_k(s,y,n)\, d\breve{M}_{r_1\Gamma(h^2)}(s,y,n)(\breve{h}^1),$$

for $\mathbb{Q}^{\tau,\breve{\mu}^1}_{r_1\Gamma(h^2)}$-a.a. $\breve{h}^1$, $\forall h^2 = \Pi(\breve{h}^2) \notin N$. Equation (4.3) then shows that

$$\sup_{\tau \le t \le k} \left| \int_\tau^t \int \theta_k(s,y,n)\, d\breve{M}^1(s,y,n)(\breve{h}^1,\breve{h}^2) - \int_\tau^t \int \theta(s,y,n)\, d\breve{M}_{r_1\Gamma(h^2)}(s,y,n)(\breve{h}^1) \right| = 0,$$

for $\mathbb{Q}^{\tau,\breve{\mu}^1}_{r_1\Gamma(h^2)}$-a.a. $\breve{h}^1$, $\forall h^2 = \Pi(\breve{h}^2) \notin N$. So, by (4.4),

$$\tilde{M}_t(\breve{h}^1) = \int_\tau^t \int \theta(s,y,n)\, d\breve{M}_{r_1\Gamma(h^2)}(s,y,n)(\breve{h}^1), \quad \forall t \ge \tau, \ \ \mathbb{Q}^{\tau,\breve{\mu}^1}_{r_1\Gamma(h^2)}\text{-a.a. } \breve{h}^1, \ \forall\, h^2 \notin N.$$

If we apply these observations to the function $\theta(t,y,n) = \phi(y)\xi(n)$ (ϕ and ξ as in the statement of the Theorem) we get from Theorem 4.5 that $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$-a.s.

$$\begin{aligned}
&\int_\tau^t \int \phi(y)\xi(n)\, d\breve{M}^1(s,y,n) \\
&= \int \phi(y)\xi(n)\, \breve{H}^1_t(d(y,n)) - \int \phi(y)\xi(n)\, \breve{\mu}^1(d(y,n)) - \int_\tau^t \int \psi(s,y)\xi(n)\, \breve{H}^1_s(d(y,n))\, ds \\
&- \int_\tau^t \iint_{[0,1]} \phi(y) \left\{ \xi(n^{s-} + \delta_{s,z}) - \xi(n^{s-}) \right\}\, m(dz)\, L(\breve{H}^1,\nu)_{|\nu = r_1\Gamma(H^2)}(d(y,n,s)).
\end{aligned}$$

A similar result holds if we interchange the roles of H^1 and H^2.

Theorem 4.4 implies for all $u > \tau$ and ϕ bounded and continuous on $C \times \mathbf{M}_\#$ for $\mathbb{Q}^{\tau,\mu^2}$-a.a. h^2,

$$\lim_{\epsilon \downarrow 0} \mathbb{Q}^{\tau,\mu^1}_{r_1\Gamma(h^2)} \Big[\sup_{\tau \le t \le u} \left| L^\epsilon_t(\breve{H}, r_1\Gamma(h^2))(\phi) - L_t(\breve{H}, r_1\Gamma(h^2))(\phi) \right|^2 \Big] = 0. \tag{4.5}$$

The expression inside the limit is bounded by

$$r_1^2 \|\phi\|_\infty^2 \mathbb{Q}^{\tau,\mu^1}_{r_1\Gamma(h^2)} \left[L^\epsilon_t(H,h^2)(1)^2 + L_t(H,h^2)(1)^2 \right]$$

which is uniformly integrable (in the parameter ϵ) under $\mathbb{Q}^{\tau,\mu^2}$ by Theorem 3.10(a). Therefore we can integrate (4.5) with respect to $\mathbb{Q}^{\tau,\mu^2}$ and conclude that

$$\sup_{\tau \le t \le u} \left| r_1 L^\epsilon_t(\breve{H}^1, \breve{H}^2)(\phi) - L_t(\breve{H}^1,\nu)_{|\nu = r_1\Gamma(H^2)}(\phi) \right|^2 \to 0 \text{ in } L^2(\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}).$$

The result follows.

$\square$

4.3 Stochastic calculus for the driving process

Lemma 4.13 *Suppose that b is bounded and $(\breve{\mathcal{D}} \times \mathcal{F}''_t)_{t\geq\tau}$–optional. Then, for $i \neq j$,*

$$\iint_\tau^t b(s,y,n)\,\ell(y,H^j)(ds)\,\breve{H}^i_t(d(y,n)) = \int_\tau^t \int \left\{ \int_\tau^s b(u,y,n)\,\ell(y,H^j)(du) \right\} d\breve{M}^i(s,y,n)$$
$$+ \int_\tau^t \int b(s,y,n) L(\breve{H}^i, \breve{H}^j)(d(s,y,n)), \quad \forall t \geq \tau,\ \breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\text{-a.s.}$$

Proof. Proceed by making minor changes in the proof of Theorem 3.10(b). Recall from Remark 3.9 that Lemma 4.11 leads to the ϵ–version of the result (that is, the analogue of (3.6)). Note that the known existence of $L(\breve{H}^i, \breve{H}^j)$ greatly simplifies the proof.

□

The following result can be proved along the same lines as Lemma 4.1 of [21].

Lemma 4.14 *Let $\{(U^k_t)_{t\geq\tau}\}_{k=1}^\infty$ be a sequence of continuous $(\mathcal{F}''_t)_{t\geq\tau}$–martingales under $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$ such that*

$$U^k_t = \int_\tau^t \int g^k(s,y,n)\,d\breve{M}^i(s,y,n)$$

for an admissible stochastic integrand g^k. If $\{U^k_t\}_{k=1}^\infty$ converges in $L^1(\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2})$ for each $t \geq \tau$, then the limit process has a continuous version $(U^\infty_t)_{t\geq\tau}$ which is a continuous $(\mathcal{F}''_t)_{t\geq\tau}$–martingale under $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$. Moreover, there exists an admissible stochastic integrand g^∞ such that

$$\lim_{k\to\infty} \int_\tau^t |g^k(s,y,n) - g^\infty(s,y,n)|^2\,\breve{H}^i_s(d(y,n))\,ds = 0$$

in $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$-probability for all $t \geq \tau$, and

$$U^\infty_t = \int_\tau^t \int g^\infty(s,y,n)\,d\breve{M}^i(s,y,n).$$

In particular, if g is such that

$$\lim_{k\to\infty} \breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\left[\int_t^\tau \int |g^k(s,y,n) - g(s,y,n)| \wedge 1\,\breve{H}^i_s(d(y,n))\,ds\right] = 0,$$

then

$$U^\infty_t = \int_\tau^t \int g(s,y,n)\,d\breve{M}^i(s,y,n).$$

Define $\breve{n}^i(ds,dz) = n(ds,dz) - r_i \ell(y,H^j)(ds) \otimes m(dz)$ $(i \neq j)$. The following result is analogous to Lemma 4.3 of [21].

Lemma 4.15 *Let f be bounded and $(\breve{\mathcal{D}}_t \times \mathcal{B}([0,1]) \times \mathcal{F}''_t)_{t\geq\tau}$–predictable.*
(a) Then $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$-a.s.

$$\begin{aligned}
&\int \left\{ \int_{]\tau,t]\times[0,1]} f(s,y,n,z)\, n(ds,dz) \right\} \breve{H}^i_t(d(y,n)) \\
&\quad - r_i \int_\tau^t \iint_{[0,1]} f(s,y,n,z)\, m(dz)\, L(\breve{H}^i,\breve{H}^j)(d(s,y,n)) \\
&\quad = \int_\tau^t \int \left\{ \int_{]\tau,s]\times[0,1]} f(u,y,n,z)\, n(du,dz) \right\} d\breve{M}^i(s,y,n)
\end{aligned}$$

for all $t \geq \tau$ and both sides are continuous $(\mathcal{F}''_t)_{t\geq\tau}$–martingales.
(b) Then $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$-a.s.

$$\begin{aligned}
&\int \left\{ \int_{]\tau,t]\times[0,1]} f(s,y,n,z)\, \breve{n}^i(ds,dz) \right\} \breve{H}^i_t(d(y,n)) \\
&\quad = \int_\tau^t \int \left\{ \int_{]\tau,s]\times[0,1]} f(u,y,n,z)\, \breve{n}^i(du,dz) \right\} d\breve{M}^i(s,y,n)
\end{aligned}$$

for all $t \geq \tau$ and both sides are continuous $(\mathcal{F}''_t)_{t\geq\tau}$–martingales.
(c) If $T \geq \tau$ is a bounded $(\mathcal{F}''_t)_{t\geq\tau}$–stopping time, then

$$(t,y,n,\breve{h}^1,\breve{h}^2) \mapsto \int_{]\tau,t\wedge T(\breve{h}^1,\breve{h}^2)]\times[0,1]} f(s,y,n,z,\breve{h}^1,\breve{h}^2)\, \breve{n}^i(ds,dz),\ t \geq \tau,$$

is a $(\breve{\mathcal{D}}_t \times \mathcal{F}''_t)_{t\geq\tau}$–martingale under the normalised Campbell measure $\bar{\mathbb{P}}^{\breve{H}^i}_T$.

Proof. a) Observe from Theorem 4.9 and Theorem 3.13(a) that for $k > 0$

$$\begin{aligned}
\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\left[\int (n(]\tau,t]\times[0,1]) \wedge k)\, \breve{H}^i_t(d(y,n)) \right] &\leq \breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\left[L_t(\breve{H}^i,\breve{H}^j)(1) \right] \\
&= \mathbb{Q}^{\tau,\mu^1} \otimes \mathbb{Q}^{\tau,\mu^2}\left[L_t(H^i,H^j)(1) \right] \\
&< \infty.
\end{aligned}$$

By monotone convergence,

$$\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\left[\int n(]\tau,t]\times[0,1])\, \breve{H}^i_t(d(y,n)) \right] < \infty.$$

Suppose first that $f(s,y,n,z,\breve{h}^1,\breve{h}^2) = g(s,y,n,z)$ with

$$g(s,y,n,z) = 1_{]v,w]}(s)\phi^*(y)\zeta^*(n(A_1),\dots,n(A_b))1_B(z),$$

where $\tau < v < w < \infty$, $\phi^* \in D_S$ is of the form $\Phi^*(y(t_1),\dots,y(t_a))$ with $\tau < t_1 < \dots < t_a < v$ and $\Phi^* \in C_K^\infty(\mathbb{R}^{ad})$, $A_1,\dots,A_b \in \mathcal{B}([0,v[\times[0,1])$, $\zeta^* : \mathbb{R}^b \to \mathbb{R}$ is bounded and continuous, and $B \in \mathcal{B}([0,1])$.

Now $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$-a.s. we have for all $t \geq \tau$ that $(y^t,n^t) = (y,n)$ for $\breve{H}_t^i$-a.e. (y,n), and therefore that

$$\begin{aligned}\int_{]\tau,t]\times[0,1]} f(s,y,n,z)\,n(ds,dz) &= \int_{]\tau,\infty[\times[0,1]} f(s,y,n,z)\,n(ds,dz)\\ &= n(]v,w]\times B)\phi^*(y)\zeta^*(n(A_1),\dots,n(A_b))\end{aligned}$$

$\breve{H}_t^i$-a.e. (y,n). The claim follows for f of this special form by first applying Theorem 4.9 with $\phi(y) = \phi^*(y)$ and $\xi(n) = (n(]v,w]\times B)\wedge k)\zeta^*(n(A_1),\dots,n(A_b))$, then letting $k\to\infty$ and applying dominated convergence and Lemma 4.14.

The claim now clearly holds for $f(s,y,n,z,\breve{h}^1,\breve{h}^2) = g(s,y,n,z)\gamma(\breve{h}^1,\breve{h}^2)$, where g is as above and γ is bounded and $(\bigvee_{s<v}\mathcal{F}_s'')$-measurable. (The only thing that requires checking is that $(s,y,n,\breve{h}^1,\breve{h}^2) \mapsto \int_{]\tau,s]\times[0,1]} g(u,y,n,z)\gamma(\breve{h}^1,\breve{h}^2)\,n(du,dz)$ is an admissible stochastic integrand and that

$$\begin{aligned}&\int_\tau^t \int \left\{\int_{]\tau,s]\times[0,1]} g(u,y,n,z)\gamma\,n(du,dz)\right\} d\breve{M}^i(s,y,n)\\ &= \gamma\int_\tau^t \int \left\{\int_{]\tau,s]\times[0,1]} g(u,y,n,z)\,n(du,dz)\right\} d\breve{M}^i(s,y,n),\end{aligned}$$

but this is so because $g(u,y,n,z) = 0$ for $u \leq v$.) A monotone class argument, dominated convergence and Lemma 4.14 establish the result for general f.

b) This is clear from part (a) and Lemma 4.13.

c) Suppose that $\tau \leq u \leq t$, ξ is bounded and $\breve{\mathcal{D}}_u$-measurable, and γ is bounded and $\mathcal{F}_u''$-measurable. Then

$$\begin{aligned}&\bar{\mathbb{P}}_T^{\breve{H}^i}\left[\int_{]u\wedge T,t\wedge T]\times[0,1]} f(s,y,n,z)\,\breve{n}^i(ds,dz)\,\xi\,\gamma\right]\breve{\mu}^i(1)\\ &= \breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\left[\int\left\{\int_{]u\wedge T,t\wedge T]\times[0,1]} f(s,y,n,z)\xi(y,n)\,\breve{n}^i(ds,dz)\right\}\breve{H}_T^i(d(y,n))\gamma\right]\\ &= \breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\left[\int\left\{\int_{]\tau,T]\times[0,1]} f(s,y,n,z)1_{]u\wedge T,t\wedge T]}(s)\xi(y,n)\,\breve{n}^i(ds,dz)\right\}\breve{H}_T^i(d(y,n))\gamma\right]\\ &= \breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\left[\int\left\{\int_{]\tau,u]\times[0,1]} f(s,y,n,z)1_{]u\wedge T,t\wedge T]}(s)\xi(y,n)\,\breve{n}^i(ds,dz)\right\}\breve{H}_u^i(d(y,n))\gamma\right] = 0,\end{aligned}$$

where part (b) is used in the next to last equality. A monotone class argument completes the proof.

□

It is straightforward to use Theorem 4.9 in the proof of Theorem 2.6 (K_3) in [31] to show that for any $(\mathcal{F}''_t)_{t\geq\tau}$–stopping time $T \geq \tau$ the process $\breve{B}_t(y,n,\breve{h}^1,\breve{h}^2) = y(t) - y(\tau)$, $t \geq \tau$, is a $(\breve{\mathcal{D}}_t \times \mathcal{F}''_t)_{t\geq\tau}$–Brownian motion stopped at T under $\bar{\mathbb{P}}^{\breve{H}^i}_T$. Given a $(\breve{\mathcal{D}}_t \times \mathcal{F}''_t)_{t\geq\tau}$–predictable $\mathbb{R}^d$–valued process η such that $\int_\tau^t \int \|\eta(s,y,n)\|^2 \breve{H}^i_s(d(y,n))\,ds < \infty$ $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$-a.s. $\forall t \geq \tau$, then, as in Theorem 3.11 of [30], one may construct a $(\breve{\mathcal{D}}_t \times \mathcal{F}''_t)_{t\geq\tau}$–predictable $\mathbb{R}$–valued process $\breve{I}^i(\eta)$ such that for all bounded $(\mathcal{F}''_t)_{t\geq\tau}$–stopping times $T \geq \tau$,

$$\breve{I}^i(\eta)(t \wedge T(\breve{h}^1,\breve{h}^2), y, n, \breve{h}^1, \breve{h}^2) = \int_\tau^t \eta(s,y,n,\breve{h}^1,\breve{h}^2)\cdot dy(s)$$

$\bar{\mathbb{P}}^{\breve{H}^i}_T$-a.s. The process $\breve{I}^i(\eta)$ is unique up to $\breve{H}^i$-evanescent sets (where the latter are defined as for H^i-evanescent sets). With a slight abuse of notation, we will write $\breve{I}^i(\eta)(t,y,n,\breve{h}^1,\breve{h}^2)$ as $\int_\tau^t \eta(s,y,n,\breve{h}^1,\breve{h}^2)\cdot dy(s)$.

Lemma 4.16 *Suppose that $\phi \in D_{ST}$ with $A\phi = \psi$. Put $\Phi(t,y) = \phi(t,y) - \phi(\tau,y) - \int_\tau^t \psi(s,y)\,ds$. Let T be a $(\breve{\mathcal{D}}_t \times \mathcal{F}''_t)_{t\geq\tau}$–stopping time. Then*

$$\int \Phi(t \wedge T(y,n), y)\,\breve{H}^i_t(d(y,n)) = \int_\tau^t \int \Phi(s \wedge T(y,n), y)\,d\breve{M}^i(s,y,n)$$

for all $t \geq \tau$, $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$-a.s.

Proof. Applying Itô's lemma and the section theorem, we see that there is a bounded $(\breve{\mathcal{D}}_t \times \mathcal{F}''_t)_{t\geq\tau}$–predictable $\mathbb{R}^d$–valued process η such that

$$\Phi(s \wedge T(y,n), y) = \int_\tau^s \eta(u,y,n)\cdot dy(u) \text{ for } \tau \leq s \leq t,\ \breve{H}^i_t\text{-a.a. } (y,n)\ ,\ \forall t \geq \tau,\quad \breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\text{-a.s.}$$

As in Theorem 3.17 of [30] we have that

$$\int\int_\tau^t \eta(s,y,n)\cdot dy(s)\,\breve{H}^i_t(d(y,n)) = \int_\tau^t \int\int_\tau^s \eta(u,y,n)\cdot dy(u)\,d\breve{M}^i(s,y,n)$$

for all $t \geq \tau$, $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$-a.s. and we are done.

□

5 A strong equation for competing species

In this section we build a solution $(\hat{H}^1, \hat{H}^2)$ of the martingale problem $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ as the unique solution of a strong equation driven by a pair of marked historical Brownian motions of the type constructed in Section 4, thereby validating the intuition set out in Section 1. Throughout this section we take $r_i > 0$, $\tau \geq 0$ and $\mu^i \in \mathbf{M}_{FS}(C)^\tau$, $i = 1, 2$. Recall from Section 1 that $\hat{S}$ denotes the set of all such (τ, μ^1, μ^2). Set $\breve{\mu}^i = \mu^i \otimes \delta_0$ on $E^\# = C \times \mathbf{M}_\#$, $i = 1, 2$.

The strong equation will be driven by a pair of continuous, adapted $\mathbf{M}_F^\Delta(E^\#)$–valued stochastic processes, $(\tilde{H}^1, \tilde{H}^2)$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with $(\mathcal{F}_t)_{t\geq 0}$ right–continuous and the σ-field $\mathcal{F}$ universally complete. The pair $(\tilde{H}^1, \tilde{H}^2)$ will have law $\breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}$. In other words, the coordinate processes $(\breve{H}^1, \breve{H}^2)$ on the space $\Omega' \times \Omega'$ equipped with the measure $\breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}$ constructed in Subsection 4.2 is the canonical version of $(\tilde{H}^1, \tilde{H}^2)$. As before the r_i are implicit in the definition of $\breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}$ but dependence on r_i is suppressed. We assume that the stochastic integrals in the semimartingale decompostion in Theorem 4.9 are $(\mathcal{F}_t)_{t\geq 0}$–martingales. For example, this is true if $\mathcal{F}_t = \mathcal{H}_t$, where $(\mathcal{H}_t)_{t\geq 0}$ is the right–continuous filtration generated by $(\tilde{H}^1, \tilde{H}^2)$. Therefore the usual extension procedure shows that the associated martingale measures $(\tilde{M}^1, \tilde{M}^2)$ may be defined with $(\mathcal{F}_t)_{t\geq 0}$ in place of $(\mathcal{F}''_t)_{t\geq 0}$. As in Theorem 4.9, $(H^1, H^2) = (\Pi(\tilde{H}^1), \Pi(\tilde{H}^2))$ defines a pair of independent $(\mathcal{F}_t)$-historical Brownian motions starting at (μ^1, μ^2) and with associated martingale measures (M^1, M^2). Let $\hat{\mathcal{H}}_t = \breve{\mathcal{D}}_t \times \mathcal{F}_t$ and $\hat{\mathcal{H}}^C_t = \mathcal{C}_t \times \mathcal{F}_t$ be filtrations on $E^\# \times \Omega$ and $C \times \Omega$, respectively.

Definition 5.1 We say that $\{(\hat{H}^1_t, \hat{H}^2_t)) : t \geq 0\}$ is a solution of (SE) if the following conditions hold:
(i) $\hat{H}^i$ is an $(\mathcal{F}_t)_{t\geq 0}$–predictable, $\mathbf{M}_F^\Delta(C)$–valued process such that $\hat{H}^i_t = \Delta$, $\forall t < \tau$, $i = 1, 2$.
(ii) The PFCLT for H^i with respect to $\hat{H}^j$ $(i \neq j)$, $\ell(y, \hat{H}^j)(dt)$, exists.
(iii) If $\lambda^i(t, y)$ is the $(\hat{\mathcal{H}}^C_t)_{t\geq 0}$–predictable Radon–Nikodym derivative $\ell(y, \hat{H}^j)(dt)/\ell(y, H^j)(dt)$ (see Lemma 3.15), $J^i :]\tau, \infty[\times C \times [0, 1] \times \Omega \to \{0, 1\}$ is defined by

$$J^i(t, y, z, \omega) = \mathbf{1}\{\lambda^i(t, y) \geq z\}(\omega),$$

and $I^i :]\tau, \infty[\times C \times \mathbf{M}_\# \times \Omega \to \{0, 1\}$ is given by

$$I^i(t, y, n, \omega) = \mathbf{1}\left\{\int_{]\tau, t[} \int_{[0,1]} J^i(s, y, z)\, n(ds, dz) = 0\right\}(\omega),$$

then

$$\hat{H}^i_t(\phi) = \int I^i(t, y, n)\phi(y)\, \tilde{H}^i_t(d(y, n)), \quad \forall t \geq \tau,\ \mathbb{P}\text{-a.s. for each bounded measurable } \phi : C \to \mathbb{R}.$$

Here is the main result of this section. Note that part (b) proves part (a) of Theorem 1.4. Recall the notation Ω'_C from the Section 1.

Theorem 5.2 *(a) There is a pathwise unique solution $(\hat{H}^1, \hat{H}^2)$ of (SE). More precisely the uniqueness means that if $(\hat{G}^1, \hat{G}^2)$ also solves (SE) then $\hat{H}^i_t = \hat{G}^i_t$, $\forall t \geq 0$, $\mathbb{P}$-a.s.*
(b) The solution to (SE) is a.s. continuous on $[\tau, \infty[$ (and therefore has paths in $\Omega'_C \times \Omega'_C$) and satisfies $(\widehat{MP})(\tau, \mu^1, \mu^2)$.
(c) The law, $\hat{\mathbb{P}}^{\tau,\mu^1,\mu^2}$, of the solution is unique, that is, it depends only on (τ, μ^1, μ^2) (and (r_1, r_2)) and not the choice of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ or $(\tilde{H}^1, \tilde{H}^2)$.
(d) For $A \in \mathcal{F}'_C \times \mathcal{F}'_C$, the map $(\tau, \mu^1, \mu^2) \mapsto \hat{\mathbb{P}}^{\tau,\mu^1,\mu^2}(A)$ is Borel from $\hat{S}$ to $\mathbb{R}$.

Proof. We begin with the existence claim. Throughout this section ϕ_i denotes a function in D_{ST} and $\psi_i = A\phi_i$, $i = 1, 2$. Define $\hat{H}^{1,0}$ and $((\hat{H}^{1,k}, \hat{H}^{2,k}))_{k=1}^\infty$ inductively as follows.

Put $\hat{H}^{1,0} = H^1$. For $k = 1, 2, \ldots$ suppose that $\hat{H}^{1,0}$ and $((\hat{H}^{1,h}, \hat{H}^{2,h}))_{h=1}^{k-1}$ have been defined and are such that

$$\hat{H}^{i,h}_t(\phi) = \int I^{i,h}(t, y, n)\phi(y)\, \tilde{H}^i_t(d(y, n)),\ t \geq \tau$$

for $(\hat{\mathcal{H}}_t)_{t\geq\tau}$–predictable functions $I^{i,h} :]\tau, \infty[\times \check{E}^{\#} \times \Omega \to \{0, 1\}$. Assume also that for $h \leq k - 1$,

$$\begin{aligned}
\hat{M}^{1,h}_t(\phi_1) \equiv \hat{H}^{1,h}_t(\phi_1(t)) - \mu^1(\phi_1(\tau)) - \int_\tau^t \hat{H}^{1,h}_s(\psi_1(s))\, ds \\
+ \int_\tau^t \int_C \phi_1(s, y)\, L(\hat{H}^{1,h}, \hat{H}^{2,h})(ds, dy) \\
= \int_\tau^t \int_{E^{\#}} \phi_1(s, y)\, I^{1,h}(t, y, n)\, d\tilde{M}^1(s, y, n)
\end{aligned}$$

and

$$\begin{aligned}
\hat{M}^{2,h}_t(\phi_2) \equiv \hat{H}^{2,h}_t(\phi_2(t)) - \mu^2(\phi_2(\tau)) - \int_\tau^t \hat{H}^{2,h}_s(\psi_2(s))\, ds \\
+ \int_\tau^t \int_C \phi_2(s, y)\, L(\hat{H}^{2,h}, \hat{H}^{1,h-1})(ds, dy) \\
= \int_\tau^t \int_{E^{\#}} \phi_2(s, y)\, I^{2,h}(s, y, n)\, d\tilde{M}^2(s, y, n)
\end{aligned}$$

so that

$$\langle \hat{M}^{i,h}(\phi_i) \rangle_t = \int_\tau^t \int_C \phi_i(s, y)^2\, \hat{H}^{i,h}_s(dy)\, ds$$

and

$$\langle M^i(\phi_i), \hat{M}^{j,h}(\phi_j) \rangle_t = \langle \hat{M}^{i,h}(\phi_i), \hat{M}^{j,h}(\phi_j) \rangle_t = 0\ (i \neq j).$$

Therefore, the $(\mathcal{H}_t)_{t\geq\tau}$–predictable process $(H^{1,k-1}, H^{2,k-1})$ is in $\mathcal{M}(H^1, H^2)$, the PFCLT $\ell(y, \hat{H}^{1,k-1})$ is defined (Theorem 3.7), and we can choose $\lambda^{2,k}$ to be the $(\hat{\mathcal{H}}_t)_{t\geq\tau}$–predictable version of

$$d\ell(y, \hat{H}^{1,k-1})/d\ell(y, H^1)$$

guaranteed by Lemma 3.15. Define $J^{2,k} :]\tau, \infty[\times C \times [0,1] \times \Omega \to \{0,1\}$ by

$$J^{2,k}(t,y,z,\omega) = \mathbf{1}\{\lambda^{2,k}(t,y) \geq z\}(\omega)$$

and $I^{2,k} :]\tau, \infty[\times E^{\#} \times \Omega \to \{0,1\}$ by

$$I^{2,k}(t,y,n,\omega) = \mathbf{1}\left\{\int_{]\tau,t[}\int_{[0,1]} J^{2,k}(s,y,z)\, n(ds,dz) = 0\right\}(\omega).$$

Set

$$\hat{H}_t^{2,k}(\phi) = \int I^{2,k}(t,y,n)\phi(y)\, \tilde{H}_t^2(d(y,n)),\ t \geq \tau.$$

We will show below that

$$\begin{aligned} \hat{M}_t^{2,k}(\phi_2) \equiv \hat{H}_t^{2,k}(\phi_2(t)) - \mu^2(\phi_2(\tau)) - \int_\tau^t \hat{H}_s^{2,k}(\psi_2(s))\, ds \\ + \int_\tau^t \int_C \phi_2(s,y)\, L(\hat{H}^{2,k}, \hat{H}^{1,k-1})(ds,dy) \\ = \int_\tau^t \int_{E^{\#}} \phi_2(s,y)\, I^{2,k}(s,y,n)\, d\tilde{M}^2(s,y,n) \end{aligned} \tag{5.1}$$

so that

$$\langle \hat{M}^{2,k}(\phi_2)\rangle_t = \int_\tau^t \int_C \phi_2(s,y)^2 \hat{H}_s^{2,k}(dy)\, ds,$$

and

$$\langle M^1(\phi_1), \hat{M}^{2,k}(\phi_2)\rangle_t = \langle \hat{M}^{1,k-1}(\phi_1), \hat{M}^{2,k}(\phi_2)\rangle_t = 0.$$

Therefore $H^{2,k}$ is $(\mathcal{H}_t)_{t\geq\tau}$–predictable, $(H^{1,k-1}, H^{2,k})$ is in $\mathcal{M}(H^1, H^2)$, the PFCLT $\ell(y, \hat{H}^{2,k})$ is defined, and we can choose $\lambda^{1,k}$ to be the $(\hat{\mathcal{H}}_t)_{t\geq\tau}$–predictable version of

$$d\ell(y, \hat{H}^{2,k})/d\ell(y, H^2)$$

guaranteed by the Lemma 3.15. Define $J^{1,k} :]\tau, \infty[\times C \times [0,1] \times \Omega \to \{0,1\}$ by

$$J^{1,k}(t,y,z,\omega) = \mathbf{1}\{\lambda^{1,k}(t,y) \geq z\}(\omega)$$

and $I^{1,k} :]\tau, \infty[\times E^{\#} \times \Omega \to \{0,1\}$ by

$$I^{1,k}(t,y,n,\omega) = \mathbf{1}\left\{ \int_{]\tau,t[} \int_{[0,1]} J^{1,k}(s,y,z)\, n(ds,dz) = 0 \right\}(\omega).$$

Set

$$\hat{H}_t^{1,k}(\phi) = \int I^{1,k}(t,y,n)\phi(y)\, \tilde{H}_t^1(d(y,n)),\ t \geq \tau.$$

We will show below that

$$\begin{aligned} \hat{M}_t^{1,k}(\phi_1) &= \hat{H}_t^{1,k}(\phi_1(t)) - \mu^1(\phi_1(\tau)) - \int_\tau^t \hat{H}_s^{1,k}(\psi_1(s))\, ds \\ &\quad + \int_\tau^t \int_C \phi_1(s,y)\, L(\hat{H}^{1,k}, \hat{H}^{2,k})(ds,dy) \\ &= \int_\tau^t \int_{E^\#} \phi_1(s,y)\, I^{1,k}(s,y,n)\phi_1(s,y)\, d\tilde{M}^1(s,y,n) \end{aligned} \tag{5.2}$$

so that

$$\langle \hat{M}^{1,k}(\phi_1)\rangle_t = \int_\tau^t \int_C \phi_1(s,y)^2 \hat{H}_s^{1,k}(dy)\, ds,$$

and

$$\langle \hat{M}^{1,k}(\phi_1), M^2(\phi_2)\rangle_t = \langle \hat{M}^{1,k}(\phi_1), \hat{M}^{2,k}(\phi^2)\rangle_t = 0.$$

Hence the inductive construction of $(\hat{H}^{1,k}, \hat{H}^{2,k})$ is complete modulo the verification of (5.1) and (5.2). Both claims are a consequence of the following Lemma 5.3 or its companion obtained by interchanging the indices 1 and 2, and we interrupt the proof of the theorem to present this result.

Lemma 5.3 *Suppose that G^1 is of the form*

$$G_t^1(\phi) = \int B^1(t,y,n)\phi(y)\, \tilde{H}_t^1(d(y,n))$$

for some $(\hat{\mathcal{H}}_t)_{t\geq\tau}$–predictable $B^1 :]\tau,\infty[\times C \times \mathbf{M}_\# \times \Omega \to \{0,1\}$ and $\mathbb{P}$-a.s. $\forall t \geq \tau$,

$$\begin{aligned} G_t^1(\phi_1(t)) &= \mu^1(\phi_1(\tau)) + \int_\tau^t G_s^1(\psi_1(s))\, ds - \int_\tau^t \int_C \phi_1(s,y)\, A(ds,dy) \\ &\quad + \int_\tau^t \int_{E^\#} B^1(s,y,n)\phi_1(s,y) d\tilde{M}^1(s,y,n), \end{aligned}$$

where A is a nondecreasing, $(\mathcal{H}_t)_{t\geq\tau}$–predictable, $\mathbf{M}_F(C)$–valued processes, null at τ and with sample paths almost surely in $\Omega_H[\tau,\infty[$. Let β^2 be the $(\hat{\mathcal{H}}_t^C)_{t\geq\tau}$–predictable version of $d\ell(y,G^1)/d\ell(y,H^1)$ guaranteed by Lemma 3.15 and Theorem 3.7. Put

$$C^2(t,z,y,\omega) = \mathbf{1}\{\beta^2(t,y,\omega) \geq z\},$$

$$B^2(t,y,n,\omega) = \mathbf{1}\left\{\int_{]\tau,t[}\int_{[0,1]} C^2(s,y,z,\omega)\, n(ds,dz) = 0\right\},$$

and define a $\mathbf{M}_F(C)$*–valued process* $(G^2_t)_{t\geq\tau}$ *by*

$$G^2_t(\phi) = \int B^2(t,y,n)\phi(y)\, \tilde{H}^2_t(d(y,n)).$$

Then the FFCLT, $L(G^2,G^1)$*, exists and*

$$\begin{aligned}G^2_t(\phi_2(t)) &= \mu^2(\phi_2(\tau)) + \int_\tau^t G^2_s(\psi_2(s))\, ds - r_2\int_\tau^t\int_C \phi_2(s,y)\, L(G^2,G^1)(ds,dy) \\ &\quad + \int_\tau^t\int_{E\#} \phi_2(s,y)\, B^2(s,y,n)\, d\tilde{M}^2(s,y,n) \qquad \forall t\geq\tau \qquad a.s.\end{aligned}$$

Proof. Set

$$T(y,n,\omega) = \inf\left\{t\geq\tau : \int_{]\tau,t[}\int_{[0,1]} C^2(s,y,z,\omega)\, n(ds,dz) > 0\right\}.$$

We have

$$\begin{aligned}&\int_C \phi_2(t,y)\, G^2_t(dy) \\ &= \int_{E\#} \phi_2(t\wedge T(y,n),y)\, \tilde{H}^2_t(d(y,n)) \\ &\quad - \int_{E\#}\int_{]\tau,t]\times[0,1]} C^2(s,z,y)B^2(s,y,n)\phi_2(s,y)\, n(ds,dz)\, \tilde{H}^2_t(d(y,n)).\end{aligned} \tag{5.3}$$

First consider the first term on the right hand side of equation (5.3). Put

$$\Phi(t,y) = \phi_2(t,y) - \phi_2(\tau,y) - \int_\tau^t \psi_2(s,y)\, ds.$$

We have

$$
\begin{aligned}
&\int_{E^\#} \phi_2(t \wedge T(y,n), y)\, \tilde{H}^2_t(d(y,n)) \\
&\quad = \int_{E^\#} \Phi(t \wedge T(y,n), y)\, \tilde{H}^2_t(d(y,n)) \\
&\qquad + \int_{E^\#} \phi_2(\tau, y)\, \tilde{H}^2_t(d(y,n)) + \int_{E^\#} \int_\tau^{t\wedge T(y,n)} \psi_2(s,y)\, ds\, \tilde{H}^2_t(d(y,n)) \\
&\quad = \int_\tau^t \int_{E^\#} \Phi(s \wedge T(y,n), y)\, d\tilde{M}^2(s,y,n) \\
&\qquad + \int_{E^\#} \phi_2(\tau, y)\, \tilde{H}^2_\tau(d(y,n)) + \int_\tau^t \int_{E^\#} \phi_2(\tau, y)\, d\tilde{M}^2(s,y,n) \\
&\qquad + \int_\tau^t \int_{E^\#} (\int_\tau^s \psi_2(u,y) B^2(u,y,n)\, du)\, d\tilde{M}^2(s,y,n) \\
&\qquad + \int_\tau^t \int_{E^\#} \psi_2(s,y) B^2(s,y,n)\, \tilde{H}^2_s(d(y,n))\, ds \\
&\quad = \int_{E^\#} \phi_2(\tau, y)\, \tilde{H}^2_\tau(d(y,n)) + \int_\tau^t \int_{E^\#} \phi_2(s \wedge T(y,n), y)\, d\tilde{M}^2(s,y,n) \\
&\qquad + \int_\tau^t \int_{E^\#} \psi_2(s,y) B^2(s,y,n)\, \tilde{H}^2_s(d(y,n))\, ds, \\
&\quad = \int_C \phi_2(\tau, y)\, \mu^2(dy) + \int_\tau^t \int_{E^\#} \phi_2(s \wedge T(y,n), y)\, d\tilde{M}^2(s,y,n) + \int_\tau^t \int_C \psi_2(s,y)\, G^2_s(dy)\, ds.
\end{aligned}
\tag{5.4}
$$

where the second equality follows from Lemmas 4.16 and 4.11, and Remark 3.9.

Now consider the second term on the right hand side of equation (5.3). We have

$$
\begin{aligned}
&\int_{E^\#}\int_{]\tau,t]\times[0,1]} C^2(s,z,y)B^2(s,y,n)\phi_2(s,y)\,n(ds,dz)\,\tilde{H}^2_t(d(y,n))\\
&\quad=\int_{E^\#}\int_{]\tau,t]\times[0,1]} C^2(s,z,y)B^2(s,y,n)\phi_2(s,y)\,\check{n}^2(ds,dz)\,\tilde{H}^2_t(d(y,n))\\
&\qquad+\int_{E^\#}\int_{]\tau,t]\times[0,1]} C^2(s,z,y)B^2(s,y,n)\phi_2(s,y)\,r_2\ell(y,H^1)(ds)m(dz)\,\tilde{H}^2_t(d(y,n))\\
&\quad=\int_\tau^t\int_{E^\#}\int_{]\tau,s]\times[0,1]} C^2(u,z,y)B^2(u,y,n)\phi_2(u,y)\,\check{n}^2(du,dz)\,d\tilde{M}^2(s,y,n)\\
&\qquad+\int_{E^\#}\int_{]\tau,t]\times[0,1]} C^2(s,z,y)B^2(s,y,n)\phi_2(s,y)\,r_2\ell(y,H^1)(ds)m(dz)\,\tilde{H}^2_t(d(y,n))\\
&\qquad\qquad\text{(by Lemma 4.15(b))}\\
&\quad=\int_\tau^t\int_{E^\#}\int_{]\tau,s]\times[0,1]} C^2(u,z,y)B^2(u,y,n)\phi_2(u,y)\,n(du,dz)\,d\tilde{M}^2(s,y,n)\\
&\qquad-\int_\tau^t\int_{E^\#}\int_{]\tau,s]} \beta^2(u,y)B^2(u,y,n)\phi_2(u,y)\,r_2\ell(y,H^1)(du)\,d\tilde{M}^2(s,y,n)\\
&\qquad+\int_{E^\#}\int_\tau^t \beta^2(s,y)B^2(s,y,n)\phi_2(s,y,n)\,r_2\ell(y,H^1)(ds)\,\tilde{H}^2_t(d(y,n))\\
&\quad=\int_\tau^t\int_{E^\#}\int_{]\tau,s]\times[0,1]} C^2(u,z,y)B^2(u,y,n)\phi_2(u,y)\,n(du,dz)\,d\tilde{M}^2(s,y,n)\\
&\qquad-\int_\tau^t\int_{E^\#}\int_{]\tau,s]} B^2(u,y,n)\phi_2(u,y)\,r_2\ell(y,G^1)(du)\,d\tilde{M}^2(s,y,n)\\
&\qquad+\int_{E^\#}\int_\tau^t B^2(s,y,n)\phi_2(s,y)\,r_2\ell(y,G^1)(ds)\,\tilde{H}^2_t(d(y,n)).
\end{aligned}
$$

Now make minor changes in the proof of Theorem 3.10 to derive the existence of $L_t(\tilde{H}^2, G^1)$ and the analogue of (3.5). The latter shows the above expression equals

$$
\begin{aligned}
&\int_\tau^t\int_{E^\#}\int_{]\tau,s]\times[0,1]} C^2(u,z,y)B^2(u,y,n)\phi_2(u,y)\,n(du,dz)\,d\tilde{M}^2(s,y,n)\\
&\qquad+r_2\int_\tau^t\int_C \phi_2(s,y)\,B^2(s,y,n)\,L(\tilde{H}^2,G^1)(ds,dy,dn).
\end{aligned}
\tag{5.5}
$$

Substituting equations (5.4) and (5.5) into equation (5.3), we see that

$$
\begin{aligned}
&\int_C \phi_2(t,y)\, G_t^2(dy) \\
&= \int_C \phi_2(\tau,y)\,\mu^2(dy) + \int_\tau^t \int_{E\#} \phi_2(s\wedge T(y,n),y)\, d\tilde{M}^2(s,y,n) + \int_\tau^t \int_C \psi_2(s,y)\, G_s^2(dy)\, ds \\
&\quad - \int_\tau^t \int_{E\#} \int_{]\tau,s]\times[0,1]} C^2(u,z,y) B^2(u,y,n) \phi_2(u,y)\, n(du,dz)\, d\tilde{M}^2(s,y,n) \\
&\quad - r_2 \int_\tau^t \int_C \phi_2(s,y) B^2(s,y,n)\, L(\tilde{H}^2, G^1)(ds,dy,dn) \\
&= \int_C \phi_2(\tau,y)\,\mu^2(dy) + \int_\tau^t \int_{E\#} \phi_2(s,y) B^2(s,y,n)\, d\tilde{M}^2(s,y,n) \\
&\quad + \int_\tau^t \int_C \psi_2(s,y)\, G_s^2(dy)\, ds - r_2 \int_\tau^t \int_C \phi_2(s,y) B^2(s,y,n)\, L(\tilde{H}^2, G^1)(ds,dy,dn).
\end{aligned}
\tag{5.6}
$$

It remains to express the last expression in terms of $L(G^1, G^2)$. Let $\gamma(s,y,\omega) = \mathbf{1}_{]T_1,T_2]}(s)\,\gamma_1(y,\omega)$, where $\tau \le T_1 \le T_2$ are $(\hat{\mathcal{H}}_t^C)$–stopping times and $\gamma_1 \in b\hat{\mathcal{H}}_{T_1}^C$. Then

$$
\begin{aligned}
\int_\tau^t \int \gamma(s,y) L^\epsilon(G^2,G^1)(ds,dy) &= \int_\tau^t \int \gamma(s,y) \mathbf{1}(s < T(y,n,\omega))\, L^\epsilon(\tilde{H}^2, G^1)(ds,dy,dn) \\
&= \int \gamma_1(y) \left[\ell^\epsilon_{T\wedge T_2\wedge t}(y,G^1) - \ell^\epsilon_{T\wedge T_1\wedge t}(y,G^1)\right] \tilde{H}_t^2(dy,dn) \\
&\quad - \int_\tau^t \int \gamma_1(y) \left[\ell^\epsilon_{T\wedge T_2\wedge s}(y,G^1) - \ell^\epsilon_{T\wedge T_1\wedge s}(y,G^1)\right] \tilde{M}^2(ds,dy,dn)
\end{aligned}
$$

(see Remark 3.9 for the last equality). Now argue as in the proof of Theorem 3.10 to see that as $\epsilon \downarrow 0$ the right side converges uniformly in t in bounded sets in L^2 to

$$
\begin{aligned}
&\int \left[\int_\tau^t \gamma(s,y)\, B^2(s,y,n)\, \ell(y,G^1)(ds)\right] \tilde{H}_t^2(dy,dn) \\
&\quad - \int_\tau^t \int \left[\int_\tau^s \gamma(u,y)\, B^2(u,y,n)\, \ell(y,G^1)(du)\right] \tilde{M}^2(ds,dy,dn) \\
&= \int_\tau^t \int \gamma(s,y) B^2(s,y,n)\, L(\tilde{H}^2, G^2)(ds,dy,dn),
\end{aligned}
$$

the last by the $\tilde{H}^2$-analogue of (3.5) already used above. This together with the approximation argument used in the proof of Theorem 3.10 show that if γ is a bounded $(\hat{\mathcal{H}}_t^C)_{t\ge\tau}$–optional process with left limits, then

$$
\begin{aligned}
&\sup_{\tau\le t\le N} \left| \int_\tau^t \int \gamma(s,y)\, L^\epsilon(G^2,G^1)(ds,dy) - \int_\tau^t \int \gamma(s,y) B^2(s,y,n)\, L(\tilde{H}^2,G^1)(ds,dy,dn) \right| \\
&\quad \to 0 \text{ in } L^2 \text{ as } \epsilon \downarrow 0,\ \forall N > \tau.
\end{aligned}
$$

If $\phi : C \to \mathbb{R}$ is bounded and continuous, and $\gamma(s,y) = \phi(y^s)$ we see from the above that $L_t(G^1, G^2)(\phi)$ exists and equals $\int_\tau^t \int \phi(y^s) B^2(s,y,n)\, L(\tilde{H}^2, G^1)(ds, dy, dn)$. This and (5.6) give the required result.

□

Proof of Theorem 5.2 (continued) By construction, $\hat{H}_t^{1,0}(A) = H_t^1(A) \geq \hat{H}_t^{1,1}(A)$ for all $A \in \mathcal{C}$ and all $t \geq \tau$. Therefore, by definition, for any bounded $(\mathcal{F}_t)_{t \geq \tau}$–stopping time, T,

$$\ell_t(y, \hat{H}^{1,0}) - \ell_s(y, \hat{H}^{1,0}) \geq \ell_t(y, \hat{H}^{1,1}) - \ell_s(y, \hat{H}^{1,1}), \quad \forall \tau \leq s \leq t \leq T,\ H_T^2\text{-a.e. } y, \quad \text{a.s.}$$

and so, by definition, $\lambda^{2,1}(u,y) \geq \lambda^{2,2}(u,y)$, $\forall u \leq T$, H_T^2-a.a. y, a.s., whence $I^{2,1}(T,y,n) \leq I^{2,2}(T,y,n)$ for $\tilde{H}_T^2$-a.e. y a.s. Consequently, by the section theorem, $\mathbb{P}$-a.s., for all $t \geq \tau$, for $\tilde{H}_t^2$-a.e. (y,n), $I^{2,1}(t,y,n) \leq I^{2,2}(t,y,n)$, and so $\mathbb{P}$-a.s. for all $t \geq \tau$, $\hat{H}_t^{2,1}(A) \leq \hat{H}_t^{2,2}(A)$ for all $A \in \mathcal{C}$.

Continuing in this way, we get that $\mathbb{P}$-a.s. for all $t > \tau$,

$$1 \geq I^{1,1}(t,y,n) \geq I^{1,2}(t,y,n) \geq \dots$$

for $\tilde{H}_t^1$-a.e. (y,n), so that $\mathbb{P}$-a.s. for all $t \geq \tau$ and all $A \in \mathcal{C}$,

$$H_t^1(A) = \hat{H}_t^{1,0}(A) \geq \hat{H}_t^{1,1}(A) \geq \hat{H}_t^{1,2}(A) \geq \dots ;$$

and, $\mathbb{P}$-a.s. for all $t > \tau$,

$$I^{2,1}(t,y,n) \leq I^{2,2}(t,y,n) \leq \dots \leq 1$$

for $\tilde{H}_t^2$-a.e. (y,n), so that $\mathbb{P}$-a.s. for all $t \geq \tau$ and all $A \in \mathcal{C}$,

$$\hat{H}_t^{2,1}(A) \leq \hat{H}_t^{2,2}(A) \leq \dots \leq H_t^2(A).$$

Thus there exist $(\hat{\mathcal{H}}_t)_{t \geq \tau}$–predictable functions

$$I^{1,\infty} :]\tau, \infty[\times C \times \mathbf{M}_\# \times \Omega \to \{0,1\}$$

and

$$I^{2,\infty} :]\tau, \infty[\times C \times \mathbf{M}_\# \times \Omega \to \{0,1\}$$

such that $\mathbb{P}$-a.s. for all $t > \tau$, $I^{1,k}(t,y,n) \downarrow I^{1,\infty}(t,y,n)$ for $\tilde{H}_t^1$-a.e. (y,n) and $I^{2,k}(t,y,n) \uparrow I^{2,\infty}(t,y,n)$ for $\tilde{H}_t^2$-a.e. (y,n). In particular, $\mathbb{P}$-a.s. for all $t \geq \tau$ and all $A \in C$,

$$H_t^1(A) \geq \hat{H}_t^{1,k}(A) \downarrow \hat{H}_t^{1,\infty}(A) = \int \mathbf{1}_A(y) I^{1,\infty}(t,y,n)\, \tilde{H}_t^1(d(y,n))$$

and

$$\hat{H}_t^{2,k}(A) \uparrow \hat{H}_t^{2,\infty}(A) = \int \mathbf{1}_A(y) I^{2,\infty}(t,y,n)\, \tilde{H}_t^2(d(y,n)) \le H_t^2(A).$$

Set $\hat{H}_t^{1,\infty} = \Delta$ for $0 \le t < \tau$.

Thus $\mathbb{P}$-a.s. for all $u \ge \tau$,

$$\lim_k \sup_{\tau \le t \le u} \left| \int_\tau^t \hat{H}_s^{i,k}(\psi_i(s))\, ds - \int_\tau^t \hat{H}_s^{i,\infty}(\psi_i(s))\, ds \right| = 0$$

for $i = 1, 2$. Also, by dominated convergence, for each $t \ge \tau$,

$$\begin{aligned}
&\mathbb{P}\left[\left\{ \int_\tau^t \int_{E\#} \phi_i(s,y)\, I^{i,k}(s,y,n)\, d\tilde{M}^i(s,y,n) - \int_\tau^t \int_{E\#} \phi_i(s,y)\, I^{i,\infty}(t,y,n)\, d\tilde{M}^i(s,y,n) \right\}^2 \right] \\
&\quad = \mathbb{P}\left[\int_\tau^t \int_{E\#} \phi_i(s,y)^2 \left\{ I^{i,k}(s,y,n) - I^{i,\infty}(s,y,n) \right\}^2 \tilde{H}_s^i(d(y,n))\, ds \right] \to 0
\end{aligned}$$

for $i = 1, 2$, and so there exists a subsequence $\{k_h\}_{h=1}^\infty$ such that $\mathbb{P}$-a.s. for all $u \ge \tau$,

$$\lim_h \sup_{\tau \le t \le u} \left| \int_\tau^t \int_{E\#} \phi_i(s,y)\, I^{i,k_h}(s,y,n)\, d\tilde{M}^i(s,y,n) - \int_\tau^t \int_{E\#} \phi_i(s,y)\, I^{i,\infty}(s,y,n)\, d\tilde{M}^i(s,y,n) \right| = 0.$$

Choose countable sets $\{\phi_i^\ell\}_{\ell=1}^\infty \subset D_{ST}$ that are measure–determining in $C(\mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^d), \mathbb{R})$. From (5.1) and (5.2) we have that $\mathbb{P}$-a.s. for all $t > \tau$,

$$\lim_h \int_\tau^t \int_C \phi_1^\ell(s,y)\, L(\hat{H}^{1,k_h}, \hat{H}^{2,k_h})(ds, dy)$$

and

$$\lim_h \int_\tau^t \int_C \phi_2^\ell(s,y)\, L(\hat{H}^{2,k_h}, \hat{H}^{1,k_h-1})(ds, dy)$$

exist for all ℓ. Also, as

$$L(\hat{H}^{1,k_h}, \hat{H}^{2,k_h})(ds, dy) \le L(H^1, H^2)(ds, dy)$$

and

$$L(\hat{H}^{2,k_h}, \hat{H}^{1,k_h-1})(ds, dy) \le L(H^2, H^1)(ds, dy),$$

the two sequences of random measures $\{L(\hat{H}^{1,k_h}, \hat{H}^{2,k_h})\}_{h=1}^\infty$ and $\{L(\hat{H}^{2,k_h}, \hat{H}^{1,k_h-1})\}_{h=1}^\infty$ are $\mathbb{P}$-a.s. tight. Combining these two pairs of observations, we conclude that $\mathbb{P}$-a.s. both of these sequences converge weakly to random measures that we will denote as Λ^1 and Λ^2. Moreover, $\mathbb{P}$-a.s. the process

$t \mapsto \Lambda^i(]\tau, t] \times C)$ is continuous for $i = 1, 2$ and $t \mapsto \int_\tau^t \int_C \phi_i(s, y)\Lambda^i(ds, dy)$ may be taken to be $(\mathcal{H}_t)_{t \geq \tau}$–predictable.

Putting all of these observations together, we find that

$$\hat{H}_t^{i,\infty}(\phi_i(t)) = \mu^i(\phi_i(\tau)) + \int_\tau^t \hat{H}_s^{i,\infty}(\psi_i(s))\, ds - r_i \int_\tau^t \int_C \phi_i(s, y)\, \Lambda^i(ds, dy)$$
$$+ \int_\tau^t \int_{E^\#} \phi_i(s, y)\, I^{i,\infty}(t, y, n)\, d\tilde{M}^i(s, y, n), \ \forall t \geq \tau \text{ a.s.}$$

for $i = 1, 2$. This shows that $(\hat{H}^{1,\infty}, \hat{H}^{2,\infty}) \in \mathcal{M}(H^1, H^2)$ and so from Theorem 3.13(b) we may conclude that $\Lambda^1 = L(\hat{H}^{1,\infty}, \hat{H}^{2,\infty})$ and $\Lambda^2 = L(\hat{H}^{2,\infty}, \hat{H}^{1,\infty})$. Note we have shown that $(\hat{H}^{1,\infty}, \hat{H}^{2,\infty})$ satisfies $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ and is a.s. continuous on $[\tau, \infty[$.

We now check that the pair $(\hat{H}^{1,\infty}, \hat{H}^{2,\infty})$ solves (SE). Let $\lambda^{i,\infty}$ be the $(\hat{\mathcal{H}}_t^C)_{t \geq \tau}$–predictable version of $d\ell(y, \hat{H}^{j,\infty})/d\ell(y, H^j)$, $i \neq j$, guaranteed by Lemma 3.15. It suffices to show that $\mathbb{P}$-a.s. for all $t > \tau$, $\mathbf{1}\{\lambda^{i,\infty}(s, y) \geq z\} = \lim_k \mathbf{1}\{\lambda^{i,k}(s, y) \geq z\}$ for n-a.e. (s, z), for $\tilde{H}_t^i$-a.e. (y, n); or, equivalently, that for any bounded $(\mathcal{H}_t)_{t \geq \tau}$–stopping time T, $\mathbf{1}\{\lambda^{i,\infty}(s, y, \omega) \geq z\} = \lim_k \mathbf{1}\{\lambda^{i,k}(s, y, \omega) \geq z\}$ for n-a.e. (s, z) for $\bar{\mathbb{P}}_T^{\tilde{H}^i}$-a.e. (y, n, ω). We have

$$\left\{(s, z, y, n, \omega) : \mathbf{1}\{\lambda^{i,\infty}(s, y, \omega) \geq z\} \neq \lim_k \mathbf{1}\{\lambda^{i,k}(s, y, \omega) \geq z\}\right\}$$
$$\subseteq \left\{(s, z, y, n, \omega) : \lambda^{i,\infty}(s, y, \omega) \neq \lim_k \lambda^{i,k}(s, y, \omega)\right\} \cup \left\{(s, z, y, n, \omega) : \lambda^{i,\infty}(s, y, \omega) = z\right\}.$$

From Lemma 4.15(c), it thus suffices to show that

$$\int_{[\tau, T(\omega)] \times [0,1]} \left|\lim_k \lambda^{i,k}(s, y, \omega) - \lambda^{i,\infty}(s, y, \omega)\right| \ell(y, H^j)(ds)(\omega)\, m(dz) = 0$$

and

$$\int_{[\tau, T(\omega)] \times [0,1]} \mathbf{1}\{\lambda^{i,\infty}(s, y, \omega) = z\}\, \ell(y, H^j)(ds)(\omega)\, m(dz) = 0$$

for $\bar{\mathbb{P}}_T^{H^i}$-a.e. (y, ω). By bounded convergence (and the fact that $\lambda^{1,k} \leq \lambda^{1,\infty}$ and $\lambda^{2,k} \geq \lambda^{2,\infty}$), the first integral is

$$\lim_k \int_{[\tau, T(\omega)]} \left|\lambda^{i,k}(s, y, \omega) - \lambda^{i,\infty}(s, y, \omega)\right| d\ell(y, H^j)(ds)(\omega)$$
$$= \lim_k \left| \int_{[\tau, T(\omega)]} \{\lambda^{i,k}(s, y, \omega) - \lambda^{i,\infty}(s, y, \omega)\}\, d\ell(y, H^j)(ds)(\omega) \right|$$
$$= \lim_k \left|\ell(y, H^{j,k})([\tau, T]) - \ell(y, H^{j,\infty})([\tau, T])\right|.$$

Theorem 3.14 shows that the right-hand side is 0 for $\bar{\mathbb{P}}_T^{H^i}$-a.e. (y,ω). The claim regarding the second integral is immediate from Fubini's theorem and the existence proof is complete.

We can now turn to the proof of the uniqueness claim in (a). Let $(\hat{H}^1, \hat{H}^2)$ be any solution. We will show that $\hat{H}^1 = \hat{H}^{1,\infty}$ and $\hat{H}^2 = \hat{H}^{2,\infty}$. This will also prove (b).

We certainly have that $\mathbb{P}$-a.s. for all $t \geq \tau$, $H_t^1(A) = \hat{H}_t^{1,0}(A) \geq \hat{H}_t^1(A)$ for all $A \in \mathcal{C}$. Arguing as for $\hat{H}^{1,k}$, we then have that $\mathbb{P}$-a.s. for all $t \geq \tau$, $\hat{H}_t^{2,1}(A) \leq \hat{H}_t^2(A)$ for all $A \in \mathcal{C}$. Continuing in this manner, we find that $\mathbb{P}$-a.s. for all $t \geq \tau$, $\hat{H}_t^{1,0}(A) \geq \hat{H}_t^{1,1}(A) \geq \ldots \geq \hat{H}_t^1(A)$ and $\hat{H}_t^{2,1}(A) \leq \hat{H}_t^{2,2}(A) \leq \ldots \leq \hat{H}_t^2(A)$ for all $A \in \mathcal{C}$. Consequently,

$$\mathbb{P}\text{- a.s. for all } t \geq \tau,\ \hat{H}_t^{1,\infty}(A) \geq \hat{H}_t^1(A) \text{ and } \hat{H}_t^{2,\infty}(A) \leq \hat{H}_t^2(A) \text{ for all } A \in \mathcal{C}. \tag{5.7}$$

Repeat the above construction of $(\hat{H}^{1,\infty}, \hat{H}^{2,\infty})$ but now with the role of the indices $1,2$ reversed to construct $\vec{H}^{1,k} \uparrow \vec{H}^{1,\infty}$, $\vec{H}^{2,k} \downarrow \vec{H}^{2,\infty}$ so that $(\vec{H}^{1,\infty}, \vec{H}^{2,\infty})$ also solves (SE) and satisfies $\widehat{MP}_H(\tau, \mu^1, \mu^2)$. Arguing as in the derivation of (5.6), we see that

$$\vec{H}_t^{1,\infty}(A) \leq \hat{H}_t^1(A) \text{ and } \vec{H}_t^{2,\infty}(A) \geq \hat{H}_t^2(A), \quad \forall A \in \mathcal{C}, \quad \forall t \geq \tau,\ \mathbb{P}\text{-a.s.} \tag{5.8}$$

From (5.7) and (5.8) we see that to prove $\hat{H}_t^i = \hat{H}_t^{i,\infty}$, $\forall t \geq \tau$, a.s., it suffices to show that $\hat{H}_t^{i,\infty}(1) = \vec{H}_t^{i,\infty}(1)$, $\forall t \geq \tau$, $i = 1,2$, $\mathbb{P}$-a.s. Use the fact that $(\hat{H}^{1,\infty}, \hat{H}^{2,\infty})$ and $(\vec{H}^{1,\infty}, \vec{H}^{2,\infty})$ both satisfy $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ (with $\phi = 1$) to see that

$$r_2(\hat{H}_t^{1,\infty}(1) - \vec{H}_t^{1,\infty}(1)) + r_1(\vec{H}_t^{2,\infty}(1) - \hat{H}_t^{2,\infty}(1)), \quad t \geq \tau,$$

is a non-negative, continuous martingale starting at 0 at time τ and hence must be identically 0 a.s., as required. (We have used the fact that $L_t(\hat{H}^{1,\infty}, \hat{H}^{2,\infty})(1) = L_t(\hat{H}^{2,\infty}, \hat{H}^{1,\infty})(1)$ and similarly with $\vec{H}^{i,\infty}$ in place of $\hat{H}^{i,\infty}$.)

The explicit construction of $(\hat{H}^{1,\infty}, \hat{H}^{2,\infty}) \equiv (\hat{H}^1, \hat{H}^2)$ given above and a tedious bit of checking shows there is a Borel map $\Psi : \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbf{M}_F(E^{\#})^2) \to \mathbf{M}_F(C) \times \mathbf{M}_F(C)$, independent of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and (τ, μ^1, μ^2), such that $(\hat{H}^1_{\tau+t}, \hat{H}^2_{\tau+t}) = \Psi(t, \tilde{H}^1_{\tau+\cdot}, \tilde{H}^2_{\tau+\cdot}) \equiv \bar{\Psi}(\tau, t, \tilde{H}^1, \tilde{H}^2)$. Clearly $\bar{\Psi} : \mathbb{R}_+^2 \times \Omega' \times \Omega' \to \mathbf{M}_F(C) \times \mathbf{M}_F(C)$ is also Borel. Therefore for fixed $\tau \leq t_1 \leq t_2 \leq \cdots \leq t_n$,

$$\mathbb{P}\Big((\hat{H}^1_{\tau+t_1}, \hat{H}^2_{\tau+t_2}, \ldots, \hat{H}^2_{\tau+t_n}) \in \cdot\Big) = \breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}\Big((\bar{\Psi}(\tau, t_i, \breve{H}^1, \breve{H}^2))_{i \leq n} \in \cdot\Big) \tag{5.9}$$

and hence the law, $\hat{\mathbb{P}}^{\tau,\mu^1,\mu^2}$, of $(\hat{H}^1, \hat{H}^2)$ on $\Omega'_C \times \Omega'_C$ depends only on (τ, μ^1, μ^2). That is, (c) holds. The Borel measurability of $(\tau, \breve{\mu}^1, \breve{\mu}^2) \mapsto \breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$ (which is clear from its definition) and (5.9) show that,

$$(\tau, \mu^1, \mu^2) \mapsto \mathbb{P}^{\tau,\mu^1,\mu^2}(\phi(\hat{H}^1_{v_1}, \ldots, \hat{H}^1_{v_n}, \hat{H}^2_{v_1}, \ldots, \hat{H}^2_{v_n}))$$

is Borel measurable on $\hat{S}$ for bounded, measurable $\phi : \mathbf{M}_F^{\Delta}(C)^{2n} \to \mathbb{R}$ and for all $v_i \geq 0$, $i = 1, \ldots, n$. Claim (d) follows easily.

$\square$

6 Afterlife processes

6.1 Construction of the afterlife process

We now turn to the question of uniqueness in the martingale problem $\widehat{MP}_H(\tau,\mu^1,\mu^2)$. We will do this by showing that any solution of $\widehat{MP}_H(\tau,\mu^1,\mu^2)$ arises as a solution of the strong equation (SE), hence proving the converse of part of Theorem 5.2 and establishing Theorem 1.4(b). This requires reversing the procedure that produced $(\hat{H}^1,\hat{H}^2)$ from $(\tilde{H}^1,\tilde{H}^2)$. In essence, we have to accomplish two things. Firstly, we need to take $(\hat{H}^1,\hat{H}^2)$ and put back in the particles removed by the competitive killing (plus all the descendents they would have had if they had not been killed) to produce a pair of historical Brownian motions. Secondly, we need to mark the paths of this latter pair in such a way that we produce a pair $(\breve{H}^1,\breve{H}^2)$ with the law $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$ of Section 4 and with the property that if we use this pair to drive the strong equation (SE), then we produce $(\hat{H}^1,\hat{H}^2)$. We carry out the first step in this section and the second step in the next section.

Fix $\tau\geq 0,\ r_i>0$ and $\mu^i\in\mathbf{M}_{FS}(C)^\tau$. Dependence on (r_1,r_2) is suppressed as usual.

Recall the definition of S^o, Ω^o, $\mathcal{F}^o$, E and E_t from Section 3. Put $\tilde{E}=C\times\mathbb{R}_+$, $\tilde{S}^o=\{(t,y,\xi):y=y^t,\xi\leq t\}$, and let $\tilde{E}^\partial$, $\tilde{E}_t$, $\tilde{\Omega}^o$, $\tilde{\mathcal{F}}^o$, denote the obvious analogues of the corresponding objects for E.

If $(s,y,\xi)\in\tilde{S}^o$ and $\omega\in C$ satisfies $\omega(0)=y(s)$ define $\bar{z}(s,y,\xi,\omega)\in\tilde{\Omega}^o$ by

$$\bar{z}(s,y,\xi,\omega)(t)=\begin{cases}\partial, & \text{if } t<s,\\ (\bar{w}(s,y,\omega)(t),\xi), & \text{if } t\geq s,\end{cases}$$

where $\bar{w}$ is defined in Definition 3.11.

Recall that P^x is Wiener measure on C starting at $x\in\mathbb{R}^d$ and recall the definition of the Brownian path–process laws $P^{s,y}$ from Section 3. Define the law $P^{s,y,\xi}$ on $(\tilde{\Omega}^o,\tilde{\mathcal{F}}^o)$ for $(s,y,\xi)\in\tilde{S}^o$ by

$$P^{s,y,\xi}(B)=P^{y(s)}\{\omega:\bar{z}(s,y,\xi,\omega)(\cdot)\in B\}.$$

Thus $P^{s,y,\xi}$ is the law of the Brownian path–process labelled with a constant $\mathbb{R}_+$–valued tag. If W_t and Z_t denote the coordinate variables on Ω^o and $\tilde{\Omega}^o$, respectively, then it is easy to see that $W=(\Omega^o,\mathcal{F}^o,\mathcal{F}^o_{[s,t+]},W_t,P^{s,y})$ and $Z=(\tilde{\Omega}^o,\tilde{\mathcal{F}}^o,\tilde{\mathcal{F}}^o_{[s,t+]},Z_t,P^{s,y,\xi})$ are the canonical realisations of time-inhomogeneous Borel strong Markov processes (IBSMP's) with continuous paths in $E_t\subset E$ and $\tilde{E}_t\subset\tilde{E}$, respectively. (See Appendix A for the definitions of an IBSMP and $\mathcal{F}^o_{[s,t+]}$. The definition of $\tilde{\mathcal{F}}^o_{[s,t+]}$ is the obviously analogous one.) That this is so for W is essentially a special case of Theorem 2.2.1 of [9], and the claim for Z is an easy consequence of that result (note that (Hyp1) and (Hyp2) in Appendix A are trivial).

In the notation of Appendix A, set

$$\Omega_1'=\{\omega\in C(\mathbb{R}_+,\mathbf{M}_F^\Delta(\tilde{E})):\alpha'(\omega)<\infty,\ \beta'(\omega)=\infty\},$$

where

$$\alpha'_1(\omega) = \inf\{t : \omega(t) \neq \Delta\}$$

and

$$\beta'_1(\omega) = \inf\{t \geq \alpha'_1(\omega) : \omega(t) \notin \mathbf{M}_F(\tilde{E}_t)\}.$$

Let $\mathcal{F}'_1$ and ${\mathcal{F}^1}'_{[s,t+]}$ be σ–fields defined on Ω'_1 as in Appendix A. Following Theorem A.1, given $\mu \in \mathbf{M}_F(\tilde{E}_\tau)$ and $L \in \mathbf{M}_{LF}(\tilde{S}^o)$, let $\mathbb{Q}^{\tau,\mu;L}$ denote the probability measure on $(\Omega'_1, \mathcal{F}'_1)$ that is the law of the Z superprocess with immigration L and starting at (τ, μ).

Suppose that $(\hat{H}^1, \hat{H}^2)$ defined on $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \geq \tau}, \hat{\mathbb{P}})$ solves the martingale problem $\widehat{MP}_H(\tau, \mu^1, \mu^2)$. Set $\Omega = \hat{\Omega} \times \Omega'_1 \times \Omega'_1$, $\mathcal{F} = \hat{\mathcal{F}} \times \mathcal{F}'_1 \times \mathcal{F}'_1$, $\bar{\mathcal{F}}_t = \hat{\mathcal{F}}_t \times {\mathcal{F}^1}'_{[\tau,t+]} \times {\mathcal{F}^1}'_{[\tau,t+]}$ and $\mathcal{F}_t = \bar{\mathcal{F}}_{t+}$, and let $(\hat{\omega}, \check{H}^1, \check{H}^2)$ denote the coordinates on Ω. We can regard $L(\hat{H}^i, \hat{H}^j)$ as a random element of $\mathbf{M}_{LF}(S^o)$ by extending it to assign zero mass to subsets of $[0, \tau[\times C$. Define $\kappa : S^o \to \tilde{S}^o$ by $\kappa(s, y) = (s, y, s)$ and identify $L(\hat{H}^i, \hat{H}^j)(\hat{\omega}) \in \mathbf{M}_{LF}(S^o)$ with $L(\hat{H}^i, \hat{H}^j)(\hat{\omega}) \circ \kappa^{-1} \in \mathbf{M}_{LF}(\tilde{S}^o)$. Define $\mathbb{P}$ on $(\Omega, \mathcal{F})$ by

$$\mathbb{P}(B \times C^1 \times C^2) = \int \mathbf{1}_B(\hat{\omega}) \mathbb{Q}^{\tau,0;r_1 L(\hat{H}^1,\hat{H}^2)(\hat{\omega})}(C^1) \mathbb{Q}^{\tau,0;r_2 L(\hat{H}^2,\hat{H}^1)(\hat{\omega})}(C^2)\, d\hat{\mathbb{P}}(\hat{\omega}). \tag{6.1}$$

Theorem A.1(c) ensures that this definition makes sense. The law of the first "marginal" of $\mathbb{P}$ is just that of the pair $(\hat{H}^1, \hat{H}^2)$. Conditional on this first marginal, the second and third marginals are essentially a pair of independent historical Brownian motions with respective immigrations $r_1 L(\hat{H}^1, \hat{H}^2)$ and $r_2 L(\hat{H}^2, \hat{H}^1)$, except that each particle is equipped with a mark that records the time when its progenitor first immigrated into the population.

Write $\pi_1 : \Omega \to \hat{\Omega}$ for the projection map. As in Theorem 5.1(a) of [2] we have the following (note that Theorem A.1(c) is also used here). The extension from $\bar{\mathcal{F}}_t$ (the setting of [2]) to $\mathcal{F}_t = \bar{\mathcal{F}}_{t+}$ is trivial.

Proposition 6.1 *Let $Y \in b\hat{\mathcal{F}}$ and $t \geq \tau$, then*

$$\mathbb{P}\left[Y \circ \pi_1 \mid \mathcal{F}_t\right] = \hat{\mathbb{P}}\left[Y \mid \hat{\mathcal{F}}_t\right] \circ \pi_1, \ \mathbb{P}- \ a.s.$$

Define $\delta : \mathbb{R}_+ \to \mathbf{M}_F(\mathbb{R}_+)$ by $\delta(s) = \delta_s$, and let $\mathbf{M}_* = \delta(\mathbb{R}_+) \cup \{0\} \subset \mathbf{M}_F(\mathbb{R}_+)$ with the subspace topology. For $e \in \mathbf{M}_*$, let $e^s = e(\cdot \cap [0, s])$. Let $\bar{S}^o = \{(s, y, e) \in S^o \times \mathbf{M}_* : e = e^s\}$, $\bar{E}_t = \{(y, e) \in C \times \mathbf{M}_* : y^t = y,\ e = e^t\}$, and define $\bar{\delta} : \tilde{S}^o \to \bar{S}^o$ by $\bar{\delta}(s, y, \xi) = (s, y, \delta_\xi)$. As usual we identify $\mathbf{M}_F(\bar{E}_t)$ with the measures in $\mathbf{M}_F(C \times \mathbf{M}_*)$ that are supported by $\bar{E}_t$. Define processes $(\bar{H}^i_t)_{t \geq \tau}$, $i = 1, 2$, on $(\Omega, \mathcal{F})$ with continuous, $\mathbf{M}^\Delta_F(C \times \mathbf{M}_*)$–valued paths satisfying $\bar{H}^i_t \in \mathbf{M}_F(\bar{E}_t)$ for $t \geq \tau$ by

$$\int \phi(y, e)\, \bar{H}^i_t(dy, de) = \int \phi(y, \delta_\xi)\, \check{H}^i_t(dy, d\xi) + \int \phi(y, 0) \hat{H}^i_t \circ \pi_1(dy),\ t \geq \tau$$

and set $\bar{H}_t^i = \Delta$ for $t < \tau$. Thus the the mass of $\bar{H}_t^i$ is of two types. Points of the form $(y, 0)$ represent particles with progenitors present in the original population at time τ. Points of the form (y, e) with $e = \delta_s$ represent particles descended from progenitors who immigrated into the population at time s. These latter particles are the ghostly descendents of individuals killed by inter–species competition that we need to reintroduce in order to produce two independent historical Brownian motions.

Define the semigroups for the space–time processes associated with $P^{s,y}$ and $P^{s,y,\xi}$ by

$$P_t^W \phi(s, y) = P^{s,y}\left[\phi(s+t, W_{s+t})\right], \ (s, y) \in S^o, \ \phi \in b\mathcal{B}(S^o), \ t \geq 0,$$

$$P_t^Z \phi(s, y, \xi) = P^{s,y,\xi}\left[\phi(s+t, Z_{s+t})\right] = P_t^W(\phi^\xi)(s, y), \ (s, y, \xi) \in \tilde{S}^o, \ \phi \in b\mathcal{B}(\tilde{S}^o), \ t \geq 0, \tag{6.2}$$

where $\phi^\xi(s, y) = \phi(s, y, \xi \wedge s)$ for $(s, y) \in S^o$, $\xi \geq 0$. Recall from Appendix A that the weak generator, $\tilde{A}$, associated with Z is the set of pairs $(\phi, \psi) \in b\mathcal{B}(\tilde{S}^o) \times b\mathcal{B}(\tilde{S}^o)$ such that $t \mapsto P_t^Z \psi(s, y, \xi)$ is right–continuous on $\mathbb{R}_+$ and $P_t^Z \phi = \phi + \int_0^t P_u^Z \psi \, du$, $\forall t \geq 0$.

If $\phi \in b\mathcal{B}(\bar{S}^o)$, define $\phi^0 \in b\mathcal{B}(S^o)$ by $\phi^0(s, y) = \phi(s, y, 0)$. Let $\bar{A}$ denote the set of pairs $(\phi, \psi) \in b\mathcal{B}(\bar{S}^o) \times b\mathcal{B}(\bar{S}^o)$ such that $\phi^0 \in D_{ST}$, $(\phi \circ \bar{\delta}, \tilde{\psi}) \in \tilde{A}$ for some $\tilde{\psi}$ and

$$\psi(s, y, e) = \begin{cases} A(\phi^0)(s, y), & \text{if } e = 0, \\ \tilde{\psi}(s, y, \xi), & \text{if } e = \delta_\xi, \end{cases}$$

where A is defined in (1.1). If we set $\phi_\xi(s, y) = \phi(s, y, \delta_\xi)$ for $\phi \in b\mathcal{B}(\bar{S}^o)$, then (6.2) implies that $(\phi, \psi) \in \bar{A}$ if and only if $\phi^0 \in D_{ST}$ and for each $(s, y, \delta_\xi) \in \bar{S}^o$ the map $t \mapsto P_t^W \psi_\xi(s, y)$ is right–continuous on $\mathbb{R}_+$ and

$$P_t^W \phi_\xi(s, y) = \phi_\xi(s, y) + \int_0^t P_u^W \psi_\xi(s, u) \, du. \tag{6.3}$$

Theorem 6.2 *(a) Under $\mathbb{P}$, $\bar{H}^i$ almost surely has sample paths in*

$$\left\{\omega \in C(\mathbb{R}_+, \mathbf{M}_F^\Delta(C \times \mathbf{M}_*)) : \omega_t = \Delta \ \text{if}\ t < \tau, \ \omega_\tau = \mu^i \otimes \delta_0, \ \omega_t \in \mathbf{M}_F(\bar{E}_t) \ \text{if}\ t \geq \tau\right\}.$$

(b) Under $\mathbb{P}$, if $(\phi^i, \psi^i) \in \bar{A}$, then

$$\bar{M}_t^i(\phi^i) = \bar{H}_t^i(\phi_t^i) - \mu^i \otimes \delta_0(\phi_\tau^i) - \int_\tau^t \bar{H}_s^i(\psi_s^i) \, ds$$
$$- r_i \int_\tau^t \int \left[\phi^i(s, y, \delta_s) - \phi^i(s, y, 0)\right] \, L(\hat{H}^i, \hat{H}^j)(ds, dy) \circ \pi_1, \ t \geq \tau,$$

is a continuous $(\mathcal{F}_t)_{t \geq \tau}$–martingale such that $\bar{M}_\tau^i(\phi^i) = 0$, $\langle \bar{M}^i(\phi_i) \rangle_t = \int_\tau^t \bar{H}_s^i((\phi_s^i)^2) \, ds$, and $\langle \bar{M}^1(\phi^1), \bar{M}^2(\phi^2) \rangle_t = 0$ almost surely.

(c) Under $\mathbb{P}$,

$$(\hat{H}_t^i \circ \pi_1)(\phi) = \int \phi(y) \mathbf{1}\{e = 0\} \, \bar{H}_t^i(dy, de),$$

$\forall t \geq \tau$ *a.s. for each* $\phi \in b\mathcal{B}(C)$, *and*

$$\int \phi(y, \delta_\xi) \check{H}_t^i(dy, d\xi) = \int \phi(y, e) \mathbf{1}\{e \neq 0\} \, \bar{H}_t^i(dy, de)$$

$\forall t \geq \tau$, *a.s. for each* $\phi \in b\mathcal{B}(C \times \mathbf{M}_*)$.

Proof. Parts (a) and (c) are immediate from the definitions of $\bar{H}^i$ and $\mathbb{P}$. Consider part (b). By definition, there exists $\tilde{\psi}^i$ such that $(\phi^i \circ \bar{\delta}, \tilde{\psi}^i) \in \tilde{A}$. Set

$$\check{M}_t^i(\phi^i) = \check{H}_t^i((\phi^i \circ \bar{\delta})_t) - \int_\tau^t \check{H}_s^i(\tilde{\psi}_s^i) \, ds - r_i \int_\tau^t \int (\phi^i \circ \bar{\delta})(s, y, s) \, L(\hat{H}^i, \hat{H}^j)(ds, dy) \circ \pi_1, \; t \geq \tau.$$

The definition of $\bar{H}^i$ and $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ imply that for $t \geq \tau$, $\bar{M}_t^i(\phi^i) = \check{M}_t^i(\phi^i) + \hat{M}_t^i((\phi^i)^0) \circ \pi_1$. Proposition 6.1 and $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ imply that $(\hat{M}^i((\phi^i)^0) \circ \pi_1)_{t \geq \tau}$ is a continuous $(\mathcal{F}_t)_{t \geq \tau}$–martingale such that $\langle \hat{M}^i((\phi_i)^0) \circ \pi_1 \rangle_t = \int_\tau^t \hat{H}_s^i(\{(\phi_s^i)^0\}^2) \circ \pi_1 \, ds$ and $\langle \hat{M}^1((\phi_1)^0) \circ \pi_1, \hat{M}^2((\phi_2)^0) \circ \pi_1 \rangle_t = 0$. Using Theorem A.3, we can argue exactly as in Theorem 5.1(b) of [2] to see that $(\check{M}^i(\phi^i))_{t \geq \tau}$ is a continuous $(\mathcal{F}_t)_{t \geq \tau}$–martingale that is orthogonal to $\hat{M}^k((\phi_k)^0) \circ \pi_1$ for $k = 1, 2$, $\langle \check{M}^i(\phi^i) \rangle_t = \int_\tau^t \check{H}_s^i(\{(\phi^i \circ \bar{\delta})_s\}^2) \, ds$ and $\langle \check{M}^1(\phi^1), \check{M}^2(\phi^2) \rangle_t = 0$. The result now follows.

□

Remark 6.3 (a) We will refer to conditions (a) and (b) of Theorem 6.2 as $\overline{MP}(\tau, \mu^1, \mu^2)$.
(b) Proposition 6.1 implies that

$$(\hat{H}^1, \hat{H}^2, L(\hat{H}^1, \hat{H}^2), L(\hat{H}^2, \hat{H}^1)) \text{ defined on } (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}})$$

and

$$(\hat{H}^1, \hat{H}^2, L(\hat{H}^1, \hat{H}^2), L(\hat{H}^2, \hat{H}^1)) \circ \pi_1 \text{ defined on } (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq \tau}, \mathbb{P})$$

have the same law on

$$\Omega_H[\tau, \infty[\times \Omega_H[\tau, \infty[\times \mathbf{M}_{LF}(S^o) \times \mathbf{M}_{LF}(S^o).$$

Moreover, they have the same adapted distribution in the sense of [24]. This implies, for example, that $(\hat{H}^1, \hat{H}^2) \circ \pi_1$ satisfies $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq \tau}, \mathbb{P})$. As any properties we establish for $(\hat{H}^1, \hat{H}^2) \circ \pi_1$ will immediately transfer over to $(\hat{H}^1, \hat{H}^2)$, we may, and shall, use $(\hat{H}^1, \hat{H}^2, L(\hat{H}^1, \hat{H}^2), L(\hat{H}^2, \hat{H}^1)) \circ \pi_1$ as an effective substitute for $(\hat{H}^1, \hat{H}^2, L(\hat{H}^1, \hat{H}^2), L(\hat{H}^2, \hat{H}^1))$. Therefore, we will write $\hat{H}^i$ in place of $\hat{H}^i \circ \pi_1$ and $L(\hat{H}^i, \hat{H}^j)$ in place of $L(\hat{H}^i, \hat{H}^j) \circ \pi_1$ and assume our original solution of $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ is defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq \tau}, \mathbb{P})$.

The next lemma gives a large class of examples of pairs in $\bar{A}$.

Lemma 6.4 *(a) If* $\phi_1 \in D_{ST}$ *and* $\phi_2 \in b\mathcal{B}(\mathbf{M}_*)$, *then* $(\phi, \psi) \in \bar{A}$, *where* $\phi(s, y, e) = \phi_1(s, y)\phi_2(e)$ *and* $\psi(s, y, e) = A\phi_1(s, y)\phi_2(e)$.
(b) If $f \in b\mathcal{B}(\bar{S}^o)$, *then* $(\phi, 0) \in \bar{A}$, *where* $\phi(t, y, e) = \int \mathbf{1}\{s \le t\} f(s, y, e)\, e(ds)$. *Moreover,*

$$\bar{H}_t^i(\phi_t) = \bar{M}_t^i(\phi) + r_i \int_\tau^t \int f(s, y, \delta_s)\, L(\hat{H}^i, \hat{H}^j)(ds, dy), \ \forall t \ge \tau.$$

Proof. (a) Observe that $\phi^0(s, y) = \phi_1(s, y)\phi_2(0) \in D_{ST}$ and $A\phi^0(s, y) = A\phi_1(s, y)\phi_2(0)$. Itô's lemma shows that

$$P_t^W \phi_\xi(s, y) = \phi_\xi(s, y) + \int_0^t P_u^W(A\phi_1)(s, y)\phi_2(\delta_\xi)\, du$$

for $(s, y, \delta_\xi) \in \bar{S}^o$. The result now follows from (6.3).
(b) Observe that $\phi(s, y, 0) \equiv 0 \in D_{ST}$. If $(s, y, \delta_\xi) \in \bar{S}^o$, then $\phi_\xi(s, y) = f(\xi, y, \delta_\xi) \equiv g(\xi, y^\xi)$ and so

$$\begin{aligned} P_t^W \phi_\xi(s, y) &= P^{s,y}\left[g(\xi, W_{s+t}^\xi)\right] \\ &= g(\xi, y^\xi) = \phi_\xi(s, y). \end{aligned}$$

The first assertion now follows from (6.3). The second assertion follows from $\overline{MP}(\tau, \mu^1, \mu^2)$ because $\phi(s, y, 0) = 0$ and $\phi(s, y, \delta_s) = f(s, y, \delta_s)$.

□

Define processes $(H_t^i)_{t \ge \tau}$, $i = 1, 2$, with sample paths in $\Omega_H[\tau, \infty[$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ by

$$H_t^i(\phi) = \int \phi(y) \bar{H}_t^i(dy, de), \ \phi \in b\mathcal{B}(C).$$

Corollary 6.5 *The processes* H^1 *and* H^2 *are independent historical Brownian motions on the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge \tau}, \mathbb{P})$, *starting at* (τ, μ^1) *and* (τ, μ^2), *respectively, and the pair* $(\hat{H}^1, \hat{H}^2)$ *belongs to* $\mathcal{M}(H^1, H^2)$.

Proof. Apply Lemma 6.4(a) with $\phi_2 \equiv 1$ and use Theorem 1.3 of [31] to see that H^1 and H^2 are historical Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge \tau}, \mathbb{P})$. From Theorem 4.7 of [21] we know that any functional of H^i can be written as a stochastic integral against the associated orthogonal martingale measure. This combined with the orthogonality of $\bar{M}^1$ and $\bar{M}^2$ gives the independence of H^1 and H^2. The last assertion is obvious.

$\square$

Recall that the set of ϕ such that $(\phi, \psi) \in \tilde{A}$ for some ψ is bounded–pointwise dense in $b\mathcal{B}(\tilde{S}^o)$ (see the discussion before Theorem A.1). Also, D_{ST} is bounded–pointwise dense in $b\mathcal{B}(S^o)$ (as usual we identify D_{ST} with $\{\phi|_{S^o} : \phi \in D_{ST}\}$). It follows (for example, from Lemma 6.4 (a)) that the set of ϕ such that $(\phi, \psi) \in \bar{A}$ for some ψ is bounded–pointwise dense in $b\mathcal{B}(\bar{S}^o)$. This allows us to extend $\{\bar{M}_t^i(\phi) : \exists \psi,\ (\phi, \psi) \in \bar{A}\}$ to an orthogonal measure $\{\bar{M}_t^i(\phi) : \phi \in b\mathcal{B}(\bar{S}^o)\}$, $i = 1, 2$. If $\phi \in b\mathcal{B}(\mathbb{R}_+ \times C \times \mathbf{M}_*)$, identify ϕ with its restriction to $\bar{S}^o$ to define $\bar{M}_t^i(\phi)$. Then for $\phi_i \in b\mathcal{B}(\mathbb{R}_+ \times C \times \mathbf{M}_*)$, $(\bar{M}_t^i(\phi_i))_{t \geq \tau}$ is an almost surely continuous $(\mathcal{F}_t)_{t \geq \tau}$–martingale such that

$$\langle \bar{M}^i(\phi_i) \rangle_t = \int_\tau^t \int \phi_i(s, y, e)^2\, \bar{H}_s^i(dy, de)\, ds,\ \forall t \geq \tau,$$

and $\langle \bar{M}^1(\phi_1), \bar{M}^2(\phi_2) \rangle = 0$. We will omit the simple L^2 calculations that show these are, in fact, L^2 martingales.

As in Section 2 of [32], we may now extend $\bar{M}_t^i(\phi_i)$ to ϕ_i that are $\mathcal{P} \times \mathcal{B}(C \times \mathbf{M}_*)$–measurable, where $\mathcal{P}$ is the predictable σ–field on $\mathbb{R}_+ \times \Omega$, and satisfy $\mathbb{P}\left[\int_\tau^t \int \phi_i(s, y, e)^2\, \bar{H}_s^i(dy, de)\, ds\right] < \infty$, $\forall t \geq \tau$, (respectively, $\int_\tau^t \int \phi_i(s, y, e)^2\, \bar{H}_s^i(dy, de)\, ds < \infty$, $\forall t \geq \tau$, $\mathbb{P}$-a.s.), in which case the $(\bar{M}_t^i(\phi_i))_{t \geq \tau}$ are almost surely continuous square - integrable $(\mathcal{F}_t)_{t \geq \tau}$–martingales (respectively, $(\mathcal{F}_t)_{t \geq \tau}$–local martingales) satisfying the obvious extensions of the above quadratic variation and covariation relations.

There is a minor technical point here. The construction in Walsh specialised to our situation would start with integrands in $b\mathcal{B}(C \times \mathbf{M}_*)$, and so one should check that the extension obtained by Walsh's procedure agrees with the one defined above for integrands in $b\mathcal{B}(\mathbb{R}_+ \times C \times \mathbf{M}_*)$. This is easy to check by starting with $\phi(s, y, e) = \mathbf{1}_{[u,v[}(s)\psi(y, e)$ for $\psi \in b\mathcal{B}(C \times \mathbf{M}_*)$ and taking limits of linear combinations of such functions.

Let $\bar{\mathcal{D}}$ be the Borel σ–field on $C \times \mathbf{M}_*$, and for $t \geq 0$ define $\bar{\mathcal{D}}_t$ to be the sub–σ–field generated by the map $(y, e) \mapsto (y^t, e^t)$. A simple but useful consequence of Theorem 6.2 and Lemma 6.4 is that if ρ is a bounded, $\bar{\mathcal{D}}_q$-measurable function, then

$$\bar{H}_u^i(\rho) = \bar{H}_q^i(\rho) + \int_q^u \int \rho(y, e)\, d\bar{M}^i(v, y, e),\ u \geq q. \tag{6.4}$$

As in the discussion prior to Lemma 4.16, for any $(\mathcal{F}_t)_{t \geq \tau}$–stopping time T, $(y_t - y_\tau, t \geq \tau)$ is a $(\bar{\mathcal{D}}_t \times \mathcal{F}_t)_{t \geq \tau}$ Brownian motion stopped at T under the normalised Campbell measure $\bar{\mathbb{P}}_T^{\bar{H}^1}$. Given a $(\bar{\mathcal{D}}_t \times \mathcal{F}_t)_{t \geq \tau}$–predictable, $\mathbb{R}^d$–valued function η, such that

$$\mathbb{P}\left[\int_\tau^t \int |\eta(s, y, e)|^2\, \bar{H}_s^1(dy, de) ds\right] < \infty,\ \forall t \geq \tau, \tag{6.5}$$

we can follow the proof of Theorem 3.11 in [30] to construct a $(\bar{\mathcal{D}}_t \times \mathcal{F}_t)_{t\geq 0}$–predictable, $\mathbb{R}$–valued process $I(\eta)$ such that for all bounded $(\mathcal{F}_t)_{t\geq\tau}$–stopping times T,

$$I(\eta)(t \wedge T(\omega), y, e, \omega) = \int_\tau^t \eta(s, y, e, \omega) \cdot dy(s)(\omega),$$

for all $t \geq \tau$, $\bar{\mathbb{P}}_T^{\bar{H}^i}$-a.s. The process $I(\eta)$ is unique up to $\bar{H}^1$-evanescent sets. With a slight abuse of notation, we will write $\int_\tau^t \eta(s, y, e, \omega) \cdot dy(s)(\omega)$ for $I(\eta)(t, y, e, \omega)$.

Corollary 6.6 *(a) Let γ be a bounded, $(\bar{\mathcal{D}}_t \times \mathcal{F}_t)_{t\geq\tau}$–predictable function. Then $\mathbb{P}$-a.s. for all $t \geq \tau$,*

$$\int \left\{ \int_{]\tau,t]} \gamma(s, y, e) \left[e(ds) - r_1 \mathbf{1}\{e(]\tau, s]) = 0\} \ell(y, \hat{H}^2)(ds) \right] \right\} \bar{H}_t^1(dy, de)$$
$$= \int_\tau^t \int \left\{ \int_{]\tau,s]} \gamma(u, y, e) \left[e(du) - r_1 \mathbf{1}\{e(]\tau, u]) = 0\} \ell(y, \hat{H}^2)(du) \right] \right\} d\bar{M}^1(s, y, e),$$

and both sides are continuous $(\mathcal{F}_t)_{t\geq\tau}$–martingales.
(b) Let η be a $(\bar{\mathcal{D}}_t \times \mathcal{F}_t)_{t\geq\tau}$–predictable, $\mathbb{R}^d$–valued function satisfying (6.5). Then $\mathbb{P}$-a.s. for all $t \geq \tau$,

$$\int \left\{ \int_\tau^t \eta(s, y, e) \cdot dy(s) \right\} \bar{H}_t^1(dy, de) = \int_\tau^t \left\{ \int_\tau^s \eta(u, y, e) \cdot dy(u) \right\} d\bar{M}^1(s, y, e),$$

and both sides are continuous, square–integrable $(\mathcal{F}_t)_{t\geq\tau}$–martingales.
(c) Let γ be as in (a) and η be as in (b). Assume also that there is a constant c such that for any bounded $(\mathcal{F}_t)_{t\geq\tau}$–stopping time T,

$$\sup_{\tau\leq t\leq T} \left| \int_\tau^t \eta(s, y) \cdot dy(s) \right| + \int_\tau^t |\gamma(s, y, e)| \ell(y, \hat{H}^2)(ds) \leq c, \quad \bar{\mathbb{P}}_T^{\bar{H}^1} - a.a. \ (y, e, \omega).$$

Then $\mathbb{P}$-a.s. for all $t \geq \tau$,

$$\int \left\{ \int_\tau^t \eta(s, y, e) \cdot dy(s) \times \int_{]\tau,t]} \gamma(s, y, e) \left[e(ds) - \mathbf{1}\{e(]\tau, s]) = 0\} r_1 \ell(y, \hat{H}^2)(ds) \right] \right\} \bar{H}_t^1(dy, de)$$
$$= \int_\tau^t \int \left\{ \int_\tau^s \eta(u, y, e) \cdot dy(u) \right.$$
$$\left. \times \int_{]\tau,s]} \gamma(u, y, e) \left[e(du) - \mathbf{1}\{e(]\tau, u]) = 0\} r_1 \ell(y, \hat{H}^2)(du) \right] \right\} d\bar{M}^1(s, y, e),$$

and both sides are continuous $(\mathcal{F}_t)_{t\geq\tau}$–martingales.

Proof. (a) The existence of $\ell(y, \hat{H}^2)$ follows from Theorem 3.7. As usual it suffices to consider $\gamma(s, y, e, \omega) = \gamma_1(y, e)\gamma_2(\omega)\mathbf{1}_{[u,v[}(s)$, where $\tau \leq u < v$, $\gamma_1 \in b\bar{\mathcal{D}}_u$, is continuous and $\gamma_2 \in b\mathcal{F}_u$. Equation

(6.4) allows us to follow the derivation of Theorem 3.10(b) (simplified significantly as we know that $L(\hat{H}^1, \hat{H}^2)$ exists) to see that

$$\begin{aligned}
&\int \int_\tau^t \gamma(s,y,e)\mathbf{1}\{e(]\tau,s]) = 0\}\ell(y,\hat{H}^2)(ds)\, \bar{H}^1_t(dy,de) \\
&\quad = \int_\tau^t \int \left[\int_\tau^s \gamma(u,y,e)\mathbf{1}\{e(]\tau,u]) = 0\}\ell(y,\hat{H}^2)(du)\right] d\bar{M}^1(s,y,e) \\
&\qquad + L^2 - \lim_{\epsilon\downarrow 0} \int_{u\wedge t}^{v\wedge t} \iint \gamma(s,y,e)\mathbf{1}\{e(]\tau,s]) = 0\}\, p_\epsilon(y_s - y'_s)\hat{H}^2_s(dy')\, \bar{H}^1_s(dy,de)\, ds.
\end{aligned} \tag{6.6}$$

Since $\gamma_1(y,e) = \gamma_1(y^u, e^u)$ we have

$$\gamma(s,y,e)\mathbf{1}\{e(]\tau,s]) = 0\} = \gamma(s,y,0)\mathbf{1}\{e(]\tau,s]) = 0\} \text{ for } s \geq u,$$

and so by Theorem 6.2(c) the last term in (6.6) is

$$\begin{aligned}
L^2 - &\lim_{\epsilon\downarrow 0} \int_{u\wedge t}^{v\wedge t} \int \gamma_1(y,0)\gamma_2\, L^\epsilon(\hat{H}^1,\hat{H}^2)(ds,dy) \\
&= \int_\tau^t \int \gamma(s,y,0)L(\hat{H}^1,\hat{H}^2)(ds,dy) \qquad \text{(Theorem 3.10(a))} \\
&= \int_\tau^t \int \gamma(s,y,\delta_s)\, L(\hat{H}^1,\hat{H}^2)(ds,dy),
\end{aligned}$$

the last by the fact that $\gamma(s,y,e) = \gamma(s,y,e^u)$ and $(\delta_s)^u = 0^u$ for $s > u$. Substitute this into (6.6) and combine this with Lemma 6.4(b) to complete the proof.

(b) This is is proved by making simple changes to the proof of Theorem 3.17 of [30]. For example, the proof of the key result (corresponding to Lemma 3.16 of [30]) turns on (6.4).

(c) If T is a bounded $(\mathcal{F}_t)_{t\geq\tau}$–stopping time then

$$\begin{aligned}
&\int_\tau^t \eta(s,y,e)\cdot dy(s) \times \int_{]\tau,t]} \gamma(s,y,e)\left[e(ds) - \mathbf{1}\{e(]\tau,s]) = 0\}r_1\ell(y,\hat{H}^2)(ds)\right] \\
&\quad = \int_\tau^t \eta(s,y,e) \int_{]\tau,s[} \gamma(u,y,e)\left[e(du) - \mathbf{1}\{e(]\tau,u]) = 0\}r_1\ell(y,\hat{H}^2)(du)\right]\cdot dy(s) \\
&\qquad + \int_{]\tau,t]} \int_\tau^s \eta(u,y,e)\cdot dy(u)\gamma(s,y,e)\left[e(ds) - \mathbf{1}\{e(]\tau,s]) = 0\}r_1\ell(y,\hat{H}^2)(ds)\right]
\end{aligned}$$

for all $T \geq t \geq \tau$, $\bar{\mathbb{P}}^{\bar{H}^1}_T$-a.s. This follows from ordinary stochastic calculus under the Campbell measure (as in Lemma 3.18 of [30]). The section theorem shows that the above equality holds $\forall t \leq u$ for $\bar{H}^1_u$-a.a. (y,e), $\forall u \geq \tau$, a.s. Now integrate both sides with respect to $\bar{H}^1_t$ and use (a) and (b) to complete the proof. The boundedness hypothesis on the stochastic integrals and another application of the section theorem is used here to ensure the hypotheses of (a) and (b) hold. Note, for example, in the second term by truncation we may assume $\sup_{\tau\leq s\leq t} |\int_\tau^s \eta(u,y,e)\cdot dy(u)| \leq c$.

□

6.2 Simplicity and faithfulness of the afterlife marks

Recall from Section 4 the picture of the process constructed there as a pair historical Brownian motions decorated with Poisson marks that are "faithful" to the historical branching structure. The pair $(\bar{H}^1, \bar{H}^2)$ is also a pair of marked historical Brownian motions, with the location of the mark along a path indicating when the ancestor of that particle was killed. Thus, if $(\hat{H}^1, \hat{H}^2)$ really does arise as a solution of (SE) against some process $(\tilde{H}^1, \tilde{H}^2)$ constructed as in Section 4, then the locations of marks for $(\bar{H}^1, \bar{H}^2)$ should look like a subset of the locations of marks for $(\tilde{H}^1, \tilde{H}^2)$. This imposes some constraints on the location of the marks for $(\bar{H}^1, \bar{H}^2)$, arising from the fact that two independent Poisson processes with diffuse intensities don't have coincident atoms. The following result verifies that these constraints hold.

Proposition 6.7 *The following hold $\mathbb{P}$-a.s. for all $t \geq \tau$.*
(a) $\bar{H}^1_t \otimes \bar{H}^2_t\,(\{((y_1, e_1), (y_2, e_2)) : e_1 = e_2 \neq 0\}) = 0.$
(b) $\bar{H}^i_t \otimes \bar{H}^i_t\,(\{((y_1, e_1), (y_2, e_2)) : \exists s < t,\ y_1^s = y_2^s,\ e_1^s \neq e_2^s\}) = 0.$
(c) $\bar{H}^i_t \otimes \bar{H}^i_t\,(\{((y_1, e_1), (y_2, e_2)) : \exists s \leq t,\ y_1^s \neq y_2^s,\ e_1 = e_2,\ e_1(]s, \infty[) = 1\}) = 0.$

Proof. (a) Let $\phi_i \in b\mathcal{B}(\mathbf{M}_*)$ be such that $\phi_i(0) = 0$. Lemma 6.4(a) and Theorem 6.2 show that if we set $\bar{\phi}_i(s, y, e) = \phi_i(e)$, then $(\bar{\phi}_i, 0) \in \bar{A}$. Now $\overline{MP}(\tau, \mu^1\mu^2)$ and Itô's lemma imply that for $\phi \in b\mathcal{B}(\mathbf{M}_* \times \mathbf{M}_*)$ given by $\phi(e_1, e_2) = \phi_1(e_1)\phi_2(e_2)$

$$\begin{aligned}
&\iint \phi(e_1, e_2)\, \bar{H}^1_t(dy, de)\bar{H}^2_t(dy, de) \\
&= \int_\tau^t \iint \phi(\delta_{\xi_1}, e_2)\check{H}^1_s(dy_1, d\xi_1)d\bar{M}^2(s, y_2, e_2) \\
&+ \int_\tau^t \iint \phi(e_1, \delta_{\xi_2})\check{H}^2_s(dy_2, d\xi_2)d\bar{M}^1(s, y_1, e_1) \\
&+ r_1 \int_\tau^t \iint \phi(\delta_s, \delta_{\xi_2})\check{H}^2_s(dy_2, d\xi_2)L(\hat{H}^1, \hat{H}^2)(ds, dy_1) \\
&+ r_2 \int_\tau^t \iint \phi(\delta_{\xi_1}, \delta_s)\check{H}^1_s(dy_1, d\xi_1)L(\hat{H}^2, \hat{H}^1)(ds, dy_2).
\end{aligned} \tag{6.7}$$

By a monotone class argument, equation (6.7) extends to all $\phi \in b\mathcal{B}(\mathbf{M}_* \times \mathbf{M}_*)$ such that $\phi(e_1, e_2) = 0$ whenever $e_1 = 0$ or $e_2 = 0$. Let $\phi(e_1, e_2) = \mathbf{1}\{e_1 = e_2 \neq 0\}$ in (6.7) to see that the process in part (a) is almost surely continuous in $t \geq \tau$. It therefore suffices to prove the claim for $t \geq \tau$ fixed.

Note that by (A.14)

$$\begin{aligned}
&\mathbb{P}\left[\int_\tau^t \iint \mathbf{1}\{\delta_{\xi_1}=\delta_s\}\,\check{H}^1_s(dy_1,d\xi_1)\,L(\hat{H}^2,\hat{H}^1)(ds,dy_2)\ \mid (\hat{H}^1,\hat{H}^2)\right]\\
&=\int_\tau^t \int \mathbb{Q}^{\tau,0;r_1L(\hat{H}^1,\hat{H}^2)}\left[\int \mathbf{1}\{\delta_{\xi_1}=\delta_s\}\,\check{H}^1_s(dy_1,d\xi_1)\right] L(\hat{H}^2,\hat{H}^1)(ds,dy_2)\\
&=\int_\tau^t \iint_\tau^{s_2}\int \mathbf{1}\{\delta_{s_1}=\delta_{s_2}\}\,L(\hat{H}^1,\hat{H}^2)(ds_1,dy_1)\,L(\hat{H}^2,\hat{H}^1)(ds_2,dy_2)\\
&=0,
\end{aligned}$$

by the continuity of $t\mapsto L(\hat{H}^1,\hat{H}^2)([0,t]\times C)$. This shows the last term in (6.7) with $\phi(e_1,e_2)=\mathbf{1}\{e_1=e_2\neq 0\}$ is almost surely 0, and the same is true of the term before it. To conclude the proof, take expectations in (6.7) (using the fact that $(\bar{H}^i_s(1))_{s\geq\tau}=(H^i_s(1))_{s\geq\tau}$, $i=1,2$, are independent continuous state branching processes to infer the necessary integrability of the martingale terms).
(b) Fix $i=1$. It suffices to check for each rational $s>\tau$ that $\mathbb{P}$-a.s.

$$\bar{H}^1_t\otimes\bar{H}^1_t\left(\{((y_1,e_1),(y_2,e_2)):\exists u\leq s,\ y_1^u=y_2^u,\ e_1(\{u\})\neq e_2(\{u\})\}\right)=0,\ \forall t>s.$$

Take $f(s,y,e)=\mathbf{1}\{s\leq\tau\}$ in Lemma 6.4(b) to see that $(\int e([0,\tau])\,\bar{H}^1_t(dy,de))_{t\geq\tau}$ is a non-negative, continuous martingale starting at 0, and hence is almost surely identically zero for $t\geq\tau$. It is now easy to see that it suffices to fix a rational $s>\tau$ and show that $\mathbb{P}$-a.s.

$$\bar{H}^1_t\otimes\bar{H}^1_t\left(\{((y_1,e_1),(y_2,e_2)):\exists\tau<u\leq s,\ y_1^u=y_2^u,\ e_1(\{u\})\neq e_2(\{u\})\}\right)=0,\ \forall t>s. \tag{6.8}$$

For $\phi_1\in D_S$ and $\phi_2\in b\mathcal{B}(\mathbf{M}_*)$ set $\phi(r,y,e)=\phi_1(y^{r\wedge s})\phi_2(e)$. Observe that $\phi_3(y)=\phi_1(y^s)$ is also in D_S and $\frac{\bar{\Delta}}{2}\phi_3(r,y)=\mathbf{1}\{r<s\}\frac{\bar{\Delta}}{2}\phi_1(r,y)$. Lemma 6.4(a) shows that $(\phi,\psi)\in\bar{A}$, where $\psi(r,y,e)=\mathbf{1}\{r<s\}\frac{\bar{\Delta}}{2}\phi_1(r,y)\phi_2(e)$. Therefore, from $\overline{MP}(\tau,\mu^1,\mu^2)$ we have

$$\bar{H}^1_t(\phi_t)=\bar{H}^1_s(\phi_s)+(\bar{M}^1_t(\phi)-\bar{M}^1_s(\phi))+r_1\int_s^t\int\phi(r,y,\delta_r)-\phi(r,y,0)L(\hat{H}^1,\hat{H}^2)(dr,dy),\ t\geq s.$$

Take bounded–pointwise limits to see that the above continues to hold if ϕ is of the form $\phi(r,y,e)=\zeta(y^{r\wedge s},e)$ for $\zeta\in b\mathcal{B}(C\times\mathbf{M}_*)$. Use Itô's lemma and another passage to the bounded–pointwise closure to see that if $\gamma\in b\mathcal{B}((C\times\mathbf{M}_*)^2)$ is of the form $\gamma((y_1,e_1),(y_2,e_2))=\rho(y_1^s,e_1,y_2^s,e_2)$ for

$\rho \in b\mathcal{B}((C \times \mathbf{M}_*)^2)$, then

$$
\begin{aligned}
&\bar{H}_t^1 \otimes \bar{H}_t^1(\gamma) \\
&\quad = \bar{H}_s^1 \otimes \bar{H}_s^1(\gamma) \\
&\quad + \int_s^t \iint \gamma((y_1, e_1), (y_2, e_2)) \left[\bar{H}_u^1(dy_1, de_1) d\bar{M}^1(u, y_2, e_2) + \bar{H}_u^1(dy_2, de_2) d\bar{M}^1(u, y_1, e_1)\right] \\
&\quad + \int_s^t \int \gamma((y, e), (y, e)) \bar{H}_u^1(dy, de) du \\
&\quad + r_1 \int_s^t \iint [\gamma((y_1, \delta_r), (y_2, e_2)) - \gamma((y_1, 0), (y_2, e_2))] \bar{H}_r^1(dy_2, de_2) L(\hat{H}^1, \hat{H}^2)(dr, dy_1) \\
&\quad + r_1 \int_s^t \iint [\gamma((y_1, e_1), (y_2, \delta_r)) - \gamma((y_1, e_1), (y_2, 0))] \bar{H}_r^1(dy_1, de_1) L(\hat{H}^1, \hat{H}^2)(dr, dy_2), \ \forall t \geq s.
\end{aligned}
\tag{6.9}
$$

By setting

$$
\gamma((y_1, e_1), (y_2, e_2)) = \mathbf{1}\left\{\int_{]\tau, s]} 1(y_1^u = y_2^u) |e_1(du) - e_2(du)| > 0\right\}
$$

we can conclude from (6.9) that for $t \geq s$

$$
\begin{aligned}
&\bar{H}_t^1 \otimes \bar{H}_t^1 \left(\{((y_1, e_1), (y_2, e_2)) : \exists \tau < u \leq s, \ y_1^u = y_2^u, \ e_1(\{u\}) \neq e_2(\{u\})\}\right) \\
&\overset{m}{=} \bar{H}_s^1 \otimes \bar{H}_s^1 \left(\{((y_1, e_1), (y_2, e_2)) : \exists \tau < u \leq s, \ y_1^u = y_2^u, \ e_1(\{u\}) \neq e_2(\{u\})\}\right)
\end{aligned}
\tag{6.10}
$$

where $=^m$ means that that for $t \geq s$ the two sides differ by a continuous martingale that is null at $t = s$. It therefore suffices to show that the right-hand side of (6.10) is 0, $\mathbb{P}$-a.s.

By definition of $\bar{H}^1$ it will in turn suffice to show that

$$
\check{H}_s^1 \otimes \hat{H}_s^1 \left(\{((y_1, u_1), y_2) : \ y_1^{u_1} = y_2^{u_1}\}\right) = 0 \tag{6.11}
$$

and

$$
\check{H}_s^1 \otimes \check{H}_s^1 \left(\{((y_1, u_1), (y_2, u_2)) : u_1 < u_2, \ y_1^{u_1} = y_2^{u_1}\}\right) = 0. \tag{6.12}
$$

It follows from (A.14) that if $\phi(s, y, u)$ is of the form $\phi(s, y, u) = \mathbf{1}\{u \leq s\} f(u, y^u)$, then

$$
\mathbb{P}\left[\int \phi(s, y, u) \, \check{H}_s^1(d(y, u)) \mid (\hat{H}^1, \hat{H}^2)\right] = r_1 \int_\tau^s \int f(u, y) \, L(\hat{H}^1, \hat{H}^2)(du, dy)
$$

(cf. the proof of Lemma 6.4(b)). Thus,

$$
\begin{aligned}
&\mathbb{P}\left[\check{H}_s^1 \otimes \hat{H}_s^1 \left(\{((y_1, u_1), y_2) : \ y_1^{u_1} = y_2^{u_1}\}\right)\right] \\
&= \mathbb{P}\left[\iint \mathbf{1}\{u_1 \leq s\} \mathbf{1}\{y_1^{u_1} = y_2^{u_1}\} \, \check{H}_s^1(d(y_1, u_1)) \, \hat{H}_s^1(dy_2)\right] \\
&= \mathbb{P}\left[r_1 \int \int_\tau^s \int \mathbf{1}\{y_1 = y_2^{u_1}\} \, L(\hat{H}^1, \hat{H}^2)(du_1, dy_1) \, \hat{H}_s^1(dy_2)\right] \\
&= 0,
\end{aligned}
$$

by Lemma 3.17(a) and this establishes (6.11).

Recall the Brownian path process laws $P^{u,y}$ defined for $(u,y) \in \mathbb{R}_+ \times C$ such that $y^u = y$. For $\tau \le u \le v \le s$ and $y \in C^u$, define a probability measure $p^{u,v,y}$ on $C^s \times C^s$ by

$$p^{u,v,y}(\psi) = \int \left[\int \psi(\omega_1(s), \omega_2(s))\, P^{v,\omega(v)} \otimes P^{v,\omega(v)}(d\omega_1, d\omega_2) \right] P^{u,y}(d\omega).$$

It follows from (A.14) and (A.15) that

$$\begin{aligned} &\mathbb{P}\left[\iint \phi(y_1,u_1,y_2,u_2)\, \check{H}^1_s(d(y_1,u_1))\, \check{H}^1_s(d(y_2,u_2)) \mid (\hat{H}^1, \hat{H}^2) \right] \\ &= r_1^2 \int_\tau^s \int \int_\tau^s \int \phi(y_1,u_1,y_2,u_2)\, L(\hat{H}^1,\hat{H}^2)(du_1,dy_1)\, L(\hat{H}^1,\hat{H}^2)(du_2,dy_2) \\ &\quad + r_1 \int_\tau^s \left[\int_\tau^v \int \left[\int \phi(y_1,u,y_2,u)\, p^{u,v,y}(dy_1,dy_2) \right] L(\hat{H}^1,\hat{H}^2)(du,dy) \right] dv. \end{aligned}$$

Thus

$$\begin{aligned} &\mathbb{P}\left[\check{H}^1_s \otimes \check{H}^1_s\left(\{((y_1,u_1),(y_2,u_2)) : u_1 < u_2,\ y_1^{u_1} = y_2^{u_1}\} \right) \right] \\ &= \mathbb{P}\left[r_1^2 \int_\tau^s \int \int_\tau^{u_2} \int \mathbf{1}\{y_1 = y_2^{u_1}\}\, L(\hat{H}_1,\hat{H}_2)(du_1,dy_1)\, L(\hat{H}_1,\hat{H}_2)(du_2,dy_2) \right] \\ &= 0, \end{aligned}$$

by Lemma 3.17(b) and this establishes (6.12).

(c) It suffices to prove for a fixed rational $s > \tau$ that $\mathbb{P}$-a.s.

$$\bar{H}^1_t \otimes \bar{H}^1_t\left(\{((y_1,e_1),(y_2,e_2)) : y_1^s \ne y_2^s,\ e_1 = e_2,\ e_1(]s,\infty[) = 1\} \right) = 0,\ \forall t \ge s. \tag{6.13}$$

Let $\gamma((y_1,e_1),(y_2,e_2)) = \mathbf{1}\{y_1^s \ne y_2^s,\ e_1 = e_2,\ e_1(]s,\infty[) = 1\}$ in (6.9) to see that $\mathbb{P}$-a.s. for all $t \ge s$

$$\begin{aligned} &\bar{H}^1_t \otimes \bar{H}^1_t\left(\{((y_1,e_1),(y_2,e_2)) : y_1^s \ne y_2^s,\ e_1 = e_2,\ e_1(]s,\infty[) = 1\} \right) \\ &\overset{m}{=} \bar{H}^1_s \otimes \bar{H}^1_s\left(\{((y_1,e_1),(y_2,e_2)) : y_1 \ne y_2,\ e_1 = e_2,\ e_1(]s,\infty[) = 1\} \right) \\ &\quad + 2r_1 \int_s^t \iint \mathbf{1}\{y_1^s \ne y_2^s\} \mathbf{1}\{e_2 = \delta_r\}\, \bar{H}^1_r(dy_2,de_2) L(\hat{H}^1,\hat{H}^2)(dr,dy_1) \\ &= 2r_1 \int_s^t \iint \mathbf{1}\{y_1^s \ne y_2^s\} \mathbf{1}\{\delta_{\xi_2} = \delta_r\}\, \check{H}^1_r(dy_2,d\xi_2)\, L(\hat{H}^1,\hat{H}^2)(dr,dy_1) \end{aligned} \tag{6.14}$$

because $\bar{H}^1_s\left(\{(y,e) : e(]s,\infty[\ne 0\} \right) = 0$ almost surely.

Take the conditional expectation of the last element in (6.14) with respect to $(\hat{H}^1, \hat{H}^2)$ and argue as in the proof of part (a) to see that the conditional expectation is 0 almost surely. This proves that (6.14) is 0 almost surely for each $t \ge s$, and as it is almost surely continuous in t, (6.13) follows.

□

Remark 6.8 It can be shown that $\mathbb{P}$-a.s. for all $t \geq \tau$,

$$\bar{H}_t^i\left(\{(y,e) \ : \ e(\{t\}) > 0\}\right) = 0$$

(cf. the proof of Lemma 4.12). Therefore the claim in part (b) of the above proposition can be strengthened to: $\mathbb{P}$-a.s. for all $t \geq \tau$,

$$\bar{H}_t^i \otimes \bar{H}_t^i\left(\{((y_1,e_1),(y_2,e_2)) : \exists s \leq t,\ y_1^s = y_2^s,\ e_1^s \neq e_2^s\}\right) = 0.$$

However, because we do not require the stronger result we leave the details to the reader.

6.3 Afterlife and driving support processes

For $\tau \leq r < t$ define measures $H_{r,t}^i$, $\bar{H}_{r,t}^i$ and $\breve{H}_{r,t}^i$ by $H_{r,t}^i(\phi) = \int \phi(y^r)\, H_t^i(dy)$, $\bar{H}_{r,t}^i(\psi) = \int \psi(y^r, e^r)\, \bar{H}_t^i(d(y,e))$, and $\breve{H}_{r,t}^i(f) = \int f(y^r, n^r) \breve{H}_t^i(d(y,n))$. Let $S_{r,t}^i$, $\bar{S}_{r,t}^i$ and $\breve{S}_{r,t}^i$ denote, respectively, the supports of $H_{r,t}^i$, $\bar{H}_{r,t}^i$ and $\breve{H}_{r,t}^i$.

Lemma 6.9 *(a) Almost surely under $\mathbb{P}$ for all $\tau \leq r < s < t$ the sets $S_{r,s}^i$ (resp. $\bar{S}_{r,s}^i$) are finite, and $S_{r,s}^i \supseteq S_{r,t}^i$ (resp. $\bar{S}_{r,s}^i \supseteq \bar{S}_{r,t}^i$).*
(b) Almost surely under $\mathbb{P}$ for all $\tau \leq r < s$ and all bounded $\mathcal{C}$-measurable (resp. $\bar{\mathcal{D}}$-measurable) ϕ, $\lim_{t\to s} H_{r,t}^i(\phi) = H_{r,s}^i(\phi)$ (resp. $\lim_{t\to s} \bar{H}_{r,t}^i(\phi) = \bar{H}_{r,s}^i(\phi)$).
(c) Almost surely under $\mathbb{P}$ for all $\tau \leq r < t$, if $\bar{S}_{r,t}^i = \{(y_1,e_1),\ldots,(y_m,e_m)\}$, then $y_1,\ldots,y_m$ are distinct and $S_{r,t}^i = \{y_1,\ldots,y_m\}$.
(d) The analogues of (a) and (b) are valid for $\breve{H}_{r,s}^i$ under $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$, where $\breve{\mu}^i \in \mathbf{M}_F(C \times \mathbf{M}_\#)^\tau$ satisfies $\pi(\breve{\mu}^i) \in \mathbf{M}_{FS}(C)^\tau$.

Proof. (a) Fix $\tau \leq p < q$. Proposition 3.5 of [9] shows the set $S_{p,q}^i$ is $\mathbb{P}$-a.s. finite. It follows from Proposition 6.7(b) that $\bar{S}_{p,q}^i$ is also $\mathbb{P}$-a.s. finite and, in fact, if $S_{p,q}^i = \{y_1,\ldots,y_m\}$, then $\bar{S}_{p,q}^i = \{(y_1,e_1),\ldots,(y_m,e_m)\}$ for some $e_1,\ldots,e_m \in \mathbf{M}_*$.

It follows from (6.4) and a monotone class argument that if ϕ is a bounded $\bar{\mathcal{D}}_q \times \mathcal{F}_q$-measurable function, then

$$\bar{H}_u^i(\phi) = \bar{H}_q^i(\phi) + \int_q^u \int \phi(y,e)\, d\bar{M}^i(v,y,e),\ u \geq q. \tag{6.15}$$

Apply (6.15) with $\phi(y,e) = 1\{(y^p,e^p) \notin \bar{S}_{p,q}^i\}$ to conclude that $\mathbb{P}$-a.s. $\bar{S}_{p,q}^i \supseteq \bar{S}_{p,u}^i$, $u \geq q$. Construct a sequence $(Y_1,E_1),(Y_2,E_2),\ldots$ of $\mathcal{F}_q$-measurable $(C \times \mathbf{M}_*)$–valued random variables such that $\bar{S}_{p,q}^i = \{(Y_1,E_1),(Y_2,E_2),\ldots\}$. Apply (6.15) with $\phi(y,e) = 1\{(y^p,e^p) = (Y_k,E_k)\}$ and the optional sampling theorem to see that if $\bar{H}_{p,u}^i(\{(Y_k,E_k)\}) = 0$ for some $u \geq q$, then $\bar{H}_{p,v}^i(\{(Y_k,E_k)\}) = 0$ for all $v \geq u$.

Combining all of the observations above, we conclude that $\mathbb{P}$-a.s. for all rational numbers $\tau \le p < q$ the set $\bar{S}^i_{p,q}$ is finite and the set-valued map $u \mapsto \bar{S}^i_{p,u}$, $u \ge q$, is nonincreasing. Given arbitrary $\tau \le r < s < t$ take rationals p, q such that $r < p < q < s$ and conclude that $\bar{S}^i_{p,s} \supseteq \bar{S}^i_{p,t}$ and both sets are finite. Thus $\bar{S}^i_{r,s} = \{(y^r, e^r) : (y, e) \in \bar{S}^i_{p,s}\} \supseteq \{(y^r, e^r) : (y, e) \in \bar{S}^i_{p,t}\} = \bar{S}^i_{r,t}$ and the claim for $\bar{H}^i$ follows. The claim for H^i is immediate.

(b) It is apparent from the proof of part (a) that $\mathbb{P}$-a.s. for all rationals $\tau \le p < q$ the map $u \mapsto \bar{H}^i_{p,u}(\{(y, e)\})$, $u \ge q$, is continuous for all $(y, e) \in \bar{S}^i_{p,q}$, and the result follows easily from the fact that $\bar{H}^i_{r,u}(\{(y^r, e^r)\}) = \bar{H}^i_{p,u}(\{(y, e)\})\ \forall (y, e) \in \bar{S}^i_{p,u}$ for $r < p < u$.

(c) This is immediate from Proposition 6.7(b).

(d) By the construction of $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$, it suffices to prove the analogous results for $\breve{H}_{s,t}$ under $\breve{\mathbb{Q}}^{\tau,\breve{\mu}}_{\nu}$, for a fixed $\nu \in \Omega_{XS}[\tau, \infty[$. The process $\breve{H}$ is not a historical process in the sense of [9]. However, as with a historical process, $\breve{H}$ is a superprocess built over a process with the property that the value at an earlier time is obtained by "truncating" the value of the process at a later time. This shared property means that many of the results obtained in the literature for historical processes have obvious analogues that hold for $\breve{H}$. In particular, for fixed $\tau \le p < q$, the measure $\breve{H}_{p,q}$ is almost surely discrete with a finite number of atoms, by an analogue of Proposition 3.5 of [9]. The proof is now completed as in (a) and (b) using Lemma 4.2 to obtain the analogue of (6.15).

□

We will need a stronger result for the driving processes from Section 4. Let $\breve{\mu}^i$ be as in (d) of the above. For $\epsilon > 0$ and $t \ge \tau + \epsilon$ put $\breve{H}^{i,\epsilon}_t = \breve{H}^i_{t-\epsilon,t}$, and for $u \ge 0$ let $\mathbf{M}^u_\# = \{n \in \mathbf{M}_\# : n^u = n\}$. Equip C^u with the metric $d_{C^u}(y, y') = \sup_{0 \le t \le u} |y_t - y'_t| \wedge 1$ and equip $\mathbf{M}^u_\#$ with the Vasershtein metric $d^u_{\mathbf{M}_\#}$ that comes from regarding $\mathbf{M}^u_\#$ as a subset of the finite measures on $[0, u] \times [0, 1]$. Equip $C^u \times \mathbf{M}^u_\#$ with the metric $d_{C^u \times \mathbf{M}^u_\#} = d_{C^u} + d_{\mathbf{M}^u_\#}$. Equip the space of finite measures on $C^u \times \mathbf{M}^u_\#$ with the corresponding Vasershtein metric $d_{\mathbf{M}_F(C^u \times \mathbf{M}^u_\#)}$. In these settings the Vasershtein metric is a complete separable metric inducing the topology of weak convergence (see [18] (p.152, Ex.2) and [31] (p.48)).

Lemma 6.10 *Fix $u \ge \tau$.*
(a) The process $(\breve{H}^{i,\epsilon}_t)_{\tau+\epsilon \le t \le u}$, is a càdlàg, $\mathbf{M}_F(C^u \times \mathbf{M}^u_\#)$–valued process, $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$-a.s.
(b) We have

$$\lim_{\epsilon \downarrow 0} \sup_{\tau+\epsilon \le t \le u} d_{\mathbf{M}_F(C^u \times \mathbf{M}^u_\#)}(\breve{H}^{i,\epsilon}_t, \breve{H}^i_t) = 0, \quad \breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}\text{-a.s.}$$

(c) The set-valued process $(\breve{S}^{i,\epsilon}_t)_{\tau+\epsilon \le t \le u}$, is càdlàg $\breve{\mathbb{P}}^{\tau,\breve{\mu}^1,\breve{\mu}^2}$-a.s. in the Hausdorff metric on finite subsets of $C^u \times \mathbf{M}^u_\#$ corresponding to $d_{C^u \times \mathbf{M}^u_\#}$.

Proof. (a) For $\tau + \epsilon \le s < t$ and ϕ bounded and continuous on $C^u \times \mathbf{M}^u_\#$ write

$$\breve{H}^{i,\epsilon}_t(\phi) - \breve{H}^{i,\epsilon}_s(\phi) = \{\breve{H}^i_{t-\epsilon,t}(\phi) - \breve{H}^i_{s-\epsilon,t}(\phi)\} + \{\breve{H}^i_{s-\epsilon,t}(\phi) - \breve{H}^i_{s-\epsilon,s}(\phi)\}. \tag{6.16}$$

We will show that $\breve{H}^{i,\epsilon}$ is almost surely right continuous by showing that almost surely for all such s both bracketed terms on the right hand side converge to zero as $t \downarrow s$.

Consider the first bracketed term on the right hand side of (6.16). Take $s - \epsilon < r < s$. From Lemma 6.9(d) we see that $\breve{H}^i_{r,s}$ is a discrete measure with a finite number of atoms and $\breve{S}^i_{r,t} \subseteq \breve{S}^i_{r,s}$. Note that if $(y,n) \in \breve{S}^i_{r,s}$, then $n^{t-\epsilon} = n^{s-\epsilon}$ for t sufficiently close to s (depending on (n,ω)). Thus

$$\begin{aligned}\lim_{t\downarrow s} \breve{H}^i_{t-\epsilon,t}(\phi) - \breve{H}^i_{s-\epsilon,t}(\phi) &= \lim_{t\downarrow s} \int \phi(y^{t-\epsilon}, n^{t-\epsilon}) - \phi(y^{s-\epsilon}, n^{s-\epsilon})\, \breve{H}^i_{r,t}(d(y,n)) \\ &= \lim_{t\downarrow s} \int \phi(y^{t-\epsilon}, n^{s-\epsilon}) - \phi(y^{s-\epsilon}, n^{s-\epsilon})\, \breve{H}^i_t(d(y,n)) \\ &= 0.\end{aligned}$$

The convergence to 0 as $t \downarrow s$ of the second bracketed term on the right hand side of (6.16) is immediate from Lemma 6.9(d).

The proof that $\breve{H}^{i,\epsilon}$ has left limits with $\lim_{t\uparrow s} \breve{H}^{i,\epsilon}_t(\phi) = \int \phi((y^{(s-\epsilon)}, n^{(s-\epsilon)-})\, \breve{H}^i_s(d(y,n))$ is similar and is omitted.

(b) Let ϕ be a bounded Lipschitz continuous function of Lipschitz norm 1 on the metric space $(C^u \times \mathbf{M}^u_\#, d_{C^u \times \mathbf{M}^u_\#})$.

For $t \ge \tau + \epsilon$ we have

$$\begin{aligned}|\breve{H}^i_t(\phi) - \breve{H}^{i,\epsilon}_t(\phi)| &\le \int |\phi(y,n) - \phi(y^{t-\epsilon}, n^{t-\epsilon})|\, \breve{H}^i_t(d(y,n)) \\ &\le \int d_{C^u \times \mathbf{M}^u_\#}((y,n),(y^{t-\epsilon}, n^{t-\epsilon}))\, \breve{H}^i_t(d(y,n)) \\ &= \int d_{C^u}(y, y^{t-\epsilon})\, \breve{H}^i_t(d(y,n)) + \int d_{\mathbf{M}^u_\#}(n, n^{t-\epsilon})\, \breve{H}^i_t(d(y,n)) \\ &\le \int d_{C^u}(y, y^{t-\epsilon})\, \breve{H}^i_t(d(y,n)) + \int n(]t-\epsilon, t] \times [0,1])\, \breve{H}^i_t(d(y,n)) \\ &\le 2 \int \sup_{t-\epsilon \le u \le t} |y(u) - y(t)| \wedge 1\, \breve{H}^i_t(d(y,n)) + \int n(\gamma_{t,\epsilon})\, \breve{H}^i_t(d(y,n)),\end{aligned}$$

where

$$\gamma_{t,\epsilon}(u,z) = \begin{cases} 0, & \text{if } u \le t - 2\epsilon, \\ 1, & \text{if } u \ge t - \epsilon, \\ (u - t + 2\epsilon)/\epsilon, & \text{if } t - 2\epsilon < u < t - \epsilon. \end{cases}$$

The last two integrals are continuous as functions of $t \ge \tau + \epsilon$ by the weak continuity of $\breve{H}^i$. Moreover, almost surely both integrals converge monotonically to 0 for all $t > \tau$ as $\epsilon \downarrow 0$ by Lemma 4.12. The result now follows from Dini's theorem.

(c) The proof is similar to that of part (a) and is omitted.

□

Remark 6.11 It is apparent from the proof of Lemma 6.10 that analogous results hold with $\breve{H}^i$ replaced by H^i. In fact, $(H_t^{i,\epsilon})_{t \geq \tau+\epsilon}$ is continuous.

7 The martingale problem and the strong equation

In this section we complete the task begun in Section 6 of proving uniqueness in $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ (that is, Theorem 1.4(b)). As explained in Section 6, we do this by reversing the procedure in Section 5 that used the strong equation (SE) to produce a solution $(\hat{H}^1, \hat{H}^2)$ from a pair $(\tilde{H}^1, \tilde{H}^2)$ with the law $\breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}$ constructed in Section 4.

In Section 6 we took a solution $(\hat{H}^1, \hat{H}^2)$ of $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ and produced the pair $(\bar{H}^1, \bar{H}^2)$. We picture points of the form $(y, 0)$ in the support of $\bar{H}_t^i$ as corresponding to points y in the support of $\hat{H}_t^i$, and picture points of the form (y, δ_s) in the support of $\bar{H}_t^i$ as being "ghostly descendents" of the path y^s killed–off at times $s < t$. If we forget about the marks, then the totality of "real" and "ghost" particles form a pair (H^1, H^2) of independent historical Brownian motions.

The intuitive description presented in Section 1 says for a point (y, δ_s) in the support of $\bar{H}_t^i$ that, conditional on $(\hat{H}^1, \hat{H}^2)$, the killing time s is time of the first arrival of a Poisson process on $[\tau, \infty[$ with intensity $r_i \ell(y, \hat{H}^j)$, $j = 3 - i$. The strong equation construction (SE) of Section 5 dictates that s is the first time a Poisson process on $[\tau, \infty[\times[0,1]$ with intensity $r_i \ell(y, H^j) \otimes m$ has a mark (u, z) with $\lambda^i(u, y) > z$.

Remark 7.1 The prototype for the problem of constructing $(\tilde{H}^1, \tilde{H}^2)$ from $(\bar{H}^1, \bar{H}^2)$ is therefore the following. Suppose we have measures ξ and ζ on $\mathbb{R}_+$ with $\xi \leq \zeta$ and a random time S that has the distribution of the first arrival of a Poisson process on $\mathbb{R}_+$ with intensity ξ. How, by introducing extra randomness, do we construct a Poisson process on $\mathbb{R}_+ \times [0,1]$ with intensity $\zeta \otimes m$ such that S is the time of the first point (u, z) with $(d\xi/d\zeta)(u) > z$? The answer, of course, is that:

- Prior to S, we lay down an independent Poisson process on $\mathbb{R}_+$ with intensity $\xi(\cdot \cap [0, S[)$ and equip each point u in this process with an independent mark that is uniformly distributed on $[(d\xi/d\zeta)(u), 1]$.
- We equip the point S with an independent mark that is uniformly distributed on $[0, (d\xi/d\zeta)(S)[$.
- After S we lay down an independent Poisson process with intensity $\zeta(\cdot \cap]S, \infty[)$ and equip each point with an independent mark that is uniformly distributed on $[0, 1]$.

The reader should keep this prototype in mind as they follow the construction in this section.

Throughout this section, $\tau \geq 0$, $r_i > 0$ and $\mu^i \in \mathbf{M}_{FS}(C)^\tau$ will be fixed. The proof of the following main result of this section will be given after a number of preliminaries. We remark that Theorem 1.4(b) will be immediate from this result and Theorem 5.2(c).

Theorem 7.2 *Let $(\hat{H}^1, \hat{H}^2)$ be a solution of $\widehat{MP}_H(\tau, \mu^1, \mu^2)$. There is a filtered probability space carrying a pair of $\mathbf{M}_F(C \times \mathbf{M}_\#)$–valued processes $(\tilde{H}^1, \tilde{H}^2)$, as in Section 5, and a pair of processes with the same law as $(\hat{H}^1, \hat{H}^2)$, and which we still denote as $(\hat{H}^1, \hat{H}^2)$, such that $(\hat{H}^1, \hat{H}^2)$ solves (SE).*

We continue to work in the setting of the Section 6. We described above verbally what we did there. More precisely, we started with a solution $(\hat{H}^1, \hat{H}^2)$ of $\widehat{MP}_H(\tau, \mu^1\mu^2)$, and, by enlarging the probability space, we constructed a pair $(\bar{H}^1, \bar{H}^2)$ solving $\overline{MP}(\tau, \mu^1, \mu^2)$ such that

$$\hat{H}^i_t(\phi) = \int \phi(y)\mathbf{1}\{e = 0\}\, \bar{H}^i_t(d(y,e)) = \int \phi(y)\mathbf{1}\{e(]\tau, t]) = 0\}\, \bar{H}^i_t(d(y,e)). \tag{7.1}$$

Moreover, if we set

$$H^i_t(\phi) = \int \phi(y)\, \bar{H}^i_t(d(y,e)). \tag{7.2}$$

then (H^1, H^2) is a pair of independent historical Brownian motions.

As $\tau \geq 0$ is fixed and $\hat{H}^i, \tilde{H}^i$ will be Δ for $t < \tau$ we consider only $t \geq \tau$ in what follows. Recall the notation $\lambda^i(t,y)$ and $I^i(t,y,n,\omega)$ from Section 5.

Our first step will be to construct a pair of càdlàg processes $(\tilde{H}^{1,\epsilon}_t, \tilde{H}^{2,\epsilon}_t)_{t \geq \tau+\epsilon}$ with the same law as the processes $(\breve{H}^{1,\epsilon}_t, \breve{H}^{2,\epsilon}_t)_{t \geq \tau+\epsilon}$ under $\breve{\mathbb{P}}^{\tau, \breve{\mu}^1, \breve{\mu}^2}$. For $x = (y, n)$, set $x^u = (y^u, n^u)$. The processes $(\tilde{H}^{1,\epsilon}, \tilde{H}^{2,\epsilon})$ will have the property that if, for $t \geq \tau + \epsilon$ we set

$$\bar{H}^{i,\epsilon}_t(\phi) = \int \phi(x^{t-\epsilon})\, \bar{H}^i_t(dx),$$
$$H^{i,\epsilon}_t(\phi) = \int \phi(y^{t-\epsilon})\, H^i_t(dy) = \int \phi(y)\, \bar{H}^{i,\epsilon}_t(d(y,e)),$$

and

$$\hat{H}^{i,\epsilon}_t(\phi) = \int \phi(y)\mathbf{1}\{e = 0\}\, \bar{H}^{i,\epsilon}_t(d(y,e)) = \int \phi(y^{t-\epsilon})\mathbf{1}\{e(]\tau, t-\epsilon]) = 0\}\, \bar{H}^i_t(d(y,e)),$$

then for $\phi \in b\mathcal{C}$ we have almost surely, $\forall t \geq \tau + \epsilon$,

$$\hat{H}^{i,\epsilon}_t(\phi) = \int I^i(t,y,n)\phi(y)\, \tilde{H}^{i,\epsilon}_t(d(y,n)) \tag{SE$_\epsilon$}$$

and

$$H^{i,\epsilon}_t(\phi) = \int \phi(y) \tilde{H}^{i,\epsilon}_t(d(y,n)).$$

We stress that $\hat{H}^{i,\epsilon}$ is **NOT** defined by analogy with $H^{i,\epsilon}$. That is, $\hat{H}_t^{i,\epsilon}(\phi) \neq \int \phi(y^{t-\epsilon})\,\hat{H}_t^i(dy)$. Rather, $\hat{H}_t^{i,\epsilon}$ dominates (possibly strictly) the measure defined by the right-hand side.

Let us first look at the conditional distributions of $\breve{H}^{i,\epsilon}$ given (H^1, H^2) a little more closely. Take $i = 1$. Hence we are considering the conditional distribution of $\breve{H}^\epsilon$ given H under $\mathbb{Q}_\nu^{\tau,\breve{\mu}^1}$, where $\nu = r_1 H^2$. Here and in what follows we use the obvious analogues of the notation described above without the superscript i. As we remarked in Section 4, we are not strictly dealing with an historical process but most of the results in Section 3 of [9] still hold in our setting. Using these results (especially Theorem 3.9 of [9]), the construction, and Lemmas 6.9 and 6.10, we have the following. Under $\mathbb{Q}_\nu^{\tau,\breve{\mu}^1}$, $\breve{H}_{t+\epsilon}^\epsilon$ is purely atomic with a finite number of atoms for all $t \geq \tau$. These atoms evolve as a càdlàg branching particle system starting at $t = \tau$ with a Poisson random measure with intensity $\breve{\mu}^1/2\epsilon$. With rate $(2/\epsilon)$ each particle dies or splits into two with equal probability. Between branch times, the particles follow independent copies of the $C \times \mathbf{M}_\#$-valued IHP with laws $\{P_\nu^{s,y,n}\}$.

We can express the content of this observation as follows. Let S_t^ϵ and $\breve{S}_t^\epsilon$ denote, respectively, the supports of H_t^ϵ and $\breve{H}_t^\epsilon$. For each atom y in S_t^ϵ there is a unique atom $(y, n(t,y))$ in $\breve{S}_t^\epsilon$ and $\breve{H}_t^\epsilon$ assigns mass $H_t^\epsilon(\{y\})$ to $(y, n(t,y))$. That is,

$$\breve{H}_t^\epsilon = \sum\nolimits_{y \in S_t^\epsilon} \delta_{(y,n(t,y))} H_t^\epsilon(\{y\}).$$

For all $\tau + \epsilon \leq t_1 \leq t_2$, if $y_1 \in S_{t_1}^\epsilon$ and $y_2 \in S_{t_2}^\epsilon$ are such that $y_1^s = y_2^s$ for some $s \leq t_1$, then $(n(t_1, y_1))^s = (n(t_2, y_2))^s$. Consequently, if we write $\tau + \epsilon = v_0 < v_1 < v_2 < \ldots$ for the branch or death times of the particle system $(S_t^\epsilon)_{t \geq \tau+\epsilon}$, then for each atom $y \in S_{v_{k+1}-}^\epsilon$ there is a unique atom $(y, n_-(v_{k+1}, y)) = (y, \lim_{s \uparrow v_{k+1}} n(s, y^s)) \in \breve{S}_{v_{k+1}-}^\epsilon$; and, moreover, for $v_k \leq t < v_{k+1}$ each $(y', n(t,y')) \in \breve{S}_t^\epsilon$ is of the form $(y^{t-\epsilon}, n_-(v_{k+1}, y)^{t-\epsilon})$ for a unique $y \in S_{v_{k+1}-}^\epsilon$. Conditional on H^ϵ, the random measures

$$\left\{ n_-(v_{k+1}, y)(\cdot \cap [v_k - \epsilon, v_{k+1} - \epsilon[\times [0,1]) \,:\, k = 0, 1, \cdots,\ y \in S_{v_{k+1}-}^\epsilon \right\}$$

are independent, and for $y \in S_{v_{k+1}-}^\epsilon$ the random measure $n_-(v_{k+1}, y)$ restricted to $[v_k - \epsilon, v_{k+1} - \epsilon[\times [0,1]$ is a Poisson random measure with intensity $(\ell(y,\nu) \otimes m)(\cdot \cap [v_k - \epsilon, v_{k+1} - \epsilon[\times [0,1])$.

We claim that the conditional distribution of $\breve{H}^\epsilon$ given H has the same description. To see this, note that if $0 < \delta < \epsilon$, then $\breve{H}_t^\epsilon$ may be recovered from $\breve{H}_t^\delta$ by truncating the atoms of $\breve{H}_t^\delta$ at time $t - \epsilon$. Applying the above description of the conditional law of $\breve{H}^\delta$ given H^δ and truncating to get $\breve{H}^\epsilon$, we see that the conditional distribution of $\breve{H}^\epsilon$ given H^δ is the same as its conditional distribution given H^ϵ described above. Now let $\delta \downarrow 0$ and use a martingale convergence argument to see that the conditional distribution of $\breve{H}^\epsilon$ given H is also the same as its conditional distribution given H^ϵ described above.

We now turn to the definition of $(\tilde{H}^{1,\epsilon}, \tilde{H}^{2,\epsilon})$. Until further notice ϵ will be fixed, and we will sometimes use notation that does not explicitly record the dependence on ϵ of various objects we consider.

Let $\ell(H^{1,\epsilon}, H^2)(dt, dy)$ denote the random measure on $[\tau, \infty[\times C$ given by

$$\sum_{y \in S^{1,\epsilon}_{t+\epsilon}} \delta_y \ell(y, H^2)(dt).$$

More precisely, for all i,

$$\ell(H^{1,\epsilon}, H^2)\left((A \cap [v_i - \epsilon, v_{i+1} - \epsilon[) \times B\right) = \sum \left\{ \int_{A \cap [v_i - \epsilon, v_{i+1} - \epsilon[} \ell(y, H^2)(dt) \mathbf{1}_B(y) \; : \; y \in S^{1,\epsilon}_{v_{i+1}-} \right\}.$$

We will define $\ell(\bar{H}^{1,\epsilon}, H^2)$, $\ell(\bar{H}^{1,\epsilon}, \hat{H}^2)$, etc. in the analogous manner. (Lemma 6.9 shows that $\bar{H}^{1,\epsilon}$ is purely atomic and inherits the branching structure of $H^{1,\epsilon}$.)

Define random times $\tau + \epsilon = T^i_0 \leq T^i_1 \leq T^i_2 \leq \ldots$ by

$$T^i_{k+1} = \inf\left\{ t > T^i_k : \bar{H}^{i,\epsilon}_t(\{(y, e) \; : \; e(\{t - \epsilon\}) = 1\}) > 0 \right\}, \quad k = 0, 1, 2, \ldots.$$

The time $T^i_k - \epsilon$ is an instant at which a particle is killed off and then resurrected in the afterlife to have ghostly descendents that survive for at least another ϵ units of time.

Lemma 7.3 *(a) If $T^i_k < \infty$, then almost surely $T^i_k < T^i_{k+1}$.*
(b) Almost surely, the intersection of the set $\{T^i_k \; : \; T^i_k < \infty\}$ with any compact interval is finite.

Proof. In order to prove both parts, it suffices to show that $\mathbb{P}$-a.s. for all intervals $[a, b]$ with $\tau + \epsilon \leq a < b < a + \epsilon$ the set

$$\left\{ t \in [a, b] : \bar{H}^{i,\epsilon}_t(\{(y, e) \; : \; e(\{t - \epsilon\}) = 1\}) > 0 \right\}$$

is finite. However, it follows from Lemma 6.9(a) that for all such intervals $[a, b]$ and all $t \in [a, b]$

$$\bar{S}^{i,\epsilon}_t = \bar{S}^i_{t-\epsilon,t} \subseteq \bar{S}^i_{t-\epsilon,a} = \{(y^{t-\epsilon}, e^{t-\epsilon}) \; : \; (y, e) \in \bar{S}^i_{b-\epsilon,a}\}$$

and the last set is finite. As $e(\{s\}) > 0$ for at most one value of s for all (y, e) in $\bar{S}^i_{b-\epsilon,a}$, the finiteness of the above set of times is clear.

□

We may enlarge the filtered probability space on which $(\bar{H}^1, \bar{H}^2)$ is defined (but still denote it by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq \tau}, \mathbb{P})$ and still require that $(\mathcal{F}_t)_{t \geq \tau}$ is right–continuous) so that on it there are also defined simple point processes Π^1, Π^2, Ξ^1, Ξ^2 on $]\tau, \infty[\times C \times [0, 1]$ with the following properties.

(i) The random measures Π^1, Π^2, Ξ^1, Ξ^2 are conditionally independent given $(\bar{H}^1, \bar{H}^2)$.

(ii) Conditional on $(\bar{H}^1, \bar{H}^2)$, Π^i is a Poisson process with intensity

$$\begin{aligned} dt \times dy \times dz \mapsto & \mathbf{1}\{e(]\tau, t[) = 0\} \, (1 - \lambda^i(t,y))^{-1} \mathbf{1}\{z \in]\lambda^i(t,y), 1]\} \, m(dz) \\ & \times r_i \left[\ell(\bar{H}^{i,\epsilon}, H^j) - \ell(\bar{H}^{i,\epsilon}, \hat{H}^j) \right] (dt, dy, de) \\ & + \mathbf{1}\{e(]\tau, t[) = 1\} \, r_i \ell(\bar{H}^{i,\epsilon}, H^j)(dt, dy, de) \, m(dz), \end{aligned}$$

where $j = 3 - i$.

(iii) Conditional on $(\bar{H}^1, \bar{H}^2)$, Ξ^i is counting measure on the set of points $\{(T_k^i - \epsilon, Y_k^i, U_k^i) : T_k^i < \infty\}$, where Y_k^i is the unique $y \in C$ such that $(y, e) \in \bar{S}^{i,\epsilon}_{T_k^i}$ and $e(\{T_k^i - \epsilon\}) = 1$ (y is unique by Proposition 6.7(c)) and the $\{U_k^i\}$ are independent with U_k^i uniformly distributed on the interval $[0, \lambda^i(T_k^i - \epsilon, Y_k^i)]$.

(iv) For all $A \in \mathcal{C} \times \mathcal{B}([0,1])$ the processes $(\Pi^i(]\tau, t] \times A))_{t \geq \tau}$ and $(\Xi^i(]\tau, t] \times A))_{t \geq \tau}$ are $(\mathcal{F}_t)_{t \geq \tau}$–adapted.

Remark 7.4 It is easy to check (for example, using Laplace functionals) that the point processes Π^1, Π^2, Ξ^1, Ξ^2 restricted to $]\tau, t-\epsilon] \times C \times [0,1]$ are conditionally independent given $\sigma\{(\bar{H}_s^1, \bar{H}_s^2) : \tau \leq s \leq t\}$, and the description of the conditional laws is that given above, restricted to $]\tau, t - \epsilon] \times C \times [0,1]$.

In terms of the prototype described in Remark 7.1, the random measures Π^i play the role of both the pre-S and post-S marked Poisson processes and the random measures Ξ^i play the role of the marked point mass at S. The prototype construction involves point processes on $\mathbb{R}_+ \times [0,1]$, but we need instances of the prototype along the path of each particle. This extra "indexing" is accomplished by the artifice of building Π^i and Ξ^i as measures on $]\tau, \infty[\times C \times [0,1]$, with the C-valued component of an atom indicating to which path the $]\tau, \infty[\times [0,1]$-valued components should be attached. This idea is made precise below.

Set

$$\tilde{H}_t^{i,\epsilon}(\phi) = \int \phi(y, N^i(t,y)) \, H_t^{i,\epsilon}(dy),$$

where $N^i(t,y) \in \mathbf{M}_\#$ is given by

$$N^i(t,y)(A \times B) = \int_{A \cap]\tau, t-\epsilon]} \iint_B \mathbf{1}\{y'^s = y^s\} \left[\Pi^i(ds, dy', dz) + \Xi^i(ds, dy', dz) \right].$$

Lemma 7.5 *(a) The process $\tilde{H}^{i,\epsilon}$ is càdlàg.*
(b) The pair $(\tilde{H}^{1,\epsilon}, \tilde{H}^{2,\epsilon})$ satisfies (SE_ϵ).

Proof. (a) By construction, $\tilde{H}_t^{i,\epsilon}(\phi) = \int \phi(y, N^i(T_k^i, y^{T_k^i}))\, H_t^{i,\epsilon}(dy)$ for $T_k^i \le t < T_{k+1}^i$, and the claim follows from Remark 6.11.
(b) We must check for $\phi \in b\mathcal{C}$ that we have almost surely for all $t \ge \tau + \epsilon$

$$\hat{H}_t^{i,\epsilon}(\phi) = \int I^i(t, y, n)\phi(y)\, \tilde{H}_t^{i,\epsilon}(d(y,n)).$$

Moreover, by an argument similar to that in the proof of Corollary 3.6 of [30], both sides of the above are $(\mathcal{F}_t)_{t \ge \tau+\epsilon}$–predictable processes. It therefore suffices by the section theorem to check for each bounded $(\mathcal{F}_t)_{t \ge \tau+\epsilon}$–stopping time $T \ge \tau + \epsilon$ that

$$\hat{H}_T^{i,\epsilon}(\phi) = \int I^i(T, y, n)\phi(y)\, \tilde{H}_T^{i,\epsilon}(d(y,n)).$$

Consider $y \in S_T^{i,\epsilon}$. By Lemma 6.9(c) there is a unique $e \in \mathbf{M}_*$ such that $(y, e) \in \bar{S}_T^{i,\epsilon}$ and we have $\bar{H}_T^{i,\epsilon}(\{(y,e)\}) = H_T^{i,\epsilon}(\{y\})$. If $e([0, T-\epsilon]) = 1$, then $y \notin \hat{S}_T^{i,\epsilon}$ (see (7.1)). If $e([0, T-\epsilon]) = 0$, then $y \in \hat{S}_T^{i,\epsilon}$ and $\hat{H}_T^{i,\epsilon}(\{y\}) = \bar{H}_T^{i,\epsilon}(\{(y,e)\}) = H_T^{i,\epsilon}(\{y\})$. On the other hand, there is a unique $n \in \mathbf{M}_\#$ (namely, $N^i(T, y)$) such that $(y, n) \in \tilde{S}_T^{i,\epsilon}$ and we have $\tilde{H}_T^{i,\epsilon}(\{(y,n)\}) = H_T^{i,\epsilon}(\{y\})$. Hence, by (7.1) and (SE$_\epsilon$) we must prove that $e([0, T-\epsilon]) = 1$ if and only if $I^i(T, y, N^i(T, y)) = 0$.

Suppose that $e([0, T-\epsilon]) = 1$, so that $e(\{s-\epsilon\}) = 1$ for some $\tau + \epsilon \le s \le T$. By Lemma 6.9(a), we have that $(y^{s-\epsilon}, e^{s-\epsilon}) \in \bar{S}_s^{i,\epsilon}$. From the construction,

$$\Xi^i(\{s-\epsilon\} \times \{y^{s-\epsilon}\} \times [0, \lambda^i(s-\epsilon, y^{s-\epsilon})]) = 1$$

and hence

$$N^i(T, y)(\{s-\epsilon\} \times [0, \lambda^i(s-\epsilon, y^{s-\epsilon})]) = 1.$$

Therefore, $I^i(T, y, N^i(T, y)) = 0$.

Conversely, suppose that $e([0, T-\epsilon]) = 0$. By Lemma 6.9(a), we have $\{(y', e') \in \bar{S}_s^{i,\epsilon} : y' = y^{s-\epsilon}\} = \{(y^{s-\epsilon}, 0)\}$ for all $\tau + \epsilon \le s \le T$. By construction,

$$N^i(T, y)(\{(u, z) : \tau \le u \le T - \epsilon,\ z \le \lambda^i(u, y^u)\} = 0.$$

Therefore, $I^i(T, y, N^i(T, y)) = 1$.

$\square$

Lemma 7.6 *In order to check that $\tilde{H}^{1,\epsilon}$ and $\tilde{H}^{2,\epsilon}$ are conditionally independent given (H^1, H^2) and that the conditional law of $\tilde{H}^{i,\epsilon}$ given (H^1, H^2) is the same as the conditional law of $\breve{H}^{i,\epsilon}$ given (H^1, H^2), it suffices to verify for any $t \ge \tau + \epsilon$ and any bounded Borel function $F :]\tau, \infty[\times C \times [0,1] \to \mathbb{R}$, that the process*

$$s \mapsto \int \left(\int_{]\tau, s\wedge(t-\epsilon)]} \int_{[0,1]} F(u, y^u, z)\, \left[n(du, dz) - r_i \ell(y, H^j)(du) m(dz)\right] \right)\, \tilde{H}_t^{i,\epsilon}(d(y,n))$$

$(j = 3 - i)$ is a martingale with respect to the filtration $(\mathcal{H}^0 \vee \mathcal{P}_s)_{s \geq \tau}$, where $\mathcal{H}^0$ is the sub–σ–field generated by (H^1, H^2) and $\mathcal{P}_s$ is the sub–σ–field generated by the restriction of the point processes $(\Pi^1, \Pi^2, \Xi^2, \Xi^2)$ to $]\tau, s - \epsilon] \times C \times [0,1]$.

Proof. We will first show that the conditional law of $\tilde{H}^{i,\epsilon}$ given (H^1, H^2) is the same as the conditional law of $\breve{H}^{i,\epsilon}$ given (H^1, H^2). For the moment i will be fixed and we will use notation that does not record the dependence on i of some of the objects we introduce.

It is clear by construction that almost surely for all $t \geq \tau + \epsilon$ and for each atom y in $S_t^{i,\epsilon}$ there is a unique atom of the form (y, n) in $\tilde{S}_t^{i,\epsilon}$ (namely, $(y, N^i(t,y))$) and $\tilde{H}_t^{i,\epsilon}$ assigns mass $H_t^{i,\epsilon}(\{y\})$ to $(y, N^i(t,y))$. Moreover, almost surely for all $\tau + \epsilon \leq t_1 \leq t_2$, if $y_1 \in S_{t_1}^{i,\epsilon}$ and $y_2 \in S_{t_2}^{i,\epsilon}$ are such that $y_1^s = y_2^s$ for some s, then $(N^i(t_1, y_1))^s = (N^i(t_2, y_2))^s$.

Let P be the total number of branch or death times of the particle system $(S_t^{i,\epsilon})_{t \geq \tau+\epsilon}$. Define $\mathcal{H}^0$–measurable random variables $\tau + \epsilon = V_0 \leq V_1 \leq V_2 \leq \ldots$ by setting $V_1 < V_2 < \ldots < V_P$ to be the successive branch or death times and putting $V_j = V_P$ for $j > P$. Let $Q_p = \mathrm{card} S_{V_p-}^{i,\epsilon}$. Adjoin a fictitious point $\dagger$ to C to form $C^\dagger$. Construct $\mathcal{H}^0$–measurable $C^\dagger$–valued random variables $W_{p,q}$, $p \geq 1$, $q \geq 1$, such that for $1 \leq p \leq P$ the set $\{W_{p,q} \;:\; 1 \leq q \leq Q_p\} = S_{V_p-}^{i,\epsilon}$, and for either $1 \leq p \leq P$ and $q \geq Q_p + 1$, or $p \geq P + 1$ we have $W_{p,q} = \dagger$. (To construct these random variables one may use the Borel isomorphism between C and $[0,1]$ and then invoke the total order on the latter set to effectively list the atoms.)

Set

$$N_{p,q} = N^i(V_p, W_{p,q})(\cdot \cap [V_{p-1} - \epsilon, V_p - \epsilon[\times [0,1])$$

for $1 \leq p \leq P$, $q \leq Q_p$, and put $N_{p,q} = 0$ otherwise. We need to show that conditional on (H^1, H^2) the collection $\{N_{p,q}\}$ is independent, and that the conditional distribution of $N_{p,q}$ is that of a Poisson process on $\mathbb{R}_+ \times [0,1]$ with intensity $\Lambda_{p,q}$, where $\Lambda_{p,q} = (r_i \ell(W_{p,q}, H^j) \otimes m)(\cdot \cap [V_{p-1} - \epsilon, V_p - \epsilon[\times [0,1])$ for $1 \leq p \leq P$, $q \leq Q_p$, and $\Lambda_{p,q} = 0$ otherwise. Here as usual we assume $i \neq j$.

Consider a grid of constant times $\tau + \epsilon = t_0 < t_1 < t_2 < \ldots$ with $\lim_{h \to \infty} t_h = \infty$. It is clear from the above description of the branching particle system $(S_t^{i,\epsilon})_{t \geq \tau+\epsilon}$ that the distribution of each V_p, $p \geq 1$, is diffuse and hence the sets $\{t_h\}_{h=1}^\infty$ and $\{V_p\}_{p=1}^\infty$ are almost surely disjoint. Let

$$N_{h,p,q} = N_{p,q}(\cdot \cap [t_{h-1} - \epsilon, t_h - \epsilon[\times [0,1]) \mathbf{1}\{V_{p-1} \leq t_h < V_p\}$$

for $h \geq 1$, $p \geq 1$, $q \geq 1$. Observe that

$$N_{p,q}(\cdot \cap [0, \sup\{t_h : t_h < V_p\}[\times [0,1]) = \sum_{h=1}^{\infty} N_{h,p,q}.$$

By considering a sequence of such grids with successively finer mesh, it suffices to show that conditional on (H^1, H^2) the collection $\{N_{h,p,q}\}$ is independent, and that the conditional distribution of $N_{h,p,q}$ is that of a Poisson process with intensity $\Lambda_{h,p,q}$, where

$$\Lambda_{h,p,q} = \Lambda_{p,q}(\cdot \cap [t_{h-1} - \epsilon, t_h - \epsilon[\times[0,1])\mathbf{1}\{V_{p-1} \le t_h < V_p\}.$$

We will accomplish this using Proposition B.1 and Corollary B.2 from Appendix B.

It is clear from the construction that if $B_1, \ldots, B_K$ are disjoint Borel subsets of $[0,1]$, then for all h, p, q the sets $\{s : N_{h,p,q}(\{s\} \times B_k\}) \neq 0\}$, $1 \le k \le K$, are almost surely disjoint. It is also clear from the construction and Lemma 6.7(c) that for any h, p and and any $q \neq q'$ the sets $\{s : N_{h,p,q}(\{s\} \times [0,1]\}) \neq 0\}$ and $\{s : N_{h,p,q'}(\{s\} \times [0,1]\}) \neq 0\}$ are almost surely disjoint. Trivially, for any $(h,p) \neq (h',p')$ and any q, q' the sets $\{s : N_{h,p,q}(\{s\} \times [0,1]\}) \neq 0\}$ and $\{s : N_{h',p',q'}(\{s\} \times [0,1]\}) \neq 0\}$ are almost surely disjoint.

Put $X_{h,p,q} = H_{t_h}^{i,\epsilon}(\{(W_{p,q})^{t_h - \epsilon}\})$ for $1 \le p \le P$, $q \le Q_p$ and $V_{p-1} \le t_h < V_p$. Set $X_{h,p,q} = 1$ otherwise. In order to apply Proposition B.1 and Corollary B.2, it suffices to show that

$$\mathbb{P}\left[X_{h,p,q}\Lambda_{h,p,q}(]\tau, s] \times [0,1])\right] < \infty$$

for all $s \ge \tau$ and that for any $B \in \mathcal{B}([0,1])$ the process

$$s \mapsto X_{h,p,q}(N_{h,p,q}(]\tau, s] \times B) - \Lambda_{h,p,q}(]\tau, s] \times B))$$

is a $(\mathcal{H}^0 \vee \mathcal{P}_s)_{s \ge \tau}$–martingale.

Note that

$$\begin{aligned}
\mathbb{P}\left[X_{h,p,q}\Lambda_{h,p,q}(]\tau, s] \times [0,1])\right] &\le \mathbb{P}\left[\sum_{r=1}^{\infty} X_{h,p,r}\Lambda_{h,p,r}(]\tau, t_h - \epsilon] \times [0,1])\right] \\
&= \mathbb{P}\left[\sum_{r=1}^{Q_p} H_{t_h}^{i,\epsilon}(\{(W_{p,r})^{t_h-\epsilon}\})\mathbf{1}\{V_{p-1} \le t_h < V_p\} r_i \ell_{t_h - \epsilon}(W_{p,r}, H^j)\right] \\
&= \mathbb{P}\left[\sum_{r=1}^{Q_p} H_{t_h}^{i,\epsilon}(\{(W_{p,r})^{t_h-\epsilon}\})\mathbf{1}\{V_{p-1} \le t_h < V_p\} r_i \ell_{t_h - \epsilon}((W_{p,r})^{t_h-\epsilon}, H^j)\right] \\
&\le \mathbb{P}\left[\int r_i \ell_{t_h}(y, H^j)\, H_{t_h}^i(dy)\right] \\
&< \infty,
\end{aligned}$$

by Theorem 3.7(c).

A monotone class argument starting with the hypothesis of the lemma gives that if $F :]\tau, \infty[\times C \times$

$[0,1] \times \Omega \to \mathbb{R}$ is bounded and $\mathcal{B}(]\tau,\infty[) \times \mathcal{C} \times \mathcal{B}([0,1]) \times \mathcal{H}^0$-measurable, then the process

$$\begin{aligned} s \mapsto & \int \left(\int_{]\tau, s\wedge(t_h-\epsilon)]} \int_{[0,1]} F(u, y^u, z) \, \left[n(du,dz) - r_i \ell(y, H^j)(du) m(dz) \right] \right) \tilde{H}^{i,\epsilon}_{t_h}(d(y,n)) \\ &= \sum \Big\{ H^{i,\epsilon}_{t_h}(\{y\}) \Big(\int_{]\tau, s\wedge(t_h-\epsilon)]} \int_{[0,1]} F(u, y^u, z) \\ &\quad \times \left[N^i(t_h, y)(du,dz) - r_i \ell(y, H^j)(du) m(dz) \right] \Big) : y \in S^{i,\epsilon}_{t_h} \Big\} \end{aligned}$$

is a martingale with respect to the filtration $(\mathcal{H}^0 \vee \mathcal{P}_s)_{s\geq\tau}$. Apply this with

$$\begin{aligned} F(u,y,z) = & \mathbf{1}\{u \in [V_{p-1} - \epsilon, V_p - \epsilon[\cap [t_{h-1} - \epsilon, t_h - \epsilon[\} \\ & \times \mathbf{1}\{y = (W_{p,q})^{t_h - \epsilon}\} \, \mathbf{1}_B(z) \, \mathbf{1}\{V_{p-1} \leq t_h < V_p\} \end{aligned}$$

to complete the proof that the conditional distribution of $\tilde{H}^{i,\epsilon}$ given (H^1, H^2) is the same as the conditional law of $\breve{H}^{1,\epsilon}$ given (H^1, H^2).

An elaboration of the above argument also handles the conditional independence of $(\tilde{H}^{1,\epsilon}, \tilde{H}^{2,\epsilon})$ given (H^1, H^2). The only new point to check is that if two point processes are constructed from $\tilde{H}^{1,\epsilon}$ and $\tilde{H}^{2,\epsilon}$ in the same manner as each of the $N_{h,p,q}$ above were defined, then the two point processes don't have atoms occurring at the same time. However, this is clear from the construction and Proposition 6.7(a). We leave the details to the reader.

□

We now proceed to verify the condition of Lemma 7.6. It suffices to consider F of the form $F(u,y,z) = f(u,y)g(z)$, where g and f are bounded Borel. We wish to show that for $t \geq \tau + \epsilon$ and $\tau + \epsilon < r < s$ we have

$$\mathbb{P}\left[\tilde{H}^{i,\epsilon}_t(A^i_t) \mid \mathcal{H}^0 \vee \mathcal{P}_r \right] = 0, \tag{7.3}$$

where

$$A^i_t(y,n) = \int_{]r\wedge(t-\epsilon), s\wedge(t-\epsilon)]} \int_{[0,1]} f(u, y^u) g(z) \, \left[n(du,dz) - r_i \ell(y, H^j)(du) m(dz) \right] \quad (j = 3-i).$$

We may assume that $s \leq t - \epsilon$. As a first step in showing equation (7.3) we will consider

$$\mathbb{P}\left[\tilde{H}^{i,\epsilon}_t(A^i_t) \mid \mathcal{H}^0 \vee \bar{\mathcal{H}}_t \vee \mathcal{P}_r \right],$$

where $\bar{\mathcal{H}}_v = \sigma\{(\bar{H}^1_u, \bar{H}^2_u) : \tau \leq u \leq v\}$, $v \geq \tau$.

Now, by a slight extension of Theorem 4.7 of [21], any bounded $\mathcal{H}^0$–measurable random variable, Γ can be written as

$$\Gamma = \mathbb{P}[\Gamma] + \int_\tau^\infty \int \Phi^1 \, dM^1 + \int_\tau^\infty \int \Phi^2 \, dM^2$$

for a suitable pair of $(\mathcal{C} \times \mathcal{H}_u^0)_{u \geq \tau}$–predictable integrands Φ^1, Φ^2, where

$$\mathcal{H}_v^0 = \sigma\{(H_u^1, H_u^2) : \tau \leq u \leq v\}, \quad v \geq \tau.$$

Now $v \mapsto \int_\tau^v \int \Phi^i \, dM^i$, $v \geq \tau$, is a martingale with respect to the filtration $(\bar{\mathcal{H}}_v)_{v \geq \tau}$ (recall that (H^1, H^2) are independent historical Brownian motions with respect to this filtration by Corollary 6.5, and so, by Remark 7.4, it follows that this stochastic integral is also a martingale with respect to the filtration $(\bar{\mathcal{H}}_v \vee \mathcal{P}_v)_{v \geq \tau}$. We can use this fact to conclude (cf. the proof of Theorem 4.9 of [20]) that

$$\mathbb{P}\left[\tilde{H}_t^{i,\epsilon}(A_t^i) \mid \mathcal{H}^0 \vee \bar{\mathcal{H}}_t \vee \mathcal{P}_r\right] = \mathbb{P}\left[\tilde{H}_t^{i,\epsilon}(A_t^i) \mid \mathcal{H}_t^0 \vee \bar{\mathcal{H}}_t \vee \mathcal{P}_r\right] = \mathbb{P}\left[\tilde{H}_t^{i,\epsilon}(A_t^i) \mid \bar{\mathcal{H}}_t \vee \mathcal{P}_r\right].$$

By the definition of $(\Pi^1, \Pi^2, \Xi^1, \Xi^2)$ and some thought, it follows that

$$\begin{aligned}
&\mathbb{P}\left[\tilde{H}_t^{i,\epsilon}(A_t^i) \mid \bar{\mathcal{H}}_t \vee \mathcal{P}_r\right] \\
&= \int \Big[\int_{]r \wedge (t-\epsilon), s \wedge (t-\epsilon)]} f(u, y^u) \left\{ \int_{]\lambda^i(u,y),1]} g(z) \, m(dz) \right\} \mathbf{1}\{e(]\tau, u]) = 0\} r_i \ell(y, H^j)(du) \\
&\quad + \int_{]r \wedge (t-\epsilon), s \wedge (t-\epsilon)]} f(u, y^u) \left\{ \int_{[0,\lambda^i(u,y)]} g(z) \, m(dz) \right\} \lambda^i(u,y)^{-1} e(du) \\
&\quad + \int_{]r \wedge (t-\epsilon), s \wedge (t-\epsilon)]} f(u, y^u) \left\{ \int_{[0,1]} g(z) \, m(dz) \right\} \mathbf{1}\{e(]\tau, u]) = 1\} r_i \ell(y, H^j)(du) \\
&\quad - \int_{]r \wedge (t-\epsilon), s \wedge (t-\epsilon)]} f(u, y^u) \left\{ \int_{[0,1]} g(z) \, m(dz) \right\} r_i \ell(y, H^j)(du) \Big] \, \bar{H}_t^i(d(y,e)) \\
&= \int \Big(\int_{]\tau, (t-\epsilon)]} \mathbf{1}\{r < u \leq s\} f(u, y^u) \left\{ \int_{[0,\lambda^i(u,y)]} g(z) \, m(dz) \right\} \\
&\quad \times \lambda^i(u,y)^{-1} \left[e(du) - \mathbf{1}\{e(]\tau, u]) = 0\} r_i \ell(y, \hat{H}^j)(du) \right] \Big) \, \bar{H}_t^i(d(y,e)) \\
&= \bar{H}_t^i(B_{t-\epsilon}^i),
\end{aligned}$$

say.

To recapitulate, we want to establish equation (7.3), that is, that

$$\mathbb{P}\left[\tilde{H}_t^{i,\epsilon}(A_t^i) \mid \mathcal{H}^0 \vee \mathcal{P}_r\right] = 0;$$

and we have shown that

$$\mathbb{P}\left[\tilde{H}_t^{i,\epsilon}(A_t^i) \mid \mathcal{H}^0 \vee \bar{\mathcal{H}}_t \vee \mathcal{P}_r\right] = \bar{H}_t^i(B_{t-\epsilon}^i).$$

It will therefore certainly suffice to prove that

$$\mathbb{P}\left[\bar{H}_t^i(B_{t-\epsilon}^i) \mid \mathcal{H}^0 \vee \bar{\mathcal{H}}_r \vee \mathcal{P}_r\right] = 0.$$

Using the stochastic integral representation of bounded, $\mathcal{H}^0$–measurable random variables and the definition of $(\Pi^1, \Pi^2, \Xi^1, \Xi^2)$ as above, we find that

$$\mathbb{P}\left[\bar{H}_t^i(B_{t-\epsilon}^i) \mid \mathcal{H}^0 \vee \bar{\mathcal{H}}_r \vee \mathcal{P}_r\right] = \mathbb{P}\left[\bar{H}_t^i(B_{t-\epsilon}^i) \mid \mathcal{H}_t^0 \vee \bar{\mathcal{H}}_r \vee \mathcal{P}_r\right] = \mathbb{P}\left[\bar{H}_t^i(B_{t-\epsilon}^i) \mid \mathcal{H}_t^0 \vee \bar{\mathcal{H}}_r\right].$$

Thus, in order to obtain equation (7.3), it suffices (by an easy monotone convergence argument) to verify for each $K > 0$ that that

$$\mathbb{P}\left[\bar{H}_t^i(B_{(t-\epsilon)\wedge S_K^i}^i) \mid \mathcal{H}_t^0 \vee \bar{\mathcal{H}}_r\right] = 0, \tag{7.4}$$

where $S_K^i(y, e, \omega) = \inf\{u \geq \tau : \ell_u(y, \hat{H}^j)(\omega) > K\}$. Equation (7.4) is equivalent to Lemma 7.8 below, which in turn will be a fairly immediate consequence of the following.

Lemma 7.7 *Suppose that $\phi, f_1, \ldots, f_\alpha, g_1, \ldots, g_\beta \in bp\mathcal{C}$, $\tau \leq u_1 \leq \ldots \leq u_\alpha \leq t$, $\tau \leq v_1 \leq \ldots \leq v_\beta \leq t$, and $\bar{\Gamma} \in bp\bar{\mathcal{H}}_r$. Then*

$$\mathbb{P}\left[\int B_{(t-\epsilon)\wedge S_K^i}^i(y, e)\phi(y)\bar{H}_t^i(d(y, e)) \prod_{a=1}^{\alpha} H_{u_a}^i(f_a) \prod_{b=1}^{\beta} H_{v_b}^j(g_b)\bar{\Gamma}\right] = 0.$$

Proof. We first comment that $\int B_{(t-\epsilon)\wedge S_K^i}^i(y, e)\phi(y)\bar{H}_t^i(d(y, e))$, $H_{u_a}^i(f_a)$ and $H_{v_b}^j(g_b)$ have finite moments of all orders, so the expectation in the statement of the claim is well-defined.

We will proceed by induction on α and β. Consider first the case $\alpha = \beta = 0$. By Theorem 2.6 of [30] and Itô's representation of square–integrable Brownian functionals as the sum of a constant and a stochastic integral, we have $\phi(y) = \gamma + \int_\tau^t \psi(w, y) \cdot dy_w$, $\bar{\mathbb{P}}_t^{\bar{H}^i}$-a.e. (y, e, ω), for some constant γ and some square–integrable $(\mathcal{C}_w \times \bar{\mathcal{H}}_w)_{w \geq \tau}$–predictable integrand ψ. Set

$$\begin{aligned}\tilde{B}_\sigma^i(y, e) = &\int_{]\tau,\sigma]} \mathbf{1}\{r < u \leq s \wedge S_K^i \wedge (t-\epsilon)\} f(u, y^u) \left\{\int_{[0,\lambda^i(u,y)]} g(z)m(dz)\right\} \\ &\times \lambda^i(u, y)^{-1}\left[e(du) - \mathbf{1}\{e(]\tau, u]) = 0\} r_i \ell(y, \hat{H}^j)(du)\right].\end{aligned}$$

By Corollary 6.6(c), for $\sigma \in [r, t]$,

$$\sigma \mapsto \int \tilde{B}_\sigma^i(y, e)\{\gamma + \int_\tau^\sigma \psi(w, y)dy_w\}\, \bar{H}_\sigma^i(d(y, e))$$

is an $(\bar{\mathcal{H}}_\sigma)$–martingale that is null at $\sigma = r$. To see that ψ satisfies the hypotheses of Corollary 6.6(c) note that

$$\mathbb{P}\left[\int_\tau^t H_s(|\psi_s|^2)ds\right] = \mathbb{P}\left[\int_\tau^t |\psi_s(W)|^2 ds\right] = \mathbb{P}\left[(\phi(W^t) - \gamma)^2\right] < \infty,$$

where W is a Brownian motion with the appropriate initial measure at time τ. Thus, since $\tilde{B}^i_t = B^i_{(t-\epsilon)\wedge S^i_K}$, we have

$$\mathbb{P}\left[\int B^i_{(t-\epsilon)\wedge S^i_K}(y,e)\phi(y)\bar{H}^i_t(d(y,e))\bar{\Gamma}\right] = \mathbb{P}\left[\int \tilde{B}^i_r(y,e)\{\gamma + \int_\tau^r \psi(w,y)\cdot dy_w\}\bar{H}^i_r(d(y,e))\bar{\Gamma}\right] = 0.$$

In order to completely describe the verification of the inductive step, we would need to introduce a significant amount of notation and write down some rather lengthy expressions. Instead, we will show how the two cases $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$ follow from the case $\alpha = \beta = 0$ and just sketch the proof of the general inductive step. Consider first $\alpha = 1$, $\beta = 0$. Arguing as in the previous paragraph, we have $f_1(y) = c_1 + \int_\tau^{u_1} F_1(w,y)\cdot dy_w$, $\bar{\mathbb{P}}^{H^i}_{u_1}$-a.s., for some constant c_1 and some stochastic integrand F_1, and

$$\sigma \mapsto \int \{c_1 + \int_\tau^{\sigma\wedge u_1} F_1(w,y)\cdot dy_w\} H^i_{\sigma\wedge u_1}(dy)$$

is a $(\bar{\mathcal{H}}_\sigma)_{\sigma\geq\tau}$–martingale. Moreover, by Corollary 6.6,

$$\begin{aligned}
&\langle \int \tilde{B}^i_\cdot(y,e)\{\gamma + \int_\tau^{\cdot\wedge t} \psi(w,y)\cdot dy_w\}\, \bar{H}^i_{\cdot\wedge t}(d(y,e)), \int \{c_1 + \int_\tau^{\cdot\wedge u_1} F_1(w,y)\cdot dy_w\} H^i_{\cdot\wedge u_1}(dy)\rangle_\sigma \\
&= \int_r^{\sigma\wedge u_1} \int \tilde{B}^i_\rho(y,e)\{\gamma + \int_\tau^\rho \psi(w,y)\cdot dy_w\}\{c_1 + \int_\tau^\rho F_1(w,y)\cdot dy_w\}\, \bar{H}^i_\rho(d(y,e))\, d\rho.
\end{aligned}$$

Thus, by integration by parts, we have

$$\begin{aligned}
&\mathbb{P}\left[\int B^i_{(t-\epsilon)\wedge S^i_K}(y,e)\phi(y)\bar{H}^i_t(d(y,e)) H^i_{u_1}(f_1)\bar{\Gamma}\right] \\
&= \mathbb{P}\left[\int \tilde{B}^i_r(y,e)\{\gamma + \int_\tau^r \psi(w,y)\cdot dy_w\}\bar{H}^i_r(d(y,e)) \int \{c_1 + \int_\tau^{r\wedge u_1} F_1(w,y)\cdot dy_w\} H^i_{r\wedge u_1}(dy)\bar{\Gamma}\right] \\
&\quad + \mathbb{P}\left[\int_r^{u_1} \int \left(\tilde{B}^i_\rho(y,e)\{\gamma + \int_\tau^\rho \psi(w,y)\cdot dy_w\}\{c_1 + \int_\tau^\rho F_1(w,y)\cdot dy_w\}\right) \bar{H}^i_\rho(d(y,e))\, d\rho\bar{\Gamma}\right] \\
&= \int_r^{u_1} \mathbb{P}\left[\int \tilde{B}^i_\rho(y,e)\tilde{\phi}(\rho,y)\bar{H}^i_\rho(d(y,e))\bar{\Gamma}\right] d\rho,
\end{aligned}$$

where $\tilde{\phi}(\rho, y)$ is the product of the stochastic integrals up to time ρ. The boundedness of ϕ and the fact that y is a Brownian motion stopped at ρ under $\bar{\mathbb{P}}^{\bar{H}^i}_\rho$ for any $\rho \geq 0$ shows that

$$|\gamma + \int_\tau^\rho \psi(w,y)dy(w)| \leq \|\phi\|_\infty \quad \bar{\mathbb{P}}^{\bar{H}^i}_\rho - a.e.$$

and so $\tilde{\phi}$ is bounded (by a symmetric argument for f_1). Hence we may apply the $\alpha = \beta = 0$ case to see the above expectation is zero.

Now consider the case $\alpha = 0$, $\beta = 1$. We have $g_1(y) = d_1 + \int_\tau^{v_1} G_1(w,y) \cdot dy_w$, $\bar{\mathbb{P}}_{v_1}^{H^j}$ almost surely, for some constant d_1 and some stochastic integrand G_1, and

$$\sigma \mapsto \int \{d_1 + \int_\tau^{\sigma \wedge v_1} G_1(w,y) \cdot dy_w\} H^j_{\sigma \wedge v_1}(dy)$$

is a $(\bar{\mathcal{H}}_\sigma)_{\sigma \geq \tau}$–martingale. Moreover, since $i \neq j$,

$$\langle \int \tilde{B}^i_\cdot(y,e)\{\gamma + \int_\tau^{\cdot \wedge t} \psi(w,y) \cdot dy_w\} \, \bar{H}^i_{\cdot \wedge t}(d(y,e)), \int \{d_1 + \int_\tau^{\cdot \wedge v_1} G_1(w,y) \cdot dy_w\} H^j_{\cdot \wedge u_1}(dy) \rangle_\sigma = 0.$$

Thus, as above we have

$$\begin{aligned}
&\mathbb{P}\left[\int B^i_{(t-\epsilon)\wedge S^i_K}(y,e)\phi(y)\bar{H}^i_t(d(y,e))H^j_{u_1}(g_1)\bar{\Gamma}\right] \\
&= \mathbb{P}\left[\int \tilde{B}^i_r(y,e)\{\gamma + \int_\tau^r \psi(w,y) \cdot dy_w\}\bar{H}^i_r(d(y,e)) \int \{d_1 + \int_\tau^{r \wedge v_1} G_1(w,y) \cdot dy_w\} H^j_{r \wedge v_1}(dy)\bar{\Gamma}\right] \\
&= 0.
\end{aligned}$$

For general α, β we can follow a similar strategy to reduce to earlier levels in the inductive hierarchy. We first write $\int B^i_{(t-\epsilon)\wedge S^i_K}(y,e)\phi(y)\bar{H}^i_t(d(y,e))$ and each of the terms $H^i_{u_a}(f_a)$ and $H^j_{v_b}(g_b)$ as the value at t of a certain martingale. We observe that, as in the proof of the case $\alpha = 1$, $\beta = 0$, the covariation of the martingale associated with $\int B^i_{(t-\epsilon)\wedge S^i_K}(y,e)\phi(y)\bar{H}^i_t(d(y,e))$ and the martingale associated with $H^i_{u_a}(f_a)$ is a time integral of $\bar{H}^i$ integrals of the special form we are considering in the statement of the claim. The covariation of the martingale associated with $H^i_{u_a}(f_a)$ and the martingale associated with $H^i_{u_{a'}}(f_{a'})$ is a time integral of H^i integrals. Here, as for the $\alpha = 1, \beta = 0$ case, we again use the fact that the stochastic integrals up to any earlier time ρ are essentially bounded. Similarly, the covariation of the martingale associated with $H^j_{v_b}(g_b)$ and the martingale associated with $H^j_{v_{b'}}(g_{b'})$ is a time integral of H^j integrals. All other covariations between the associated martingales are 0. We now use Itô's lemma to represent the product of these associated martingales as a martingale that is null at r plus a sum of terms arising from the covariations. From our observations about the covariations, each of these latter terms will be the time integral of the product of a $\bar{H}^i$ integral of the special form we are considering in the statement of the claim with a collection of H^i and H^j integrals that number $\alpha + \beta - 1$ in total, and so we may apply the induction hypothesis to complete the argument.

$\square$

Lemma 7.8 *Suppose that* $\Gamma \in b\mathcal{H}^0_t$ *and* $\bar{\Gamma} \in bp\bar{\mathcal{H}}_r$. *Then* $\mathbb{P}\left[\bar{H}^i_t(B^i_{(t-\epsilon)\wedge S^i_K})\Gamma\bar{\Gamma}\right] = 0.$

Proof. Let $f_1, \ldots, f_\alpha$, $g_1, \ldots, g_\beta$, $u_1, \ldots, u_\alpha$, $v_1, \ldots, v_\beta$ be as in the statement of Lemma 7.7. As $(H_t^i(1))_{t \geq \tau}$ is a continuous state branching process, $\mathbb{P}\left[\exp(\chi H_{u_a}^i(f_a))\right] < \infty$ and $\mathbb{P}\left[\exp(\chi H_{v_b}^j(g_b))\right] < \infty$ for χ in a neighbourhood of 0, $1 \leq a \leq \alpha$, $1 \leq b \leq \beta$. We conclude from Hölder's inequality and Lemma 7.7 (upon expanding the exponentials in a multivariable Taylor series) that the expectations

$$\mathbb{P}\left[\exp(\sum_a \xi_a H_{u_a}^i + \sum_b \zeta_b H_{v_b}^j) \bar{H}_t^i(B_{(t-\epsilon)\wedge S_K^i}^i)^+ \bar{\Gamma}\right]$$

and

$$\mathbb{P}\left[\exp(\sum_a \xi_a H_{u_a}^i + \sum_b \zeta_b H_{v_b}^j) \bar{H}_t^i(B_{(t-\epsilon)\wedge S_K^i}^i)^- \bar{\Gamma}\right]$$

exist and are equal for $(\xi, \zeta) \in \mathbb{R}^{\alpha+\beta}$ in a neighbourhood of 0.

The uniqueness theorem for moment generating functions gives that

$$\mathbb{P}\left[\Lambda((H_{u_a}^i)_{a=1}^\alpha, (H_{v_b}^j)_{b=1}^\beta) \bar{H}_t^i(B_{(t-\epsilon)\wedge S_K^i}^i)^+ \bar{\Gamma}\right] = \mathbb{P}\left[\Lambda((H_{u_a}^i)_{a=1}^\alpha, (H_{v_b}^j)_{b=1}^\beta) \bar{H}_t^i(B_{(t-\epsilon)\wedge S_K^i}^i)^- \bar{\Gamma}\right]$$

for all bounded, Borel functions $\Lambda : \mathbb{R}^{\alpha+\beta} \to \mathbb{R}$.

A monotone class argument now establishes that

$$\mathbb{P}\left[\Gamma \bar{H}_t^i(B_t^i)^+ \bar{\Gamma}\right] = \mathbb{P}\left[\Gamma \bar{H}_t^i(B_t^i)^- \bar{\Gamma}\right],$$

as required.

$\square$

Proof of Theorem 7.2 We can now complete the construction of $(\tilde{H}^1, \tilde{H}^2)$ and the verify that this pair has the same law as $(\breve{H}^1, \breve{H}^2)$ and satisfies (SE).

Observe that if $\epsilon' > \epsilon$, then

$$\breve{H}_t^{i,\epsilon'}(\phi) = \int \phi(x^{t-\epsilon'})\, \breve{H}_t^{i,\epsilon}(dx), \quad t \geq \tau + \epsilon'.$$

Therefore, if for some integer $k \geq 0$ we start with $(\bar{H}^1, \bar{H}^2)$ and build $(\tilde{H}^{1,2^{-k}}, \tilde{H}^{2,2^{-k}})$ by the above construction and set (with a slight abuse of notation)

$$\tilde{H}_t^{i,2^{-h}}(\phi) = \int \phi(x^{t-2^{-h}})\, \tilde{H}_t^{i,2^{-k}}(dx), \quad t \geq \tau + 2^{-h},$$

for $0 \leq h < k$, then $(\tilde{H}^{1,2^{-h}}, \tilde{H}^{2,2^{-h}})_{h=0}^k$ has the same joint law as $(\breve{H}^{1,2^{-h}}, \breve{H}^{2,2^{-h}})_{h=0}^k$. Moreover, with this new use of notation we still have

$$\hat{H}_t^{i,2^{-h}}(\phi) \equiv \int \phi(y^{2^{-h}}) \mathbf{1}\{e(]\tau, t-2^{-h}]) = 0\} \bar{H}_t^i(d(y,e)) = \int I^i(t,y,n)\phi(y)\, \tilde{H}_t^{i,2^{-h}}(d(y,n)) \quad (\mathrm{SE}_h)$$

and

$$H_t^{i,2^{-h}}(\phi) = \int \phi(y)\, \tilde{H}_t^{i,2^{-h}}(d(y,n))$$

for $0 \le h \le k$.

Applying Kolmogorov's extension theorem (and another slight abuse of notation) we can construct on some probability space a pair with the same law as $(\bar{H}^1, \bar{H}^2)$ (we also call this pair $(\bar{H}^1, \bar{H}^2)$) and a sequence of pairs which we denote by $(\tilde{H}^{1,2^{-h}}, \tilde{H}^{2,2^{-h}})_{h=0}^{\infty}$ such that $(\bar{H}^1, \bar{H}^2, (\tilde{H}^{1,2^{-h}}, \tilde{H}^{2,2^{-h}})_{h=0}^{k})$ has the same joint law as $(\bar{H}^1, \bar{H}^2, (\breve{H}^{1,2^{-h}}, \breve{H}^{2,2^{-h}})_{h=0}^{k})$ for every k. Moreover, (SE$_h$) is satisfied for $0 \le h < \infty$.

It follows from Lemma 6.10 that, on the same probability space that $(\bar{H}^1, \bar{H}^2)$ and $(\tilde{H}^{1,2^{-h}}, \tilde{H}^{2,2^{-h}})_{h=0}^{\infty}$ are defined, there is a process $(\tilde{H}_t^1, \tilde{H}_t^2)_{t \ge \tau}$ with the same law as $(\breve{H}_t^1, \breve{H}_t^2)_{t \ge \tau}$ such that as $h \to \infty$, $(\tilde{H}^{1,2^{-h}}, \tilde{H}^{2,2^{-h}})$ almost surely converges uniformly on compact intervals of $]\tau, \infty[$ to $(\tilde{H}^1, \tilde{H}^2)$. Moreover,

$$\tilde{H}_t^{i,2^{-h}}(\phi) = \int \phi(x^{t-2^{-h}})\, \tilde{H}_t^i(dx), \quad t \ge \tau + 2^{-h},$$

for $0 \le h < \infty$.

Thus

$$\begin{aligned}\int \phi(y^{2^{-h}}) \mathbf{1}\{e(]\tau, t-2^{-h}]) = 0\}\, \bar{H}_t^i(d(y,e)) &= \hat{H}_t^{i,2^{-h}}(\phi) \\ &= \int I^i(t, y^{2^{-h}}, n^{2^{-h}}) \phi(y^{2^{-h}})\, \tilde{H}_t^i(d(y,n))\end{aligned}$$

and

$$H_t^{i,2^{-h}}(\phi) = \int \phi(y^{t-2^{-h}})\, \tilde{H}_t^i(dy, dn)$$

for $t \ge \tau + 2^{-h}$ and $0 \le h < \infty$ a.s. Letting $h \to \infty$ and using Lemma 4.12, we see that if ϕ is continuous then, almost surely for all $t > \tau$,

$$\hat{H}_t^i(\phi) = \int \phi(y) \mathbf{1}\{e(]\tau, t]) = 0\}\, \bar{H}_t^i(d(y,e)) = \int I^i(t, y, n) \phi(y)\, \tilde{H}_t^i(d(y,n))$$

and

$$H_t^i(\phi) = \int \phi(y)\, \tilde{H}_t^i(d(y,n)).$$

These equations are trivial for $t = \tau$ Therefore (SE) holds for such ϕ. A monotone class argument shows that (SE) holds for all $\phi \in b\mathcal{C}$.

□

Proof of Theorem 1.6 It suffices to prove part (b). Let $\epsilon_n \downarrow 0$ and assume $(\hat{H}^{1,n}, \hat{H}^{2,n})$ satisfy $\widehat{MP}_H^{\epsilon_n}(0, \nu^1, \nu^2)$. As in [2] or Section 6 we may assume there are a pair of independent historical Brownian motions $(H^{1,n}, H^{2,n})$ with law $\mathbb{Q}^{0,\nu^1} \times \mathbb{Q}^{0,\nu^2}$ such that $\hat{H}^{i,n} \leq H^{i,n}$ a.s. The tightness of the laws of $\{(\hat{H}^{1,n}, \hat{H}^{2,n}) \ : \ n \in \mathbb{N}\}$ on $\Omega_H \times \Omega_H$ now follows by making minor changes in the proof of Theorem 3.6 of [20] (note that the required compact containment condition is immediate from the above domination by historical Brownian motion). The uniformity in Theorem 3.13(a) also allows us to trivially modify the argument in Theorem 3.6 of [20] to see that every weak limit point of $\{(\hat{H}^{1,n}, \hat{H}^{2,n} : n \in \mathbb{N}\}$ solves $\widehat{MP}_H(0, \nu^1, \nu^2)$. Theorem 1.4(b) now completes the proof of (b).

8 Markov property

In this section we complete the proof of Theorem 1.4 by establishing part (c) of that result and derive the corresponding result for solutions of $\widehat{MP}_X(\tau, \nu^1, \nu^2)$.

Proof of Theorem 1.4(c) We first show that the uniqueness in part (b) extends to random space–time initial conditions. Let ζ be a probability measure on $\hat{S}$. We will say that a process $\hat{H} = (\hat{H}^1, \hat{H}^2)$ on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ with sample paths a.s. in

$$\{(\omega_1, \omega_2) \in \Omega'_C \times \Omega'_C : \alpha_C(\omega_1) = \alpha_C(\omega_2)\}$$

is a solution of $\widehat{MP}_H(\zeta)$ if the following conditions hold:

(M1) If $A = \alpha_C(\hat{H}^1) \vee \alpha_C(\hat{H}^2)$ $(= \alpha_C(\hat{H}^i)$ a.s.), then $(A, \hat{H}^1_A, \hat{H}^2_A)$ has law ζ.

(M2) The field–field collision local times $L(\hat{H}^i, \hat{H}^j)$ exist for $i \neq j$.

(M3) If $\mathcal{H}_t^{(A)} = \bigcap_n \left(\sigma(\hat{H}_{A+s} : 0 \leq s \leq t + \frac{1}{n}) \vee \sigma(A)\right)$, then for all $\phi_1, \phi_2 \in D_S$ the process

$$\hat{M}_t^i(\phi^i) = \hat{H}^i_{A+t}(\phi_i) - \hat{H}^i_A(\phi_i) - \int_A^{A+t} \hat{H}^i_s(\frac{\bar{\Delta}}{2}\phi_i)\, ds + r_i \int_A^{A+t} \int \phi_i(y) L(\hat{H}^i, \hat{H}^j)(ds, dy), \quad t \geq 0,$$

$(i \neq j)$ is a continuous $(\mathcal{H}_t^A)_{t\geq 0}$–martingale such that

$$\left\langle \hat{M}^i(\phi_i), \hat{M}^j(\phi_j) \right\rangle_t = \delta_{ij} \int_A^{A+t} \hat{H}^i_s((\phi_i)^2)\, ds, \quad \forall t \geq 0, \text{a.s.}$$

for all $\phi_i, \phi_j \in D_*$ and all $t \geq 0$.

Clearly, if the filtration appearing in the definition of $\widehat{MP}(\tau, \mu^1, \mu^2)$ is the canonical right–continuous filtration, then $\widehat{MP}_H(\delta_\tau \otimes \delta_{\mu^1} \otimes \delta_{\mu^2})$ and $\widehat{MP}(\tau, \mu^1, \mu^2)$ are equivalent martingale problems.

We claim that the law on $\Omega'_C \times \Omega'_C$ of any solution to $\widehat{MP}_H(\zeta)$ is

$$\hat{\mathbb{P}}^\zeta \equiv \int \hat{\mathbb{P}}^{\tau,\mu^1,\mu^2} \zeta(d(\tau,\mu^1,\mu^2)). \tag{8.1}$$

To see this, assume $(\hat{H}^1, \hat{H}^2)$ solves $\widehat{MP}(\zeta)$ and let $\hat{\mathbb{P}}_{\tau,\mu^1,\mu^2}$ be a regular conditional probability for $(\hat{H}^1, \hat{H}^2)$ given $(A, \hat{H}^1_A, \hat{H}^2_A) = (\tau, \mu^1, \mu^2)$. Let D_* be a countable subset of D_S such that $1 \in D_*$ and the bounded–pointwise closure of $\{(\phi, \frac{\bar{\Delta}}{2}\phi) : \phi \in D_*\}$ contains $\{(\phi, \frac{\bar{\Delta}}{2}\phi) : \phi \in D_S\}$. Let K_* be a countable collection of bounded, measurable functions on $\mathbf{M}_F(C) \times \mathbf{M}_F(C)$ whose bounded–pointwise closure is the set of all bounded, measurable functions on $\mathbf{M}_F(C) \times \mathbf{M}_F(C)$. For $u \geq 0$, let $\Psi_*(u)$ be the countable set of random variables of the form

$$\psi = \prod_{i=1}^m \psi_i(\hat{H}^1_{A+v_i}, \hat{H}^2_{A+v_i}), \quad \psi_i \in K_*,\ v_i \in \mathbb{Q} \cap [0,u],\ m \in \mathbb{N}.$$

If $0 \leq u < v$, $\phi_i \in D_*$ for $i = 1, 2$, and $\psi \in \Psi_*(u)$, then

$$\hat{\mathbb{P}}\left[(\hat{M}^i_v(\phi_i) - \hat{M}^i_u(\phi_i))\psi \mid A, \hat{H}^1_A, \hat{H}^2_A\right] = 0, \quad \text{a.s.}$$

Therefore, we may fix (τ, μ^1, μ^2) outside a ζ–null set N so that for all $\phi_1, \phi_2 \in D_*$, rational $0 \leq u \leq v$ and $\psi \in \Psi_*(u)$ we have

(i) $s \mapsto \hat{H}^i_s(\phi_i)$ is continuous $\hat{\mathbb{P}}_{\tau,\mu^1,\mu^2}$-a.s.

(ii) $\hat{H}^i_A = \mu^i$ and $A = \tau$, $\hat{\mathbb{P}}_{\tau,\mu^1,\mu^2}$-a.s.

(iii) $\hat{\mathbb{P}}_{\tau,\mu^1,\mu^2}\left[(\hat{M}^i_v(\phi_i) - \hat{M}^i_u(\phi_i))\psi\right] = 0.$

This implies for $\phi_i \in D_*$ that

$$\hat{\mathbb{P}}_{\tau,\mu^1,\mu^2}\left[\hat{M}^i_v(\phi_i) \mid \mathcal{H}_{\tau+s},\ 0 \leq s \leq u\right] = \hat{M}^i_u(\phi_i), \quad \forall \text{ rational } 0 \leq u \leq v,\ \hat{\mathbb{P}}_{\tau,\mu^1,\mu^2}\text{-a.s.}$$

Taking limits from above in u and v and applying continuity and reverse martingale convergence, we see that the processes $(\hat{M}^i_t(\phi_i))_{t\geq 0}$ are $(\mathcal{H}^{(\tau)}_t)_{t\geq 0}$–martingales under $\hat{\mathbb{P}}_{\tau,\mu^1,\mu^2}$ for $\phi_i \in D_*$.

By enlarging N, we may also assume (again for $\phi_i \in D_*$) that

$$\langle \hat{M}^i(\phi_i), \hat{M}^j(\phi_j)\rangle_t = \delta_{ij} \int_\tau^{\tau+t} \hat{H}^i_s(\phi_i^2)\, ds, \quad \forall t \geq 0,\ \hat{\mathbb{P}}_{\tau,\mu^1,\mu^2} - \text{a.s.}$$

As usual, we can extend $\hat{M}^i$ to an orthogonal martingale measure under $\hat{\mathbb{P}}_{\tau,\mu^1\mu^2}$ so that the above quadratic variation extends in usual way. By starting with (M3) above for ϕ_i in D_* and taking limits in probability under $\hat{\mathbb{P}}_{\tau,\mu^1\mu^2}$, we see that the semimartingale representation in (M3) remains valid under $\hat{\mathbb{P}}_{\tau,\mu^1\mu^2}$ for $\phi_i \in D_S$, with $\hat{M}^i_t(\phi_i)$ an a.s. continuous $(\mathcal{H}^{(\tau)}_t)_{t\geq 0}$–martingale.

There is one remaining point to verify before we can conlude that under $\hat{\mathbb{P}}_{\tau,\mu^1\mu^2}$ the process $(\hat{H}^1, \hat{H}^2)$ solves $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ with respect to $(\hat{\mathcal{H}}_t^{(0)})_{t\geq\tau}$. We need to check that FFCLT's exist under $\hat{\mathbb{P}}_{\tau,\mu^1\mu^2}$ and that these are the measure–valued process appearing in the semimartingale decomposition of $\hat{H}^i(\phi_i)$ under $\hat{\mathbb{P}}_{\tau,\mu^1,\mu^2}$. The latter decomposition and the construction of the majorising historical Brownian motions (H^1, H^2) in Section 6 shows that under $\hat{\mathbb{P}}_{\tau,\mu^1\mu^2}$ we have $(\hat{H}^1, \hat{H}^2) \in \mathcal{M}(H^1, H^2)$ for suitable (H^1, H^2) (providing $(\tau, \mu^1, \mu^2) \notin N$), and so the existence of FFCLT's, call them $L(\hat{H}^i, \hat{H}^j; \tau, \mu^1, \mu^2)$, under $\hat{\mathbb{P}}_{\tau,\mu^1\mu^2}$ follows from Theorem 3.13(a). A Borel–Cantelli argument gives that if $\phi \in C_b(C)$ and $S > \tau$, then there is a sequence $\epsilon_n \downarrow 0$ such that

$$\sup_{\tau\leq t\leq S} \left| L_t^{\epsilon_n}(\hat{H}^i, \hat{H}^j)(\phi) - L_t(\hat{H}^i, \hat{H}^j)(\phi) \right| \to 0, \quad \hat{\mathbb{P}}_{\tau,\mu^1\mu^2}\text{-a.s., } \zeta\text{-a.e. } (\tau, \mu^1, \mu^2)$$

Hence $L(\hat{H}^1, \hat{H}^2)$ is a version of $L(\hat{H}^i, \hat{H}^j; \tau, \mu^1, \mu^2)$ for ζ-a.e. (τ, μ^1, μ^2), and the proof that for ζ-a.e. (τ, μ^1, μ^2) the process $(\hat{H}^1, \hat{H}^2)$ under $\hat{\mathbb{P}}_{\tau,\mu^1\mu^2}$ solves $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ is complete. The uniqueness in Theorem 1.4(b) shows that the law of $(\hat{H}^1, \hat{H}^2)$ under $\hat{\mathbb{P}}$ is given by (8.1).

To prove Theorem 1.4(c), we can now follow the proof of Theorem 4.4.2 of [18]. Let $F \in \hat{\mathcal{F}}_T$ satisfy $\hat{\mathbb{P}}(F) > 0$. For $B \in \mathcal{F}'_C \times \mathcal{F}'_C$ set

$$P_1(B) = \hat{\mathbb{P}}\left[\mathbf{1}_F \hat{\mathbb{P}}^{T,\hat{H}_T^1,\hat{H}_T^2}(B)\right] / \hat{\mathbb{P}}(F)$$

and

$$P_2(B) = \hat{\mathbb{P}}\left[\mathbf{1}_F \hat{\mathbb{P}}((\hat{H}^{1,T}, \hat{H}^{2,T}) \in B \mid \hat{\mathcal{F}}_T)\right] / \hat{\mathbb{P}}(F),$$

where

$$\hat{H}_t^{i,T} = \begin{cases} \Delta, & \text{if } t < T, \\ \hat{H}_t^i, & \text{otherwise.} \end{cases}$$

As in Theorem 4.4.2 of [18], one readily checks that the coordinate variables on $\Omega'_C \times \Omega'_C$ solve $\widehat{MP}_H(\zeta)$ under P_1 and P_2, where $\zeta(G) = \hat{\mathbb{P}}\left[\mathbf{1}_F \mathbf{1}((T, \hat{H}_T^1, \hat{H}_T^2) \in G)\right] / \hat{\mathbb{P}}(F)$. The uniqueness proved above implies that $P_1 = P_2$. As F is arbitrary, this gives $\hat{\mathbb{P}}^{T,\hat{H}_T^1,\hat{H}_T^2}(B) = \mathbb{P}((\hat{H}^{1,T}, \hat{H}^{2,T}) \in B \mid \hat{\mathcal{F}}_T)$ $\hat{\mathbb{P}}$-a.s. for all $B \in \mathcal{F}'_C \times \mathcal{F}'_C$, as required.

□

Let $(\theta_t)_{t\geq 0}$ denote the usual family of shift operators on C; that is, $(\theta_t y)(s) = y(s + t)$. Given $\mu \in \mathbf{M}_F(C)$, write $\theta_t\mu$ for the push-forward of μ by θ_t. Note that θ_τ maps $\mathbf{M}_{FS}(C)^t$ into $\mathbf{M}_{FS}(C)^{t-\tau}$ if $t \geq \tau$. Extend the definition of θ_t to $\mathbf{M}_F^\Delta(C)$ by setting $\theta_t\Delta = \Delta$

Proposition 8.1 *For* $(\tau, \mu^1, \mu^2) \in \hat{S}$, *let* $(\hat{H}^1, \hat{H}^2)$ *have the law* $\hat{\mathbb{P}}^{\tau,\mu^1,\mu^2}$ *described in Theorem 1.4 Then the law of* $((\theta_\tau \hat{H}^1_{\tau+t}, \theta_\tau \hat{H}^2_{\tau+t}))_{t\geq 0}$ *is* $\hat{\mathbb{P}}^{0,\theta_\tau\mu^1,\theta_\tau\mu^2}$.

Proof. By definition, $(\hat{H}^1, \hat{H}^2)$ solves $\widehat{MP}_H(\tau, \mu^1, \mu^2)$ with respect to the canonical right–continuous filtration $(\mathcal{H}_t)_{t\geq\tau}$, say. It is straightforward to check this implies that $((\theta_\tau \hat{H}^1_{\tau+t}, \theta_\tau \hat{H}^2_{\tau+t}))_{t\geq 0}$ solves $\widehat{MP}_H(0, \theta_\tau\mu^1, \theta_\tau\mu^2)$ with respect to the right–continuous filtration $(\mathcal{H}_{\tau+t})_{t\geq 0}$. An application of Theorem 1.4(c) completes the proof.

□

Define $\kappa : \mathbb{R}^d \to C$ by $(\kappa\xi)(s) = \xi$; that is, $\kappa\xi$ is the constant function with value ξ. For $\nu \in M_F(\mathbb{R}^d)$ let $\kappa\nu \in M_F(C)$ be the push–forward of ν by κ. Note that if $\nu \in \mathbf{M}_{FS}(\mathbb{R}^d)$, then $\kappa\nu \in \mathbf{M}_{FS}(C)^0$. Suppose that the process $(\hat{H}^1, \hat{H}^2)$ on $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}})_{t\geq 0}, \hat{\mathbb{P}})$ solves the martingale problem $\widehat{MP}_H(0, \kappa\nu^1, \kappa\nu^2)$ for $\nu^1, \nu^2 \in \mathbf{M}_{FS}(\mathbb{R}^d)$ and hence has law $\hat{\mathbb{P}}^{0,\kappa\nu^1,\kappa\nu^2}$. Set $(\hat{X}^1, \hat{X}^2) = (\Gamma(\hat{H}^1), \Gamma(\hat{H}^2))$, where we recall from Section 1 that $\Gamma(h)_t(\phi) = \int \phi(y_t)\, h_t(dy)$ for $h \in \Omega_H[0, \infty[$. Write $\hat{\mathbb{Q}}^{\nu^1,\nu^2}$ for the law of the continuous $\mathbf{M}_{FS}(\mathbb{R}^d) \times \mathbf{M}_{FS}(\mathbb{R}^d)$–valued process $(\hat{X}^1, \hat{X}^2)$. The following result is immediate from Remark 1.2, Theorem 1.4 and Proposition 8.1. (Note that that $\Gamma(h_{s+\cdot}) = \Gamma(\theta_s h_{s+\cdot})$ for all $s \geq 0$.)

Theorem 8.2 *(a) The pair* $\hat{X} = (\hat{X}^1, \hat{X}^2)$ *solves the martingale problem* $\widehat{MP}_X(0, \nu^1, \nu^2)$ *of Remark 1.2.*
(b) For any Borel set $A \subseteq C(\mathbb{R}_+, \mathbf{M}_{FS}(\mathbb{R}^d) \times \mathbf{M}_{FS}(\mathbb{R}^d))$, *the map* $(\nu^1, \nu^2) \mapsto \mathbb{Q}^{\nu^1,\nu^2}(A)$ *is Borel.*
(c) If T *is a* $(\hat{\mathcal{F}}_t)_{t\geq 0}$*–stopping time, then*

$$\hat{\mathbb{P}}\left[\phi(\hat{X}_{T+\cdot}) \mid \hat{\mathcal{F}}_T\right](\omega) = \mathbb{Q}^{\hat{X}_T(\omega)}(\phi), \quad \text{for } \hat{\mathbb{P}}\text{-a.e. } \omega \in \hat{\Omega}$$

for any bounded measurable function ϕ *on* $C(\mathbb{R}_+, \mathbf{M}_F(\mathbb{R}^d) \times \mathbf{M}_F(\mathbb{R}^d))$.

A Superprocesses with immigration

Let $(E, \mathcal{E})$ be a metrisable Lusin space (that is, E is homeomorphic to a Borel subset of a compact metric space and $\mathcal{E}$ is its Borel σ-field). Suppose that $S^o \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{E}$, and for $t \in \mathbb{R}_+$ set $E_t = \{z \in E : (t, z) \in S^o\} \in \mathcal{E}$. Adjoin an isolated point ∂ to E to form E^∂. Let $D(\mathbb{R}_+, E^\partial)$ denote the Skorohod space of càdlàg, E^∂–valued paths, equipped with its Borel σ–field, and set

$$\Omega^o = \{\omega \in D(\mathbb{R}_+, E^\partial) : \alpha^o(\omega) < \infty,\ \beta^o(\omega) = \infty\},$$

where

$$\alpha^o(\omega) = \inf\{t : \omega(t) \neq \partial\}$$

and

$$\beta^o(\omega) = \inf\{t \geq \alpha^o(\omega) : (t, \omega(t)) \notin S^o\}.$$

The set Ω^o is a universally measurable subset of $D(\mathbb{R}_+, E^\partial)$ (for example, by p. 64 of [11] and the Borel measurability of $(t,\omega) \mapsto (t, \omega(t))$). Let $\mathcal{F}^o$ be the trace of the universally measurable subsets of $D(\mathbb{R}_+, E^\partial)$ on Ω^o. Write $Z_t(\omega) = \omega(t)$ for the coordinate random variables on Ω^o and $\mathcal{F}^o_{[s,t+]}$ for the universal completion of $\bigcap_n \sigma\{Z_r : s \leq r \leq t + n^{-1}\}$ for $0 \leq s \leq t$. One readily checks that $\mathcal{F}^o_{[s,t+]}$ is right–continuous in $t > s$.

If $\{P^{s,z} : (s,z) \in S^o\}$ is a collection of probabilities on $(\Omega^o, \mathcal{F}^o)$, consider the following conditions:

(Hyp 1) For each $(s,z) \in S^o$, $P^{s,z}(\alpha^o = s,\ Z_s = z) = 1$.

(Hyp 2) For each $A \in \mathcal{F}^o$, the map $(s,z) \mapsto P^{s,z}(A)$ is $\mathcal{B}(S^o)$–measurable.

(Hyp 3) If $(s,z) \in S^o$, $\phi \in b(\mathcal{B}(\mathbb{R}_+) \times \mathcal{F}^o)$, and $T \geq s$ is a $(\mathcal{F}^o_{[s,t+]})_{t \geq s}$–stopping time, then

$$P^{s,z}\left[\phi(T, Z(T + \cdot)) \mid \mathcal{F}^o_{[s,T+]}\right](\omega) = P^{T(\omega), Z(T)(\omega)}\left[\phi(T(\omega), Z(T(\omega) + \cdot))\right]$$

for $P^{s,z}$-a.e. ω such that $T(\omega) < \infty$.

(Hyp 4) If $(s,z) \in S^o$, and $\{T_n\}$ are $(\mathcal{F}^o_{[s,t+]})_{t \geq s}$–stopping times such that $s \leq T_n \uparrow T$ with $P^{s,z}\{T < \infty\} = 1$, then $Z(T_n) \to Z(T)$, $P^{s,z}$-a.s.

If we were to alter the definition of $P^{s,z}$ by restricting it to $\mathcal{F}^o_{[s,\infty[}$, then conditions (Hyp1) — (Hyp3) imply that $Z = (\Omega^o, \mathcal{F}^o, \mathcal{F}^o_{[s,t+]}, Z_t, P^{s,z})$ is an inhomogeneous Borel strong Markov process (IBSMP) with càdlàg paths in $E_t \subset E$, in the sense of p. 12 of [9]. In this case we call Z the canonical realisation of the IBSMP with càdlàg paths in $E_t \subset E$.

With a slight abuse of the usual notation, let $C(\mathbb{R}_+, E^\partial)$ be the subspace of $D(\mathbb{R}_+, E^\partial)$ consisting of functions f with the property that $f(t) = f(t-)$ unless $\partial \in \{f(t), f(t-)\}$. If $D(\mathbb{R}_+, E^\partial)$ is replaced by $C(\mathbb{R}_+, E^\partial)$ in the above, then we call Z the canonical realisation of an IBSMP with continuous paths in $E_t \subset E$. Similarly, if in the above $D(\mathbb{R}_+, E^\partial)$ is replaced by $D_r(\mathbb{R}_+, E^\partial)$, the space of right–continuous E^∂–valued paths (equipped with the σ–field generated by the coordinate maps), then we call Z the canonical realisation of an IBSMP with right–continuous paths in $E_t \subset E$. Finally, if conditions (Hyp1) - (Hyp4) hold, Z is the canonical realisation of an inhomogeneous Hunt process (IHP) with càdlàg paths in $E_t \subset E$.

If Z is the canonical realisation of an IBSMP with right–continuous paths in $E_t \subset E$, it is easy to check that Z is a right process in the sense of [15]. For example, since S^o is also a metrisable Lusin space, it may be embedded as a dense Borel subset of a compact metric space $\bar{S}$. Let $\{f_n\}$ be a countable dense subset of $C(\bar{S}, \mathbb{R})$. Then $\{f_n|_{S^o}\}$ generates $\mathcal{B}(S^o)$ and obviously $f_n(t, Z_t(\omega))$ is right–continuous for each ω, so that 1.2.B of [15] holds.

Assume conditions (Hyp1) - (Hyp4). If $f \in b\mathcal{B}(S^o)$, $(s,z) \in S^o$, and $t \geq 0$, define

$$P_{s,t}f(z) = P^{s,z}(f(s \vee t, Z(s \vee t))),$$

where we set $f(u,\partial) = 0$. If $f \in bp\mathcal{B}(S^o)$, let $V_{s,t}f(z)$ denote the unique Borel measurable solution of

$$V_{s,t}f(z) = P_{s,t}f(z) - \frac{1}{2}\int_{s \wedge t}^{t} P_{s,r}(\{V_{r,t}f(\cdot)\}^2)(z)\,dr. \tag{A.1}$$

Note that if $t \leq s$, then $V_{s,t}f(z) = P_{s,t}f(z) = f(s,z)$. We also define the corresponding space–time semigroups for $f \in b\mathcal{B}(S^o)$ (respectively, $f \in bp\mathcal{B}(S^o)$) by $P_t f(s,z) = P_{s,s+t}f(z)$ (respectively, $V_t f(s,z) = V_{s,s+t}f(s,z)$), where $(s,z) \in S^o$ and $t \geq 0$. The existence of a unique solution to (A.1) is a standard application of Picard iteration (see, for example, Theorem 3.1 of [13]).

We abuse notation slightly and use $\mathbf{M}_F(B)$ for $B \subseteq E$ to denote the set of measures $\mu \in \mathbf{M}_F(E)$ such that $\mu(E \backslash B) = 0$. Let

$$S' = \{(t,\mu) \in \mathbb{R}_+ \times \mathbf{M}_F(E) : \mu \in \mathbf{M}_F(E_t)\} = \{(t,\mu) : \mu\{z : (t,z) \notin S^o\} = 0\} \in \mathcal{B}(\mathbb{R}_+ \times \mathbf{M}_F(E)).$$

Adjoin Δ as an isolated point to $\mathbf{M}_F(E)$ to form $\mathbf{M}_F^\Delta(E)$. Set

$$\Omega' = \left\{\omega \in C(\mathbb{R}_+, \mathbf{M}_F^\Delta(E)) : \alpha'(\omega) < \infty,\ \beta'(\omega) = \infty\right\},$$

where

$$\alpha'(\omega) = \inf\{t : \omega(t) \neq \Delta\}$$

and

$$\beta'(\omega) = \inf\{t \geq \alpha'(\omega) : \omega(t) \notin \mathbf{M}_F(E_t)\}.$$

Then, as before, Ω' is a universally measurable subset of $C(\mathbb{R}_+, \mathbf{M}_F^\Delta(E))$. Let $\mathcal{F}'$ be the trace of the universally measurable subsets of $C(\mathbb{R}_+, \mathbf{M}_F^\Delta(E))$ on Ω'. Write $X_t(\omega) = \omega(t)$ for the coordinate random variables on Ω' and $\mathcal{F}'_{[s,t+]}$ for the universal completion of $\bigcap_n \sigma\{X_r : s \leq r \leq t + n^{-1}\}$ for $0 \leq s \leq t$.

Let

$$\mathbf{M}_{LF}(S^o) = \{\mu \in M(S^o) : \mu(([0,t] \times E) \cap S^o) = \mu(([0,t[\times E) \cap S^o) < \infty,\ \forall t \in \mathbb{R}_+\}.$$

Equip $\mathbf{M}_{LF}(S^o)$ with a metric d such that $d(\mu_n,\mu) \to 0$ if and only if for each $k \in \mathbb{N}$ $\mu_n|_{S_k^o} \to \mu|_{S_k^o}$ in the topology of weak convergence of finite measures, where $S_k^o = S^o \cap ([0,k] \times E)$.

We will use an appropriate weak generator for Z. Let A be the set of pairs $(\phi,\psi) \in b\mathcal{B}(S^o) \times b\mathcal{B}(S^o)$ such that $P_t(\psi)(s,z)$ is right–continuous in t for each $(s,z) \in S^o$ and

$$P_t\phi(s,z) = \phi(s,z) + \int_0^t P_r(\psi)(s,z)\,dr$$

$\forall (s,z) \in S^o$, $t \geq 0$.

It is easy to see that $(\phi, \psi) \in A$ if and only if $P_t(\psi)(s,z)$ is right–continuous in t for each $(s,z) \in S^o$ and $(\phi(t, Z_t) - \phi(s,z) - \int_s^t \psi(u, Z_u)\, du)_{t \geq s}$ is an almost surely right–continuous $\mathcal{F}^o_{[s,t+]}$–martingale under $P^{s,z}$ for each $(s,z) \in S^o$. In particular, $(\phi(t, Z_t))_{t \geq s}$ has càdlàg paths almost surely. If $\theta \in b\mathcal{B}$, set

$$R_\lambda \theta(s,z) = \int_0^\infty e^{-\lambda t} P_t \theta(s,z)\, dt$$

for $\lambda > 0$. One can easily check that $(R_\lambda \theta, \lambda R_\lambda \theta - \theta) \in A$. If $\theta \in C_b(S^o)$ (or, more generally, if θ is finely continuous with respect to the space–time process associated with Z), then $\lambda R_\lambda \theta \to \theta$ in the bounded pointwise sense as $\lambda \to \infty$. Hence the set $D(A)$ of $\phi \in b\mathcal{B}(S^o)$ such that $(\phi, \psi) \in A$ for some ψ, is bounded–pointwise dense in $b\mathcal{B}(S^o)$.

Let $\mathcal{M}_{[s,t]}$ be the σ–field of subsets of $\mathbf{M}_{LF}(S^o)$ generated by the maps $\nu \to \nu(A)$, where A is a Borel subset of $S^o \cap ([s,t] \times E)$.

Theorem A.1 *Assume that Z is the canonical realisation of an IHP.*
(a) For each $L \in \mathbf{M}_{LF}(S^o)$ there is a unique collection of probabilities $\{\mathbb{Q}^{\tau,\mu;L} : (\tau,\mu) \in S'\}$ on $(\Omega', \mathcal{F}')$ such that:

- *(i) $X = (\Omega', \mathcal{F}', \mathcal{F}'_{[\tau,t+]}, X_t, \mathbb{Q}^{\tau,\mu;L})$ is the canonical realisation of an IBSMP with continuous paths in $\mathbf{M}_F(E_t) \subset \mathbf{M}_F(E)$. (Here we indentify $\mathbb{Q}^{\tau,\mu;L}$ with its restriction to $\mathcal{F}'_{[\tau,\infty[}$.)*
- *(ii) For each f in $bp\mathcal{B}(S^o)$*

$$\mathbb{Q}^{\tau,\mu;L}\left[\exp(-X_t(f_t))\right] = \exp(-\mu(V_{\tau,t} f) - \int_\tau^t \int V_{r,t} f(z) L(d(r,z)) \qquad (A.2)$$

$\forall (\tau,\mu) \in S'$, $t \geq \tau$.

(b) For each $\phi \in D(A)$, $\{X_t(\phi_t) : t \geq \tau\}$ is continuous $\mathbb{Q}^{\tau,\mu;L}$-a.s. for all $(\tau,\mu) \in S'$.
(c)If $\Psi \in b\sigma\{X_r : \tau \leq r \leq t\}$ and $(\tau,\mu) \in S'$, then $L \mapsto \mathbb{Q}^{\tau,\mu;L}[\Psi]$ is $\mathcal{M}_{[\tau,t]}$–measurable. If $\Psi \in b\mathcal{F}'$, then $(\tau,\mu,L) \mapsto \mathbb{Q}^{\tau,\mu;L}[\Psi]$ is Borel on $S' \times \mathbf{M}_{LF}(S^o)$. We call $\mathbb{Q}^{\tau,\mu;L}$ the law of the Z–superprocess with L–immigration starting at (τ,μ).

Proof. In the notation of Sections 2 and 3 of [15], let $\bar{E}_s = \{(r,z) \in S^o : r \geq s\}$, $\bar{E}_{s+} = \{(r,z) \in S^o : r > s\}$, $\tilde{S} = \{(s,\rho) \in \mathbb{R}_+ \times \mathbf{M}_F(S^o) : \rho \in \mathbf{M}_F(\bar{E}_s)\} \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbf{M}_F(S^o))$, $\kappa(dt) = dt$, $\tau_t = t$, and $\Psi(f)(s,z) = f(s,z)^2/2$.

Theorem 2.1 of [15] gives the existence of a right Markov process (see [15] for the relevant definition of right process) $\bar{X}_t \in \mathbf{M}_F(\bar{E}_t)$ with transition probabilities $\{\bar{P}^{s,\rho} : (s,\rho) \in \tilde{S}\}$ defined on some measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ such that

$$\bar{P}^{s,\rho}\left[\exp(-\bar{X}_t(f))\right] = \exp(-\int_{\bar{E}_{t+}} f(r,z)\, \rho(dr,dz) - \int V_{r,t} f(z) \mathbf{1}\{s \leq r \leq t\}\, \rho(dr,dz)) \qquad (A.3)$$

$\forall f \in bp\mathcal{B}(S^o)$, $t \geq s$, $(s,\rho) \in \tilde{S}$.

Let $\bar{S}$ be the compact metric space and $\{f_n : n \in \mathbb{N}\}$ be the dense subset of $C(\bar{S},\mathbb{R})$ described above. Theorem 3.1 of [15] shows that $(\bar{X}_t(f_n))_{t\geq s}$ is right–continuous $\bar{P}^{s,\rho}$-a.s. for each $(s,\rho) \in \tilde{S}$. The restrictions to S^o of the continuous functions on $\bar{S}$ are convergence determining (they are the uniformly continuous functions on S^o) and therefore it follows that $(\bar{X}_t)_{t\geq s}$ is right–continuous in $\mathbf{M}_F(S^o)$, $\bar{P}^{s,\rho}$-a.s.

Define $X'_t \in \mathbf{M}_F(E_t) \subset \mathbf{M}_F(E)$ and $X''_t \in \mathbf{M}_F(\bar{E}_{t+}) \subset \mathbf{M}_F(S^o)$ by

$$X'_t(g) = \int g(z)\mathbf{1}\{r=t\}\,\bar{X}_t(dr,dz)$$

and

$$X''_t(f) = \int f(r,z)\mathbf{1}\{r>t\}\,\bar{X}_t(dr,dz).$$

If $f^t(r,z) = \mathbf{1}\{r>t\}f_n(r,z)$ in the notation of Theorem 3.1 of [15], then $f^t(t,Z_t) \equiv 0$ and so that result implies $(X''_t)_{t\geq s}$ is right–continuous $\bar{P}^{s,\rho}$-a.s. (as an $\mathbf{M}_F(S^o)$–valued process). Note that if $f(t,z) = 0$ for $z \in E_t$, then $V_{r,t}f(z) = 0$ for $(r,z) \in S^o \cap ([0,t]\times E)$ and so (A.3) implies $X''_t = \rho|_{\bar{E}_{t+}}$, $\forall t \geq s$, $\bar{P}^{s,\rho}$-a.s., $\forall (s,\rho) \in \tilde{S}$. We have used right–continuity to get equality for all $t \geq s$ simultaneously. Therefore

$$X'_t(g) = \int g(z)\,\bar{X}_t(dr,dz) - \int \mathbf{1}\{r>t\}g(z)\,\rho(dr,dz),$$

$\forall t \geq s$, $\bar{P}^{s,\rho}$-a.s., $\forall (s,\rho) \in \tilde{S}$, and so $(X'_t)_{t\geq s}$ is right–continuous in $\mathbf{M}_F(E)$, $\bar{P}^{s,\rho}$-a.s. Equation (A.3) now implies

$$\bar{P}^{s,\rho}\left[\exp(-X'_t(f_t))\right] = \exp(-\int V_{r,t}f(z)\mathbf{1}\{s\leq r\leq t\}\rho(dr,dz)) \tag{A.4}$$

$\forall f \in bp\mathcal{B}(S^o)$.

We abuse notation and let X be as in the statement of the theorem, but with $D_r(\mathbb{R}_+,\mathbf{M}_F^\Delta(E))$ in place of $C(\mathbb{R}_+,\mathbf{M}_F^\Delta(E))$. Assume first that $L \in \mathbf{M}_{LF}(S^o)$ is finite. If $(\tau,\mu) \in S'$, let $L^{\tau,\mu} = \delta_\tau \otimes \mu + L|_{\bar{E}_\tau}$ and define probabilities $\{\mathbb{Q}^{\tau,\mu;L} : (\tau,\mu) \in S'\}$ on $(\Omega',\mathcal{F}')$ by $\mathbb{Q}^{\tau,\mu;L} = \bar{P}^{\tau,L^{\tau,\mu}} \circ (X')^{-1}$. Equation (A.2) and conditions (Hyp1) and (Hyp3) hold for $X = (\Omega',\mathcal{F}',\mathcal{F}'_{[s,t+]},X_t,\mathbb{Q}_{\tau,\mu;L})$ because of the analogous properties for $(\bar{X}_t,\bar{P}^{\tau,\rho})$ and (A.4). Condition (Hyp2) follows from (A.2). Therefore X is the canonical realisation of an IBSMP with right–continuous paths in $\mathbf{M}_F(E_t) \subset \mathbf{M}_F(E)$.

We now follow [22]to show X has left limits a.s. Let $Ug(s,z) = \int_0^\infty e^{-t}P_tg(s,z)\,dt$, $g \in b\mathcal{B}(S^o)$, denote the 1–potential for the associated space–time process. Let $\{T_n\}$ and T_∞ be $\mathcal{F}'_{[s,t+]}$–stopping times such that $s \leq T_n \uparrow T_\infty \leq t_0$ for some $s \leq t_0 < \infty$. If $f \in b\mathcal{B}(S^o)$ and $n \in \mathbb{N}\cup\{\infty\}$, define

$$\rho_n(f) = \mathbb{Q}^{\tau,\mu;L}\left[e^{-T_n}X_{T_n}(f_{T_n}) + \iint e^{-r}\mathbf{1}\{r\geq T_n\}f(r,z)L(dr,dz)\right]. \tag{A.5}$$

A simple expression for the mean measure of X_t (which is clear from (A.2) — see (A.14) below) and the strong Markov property show that for $f \in b\mathcal{B}(S^o)$

$$\begin{aligned}\mathbb{Q}^{\tau,\mu;L}\left[X_{t+T_n}(f_{t+T_n})|\mathcal{F}'_{[s,T_n+]}\right](\omega) &= X_{T_n(\omega)}(P_t f(T_n(\omega),\cdot)) \\ &\quad + \int_{T_n(\omega)}^{t+T_n(\omega)}\int P_{s,T_n(\omega)+t}f(z)\,L(ds,dz).\end{aligned} \tag{A.6}$$

Therefore, by (A.5) and (A.6),

$$\begin{aligned}\rho_n(Uf) &= \int_0^\infty \mathbb{Q}^{\tau,\mu;L}\left[e^{-(t+T_n)}X_{T_n}(P_t(f(T_n,\cdot)))\right]dt \\ &\quad + \mathbb{Q}^{\tau,\mu;L}\left[\iint e^{-r}\mathbf{1}\{r \geq T_n\}Uf(r,z)\,L(dr,dz)\right] \\ &= \mathbb{Q}^{\tau,\mu;L}\left[\int_{T_n}^\infty e^{-u}X_u(f_u)\,du\right] - \mathbb{Q}^{\tau,\mu;L}\left[\int_{T_n}^\infty e^{-u}\int_{T_n}^u\int P_{s,u}f(z)\,L(ds,dz)\,du\right] \\ &\quad + \mathbb{Q}^{\tau,\mu;L}\left[\int_{T_n}^\infty\iint_r^\infty e^{-u}P_{r,u}f(z)\,du\,L(dr,dz)\right] \\ &= \mathbb{Q}^{\tau,\mu;L}\left[\int_{T_n}^\infty e^{-u}X_u(f_u)\,du\right].\end{aligned}$$

Hence $\rho_n U \downarrow \rho_\infty U$ "setwise".

Assume $f \in C_b(S^o)$. By (3.4)(b) of [22] and the fact that $L(\{t\}\times E) = 0\ \forall t$, we have

$$\lim_{n\to\infty}\mathbb{Q}^{\tau,\mu;L}\left[e^{-T_n}X_{T_n}(f_{T_n})\right] = \mathbb{Q}^{\tau,\mu;L}\left[e^{-T_\infty}X_{T_\infty}(f_{T_\infty})\right]; \tag{A.7}$$

and hence (by Theorem VI.48 of [12]), $t \mapsto X_t(f_t)$ has left limits $\mathbb{Q}^{\tau,\mu;L}$-a.s. It is now easy to modify the proof of Theorem 2.1.3(d) in [9] to see that $t \mapsto X_t$ has left limits in $\mathbf{M}_F(E)$, $\mathbb{Q}^{\tau,\mu;L}$-a.s.

Remark A.2 Equation (A.7) shows that that $\forall f \in C_b(E)$, $X_{t-}(f)$ is the predictable projection of $X_t(f)$.

To complete the proof of (a), we must show that $t \mapsto X_t$ is continuous on $[\tau,\infty[$, $\mathbb{Q}^{\tau,\mu}_\nu$-a.s. This is an easy modification of a standard Laplace functional calculation (see, for example, Section 4.7 of [5]) which we now sketch. Let $\hat{X}_t(\phi) = X_t(\phi_t) - \int \mathbf{1}\{r \leq t\}\phi(r,z)\,L(dr,dz)$ and use (A.2) to see that for $\phi \in bp\mathcal{B}(S^o)$ and $\lambda \geq 0$,

$$\begin{aligned}&\mathbb{Q}^{\tau,\mu;L}\left[\exp(-(\hat{X}_t(\lambda\phi) - \hat{X}_\tau(\lambda\phi)))\right] \\ &\quad = \exp(-\int \mathbf{1}\{\tau \leq s \leq t\}(V_{s,t}(\lambda\phi(z)) - \lambda\phi(s,z))L^{\tau,\mu}(ds,dz),\ t \geq \tau,\end{aligned} \tag{A.8}$$

and, in particular,

$$\mathbb{Q}^{s,\delta_z;0}\left[\exp(-X_t(\lambda\phi_t))\right] = \exp(-V_{s,t}(\lambda\phi)(z)),\ z \in E_s,\ s \leq t. \tag{A.9}$$

Equation (A.9) and standard bounds on $\mathbb{Q}^{s,\delta_z;0}[X_t(1)^n]$ allow us to differentiate $V_{s,t}(\lambda\phi)$ n times with respect to $\lambda > 0$ and conclude that the resulting n^{th} derivative $V_{s,t}^{(n)}(\lambda,\phi)(z)$ extends continuously to $\lambda \geq 0$ and satisfies

$$0 \leq V_{s,t}^{(n)}(\lambda,\phi)(z) \leq c(T, \|\phi\|_\infty, n), \tag{A.10}$$

$\forall \lambda \in [0,1]$, $z \in E_s$, and $0 \leq t-s \leq T$. This allows us to set $f = \lambda\phi$ in (A.1), differentiate n times with respect to λ, let $\lambda \downarrow 0$, and use induction on n to see that if $V_{s,t}^{(n)}(\phi)(z) = V_{s,t}^{(n)}(0,\phi)(z)$, then there are constants $\{c_n(\|\phi\|_\infty)\}$ such that

$$0 \leq V_{s,t}^{(n)}(\phi)(z) \leq c_n(t-s)^{n-1},\ \forall x \in E_s,\ s \leq t,\ n \in \mathbb{N}. \tag{A.11}$$

Equation (A.10) also allows us to differentiate (A.8) n times with respect to λ, use Leibniz's rule and let $\lambda \downarrow 0$ to get for $n \in \mathbb{N}$

$$\begin{aligned} m_n(\tau,t) &\equiv \left|\mathbb{Q}^{\tau,\mu;L}\left[(\hat{X}_t(\phi) - \hat{X}_\tau(\phi))^n\right]\right| \\ &= \left|\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{n-1-k} m_k(\tau,t)\int_\tau^t\int V_{s,t}^{(n-k)}(\phi)(z) - \mathbf{1}\{n-k=1\}\phi(s,z)\, L^{\tau,\mu}(ds,dz)\right|, \end{aligned}$$

and so, by (A.11), for all $\tau \leq t \leq T$,

$$\begin{aligned} m_n(\tau,t) \leq & \sum_{k=0}^{n-2}\binom{n-1}{k} c_{n-k}(\mu(E) + L([0,T]\times E)) m_k(\tau,t)(t-\tau)^{n-k-1} \\ & + \left|\int_\tau^t\int P_{s,t}\phi(z) - \phi(s,z) L^{\tau,\mu}(ds,dz)\right| m_{n-1}(\tau,t). \end{aligned} \tag{A.12}$$

Assume $(\phi,\psi) \in A$. Then $|P_{s,t}\phi(z) - \phi(s,z)| \leq \|\psi\|_\infty |t-s|$ for $s \leq t$, and an induction using (A.12) shows that $m_n(\tau,t) \leq c_n'(t-\tau)^{n/2}$ for $\tau \leq t \leq T$ and $c_n' = c_n'(T,\phi,\mu,\nu)$. Since $X_t(\phi_t)$ is right–continuous on $[\tau,\infty[$, $\mathbb{Q}^{\tau,\mu;L}$-a.s. (use Theorem 3.1 of [15] as before), Kolmogorov's continuity criterion implies that $X_t(\phi_t)$ is continuous on $[\tau,\infty[$, $\mathbb{Q}^{\tau,\mu;L}$-a.s.

Taking bounded–pointwise limits, we see that $X_t(\phi_t)$ is $\mathcal{F}_{[\tau,t+]}$–predictable on $[\tau,\infty[\times\Omega'$ for all $\phi \in b\mathcal{B}(S^o)$. Remark A.2 therefore implies that $X_{t-}(f) = X_t(f)$, $\forall t \geq \tau$, $\mathbb{Q}^{\tau,\mu;L}$-a.s., $\forall f \in C_b(E)$, and the $\mathbb{Q}^{\tau,\mu;L}$-a.s. continuity of $(X_t)_{t\geq\tau}$ in $\mathbf{M}_F(E)$ follows. This completes the proof of claims (a) and (b) for L finite.

For general L, let $L^T = L(\cdot \cap ([0,T]\times E))$. Property (A.2) and the Markov property show that $\mathbb{Q}^{\tau,\mu;L^{T_1}} = \mathbb{Q}^{\tau,\mu;L^{T_2}}$ on $\mathcal{F}_{[0,(T_1\wedge T_2-1)+]}$ for $T_1 \wedge T_2 > 1$. Therefore there is a unique law $\mathbb{Q}^{\tau,\mu;L}$ on $(\Omega',\mathcal{F}')$ such that $\mathbb{Q}^{\tau,\mu;L} = \mathbb{Q}^{\tau,\mu;\nu^T}$ on $\mathcal{F}'_{[0,(T-1)+]}$ for all $T > 1$. It is easy to check that $\mathbb{Q}^{\tau,\mu;L}$ satisfies conditions (i) and (ii) of part (a) and part (b). Moreover, uniqueness in (a) is obvious.

Finally, part (c) is an easy consequence of (A.2), the Markov property, and a monotone class argument.

□

Theorem A.3 *Given $L \in \mathbf{M}_{LF}(S^o)$, $(\tau, \mu) \in S'$, and $(\phi, \psi) \in A$, put*

$$M_t(\phi) = X_t(\phi_t) - \mu(\phi_\tau) - \int_\tau^t \int \phi \, dL - \int_\tau^t X_s(\psi_s) \, ds \; t \geq \tau. \tag{A.13}$$

Under $\mathbb{Q}^{\tau,\mu;L}$, $M_t(\phi)$ is a continuous $\mathcal{F}'_{[\tau,t+]}$–martingale for which

$$\langle M(\phi) \rangle_t = \int_\tau^t X_s(\phi_s^2) \, ds, \; \forall t \geq \tau, \; \mathbb{Q}^{\tau,\mu;L} - \; a.s.$$

Proof. The derivation of this result from Theorem A.1 is a minor variant of the standard $L \equiv 0$ case (see [16], [22], [5]). We give only a brief sketch. Take $f = \lambda\phi$ in (A.2), differentiate with respect to λ, and let $\lambda \downarrow 0$ to see that

$$\mathbb{Q}^{\tau,\mu;L}[X_t(\phi_t)] = \mu(P_{\tau,t}(\phi)) + \int_\tau^t \int P_{s,t}\phi(z) \, L(ds, dz), \; \forall t \geq \tau, \; \phi \in b\mathcal{B}(S^o). \tag{A.14}$$

The Markov property now easily shows that $M_t(\phi)$ is a martingale. It is a.s. continuous by Theorem A.1(b).

Differentiating (A.2) twice leads to

$$\begin{aligned} \mathbb{Q}^{\tau,\mu;L}\left[X_t(\phi_t)^2\right] &= \mathbb{Q}^{\tau,\mu;L}[X_t(\phi_t)]^2 + \int_\tau^t \mu(P_{\tau,s}((P_{s,t}\phi)^2)) \, ds \\ &+ \int_\tau^t \left[\int_\tau^u \int P_{s,u}((P_{u,t}\phi)^2)(z) \, L(ds, dz)\right] du, \; \phi \in b\mathcal{B}(S^o). \end{aligned} \tag{A.15}$$

This, the fact that $M_t(\phi)$ is a martingale and a short calculation give, for $(\phi, \psi) \in A$,

$$\begin{aligned} \mathbb{Q}^{\tau,\mu;L}\left[X_t(\phi_t)^2\right] &= \mu(\phi_\tau)^2 + \int_\tau^t \int 2\phi(s,z) \mathbb{Q}^{\tau,\mu;L}[X_s(\phi_t)] \, L(ds, dz) \\ &+ \int_\tau^t \mathbb{Q}^{\tau,\mu;L}\left[2X_s(\phi_s)X_s(\psi_s) + X_s(\phi_s^2)\right] ds. \end{aligned}$$

It is now easy to use the Markov property to obtain the canonical decomposition of the continuous semimartingale $X_t(\phi_t)^2$. Compare this with that obtained from (A.13) by Itô's lemma to derive the required expression for $\langle M(\phi) \rangle_t$.

□

B Conditional Poisson point processes

The following result can be proved using an extension of the ideas in the stochastic calculus proofs of the martingale problem characterisation of Poisson point processes (see, for example, Theorem II.6.2 of [25]).

Proposition B.1 *Let $(E, \mathcal{E})$ be a measurable space and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space with right–continuous filtration. Suppose that X is a random variable and N and L are random measures on $]0, \infty[\times E$ such that:*
(i) almost surely N is integer-valued and all the atoms of N have mass 1;
(ii) for any $B \in \mathcal{E}$ the process $t \mapsto N(]0,t] \times B)$ is $(\mathcal{F}_t)_{t \geq 0}$-adapted;
(iii) if $B_1, \ldots, B_n \in \mathcal{E}$ are disjoint, then almost surely the sets $\{t > 0 : N(\{t\} \times B_i) > 0\}$, $1 \leq i \leq n$, are disjoint;
(iv) the random variable X is $\mathcal{F}_0$-measurable and $X > 0$ a.s.;
(v) for each $t > 0$ and $B \in \mathcal{E}$ the random variable $L(]0,t] \times B)$ is $\mathcal{F}_0$-measurable;
(vi) for any $B \in \mathcal{E}$ the process $t \mapsto L(]0,t] \times B)$ is almost surely continuous;
(vii) $\mathbb{P}\left[X L(]0,t] \times E)\right] < \infty$ for all $t > 0$;
(viii) for any $B \in \mathcal{E}$ the process $t \mapsto X(N(]0,t] \times B) - L(]0,t] \times B))$ is an $(\mathcal{F}_t)_{t \geq 0}$–martingale.
Then the conditional distribution of N given $\mathcal{F}_0$ is that of a Poisson point process with intensity L. That is, for a bounded $\mathcal{F}_0$-measurable random variable Y, disjoint sets $A_1, \ldots, A_n \in \mathcal{B}(]0,\infty[) \times \mathcal{E}$, and constants $\lambda_1, \ldots \lambda_n \geq 0$ we have

$$\mathbb{P}\left[Y \exp(-\sum_i \lambda_i N(A_i))\right] = \mathbb{P}\left[Y \exp(-\sum_i L(A_i)(1 - \exp(-\lambda_i)))\right].$$

Corollary B.2 *Suppose that $(E, \mathcal{E})$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ are as in Proposition B.1 and (X_k, N_k, L_k), $1 \leq k \leq m$, is a collection of triples that each satisfy the conditions of that result. Suppose further that almost surely the sets $\{t > 0 : N_k(\{t\} \times E) > 0\}$, $1 \leq k \leq m$, are disjoint. Then $N_1, \ldots, N_m$ are conditionally independent given $\mathcal{F}_0$.*

Proof. This follows by applying the previous result with E replaced by the disjoint union of m copies of E, and with N and L being the measures that coincide with N_k and L_k when restricted to the product of $]0,\infty[$ and the k^{th} copy of E.

□

C Parametrised conditional probabilities

The following lemma shows that if we have a measurably parametrised family of probability measures, then it is possible to choose a measurably parametrised family of regular conditional probabilities given some random variable.

Proposition C.1 *Let $(S, \mathcal{S})$ be a Lusin measurable space, $(T, \mathcal{T})$ be a Polish space equipped with its Borel σ–field, and $(E, \mathcal{E})$ be a separable metric space equipped with its Borel σ–field. Let $(M_1, \mathcal{M}_1)$ denote the space of probability measures on T equipped with the Borel σ–field induced by the topology of weak convergence. Suppose that $s \mapsto \mathbb{Q}_s$ is a $\mathcal{S}\backslash\mathcal{M}_1$–measurable map from S to M_1, and Z is a $\mathcal{T}\backslash\mathcal{E}$–measurable map from T to E. Then there exists a $(\mathcal{S} \times \mathcal{E})\backslash\mathcal{M}_1$–measurable map from $S \times E$ to M_1, which we will denote by $(s, e) \mapsto \mathbb{Q}_s(\cdot \mid Z = e)$, such that for all $s \in S$, $A \in \mathcal{T}$, and $B \in \mathcal{E}$, we have*

$$\int_B \mathbb{Q}_s(A \mid Z = e)\mathbb{Q}_s(Z \in de) = \mathbb{Q}(A \cap \{Z \in B\}).$$

Proof. Our proof is basically an elaboration of one of the usual proofs of the existence of regular conditional probabilities using the martingale convergence theorem. As T is Borel isomorphic to $\mathbb{R}, \mathbb{N}$ or $\{1, \cdots, n\}$ for some $n \in \mathbb{N}$, we may suppose without loss of generality that $T = \mathbb{R}$.

Let $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$ be an increasing sequence of finite sub–σ–fields of $\mathcal{E}$ such that $\mathcal{E} = \bigvee_n \mathcal{E}_n$, and let $e \mapsto [e]_n$ be an $\mathcal{E}_n\backslash\mathcal{E}$–measurable map from E to E such that e and $[e]_n$ belong to the same atom of $\mathcal{E}_n$. Put

$$F_{s,e,n}(r) = \begin{cases} \mathbb{Q}_s(]-\infty, r] \cap \{[Z]_n = [e]_n\})/\mathbb{Q}_s(\{[Z]_n = [e]_n\}), & \text{if } \mathbb{Q}_s(\{[Z]_n = [e]_n\}) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that for each fixed n and r, $(s, e) \mapsto F_{s,e,n}(r)$ is $\mathcal{S} \times \mathcal{E}_n$–measurable.

Put

$$F_{s,e}(r) = \begin{cases} \lim_{n \to \infty} F_{s,e,n}(r), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus $(s, e) \mapsto F_{s,e}(r)$ is $\mathcal{S} \times \mathcal{E}$–measurable. Let $\Lambda \subset S \times E$ denote the set of (s, e) such that: for all $r \in \mathbb{Q}$, $F_{s,e}(r) = \lim_{n \to \infty} F_{s,e,n}(r)$; for all $q, r \in \mathbb{Q}$ with $q < r$, $F_{s,e}(q) \leq F_{s,e}(r)$; $\lim_{r \to \infty, r \in \mathbb{Q}} F_{s,e}(r) = 1$; and $\lim_{r \to -\infty, r \in \mathbb{Q}} F_{s,e}(r) = 0$. By the martingale convergence theorem, for all $s \in S$ the set $\{e \in E : (s, e) \in \Lambda\}$ has full measure with respect to the measure $\mathbb{Q}_s(Z \in de)$, and for all $r \in \mathbb{Q}$ and $B \in \mathcal{E}$

$$\mathbb{Q}_s(]-\infty, r] \cap \{Z \in B\}) = \int_B F_{s,e}(r)\mathbb{Q}_s(\{Z \in de\}).$$

For $(s,e) \in S \times E$ and $x \in \mathbb{R}$ set

$$G_{s,e}(x) = \begin{cases} \lim_{r \downarrow x,\, r \in \mathbb{Q}} F_{s,e}(r), & \text{if } (s,e) \in \Lambda, \\ \mathbf{1}\{x \geq 0\}, & \text{otherwise.} \end{cases}$$

Then $G_{s,e}$ is a distribution function for all (s,e). Let $P_{s,e}$ denote the corresponding measure on $\mathbb{R}$. A monotone class argument shows that $(s,e) \mapsto P_{s,e}(A)$ is $\mathcal{S} \times \mathcal{E}$–measurable for all Borel $A \subset \mathbb{R}$. Thus $(s,e) \mapsto P_{s,e}$ is $(\mathcal{S} \times \mathcal{E}) \backslash \mathcal{M}_1$–measurable.

By construction and the bounded convergence theorem, for all $x \in \mathbb{R}$ and $B \in \mathcal{E}$

$$\begin{aligned} \int_B P_{s,e}(]-\infty, x])\mathbb{Q}_s(\{Z \in de\}) &= \lim_{r \downarrow x,\, r \in \mathbb{Q}} \int_B F_{s,e}(r) \mathbb{Q}_s(\{Z \in de\}) \\ &= \lim_{r \downarrow x,\, r \in \mathbb{Q}} \mathbb{Q}_s(]-\infty, r] \cap \{Z \in B\}) \\ &= \mathbb{Q}_s(]-\infty, x] \cap \{Z \in B\}). \end{aligned}$$

A monotone class argument shows that for all Borel $A \subset \mathbb{R}$ and $B \in \mathcal{E}$

$$\int_B P_{s,e}(A) \mathbb{Q}_s(\{Z \in de\}) = \mathbb{Q}_s(A \cap \{Z \in B\}).$$

We can therefore set $\mathbb{Q}_s(\cdot \mid Z = e) = P_{s,e}$.

$\square$

D Partial list of notation

$\bar{\Delta}$	Subsection 1.2
D_S, D_1, D_{ST}	Subsection 1.2
A	Subsection 1.2
S^o	Subsection 1.2
$\mathbf{M}_F$, $\mathbf{M}_F(C)^t$	Subsection 1.2
$\Omega_H[\tau,\infty[$, $\Omega_X[\tau,\infty[$	Subsection 1.2
$\mathbb{Q}^{\tau,\mu}$, $\mathbb{P}^{\tau,m}$	Subsection 1.2
Γ	Subsection 1.2
$L_t(K^1,K^2)$	Subsection 1.3
$\mathbf{M}_{FS}(\mathbb{R}^d)$, $\mathbf{M}_{FS}(C)^t$	Subsection 1.4
S'_C, $\hat{S}$, Ω'_C, α_C, β_C	Subsection 1.4
$\widehat{MP}_H(\tau,\mu^1,\mu^2)$, $\widehat{MP}_X(\tau,\nu^1,\nu^2)$	Subsection 1.5
$\hat{\mathbb{P}}^{\tau,\mu^1,\mu^2}$	Subsection 1.5
$\hat{\Omega}_X[\tau,\infty[$, $\Omega_{XS}[\tau,\infty[$, $\Omega^g_{XS}[\tau,\infty[$	Section 2
$\mathbf{M}^g_{FS}(\mathbb{R}^d)$	Section 2
$W=(T,B_0)$, $Q^{s,x}$, P^x	Section 2
$\hat{\ell}_t(y,\nu)$, $\ell_t(y,\nu)$	Section 2
$h_\alpha(x)$, $p_s(x)$	Section 2
$\tilde{\Delta}$, $\vec{\nabla}$	Section 2
$\ell_t(y,X)$	Subsection 3.1
Ω^o, α^o, β^o	Subsection 3.2
$(y/s/\omega)$, $\bar{w}(s,y,\omega)(t)$	Subsection 3.2
$P^{s,y}$	Subsection 3.2
S^2, P^2_t	Subsection 3.2
$\mathbf{M}_\#$, $E^\#$, $S^\#$, $E^\#_t$, $\Omega^\#$	Subsection 4.1
S', Ω', α', β', $\mathcal{F}'$, $\mathcal{F}'_{[s,t+]}$, $\mathcal{F}'_t$	Subsection 4.1
$\breve{\mathbb{Q}}^{\tau,\breve{\mu}}_\nu$	Subsection 4.1
$\breve{H}_t$	Subsection 4.1
$\breve{\mathcal{D}}$, $\breve{\mathcal{D}}_t$	Subsection 4.1
$L_t(\breve{H},\nu)$	Subsection 4.1
Ω'_H, $\mathcal{F}^H_t$	Subsection 4.2
π, Π	Subsection 4.2
$(\breve{H}^1,\breve{H}^2)$	Subsection 4.2
$\mathcal{F}''_t$	Subsection 4.2
$L_t(\breve{H}^1,\breve{H}^2)$	Subsection 4.2
$(\tilde{H}^1,\tilde{H}^2)$	Section 5
$\hat{\mathcal{H}}_t$, $\hat{\mathcal{H}}^C_t$	Section 5
I^i	Section 5
$\tilde{E}$, $\tilde{S}^o$, $\tilde{E}^\partial$, $\tilde{E}_t$, $\tilde{\Omega}^o$, $\tilde{F}^o$	Subsection 6.1
$\bar{z}(s,y,\xi,\omega)(t)$	Subsection 6.1
$P^{s,y,\xi}$	Subsection 6.1
Ω'_1, α'_1, β'_1, $\mathcal{F}'_1$, ${\mathcal{F}^1}'_{[s,t+]}$	Subsection 6.1
$\mathbf{M}_*$	Subsection 6.1
$\bar{S}^o$, $\bar{E}$, $\bar{E}_t$, $(\bar{H}^1,\bar{H}^2)$	Subsection 6.1
P^W_t, P^Z_t	Subsection 6.1
$H^i_{r,t}$, $\bar{H}^i_{r,t}$, $\breve{H}^i_{r,t}$	Subsection 6.3
$S^i_{r,t}$, $\bar{S}^i_{r,t}$, $\breve{S}^i_{r,t}$	Subsection 6.3
$\breve{H}^{i,\epsilon}_t$	Subsection 6.3
$\mathbf{M}^u_\#$	Subsection 6.3
$H^{i,\epsilon}_t$, $\bar{H}^{i,\epsilon}_t$, $\hat{H}^{i,\epsilon}_t$	Section 7
$\mathcal{H}^0$, $\mathcal{H}^0_t$	Section 7

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